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The tail term of the generalized Lorentz-Abraham-Dirac equation

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Preface

I here present several minor, but new (to my knowledge) results pertaining to the generalized Lorentz-Abraham-Dirac equation and its tail term. These are:

- The tail term for a hyperbolically accelerated particle in the Rindler frame in a universe with cylindrical topology.
- The generalized Lorentz-Abraham-Dirac equation does not work in 4+1 dimensions.
- An equation of motion for a hyperbolically accelerated charge in 4+1 dimensions.
- A derivation of the electromagnetic wave equation in Schwarzschild coordinates.
- A graphical illustration of the Green function near a black hole.

I have also performed new derivations of some already known results:

- The Green function in 4+1-dimensional flat spacetime.
- The scalar Green function in dust, radiation and LIVE dominated universes.

Conceptually, the most important points are

- The splitting of the tail term into a topological and "tail" part.
- Both parts of the tail term can occur without (local) curvature.

Some results that I originally believed to be new, later turned out to be known already. Most notable of these, is the equation of motion for a hyperbolically accelerated charge in 4+1 dimensions. I here get a somewhat worrying divergent result, but later found that this is corroborated by Gal'tsov [10].

Acknowledgments

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Chapter 1

Introduction

The motion of charged particles is a complicated matter, and we have several equations to describe this motion, with varying levels of explicitness and generality. In its most basic form, the motion is governed by the Lorenz force

$$f^{\mu} = F^{\mu}{}_{\nu} j^{\nu} \tag{1.1}$$

where f^{μ} are the components of the 4-force, $F^{\mu\nu}$ is the electromagnetic field tensor, and $j^{\nu} = qu^{\nu}$ is the 4-current. The simplicity of this equation is deceptive, much like the famous $G_{\mu\nu} = 8\pi T_{\mu\nu}$, as the particle itself is a source of the electromagnetic field. This complexity is made explicit and solved in the Lorentz-Abraham-Dirac equation.

$$f^{\mu} = qF^{\mathrm{ext.}\mu}_{\nu} u^{\nu} + \frac{2}{3}q^2 \left(\frac{Da^{\mu}}{D\tau} - u^{\mu} a^{\nu} a_{\nu} \right) \tag{1.2}$$

where $\frac{Da^{\mu}}{D\tau} \equiv a^{\mu}_{;\nu} u^{\nu}$, the directional derivative of the acceleration a along the world line, is called the absolute derivative of a. The part of $F^{\mu\nu}$ arising from the particle itself has here been separated from the external field, and its effect calculated. Equation (1.2) was originally found by Dirac [7]. A non-covariant version of this equation was first found by Abraham, and the term in the parentheses is therefore called the Abraham vector.

Equation (1.2) includes derivatives of the acceleration, and is therefore a third order differential equation, in contrast to most equations of motion, which are first or second order, and this causes it to possess runaway solutions, where a particle ends up accelerating exponentially even without the presence of an external force[7]. However, Flanagan and Wald [8] have shown that it is consistent with the limits of validity for classical electrodynamics to replace the occurrences of the acceleration on the right hand side with $\frac{f^{\mu}_{\mathrm{ext.}}}{m}$, yielding the second order equation

$$f^{\mu} = f^{\mu}_{\mathrm{ext.}} + \frac{2q^2}{3m}(\delta^{\mu}_{\nu} + u^{\mu}u_{\nu}) \frac{Df^{\nu}_{\mathrm{ext.}}}{D\tau} \tag{1.3}$$

Let us consider the implications of this equation for an example. A particle in a flat background is kept in a circular orbit $x^\mu = (t, r_0, \frac{\pi}{2}, \omega t)$, giving it a velocity of $u^\mu = \gamma(1,0,0,\omega)$ with $\gamma = (1 - r_0^2\omega^2)^{-\frac{1}{2}}$. The acceleration corresponding to this is $a^\mu = \frac{Du^\mu}{D\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\beta}u^\alpha u^\beta = \gamma^2(0, -r_0\omega^2, 0, 0)$, using the Christoffel symbols for a polar coordinate system (these can be found in the appendix section 9.1). This acceleration is provided by the external force, so $f^\mu_{\text{ext.}} = ma^\mu$, and $\frac{Df^\mu_{\text{ext.}}}{D\tau} = m\left(\frac{da^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\beta}a^\alpha u^\beta\right) = (0,0,0,-m\gamma^3\omega^3)$. The correction to the force, which is the second term in equation (1.3), is then $\Delta f^\mu = \frac{2q^2}{3}\gamma^3\omega^3(r_0^2\gamma\omega, 0, 0, -(1 + r_0^2\gamma^2\omega^2))$.

Thus, for a circular orbit caused by some external centripetal force, the total force gets a small component pointing in the opposite direction of the orbit, leading the particle to slowly spiral inwards. The energy thus lost by the particle is emitted as radiation. Qualitatively, the mechanism for this emission can be understood as follows: A particle at rest with respect to an observer sets up a Coulomb field, with field lines pointing radially outwards. Let us now give the particle a small jerk, shifting its position. A Coulomb field centered on the new location will start radiating out from the particle. However, the field only moves with the speed of light, and so further away from the particle we must still have the old field. At the border between these to regions, there will be an area where the field changes rapidly. Associated with this rapid change is an energy density, which thus moves outwards together with this area. This is radiation. This is illustrated in figure 1.1

In the present case, the circular motion of the charged particle causes similar readjustment of the field lines. So far, there is no problem. However, what happens if the circular motion is caused by gravity? The qualitative argument for the existence of radiation is unchanged by this, but as gravity is not a force in general relativity, the particle is now in free fall, $a^\mu = 0$, and $f^\mu_{\text{ext.}} = 0$, so the Abraham vector vanishes too, so the total force would seem to be

$$f^\mu = 0 \tag{1.4}$$

which means that the particle should follow geodetic motion, for example the circular orbit we consider. This is a recipe for a perpetuum mobile: The charge returns to the same state after each orbit, but has during the same period radiated a nonzero amount of energy.

An obvious weakness in this argument is the part regarding the radiation, which is by no means rigid. Have we not, one might ask, a proper equation for the radiated energy, so that one can do away with all the hand-waving? The Larmor equation is such an equation:

$$P_{EM} = \frac{2}{3}q^2 a^\mu a_\mu \tag{1.5}$$

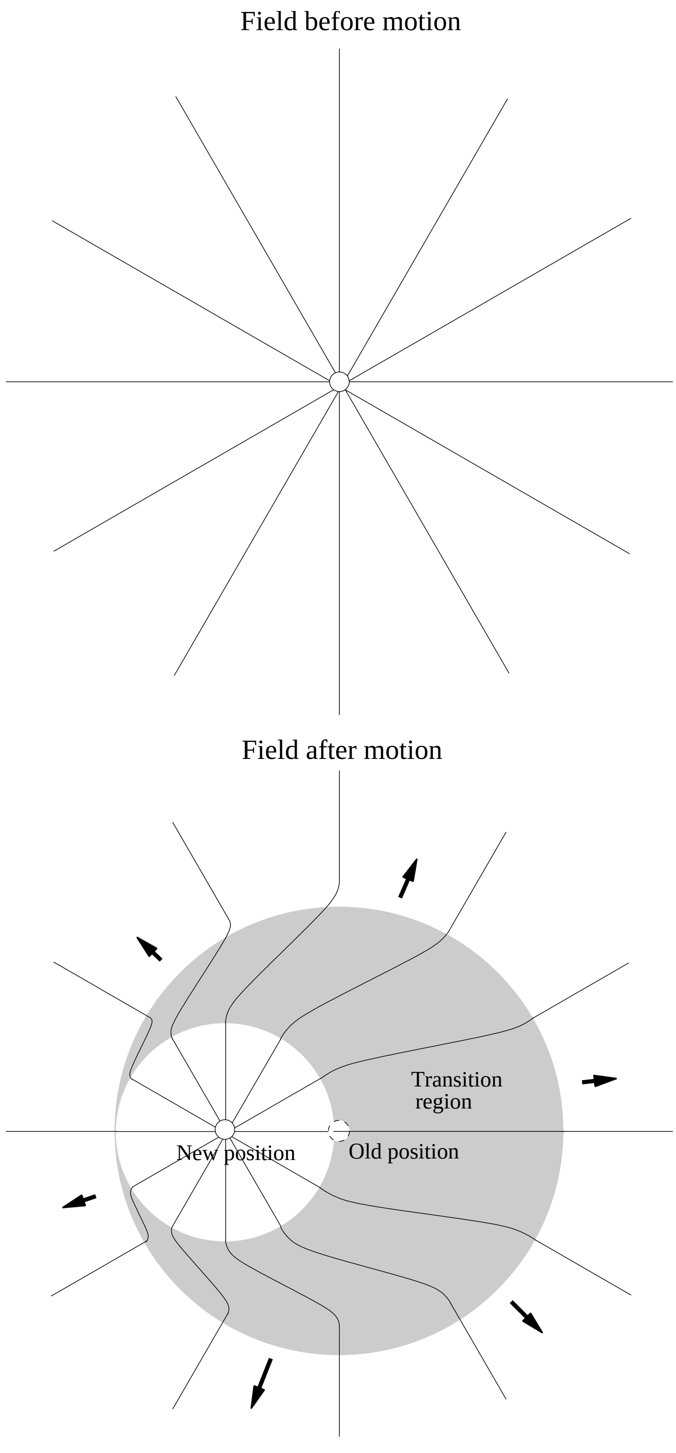

Figure 1.1: The radiation caused by an accelerated charge lies in the rearrangement of the field lines to compensate for the new position.

Applying this to the problem at hand, we immediately find a radiated effect of zero, which is consistent with the equation of motion.

Is this, then, the solution to our problem? No. Both the LAD equation and the Larmor equations were derived in special relativity, and it is inconsistent to use them together with gravity. To resolve the problem properly, we need generalized versions of these equations.

In the sixties, DeWitt and Brehme [5], Hobbs [12] generalized the LAD equation to general relativity, and found

$$\begin{aligned} f^{\mu} =& qF^{\text{in}\mu}_{\nu}u^{\nu} + \frac{2}{3}q^2\left(\frac{Da^{\mu}}{D\tau} - u^{\mu}a^{\nu}a_{\nu}\right) \\ &-\frac{1}{3}q^2\left(R^{\mu}{}_{\nu}u^{\nu} + R_{\alpha\beta}u^{\alpha}u^{\beta}u^{\nu}\right) + q^2u^{\nu}\int_{-\infty}^{\tau} f^{\mu}_{\nu\gamma'}u^{\gamma'}d\tau' \end{aligned} \tag{1.6}$$

where $f^{\mu}{}_{\nu\gamma'} \equiv G_{\nu\gamma'}{}^{;\mu} - G^{\mu}{}_{\nu\gamma'}$. There are two changes here compared to the normal LAD equation: There is an extra term containing the Ricci tensor, which does not present too much of a problem, and a more notable term, called the "tail term", containing an integral over the past history of the particle.

A weaker generalization exists for the Larmor formula. Fugmann and Kretzschmar [9] have generalized it to hold for non-inertial observers, but only in flat spacetime. They find that the relative acceleration between emitter and observer, and not the four-acceleration, is what matters for the radiated energy. Their formula is not applicable to curved spacetime, and to my knowledge a usable formula has yet to be found. Due to the connection between the equation and motion and the emitted energy, one can expect such a generalized formula to contain a "tail term" of its own.

The "tail term", not only introduces immense practical difficulties in calculating the motion of a charged particle, but also presents a difficulty for intuitive understanding. The focus of this work will be to investigate and explain this term, and to do this, I will examine several cases which each highlight different aspects of the tail term, while staying as simple as possible. These cases are:

- Tail term from topology
- Tail term from extra dimensions
- Tail term from curvature

At the end, I will also touch briefly on the case of the tail term for a particle near a black hole, which exhibits properties from both the first and last of the simpler examples, at the cost of having to be dealt with numerically.

Before we can start with this, though, it is necessary with an introduction to a few important concepts.

Chapter 2

Background theory

2.1 Tensor comparisons and bitensors

When specifying a tensor in a given coordinate system, it is not only necessary to give its components, but also the position of the tensor. To illustrate why, consider two particles at opposite points of a circular orbit, as seen in a co-rotating polar coordinate system. The particles' 4-velocity will then have identical components, and thus if one ignored the position of the vectors, one would wrongly conclude that the vectors themselves are equal.

To compare two tensors at different points, we need to transport one of them along some path to the other, and the result of the comparison will in fact depend on the chosen path. This non-definiteness may seem problematic, but since information has to get from one point in spacetime to another by traversing the interval, it is clear that the path to be used is dictated by the interaction mediating the information. For example, if an electromagnetic field carries the information, the path used will be the null geodesic connecting the two points, and the transport involved would be parallel transport. If the interaction involved does not follow geodesics, the more general Fermi-Walker transport would have to be used.

In some cases, information will get from one point to another along several different paths, and in these cases one simply uses a weighed sum over all the paths.

The most commonly considered case, is when the information follows a unique geodesic from one point to another. In these special circumstances, it is possible to define a quantity that lets one directly compare components of tensors at different positions.

Given an orthonormal set of basis vectors e_a^{μ} defined at the point $\mathbf{x}$, one parallel transports them along the unique geodesic to the point $\mathbf{x}'$, obtaining $e_a^{\mu'}$, where the primed indices indicate that they belong at the primed point. The index a is here not a normal index, but just a label for which basis vector we are dealing with. It is still convenient, however, to be able to raise and lower these labels, and we therefore choose (quite arbitrarily, but following the norm) to use the Minkowski metric η for this, giving $e^a_\mu = g_{\mu\nu}\eta^{ab}e^\nu_b$. We can now decompose an arbitrary vector A in this basis: $A^\mu = A^a e^\mu_a$, and parallel transport of A from $\mathbf{x}$ to $\mathbf{x}'$ is achieved simply by parallel transport of the basis vectors: $A^{\mu'} = A^a e^{\mu'}_a$. A^a is given in terms of the original coordinates as $A^a = e^a_\mu A^\mu$, which lets us describe the primed coordinates as a function of the unprimed ones:

$$A^{\mu'} = e^a_\mu e^{\mu'}_a A^\mu \equiv g^{\mu'}{}_\mu \tag{2.1}$$

The object $g^{\mu'}{}_\mu$, called the parallel propagator, thus translates between indices at two different positions. It is a product of a tensor at each of those positions, and is thus itself a tensor with respect to two positions, and is therefore called a *bitensor*.

A more well-known example of a bitensor is the geodetic interval between two events, s, computed by integrating $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. In the context of bitensors, however, it is common to use the biscalar called Synge's world function σ, defined as

$$\sigma \equiv \frac{1}{2}s^2 \tag{2.2}$$

as this avoids taking the square root of a negative number.

2.2 The local convex neighborhood

As mentioned, the parallel propagator can only be defined uniquely in the area where the path information takes between two points is unique, and this is in general the case for all bitensors. When the path is a geodesic, this set of points $\mathbf{x}$ that are connected with a unique path to a fixed point $\mathbf{x}'$, is called the local convex neighborhood of $\mathbf{x}'$ (see figure (2.1)).

No information can propagate faster than the speed of light, and thus the definition of the local convex neighborhood (LCN) I gave above, which is based on the path information takes from one point to another, would exclude space-like geodesics from consideration. This results in a larger LCN than one would get if space-like geodesics are included.

To illustrate, consider the LCN of a particle in 1+1-dimensional flat spacetime, where the spatial dimension is periodic, giving a cylinder-shaped spacetime. As figure (2.2) illustrates, the LCN is part of the area inside the light cone if space-like paths are excluded, and the empty set otherwise. I will be excluding light-like paths unless otherwise noted in this paper.

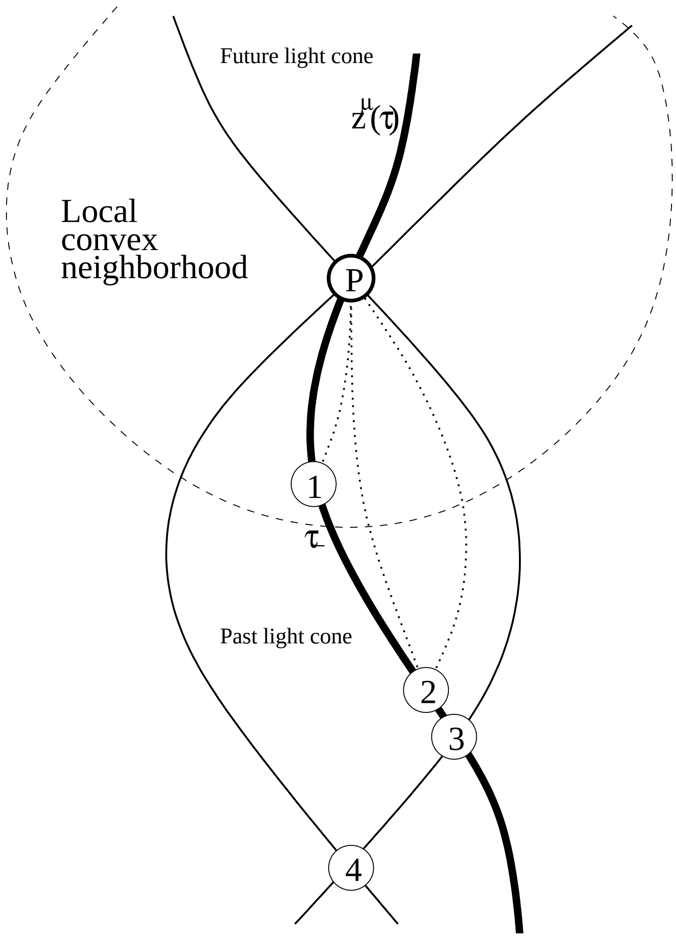

Figure 2.1: Illustration of the local convex neighborhood around the point P. The central, thick line is the world line of a particle, and the local convex neighborhood is the area within the dotted enclosure. Within this area, there is a unique geodesic between P and any point, illustrated with point 1. Outside it, this no longer holds, and in this stylized example, there are two geodesics between P and 2. The figure also illustrates phenomena that may occur outside the LCN: the overlap of different parts of the light cone, at 4, and the intersection of the light cone with the world line of the particle, at 3.

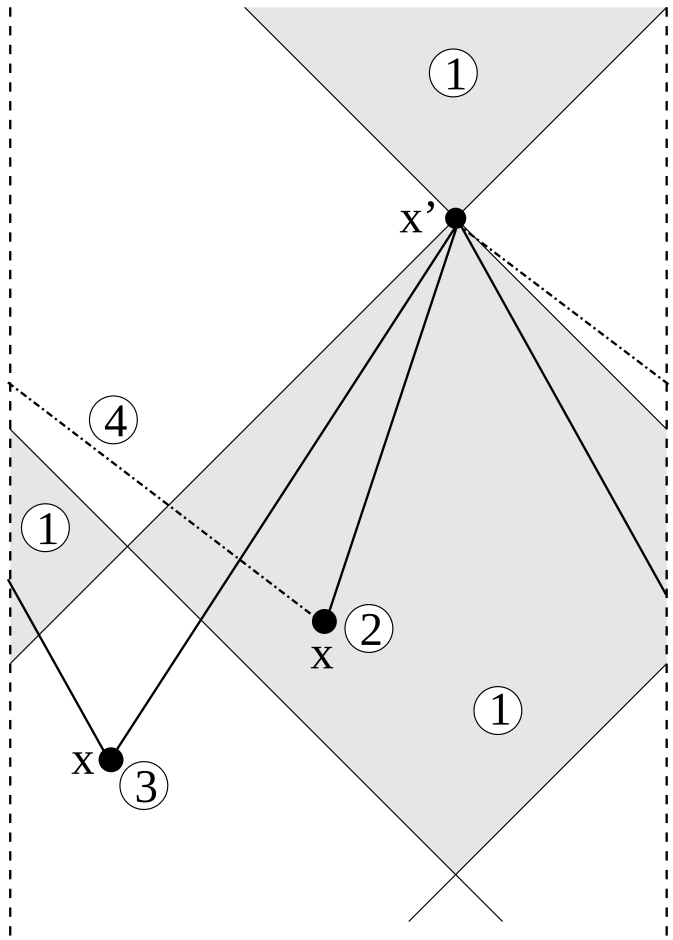

Figure 2.2: Illustration of the effect of excluding or including space-like geodesics when computing the LCN. The gray areas marked 1 make out the LCN with space-like geodesics excluded. There is only one non-space-like geodesic connecting $\mathbf{x}$ at 2 to $\mathbf{x}'$, and so 2 is in $\mathbf{x}'$'s LCN. $\mathbf{x}$ at 3, however, is connected to $\mathbf{x}'$ by two different non-space-like geodesics, and is thus outside the LCN. The dotted line at 4 shows what happens when space-like geodesics are included: 2 is now outside the LCN, as there are more than one geodesic connecting it to $\mathbf{x}'$. In this case there is actually an infinity of space-like geodesics connecting any two points, and so the LCN is the empty set if these are allowed.

2.3 Coordinate systems and notation

I will make use of several coordinate systems. Notation-wise, these will be distinguished by text decoration like overlines and hats, which will be used not only for the coordinates themselves, but also for the corresponding indices. I will sometimes be using more indices on a tensor than it really has; these should be read as covariant derivatives. Partial derivatives will use the normal comma notation. Thus, since σ is a scalar, $\sigma_\mu = \sigma_{;\mu}$, and for a vector A^μ, $A^\mu{}_{\nu\rho} = A^\mu{}_{;\nu\rho}$, and so on. Additional text decoration will be used to refer to different locations in spacetime. For example, A^x would be the x component of $\mathbf{A}$ at the point $\mathbf{x}$, while $A^{x'}$ is the x component at the point $\mathbf{x}'$. When the coordinate system does not matter, undecorated letters will be used.

2.3.1 Minkowski coordinates

Minkowski coordinates are rectangular (Cartesian) coordinates in Minkowski spacetime. These coordinates will be indicated with an overline: $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$. The metric here is $g_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$, and all Christoffel symbols are 0. In these coordinates, the parallel propagator $g^{\mu'}{}_\mu$ is simply the metric, as parallel transport has no effect there. Synge's world function $\sigma(\mathbf{x}, \mathbf{x}') = \frac{1}{2}(\mathbf{x} - \mathbf{x}')^2$, and its derivatives are $\sigma_\mu(\mathbf{x}, \mathbf{x}') = x_\mu - x'_\mu$, $\sigma_{\mu'}(\mathbf{x}, \mathbf{x}') = x'_{\mu'} - x_{\mu'}$. Primed and unprimed coordinates are interchangeable in these coordinates.

2.3.2 Rindler coordinates

Rindler coordinates are coordinates in a rectangular coordinate system in a Rindler frame, which is a Born-rigid frame of reference where each reference particle has constant acceleration, and is the natural frame of reference when studying the dynamics of a hyperbolically accelerated particle. Rindler coordinates will be denoted using undecorated letters, so rectangular coordinates in the Rindler frame are (t, x, y, z).

The connection between Minkowski coordinates and rectangular coordinates in the Rindler frame is given by

$$\begin{aligned} t &= x_0 \tanh^{-1} \tfrac{\bar{t}}{\bar{x}} & x &= \sqrt{\bar{x}^2 - \bar{t}^2} \\ \bar{t} &= x \sinh \tfrac{t}{x_0} & \bar{x} &= x \cosh \tfrac{t}{x_0} \end{aligned} \tag{2.3}$$

the other coordinates being the same, x_0 is a scale factor between the two sets of coordinates. To keep the reference frame stiff, the reference particles are accelerating more strongly at low values of x; in fact, the acceleration at a given point (in units where $c = 1$) is simply x^{-1}. For particles in hyperbolic motion, I will use $x = x_0$ at the particle position for simplicity, so the acceleration of such a particle is x_0^{-1} The line element is $ds^2 = -\frac{x^2}{x_0^2}dt^2 + dx^2 + dy^2 + \ldots$. For tensor transformations between Rindler and Minkowski, we need

$$\begin{array}{rlrlrlrl}
\frac{\partial t}{\partial \bar{t}} = & \frac{x_0}{x}\cosh\frac{t}{x_0} & \frac{\partial t}{\partial \bar{x}} = & -\frac{x_0}{x}\sinh\frac{t}{x_0} & \frac{\partial x}{\partial \bar{t}} = & -\sinh\frac{t}{x_0} & \frac{\partial x}{\partial \bar{x}} = & \cosh\frac{t}{x_0} \\
\frac{\partial \bar{t}}{\partial t} = & \frac{x}{x_0}\cosh\frac{t}{x_0} & \frac{\partial \bar{t}}{\partial x} = & \sinh\frac{t}{x_0} & \frac{\partial \bar{x}}{\partial t} = & \frac{x}{x_0}\sinh\frac{t}{x_0} & \frac{\partial \bar{x}}{\partial x} = & \cosh\frac{t}{x_0}
\end{array} \tag{2.4}$$

In Rindler coordinates, we find the parallel propagator by using $g^{\mu}{}_{\mu'} = e^{\mu}_{a} e^{a}_{\mu'}$. The simplest way of doing this, is to start with the basis vectors of Minkowski coordinates, transport them to $\mathbf{x}$ and $\mathbf{x}'$ (which does not change them), and then transform to Rindler coordinates. We have

$$e^{\overline{\mu}}_{a} = \delta^{\overline{\mu}}_{a} \qquad\qquad e^{\overline{\mu}'}_{a} = \delta^{\overline{\mu}'}_{a} \tag{2.5}$$

where $\delta^{\overline{\mu}}_{a}$ is the Kronecker delta for the coordinate index $\overline{\mu}$ and the label a, and is 1 if they have the same value, and 0 otherwise. This gives

$$e^{\mu}_{t} = \left(\frac{x_0}{x}\cosh\frac{t}{x_0}, -\sinh\frac{t}{x_0}, 0, 0\right) \tag{2.6}$$

$$e^{\mu}_{x} = \left(-\frac{x_0}{x}\sinh\frac{t}{x_0}, \cosh\frac{t}{x_0}, 0, 0\right) \tag{2.7}$$

$$e^{\mu}_{y|z} = e^{\mu'}_{y|z} \tag{2.8}$$

for the point $\mathbf{x}$ and the same, with x, t replaced with x', t' at the point $\mathbf{x}'$. Contracting, we find

$$g^{t}{}_{t'} = \frac{x'}{x}\left(\cosh\frac{t}{x_0}\cosh\frac{t'}{x_0} - \sinh\frac{t}{x_0}\sinh\frac{t'}{x_0}\right) \tag{2.9}$$

$$g^{t}{}_{x'} = \frac{x_0}{x}\left(\cosh\frac{t}{x_0}\sinh\frac{t'}{x_0} - \sinh\frac{t}{x_0}\cosh\frac{t'}{x_0}\right) \tag{2.10}$$

$$g^{x}{}_{t'} = -\frac{x'}{x_0}\left(\sinh\frac{t}{x_0}\cosh\frac{t'}{x_0} - \cosh\frac{t}{x_0}\sinh\frac{t'}{x_0}\right) \tag{2.11}$$

$$g^{x}{}_{x'} = -\left(\sinh\frac{t}{x_0}\sinh t' x_0 - \cosh\frac{t}{x_0}\cosh t' x_0\right) \tag{2.12}$$

$$g^{y|z}{}_{y'|z'} = \delta^{y|z}{}_{y'|z'} \tag{2.13}$$

which simplifies to

$$g^{\mu}{}_{\mu'} = \begin{pmatrix} \frac{x'}{x}\cosh\frac{t-t'}{x_0} & -\frac{x_0}{x}\sinh\frac{t-t'}{x_0} & 0 & 0 \\ -\frac{x'}{x_0}\sinh\frac{t-t'}{x_0} & \cosh\frac{t-t'}{x_0} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.14}$$

The geodetic interval is a biscalar, we therefore just need to insert the expression for the coordinates into that for σ:

$$\sigma = \frac{1}{2}(-\bar{t}^2 + \bar{x}^2 + \overbrace{\bar{y}^2 + \bar{z}^2}^{\rho^2}) \tag{2.15}$$

$$= \frac{1}{2}\left(x^2 + x'^2 + \rho^2 - 2xx' \cosh \frac{t - t'}{x_0}\right) \tag{2.16}$$

The derivatives are $\sigma_\mu = (-\frac{xx'}{x_0} \sinh \frac{t-t'}{x_0}, x - x' \cosh \frac{t-t'}{x_0}, y, z)$.

In general, the derivative will be will be a vector pointing towards the point where the derivative is being taken, and a squared length equal to twice the geodetic interval itself: $\sigma_\mu \sigma^\mu = 2\sigma$. The derivatives of σ can therefore be used as position vectors of one point, relative to another. This is exploited in Fermi normal coordinates.

2.3.3 Fermi normal coordinates

These somewhat complicated coordinates are well-suited for observers that follow a particle, no matter what its motion is. They have the properties that the metric is always of Minkowski form at the world line they follow (which will always have spatial coordinates of 0), and that areas of constant time coordinate correspond to the surface of simultaneity for an instantly co-moving observer. Additionally, the spatial coordinates correspond to spatial distances as measured along geodesics of simultaneity. Fermi normal coordinates will be denoted by underlines.

To construct Fermi normal coordinates, one first chooses some point $\mathbf{x}'$ along the world line of interest. One here introduces a set of orthonormal basis vectors $e^\mu_{\underline{\mu}}$, with the restriction that $\mathbf{e}_{\underline{t}} = \mathbf{u}$, the four-velocity at the chosen point. Note that the underlined indices are not covariant or contravariant indices, but just labels. Basis vectors for every other point on the world line are then constructed using Fermi-Walker transport:

$$\frac{De^\mu_{\underline{\mu}}}{D\tau} = e_{\underline{\mu}\nu} a^\nu u^\mu - e_{\underline{\mu}\nu} u^\nu a^\mu \tag{2.17}$$

where $\frac{De^\mu_{\underline{\mu}}}{D\tau} = \frac{\partial e^\mu_{\underline{\mu}}}{d\tau} + \Gamma^\mu{}_{\alpha\beta} e^\alpha_{\underline{\mu}} u^\beta$.

Fermi normal coordinates are then defined as

$$x^{\underline{\mu}} = -\sigma_{\mu''} e^{\underline{\mu}}_{\mu''} \tag{2.18}$$

where $\mathbf{x}''$ is the point on the world line with the property that the geodesic from $\mathbf{x}$ to it intersects it orthogonally.

We will later need these coordinates for a particle with constant acceleration in flat spacetime. In Rindler coordinates, its motion is given by $x^\mu = (t, t_0, 0, \ldots)$, $u^\mu = \delta^\mu_t$, $a^\mu = x_0^{-1}\delta^\mu_x$. The basis vectors at $\mathbf{x}'$ can be chosen as $e^\mu_{\underline{\mu}} = \delta^\mu_{\underline{\mu}}$. Fermi-Walker transport has no effect

$$\frac{de^t_{\underline{\mu}}}{d\tau} = e_{\underline{\mu}x}x_0^{-1} - x_0^{-1}e^x_{\underline{\mu}} = 0 \tag{2.19}$$

$$\frac{de^x_{\underline{\mu}}}{d\tau} = -e_{\underline{\mu}t}x_0^{-1} - x_0^{-1}e^t_{\underline{\mu}} = 0 \tag{2.20}$$

$$\frac{de^\rho_{\underline{\mu}}}{d\tau} = 0 \tag{2.21}$$

where ρ is any of the other coordinates. We then examine the space spanned by parallel transport of all linear combinations ξ^μ of the spatial basis vectors, along themselves.

$$\xi^\mu{}_{;\nu}\xi^\nu = 0 \Leftrightarrow \tag{2.22}$$

$$\xi^\mu{}_{,\nu}\xi^\nu = -\Gamma^\mu{}_{\nu\rho}\xi^\nu\xi^\rho \tag{2.23}$$

All nonzero Christoffel symbols have at least one t among the two last indices, and since ξ^μ does not point in the t direction initially, we get

$$\xi^\mu{}_{,\nu}\xi^\nu = 0 \tag{2.24}$$

so ξ^μ will not be changed during the parallel transport. Thus, any point $\mathbf{x}$ will have a geodesic that intersects the world line at a point $\mathbf{x}''$ with the same t. Recall that we in Rindler coordinates have $\sigma(\mathbf{x}, \mathbf{x}'') = \frac{1}{2}(x^2 + x''^2 + \rho^2 - 2xx''\cosh\frac{\Delta t}{x_0})$. Since we have just established that $\Delta t = 0$, this is simply $\sigma = \frac{1}{2}((x - x'')^2 + (y - y'')^2 + \ldots)$, which has derivatives of $-\sigma_{x''} = x - x'' \equiv \Delta x$, $-\sigma_{y''} = (y - y'')$, and so on. If we let the world line have zero $y, z, \ldots$ coordinates, and insert into the formula for the Fermi normal coordinates, and use the proper time τ, which is equal to the Rindler time t as the otherwise unspecified time coordinate, we get

$$\underline{t} = t \qquad \underline{x} = \Delta x \qquad \underline{\rho} = \rho \tag{2.25}$$

where ρ again is any of the remaining coordinates. Thus, Fermi normal coordinates for a particle that is stationary in Rindler frame are basically equivalent to Rindler coordinates.

Rindler coordinates are only single-valued when the geodesic used is unique. Since the geodesics used are space-like, this is only valid in the LCN for space-like geodesics.

2.3.4 Polar coordinates

Polar coordinates for various numbers of dimensions will also be useful. I will here introduce polar coordinates in d spatial dimensions, with and without time. This will be done in two steps: first d-dimensional spatial polar coordinates are derived, after which time is added.

Spatial polar coordinates

I will build the d-dimensional polar coordinates by gradually adding dimensions. The radial coordinate is ρ, and the angles run from α_1 to α_{d-1}. The expression for the rectangular coordinates and line element as a function of the polar coordinates, will vary with changing d, so these have an additional index of d: ds_d^2, and $x_{d,1}$ to $x_{d,d}$. Regardless of d, it always holds that $\rho^2 = \sum_{n=1}^{d} x_{d,n}^2$, which means that $\rho d\rho = \sum_{n=1}^{d} x_{d,n} dx_{d,n}$.

One dimension In this trivial case, we have

$$x_{1,1} = \rho \Rightarrow dx_{1,1} = d\rho \Rightarrow ds_1^2 = d\rho^2 \tag{2.26}$$

Two dimensions

$$x_{2,1} = \rho \sin(\alpha_1) = x_{1,1} \sin(\alpha_1) \tag{2.27}$$
$$x_{2,2} = \rho \cos(\alpha_1) \tag{2.28}$$

Three dimensions

$$x_{3,1} = \rho \sin(\alpha_1) \sin(\alpha_2) = x_{2,1} \sin(\alpha_2) \tag{2.29}$$
$$x_{3,2} = \rho \cos(\alpha_1) \sin(\alpha_2) = x_{2,2} \sin(\alpha_2) \tag{2.30}$$
$$x_{3,3} = \rho \cos(\alpha_2) \tag{2.31}$$

d dimensions We see that we in general have

$$x_{d,i} = x_{d-1,i} \sin(\alpha_{d-1}) \qquad i < d \tag{2.32}$$
$$x_{d,d} = \rho \cos(\alpha_{d-1}) \tag{2.33}$$

To find the line element, we need

$$dx_{d,i} = dx_{d-1,i} \sin(\alpha_{d-1}) + x_{d-1,i} \cos(\alpha_{d-1}) d\alpha_{d-1} \qquad i < d \tag{2.34}$$
$$dx_{d,d} = d\rho \cos(\alpha_{d-1}) - \rho \sin(\alpha_{d-1}) d\alpha_{d-1} \tag{2.35}$$

giving

$$
\begin{aligned}
ds_d^2 = &\overbrace{\sum_{i=1}^{d-1} dx_{d-1,i}^2}^{ds_{d-1}^2} \sin^2(\alpha_{d-1}) + \overbrace{\sum_{i=1}^{d-1} x_{d-1,i}^2}^{\rho^2} \cos^2(\alpha_{d-1}) d\alpha_{d-1}^2 \\
&+ d\rho^2 \cos^2(\alpha_{d-1}) + \rho^2 \sin^2(\alpha_{d-1}) d\alpha_{d-1}^2 \\
&+ 2\sin(\alpha_{d-1})\cos(\alpha_{d-1}) d\alpha_{d-1} \left(\overbrace{\sum_{i=1}^{d-1} x_{d-1,i} dx_{d-1,i} - \rho d\rho}^{0} \right) \\
= &\sin^2(\alpha_{d-1}) ds_{d-1}^2 + \rho^2 d\alpha_{d-1}^2 + \cos^2(\alpha_{d-1}) d\rho^2 \qquad (2.36) \\
= &d\rho^2 + \rho^2 \sum_{i=1}^{d-1} \left(\prod_{j=1}^{i-1} \sin^2(\alpha_j) \right) d\alpha_i^2 \qquad (2.37)
\end{aligned}
$$

Space-time polar coordinates

We now wish to include time as one of the axes in a polar coordinate system. This means introducing a new angle, the angle between the point whose coordinates we are trying to find, and the time axis, analogous to when we introduced all the spatial axes. To differentiate this angle from the spatial ones, we will call this one ω. Using the general formula, we get

$$
\begin{aligned}
x_{d+t,i} =& x_{d,i} \sin(\omega), i \leq d \qquad (2.38) \\
t =& \rho \cos(\omega) \qquad (2.39) \\
dx_{d+t,i} =& dx_{d,i} \sin(\omega) + x_{d,i} \cos(\omega) d\omega \\
dt =& d\rho \cos(\omega) - \rho \sin(\omega) d\omega \\
ds_{d+t}^2 =& ds_d^2 \sin^2 \omega + \rho^2 (\cos^2(\omega) - \sin^2(\omega)) d\omega^2 \\
&- \cos^2(\omega) d\rho^2 + 4\rho \sin(\omega)\cos(\omega) d\omega d\rho \\
=& \sin^2(\omega) ds_d^2 - \cos^2(\omega) d\rho^2 + \rho^2 \cos(2\omega) d\omega^2 + 2\sin(2\omega) d\omega d\rho \qquad (2.40) \\
=& -\cos^2(\omega) d\rho^2 + \sin(2\omega) d\omega d\rho \\
&+ \rho^2 \left(\cos(2\omega) + \sin^2(\omega) \sum_{i=1}^{d-1} \left[\prod_{j=1}^{i-1} \sin^2(\alpha_j) \right] d\alpha_i^2 \right) \qquad (2.41)
\end{aligned}
$$

This corresponds to a metric of

$$g_{\rho\rho} = -\cos(2\omega) \tag{2.42}$$
$$g_{\rho\omega} = \rho\sin(2\omega) \tag{2.43}$$
$$g_{\omega\omega} = \rho^2\cos(2\omega) \tag{2.44}$$
$$g_{\alpha_i\alpha_i} = \rho^2\sin^2(\omega)\prod_{n=1}^{i-1}\sin^2(\alpha_n) \tag{2.45}$$

Since this is block diagonal, it is easy to invert to find $g^{\mu\nu}$. The $\rho\omega$ block has a determinant of $-\rho^2\cos^2(2\omega) - \rho^2\sin^2(2\omega) = -\rho^2$, so this block has an inverse of

$$g^{\rho\rho} = -\cos(2\omega) \tag{2.46}$$
$$g^{\rho\omega} = \rho^{-1}\sin(2\omega) \tag{2.47}$$
$$g^{\omega\omega} = \rho^{-2}\cos(2\omega) \tag{2.48}$$

and since the rest of the metric is diagonal, its inverse is immediately seen to be

$$g^{\alpha_i\alpha_i} = \rho^{-2}\sin^{-2}(\omega)\prod_{n=1}^{i-1}\sin^{-2}(\alpha_n) \tag{2.49}$$

Finally, metric determinant is

$$\sqrt{|g|} = \rho^d\sin^{d-1}(\omega)\prod_{i=1}^{d-2}\sin^{d-i-1}(\alpha_i) \tag{2.50}$$

2.4 Units

I will use units with $c = 1$ unless otherwise noted. Since most of the literature in this field uses cgs units, these will see the most use here. Any conversion between the two will be mentioned explicitly, and takes the form $A^\mu_{\mathrm{SI}} = \frac{A^\mu_{\mathrm{cgs}}}{\sqrt{4\pi\epsilon_0}}$ (which means that E^μ and $F^{\mu\nu}$ transform the same way) and $q_{\mathrm{SI}} = \sqrt{4\pi\epsilon_0}q_{\mathrm{cgs}}$. The 4π factor has its origin in the area of the unit sphere, so for space with another number of dimensions $d \neq 3$, we must replace this factor by the corresponding surface area S_d. Useful values of S_d are

$$S_1 = 2 \qquad S_2 = 2\pi \qquad S_3 = 4\pi \qquad S_4 = 2\pi^2 \tag{2.51}$$

2.5 The electromagnetic wave equation

In the form language of differential geometry, Maxwell's four equations reduce to the two equations

$$\underline{d} * \underline{F} = * \mu_0 \underline{j} \tag{2.52}$$

$$\underline{dF} = 0 \tag{2.53}$$

where $\underline{F}$ and $\underline{j}$ are forms corresponding to the electromagnetic field tensor and four-current respectively. In form language, the field potential is related to the field itself by $\underline{dA} = \underline{F}$. Using this, and the identities $\underline{dd} = 0$ and $* * \underline{dT} = s(-1)^{p(n-p)}\underline{T}$, where $\underline{T}$ is some form of order p, where s is the sign of the metric determinant (-1 in this case), and n is the number of dimensions, Maxwell's equations become

$$*\underline{d} * \underline{dA} = \mu_0 \underline{j} \tag{2.54}$$

When in Minkowski coordinates and using the Lorenz condition $A^\mu{}_\mu = 0$, this reduces to the familiar

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A^\mu = g^{\alpha\beta} A^\mu{}_{,\alpha\beta} = -\mu_0 j^\mu \tag{2.55}$$

(for a thorough calculation of this kind, see chapter 7). In cgs units, the equation becomes

$$\begin{aligned} g^{\alpha\beta} A^\mu{}_{,\alpha\beta} &= - S_d \epsilon_0 \mu_0 j^\mu \\ &= - S_d c^{-2} j^\mu = -S_d j^\mu \end{aligned} \tag{2.56}$$

When generalized to an arbitrary metric, this becomes[13, page 10]:

$$A^{\mu;\alpha}{}_\alpha - R^\mu_\nu A^\nu = - S_d j^\mu \tag{2.57}$$

This equation governs the behavior of the field potential, and thus that of the field itself. Solving this equation is a difficult task. One of the tools used for solving it is the Green function, which is also an interesting function in its own right.

2.6 Green functions and the wave equation

Given a differential equation

$$L(\phi(\mathbf{x})) = f(\mathbf{x}) \tag{2.58}$$

where L is a general differential operator (for example $L = \frac{\partial^2}{dx^2} + y\frac{\partial}{\partial y}$), a Green function defined by

$$L(G(\mathbf{x}, \mathbf{x}')) = \delta(\mathbf{x} - \mathbf{x}') \tag{2.59}$$

where $\delta(\mathbf{x} - \mathbf{x}')$ is the Dirac delta function, solves the original differential equation through

$$\phi(\mathbf{x}) = \int G(\mathbf{x}, \mathbf{x}') f(\mathbf{x}') d\mathbf{x}' + S \tag{2.60}$$

where S is a surface term that can be taken to be zero when the integral goes over all of spacetime.

In many cases, it is easier to find the Green function than the unknown function, and so it is mainly used for convenience, but it also serves to make the causal connections inherent in the differential equation explicit. From the definition, we see that the $G(\mathbf{x}, \mathbf{x}')$ is the effect on the spacetime point $\mathbf{x}$ of the point $\mathbf{x}'$, and thus can be seen as the degree to which the two points are causally connected (as far as the differential equation is concerned). For example, in flat, 3+1-dimensional spacetime, electromagnetic radiation moves only on the light cone, and the Green function for electromagnetic radiation is therefore zero everywhere outside of the light cone, so in this case $G \propto \delta(\sigma)$.

The Green function described above is a scalar quantity. For differential operators involving vectors, one can readily generalize equation (2.59) to get

$$L^{\mu}{}_{\nu}(G^{\nu}{}_{\mu'}(\mathbf{x}, \mathbf{x}')) = \delta^{\mu}{}_{\mu'} \delta(\mathbf{x} - \mathbf{x}') \tag{2.61}$$

$$A^{\mu} = \int G^{\mu}{}_{\mu'} f^{\mu'} d\mathbf{x}' \tag{2.62}$$

However, the fact that indices belonging to different points are being compared directly in $\delta^{\mu}{}_{\mu'}$, tells us that this cannot be the correct expression when we introduce curvature, or indeed any non-constant basis vectors. Additionally, we lack the volume element $\sqrt{-g}$, and the δ-function needs to be turned into the invariant delta function $\delta(\mathbf{x}, \mathbf{x}') \equiv \sqrt{-g}^{-1} \delta(\mathbf{x} - \mathbf{x}')$, where $g \equiv |g_{\mu\nu}|$, i.e. g is the determinant of the metric with lower indices. Rectifying this, we arrive at

$$L^{\mu}{}_{\nu}(G^{\nu}{}_{\mu'}(\mathbf{x}, \mathbf{x}')) = g^{\mu}{}_{\mu'} \delta(\mathbf{x}, \mathbf{x}') \tag{2.63}$$

$$\phi^{\mu} = \int G^{\mu}{}_{\mu'} f^{\mu'} \sqrt{-g} d\mathbf{x}' \tag{2.64}$$

where the indices now are compared using the parallel propagator.

Applying this to the electromagnetic wave function, we would get

$$\Box G^{\mu}{}_{\mu'} - R^{\mu}{}_{\nu} G^{\nu}{}_{\mu'} = g^{\mu}{}_{\mu'} \delta(\mathbf{x}, \mathbf{x}')$$

but the convention is to define it slightly differently, as

$$\Box G^{\mu}{}_{\mu'} - R^{\mu}{}_{\nu} G^{\nu}{}_{\mu'} = -4\pi g^{\mu}{}_{\mu'} \delta(\mathbf{x}, \mathbf{x}') \tag{2.65}$$

and we will stay with this definition. I will refer to this equation as the electromagnetic Green equation or, when comparing it to the homogeneous version of the same equation, it will be called the inhomogeneous equation.

2.6.1 Properties of the electromagnetic Green functions

The Green equations above do not give unique solutions before we specify initial conditions. The question we usually try to solve with the Green equations, is "how does the field respond to there being a charge at a certain event?". We therefore let the initial value for the field be zero, so that any resulting value for the field is due entirely to the charge. The resulting Green function is called the *retarded* Green function, since it represents a perturbation of the field that propagates *after* the event that generated it. I will label this solution G_+.

Less physically relevant, but sometimes useful, is the solution to the question "what does a field that exactly cancels the effect of there being a charge at a certain event look like?". The solution to this is called the *advanced* Green function G_-, and represents a field propagating *before* the event that it is defined in relation to. It can also be interpreted as a field propagating backwards in time from the event.

Any linear combination $\alpha G_+ + \beta G_-$ with $\alpha + \beta = 1$ will produce another solution to the inhomogeneous equation. Similarly, $\alpha + \beta = 0$ yields solutions to the homogeneous equation. Two common Green functions are defined this way: The singular Green function $G_S \equiv \frac{1}{2}(G_+ + G_-)$, and the radiative Green function $G_R \equiv \frac{1}{2}(G_+ - G_-)$.

Since the wave equation is linear, it is always possible to add a homogeneous solution to an inhomogeneous one, and still have a solution to the inhomogeneous equation. Still, the advanced and retarded Green functions are unique: It is not possible to add a homogeneous solution to $G_\pm$, and still have the correct behavior in the past or future. In other words, no homogeneous solution can be zero everywhere before (after) some space-like surface separating past and future, and nonzero after (before) it. This is trivial to see, since every term in the homogeneous equation is proportional to G or its derivatives, so if these are all zero at some time, the equation gives a time-derivative of zero. This is also reasonable from an energy and charge conservation point of view: If we start out with space devoid of an electromagnetic field, we do not expect radiation to suddenly spring into existence without any charges.

Since Green functions are defined as the result of a charge at a single point in space-time, they suffer from a problem of infinities. The charge density of a finite charge confined to a point will necessarily be infinite. The resulting Green function will therefore also involve infinities. However, the resulting field will in almost all cases end up being finite — the Green function is integrated to produce the scalar or electromagnetic field potential, and the infinities in the Green function must be of the kind that still yield finite integrals. The exception is the point of emission, where the field ends up being infinitely strong. When I mention the singular or divergent behavior of the Green functions, it will be this last kind of infinity

I mean.

Green functions in flat spacetime

In flat spacetime, the Ricci term of the electromagnetic Green equation disappears. Furthermore, one can always find a global coordinate system where $\Box$ does not mix the components of the Green function, and the parallel propagator reduces to the Kronecker delta. These effects conspire to reduce the equation to a set of identical, scalar equations: $\Box G = -S_d \delta(\mathbf{x}, \mathbf{x}')$, with $G^{\mu}{}_{\mu'} = g^{\mu}{}_{\mu'} G$.

Since this is a scalar equation, it must have scalar solutions, i.e. they must be Lorentz invariant. This requires that the solutions depend only on the invariant interval within each half of the light cone. But since any (subluminal) observer capable of observing the solution will agree on which part of the light cone is past and future, it is possible to have separate functions for each of these halves. Thus, the general solution to the scalar Green equation in flat spacetime must be of the form $G = a(\sigma)\theta(\Delta\bar{t}) + b(\sigma)\theta(-\Delta\bar{t})$. The retarded and advanced solutions must therefore be of the form

$$G_{\pm} = g(\sigma)\theta(\pm\Delta\bar{t}) \tag{2.66}$$

In the case of spacetime with 3 spatial dimensions, the scalar equation can be solved by Fourier transforming it. First we perform the transformation $F_{\omega}(f) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-i\omega t} f dt \equiv \widetilde{f}$, followed by $F_k(f) = (2\pi)^{-\frac{3}{2}} \int e^{-i\vec{k}\cdot\vec{x}} f d\vec{k} \equiv \overline{f}$, resulting in

$$\left(\frac{\partial^2}{\partial \bar{t}^2} - \nabla^2\right) G = -4\pi\delta(\mathbf{x} - \mathbf{x}') \Leftrightarrow \tag{2.67}$$

$$(\omega^2 - k^2)\widetilde{\overline{G}} = -\pi^{-1} \Leftrightarrow \tag{2.68}$$

$$\overline{G} = -\pi^{-1}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{(\omega - k)(\omega + k)} d\omega \tag{2.69}$$

This integral is problematic, as it crosses two poles. If the poles had been slightly shifted by a small imaginary number, this integral could be easily evaluated by making a closed curve and using the residue theorem. We will still try to use this method by introducing such shifts, but this introduces a sense of arbitrariness, as both poles can be shifted either up or down, producing four possibilities: up and up, down and down, up and down, and down and up. It will turn out that the first two of these correspond to the retarded and advanced solution respectively. Choosing to shift the poles up, thereby finding G_+, we see that the integral must be completed in the upper ω-plane when $\bar{t} > 0$, and lower ω-plane otherwise, and using

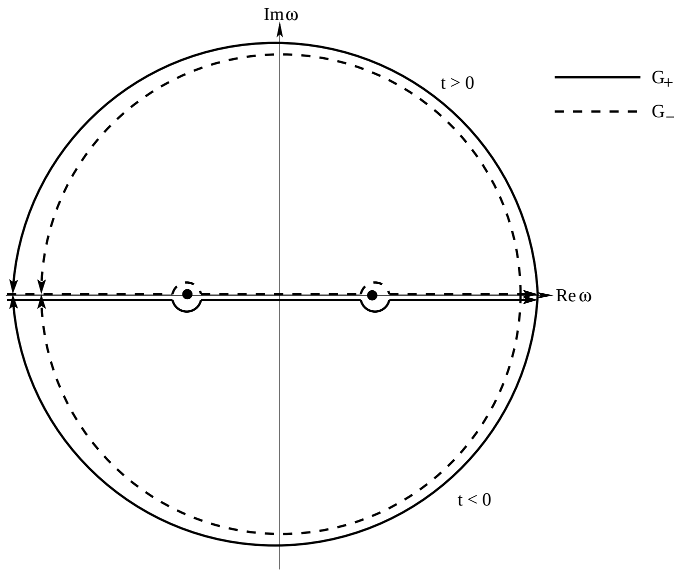

Figure 2.3: The retarded and advanced solutions result from integrating respectively under or over the poles along the real axis of ω. In each case, the sign of $\bar{t}$ determines which half-plane must be used to complete the integral without altering its value. It is easy to see that integrating under both poles will lead to an integration around no poles in the case of $\bar{t} < 0$, and around two poles for $\bar{t} > 0$, giving the $\theta(\bar{t})$-dependence of G_+.

the residue theorem, we find

$$\overline{G}_+ = \sqrt{\frac{2}{\pi}} k^{-1} \sin(k\bar{t})\theta(\bar{t}) \tag{2.70}$$

We should check that this is still a solution to the differential equation for $\overline{G}_+$: $\left(\frac{\partial^2}{\partial \bar{t}^2} + k^2\right)\overline{G} = \sqrt{2\pi^{-1}}\delta(\bar{t})$:

$$\frac{\partial^2 \overline{G}_+}{\partial \bar{t}^2} = -\sqrt{2\pi^{-1}} k \sin(k\bar{t})\theta(\bar{t}) + \sqrt{8\pi^{-1}}\delta(\bar{t}) + \sqrt{2\pi^{-1}}\delta'(\bar{t}) \tag{2.71}$$

Using $f(x)\delta(x) = f(0)\delta(x)$ and $f(x)\delta'(x) = f(0)\delta'(x) - f'(0)\delta(x) = -f'(0)\delta(x)$, this simplifies to

$$\frac{\partial^2 \overline{G}_+}{\partial \bar{t}^2} = -\sqrt{2\pi^{-1}} k \sin(k\bar{t})\theta(\bar{t}) + \sqrt{2\pi^{-1}}\delta(\bar{t}) \tag{2.72}$$

and we see that the differential equation is satisfied. The untransformed Green function can now be recovered by applying F_k^{-1}. I will use polar coordinates (k, θ, ϕ) aligned along $\vec{x} = (\bar{x}, \bar{y}, \bar{z})$, for the $\vec{k}$ integral.

$$G_+ = 2(2\pi)^{-1} \int d\vec{k} k^{-1} \sin(k\bar{t})\theta(\bar{t}) e^{i\vec{k}\cdot\vec{x}} \tag{2.73}$$

$$= 2(2\pi)^{-1} \int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\phi k \sin(\theta)\sin(k\bar{t})\theta(\bar{t}) e^{ikr\cos(\theta)} \tag{2.74}$$

$$= \frac{1}{\pi r} \int_{-\infty}^\infty dk \sin(kr)\sin(k\bar{t}) \tag{2.75}$$

$$= \frac{\theta(\bar{t})}{r}\left(\delta(r - \bar{t}) + \delta(r + \bar{t})\right) \tag{2.76}$$

$$= \frac{\theta(\bar{t})\delta(r - \bar{t})}{r} \tag{2.77}$$

since $r = |\vec{x}| \geq 0$. Since $\delta(f(x)) = \sum_n \frac{\delta(x - x_n)}{|f'(x_n)|}$, where x_n are the zeros of the function f, we see that $\delta(\sigma) = r^{-1}(\delta(r - t) - \delta(r + t))$, so we finally get

$$G_\pm = \theta(\pm\bar{t})\delta(\sigma) \tag{2.78}$$

where the advanced part results from a completely analogous derivation. Due to the simple relation between the scalar and electromagnetic Green functions in flat spacetime, the electromagnetic Green functions are

$$G_{\pm}{}^{\mu}{}_{\mu'} = \theta(\pm\bar{t})\delta(\sigma) g^{\mu}{}_{\mu'} \tag{2.79}$$

2.7 The Generalized Lorentz-Abraham-Dirac equation and the Tail Term

The Lorentz-Abraham-Dirac, or LAD, equation is the equation of motion for a charged particle in 3+1-dimensional flat spacetime, and is of the form

$$f^\mu = qF^{\text{in}\mu}{}_\nu u^\nu + \frac{2}{3}q^2 \left(\frac{Da^\mu}{D\tau} - u^\mu a^\nu a_\nu \right) \tag{2.80}$$

It expresses the total force a charged particle feels due to the electromagnetic interaction as the sum of an external force ($qF^{\text{in}\mu}{}_\nu u^\nu$) and a self-interaction term: a force due to the interaction between the particle and its field.

To be valid outside the confines of flat spacetime, the equation was extended by DeWitt and Brehme [5] and Hobbs [12], producing what I will call the generalized LAD equation, or GLAD equation, and has the form

$$f^\mu = qF^{\text{in}\mu}_\nu u^\nu + \frac{2}{3}q^2 \left(\frac{Da^\mu}{D\tau} - u^\mu a^\nu a_\nu \right) \tag{2.81}$$

$$-\frac{1}{3}q^2 \left(R^\mu{}_\nu u^\nu + R_{\alpha\beta} u^\alpha u^\beta u^\nu \right) + q^2 u^\nu \int_{-\infty}^{\tau} f^\mu_{\nu\gamma'} u^{\gamma'} d\tau' \tag{2.82}$$

where $f^\mu{}_{\nu\gamma'} \equiv G_{\nu\gamma'}{}^{;\mu} - G^\mu{}_{\gamma';\nu}$. The self-interaction of the LAD equation is purely local: the only point in space-time at which a particle's own field may interact with it, is at the point where the field was emitted, i.e. at the origin of the light cone.

The GLAD equation adds two new terms. One is a correction due to the local curvature, as can be seen from the Ricci tensor. The other is an integral over the whole past history of the particle, and is as such a nonlocal effect. It is, of course, not nonlocal in the meaning of instant action at a distance, but nonlocal in the same sense as the external force is: the field is not absorbed and emitted at the same point, and has to propagate between them. This last term is called the "tail term" of the GLAD equation.

What, then, is this "tail"? Consider the electromagnetic field potential as expressed by the retarded Green function:

$$A^\mu(x) = \int G^\mu{}_{\alpha'}(x, x') j^{\alpha'}_{\text{source}}(x') \sqrt{-g'} d^4x' \tag{2.83}$$

The corresponding field sets up a force (the Lorenz force)

$$f^\mu = g^{\mu\nu} F_{\nu\rho} j^\rho = g^{\mu\nu} A_{[\rho;\nu]} j^\rho_{\text{particle}} \tag{2.84}$$

Inserting the field potential, we find

$$f^\mu = g^{\mu\nu}(x) j^\rho_{\text{particle}}(x) \int \left(G_{+\rho\alpha';\nu}(x, x') - G_{+\nu\alpha';\rho}(x, x') \right) j^{\alpha'}_{\text{source}}(x') \sqrt{-g'} d^4x' \tag{2.85}$$

If the only source is the point particle itself, and this has charge q, this reduces to

$$f^{\mu}(\tau) = q^2 \int_{-\infty}^{\infty} d\tau' \left(G_{+\rho\alpha'}^{;\mu}(\tau,\tau') - G^{\mu}_{+\alpha';\rho}(\tau,\tau') \right) u^{\alpha'}(\tau') u^{\rho}(\tau) \quad (2.86)$$

which is of the same form as the tail term.

Since this is the retarded Green function, there is no contribution from the $\tau' > \tau$ parts of the integral. There is, however, a contribution from $\tau' = \tau$, where the Green function diverges. This requires a careful analysis of G_+'s behavior close to this point.

Thankfully, things simplify quite a bit when the emitting and absorbing points are close to each other. One is then in the local convex neighborhood (see figure (2.1)), so there is only one geodesic between the two points, and one can then use a Hadamard decomposition[11, 5] of the Green function into a on-light-cone ($U^{\mu}_{\alpha'}$) and inside-light-cone part ($V^{\mu}_{\alpha'}$):

$$G^{\mu}_{+\alpha'}(x,x') = U^{\mu}{}_{\alpha'}(x,x')\delta_+(\sigma) + V^{\mu}{}_{\alpha'}(x,x')\theta_+(-\sigma) \quad (2.87)$$

where σ is half the squared distance between the two points as measured along the unique geodesic connecting them, and δ_+ and θ_+ are δ and step functions that are turned off when x' is in the future of x. $U^{\mu}_{\alpha'}$ represents propagation of the field along the light cone (so this part of the signal travels with speed c, and does not disperse), while $V^{\mu}_{\alpha'}$ represents propagation at all speeds slower than c, leading to dispersion. This can be called the tail part of the Green function.

After a long, tricky calculation, DeWitt and Brehme [5] and Hobbs [12] found that the contribution from $\tau = \tau'$ in the integral in the expression for the force, is

$$f^{\mu}{}_{\tau=\tau'} = \frac{2}{3}q^2 \left(\frac{Da^{\mu}}{D\tau} - u^{\mu}a^{\nu}a_{\nu} \right) - \frac{1}{3}q^2 \left(R^{\mu}{}_{\nu}u^{\nu} + R_{\alpha\beta}u^{\alpha}u^{\beta}u^{\nu} \right) \quad (2.88)$$

The rest of the expression is left uncomputed, and is called the tail term:

$$f^{\mu}_{\text{tail}}(\tau) = q^2 \lim_{\tau''\to\tau} \int_{-\infty}^{\tau''} d\tau' \left(G_{+\rho\alpha'}^{;\mu}(\tau,\tau') - G^{\mu}_{+\alpha';\rho}(\tau,\tau') \right) u^{\alpha'}(\tau') u^{\rho}(\tau) \quad (2.89)$$

Furthermore, aside from the point $\tau = \tau'$, there is no way for the particle to catch up to its own light cone, as long as one stays within the local convex neighborhood, so only V contributes here. If the world line of the particle crosses the LCN in the past at τ_-, we can split up the tail term in the two parts

$$\begin{aligned} f^{\mu}{}_{\text{tail}}(\tau) = & q^2 \lim_{\tau''\to\tau} \int_{\tau_-}^{\tau''} d\tau' \left(V_{+\rho\alpha'}^{;\mu}(\tau,\tau') - V^{\mu}_{+\alpha';\rho}(\tau,\tau') \right) u^{\alpha'}(\tau') u^{\rho}(\tau) \\ & + q^2 \int_{-\infty}^{\tau_-} d\tau' \left(G_{+\rho\alpha'}{}^{;\mu}(\tau,\tau') - G^{\mu}_{+\alpha';\rho}(\tau,\tau') \right) u^{\alpha'}(\tau') u^{\rho}(\tau) \end{aligned} \quad (2.90)$$

Using only the first of these, and extending its lower limit to $-\infty$, seems to be a common confusion, but it must be stressed that the first part only is valid within the LCN. Point 3 of figure (2.1) illustrates the problem of continuing to use only $V^{\mu}_{\alpha'}$ outside of the LCN, while point 4 illustrates that the concept of a given point being unambiguously either on or inside the light cone may break down outside this neighborhood.

To summarize, the tail term is the force on the particle due to it reencountering its own field after it has been emitted. This can either be because some parts of the field travel slower than the speed of light, permitting the particle to catch up to them, that is: due only to the V part of the Green function, or it can be due to the light cone becoming deformed, letting it intersect the path of the particle. In the latter case, the U part will also contribute. In any case, the force involved is just the Lorenz force.

In one sense, Lorentz-Abraham-Dirac equation,

$$f^{\mu} = qF^{\mu}_{\text{ext}\ \nu} + f^{\mu}_{\tau=\tau'} + f^{\mu}_{\text{tail}} \tag{2.91}$$

leaves most of the calculation to the user. After all, it only takes care of 1 point on the past world line of the particle, leaving the rest uncomputed as the "tail". On the other hand, that 1 point is by far the most problematic, as we shall see in chapter 4

2.8 Huygens' principle

Huygens' principle is the observation that any point in an electromagnetic field may be viewed as the source of a new spherical wave, the sum of which builds up the next step in the propagation of the field. This follows from the fact that the wave equation is linear, making its solutions obey the superposition principle. Due to the superposition principle, we may find the total time evolution of the field by dividing it into small regions, solving for each of these regions as if all the other regions were empty, and then adding all these contributions together. Each of these contributions will, for distances that are large compared to the diameter of the regions, be spherical, so in the limit where we choose infinitesimally small regions, we effectively decompose the field into spherical wavelets. This principle follows directly from the linearity of the wave equation, and thus holds for any spacetime.

This is, however, not the only version of Huygens' principle. Brown [1] writes:

> ...Huygens' Principle is understood to apply equally to any locus of constant phase (not just the leading edge of the disturbance), all propagating at the same characteristic wave speed. This implies that a wave doesn't get "thicker" as it propagates, i.e., there is no diffusion of waves.

This principle, simply stated "there is no diffusion of electromagnetic waves in vacuum", is called the strong version of Huygens' principle. This version puts stringent limits on the shape of the Green function. Since the field moves with just one speed, the speed of light, it will be confined to the light cone. The Green function will therefore be shaped as a Dirac delta function or one of its derivatives. Any Green function that is nonzero for nonzero σ breaks the strong version of the principle.

Chapter 3

The tail term in a Rindler metric with cylindrical topology

Perhaps the simplest system where a charge will feel a nonlocal self force due to the topology of space is the (admittedly somewhat contrived) case of a charge with constant proper acceleration in a locally flat spacetime where one of the dimensions perpendicular to the acceleration is cyclical. This is just normal hyperbolic motion in flat space, which is known not to have any radiation reaction, but with the small change that the y dimension is periodic with period L, such that the point (t, x, y, z) is identical to $(t, x, y + nL, z)$. The case of one particle moving hyperbolically along the x axis in this space, is obviously identical to having infinitely many particles moving along in the x direction separated by an interval of L in the y direction in normal space.

I will attempt to find an explicit expression for the tail term of the generalized LAD (GLAD) equation for this special case. This will be done first by calculating the force on the particle directly, and then comparing with the GLAD equation to find the tail term, and second by direct calculation of the tail term.

Throughout this chapter, I will use bars to indicate the lab system, primes to indicate the instantly co-moving system, and the absence of these to indicate the co-moving Rindler frame. $'$ will be used to indicate the charge, so (for example) $\bar{t}'$ is the time coordinate of the charge in the lab system.

3.1 Indirect evaluation of the tail term

My goal is to find the force on the particle in its co-moving Rindler frame by using the Lorentz force equation $f^\mu = F^\mu_\nu j^\nu$, but for this the field tensor is needed. I will find this first in an inertial frame, the lab system, and then transform it to the co-moving Rindler frame. The effects of the cyclic dimension can be accounted for by simply superpositioning fields with various y-offsets at any point in this process, and I will do it in the end, for simplicity.

3.1.1 The vector potential in the lab system

In the lab system the coordinates are $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$, and the charge moves according to

$$\bar{x}'^2 - \bar{t}'^2 = \bar{x}_0^2 \quad \bar{y}' = \bar{z}' = 0 \tag{3.1}$$

The field from the charge at a general position $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ is produced by the charge at $\bar{t}'$ given by $(\bar{t} - \bar{t}')^2 = (\bar{x} - \bar{x}')^2 + \rho^2$, where $\rho^2 = \bar{y}^2 + \bar{z}^2$. We find, using (3.1)

$$\begin{aligned}
\bar{t}^2 + \bar{t}'^2 - 2\bar{t}\bar{t}' &= \bar{x}'^2 + \bar{x}^2 - 2\bar{x}\bar{x}' + \rho^2 \Leftrightarrow \\
\bar{x}\bar{x}' - \bar{t}\bar{t}' &= \frac{1}{2}\left(\bar{x}_0^2 + \bar{x}^2 + \rho^2 - t^2\right) \equiv a \Leftrightarrow \qquad (3.2)\\
0 &= (\bar{x}^2 - \bar{t}^2)\bar{t}'^2 - 2a\bar{t}\bar{t}' - a^2 + \bar{x}^2\bar{x}_0^2 \Leftrightarrow \\
\bar{t}' &= \frac{a\bar{t} - \bar{x}\sqrt{a^2 - (\bar{x}^2 - \bar{t}^2)\bar{x}_0^2}}{\bar{x}^2 - \bar{t}^2} \\
&= \frac{a\bar{t} - \bar{x}b}{\bar{x}^2 - \bar{t}^2} \qquad (3.3)
\end{aligned}$$

where $b = \sqrt{a^2 - (\bar{x}^2 - \bar{t}^2)\bar{x}_0^2}$. The negative sign has been chosen because $\bar{t}'$ shouldn't be greater than $\bar{t}$. We also find $\bar{x}'$:

$$\begin{aligned}
\bar{x}\bar{x}' - \bar{t}\bar{t}' &= a \Leftrightarrow \\
0 &= (\bar{x}^2 - \bar{t}^2)\bar{x}'^2 - 2a\bar{x}\bar{x}' + a^2 + \bar{t}^2\bar{x}_0^2 \Leftrightarrow \\
\bar{x}' &= \frac{a\bar{x} - \bar{t}\sqrt{a^2 - (\bar{x}^2 - \bar{t}^2)\bar{x}_0^2}}{\bar{x}^2 - \bar{t}^2} \\
&= \frac{a\bar{x} - \bar{t}b}{\bar{x}^2 - \bar{t}^2} \qquad (3.4)
\end{aligned}$$

In the inertial system that is instantly co-moving with the charge at $\bar{t} = \bar{t}'$ the field at the given point is simply

$$\mathbf{A}^{\cdot} = \frac{q}{R}(1, 0, 0, 0) \tag{3.5}$$

where objects in this system are marked with $\cdot$, and $R = \sqrt{(x^{\cdot} - x^{\cdot\prime})^2 + \rho^2} = t^{\cdot} - t^{\cdot\prime}$. To transform back to the lab system, we need an expression for the velocity and gamma factor of the particle at the emission point. This is:

$$v = \frac{d\bar{x}^{\cdot}}{d\bar{t}^{\cdot}} = \frac{\bar{t}^{\cdot}}{\bar{x}^{\cdot}} \tag{3.6}$$

$$\gamma = \left(1 - v^2\right)^{-\frac{1}{2}} = \frac{\bar{x}^{\cdot}}{\bar{x}_0} \tag{3.7}$$

We also need to express R using lab coordinates:

$$\begin{aligned}
R^2 &= \left(x^{\cdot} - x^{\cdot\prime}\right)^2 + \rho^2 \\
&= \left(\gamma(\bar{x} - \bar{x}' - v(\bar{t} - \bar{t}'))\right)^2 + \rho^2 \\
&= \left(\frac{1}{x_0}(\bar{x}'\bar{x} - \bar{t}'\bar{t}) - \bar{x}_0\right)^2 + \rho^2 \\
&= \left(\frac{a}{\bar{x}_0} - \bar{x}_0\right)^2 + \rho^2 \\
&= \frac{1}{\bar{x}_0^2}\left(a^2 + \bar{x}_0^4 + \rho^2\bar{x}_0^2 - 2a\bar{x}_0^2\right) \\
&= \frac{1}{\bar{x}_0^2}\left(a^2 - (\bar{x}^2 - \bar{t}^2)\bar{x}_0^2\right) = \left(\frac{b}{\bar{x}_0}\right)^2
\end{aligned} \tag{3.8}$$

With this, we get

$$A^{\bar{t}} = \gamma(A^{t^{\cdot}} + vA^{x^{\cdot}}) = \frac{q}{R\bar{x}_0}\frac{a\bar{x} - \bar{t}b}{\bar{x}^2 - \bar{t}^2} \tag{3.9}$$

$$A^{\bar{x}} = \gamma(A^{x^{\cdot}} + vA^{t^{\cdot}}) = \frac{q}{R\bar{x}_0}\frac{a\bar{t} - \bar{x}b}{\bar{x}^2 - \bar{t}^2} \tag{3.10}$$

$$A^{\bar{y}} = A^{\bar{z}} = 0 \tag{3.11}$$

3.1.2 The vector potential in the Rindler system

The co-moving system of the accelerating charge has coordinates (t, x, y, z), connected to the lab system by equation (2.3). The line element is $ds^2 = -\frac{x^2}{x_0^2}dt^2 + dx^2$. Equation (2.4) may be used to transform **A** from Minkowski to Rindler coordinates. Additionally

$$a = \frac{1}{2}\left(x^2 + x_0^2 + \rho^2\right) \tag{3.12}$$

$$b = x_0 R = \sqrt{a^2 - x^2 x_0^2} \tag{3.13}$$

With this, we find

$$
\begin{aligned}
A^t =& \frac{\partial t}{\partial \bar{t}} A^{\bar{t}} + \frac{\partial t}{\partial \bar{x}} A^{\bar{x}} \\
=& \frac{q}{Rx^2 x_0} \left(\frac{x_0}{x} \cosh \frac{t}{x_0} \left(ax \cosh \frac{t}{x_0} - bx \sinh \frac{t}{x_0} \right) \right. \\
& \left. - \frac{x_0}{x} \sinh \frac{t}{x_0} \left(ax \sinh \frac{t}{x_0} - bx \cosh \frac{t}{x_0} \right) \right) \\
=& \frac{qa}{Rx^2} \qquad (3.14) \\
A^x =& \frac{\partial x}{\partial \bar{t}} A^{\bar{t}} + \frac{\partial x}{\partial \bar{x}} A^{\bar{x}} \\
=& \frac{q}{Rx^2 x_0} \left(- \sinh \frac{t}{x_0} \left(ax \cosh \frac{t}{x_0} - bx \sinh \frac{t}{x_0} \right) \right. \\
& \left. + \cosh \frac{t}{x_0} \left(ax \sinh \frac{t}{x_0} - bx \cosh \frac{t}{x_0} \right) \right) \\
=& \frac{-qb}{x_0 Rx} = \frac{-q}{x} \qquad (3.15)
\end{aligned}
$$

We see that A^x only depends on x. It will therefore not contribute to $\mathbf{F}$, and we can just as well set it to zero without any physical change. This corresponds to going to the Lorentz gauge, as it results in $A^{\mu}{}_{,\mu} = 0$. We therefore only need to concern ourselves with A^t. The field tensor is $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. To evaluate it, we need

$$A_t = -\frac{x^2}{x_0^2} A^t = \frac{-qa}{Rx_0^2} \qquad (3.16)$$

$$\tfrac{\partial a}{\partial x} = x \quad \tfrac{\partial a}{\partial \rho} = \rho \quad \tfrac{\partial b}{\partial x} = \left(a - x_0^2\right) \tfrac{x}{b} \quad \tfrac{\partial b}{\partial \rho} = \tfrac{a\rho}{b} \qquad (3.17)$$

giving

$$\begin{aligned}
F_{tx} =& - F_{xt} = A_{x,t} - A_{t,x} = -A_{t,x} & (3.18)\\
=& \frac{q}{x_0^2}\left(\frac{x}{R} - \frac{a}{R^2}\frac{a-x_0^2}{b}\frac{x}{x_0}\right)\\
=& \frac{qx}{x_0^2 R^3}\left(a - x^2\right) & (3.19)\\
F_{t\rho} =& - F_{\rho t} = A_{\rho,t} - A_{t,\rho} = -A_{t,\rho} & (3.20)\\
=& \frac{q}{x_0^2}\left(\frac{\rho}{R} - \frac{a}{R^2}\frac{a\rho}{bx_0}\right)\\
=& \frac{-q\rho x^2}{x_0^2 R^3} & (3.21)\\
F^{tx} =& - \frac{x_0^2}{x^2}F_{tx} = \frac{q}{xR^3}\left(x^2 - a\right) = E^x & (3.22)\\
F^{t\rho} =& - \frac{x_0^2}{x^2}F_{T\rho} = \frac{q\rho}{R^3} = E^\rho & (3.23)
\end{aligned}$$

To be able to compare this with a normal Coulomb field, we want Minkowski metric in the vicinity of the charge, and therefore introduce orthonormal coordinates. Because of the simple form of the metric, this is only a rescaling of T-components

$$\frac{\partial \hat{t}}{\partial t} = \frac{x}{x_0} \quad \frac{\partial \hat{x}_i}{\partial x_i} = 1 \tag{3.24}$$

with the others equal to 0. This gives

$$\begin{aligned}
F^{\hat{t}x} =& \frac{x}{x_0}F^{tx} = \frac{q}{x_0 R^3}\left(x^2 - a\right) & (3.25)\\
F^{\hat{t}\rho} =& \frac{x}{x_0}F^{t\rho} = \frac{qx\rho}{x_0 R^3} & (3.26)
\end{aligned}$$

Introducing coordinates relative to the charge, $\xi = x - x_0$ and $r = \sqrt{\rho^2 + \xi^2}$, we get

$$\begin{aligned}
a =& \frac{1}{2}r^2 + x_0^2 + x_0\xi & (3.27)\\
R =& r\sqrt{1 + \frac{\xi}{x_0} + \frac{r^2}{4x_0^2}} & (3.28)\\
F^{\hat{t}\xi} =& \frac{q\xi}{r^3}\frac{1 + \frac{\xi^2-\rho^2}{2x_0\xi}}{\sqrt{1 + \frac{\xi}{x_0} + \frac{r^2}{4x_0^2}}^3} = E^\xi & (3.29)\\
F^{\hat{t}\rho} =& \frac{q\rho}{r^3}\frac{1 + \frac{\xi}{x_0}}{\sqrt{1 + \frac{\xi}{x_0} + \frac{r^2}{4x_0^2}}^3} = E^\rho & (3.30)
\end{aligned}$$

3.1.3 The singular fields and how to extract the motion of the particle

As r, ξ and ρ go to zero, E^{ξ} and E^{ρ} become arbitrarily close to the Coulomb field $E^{\xi} = \frac{kq\xi}{r^3}$, $E^{\rho} = \frac{kq\rho}{r^3}$. One might therefore conclude that the particle therefore cannot distinguish this field from a Coulomb field. However, since the contribution from a Coulomb field to the equation of motion for the particle is zero, it is the comparatively small deviation from a Coulomb field that governs the particle's motion, and we have not shown that this is zero.

The proper way of analyzing this is the world tube formalism used by DeWitt and Brehme [5], Hobbs [12]. Here one constructs a surface encompassing a vanishing section of the world line of the particle, and analyzes the energy-momentum balance there, equating net momentum loss through the surface with momentum loss of the particle.

In his rederivation of the equation of motion of a charged particle, Poisson [13, sect. 5.1.4] uses a shortcut which I will call smoothing. This technique involves constructing a surface with constant σ and time coordinate in Fermi normal coordinates, i.e. a sphere in space centered on the particle, and averaging the field over this surface. Since this surface is symmetrical in space, averaging over it will eliminate the antisymmetric parts of the field, leaving the symmetrical part which hopefully is more well-behaved.

Thus, given a field $F^{\mu\nu}$, we calculate

$$\begin{aligned} \langle F^{\mu\nu} \rangle \equiv& \frac{1}{S_d} \int_{S_d} F^{\mu\nu} dS_d \\ =& \frac{S_{d-1} r}{S_d} \int_0^{\omega} \sin^{d-2} \omega F^{\mu\nu} d\omega \end{aligned} \tag{3.31}$$

where d is the number of spatial dimensions, and use this, instead of $F^{\mu\nu}$, at the limit where the radius of the sphere of averaging shrinks to 0.

In this case, only E^{ξ} is interesting, as the symmetry considerations do not allow any force in non-ξ directions. At the limit of low r, we can simplify the expression for E^{ξ} considerably by expanding it in a series of powers of r, since any positive powers of r will disappear when we take the limit $r \to 0$. What prevents E^{ξ} from already being in the form of a series, is the square root in the denominator, so we will expand this first. To determine how far we will have to expand it, we observe that the rest of the expression is of order r^{-2}, so terms up to and including r^2 will give contributions.

$$\left(1 + \cos\omega x_0^{-1} r + \frac{1}{4} x_0^{-2} r^2\right)^{-\frac{3}{2}} = 1 - \frac{3}{2}\cos\omega x_0^{-1} r - \frac{3}{8}\left(1 - 5\cos^2\omega\right) x_0^{-2} r^2 + O(r^2) \tag{3.32}$$

Combining this with the rest, we find

$$\begin{aligned}E^{\xi} =& q\left(\cos\omega r^{-2} - \frac{1}{2}\left(\cos^2\omega + 1\right)x_0^{-1}r^{-1}\right.\\ &\left.-\frac{3}{8}\cos\omega\left(\cos^2\omega+1\right)x_0^{-2}\right) + O(r) \end{aligned} \tag{3.33}$$

When we integrate over ω during the smoothing, all terms of odd order of $\cos\omega$ will disappear. Using $\int_0^{\pi}\sin\omega d\omega = 2$ and $\int_0^{\pi}\sin\omega\cos^2\omega d\omega = \frac{2}{3}$, we find

$$\begin{aligned}\langle E^{\xi}\rangle =& \frac{2\pi r^2}{4\pi r^2}\int_0^{\pi}\sin\omega E^{\xi} d\omega\\ =& -\frac{1}{k}q x_0^{-1}r^{-1} + O(r)\end{aligned} \tag{3.34}$$

Smoothing got rid of the worst divergence, but this result is still infinite. Does this, then, mean that the particle feels an infinite force? Since the particle is stationary in our reference system,

$$f^x = f^{\xi} = ma^x = mx_0^{-1} = q\langle E^{\xi}\rangle \tag{3.35}$$

$$= -\frac{1}{3}q^2 x_0^{-1}r^{-1} + O(r) \tag{3.36}$$

Here I have used the fact that the acceleration of a stationary particle in Rindler coordinates is the inverse of its x coordinate. Observe that the problematic divergent term behaves the same way as the mass term. The technique of combining the plain mass with the divergent term with the same behavior, and claiming that the sum is the mass we observe, is called mass renormalization. This works as long as these two terms always appear together, never alone. I have not shown this, as this equation is derived under the assumption of hyperbolic motion, so other forms of motion could have other factors. However, I will still assume that this is valid, as this is consistent with the general result by DeWitt and Brehme [5].

After mass renormalization, the remaining force is just $f^x = O(r)$, so in the limit $r \to 0$, this goes to zero. We have therefore regained the well-known result that a hyperbolically accelerated charged particle feels no radiation reaction: The force needed to keep a charged particle at constant proper acceleration is the same as that which is needed for a neutral particle.

3.1.4 Cyclic dimension

The above conclusion was reached without taking the cylindrical nature of the space considered into account. This is easily remedied thanks to the superposition principle. If we take y to be the cyclic dimension, with length L, then this is equivalent to there being an identical copy of the charge at $y = nL$, $n \in Z_+ \cup Z_-$. Relative to mirror image n, the charge has $y = -nL$, so at the charge's position, we have

$$r = |n|L \tag{3.37}$$

$$x = x_0 \tag{3.38}$$

$$a = \frac{1}{2}n^2L^2 + x_0^2 \tag{3.39}$$

$$R = |n|L\sqrt{1 + \frac{n^2L^2}{4x_0^2}} \tag{3.40}$$

$$E_n^{\xi} = \frac{-q}{2x_0L|n|\sqrt{1 + \frac{n^2L^2}{4x_0^2}}^3} \tag{3.41}$$

E^y and E^z must be 0 at the charge's location due to symmetry. The total field from all the mirror images is

$$E_{\Sigma}^{\xi} = \frac{-q}{x_0L}\sum_{n=1}^{\infty}\left(n\sqrt{1 + \frac{1}{4}n^2\frac{L^2}{x_0^2}}^3\right)^{-1} \tag{3.42}$$

Let us examine this sum in the limit of $\frac{L^2}{4x_0^2} \gg 1$:

$$E_{\Sigma}^{\xi} \approx \frac{-8qx_0^2}{L^4}\sum_{n=1}^{\infty} n^{-4} \tag{3.43}$$

$$= \frac{-4qx_0^2\pi^4}{45L^4} \tag{3.44}$$

In general, we can approximate the sum into the part where 1 dominates, and the part where the rest dominates. The border between these is at $n_b = 2\frac{x_0}{L}$, and the sum becomes

$$\sum_{n=1}^{\infty}\left(n\sqrt{1 + \frac{1}{4}n^2\frac{L^2}{x_0^2}}^3\right)^{-1} \approx \sum_{n=1}^{2\frac{x_0}{L}}\frac{1}{n} + \left(\frac{2x_0}{L}\right)^3\sum_{n=2\frac{x_0}{L}}^{\infty} n^{-4} \tag{3.45}$$

The first term diverges as L goes to 0, so the field can be made arbitrarily large. The force acting on the particle is $f^{\mu} = qg_{\alpha\nu}F^{\mu\alpha}u^{\nu}$, the only nonzero component being

$$f^{\xi} = g_{tt}F^{\xi t}qu^t = qE_{\Sigma}^{\xi} \tag{3.46}$$

which acts in the opposite direction of the acceleration.

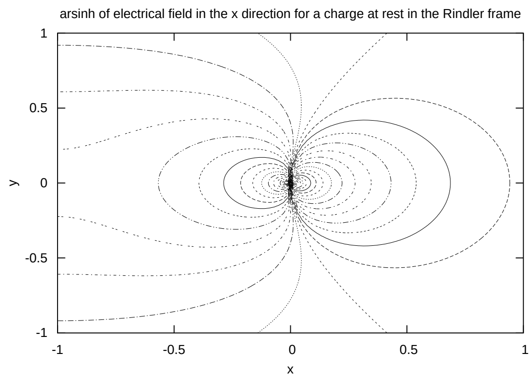

Figure 3.1: E^{ξ} for charge at rest at $\xi = 1$ in the Rindler system with normal topology. The curve between the two lobes is $E^{\xi} = 0$. The area left of this has field pointing in the negative x direction, the area to the right has field pointing in the positive x direction.

3.1.5 The physical picture

Figure 3.1 illustrates the situation before topology is taken into account. Very near the charge, we have a normal dipole field, with two lobes of opposite sign, one on each side of the charge, and of equal size. As we leave the close vicinity of $(0, 0)$, the negative backwards lobe grows, while the forward lobe shrinks. The border between these areas, where $E^{\xi} = 0$, which starts out as a vertical line, therefore curves off to the right. This means that any particle with $\xi = 0$, but $y \neq 0$ is in the backwards lobe of the field, and will (if it has charge of equal sign as the source) be accelerated backwards towards lower values of ξ. The source is thus effectively in front of it, even though they have the same ξ.

The cylindrical topology we are considering is equivalent to there being an infinite array of equal charges accelerating uniformly in the ξ direction, with the same ξ position, but with y differing in steps of L, each of them feeling the field from all the others. The total field one of these charges feels, is therefore the sum of many fields just like the one it itself creates, but offset in the y direction. But as we have seen, this means that it is effectively behind these fields, and will be accelerated in the negative ξ direction by them. This is illustrated by figure 3.2. We see that E^{ξ}_{Σ} at $\xi = 0$, $y = 0$, the position of the charge, is finite and negative.

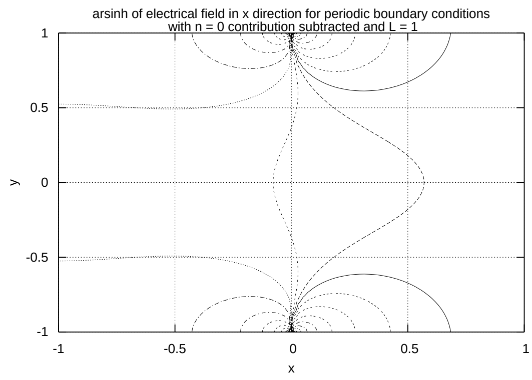

Figure 3.2: E^{ξ}_{Σ} centered on the position of the charge. We see that $(0,0)$ lies to the left of the $E^{\xi}_{\Sigma} = 0$ line, which is the dotted line that extends the furthest to the right. The field here is negative, and so the charge will be accelerated towards the left.

3.1.6 Connection to the tail term in the GLAD equation

The GLAD equation is

$$\begin{aligned} f^{\mu} =& qF^{\text{in}\mu}_{\nu}u^{\nu} + \frac{2}{3}q^2\left(\frac{Da^{\mu}}{D\tau} - u^{\mu}a^{\nu}a_{\nu}\right) \\ &-\frac{1}{3}q^2\left(R^{\mu}_{\nu}u^{\nu} + R_{\alpha\beta}u^{\alpha}u^{\beta}u^{\nu}\right) + q^2u^{\nu}\int_{-\infty}^{\tau^-} f^{\mu}_{\nu\gamma}u^{\gamma}(\tau_Q)d\tau_Q \end{aligned} \tag{3.47}$$

In our case, the external field term is zero, since the only field here comes from the charge itself. The explicit curvature term $R^{\mu}_{\nu}u^{\nu} + R_{\alpha\beta}u^{\alpha}u^{\beta}u^{\nu}$ is zero since $R^{\mu}_{\nu} = 0$ in the absence of mass (our particle and field are considered to be too weak to have any effect on curvature). For the second term, we need $a^{\mu} = (0, g, 0, 0)$ (from $a^{\mu}a_{\mu} = g^2$, $u^{\mu}a_{\mu} = 0$ and since $a^y = a^z = 0$ due to symmetry), where $g = \frac{1}{x_0}$, and get

$$\begin{aligned} \frac{Da^{\mu}}{D\tau} - u^{\mu}a^{\nu}a_{\nu} =& \frac{\partial a^{\mu}}{\partial\tau} + \Gamma^{\mu}_{\alpha\beta}a^{\alpha}u^{\beta} - u^{\mu}g^2 \\ =& \Gamma^{\mu}_{\xi,t}g - u^{\mu}g^2 \end{aligned} \tag{3.48}$$

the only not obviously nonzero component of which is

$$\frac{Da^{t}}{D\tau} - u^{t}a^{\nu}a_{\nu} = \Gamma^{t}_{\xi,t}g - g^2 = \frac{g}{x_0 + \xi} - g^2 = 0 \tag{3.49}$$

where I have used $\xi = 0$ at the particle's position. The GLAD equation thus reduces to an equation for the tail term, and we will use this to find an explicit expression for this term.

Let us first consider what we would get if we had had normal topology. As we saw in section 3.1.3, this case gives $f^\mu = 0$, which means that the tail term is zero in this case. With tubular topology, we get a tail term of qE^ξ_Σ, so

$$q^2 u^\nu \int_{-\infty}^{\tau} f^\xi_{\nu\gamma} u^\gamma(\tau_Q) d\tau_Q = \frac{-q^2}{x_0 L} \sum_{n=1}^{\infty} \left(n \sqrt{1 + \frac{1}{4} n^2 \frac{L^2}{x_0^2}}^3 \right)^{-1} \tag{3.50}$$

$$\approx \frac{-4q^2 x_0^2 \pi^4}{45 L^4} \tag{3.51}$$

In this case, the tail term is simply the effect of the particle's world line intersecting its light cone multiple times. The part of the field headed in the y direction encounters the particle after looping around the circular y dimension, then loops around again and repeats the encounter, and so on, each time imparting a small force in the negative x direction on the particle.

3.2 Direct evaluation of the tail term

The method used for calculating the tail term in the previous section was rather roundabout, and the tail term was never explicitly evaluated. In this section I will derive the same result by explicit calculation of the tail term.

The tail term is

$$\mathrm{tail}_\mu = q^2 u^\nu \int_{-\infty}^{0^-} \left(G_{+\nu\rho';\mu} - G_{+\mu\rho';\nu} \right) u^{\rho'} dt' \tag{3.52}$$

where I have used $d\tau = dt'$, which is due to the definition of the time coordinate in Rindler coordinates. Our particle is stationary in the Rindler system, $u^\mu \propto \delta^\mu_t$, and using the four-velocity identity, we find $g_{tt} {u^t}^2 = -1 \Rightarrow u^t = 1$ for a particle positioned at x_0. Using this, and introducing $\Delta t = t - t'$, we find

$$\begin{aligned} \mathrm{tail}_\mu &= q^2 \int_{0^+}^{\infty} \left(G_{+tt';\mu} - G_{+\mu t';t} \right) d\Delta t \\ &= q^2 \int_{0^+}^{\infty} \left(G_{+tt',\mu} - G_{+\mu t',t} \right) d\Delta t \end{aligned} \tag{3.53}$$

Inserting the Green function from equation (2.79), this becomes

$$\mathrm{tail}_\mu = q^2 \int_{0^+}^{\infty} \Big([\theta(\Delta t)\delta(\sigma) g_{tt'}]_{,\mu} - [\theta(\Delta t \delta(\sigma) g_{\mu t'}]_{,t} \Big) d\Delta t' \tag{3.54}$$

Since the point of coincidence $\Delta t = 0$ is excluded from the integral (the contributions from that point are taken into account in the other parts of the GLAD equation), the step functions have no effect, and can be removed. Further, since the integration limits are constant, we can switch the order of integration and derivation:

$$\mathrm{tail}_\mu = q^2 \left(\partial_\mu \int_{0^+}^{\infty} \delta(\sigma) g_{tt'} d\Delta t + \partial_{\Delta t} \int_{0^+}^{\infty} \delta(\sigma) g_{\mu t'} d\Delta t \right) \tag{3.55}$$

Due to the presence of $\delta(\sigma)$ it is best to switch from Δt to σ as the independent variable in the integral. From equation (2.3.2) we see that $\Delta t(\sigma)$ is a double-valued function, but the integration limits make it clear that the positive branch should be used

$$\Delta t = x_0 \operatorname{acosh} \frac{a - \sigma}{x x_0} \tag{3.56}$$

with $a = \frac{1}{2}\left(x^2 + x'^2 + \rho^2\right)$. We see that $\Delta t = 0$ corresponds to $\sigma = a - x x_0$, and $\Delta t = \infty$ corresponds to $\sigma = -\infty$. We will also need

$$\sigma_{\Delta t} = -\, x \sinh \frac{\Delta t}{x_0} \tag{3.57}$$

$$= -\, x \sqrt{\cosh \left(\frac{\Delta t}{x_0} \right)^2 - 1} \tag{3.58}$$

Since $\cosh \frac{\Delta t}{x_0} = \frac{a - \sigma}{x x_0}$, we get

$$\sigma_{\Delta t} = -\, \frac{1}{x_0} \sqrt{(a - \sigma)^2 - x^2 x_0^2} \tag{3.59}$$

Performing the change in variable, the expression for the tail term becomes

$$\begin{aligned}\mathrm{tail}_\mu &= -\, q^2 \left(\partial_\mu \int_{-\infty}^{(a - x x_0)^-} \delta(\sigma) g_{tt'} \sigma_{\Delta t}^{-1} d\sigma + \partial_{\Delta t} \int_{-\infty}^{(a - x x_0)^-} \delta(\sigma) g_{\mu t'} \sigma_{\Delta t}^{-1} d\sigma \right) \\ &= -\, q^2 \left(\partial_\mu \left[\frac{g_{tt'}}{\sigma_{\Delta t}} \right]_{\sigma = 0} + \partial_{\Delta t} \left[\frac{g_{\mu t'}}{\sigma_{\Delta t}} \right]_{\sigma = 0} \right) \end{aligned} \tag{3.60}$$

We here assume that the δ-function has its zero within the limits of the integral, which is equivalent to assuming $(a - x x_0)^- > 0$. The particle is positioned at $x = x_0$, so $a = x_0^2 + \frac{1}{2}\rho$, and the assumption becomes $\rho^- > 0$. Of course, the particle also has $\rho = 0$, but as we saw earlier, the cylinder topology will introduce mirror images of the particle with $\rho = -nL$. The assumption is valid for all $n \neq 0$. For the remaining $n = 0$ term, the δ-functions never turn on, and we are left with 0.

Symmetry confines tail_μ to be pointing in the t or x direction, and inspection of equation (3.53) immediately gives $\mathrm{tail}_t = 0$. We are left with the x direction. Introducing the shortcuts $c \equiv \cosh \frac{\Delta t}{x_0}$ and $s \equiv \sinh \frac{\Delta t}{x_0}$, we have

$$\begin{aligned}
g_{tt'} &= -\frac{x}{x_0}c = \frac{\sigma - a}{x_0^2} && \Rightarrow \quad [g_{tt'}]_{\sigma=0} = -\frac{a}{x_0^2} \\
g_{xt'} &= -\frac{x_0}{x}s = -\frac{1}{x^2}\sqrt{(a-\sigma)^2 - x^2x_0^2} && \Rightarrow \quad [g_{xt'}]_{\sigma=0} = -\frac{1}{x^2}\sqrt{a^2 - x^2x_0^2} \\
[\sigma_{\Delta t}]_{\sigma=0} &= -\frac{1}{x_0}\sqrt{a^2 - x^2x_0^2}
\end{aligned}$$

Inserting this, we find

$$\begin{aligned}
\mathrm{tail}_\mu &= -q^2\left(\partial_x\left[\frac{a}{x_0\sqrt{a^2 - x^2x_0^2}}\right] + \overbrace{\partial_{\Delta t}\left[\frac{x_0}{x^2}\right]}^{0}\right) \\
&= -q^2\left[\frac{x}{x_0\sqrt{a^2 - x^2x_0^2}} - \frac{a(2ax - 2xx_0^2)}{2x_0\sqrt{a^2 - x^2x_0^2}^3}\right] \\
&= \frac{-q^2x}{x_0\sqrt{a^2 - x^2x_0^2}^3}(-x^2x_0^2 + x_0^2a) \\
&= \frac{-q^2}{x_0R^3}(a - x_0^2) = \mathrm{tail}^\mu
\end{aligned} \tag{3.61}$$

where I have used $R = x_0\sqrt{a^2 - x^2x_0^2}$ and $x = x_0$ at the location of the particle.

Applying the effects of the cylinder topology, we find that the contribution to the tail term from the n'th mirror image of the particle is

$$\mathrm{tail}_n^\mu = \begin{cases} \frac{-q^2}{2x_0^2|n|L\sqrt{1+\left(\frac{nL}{2x_0}\right)^2}^3} & n \neq 0 \\ 0 & n = 0 \end{cases} \tag{3.62}$$

When we collect these we find the total force from the tail term to be

$$\mathrm{tail}^\mu = \frac{-q^2}{x_0L}\sum_{n=1}^{\infty}\left(n\sqrt{1 + \frac{n^2L^2}{4x_0^2}}^3\right)^{-1} \tag{3.63}$$

This is consistent with equation (3.42), as $f^\mu = qF^\mu_\nu u^\nu = qF^\mu_t = qE^\mu$ for a stationary particle. We did not have to worry about divergence this time, as the GLAD equation takes care of this.

3.3 Radiated energy

The radiation reaction is the extra force needed to make a charged particle behave like an equivalent uncharged particle. This extra force does extra work on the particle, but since the particle still only receives as much kinetic energy as the uncharged one (since we require them to follow the same path) that extra energy must go somewhere else than the particle, and the only place available is the field. We do not know a priori what happens to this energy in the field. A natural guess would be that it is radiated away to be observed by remote observers, but this is not necessarily the case. In our case, we absorbed an infinite term proportional to the acceleration into the mass of the particle when calculating the equation of motion for a hyperbolically accelerated charged particle. If the energy associated with this term had been radiated away, there would be infinite radiation, which we do not observe. The term must therefore correspond to non-radiative field energy.

This motivates the division of the radiation reaction into two forces: A conservative force, storing energy in the field without radiating it away, and a nonconservative force, which leads to radiation.

3.3.1 Radiated effect in Rindler frame with normal topology

The basic procedure for calculating the energy per time radiated by a particle, is to calculate the flow of four-momentum through a closed surface containing the particle. The surface is taken to be far away in order to find only the energy that escapes to infinity. Finding $F^{\mu\nu}$ on this surface, one can calculate the stress-energy tensor using

$$T^{\mu}{}_{\nu} = \frac{1}{4\pi}\left(F^{\mu\rho}F_{\nu\rho} - \frac{1}{4}\delta^{\mu}_{\nu}F^{\rho\sigma}F_{\rho\sigma}\right) \tag{3.64}$$

which is valid for 3+1 dimensions. The flow of momentum through an element of the surface is then given by

$$p^{\mu} = T^{\mu\nu}d\Sigma_{\nu} \tag{3.65}$$

where $d\Sigma_{\nu}$ is the one-form of the surface element.

In the case of a particle in the Rindler frame, it is sufficient to use the horizon (or really a plane infinitely close to it) as the surface. This is somewhat surprising, as the horizon does not form a closed surface, but it is not hard to see that any geodesic on and inside the light cone, with one exception, will approach the horizon sooner or later. Figure 3.3 shows the situation in both the Minkowski and Rindler frame. In the Minkowski frame, the horizon of a particle moving according to $\overline{x}^2 = x_0^2 + \overline{t}^2$ is the plane with $\overline{x} = \overline{t}$. Any geodesic (which is a straight line in this case) with

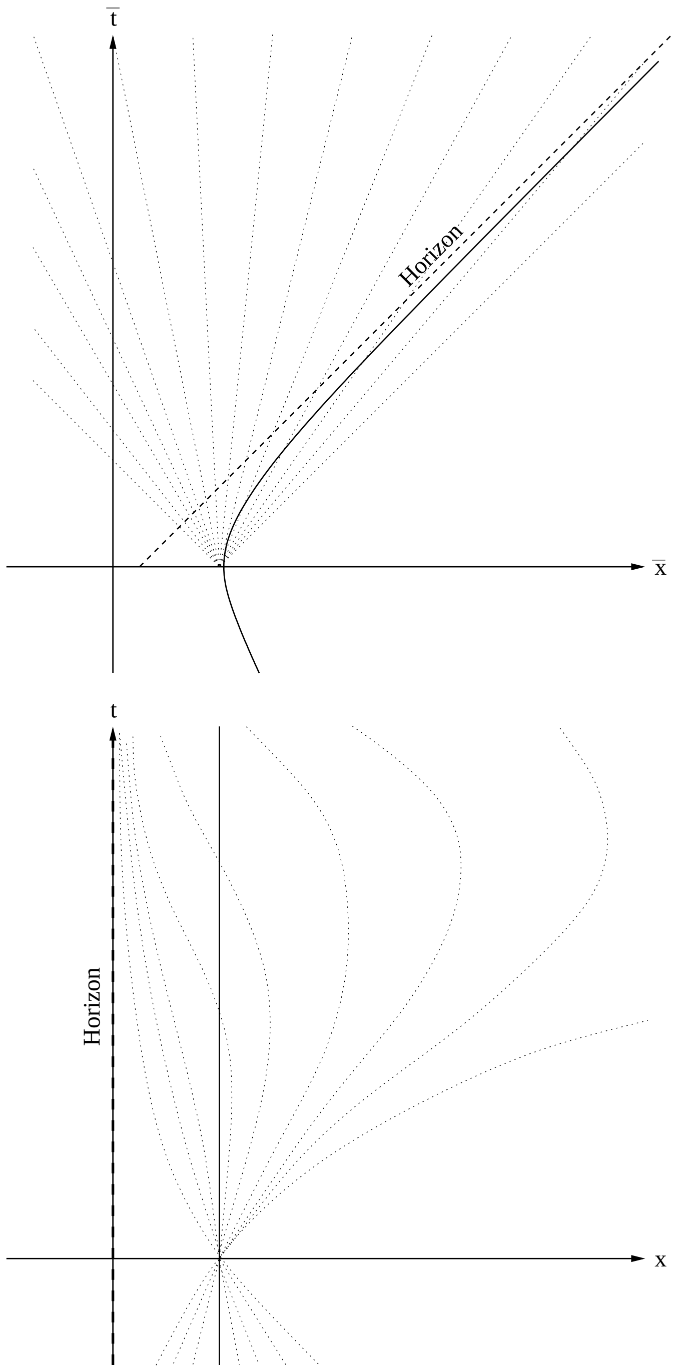

Figure 3.3: All geodesics with effective velocity in the x direction less than that of light, will eventually cross the horizon (Minkowski, top) or approach it (Rindler, bottom).

$v_x < c$ starting to the right of this plane, is going to cross it eventually. Note that this also applies to light-like geodesics, as long as some of their velocity is in other directions than x. The exception mentioned is the case of a geodesic pointing exactly in the x direction, which is negligible compared to all the others.

To find the stress-energy tensor near the horizon, we need the nonzero components of the electromagnetic field tensor there, and these are listed in equations (3.22) and (3.23):

$$F^{tx} = -F^{xt} = \frac{q}{xR^3}(x^2 - a) \tag{3.66}$$
$$F^{ty} = -F^{yt} = \frac{qy}{R^3} \tag{3.67}$$
$$F^{tz} = -F^{zt} = \frac{qz}{R^3} \tag{3.68}$$

where I have decomposed $F^{t\rho}$ into F^{ty} and F^{tz}. When we approach the horizon ($x \to 0$), we find, using equation (3.12) and (3.13),

$$a \to \frac{1}{2}\left(x_0^2 + \rho^2\right) \tag{3.69}$$
$$R \to \frac{a}{x_0} \tag{3.70}$$

To construct the surface element 1-form, we let the element be spanned by $dy\mathbf{e}_y$, $dz\mathbf{e}_z$, $dt\mathbf{u}_{\text{particle}} = -dt\mathbf{e}_t$, giving $d\Sigma_\mu = -\epsilon_{\mu\alpha\beta\gamma} dy\mathbf{e}_y^\alpha dz\mathbf{e}_z^\beta dt e_t^\gamma = dydzdt\delta_\mu^x$. With this, $T^{\mu\nu}d\Sigma_\nu$ will be the momentum an observer at the particle's position sees cross the surface element during the period dt. We see that only $T^{\mu x}$ is needed, and find

$$T^t{}_x = \frac{1}{4\pi}\left(F^{t\mu}F_{x\mu} - \frac{1}{4}\delta_x^t F^{\mu\nu}F_{\mu\nu}\right) = 0 \tag{3.71}$$
$$\begin{aligned} T^x{}_x &= \frac{1}{4\pi}\left(F^{xt}F_{xt} - \frac{1}{2}\left[F^{tx}F_{tx} + F^{ty}F_{ty} + F^{tz}F_{tz}\right]\right) \\ &\to \frac{-q^2x^2}{8\pi R^6 x_0^2}\left(x^{-2}a^2 - y^2 - z^2\right) \\ &\to -2\pi^{-1}q^2x_0^4\left(x_0^2 + \rho^2\right)^{-4} \end{aligned}$$
$$T^y{}_x = T^y{}_z = 0 \tag{3.72}$$

The total momentum flow through the horizon (or really a surface very close to it) per unit time, is given by

$$
\begin{aligned}
dp^t =& dp^y = dp^z = 0 && (3.73)\\
dp^x =& \int_S T^{xx} dydzdt \\
=& 2\pi dt \int_0^\infty \rho T^x x d\rho \\
=& -4q^2 x_0^4 dt \int_0^\infty \rho (x_0^2 + \rho^2)^{-4} d\rho \\
=& -2q^2 x_0^4 dt \int_{x_0^2}^\infty \eta^{-4} d\eta \\
=& -\frac{2q^2}{3x_0^2} dt && (3.74)
\end{aligned}
$$

Since the time-component of the momentum flow through the horizon is zero, no energy is escaping to infinity from the particle, so there is no radiation. That a constantly accelerating particle does not radiate according to a co-accelerating observer is well known, and also quite necessary. Any particle that is kept stationary on the surface of the Earth is (disregarding the effects of rotation) subject to a constant acceleration keeping it from falling towards the Earth's center. Thus, if constantly accelerated charged particles were to radiate in a co-accelerating frame, every charged particle on the Earth would be an infinite source of radiation.

That the x-component of the momentum flow is nonzero is difficult to understand. It means that the field is exerting a force in the negative x direction on the horizon, which must be offset by a corresponding force on the field or the particle. Since this force is proportional to the acceleration squared, it is not affected by mass renormalization, and we can therefore use the fact that the particle does not feel a radiation reaction to conclude that the opposing force must affect only the field. I have, however, not come up with an interpretation of this force.

3.4 The relevance of this case

A hyperbolically accelerated charge in a universe with cylinder-topology is really a trivial modification of the well-known example of a plain hyperbolically accelerated charge. Most of this chapter was devoted to rederiving that result, and the boundary conditions could be implemented quite trivially. Nevertheless, this chapter illustrates an important point about the tail term: It does not only get contributions from the "tail" of the Green function, even though that is what gives it its name. This tail is the part of the Green function that is nonzero inside the light cone. In our case, there is no such component, $G^{\mu}_{\mu'} \propto \delta(\sigma)$, but as shown we still get a nonzero contribution from the tail term with this topology.

The tubular topology used here is admittedly a bit strange. For small values of L, it could perhaps represent the case of a "rolled up" dimension. Regardless of the direct applicability of this case, it illustrates an effect that also occurs in more complicated geometries. In Schwarzschild geometry, for example, a light ray sent in the correct angle towards a black hole will be directed back at the emitter, meaning that one would find a tail term for a charged particle in the vicinity of a black hole even if there had not been a dispersive tail in the Green function for this case.

3.4.1 Example: Electron feeling a constant acceleration of $g = 9.82\frac{m}{s^2}$

It is interesting to see how large the force considered in this chapter actually would be. To get results in SI units, we need to convert equation (3.42) from cgs units with $c = 1$. This gives $q \to \frac{q}{\sqrt{4\pi\epsilon_0}}$, and since $\frac{1}{x_0}$ should be an acceleration, we get $\frac{1}{x_0} = \frac{g}{c^2}$, which gives

$$f^x = f^\xi = -\frac{q^2 g}{c^2 L}\sum_{n=1}^{\infty}\left(n\sqrt{1+\frac{1}{4}n^2\frac{L^2g^2}{c^4}}\right)^{-1} \tag{3.75}$$

Choosing the case of an electron feeling a constant acceleration equal to that on the Earth's surface, we have $q = e = 1.60 \cdot 10^{-19}C$, $g = 9.82\frac{m}{s^2}$, which gives $L_0 = \frac{2c^2}{g} \approx 1.9$ light years. When $L \gg L_0$, we may use the simplified equation (3.44), and choosing $L = 10$ light years, we find

$$\begin{aligned} f^x &= -\frac{q^2c^4\pi^3}{45g^2\epsilon_0 L^4} \\ &= -2.1 \cdot 10^{-67}N \end{aligned} \tag{3.76}$$

corresponding to an acceleration of $2.3 \cdot 10^{-37}\frac{m}{s^2}$, which is totally negligible. Reducing the radius, the force increases as L^{-4} while $L \gg L_0$, but it is not immediately obvious what will happen after this, as the limits of the sums in the approximated equation (3.45) vary along with the other factors. Figure 3.4 shows this force as a function of L. We see that the force goes as $\frac{1}{L}$ for $L \ll L_0$. The self force becomes notable when L reaches the nm scale, where we do not expect classical electrodynamics to work very well. Let it also be noted that only two large spatial dimensions remain after one dimension has been rolled up, so for both these reasons, this case does not cast much light on the case of small rolled-up dimensions, which appear in non-standard theories like string theory.

Since the part of the tail term illustrated in this chapter has nothing to do with (local) curvature, but arose due to the overall shape of the spacetime, I will call this a topological tail term. The next chapter will attack the other

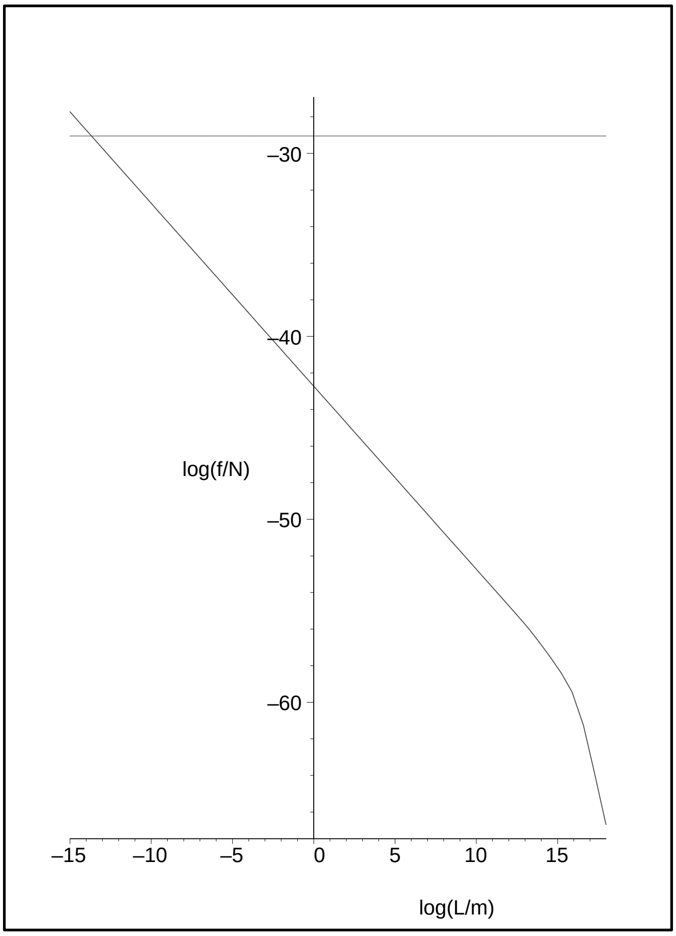

Figure 3.4: Logarithmic plot of the force an electron subject to a constant acceleration of $g = 9.82\frac{m}{s^2}$ experiences in a 3+1-dimensional universe where one spatial dimension perpendicular to the acceleration is cyclic with period L, as a function of that period. We see that the force has a slope of 1 for most values of L, so $f \propto \frac{1}{L}$. The constant line is the force $f = m_e g$ corresponding to the acceleration. The self force is negligible compared to this force for all but the smallest values of L.

possibility: a tail term due to a Green function which is nonzero inside the light cone.

Chapter 4

The tail term in a cylindrical 4+1-dimensional Rindler metric

4.1 The scalar Green function in 4+1-dimensional Minkowski spacetime

In normal 3+1-dimensional flat spacetime, the scalar Green function was found by solving

$$G(\mathbf{x},\mathbf{x}')^{;\mu}_{\mu} = -4\pi\delta_4(\mathbf{x},\mathbf{x}') \Leftrightarrow \tag{4.1}$$

$$\left(-\frac{\partial^2}{\partial\bar{t}^2} + \nabla^2\right) G(\mathbf{x},\mathbf{x}') = -4\pi\delta_4(\mathbf{x},\mathbf{x}') \tag{4.2}$$

where $\delta_4(\mathbf{x},\mathbf{x}') = (-g)^{-\frac{1}{2}}\delta_4(\mathbf{x}-\mathbf{x}')$ is the invariant delta function, and we found, in equation (2.78)

$$G_{\pm} = \theta(\bar{t})\delta(\sigma)$$

In d dimensions, the general minimally coupled scalar wave equation in Minkowski spacetime is

$$G^{;\mu}_{\mu} = \left(-\frac{\partial^2}{\partial\bar{t}^2} + \nabla^2_d\right) G = -S_d\delta_d(x,x') \tag{4.3}$$

where S_d is the surface area of the d-dimensional unit sphere. Here, the Fourier transform method used in section (2.6.1) does not in general work (one is left with indeterminate integrals).

4.1.1 Examining the radial wave equation

However, it still holds that $G_{\pm}(x,x') = \theta(\pm\Delta\bar{t})g(x,x')$, which means that $G_{\pm}$ only depends on r and $\bar{t}$. It should therefore be informative to rewrite the equation in polar coordinates in space, keeping time separate. The prescription for this can be found section (2.3.4). The scalar wave equation itself is

$$
\begin{aligned}
G^{;\mu}{}_{\mu} &= - S_d\delta(\mathbf{x},\mathbf{x}') \Leftrightarrow && (4.4)\\
g^{\mu\nu}G_{,\mu\nu} - g^{\mu\nu}\Gamma^{\rho}{}_{\mu\nu}G_{,\rho} &= - S_d\delta(\mathbf{x},\mathbf{x}') && (4.5)
\end{aligned}
$$

and using the angular symmetry and the fact that the metric is diagonal, this is

$$
g^{ii}G_{,ii} - g^{\mu\mu}\Gamma^{i}{}_{\mu\mu}G_{,i} = - 2\delta(\Delta\bar{t})\delta(r) \qquad (4.6)
$$

where i takes the values $\bar{t}, r$, and the factor of 2 is there to compensate for the fact that r only covers half the delta function.

$$
\begin{aligned}
\Gamma^{i}{}_{\mu\mu} &= - \frac{1}{2}g^{ii}g_{\mu\mu,i} && (4.7)\\
&= \begin{cases} -r^{-1}g_{\mu\mu} & i=r \qquad \mu=\alpha_1,\alpha_2,\ldots \\ 0 & \text{otherwise} \end{cases} && (4.8)
\end{aligned}
$$

Since there are $d-1$ angles, we end up with

$$
\left(-\frac{\partial^2}{\partial\bar{t}^2} + \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}\right)G = - 2\delta(\bar{t})\delta(r) \qquad (4.9)
$$

To insert $G_{\pm}$ here, we need

$$
\begin{aligned}
G_{\pm\bar{t}\bar{t}} =& \theta(\pm\bar{t})g''\sigma_{\bar{t}}^2 + \theta(\pm\bar{t})g'\sigma_{\bar{t}\bar{t}} \pm 2\delta(\bar{t})g'\sigma_{\bar{t}} \pm \delta'(\bar{t})g && (4.10)\\
G_{\pm rr} =& \theta(\pm\bar{t})(g''\sigma_r^2 + g'\sigma_{rr}) && (4.11)\\
\sigma_{rr} =& - \sigma_{\bar{t}\bar{t}} = 1 && (4.12)\\
\sigma_r^2 - \sigma_{\bar{t}}^2 =& 2\sigma && (4.13)
\end{aligned}
$$

resulting in

$$
\theta(\pm\bar{t})(2\sigma g'' + (d+1)g') \pm 2\delta(\bar{t})g'\sigma_{\bar{t}} \pm \delta'(\bar{t})g = - 2\delta(\bar{t})\delta(r) \qquad (4.14)
$$

In areas where $t \neq 0$ and $r \neq 0$, we have the homogeneous equation

$$
\begin{aligned}
2\sigma g'' + (d+1)g' =& 0 \Leftrightarrow && (4.15)\\
\int \frac{dg'}{g'} =& - \frac{d+1}{2}\int \sigma^{-1}d\sigma \Leftrightarrow && (4.16)\\
g =& K_1\sigma^{-\frac{d-1}{2}} + K_2 && (4.17)
\end{aligned}
$$

Notice that the integrand of equation (4.16) diverges at $\sigma = 0$. We are therefore prevented from integrating over it, preventing us from associating the K_1 and K_2 in the $\sigma > 0$ region with those in the $\sigma < 0$ region.

At the light cone itself ($\sigma = 0$), the expression we found for σ cannot be assumed to hold. However, any deviations from this expression would have to be confined to the light cone, and therefore be proportional to delta functions and their derivatives.

Determining the constants K_1 and K_2 and the expression for g at the light cone directly, would involve examining the equation at the points where the delta functions activate, which is a very messy business, with varying levels of infinities. Poisson [13, sect. 4.1.2] has demonstrated a way around this in 3+1 dimensions by examining the volume integral of the inhomogeneous equation over a space-time sphere centered on the emitting point $\mathbf{x}'$. Since the right hand side of the inhomogeneous equation is a delta function, the resulting equation will be constant no matter the radius of the sphere, and taking the limit as the radius goes to zero lets one determine the most divergent terms of the solution to the original equation. I will here generalize this procedure to d dimensions as far as possible, and then specialize on 4 dimensions when necessary.

The volume integral of the inhomogeneous scalar equation can be expressed as a surface integral using the divergence theorem:

$$-S_d = \lim_{\rho \to 0} \int_V G_{\pm}{}^{;\mu}_{\mu} dV \tag{4.18}$$

$$= \lim_{\rho \to 0} \int_{\partial V} G_{\pm}{}^{;\mu} d\Sigma_\mu \tag{4.19}$$

Since we are dealing with a space-time sphere, it is useful to introduce the fully polar coordinates of section (2.3.4).

4.1.2 The surface element

The surface element $d\Sigma_\mu$ is normal to the surface, and has a length equal to the volume element of the space the surface spans. In our case the surface is a sphere centered on the origin, so $d\Sigma_\mu = \delta^\rho_\mu N d\omega \prod_{i=1}^{d-1} d\alpha_i$. The surface has all the degrees of freedom of the full space, except for ρ, so it is described by a metric with ρ projected away using $\gamma_{ij} = g_{ij} - \frac{g_{\rho i} g_{\rho j}}{g_{\rho\rho}}$, giving:

$$\gamma_{\omega\omega} = \frac{\rho^2}{\cos(2\omega)} \tag{4.20}$$

$$\gamma_{\alpha_i \alpha_i} = \rho^2 \sin^2(\omega) \prod_{n=1}^{i-1} \sin^2(\alpha_n) \tag{4.21}$$

so $\sqrt{|\gamma|} = \frac{\rho^d}{\sqrt{|\cos(2\omega)|}} \sin^{d-1}(\omega) \prod_{i=1}^{d-2} \sin^{d-i}(\alpha_n)$. The normalization factor of $d\Sigma_\mu$ is

$$\begin{aligned}|\gamma| &= |d\Sigma_\mu d\Sigma^\mu| = |g^{\rho\rho}| N^2 \Leftrightarrow \\ |N| &= \sqrt{\frac{|\gamma|}{|g^{\rho\rho}|}}\end{aligned}$$

N must be positive to make $d\Sigma_\mu$ point out of the surface, and we get

$$N = \rho^d \sin^{d-1}(\omega) \prod_{i=1}^{d-2} \sin^{d-i}(\alpha_i) \Rightarrow \tag{4.22}$$

$$d\Sigma_\mu = \delta^\rho_\mu \rho^d \sin^{d-1}(\omega) d\omega \prod_{i=1}^{d-1} \sin^{d-1-i}(\alpha_i) d\alpha_i \tag{4.23}$$

4.1.3 The surface integral

Our integrand involved $G_\pm(x, x') = \theta(\pm(\bar{t} - \bar{t}'))g(\sigma)$, and this should also be expressed in the polar coordinates. Since the coordinate system is centered on $\mathbf{x}'$, and since spacetime is flat, the geodesic between $\mathbf{x}'$ and $\mathbf{x}$ is purely radial, and so σ is easily found by setting the variation of all other coordinates equal to zero in the line element, and we find

$$\sigma = \frac{1}{2}s^2 = -\frac{1}{2}\cos(2\omega)\rho^2 \Rightarrow \tag{4.24}$$

$$G_\pm = \theta(\pm\rho\cos(\omega))g\left(-\frac{1}{2}\cos(2\omega)\rho^2\right) \tag{4.25}$$

The integral becomes

$$\int_{\partial V} G_\pm^{;\mu} d\Sigma_\mu = \int_{\partial V} g^{\mu\rho} G_{\pm,\mu} d\Sigma_\rho \tag{4.26}$$

The relevant derivatives are

$$G_{\pm,\rho} = \pm\cos(\omega)\delta(\rho\cos(\omega))g(\sigma) - \rho\cos(2\omega)\theta(\pm\rho\cos(\omega))g'(\sigma) \tag{4.27}$$

$$G_{\pm,\omega} = \mp\rho\sin(\omega)\delta(\rho\cos(\omega))g(\sigma) + \rho^2\sin(2\omega)\theta(\pm\rho\cos(\omega))g'(\sigma) \tag{4.28}$$

which gives

$$g^{\mu\rho} G_{\pm,\mu} = \mp\cos(\omega)\delta(\rho\cos(\omega))g(\sigma) + \rho\theta(\pm\rho\cos(\omega))g'(\sigma) \tag{4.29}$$

Since the integrand does not depend on the spatial angles α_i, these can be integrated away immediately, producing a factor of S_d, the area of the unit sphere in d dimensions. We are left with

$$S_d \int_0^\pi d\omega \rho^d \sin^{d-1} g^{\mu\rho} G_{\pm,\mu} = \mp S_d \int_0^\pi d\omega \rho^d \sin(\omega)\cos(\omega)\delta(\rho\cos(\omega))g(\sigma) + S_d \int_0^\pi d\omega \rho^{d+1} \sin^{d-1}(\omega)\theta(\pm\rho\cos(\omega))g'(\sigma) \tag{4.30}$$

A word on distributional identities

$x\delta(x)$ is said to be distributionally equal to zero, which means that $\int_{-\infty}^{\infty} x\delta(x)f(x)dx = 0$ for any $f(x)$, provided that $f(x)$ *is sufficiently nice*, in this case meaning that $\lim_{x\to 0} xf(x) = 0$, so $f(x)$ must diverge slower than x^{-1}. The distributional identities of the delta function are useful when one knows that the delta function is going to be integrated sooner or later, and that it isn't going to be exposed to any unpleasantly divergent expressions.

In our case, it is tempting to use this to remove the first term of the integral, but that is only possible if

$$\lim_{\rho\cos(\omega)\to 0} \rho^d \cos(\omega)g(\sigma) = 0 \Leftrightarrow$$
$$\lim_{\cos(\omega)\to 0} \rho^{d-1}\cos(\omega) g\left(\frac{1}{2}\rho^2 - \rho^2\cos^2\omega\right) = 0$$

For this not to hold, g has to diverge when $\cos(\omega) = 0$, which we cannot determine if is the case a priori. Instead, we will eliminate this term through a symmetry argument. $\delta(\omega)$, $\sin(\omega)$ and σ are all symmetric about the middle point of the integration, $\omega = \frac{\pi}{2}$, while $\cos(\omega)$ is antisymmetric. The product of a symmetric and antisymmetric function is an antisymmetric function, the symmetric integral of which is 0.

The step function in the remaining integral will change the limits to $\left[0, \frac{\pi}{2}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$ respectively. However, the rest of the integrand is symmetric around $\frac{\pi}{2}$, so both of these cases are equivalent. We can therefore restrict ourselves to the first case. Within these limits, σ is a monotonous function of ω, and the remaining integral can therefore easily be expressed as a function of just σ by performing a change of variable. Since $\sigma = -\frac{1}{2}\rho^2\cos(2\omega)$, $d\sigma = \rho^2\sin(2\omega)d\omega = 2\rho^2\sin(\omega)\cos(\omega)$. We get

$$\int_{\partial V} G_{\pm}^{;\mu} d\Sigma_\mu = \frac{1}{2} S_d \int_{-\frac{1}{2}\rho^2}^{\frac{1}{2}\rho^2} \rho^{d-1} \frac{\sin^{d-2}(\omega)}{\cos(\omega)} g'(\sigma)d\sigma \tag{4.31}$$

$$= \sqrt{2}^{-d+1} S_d \rho^{d-1} \int_{-\frac{1}{2}\rho^2}^{\frac{1}{2}\rho^2} f\left(\frac{2\sigma}{\rho^2}\right) g'(\sigma)d\sigma \tag{4.32}$$

where

$$f(x) = \sqrt{\frac{(1+x)^{d-2}}{1-x}} \tag{4.33}$$

A final simplification results from introducing $\epsilon = \frac{1}{2}\rho^2$, yielding

$$\begin{aligned}\epsilon^{\frac{d-1}{2}} S_d \int_{-\epsilon}^{\epsilon} f\left(\frac{\sigma}{\epsilon}\right) g'(\sigma) d\sigma = \int_{\partial V} G_{\pm}^{;\mu} d\Sigma_\mu = -S_d \Leftrightarrow \\ \epsilon^{\frac{d-1}{2}} \int_{-\epsilon}^{\epsilon} f\left(\frac{\sigma}{\epsilon}\right) g'(\sigma) d\sigma = -1\end{aligned} \tag{4.34}$$

4.1.4 The divergent behavior of $g(\sigma)$

The right hand side of equation (4.34) is independent of ϵ. This is not possible in the limit $\epsilon \to 0$ if $g(0)$ is finite. To see this, we assume that $g(\sigma)$ can be written as a power series, with α as the lowest order exponent. Since σ and ϵ are of the same order, $f\left(\frac{\sigma}{\epsilon}\right))$ is of order 1, and furthermore has no information of the order of magnitude of ϵ.

Since $d\sigma$ is of the same order of ϵ as σ, the whole integral is of order $\epsilon^{\frac{d+2\alpha-1}{2}}$, and this can only be of order 1, like the right hand side, if $\alpha = -\frac{d-1}{2}$, which is negative.

However, a power series is not enough to represent all possible shapes of g. A series of delta functions and their derivatives is another candidate. Since $\delta^{(n)}(\sigma)$ is of order ϵ^{-n-1}, such a term in the integral would be of order $\epsilon^{\frac{d-2n-3}{2}}$, so the highest possible n is $\frac{d-3}{2}$.

In the normal case of 3 spatial dimensions, and in general for an odd number of spatial dimensions, this gives an integer value for n, and in this case the delta functions are enough to describe the singular part of g. One then makes the Hadamard ansatz

$$g = V\theta(-\sigma) + \sum_{n=0}^{\frac{d-3}{2}} U_n \delta^{(n)}(\sigma) \tag{4.35}$$

where V and U_n are a smooth (i.e. nonsingular) functions, and would be functions of σ only in our case. This is the method used by DeWitt and Brehme [5], Hobbs [12] in their generalization of the LAD equation. However, this ansatz does not work well for even d, in which case the sum either stops too early, and the needed $\frac{d-3}{2}$ part is not included, or, if we choose to include the last half-step, get a half-integer derivative of the delta function, which is actually of the same form as the power series above.

We should, to be consistent, also consider combinations of powers of σ and derivatives of $\theta(-\sigma)$. From the orders of epsilon of the two parts, we see that any $\sigma^{n-\frac{d-1}{2}}\theta^{(n)}(-\sigma)$ will give an integral of the correct order of ϵ. Our new ansatz is therefore

$$g(\sigma) = \sum_{n=0} c_n \sigma^{n-\frac{d-1}{2}} \theta(-\sigma)^{(n)} + \text{less divergent terms} \tag{4.36}$$

The sum can be stopped when σ's exponent becomes positive. To see this, consider the identity $\delta(x)f(x) = \delta(x)f(0)$, which is valid as long as $f(0)$ makes sense. Differentiating this, we get $\delta'(x)f(x) = \delta'(x)f(0) - \delta(x)f'(x)$, $\delta''(x)f(x) = \delta''(x)f(0) - 2\delta'(x)f'(x) - \delta(x)f''(x)$, and so on. As long as the coefficient in front of a derivative of δ does not diverge when the argument of δ is zero, it is possible to rewrite it using lower derivatives of δ. Assuming that $d > 1$, so that the first term with positive exponent has a derivative of δ, it is easy to see that all terms with positive exponent can be absorbed into the ones with negative exponent.

4.1.5 Restricting the most divergent part

It will here become necessary to make assumptions about d, so we now set $d = 4$. This means that our ansatz for the most divergent part is

$$A_0 \sigma^{\frac{-3}{2}} \theta(-\sigma) - A_1 \sigma^{-\frac{1}{2}} \delta(\sigma) \tag{4.37}$$

Using partial integration on equation (4.34), and inserting (4.37), we get

$$\epsilon^{\frac{3}{2}} \left(\underbrace{\left[f\left(\frac{\sigma}{\epsilon}\right) g(\sigma) \right]_{-\epsilon}^{\epsilon}}_{0} - \epsilon^{-1} \int_{-\epsilon}^{\epsilon} f'\left(\frac{\sigma}{\epsilon}\right) g(\sigma) d\sigma \right) \tag{4.38}$$

$$= - \lim_{\epsilon \to 0} \epsilon^{\frac{1}{2}} \int_{-\epsilon}^{\epsilon} f'\left(\frac{\sigma}{\epsilon}\right) \left(A_0 \sigma^{\frac{-3}{2}} \theta(-\sigma) - A_1 \sigma^{-\frac{1}{2}} \delta(\sigma) \right) d\sigma \tag{4.39}$$

This integral contains problematic divergences due to our integration over the point $\sigma = 0$, when our integral contains negative powers of σ. To show that it is possible to make these divergences cancel, I will introduce a small shift $\kappa\epsilon$ in the σ for the θ and δ function, and later let $\kappa \to 0$. The second part of the integral then becomes

$$\lim_{\epsilon \to 0} \epsilon^{\frac{1}{2}} A_1 (-\kappa\epsilon)^{-\frac{1}{2}} f'(0) = A_1 (-\kappa)^{-\frac{1}{2}} f'(0)$$

For the first part, we make the substitution $x = \frac{\sigma}{\epsilon}$, giving

$$- \lim_{\epsilon \to 0} \epsilon^{-\frac{1}{2}} \int_{-\epsilon}^{\epsilon} f'\left(\frac{\sigma}{\epsilon}\right) A_0 \sigma^{\frac{-3}{2}} \theta(-\sigma - \kappa) d\sigma = - \int_{-1}^{-\kappa} A_0 f'(x) x^{\frac{-3}{2}} dx \tag{4.40}$$

Recalling the definition of $f(x)$ from equation (4.33), we find

$$f'(x) = \frac{(1+x)^{d-3}}{2f(x)(1-x)^2}\left((d-2)(1-x)+(1+x)\right) \Rightarrow f'(0) = \frac{d-1}{2} = \frac{3}{2}$$

and using Maple, the second integral evaluates to

$$\begin{aligned} -\lim_{\kappa\to 0}\int_{-1}^{-\kappa} A_0 f'(x)x^{\frac{-3}{2}}dx =& \lim_{\kappa\to 0} A_0\left[\frac{-(5x-3)(1+x)}{(1-x)f(x)\sqrt{x}}\right]_{-1}^{-\kappa} \\ =& \lim_{\kappa\to 0} A_0\left(3(-\kappa)^{-\frac{1}{2}} - 4\sqrt{2}(-1)^{-\frac{1}{2}}\right) \end{aligned}$$

Combining the two parts, we can now finally determine A_0 and A_1:

$$\begin{aligned} -1 =& \frac{3}{2}A_1(-\kappa)^{-\frac{1}{2}} + 3A_0(-\kappa)^{-\frac{1}{2}} + 4\sqrt{2}iA_0 \Leftrightarrow \\ A_1 =& -2A_0 \wedge A_0 = \frac{i}{4\sqrt{2}} \end{aligned} \tag{4.41}$$

The most divergent part of $g(\sigma)$ must thus be

$$\underbrace{\frac{i}{4\sqrt{2}}}_{b}\left(\sigma^{-\frac{3}{2}}\theta(-\sigma) + 2\sigma^{-\frac{1}{2}}\delta(\sigma)\right) = \frac{-i}{2\sqrt{2}}\frac{d}{d\sigma}\left(\frac{\theta(-\sigma)}{\sqrt{\sigma}}\right) \tag{4.42}$$

4.2 The less divergent parts of $g(\sigma)$

Though we have found the most divergent part of $g(\sigma)$, we have not shown that this is the only part of the function. Naming the most divergent part $h(\sigma)$, we have, however, shown that

$$\int_V H_\pm{}^{;\mu}_{\mu} dV = -S_d \qquad \mathbf{x}' \in V \tag{4.43}$$

$$\int_V H_\pm{}^{;\mu}_{\mu} dV = 0 \qquad \text{else} \tag{4.44}$$

$$\int_V (G_\pm - H_\pm)^{;\mu}_{\mu} = 0 \tag{4.45}$$

where $H_\pm = \theta(\pm(\bar{t}-\bar{t}'))h(\sigma)$. The last of these equations is valid for any volume V, since $G_\pm$ also fulfills the two previous equations, and the interior of the integral must therefore also be zero:

$$(G_\pm - H_\pm)^{;\mu}_{\mu} = 0 \tag{4.46}$$

that is, the less divergent parts of the Green function must follow the homogeneous wave equation.

Since $G_\pm - H_\pm \equiv K_\pm$ is supposed to be part of either the retarded or advanced Green function, it must satisfy the correct boundary condition. For the retarded version, it must be proportional to $\theta(\bar{t})$, which means that all $\bar{t}$-derivatives are zero at the initial boundary; K_+ starts out being time-independent. Since the differential equation itself is time-independent (though derivatives of $\bar{t}$ are involved, the equation does not depend on the value of $\bar{t}$), K_+ will stay time-independent, and so must be zero for all values of $\bar{t}$. An analogous argument holds for K_-. Thus, the full solution for the inhomogeneous equation is $G_\pm = H_\pm$:

$$G_\pm = \theta(\pm\Delta\bar{t}) \frac{-i}{2\sqrt{2}} \frac{d}{d\sigma} \left(\frac{\theta(-\sigma)}{\sqrt{\sigma}} \right) \tag{4.47}$$

4.3 The electromagnetic Green function in 4+1-dimensional Minkowski spacetime

The Green equation is now

$$\Box G^\mu_{\mu'}(\mathbf{x}, \mathbf{x}') - R^\mu_\nu G^\nu_{\mu'}(\mathbf{x}, \mathbf{x}') = -4\pi g^\mu_{\mu'} \delta_5(\mathbf{x}, \mathbf{x}') \tag{4.48}$$

where R is 0 here, since spacetime is flat. The only change from the scalar case, then, is the addition of extra indices. In Minkowski coordinates, however, neither $\Box$ nor parallel transport involves any mixing of coordinates, and the equation can therefore be solved separately for each coordinate, and the equation we end up with for each coordinate, is simply the scalar equation we solved in the previous section. This means that we immediately can write up the solution for these coordinates:

$$G^\mu_{\mu'} = \delta^\mu_{\mu'} G \tag{4.49}$$

Switching to other coordinates will not change the scalar G, but $\delta^\mu_{\mu'}$, which compares coordinates at two different points directly, will in general have to be replaced by the parallel propagator:

$$G^\mu_{\mu'} = g^\mu_{\mu'} G \tag{4.50}$$

4.3.1 Comparison of the 4+1d and 3+1d Green functions

As we have seen, the Green function in normal 3+1-dimensional flat spacetime is proportional to $\delta(\sigma)$. This means that only events that are on each other's light cone, are causally connected through the field the Green function describes.

Increasing the number of spatial dimensions by one, we see that this changes dramatically. Now, every event on *and inside* an event's light cone are causally connected to it. Thus, even though an event in spacetime has infinitesimal duration, it will continue influencing points inside its light cone (with rapidly diminishing intensity) for all eternity.

The visual appearance of this would be fading and blurring: A light bulb that is turned off will not appear to wink out immediately, but rather to fade gradually into obscurity, and a walking person would appear to be under slight motion-blurring.

The fact that light is not restricted to the light cone, is a violation of the strong version of Huygens' principle (see section 2.8), and the part of the Green function that lies inside the light cone is called the tail part of the Green function. That all solutions of the electromagnetic wave equation move at the speed of light, is usually taken for granted, but we shall later see that it actually requires a coincidence that will not in general occur.

4.4 The equation of motion for a hyperbolically accelerated particle in 4+1 dimensions

A charged particle is assumed to be forced into constant acceleration by some force, and we wish to find the correction to this force due to the self-force of the particle. This is the sort of question the generalized LAD equation answers, and this gives

$$\begin{aligned} f^\mu =& \frac{2}{3}q^2\left(\frac{Da^\mu}{Dt} - u^\mu a^\nu a_\nu\right) - \frac{1}{2}q^2\left(R^\mu_\nu u^\nu + R_{\alpha\beta}u^\alpha u^\beta u^\nu\right) \\ &+ q^2 u^\nu \int_{-\infty}^{\tau^-}\left(G_{+}{}_{\nu\gamma'}{}^{;\mu} - G_{+}{}^{\mu}_{\gamma';\nu}\right)u^{\gamma'}d\tau' \end{aligned} \tag{4.51}$$

As we have seen, the first parts of this come from the point of coincidence, where the emitting and absorbing points are the same, while the tail integral stops just short of this problematic point. That the first parts of this are zero when the acceleration is constant follows from an argument identical to the one made for 3+1 dimensions, and so it seems we are left with but the tail term.

The generalized LAD equation is, however, general as it is, not immediately usable when there are more than 3 spatial dimensions. During its derivation, crucial use is made of the Hadamard ansatz, where the coefficients in front of both the δ and θ-functions are assumed to be smooth, well-behaved functions that are well-amenable to expansions. This assumption is, as we have seen, wrong for 4+1 dimensions, and it is therefore not at all given that the equation holds in this case. We must therefore derive the equation for this case ourselves.

A particle with charge q feels, when exposed to the field $F_{\mu\nu}$, a Lorenz force $f^\mu = qF^\mu_\nu u^\nu$, with $F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu}$. A_μ is given by the Green function

$$A_\mu = q \int_{-\infty}^{\infty} G^{\mu}_{\mu'} u^{\mu'} dt' \tag{4.52}$$

As we have seen, there are several Green functions that fit the wave equation. The physically relevant one, the retarded Green equation, represents waves originating from the particle and carrying its energy and momentum away, while the advanced Green function represents the opposite. Both of these lead to a divergent field at the particle's own position, making it impossible to find the equation of motion simply by using the expression for the Lorenz force directly; some technique must be used to deal with the singularity.

4.4.1 Handling the singularity

The usual method for dealing with this in flat 3+1-dimensional spacetime, due to Dirac, is to construct a new Green function, called the singular or symmetric Green function, G_s, defined as the average of the advanced and retarded green function. Since it is the average of two functions that both solve the inhomogeneous wave equation, so does it, and it therefore contains the same singular behavior as the retarded Green function. Subtracting this from the retarded Green function should therefore result in a new Green function without the singular behavior, solving the homogeneous wave equation. This is called the radiative Green function: $G_R \equiv \frac{1}{2}(G_+ + G_-)$.

The point of this exercise is that it can be argued that G_R results in the same equation of motion as the retarded function. The rationalization is, quoting Poisson [13, sect. 4.3.5], that

> The motion is not affected because it is intimately tied to the boundary conditions: If the waves are outgoing, the particle loses energy to the radiation and its motion is affected; if the waves are incoming, the particle gains energy from the radiation and the motion is affected differently. With equal amounts of outgoing and incoming radiation, the particle neither loses nor gains energy and its interaction with the scalar field cannot affect its motion. Thus, subtracting $G_s(x, x')$ from the retarded Green's function eliminates the singular part of the field without affecting the motion of the scalar charge.

Thus, for every point on the world line of the particle, one asserts that incoming waves of radiation converging on that point from past infinity, and being exactly balanced by emitted waves that escape to future infinity, can be subtracted without affecting the motion of the particle. This assumes that such a subtraction would only affect the particle *at the point* *considered*, or equivalently, that the radiation converging on one point does not intersect any other point on the world line. This is illustrated in figure 4.1. Since the Green function has a tail in our 4+1-dimensional case, these assumptions are violated, and we can therefore not use this technique.

Detweiler and Whiting [4] have generalized this technique to be usable even when the Green functions have a tail, by finding a function that, like G_S produces the same singular field at the location of the particle as the retarded Green function, but that, unlike G_S, is zero inside inside the light cone, preventing the problems detailed above. It must, however, be emphasized that this solution still fails when the world line intersects the light cone, and must therefore be restricted to the local convex neighborhood.

While this generalized technique solves the problem of the divergent field on the world line, it introduces a new problem: A function with the desired properties must be found. In our case, this problem is more difficult than dealing with the singularity through other means, so it is not useful here.

We will instead use the smoothing method introduced in section 3.1.3: We expect the field near the particle to resemble a Coulomb field, which is antisymmetric. Due to its antisymmetry, a Coulomb field pushes the particle that emits it equally in all directions, and so causes no net force. We therefore hope that averaging the field over a sphere centered on the particle, effectively removing the antisymmetric parts, will remove the divergence, leaving a finite force.

The only difference from the 3+1-dimensional case lies in the factors and exponents involved. Given a field $F^{\mu\nu}$, we find

$$\langle F^{\mu\nu}\rangle \equiv \frac{1}{S_d}\int_{S_d} F^{\mu\nu} dS_d \tag{4.53}$$

$$= \frac{S_{d-1}r}{S_d}\int_0^{\omega} \sin^{d-2}\omega F^{\mu\nu} d\omega$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin^2\omega F^{\mu\nu} d\omega \tag{4.54}$$

Notably, the exponent of $\sin\omega$ is now 2 instead of 1.

4.4.2 Finding the relevant part $F^{\mu\nu}$ near the particle

A particle of charge q stationary at $\mathbf{x}' = (t, x_0, 0, 0, 0)$ in 4+1-dimensional Rindler spacetime, has a 4-velocity $u^{\mu'} = \delta_t^{\mu'}$. It sets up a field potential A^μ given by

$$A^\mu(\mathbf{x}) = \int d\mathbf{x}' G^{\mu}_{+\mu'} j^{\mu'} \tag{4.55}$$

$$= q\int_{-\infty}^{\infty} G^{\mu}_{+t'} dt' = q\int_0^{\infty} g(\sigma) g^{\mu}{}_{t'} \Delta t \tag{4.56}$$

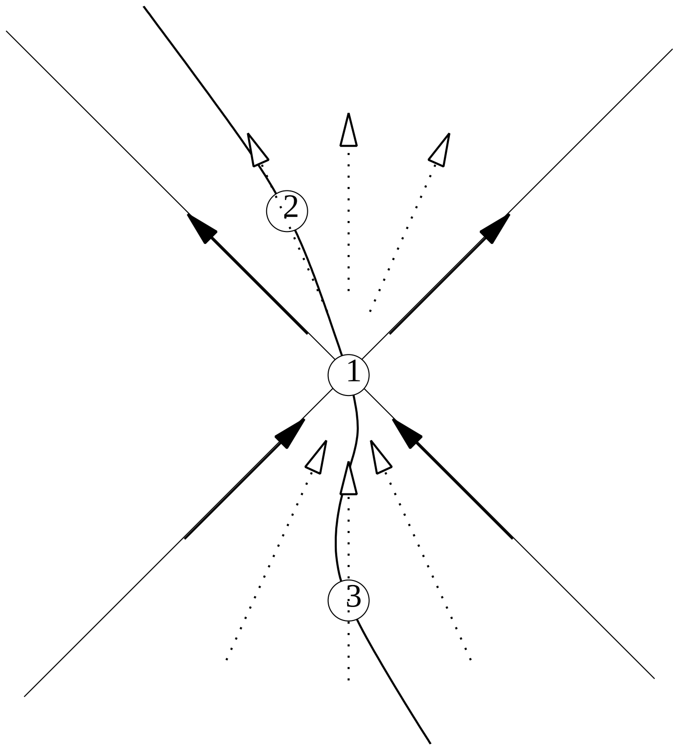

Figure 4.1: Radiation is converging on and diverging from point 1 on the world line of a particle, and is assumed to balance, having no effect on the given point. In the case of flat 3+1-dimensional space-time, this radiation follows the light cone (filled arrows), and will not intersect the world line at other points. In general, however, there will also have to be radiation inside the light cone (dotted arrows), and these may intersect the world line at other points. This happens at point 2 and 3. In this case, the radiation will disturb point 2 and 3 even though it does not disturb the point it is converging on. (In general, the light cone may end up being bent back towards the particle, and thus this effect can occur independently of whether the Green function has a tail.)

It is convenient to make σ the independent variable instead of Δt. However, Δt as a function of σ has two values. Thankfully, the negative of these is excluded by the integration limits.

$$A^\mu = -q \int_{-\infty}^{a-xx_0} g(\sigma) g^\mu{}_{t'} \sigma_{\Delta t}^{-1} d\sigma \tag{4.57}$$

Here $a \equiv \sigma(\Delta t = 0) = \frac{1}{2}\left(x^2 + x_0^2 + \rho^2\right)$. Of the five components of A^μ, only A^t is of interest. We are calculating the force on a hyperbolically accelerated charge, and symmetry only allows t and x components for the force. Since $f^\mu = qF^\mu{}_\nu u^\nu$, and u^μ only has a t component, this leaves $F^t{}_t = 0$ for f^t, and $F^x{}_t$ for f^x. So the force can only have an x component, and that is given by $F_{xt} = A_{t,x} - A_{x,t}$. The charge that sets up the field is static, as is the equation that governs it, so the resulting field must also be static. Hence $A_{x,t} = 0$. This is also easy to see by observing that none of the limits of integration depend on t, and no t's remain in the integrand since me changed the variable to σ, meaning that A_x is t-independent. We are left with A_t as the only interesting part of the field potential. Using equation (4.42) for $g(\sigma)$, we get

$$A_t = -bq \int_{-\infty}^{a-xx_0} \left(\sigma^{-\frac{3}{2}}\theta(-\sigma) + 2\sigma^{-\frac{1}{2}}\delta(\sigma)\right) \overbrace{g_{tt'}}^{-\frac{x}{x_0}c} \sigma_{\Delta t}^{-1} d\sigma \tag{4.58}$$

I have again introduced the shortcuts $c = \cosh\frac{\Delta t}{x_0}$, $s = \sinh\frac{\Delta t}{x_0}$. Since σ now is the independent variable, we should express these and $\sigma_{\Delta t}$ as functions of it instead of Δt. We have (equation (2.3.2)) $\sigma = a - xx_0 c \Leftrightarrow c = \frac{a-\sigma}{xx_0}$, and remembering that $\Delta t > 0$ we find $s = \sqrt{c^2 - 1} = \frac{1}{xx_0}\sqrt{(a-\sigma)^2 - x^2 x_0^2}$. Using this, the ts can be eliminated

$$g_{tt'} = \frac{\sigma - a}{x_0^2} \tag{4.59}$$

$$\sigma_{\Delta t} = -xs = \frac{-1}{x_0}\sqrt{(a-\sigma)^2 - x^2 x_0^2} \tag{4.60}$$

To find F_{xt}, this must be differentiated with respect to x. Since we are going to smooth F_{xt} by using averaging, this integral will never be evaluated at $\mathbf{x} = \mathbf{x}'$, and the upper limit of the integral, $a - xx_0$, will always be strictly positive. The integrand is zero for $\sigma > 0$, so the x-dependency of the upper limit can be ignored.

$$F_{xt} = A_{t,x} = -bq \int_{-\infty}^{a-xx_0} \left(\sigma^{-\frac{3}{2}}\theta(-\sigma) + 2\sigma^{-\frac{1}{2}}\delta(\sigma)\right) \frac{d}{dx}\frac{g_{tt'}}{\sigma_{\Delta t}} d\sigma \tag{4.61}$$

$$\left(\frac{g_{tt'}}{\sigma_{\Delta t}}\right)_{,x} = \frac{g_{tt',x}}{\sigma_{\Delta t}} - \frac{g_{tt'}\sigma_{\Delta t x}}{\sigma_{\Delta t}^2} \tag{4.62}$$

Using $g_{tt',x} = -\frac{x}{x_0^2}$ and $\sigma_{\Delta t x} = \frac{x}{x_0^2}\frac{a-\sigma-x_0^2}{\sigma_{\Delta t}}$, the latter simplifies to

$$\left(\frac{g_{tt'}}{\sigma_{\Delta t}}\right)_{,x} = -\frac{x}{x_0^2\sigma_{\Delta t}^3}(a-\sigma-x^2) \tag{4.63}$$

giving

$$F_{xt} = \frac{qbx}{x_0^2}\int_{-\infty}^{a-xx_0}\left(\sigma^{-\frac{3}{2}}\theta(-\sigma) + 2\sigma^{-\frac{1}{2}}\delta(\sigma)\right)\underbrace{\sigma_{\Delta t}^{-3}(a-\sigma-x^2)}_{f(\sigma)}d\sigma \tag{4.64}$$

I have here introduced the shorthand $f(\sigma) \equiv \sigma_{\Delta t}^{-3}(a-\sigma-x^2)$. As long as $\mathbf{x} \neq \mathbf{x}'$, $f(0)$ is finite, and $\delta(\sigma)f(\sigma) = \delta(\sigma)f(0)$. Using partial integration, we find

$$\int_{-\infty}^{a-xx_0} 2\sigma^{-\frac{1}{2}}\delta(\sigma)f_0 d\sigma = -\underbrace{\left[2\sigma^{-\frac{1}{2}}\theta(-\sigma)f_0\right]_{-\infty}^{a-xx_0}}_{0} - \int_{-\infty}^{a-xx_0}\sigma^{-\frac{3}{2}}\theta(-\sigma)f_0 d\sigma \tag{4.65}$$

so

$$F_{xt} = -\frac{qbx}{x_0^2}\int_{-\infty}^{a-xx_0}\sigma^{-\frac{3}{2}}(f_0 - f(\sigma))\theta(-\sigma)d\sigma \tag{4.66}$$

$$= -\frac{qbx}{x_0^2}\int_{-\infty}^{0}\sigma^{-\frac{3}{2}}(f_0 - f(\sigma))d\sigma \tag{4.67}$$

This expression bears strong similarities to that for 3 spatial dimensions. Expressed using f, we then got (equations (3.60) and (3.61)) $F_{xt} = \frac{qx}{x_0^2}\int_{-\infty}^{a-xx_0}\delta(\sigma)f(\sigma)d\sigma = \frac{qx}{x_0^2}f(0)$. This time, however, the integral is not killed by a delta function, and is in fact not analytically solvable. However, since we are going to evaluate the integral on the surface of a sphere centered on $\mathbf{x}'$, which radius is then going to go to zero, we only need the value of the integral for small values of $r = \sqrt{(x-x_0)^2 + rho^2}$. This should make some simplifications possible for this troublesome integral.

Naively expanding the integrand in powers of r is, however, not going to work. Such an expansion will only converge quickly when the expansion parameter is negligible compared to other terms, but in our case, σ takes on all values from $-\infty$ to 0, meaning that no matter how small an r is considered, one can always find values of σ which prevent the series from converging.

I will here take another approach to simplify the integral. Introducing

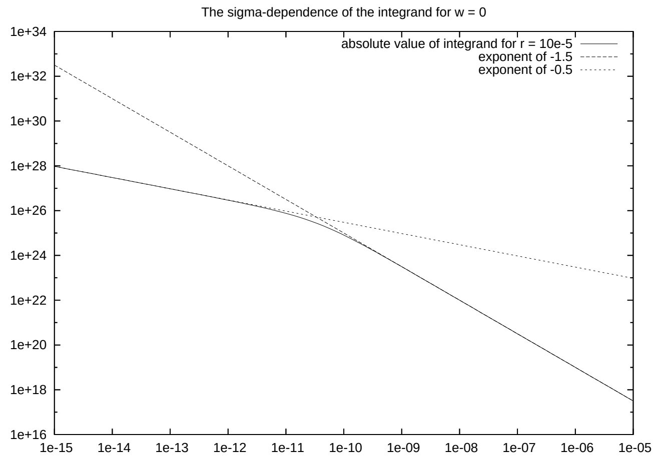

Figure 4.2: This figure shows the absolute value of $\sigma^{-\frac{3}{2}}(f(0) - f(\sigma))$ as a function of σ. We see that it goes as $\sigma^{-\frac{1}{2}}$ for low values of σ, and $\sigma^{-\frac{3}{2}}$ otherwise, the change occurring at about $|\sigma| = r^2$. It is easy to see that the integral is finite and well-defined.

polar coordinates $x = x_0 + r\cos\omega$, $\rho = r\sin\omega$, and the shorthands

$$\alpha = -\overbrace{x_0 r\cos\omega}^{1} - \overbrace{\frac{1}{2}r^2(2\cos\omega - 1)}^{2} \tag{4.68}$$

$$\beta = \overbrace{x_0^2 r^2}^{2} + \overbrace{x_0 r^3\cos\omega}^{3} + \overbrace{\frac{1}{4}r^4}^{4} \tag{4.69}$$

$$\gamma = -\overbrace{2x_0^2}^{0} - \overbrace{2x_0 r\cos\omega}^{1} \tag{4.70}$$

where the numbers indicate the order of r, we have

$$\sigma_{\Delta t} = \frac{-1}{x_0}\sqrt{\beta + \sigma\gamma + \sigma^2} \tag{4.71}$$

$$f = -x_0^3(\beta + \sigma\gamma + \sigma^2)^{-\frac{3}{2}}(\alpha - \sigma) \tag{4.72}$$

$$F_{xt} = qbxx_0\int_{-\infty}^{0}\sigma^{-\frac{3}{2}}\left(\alpha\beta^{-\frac{3}{2}} - (\beta + \gamma\sigma + \sigma^2)^{-\frac{3}{2}}(\alpha - \sigma)\right)d\sigma \tag{4.73}$$

The f_0 part of the integral is easily solved

$$-qbxx_0\int\sigma^{-\frac{3}{2}}\alpha\beta^{-\frac{3}{2}}d\sigma = 2qbxx_0\alpha\beta^{-\frac{3}{2}}\sigma^{-\frac{1}{2}} \tag{4.74}$$

The remaining, problematic part is $qbxx_0H$, where

$$H = \int \sigma^{-\frac{3}{2}}(\beta + \sigma\gamma + \sigma^2)^{-\frac{3}{2}}(\alpha - \sigma)d\sigma \tag{4.75}$$

This integral would have been solvable (though difficult) if the second order polynomial $\beta + \sigma\gamma + \sigma^2$ could be replaced by one of first order. This could be achieved either by using a series expansion with respect to σ^2, or one with respect to β. As was the case with a plain expansion with respect to r, such expansions are only usable for some values of σ, and we will now proceed to show that these two expansions compliment each other, together covering the whole range of σ.

Expanding by σ^2

We now consider σ^2 to be a small, independent variable, and expand $(\beta + \sigma\gamma + \sigma^2)^{-\frac{3}{2}}$, getting

$$(\beta + \sigma\gamma + \sigma^2)^{-\frac{3}{2}} = (\beta + \sigma\gamma)^{-\frac{3}{2}} - \frac{3}{2}(\beta + \sigma\gamma)^{-\frac{5}{2}}\sigma^2 + \frac{15}{8}(\beta + \sigma\gamma)^{-\frac{7}{2}}\sigma^4 + \cdots \tag{4.76}$$

To find which values of σ for which this converges, we make the assumption $\sigma \sim r^\nu$, and examine the leading power of r of each term. Since $\beta \sim r^2$ and $\gamma \sim r^0$, the leading power of $\beta + \sigma\gamma$ will be the least of 2 and ν. Assuming first that $\nu < 2$, we get

$$\underbrace{(\beta + \sigma\gamma)^{-\frac{3}{2}}}_{-\frac{3}{2}\nu} - \frac{3}{2}\underbrace{(\beta + \sigma\gamma)^{-\frac{5}{2}}\sigma^2}_{-\frac{1}{2}\nu} + \frac{15}{8}\underbrace{(\beta + \sigma\gamma)^{-\frac{7}{2}}\sigma^4}_{\frac{1}{2}\nu} + \cdots \tag{4.77}$$

We see that each term in the expansion is multiplied by r^ν compared to the previous term, so since r is very small, we have convergence for any $\nu > 0$.

For the case $\nu > 2$ we get

$$\underbrace{(\beta + \sigma\gamma)^{-\frac{3}{2}}}_{-3} - \frac{3}{2}\underbrace{(\beta + \sigma\gamma)^{-\frac{5}{2}}\sigma^2}_{-5+2\nu} + \frac{15}{8}\underbrace{(\beta + \sigma\gamma)^{-\frac{7}{2}}\sigma^4}_{-7+4\nu} + \cdots \tag{4.78}$$

This time, the factor between each term is $r^{2\nu-2}$, so for it to converge, we must have $\nu > 1$, which is consistent with $\nu > 2$. Thus, the expansion converges as long as $\nu > 0$, meaning that $|\sigma|$ should be less than about 1.

For an expansion to be practical, however, it is not enough that it converges; it must be enough only to consider the first few terms. As we have seen, the order of r of each term of the expansion increases, and so sooner or later, the exponent is going to become positive. Since we are interested in the limit $r \to 0$, all terms of positive power of r can be immediately discarded. It is important to note that the total order of r, including contributions from the rest of $qbxx_0F$, must be positive, not just the expansion of $(\beta + \sigma\gamma + \sigma^2)^{-\frac{3}{2}}$ in isolation. Since $qbxx_0$ is of order 1, the other contributions come from $\underbrace{d\sigma}_{\nu} \underbrace{\sigma^{-\frac{3}{2}}}_{-\frac{3}{2}\nu}(\underbrace{\alpha}_{1} - \underbrace{\sigma}_{\nu})$, which is of order $r^{\frac{1}{2}\nu}$ for $\nu < 1$ and order $r^{1-\frac{1}{2}\nu}$ for $\nu > 1$. The n'th term of the expansion, counting from 0, is of order $r^{(-\frac{3}{2}+n)\nu}$ for $\nu < 2$ and $r^{-3+2n(\nu-1)}$ for $\nu > 2$. We therefore have

$$\begin{array}{ll} \nu < 1 & r^{(n-1)\nu} \\ 1 < \nu < 2 & r^{1+(n-2)\nu} \\ 2 < \nu & r^{-2-\frac{1}{2}\nu+2n(\nu-1)} \end{array}$$

To stop the expansion after just one term, we must have positive exponents of r for $n = 1$, meaning that $\nu > \frac{8}{3}$, so the area of validity is $0 < |\sigma| < O(r^{\frac{8}{3}})$. If we include one more term, the area of validity increases to $\nu > 0$, corresponding to $0 < |\sigma| < O(1)$ and adding more terms does not increase it further.

Expanding by β

Proceeding in a similar fashion, we now consider β to be a small perturbation

$$(\beta + \sigma\gamma + \sigma^2)^{-\frac{3}{2}} = (\sigma\gamma + \sigma^2)^{-\frac{3}{2}} - \frac{3}{2}(\sigma\gamma + \sigma^2)^{-\frac{5}{2}}\beta + \frac{15}{8}(\sigma\gamma + \sigma^2)^{-\frac{7}{2}}\beta^2 + \cdots \tag{4.79}$$

For $\nu < 0$, the leading term of $\sigma\gamma + \sigma^2$ is of the order $r^{2\nu}$; otherwise it is of order r^{ν}. In the first case, the n'th term is of order $r^{2n-(3+2n)\nu}$. This gives convergence for $1 - \nu > 0$, so we have convergence for all $\nu < 0$. For $\nu > 0$, the n'th term is of order $r^{2n-\frac{3+2n}{2}\nu}$. This gives convergence for $2 - \nu > 0 \Rightarrow \nu < 2$. We thus have convergence when $O(r^2) < |\sigma| < \infty$. We also need the area of validity for when we cut the series short. The total order of r of the n'th term is

$$\begin{array}{ll} \nu < 0 & r^{2n-(\frac{5}{2}+2n)\nu} \\ 0 < \nu < 1 & r^{2n-(1+n)\nu} \\ 1 < \nu & r^{1+2n-(2+n)\nu} \end{array}$$

To stop after the lowest order term, one must have $\nu < 1$ ($O(r) < |\sigma| < \infty$), while including another term increases the area of validity to $\nu < \frac{5}{4}$ ($O(r^{\frac{5}{4}}) < |\sigma| < \infty$). We are still short of the area of convergence, so adding more terms will continue to increase the area of validity, making it approach the area of convergence.

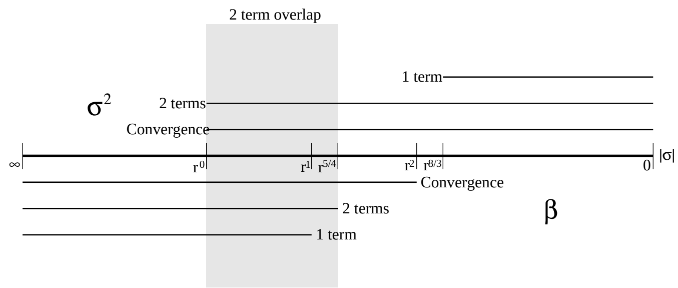

Figure 4.3: The values of σ for which each expansion converges, stops after 1 term and stops after 2 terms. We see that as long as not both expansions stop after the first term, there is an overlap. Within this overlap, both expansion are exact in the limit $r \to 0$, while only one holds outside the overlap.

The overlap

Figure 4.3 displays the areas of convergence and validity for the two expansions. We see that for expansions that include two terms, the whole range of σ is covered by the two expansions collectively. To evaluate the integral, we choose some point within the overlap, and use the β-expansion for σ of greater magnitude than this, and the σ^2 expansion otherwise. For convenience, I will choose the only whole power of r within the overlap, $\sigma = -rx_0\xi$. ξ is a free variable of the order r^0, which represents our freedom to vary the transition point. Since the integral has a definite value, the final result should not depend on ξ.

The σ^2 integrals

The first term of the σ^2 expansion gives rise to the integral

$$\begin{aligned} H^0_{\sigma^2} &= \int \sigma^{-\frac{3}{2}}(\beta+\sigma\gamma)^{-\frac{3}{2}}(\alpha-\sigma)d\sigma \\ &= -\frac{2(\alpha\beta+\sigma\beta+2\gamma\alpha\sigma)}{\beta^2(\beta+\gamma\sigma)^{\frac{1}{2}}\sigma^{\frac{1}{2}}} \end{aligned} \tag{4.80}$$

Where help from the computer algebra package Maple was enlisted to evaluate the integral. The σ^2 expansion is used in the area $(-rx_0\xi, 0)$, so we need to calculate $\left[H^0_{\sigma^2}\right]^0_{-rx_0\xi}$. In the upper limit, most of the terms fall away, and we are left with

$$\left[H^0_{\sigma^2}\right]_{\sigma=0} = \left[-2\sigma^{-\frac{1}{2}}\alpha\beta^{-\frac{3}{2}}\right]_{\sigma=0} \tag{4.81}$$

where I have used equation (4.74). This is infinite, but equal and opposite to what we get from the f_0 integral in the *same limit*, so these two infinities will cancel.

At the lower limit we are left with the whole expression, but now that the integral is out of the way, we can expand with respect to r, keeping only terms with order r^0 or lower. The easiest way to do this is to expand the factors individually, and then multiply those expansions in the end.

To see how many terms we need to keep in these partial expansions, we need to find the lowest order of r of $\left[H^0_{\sigma^2}\right]_{\sigma=-rx_0\xi}$. Using $\alpha \sim r$, $\beta \sim r^2$, $\gamma \sim r^0$ and $\sigma \sim r$, we see that $\left[H^0_{\sigma^2}\right]_{\sigma=-rx_0\xi} \sim r^{-3}$, so we need 4 terms in the partial expansions. For the various factors, we get

$$
\begin{aligned}
&\alpha\beta + \sigma\beta + 2\gamma\alpha\sigma \\
=& -4\cos\omega\xi x_0^4 r^2 \\
& x_0^3 r^3 \left(-\cos\omega - \xi - 4\xi\cos^2\omega - 2\xi(2\cos^2\omega - 1)\right) \\
& x_0^2 r^4 \left(-\frac{1}{2}(2\cos^2\omega - 1) - \cos^2\omega - \xi\cos\omega - 2\xi(\cos\omega + \cos\omega(2\cos^2\omega - 1))\right) \\
& x_0 r^5 \left(-\frac{1}{4}\cos\omega - \frac{1}{2}\cos\omega(2\cos^2\omega - 1) - \frac{1}{4}\xi + \xi(2\cos^2\omega - 1)\right) + O(r^6) \\
=& -\frac{1}{4}\xi\cos\omega x_0^4 r^2 - (\cos\omega - \xi + 8\xi\cos^2\omega)x_0^3 r^3 \\
& -\frac{1}{2}(4\cos^2\omega - 1 + 2\xi\cos\omega + 8\xi\cos^3\omega)x_0^2 r^4 \\
& -\frac{1}{4}(-\cos\omega + 4\cos^3\omega - 3\xi + 8\xi\cos^2\omega)x_0 r^5 + O(r^6)
\end{aligned}
\tag{4.82}
$$

$$
\begin{aligned}
\beta^{-2} =& \beta_0^{-2} - 2\beta_0^{-3}\beta_0' + \frac{1}{2}\left(6\beta_0^{-4}\beta_0'^2 - 2\beta_0^{-3}\beta_0''\right) \\
& + \frac{1}{6}\left(-24\beta_0^{-5}\beta_0'^3 + 18\beta_0^{-4}\beta_0'\beta_0'' - 2\beta_0^{-3}\beta_0'''\right) + O(r^0)
\end{aligned}
\tag{4.83}
$$

where $\beta_0 = r^2 x_0^2$, $\beta_0' = r^2 x_0 \cos\omega$, $\beta_0'' = \frac{1}{2}r^2$ and $\beta_0''' = 0$:

$$
\begin{aligned}
\beta^{-2} =& x_0^{-4} r^{-4} - 2\cos\omega x_0^{-5} r^{-3} + \frac{1}{2}(6\cos^2\omega - 1)x_0^{-6} r^{-2} \\
& - \frac{1}{2}\cos\omega(8\cos^2\omega - 3)x_0^{-7} r^{-1} + O(r^0)
\end{aligned}
\tag{4.84}
$$

The last factor to be expanded is found through similar calculations, and dropping the tedious details, one ends up with

$$(\beta+\gamma\sigma)^{-\frac{1}{2}} = \frac{\sqrt{2}}{2}x_0^{-\frac{3}{2}}\xi^{-\frac{1}{2}}r^{-\frac{1}{2}}\Big(1-\frac{1}{4}(\xi^{-1}+2\cos\omega)x_0^{-1}r$$
$$+\frac{1}{32}\left(3\xi^{-2}+4\xi^{-1}\cos\omega+12\cos^2\omega-8\right)x_0^{-2}r^2$$
$$-\frac{1}{128}\Big(5\xi^{-3}+6\xi^{-2}\cos\omega+12\xi^{-1}\cos^2\omega-16\xi^{-1}$$
$$-48\cos\omega+40\cos^3\omega\Big)x_0^{-3}r^3+O(r^4)\Big) \tag{4.85}$$

The partial expansions (4.82), (4.84) and (4.85) can now be combined, one by one, to find the total expansion

$$\beta^{-2}(\alpha\beta+\sigma\beta+2\gamma\sigma\alpha)$$
$$=-4\xi\cos\omega r^{-2}+(\xi-\cos\omega)x_0^{-1}r^{-1}-\frac{1}{2}(2\cos\omega\xi-1)x_0^{-2}$$
$$+\frac{1}{4}([4\cos^2\omega+1]\xi-\cos\omega)x_0^{-3}r+O(r^2) \tag{4.86}$$

$$[H^0_{\sigma^2}]_{\sigma=-rx_0\xi} = -2\sigma^{-\frac{1}{2}}(\beta+\gamma\sigma)^{-\frac{1}{2}}(\alpha\beta+\sigma\beta+2\gamma\alpha\sigma)\beta^{-2}$$
$$=\sqrt{2}i\Big\{-4\cos\omega x_0^{-2}r^{-3}+(2\cos^2\omega+1)x_0^{-3}r^{-2}$$
$$-\frac{1}{8}\left(12\cos^3\omega+[4+\xi^{-2}]\cos\omega-2\xi^{-1}\right)x_0^{-4}r^{-1}$$
$$+\frac{1}{32}\left(40\cos^4\omega+[2\xi^{-2}+12]\cos^2\omega+[2\xi^{-3}-12\xi^{-1}]\cos\omega-\xi^{-2}\right)x_0^{-5}\Big\}$$
$$+O(r) \tag{4.87}$$

We also need the next term in the σ^2 expansion: $-\frac{3}{2}H^1_{\sigma^2}$, where

$$H^1_{\sigma^2} = \int \sigma^{\frac{1}{2}}(\beta+\sigma\gamma)^{-\frac{3}{2}}(\alpha-\sigma)d\sigma$$
$$=-\frac{1}{3}\Big(3K\beta\gamma^2\sigma^2-2\gamma^{\frac{5}{2}}\sigma\alpha L-8\gamma^{\frac{3}{2}}\sigma\beta L+6\sigma\gamma\beta^2K$$
$$-6L\gamma^{\frac{1}{2}}\beta^2+3K\beta^3\Big)\sigma^{\frac{1}{2}}\gamma^{-\frac{5}{2}}\beta^{-1}L^{-1}(\beta+\gamma\sigma)^{-\frac{3}{2}} \tag{4.88}$$
$$L=\sigma^{\frac{1}{2}}(\beta+\gamma\sigma)^{\frac{1}{2}} \tag{4.89}$$
$$K=\ln\left\{\frac{\beta}{2\sqrt{\gamma}}+\sigma\sqrt{\gamma}+L\right\} \tag{4.90}$$

where help from Maple again was most welcome. Simplifying this, we get

$$H^1_{\sigma^2} = -K\gamma^{-\frac{5}{2}}+\frac{2}{3}\left(\gamma^2\sigma\alpha+4\gamma\sigma\beta+3\beta^2\right)\sigma^{\frac{1}{2}}\gamma^{-2}\beta^{-1}(\beta+\gamma\sigma)^{-\frac{3}{2}} \tag{4.91}$$

In the limit $\sigma = 0$ only the K term survives:

$$[H^1_{\sigma^2}]_{\sigma=0} = -\gamma^{-\frac{5}{2}} \ln \left\{ \frac{1}{2}\beta\gamma^{-\frac{1}{2}} \right\} \tag{4.92}$$

We only need the parts of this expression that are of order r^0 or lower. Using $\lim_{x\to 0} x^y \ln(x^z) = 0$ for all $y > 0$, and $\gamma \sim r^0$, we see that the only surviving term is

$$[H^1_{\sigma^2}]_{\sigma=0} = -\gamma_0^{-\frac{5}{2}} \ln \left\{ \frac{1}{2}\gamma_0^{-\frac{1}{2}} x_0^2 r^2 \right\} \tag{4.93}$$

$$\gamma_0 = -2x_0^2 \tag{4.94}$$

At the other end of the integral, $\sigma = -rx_0\xi$, significant simplifications are also possible. Letting the braces indicate the order of r, we have

$$H^1_{\sigma^2} = \underbrace{-K\gamma^{-\frac{5}{2}}}_{0} + \frac{2}{3}\left(\underbrace{\gamma^2\sigma\alpha}_{2} + \underbrace{4\gamma\sigma\beta}_{3} + \underbrace{3\beta^2}_{4}\right) \underbrace{\sigma^{\frac{1}{2}}\gamma^{-2}\beta^{-1}(\beta+\gamma\sigma)^{-\frac{3}{2}}}_{-3} \tag{4.95}$$

Observe first that the β^2 term can be eliminated, and second that the lowest order is r^{-1}, meaning that we this time only need to expand to 2 orders, instead of 4 orders, as we had to for $H^0_{\sigma^2}$. Dealing first with the part without K, we have

$$(\beta+\gamma\sigma)^{-\frac{3}{2}} = (2x_0^3\xi r)^{-\frac{3}{2}} \left(1 - \frac{3}{4}\xi^{-1}(1+2\xi\cos\omega)x_0^{-1}r + O(r^2)\right) \tag{4.96}$$

$$\beta^{-1} = x_0^{-2}r^{-2} - \cos\omega x_0^{-3}r^{-1} + O(r^0) \tag{4.97}$$

$$\gamma^{-2} = \frac{1}{4}x_0^{-4} - \frac{1}{2}\cos\omega x_0^{-5} r + O(r^2) \tag{4.98}$$

$$\frac{2}{3}\left(\gamma^2\sigma\alpha + 4\gamma\sigma\beta + 3\beta^2\right) = \frac{4}{3}\left(2\cos\omega\xi x_0^6 r^2 + 3[2\cos^2\omega + 1]\xi x_0^5 r^3\right) + O(r^4) \tag{4.99}$$

And, as before, we combine the partial expansions to get the full expansion.

$$\begin{aligned}&\sigma^{\frac{1}{2}}\gamma^{-2}\beta^{-1}(\beta+\gamma\sigma)^{-\frac{3}{2}} \\ =&\frac{i\sqrt{2}}{8}\left(\frac{1}{2}\xi^{-1}x_0^{-10}r^{-3} - \frac{3}{8}(6\cos\omega + \xi^{-1})\xi^{-1}x_0^{-11}r^{-2}\right) + O(r^{-1})\end{aligned} \tag{4.100}$$

$$\begin{aligned}&\frac{2}{3}\left(\gamma^2\sigma\alpha + 4\gamma\sigma\beta + 3\beta^2\right)\sigma^{\frac{1}{2}}\gamma^{-2}\beta^{-1}(\beta+\gamma\sigma)^{-\frac{3}{2}} \\ =&\frac{i\sqrt{2}}{6}\cos\omega x_0^{-4}r^{-1} - \frac{i\sqrt{2}}{8}\left(2\cos^2\omega - 2 + \xi^{-1}\cos\omega\right) + O(r)\end{aligned} \tag{4.101}$$

The other part, $-K\gamma^{-\frac{5}{2}}$, is surprisingly nontrivial to simplify. As before, we need only keep the lowest order term inside the logarithm, but this term is not of order r, even though $L \sim r$, $\gamma^{\frac{1}{2}}\sigma \sim r$ and $\beta \sim r^2$. The leading part of L is $L_0 = \sqrt{\gamma_0\sigma^2} = \gamma_0^{\frac{1}{2}}|\sigma|$, while the leading part of $\gamma^{\frac{1}{2}}\sigma$ is $\sigma\gamma_0^{-\frac{1}{2}}$, and since $\sigma < 0$, these terms cancel. Going to the next order, the same thing happens, and we shall see that the leading term is really of order r^3. Expanding first

$$\gamma^{\frac{1}{2}} = i\sqrt{2}x_0\left(1 + \frac{1}{2}\cos\omega x_0^{-1}r + \frac{1}{8}(2 - \cos^2\omega)x_0^{-2}r^2\right) + O(r^3) \tag{4.102}$$

$$\gamma^{-\frac{1}{2}} = -i\frac{\sqrt{2}}{2}x_0^{-1}\left(1 - \frac{1}{2}\cos\omega x_0^{-1}r\right) + O(r^2) \tag{4.103}$$

we find the terms inside the logarithm to be

$$\begin{aligned} L =& i\xi\sqrt{2}x_0^2 r + \frac{i\sqrt{2}}{4}(1 + 2\xi\cos\omega)x_0 r^2 \\ &+ \frac{i\sqrt{2}}{32}\left(-\xi^{-1} + 4\cos\omega - 4\cos^2\omega\xi + 8\xi\right)r^3 + O(r^4) \end{aligned} \tag{4.104}$$

$$\sigma\gamma^{\frac{1}{2}} = -i\xi\sqrt{2}x_0^2\left(r + \frac{1}{2}\cos\omega x_0^{-1}r^2 + \frac{1}{8}(2 - \cos^2\omega)x_0^{-2}r^3\right) + O(r^4) \tag{4.105}$$

$$\frac{1}{2}\beta\gamma^{-\frac{1}{2}} = \frac{-i\sqrt{2}}{4}\left(r^2 x_0 + \frac{1}{2}\cos\omega r^3\right) + O(r^4) \tag{4.106}$$

It is easy to see that most of these terms cancel, and we are left with

$$\frac{1}{2}\beta\gamma^{-\frac{1}{2}} + \sigma\gamma^{\frac{1}{2}} + L = \frac{-i\sqrt{2}}{32}\xi^{-1}r^3 \tag{4.107}$$

The part of $-\gamma^{-\frac{5}{2}}K$ that survives when $r \to 0$ is therefore $-\gamma_0^{-\frac{5}{2}}\ln\left\{\frac{-i\sqrt{2}r^3}{32\xi}\right\}$. With this, we finally have the last part of the σ^2 part of the integral.

$$\begin{aligned} [H^1_{\sigma^2}]_{\sigma=-rx_0\xi} =& \frac{i\sqrt{2}}{6}\cos\omega x_0^{-4}r^{-1} - \frac{i\sqrt{2}}{8}\left(2\cos^2\omega - 2 + \xi^{-1}\cos\omega\right) \\ &- \gamma_0^{-\frac{5}{2}}\ln\left\{\frac{-i\sqrt{2}r^3}{32\xi}\right\} + O(r) \end{aligned} \tag{4.108}$$

Pulling together the contributions from both orders of expansion in σ^2 at both limits, we find

$$\lim_{r\to 0}[H]^0_{-rx_0\xi} = \lim_{r\to 0}\left([H^0_{\sigma^2}]_{\sigma=0} - [H^0_{\sigma^2}]_{\sigma=-rx_0\xi} - \frac{3}{2}\left\{[H^1_{\sigma^2}]_{\sigma=0} - [H^1_{\sigma^2}]_{\sigma=-rx_0\xi}\right\}\right)$$
$$= \left[-2\sigma^{-\frac{1}{2}}\alpha\beta^{-\frac{3}{2}}\right]_{\sigma=0}$$
$$- \frac{i\sqrt{2}}{2}\left\{-8\cos\omega x_0^{-2}r^{-3} + (4\cos^2\omega + 2)x_0^{-3}r^{-2}\right.$$
$$- \frac{1}{4}\left(12\cos^3\omega + [6+\xi^{-2}]\cos\omega - 2\xi^{-1}\right)x_0^{-4}r^{-1}$$
$$+ \frac{1}{16}\left(40\cos^4\omega + [2\xi^{-2}+24]\cos^2\omega\right.$$
$$\left.\left.+[2\xi^{-3} - 6\xi^{-1}]\cos\omega - 12 - \xi^{-2}\right)x_0^{-5}\right\}$$
$$- \frac{3}{2}\gamma_0^{-\frac{5}{2}}\ln\left\{\frac{-r}{8\xi x_0}\right\} \tag{4.109}$$

The β integrals

So far, we have only handled the $(-rx_o\xi, 0)$ part of the integral. For the rest, we need the β expansion, and because we are using a point with $\sigma \sim r$ as the middle boundary, we will need the two first terms of this expansion. The first term gives rise to an integral that looks quite similar to the corresponding one in the σ^2 expansion, though the result is rather more complicated.

$$H^0_\beta = \int \sigma^{-\frac{3}{2}}(\sigma\gamma + \sigma^2)^{-\frac{3}{2}}(\alpha - \sigma)d\sigma \tag{4.110}$$
$$= -\frac{1}{4}\left(-5\alpha\gamma^{\frac{3}{2}}\sigma + 2\alpha\gamma^{\frac{5}{2}} - 15\sigma^2\gamma^{\frac{1}{2}}\alpha + 15\alpha(\sigma+\gamma)^{\frac{1}{2}}J\sigma^2\right.$$
$$\left.-12\sigma^2\gamma^{\frac{3}{2}} - 4\sigma\gamma^{\frac{5}{2}} + 12\sigma^2\gamma(\sigma+\gamma)^{\frac{1}{2}}J\right)\gamma^{-\frac{7}{2}}\sigma^{-\frac{3}{2}}(\sigma[\sigma+\gamma])^{-\frac{1}{2}}$$
$$= \frac{3}{4}\left(5\alpha\gamma^{-\frac{7}{2}} + 4\gamma^{-\frac{5}{2}}\right)J$$
$$+ \frac{1}{4}\left(-5\alpha\gamma\sigma + 2\alpha\gamma^2 - 15\alpha\sigma^2 - 12\sigma^2\gamma - 4\sigma\gamma^2\right)\gamma^{-3}\sigma^{-2}(\sigma+\gamma)^{-\frac{1}{2}} \tag{4.111}$$

Here $J = \operatorname{atanh}\sqrt{1+\sigma\gamma^{-1}}$. In the limit $\sigma \to -\infty$, only the J terms survive. To see what J itself becomes, we use

$$\operatorname{atanh} x = \frac{1}{2}\ln\frac{1+x}{1-x} \Rightarrow \tag{4.112}$$
$$\lim_{x\to\infty}\operatorname{atanh} x = \frac{1}{2}\ln(-1) \tag{4.113}$$

The logarithm becomes a many-valued imaginary function for negative arguments. It is multi-valued because ln is the inverse of the exponential function, and $e^{i(\theta+2\pi n)}$ all have the same value. It is possible to regard $\ln z$ as a single-valued function instead, by restricting the angles to the interval from $-\pi$ exclusive to π inclusive, giving $\ln(z) = \ln(|z|) + i\arg(z)$. Any other way of restricting the angles that would give every complex number a definite polar representation would also work, as long as we are consistent. With this, $\ln(-1) = i\pi$.

$$[H^0_\beta]_{\sigma=-\infty} = \frac{3}{2}\gamma_0^{-\frac{5}{2}} \ln(-1) + O(r) \tag{4.114}$$

At the other end, where $\sigma = -rx_0\xi$, we see that the lowest order of $\sqrt{1+\sigma\gamma^{-1}}$ is 1, which causes problems for atanh, so we shall need the next term, too:

$$\sqrt{1+\sigma\gamma^{-1}} = 1 + \frac{1}{4}\xi x_0 r + O(r) \Rightarrow \tag{4.115}$$

$$\begin{aligned} \operatorname{atanh}\sqrt{1+\sigma\gamma^{-1}} &= \frac{1}{2}\ln\frac{2+\frac{1}{4}\xi x_0 r + O(r)}{-\frac{1}{4}\xi x_0 r + O(r)} \\ &= -\frac{1}{2}\ln\frac{-\xi r}{8x_0} + \frac{i\pi}{2} + O(r) \end{aligned} \tag{4.116}$$

so the whole J part of $[H^0_\beta]_{\sigma=-rx_0\xi}$ is $-\frac{3}{2}\gamma_0^{-\frac{5}{2}}\ln\frac{-\xi r}{8x_0} + \frac{3}{2}\gamma_0^{-\frac{5}{2}} i\pi$. The rest is, denoting by the braces the order of r,

$$\underbrace{\frac{1}{4}\left(-5\underbrace{\alpha\gamma\sigma}_{2} + 2\underbrace{\alpha\gamma^2}_{1} - 15\underbrace{\alpha\sigma^2}_{3} - 12\underbrace{\sigma^2\gamma}_{2} - 4\underbrace{\sigma\gamma^2}_{1}\right)}_{U}\underbrace{\gamma^{-3}\sigma^{-2}(\sigma+\gamma)^{-\frac{1}{2}}}_{-2} \tag{4.117}$$

so one of the terms can be dismissed, and we only need two orders in the expansions here. The partial expansions are

$$\begin{aligned} \gamma^{-3} &= -\frac{1}{8}x_0^{-6}\left(1 - 3\cos\omega x_0^{-1} r\right) + O(r^2) \\ (\sigma+\gamma)^{-\frac{1}{2}} &= \gamma_0^{-\frac{1}{2}}\left(1 - \frac{1}{4}(\xi + 2\cos\omega)x_0^{-1} r\right) + O(r^2) \\ U &= (-8\cos\omega + 16\xi)x_0^{-3} r \\ &\quad + \left(24\xi^2 + 42\xi\cos\omega - 24\cos^2\omega + 4\right) x_0^{-4} r^2 + O(r^3) \end{aligned}$$

combining these, we get

$$\begin{aligned}
\gamma^{-3}\sigma^{-2}(\sigma+\gamma)^{-\frac{1}{2}} &= -\frac{1}{8}\gamma_0^{-\frac{1}{2}}\xi^{-2}x_0^{-2}r^{-2}\left(1-\frac{1}{4}(7\cos\omega+\xi)x_0^{-1}r+O(r^2)\right) \\
\frac{1}{4}U\gamma^{-3}\sigma^{-2}(\sigma+\gamma)^{-\frac{1}{2}} &= -\frac{1}{8}x_0^{-3}\xi^{-2}\gamma_0^{-\frac{1}{2}}r^{-1}\Big\{4\xi-2\cos\omega \\
&\quad +(5\xi^2-3\xi\cos\omega+\cos^2\omega+1)x_0^{-1}r+O(r^2)\Big\}
\end{aligned} \tag{4.118}$$

Combining this with the J part, we get

$$\begin{aligned}
[H_\beta^0]_{\sigma=-rx_0\xi} &= -\frac{1}{8}x_0^{-3}\xi^{-2}\gamma_0^{-\frac{1}{2}}r^{-1}\Big\{4\xi-2\cos\omega+(5\xi^2-3\xi\cos\omega \\
&\quad +\cos^2\omega+1)x_0^{-1}r+O(r^2)\Big\}-\frac{3}{2}\ln\frac{-\xi r}{8x_0}
\end{aligned} \tag{4.119}$$

The next term in the β expansion is $-\frac{3}{2}H_\beta^1$, with

$$H_\beta^1 = \int \sigma^{-\frac{3}{2}}(\sigma\gamma+\gamma^2)(\alpha-\sigma)\beta d\sigma \tag{4.120}$$

$$\begin{aligned}
&= \frac{1}{24}\beta\Big\{-420\gamma^{\frac{3}{2}}\sigma^3\alpha-63\sigma^2\sigma^{\frac{5}{2}}\alpha+18\sigma\gamma^{\frac{7}{2}}\alpha-315\sigma^4\gamma^{\frac{1}{2}}\alpha \\
&\quad -280\gamma^{\frac{5}{2}}\sigma^3-42\sigma^2\gamma^{\frac{7}{2}}+12\sigma\gamma^{\frac{9}{2}}-210\sigma^4\gamma^{\frac{3}{2}}-8\alpha\gamma^{\frac{9}{2}} \\
&\quad +J(\sigma+\gamma)^{\frac{1}{2}}\Big[315\sigma^4\alpha+315\sigma^3\alpha\gamma+210\sigma^4\gamma+210\sigma^3\gamma^2\Big]\Big\} \\
&\quad \gamma^{-\frac{11}{2}}\underbrace{\sigma^{-\frac{5}{2}}(\sigma\gamma)^{-1}(\sigma[\sigma+\gamma])^{-\frac{1}{2}}}_{-\sigma^{-3}(\sigma+\gamma)^{-\frac{3}{2}}}
\end{aligned} \tag{4.121}$$

where we again have $J = \operatorname{atanh}\sqrt{1+\sigma\gamma^{-1}}$. In the limit $\sigma\to-\infty$, all terms disappear, as they all have a negative total order of σ. At the other limit, $\sigma=-rx_0\xi$, eliminating all terms of positive order of r, we get

$$[H_\beta^1]_{\sigma=-rx_0\xi} = -\frac{1}{6}\underbrace{\beta\sigma^{-3}\gamma^{-1}(\sigma+\gamma)^{-\frac{3}{2}}}_{-1}(\underbrace{3\sigma}_{1}-\underbrace{2\alpha}_{1})+O(r) \tag{4.122}$$

This expression is of order r^0, so it is sufficient to expand it to the 0'th order in r, which is

$$[H_\beta^1]_{\sigma=-rx_0\xi} = \frac{1}{24}x_0^{-4}\xi^{-3}\gamma_0^{-\frac{1}{2}}(2\cos\omega-3\xi)+O(r) \tag{4.123}$$

Pulling together the contributions from both orders of the expansion in β at both limits, we find

$$\begin{aligned}\lim_{r\to 0}[H]_{-\infty}^{-rx_0\xi} =& \lim_{r\to 0}\left([H_\beta^0]_{\sigma=-rx_0\xi} - [H_\beta^0]_{\sigma=-\infty} - \frac{3}{2}\left\{[H_\beta^1]_{\sigma=-rx_0\xi} - [H_\beta^1]_{\sigma=-\infty}\right\}\right)\\ =& -\frac{1}{4}(2\xi - \cos\omega)\xi^{-2}x_0^{-3}\gamma_0^{-\frac{1}{2}}r^{-1} + \frac{1}{16}x_0^{-4}\xi^{-3}\gamma_0^{-\frac{1}{2}}\Big(-2\cos\omega\\ &+\xi - 2\xi\cos^2\omega + 6\xi^2\cos\omega - 10\xi^3\Big) - \frac{3}{2}\gamma_0^{-\frac{5}{2}}\ln\frac{\xi r}{8x_0}\end{aligned} \tag{4.124}$$

Combining the σ^2 and β parts

We can finally find the value of the full integral by combining the results from subsection 4.4.2 and 4.4.2.

$$\lim_{r\to 0}\int_{-\infty}^{0}\sigma^{-\frac{3}{2}}(f_0 - f)d\sigma = -x_0^3\left(\lim_{\sigma\to 0}[-2\alpha\beta^{-\frac{3}{2}}\sigma^{-\frac{1}{2}}] - [H]_{-\infty}^{-rx_0\xi} - [H]_{-rx_0\xi}^{0}\right) \tag{4.125}$$

And, inserting for the various H's, we end up with

$$\begin{aligned}x_0^3\gamma_0^{-\frac{1}{2}}\Big\{ &-8\cos\omega x_0^{-1}r^{-3} + (4\cos^2\omega + 2)x_0^{-2}r^{-2}\\ &-\frac{1}{4}\Big(12\cos^3\omega + 6\cos\omega\Big)x_0^{-3}r^{-1}\\ &+\frac{1}{16}\Big(40\cos^4\omega + 24\cos^2\omega - 24\Big)x_0^{-4}\Big\} - 3\gamma_0^{-\frac{5}{2}}\ln\frac{r}{8x_0}\end{aligned} \tag{4.126}$$

As we needed, all ξ's cancel. If this had not been the case, we would have had an indeterminate expression, since ξ is a nonphysical free variable we introduced when splitting up the integral. This expression is, however, rather infinite, with 4 divergent expressions of varying degree of r. The is the divergence inherent in dealing with point particles, and is not necessarily a problem, as some of it can be smoothed away (infinite terms pulling equally and opposite at the particle), and some can be dealt with using mass renormalization.

To finally find F_{xt}, we use equation (4.67) together with $x = x_0 + r\cos\omega$, and eliminate factors proportional to odd orders of $\cos\omega$ at the same time, as these are antisymmetric over the smoothing integral, and will disappear during the smoothing.

$$\begin{aligned}F_{xt} =& -qx_0b\gamma_0^{-\frac{1}{2}}\left\{\Big(-4\cos^2\omega + 2\Big)x_0^{-1}r^{-2} - \frac{1}{4}\left(2\cos^4\omega + 6 + 3\ln\frac{r}{8x_0}\right)x_0^{-3}\right\}\\ &+ \text{odd orders of }\cos\omega\end{aligned} \tag{4.127}$$

The only angular dependence is in $\cos\omega$, so integrating over ω is easily done. From equation (4.54) we see that we will need the integrals $\int_0^\pi \sin^2\omega d\omega = \frac{\pi}{2}$, $\int_0^\pi \sin^2\omega\cos^2\omega d\omega = \frac{\pi}{8}$ and $\int_0^\pi \sin^2\omega\cos^4\omega d\omega = \frac{\pi}{16}$. Inserting also the values $\gamma_0^{-\frac{1}{2}} = -i\frac{\sqrt{2}}{2}x_0^{-1}$ and $b = \frac{i}{4\sqrt{2}}$, we end up with

$$f^x = q\langle F^x{}_t\rangle u^t = q\langle F_{xt}\rangle \tag{4.128}$$

$$= -\frac{q^2}{8}\left\{x_0^{-1}r^{-2} - \frac{1}{16}\left(25 + 12\ln\frac{r}{8x_0}\right)x_0^{-3}\right\} \tag{4.129}$$

So two of the divergences disappeared due to symmetry. Furthermore, the r^{-2} divergence is proportional to x_0^{-1}, which is just the particle's acceleration. This term has the same effect as the mass of the particle, and we can combine these two to form the total, observed mass of the particle. This is called mass renormalization, and also appears in the much simpler case of 3+1 dimensions. We are, however, left with one divergence: $\ln r$. It is a much weaker divergence than the others, but still gives rise to an infinite force acting in the opposite direction of the driving force, that is keeping the acceleration of the particle constant. Since it is proportional to the acceleration cubed, it cannot be handled by mass renormalization, so this has to be a real, observable force.

What does this mean? Our particle is bound to have constant acceleration, and to achieve this, some external force of unspecified magnitude is being applied. Had the particle being accelerated been neutral, the force would simply be $f_{\text{ext}}^{\text{neut.}x} = mx_0^{-1}$, but the radiation reaction force opposes this, meaning that the external force must be correspondingly greater: $f_{\text{ext}} = f_{\text{ext}}^{\text{neut.}x} - f^x$. Since f^x in this case is $-\infty$, the driving force must be infinite to maintain a constant acceleration. This would seem to imply that it is impossible to cause any acceleration to the particle, but we cannot conclude this, as our derivation was made under the assumption that the acceleration was permanently constant.

An infinite radiation reaction is unexpected from our 3+1-dimensional experience, where we have no radiation reaction. It is, however, supported by a calculation by Gal'tsov [10] using a different and more general approach. He here shows that no non-divergent equation of motion is possible for charged point particles in spacetimes of $1+d$ dimensions with $d > 3$.

4.4.3 What does the GLAD equation predict?

We had good reasons not to expect the GLAD equation to work in this case: It depends on a specific form of the Green function, and the Green function in this case does not have that form. Still, it is interesting to see what the consequences of using the equation would be. The non-integral parts of the equation do not differ from the case in section 3.1.6, so they still do not contribute. The tail term remains, though, and from equation (3.53) we have

$$\text{tail}_\mu = q^2 \int_{0^+}^{\infty} \left(G_{+tt',\mu} - G_{+\mu t',t} \right) d\Delta t \tag{4.130}$$

As symmetry excludes forces in $y, z, \ldots$ directions, and the equation is trivially zero for t, we need only calculate the x component. Since we saw that $A_{x,t} = 0$ in section 4.4.2, we can use the relation between A and G (equation (2.83+)) to conclude that $G_{+xt',t} = 0$, leaving

$$\begin{aligned}
\text{tail}_x &= q^2 \partial_x \int_{0^+}^{\infty} G_{+tt'} d\Delta t \\
&= -q^2 \partial_x \int_{-\infty}^{(a-xx_0)^-} g_{tt'} g(\sigma) \sigma_{\Delta t} d\sigma
\end{aligned}$$

which is to be evaluated at the position of the particle, giving $a - xx_0 = 0$.

$$\begin{aligned}
\text{tail}_x &= -q^2 \partial_x \int_{-\infty}^{0^-} g_{tt'} g(\sigma) \sigma_{\Delta t} d\sigma \\
&= -q^2 b \int_{-\infty}^{0^-} \left(\theta(-\sigma) \sigma^{-\frac{3}{2}} + \delta(\sigma) \sigma^{-\frac{1}{2}} \right) \frac{d}{dx} \frac{g_{tt'}}{\sigma_{\Delta t}} d\sigma
\end{aligned} \tag{4.131}$$

This is (aside from a different factor in front of the integral), the same as equation (4.61), with an important difference: The point $\sigma = 0$ is excluded from this integral, but included in the other, which means that the delta function has no effect. That point is, we remember, excluded because the rest of the GLAD equation deals with it, which is the whole point of the equation. From this point on, then, we can follow the old derivation, but remove any terms containing delta functions. Equation (4.67) gets an integral of $-\int_{-\infty}^{0} \sigma^{-\frac{3}{2}} f(\sigma) d\sigma$ in stead of $\int_{-\infty}^{0} \sigma^{-\frac{3}{2}} (f_0 - f(\sigma)) d\sigma$. The disappearance of f_0 means that nothing cancels the divergent term in equation (4.81). Unlike the other divergences encountered, which were r-divergent, this is both σ- and r-divergent; a much more serious divergence than the others we have encountered, and one about which there is not much one can do.

The appearance of this severe divergence when comparing the result of using the GLAD equation with what one gets with direct calculations, confirms that it does not work in 4+1-dimensional spacetime.

4.5 Radiation

As we did in the 3+1-dimensional case, we can now integrate the energy flux through a surface very close to the horizon, to find the total energy emitted by the particle. We again expect the total energy flow to be zero, as any nonzero value would give odd results for charged particles that are kept from free fall.

To construct the stress-energy tensor for the field, we need the electromagnetic field tensor $F^{\mu\nu}$. Inserting equation (4.42) into equation (4.57), and using $a - xx_0 > 0$ at the horizon, we have

$$\begin{aligned} A_\mu =& - bq \int_{-\infty}^{0} \left(\sigma^{-\frac{3}{2}} \theta(-\sigma) + 2\sigma^{-\frac{1}{2}} \delta(\sigma) \right) g_{\mu t'} \sigma_{\Delta t} d\sigma \\ =& bq \int_{-\infty}^{0} \sigma^{-\frac{3}{2}} \left([g_{\mu t'} \sigma_{\Delta t}]_0 - g_{\mu t'} \sigma_{\Delta t} \right) d\sigma \Leftrightarrow \qquad (4.132) \\ A_{\mu,\nu} =& bq \int_{-\infty}^{0} \sigma^{-\frac{3}{2}} \Big(\underbrace{\partial_\nu [g_{\mu t'} \sigma_{\Delta t}]_0}_{h_{0\mu\nu}} - \underbrace{\partial_\nu [g_{\mu t'} \sigma_{\Delta t}]}_{h_{\mu\nu}} \Big) d\sigma \qquad (4.133) \end{aligned}$$

To calculate the hs, we need equation (2.14), $c = \frac{a-\sigma}{xx_0}$ and $\sigma_{\Delta t} = -xs$, again using c and s as shortcuts for $\cosh \frac{\Delta t}{x_0}$ and $\sinh \frac{\Delta t}{x_0}$ respectively. Letting Latin indices indicate the other dimensions than t and x, we see immediately that $h_{i\nu} = h_{0i\mu} = 0$, since $g_{\mu\mu'}$ is diagonal in these indices. When the first index is x, we find

$$\begin{aligned} h_{x\nu} &= (g_{xt'} \sigma_{\Delta t})_\nu \\ &= \left(-s(-xs)^{-1} \right)_\nu = -x^{-2} \delta^x_\nu = h_{0x\nu} \qquad (4.134) \end{aligned}$$

Since $h_{x\nu} = h_{0x\nu}$, these will cancel, giving $A_{x,\nu} = 0$. The component h_{tx} has already been calculated in equation (4.63):

$$h_{tx} = - \frac{x}{x_0^2 \sigma_{\Delta t}^3} (a^2 - \sigma - x^2) \qquad (4.135)$$

And finally, for h_{ti} we get

$$\begin{aligned} h_{ti} =& \left(- \frac{a-\sigma}{x_0^2} (-x_0) \Big(\overbrace{(a-\sigma)^2 - x^2 x_0^2}^{M} \Big)^{-\frac{1}{2}} \right)_i \\ =& \frac{1}{x_0} \left(x_i M^{-\frac{1}{2}} - (a-\sigma)^2 x_i M^{-\frac{3}{2}} \right) \\ =& \frac{x^2 x_i}{x_0^3 \sigma_{\Delta t}^3} \qquad (4.136) \end{aligned}$$

With this, the nonzero components of the electromagnetic field tensor are

$$
\begin{aligned}
F_{tx} =& - F_{xt} = A_{x,t} - A_{t,x} \\
=&\frac{bqx}{x_0^2}\int_{-\infty}^{0} \sigma^{-\frac{3}{2}} \left(\frac{a-x^2}{\sigma_{\Delta t0}^3} - \frac{a-\sigma-x^2}{\sigma_{\Delta t}^3}\right) d\sigma
\end{aligned} \tag{4.137}
$$

$$
\begin{aligned}
F_{ti} =& - F_{it} = A_{i,t} - A_{t,i} \\
=& - \frac{qbx^2x_i}{x_0^3}\int_{-\infty}^{0} \sigma^{-\frac{3}{2}} \left(\sigma_{\Delta t0}^{-3} - \sigma_{\Delta t}^{-3}\right) d\sigma
\end{aligned} \tag{4.138}
$$

This is a pure electric field, as one would expect from a stationary particle.

In 4+1 dimensions, the expression for the stress-energy tensor changes slightly: The factor 4π, which, along with similar factors appearing in formulas for the cgs system, is a normalization factor that counters the 4π factor one gets when performing an integral over all angles: $4\pi = \int d\Omega$. It is ultimately there to make the Coulomb force law look nicer. When the number of spatial dimensions increases to 4, we instead get $\int d\Omega = 2\pi^2$, so the expression becomes

$$
T^{\mu}{}_{\nu} =\frac{1}{2\pi^2}\left(F^{\mu\rho}F_{\nu\rho} - \frac{1}{4}\delta^{\mu}_{\nu}F^{\rho\sigma}F_{\rho\sigma}\right) \tag{4.139}
$$

The surface element changes slightly compared to the 3+1-dimensional case, as there now is an extra index in the Levi-Chevita symbol, and we get $d\Sigma_\mu = -dydzd\psi dt\delta^x_\mu$, where ψ is the fourth spatial dimension. The total flow of five-momentum is then

$$
dp^\mu = \int T^{\mu\nu}d\Sigma_\nu = -\int T^{\mu x}dydzd\psi dt = -4\pi\int_0^\infty T^{\mu x}\rho^2 d\rho dt \tag{4.140}
$$

We are looking for the energy flow, which is given by T^{tx}, but this is trivially zero, as there is no index ρ that makes both $F^{t\rho}$ and $F^{x}{}_{\rho}$ nonzero. So a hyperbolically accelerated particle does not radiate energy in a co-moving frame of reference in 4+1 dimensions, just as it does not in the normal 3+1-dimensional case.

As for the other components, we can immediately eliminate p^i due to symmetry, so we are left with p^x, which is given by

$$
\begin{aligned}
T^{xx} =& \frac{1}{2\pi^2}\left(F^{xt}F^{x}{}_{t} - \frac{1}{2}\left(F^{tx}F_{tx} + F^{ti}F_{ti}\right)\right) \\
=& -\frac{x_0^2}{4\pi^2x^2}\left(F_{tx}^2 - F_{ti}^2\right)
\end{aligned} \tag{4.141}
$$

As we approach the horizon, $x \to 0$, $a \to \frac{1}{2}\left(x_0^2 + \rho^2\right)$ and $\sigma_{\Delta t} \to -\frac{1}{x_0}\left|a-\sigma\right|$. Recalling that $F_{tx} \sim x$ and $F_{ti} \sim x^2$, we see that only F_{tx} will contribute in the limit

$$\lim_{x\to 0} T^{xx} = -\frac{b^2q^2x_0^4}{4\pi^2}\left(\int_{-\infty}^{0}\sigma^{-\frac{3}{2}}\left(a^{-2}-(a-\sigma)^{-2}\right)d\sigma\right)^2 \tag{4.142}$$

The integral evaluates to $\frac{3i\pi}{2a^{\frac{5}{2}}}$, giving

$$\lim_{x\to 0} T^{xx} = \frac{9b^2q^2x_0^4}{16a^5} = -\frac{(3qx_0^2)^2}{(x_0^2+\rho^2)^5} \Rightarrow \tag{4.143}$$

$$dp^x = -4\pi(3qx_0^2)^2\int_0^\infty \frac{\rho^2}{(x_0^2+\rho^2)^5}d\rho dt \tag{4.144}$$

This time the integral evaluates to $5\pi 2^{-8}x_0^{-7}$, and so

$$f^x = \frac{dp^x}{dt} = -5x_0^{-1}\left(\frac{3q\pi}{8x_0^2}\right)^2 \tag{4.145}$$

As in the 3+1-dimensional case, we find no radiation here, but still a force directed in the negative x direction at the horizon. See section 3.3.1 for a short discussion of this.

4.6 Summary

In this chapter, we have seen how the wave equation gives a Green function with a "tail" for flat 4+1-dimensional space-time. We also observed that the form of this Green function breaks with the assumptions of the GLAD equation, establishing a limit on its validity: It does not hold in 4+1 dimensions, and this sows doubt about its validity for all cases of $d \neq 3$.

Since the GLAD equation could not be used, we derived the equation of motion ourselves, going through the same basic steps involved in deriving the GLAD equation originally, major differences being the use of smoothing and the lack of generality. Going through this derivation should also enlighten us on the nature of the radiation reaction.

The result ended up being infinite and directed against the external force keeping the particle in hyperbolic motion. This is in stark contrast to the null result from 3+1 dimensions. The infinity in the result is of the form $\ln r$, which only diverges in the limit $r \to 0$ - when we let our surface of smoothing, which approximates the particle, shrink to a point. However, classical descriptions of point particles are known to be unphysical. Thus, the infinity we see here could be interpreted as a break down of the concept of a classical point particle, rather than the impossibility of having charges in higher-dimensional spacetimes.

Be that as it may, we have still not seen the connection between the tail term and curvature: The previous chapter showed how topology may result in a tail term, and this chapter showed how a tail term may result from the Green function having a tail, but still did so in flat spacetime. The next chapter will investigate why the Green function does not have a tail in flat 3+1-dimensional spacetime, and how curvature changes this.

Chapter 5

The coincidental nature of the strong Huygens' principle

5.1 From 4 to 3 spatial dimensions

As explained in section 2.8, the strong Huygens' principle requires $g(\sigma) = 0$ for $\sigma \neq 0$. In chapter 4, we found

$$g(\sigma) = \frac{-i}{2\sqrt{2}} \frac{d}{d\sigma} \left(\frac{\theta(\sigma)}{\sqrt{\sigma}} \right) \tag{5.1}$$

which is an example of a Green function that violates this requirement, so Huygens' strong principle breaks down in 4+1-dimensional spacetime. This principle is often taken for granted as an inherent property of the wave equation, so it may be surprising to see it fail here, especially since the case of 3 and 4 spatial dimensions are strongly related, and the principle holds for 3+1 dimensions.

A flat spacetime with 3+1 dimensions is mathematically equivalent to a flat spacetime with 4+1 dimensions if there is full symmetry in the extra dimension, so that nothing depends on that dimension. Hence, we should recover the 3+1 dimensional Green function, which is the vector potential generated by a charge existing at a given point in spacetime, by instead calculating the vector potential generated by a spatial line in 4+1-dimensional spacetime.

The field at a given point will have contributions from all points on the emitting line, and the 4+1-dimensional Green function G_4 describes the contribution from each of them. The sum of all these contributions, i.e. the field resulting from the appearance of the whole line, is then the 3+1-dimensional Green function G_3. Labeling the dimensions t, x, y, z, ξ, where ξ is the extra dimension, and choosing, without loss of generality, $\xi = 0$ for the point at which we calculate the field, we first find the relation between the invariant intervals in 3+1 and 4+1 dimensions. Using

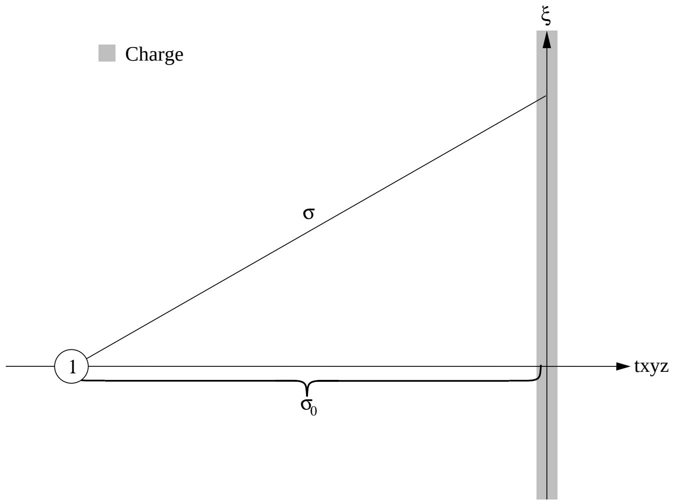

Figure 5.1: A charge existing at a spacetime point (t, x, y, z) in 3+1 dimensions is equivalent to a line of charge existing at the line formed by (t, x, y, z, ξ) when ξ takes on all values from $-\infty$ to ∞. The point in which we want to find the field also corresponds to such a line, but due to the symmetry, we may choose a single point (1) on that line as representative for our calculations. The full interval from (t, x, y, z, ξ) to the point in question is σ, while the interval in 3+1 dimensions is $\sigma_0 = [\sigma]_{\xi=0}$.

Minkowski coordinates, we have

$$\sigma_3 = \frac{1}{2}\left(-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2\right) \tag{5.2}$$

$$\begin{aligned}\sigma_4 =& \frac{1}{2}\left(-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 + \xi^2\right) \\ =& \sigma_3 + \frac{1}{2}\xi^2\end{aligned} \tag{5.3}$$

The field from a given point on the line is $G_4(\sigma_4)$, so the sum is

$$\begin{aligned}G_3 =& \int_{-\infty}^{\infty} G_4(\sigma_4) d\xi \\ =& 2\int_0^{\infty} G_4(\sigma_4) d\xi\end{aligned} \tag{5.4}$$

We now perform a change of variable to σ_4: $\xi = \pm\sqrt{2(\sigma_4 - \sigma_3)}$, $d\sigma_4 = \xi d\xi$. From the range of ξ in equation (5.4), we see that the positive choice of ξ should be used.

$$G_3 = \sqrt{2} b\theta(\Delta t) \int_{\sigma_3}^{\infty} (\sigma_4 - \sigma_3)^{-\frac{1}{2}} \left(\sigma_4^{-\frac{3}{2}}\theta(-\sigma_4) + 2\sigma_4^{-\frac{1}{2}}\delta(\sigma_4)\right) d\sigma_4 \tag{5.5}$$

In the case $\sigma_3 > 0$, this is clearly 0, as neither $\theta(-\sigma_3)$ nor $\delta(\sigma_3)$ activate. In the case $\sigma_3 < 0$, we integrate over the point $\sigma_4 = 0$, which means that the expression contains divergences. To investigate this, we introduce very small shifts ϵ in the arguments of the δ and θ functions:

$$\begin{aligned}G_3 =& \sqrt{2} b\theta(\Delta t) \int_{\sigma_3}^{\infty} (\sigma_4 - \sigma_3)^{-\frac{1}{2}} \left(\sigma_4^{-\frac{3}{2}}\theta(-\sigma_4 - \epsilon) + 2\sigma_4^{-\frac{1}{2}}\delta(\sigma_4 + \epsilon)\right) d\sigma_4 \\ =& \sqrt{2} b\theta(\Delta t) \left(\int_{\sigma_3}^{-\epsilon} (\sigma_4 - \sigma_3)^{-\frac{1}{2}} \sigma_4^{-\frac{3}{2}} d\sigma + 2(-\sigma_3)^{-\frac{1}{2}}(-\epsilon)^{-\frac{1}{2}}\right)\end{aligned} \tag{5.6}$$

The remaining integral is problematic for $\epsilon < 0$, which leads to an integral of the form $\int_-^+ x^{-\frac{3}{2}} dx$, which is undefined. The choice of $\epsilon > 0$, however, is unproblematic, as the integral then does not cross $\sigma_4 = 0$. In this case we get

$$\int_{\sigma_3}^{-\epsilon} (\sigma_4 - \sigma_3)^{-\frac{1}{2}} \sigma_4^{-\frac{3}{2}} d\sigma = 2\left[\frac{\sqrt{\sigma_4 - \sigma_3}}{\sigma_3\sqrt{\sigma_4}}\right]_{\sigma_3}^{-\epsilon} \tag{5.7}$$

$$= 2(-\sigma_3)^{-\frac{1}{2}}(-\epsilon)^{-\frac{1}{2}} \tag{5.8}$$

canceling the contribution from the delta function, giving $G_3 = 0$ here, too. Note that this null result came about as a sum of nonzero terms (G_4), so this is an example of destructive interference.

In the case $\sigma_3 = 0$, we get problems whether we shift or not, and no matter which direction, so I will leave $G_3(0)$ unevaluated. What matters is that we have rederived the most important property of G_3: it is 0 for nonzero arguments. This came about through complete destructive interference, and if G_4 did no have exactly the shape it has, there would be dispersion in 3+1 dimensions too.

5.2 Curved spacetime and the "effective number of dimensions"

The radial scalar wave equation for d+1-dimensional flat spacetime was derived in equation (4.9), and I will repeat the result here:

$$\left(-\partial_t^2 + \partial_r^2 + \frac{d-1}{r}\partial_r\right) G = -2\delta(t)\delta(r) \tag{5.9}$$

This equation would be well suited for numerical simulations if the infinitely sharp delta functions could be handled somehow, and the obvious way of doing this, is to replace them with a function that is known to approach the delta functions, for example Gaussian functions. Selected time steps from simulations with $d = 2$, $d = 3$ and $d = 4$ are plotted in figures 5.2, 5.3 and 5.4 respectively. We see that these start off similarly, but G falls off with different speeds in the area after the wave front. For $d = 2$, it falls too slowly, and so never reaches 0. For $d = 3$, G falls at just the right speed, reaching 0 behind the wave front (or inside the light cone). For $d = 4$, G falls too quickly, overshoots 0, and then does not quite reach it on the way back. The higher the number of spatial dimensions, the more the Green function overcompensates, corresponding to the factor $\frac{d-1}{r}$ in the wave equation. In general, if $d > 1$, it will cross zero $\frac{d-2}{2}$ times (rounded down), and stop at zero if d is an odd integer.

The presence of curvature has similar effects on equation (4.9) as the number of dimensions has. As a simple example, let us examine the spherically symmetric scalar wave equation in the Schwarzschild spacetime. Starting from equation (4.6)

$$g^{ii}G_{,ii} - g^{\mu\mu}\Gamma^i{}_{\mu\mu}G_{,i} = -2\delta(\Delta t)\delta(r) \tag{5.10}$$

which is the spherically symmetric scalar wave equation for diagonal metrics, where i takes the values t, r, and using $\Gamma^r{}_{tt} = \frac{R}{2r}\left(1 - \frac{R}{r}\right)$, $\Gamma^r{}_{rr} = \frac{-R}{2r\left(1-\frac{R}{r}\right)}$, $\Gamma^r{}_{\theta\theta} = -r\left(1 - \frac{R}{r}\right)$ and $\Gamma^r{}_{\phi\phi} = -r\left(1 - \frac{R}{r}\right)\sin^2\theta$, as well as $g_{tt} = -\left(1 - \frac{R}{r}\right)$, $g_{rr} = \frac{1}{1-\frac{R}{r}}$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2\sin^2\theta$, we find

$$\left(-\left(1 - \frac{R}{r}\right)\partial_t^2 + \frac{1}{1-\frac{R}{r}}\partial_r^2 + \frac{2-\frac{R}{r}}{r}\partial_r\right) G = -2\delta(t)\delta(r) \tag{5.11}$$

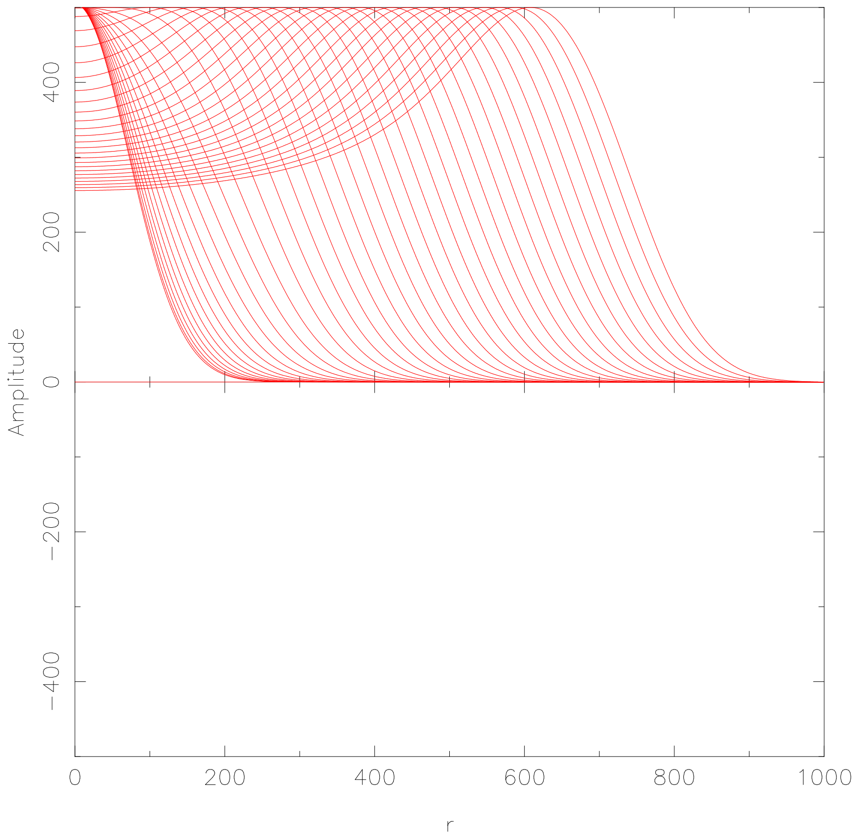

Figure 5.2: Time series for the solution of the equation $\left(-\partial_t^2 + \partial_r^2 + \frac{d-1}{r}\partial_r\right) G = -2\delta(t)\delta(r)$ with $d = 2$ and Gaussians instead of delta functions. This is a set superimposed curves, one for each time step. The general behavior is that of a pulse propagating to the right, but the amplitude behind the pulse does not reach 0, making the final result a lopsided Gaussian.

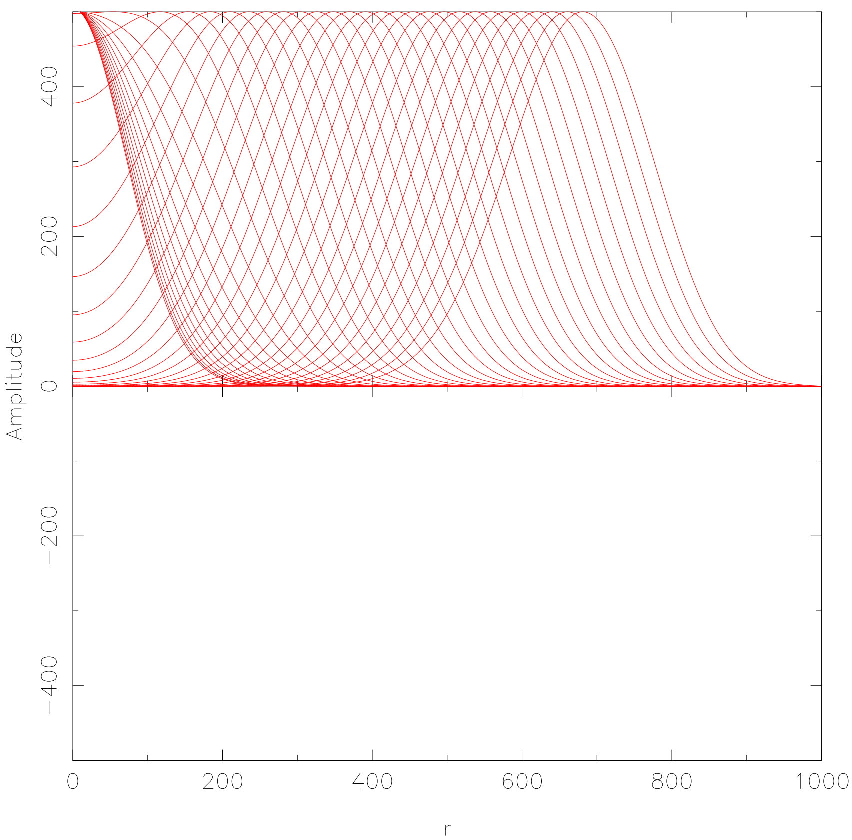

Figure 5.3: As figure 5.2, but for $d = 3$. This time the amplitude behind the pulse does reach 0, and we get a normal propagating Gaussian, corresponding to a propagating delta delta function. Since the amplitude is zero before and after the wave front, radiation propagates only on the light cone in this case.

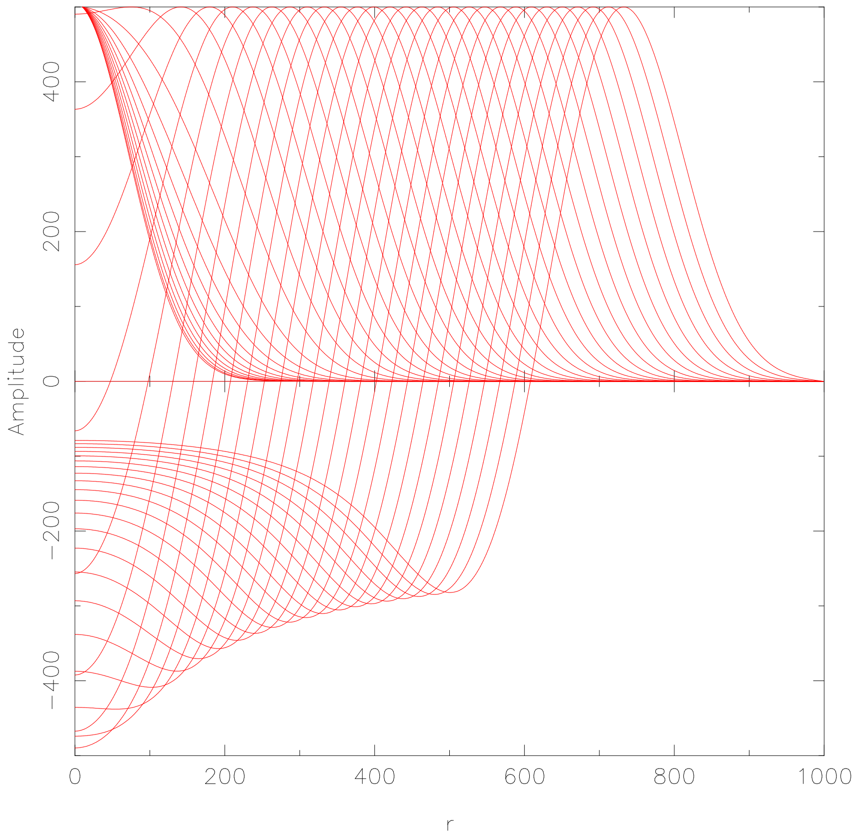

Figure 5.4: As figure 5.2, but for $d = 4$. This bears more resemblance to the case of $n = 2$ than $n = 3$, in that the amplitude behind the wave front does not reach zero, but this time it is due to it falling too rapidly, overshooting zero. Like in $d = 2$, Huygens strong principle is violated here.

To be able to compare with the flat space cases, we introduce the coordinates (τ, ρ), and tailor them to give orthonormal basis vectors at some radius r_0. It is easy to see that this requires $\tau = \sqrt{1 - \frac{R}{r_0}}t$, $\rho = \frac{r}{\sqrt{1-\frac{R}{r_0}}}$. With this, we get

$$\left(-\frac{1 - \frac{R}{r}}{1 - \frac{R}{r_0}} \partial_\tau^2 + \frac{1 - \frac{R}{r_0}}{1 - \frac{R}{r}} \partial_\rho^2 + \frac{(2 - \frac{R}{r})(1 - \frac{R}{r_0})}{\rho} \partial_\rho \right) G = -\, 2\delta(\tau)\delta(\rho)$$

We are interested in the behavior of G at the point (or rather the sphere) where our basis is orthonormal, so we set $r = r_0$, producing

$$\left(-\partial_\tau^2 + \partial_\rho^2 + \frac{2 - 3\frac{R}{r_0} + \frac{R^2}{r_0^2}}{\rho} d\rho \right) G = -\, 2\delta(\tau)\delta(\rho) \tag{5.12}$$

Meaning that the equation behaves as if space had $d = 3(1 - \frac{R}{r_0}) + \frac{R^2}{r_0^2}$ dimensions. This d is generally not an odd integer, so the Green function will have a tail here. It is reasonable to expect similar effects in other cases of curved spacetime.

The right hand side of this equation is problematic, as the source term is in the middle of the black hole. This does not matter much, though, as what we are interested in is the fate of an initially sharp wave front, and we only need the homogeneous version of the equation to find the time evolution of such a wave. One could, for example, start with an initial wave concentrated in a thin shell centered on the black hole (a shell because we assume spherical symmetry here). The homogeneous equation then tells us that this will evolve as in a d-dimensional space, and thus disperse correspondingly.

The disappearance of the Green function inside the light cone, i.e. the lack of dispersion of light, is not a general property of the wave equation, but rather is a property of a few special cases, one of which happens to be flat, 3+1-dimensional spacetime. The next chapter will give explicit examples of dispersive Green functions in curved 3+1-dimensional spacetime.

Chapter 6

Scalar Green functions in homogeneous, isotropic universes

A good example of when the curvature of space gives rise to a tail part of the Green function, is in expanding, flat, homogeneous and isotropic spacetime. Burko and A. I. Harte [3] have shown that the Green function for a minimally coupled scalar wave equation, satisfying $\Box G = -4\pi\delta(x, x')$, can be found for this case, under the assumption that there is only one matter field, with $p = \omega\rho$. They gave explicit expressions for the Green function for the cases $\omega = 0$ (dust) and $\omega = -1$ (a universe dominated by Lorenz-invariant vacuum energy, or LIVE). I will here repeat their calculation in greater detail, find the expression also for the case of $\omega = \frac{1}{3}$ (radiation dominated), and give the expressions both as a function of conformal time and cosmological time.

6.1 The metric

A flat, homogeneous and isotropic universe has a metric given by

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + dX^2 + dY^2 + dZ^2\right) \tag{6.1}$$

where η is conformal time. The form of $a(\eta)$ is dictated by the first Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G a^2 \sum_n \rho_n(a) - k \tag{6.2}$$

where dots denote differentiation by η, and which, when we assume that the universe is flat ($k = 0$) and dominated by a single ideal fluid, becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G a^2 \rho(a) \tag{6.3}$$

$\rho(a)$ is determined by the conservation of energy and momentum as the universe expands ($T^{\mu\nu}_{;\nu} = 0$), which gives

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+\omega)} \tag{6.4}$$

where zeros indicate the present time, and a_0 is fixed to 1. These combine to form

$$\dot{a}^2 = \overbrace{\frac{8}{3}\pi G \omega_0}^{H_0^2} a^{1-3\omega} \Leftrightarrow \tag{6.5}$$

$$\dot{a} = H_0 a^{\frac{1}{2}(1-3\omega)} \Leftrightarrow \tag{6.6}$$

$$a = \left(\frac{1}{2}(1+3\omega)H_0\eta + C\right)^{\frac{2}{1+3\omega}} \tag{6.7}$$

For cases $1 + 3\omega > 0$, there is a time when $a = 0$, and by fixing that time to 0, we get

$$a = \left(\frac{1}{2}(1+3\omega)H_0\eta\right)^{\frac{2}{1+3\omega}} \tag{6.8}$$

otherwise, we make $\eta = 0$ correspond to the present, getting

$$a = \left(\frac{1}{2}(1+3\omega)H_0\eta + 1\right)^{\frac{2}{1+3\omega}} \tag{6.9}$$

for the special cases we will consider, this becomes

$$\omega = -1 \qquad a = \frac{1}{1 - H_0\eta} \tag{6.10}$$

$$\omega = 0 \qquad a = \frac{1}{4}H_0^2\eta^2 \tag{6.11}$$

$$\omega = \frac{1}{3} \qquad a = H_0\eta \tag{6.12}$$

6.1.1 Connection coefficients

As we are in coordinate basis, the connection coefficients are given by

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(g_{\alpha\nu;\beta} + g_{\nu\beta;\alpha} - g_{\alpha\beta;\nu}\right) \tag{6.13}$$

making the nonzero coefficients

$$\Gamma^{\eta}_{\mu\mu} = \Gamma^{i}_{\eta i} = \frac{\dot{a}}{a} \tag{6.14}$$

6.2 The wave equation

To compare with the earlier flat-space examples, we would ideally use the electromagnetic wave equation

$$\Box A^{\mu} - R^{\mu}_{\nu} A^{\nu} = -4\pi j^{\mu} \tag{6.15}$$

Unfortunately, this is a set of four coupled second-order partial differential equations due to the mixing nature of the $\Box$ operator in curved spacetime. We therefore follow Burko and A. I. Harte [3], and limit ourselves to the minimally coupled scalar wave equation

$$\Box \phi = -4\pi j \tag{6.16}$$

with the corresponding Green equation

$$\Box G(x, x') = -4\pi\delta(x, x') = -4\pi(-g)^{-\frac{1}{2}}\delta(x - x') \tag{6.17}$$

$\delta(x, x')$ being the invariant delta function $\delta(x, x') = (-g)^{-\frac{1}{2}}\delta(x - x')$, and

$$\begin{aligned}
\Box G =& g^{\alpha\beta} G_{;\alpha\beta} && (6.18)\\
=& g^{\alpha\beta}\left(G_{,\alpha\beta} - \Gamma^{\rho}_{\alpha\beta} G_{,\rho}\right) && (6.19)\\
=& a^{-2}\left(-\frac{\partial^2}{\partial\eta^2} + \nabla^2\right) G + \frac{\dot{a}}{a^3} G_{,\eta} - 3\frac{\dot{a}}{a^3} G_{,\eta} && (6.20)\\
=& a^{-2}\left(-\frac{\partial^2}{\partial\eta^2} + \nabla^2 - \frac{2\dot{a}}{a}\frac{\partial}{\partial\eta}\right) G && (6.21)
\end{aligned}$$

It will prove advantageous to work with the reduced Green function $g = aa'G$, where primed variables correspond to the spacetime point x', and the unprimed ones to x. With this, we get

$$\begin{aligned}
\Box G =& \frac{1}{a^2 a'}\left(-\frac{\partial^2}{\partial\eta^2} + \nabla^2 - \frac{2\dot{a}}{a}\frac{\partial}{\partial\eta}\right)\frac{g}{a} && (6.22)\\
=& \frac{1}{a^3 a'}\left(-\frac{\partial^2}{\partial\eta^2} + \nabla^2 + \frac{\ddot{a}}{a}\right) g && (6.23)
\end{aligned}$$

The volume element $\sqrt{-g} = a^4$, but since it appears multiplied by the delta function, which only gives a contribution when $x = x'$, we may just as well use $a^3 a'$. With this, the a factors on both sides cancel, and we are left with

$$\left(-\frac{\partial^2}{\partial\eta^2} + \nabla^2 + \frac{\ddot{a}}{a}\right) g = -4\pi\delta(x - x') \tag{6.24}$$

The factor $\frac{\ddot{a}}{a}$ is $\frac{2H_0^2}{(1-H_0\eta)^2}$ for $\omega = -1$, $\frac{2}{\eta^2}$ for $\omega = 0$ and 0 for $\omega = \frac{1}{3}$.

We will solve the equation by Fourier transforming with respect to the three spatial dimensions. Applying the Fourier transform $\int d^3\vec{x}e^{-i\vec{k}\cdot\Delta\vec{x}}$ on both sides yields

$$\left(\frac{\partial^2}{\partial\eta^2} + k^2 - \frac{\ddot{a}}{a}\right)\tilde{g} = 4\pi\delta(\eta - \eta') \tag{6.25}$$

We know that the wave equation in general has both retarded (cause before effect) and advanced (cause after effect) solutions. To restrict ourselves to the physically relevant retarded solution, we let $\tilde{g} = \theta(\eta - \eta')\widehat{g}$, giving

$$\theta(\eta - \eta')\left(\frac{\partial^2}{\partial\eta^2} + k^2 - \frac{\ddot{a}}{a}\right)\widehat{g} + \dot{\delta}(\eta - \eta')\widehat{g} + 2\delta(\eta - \eta')\dot{\widehat{g}} = 4\pi\delta(\eta - \eta') \tag{6.26}$$

Using $f(x)\delta(x) = f(0)\delta(x)$ and differentiating, we find $(f(x)\delta(x))' = f'(x)\delta(x) + f(x)\delta'(x) = f'(0)\delta(x) + f(x)\delta'(x) = f(0)\delta'(x)$, so $\dot{\delta}(\eta - \eta')\widehat{g}(\eta, \eta') = \dot{\delta}(\eta - \eta')\widehat{g}(\eta = \eta') - \delta(\eta - \eta')\dot{\widehat{g}}(\eta = \eta')$. This gives

$$\begin{aligned}&\theta(\eta - \eta')\left(\frac{\partial^2}{\partial\eta^2} + k^2 - \frac{\ddot{a}}{a}\right)\widehat{g}(\eta, \eta')\\ &+\delta(\eta - \eta')\dot{\widehat{g}}(\eta = \eta') + \dot{\delta}(\eta - \eta')\widehat{g}(\eta = \eta') = 4\pi\delta(\eta - \eta')\end{aligned} \tag{6.27}$$

This must be fulfilled for all η, η', so we get three equations:

$$\left(\frac{\partial^2}{\partial\eta^2} + k^2 - \frac{\ddot{a}}{a}\right)\widehat{g} = 0 \tag{6.28}$$

$$\dot{\widehat{g}}(\eta = \eta') = 4\pi \tag{6.29}$$

$$\widehat{g}(\eta = \eta') = 0 \tag{6.30}$$

For all $\omega \geq -\frac{1}{3}$, $a \propto \eta^\alpha$, $\alpha = \frac{2}{1+3\omega}$. For the other values of ω, we make the change of variable $\epsilon = \eta + \frac{2}{(1+3\omega)H_0}$, giving the same differential equation, and $a \propto \epsilon^\alpha$. The general form of the equation we have to solve is, therefore

$$\left(\eta^2\frac{\partial^2}{\partial\eta^2} + \eta^2k^2 - \alpha(\alpha - 1)\right)\widehat{g} = 0 \tag{6.31}$$

If we now make the substitution $\mu = k\eta$, giving

$$\mu^2\widehat{g}'' + (\mu^2 - \alpha(\alpha - 1))\widehat{g} = 0 \tag{6.32}$$

where $' = \frac{\partial}{\partial\mu}$, we see that we are very close to the Bessel equation $x^2y'' + xy' + (x^2 - a^2)y = 0$. To find a function satisfying the Bessel equation, we make the ansatz $\widehat{g} = h(\eta)m(\eta$:

$$\mu^2 \left(h'' + \underbrace{2h'\frac{m'}{m}}_{D} + \underbrace{\frac{m''}{m}h}_{E} \right) + \left(\mu^2 - \alpha[\alpha - 1]\right) h = 0 \tag{6.33}$$

From the part marked by D, we get the requirement $2\frac{m'}{m} = \frac{1}{\mu}$. This requires $m = C\sqrt{\mu}$. The final equation for h is then

$$\mu^2 h'' + \mu h' + \left(\mu^2 - [\alpha - \frac{1}{2}]^2\right) h = 0 \tag{6.34}$$

The general solution is

$$h(\mu) = A J_{\alpha-\frac{1}{2}} + B J_{\frac{1}{2}-\alpha} \tag{6.35}$$

unless $\alpha - \frac{1}{2}$ is an integer, in which case it is

$$h(\mu) = A J_{\alpha-\frac{1}{2}} + B Y_{\alpha-\frac{1}{2}} \tag{6.36}$$

Here J denotes Bessel functions of the first kind, while Y denotes Bessel functions of the second kind.

Assuming that $\alpha - \frac{1}{2}$ is non-integral, and going back to the normal variables, we get

$$\widehat{g} = \sqrt{k\eta} \left[A J_{\alpha-\frac{1}{2}}(k\eta) + B J_{\frac{1}{2}-\alpha}(k\eta) \right] \tag{6.37}$$

Equation (6.29) and (6.30) allow us to determine A and B, but to get further, we need to know which Bessel functions we are dealing with, and so α must be specified.

6.3 Radiation dominated universe, $\omega = \frac{1}{3}$

Here $\alpha = 0$, so the functions involved are $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$. These particular Bessel functions thankfully have the simple expressions $\sqrt{\frac{2}{\pi x}} \sin x$ and $\sqrt{\frac{2}{\pi x}} \cos x$ respectively, so we have

$$\widehat{g} = \sqrt{\frac{2k}{\pi}} \left[A \cos(k\eta) + B \sin(k\eta) \right] \tag{6.38}$$

Inspection of the differential equation shows that we still have a solution even if we shift the arguments of the trigonometric functions, so we conveniently replace η by $\Delta\eta$ there.

$$\widehat{g} = \sqrt{\frac{2k}{\pi}} \left[A \sin(k\Delta\eta) + B \cos(k\Delta\eta) \right] \tag{6.39}$$

The requirement $\widehat{g}(\eta = \eta') = 0$ now takes the simple form $A = 0$, while $\dot{\widehat{g}}(\eta = \eta') = 4\pi$ becomes $B = \sqrt{\frac{8\pi^3}{k^3}}$. Thus:

$$\widehat{g} = \frac{4\pi}{k} \sin(k\Delta\eta) \tag{6.40}$$

We now recall that $\tilde{g} = \theta(\Delta\eta)\widehat{g}$, and use an inverse Fourier transform to recover the reduced Green function g.

$$\begin{aligned}
g =& (2\pi)^{-3} \int d^3\vec{k} e^{i\vec{k}\cdot\vec{x}} 4\pi\theta(\Delta\eta) k^{-1} \sin(k\Delta\eta) && (6.41)\\
=& \frac{\theta(\Delta\eta)}{\pi} \int_0^\infty dk \int_0^\pi d\theta_k k \sin\theta_k e^{ikr\cos\theta_k} \sin(k\Delta\eta) && (6.42)\\
=& \frac{\theta(\Delta\eta)}{\pi} \int_0^\infty dk \int_{-1}^1 dmk e^{ikrm} \sin(k\Delta\eta) && (6.43)\\
=& \frac{2}{\pi} \frac{\theta(\Delta\eta)}{r} \int_0^\infty \sin(k\Delta\eta) \sin(kr) && (6.44)\\
=& \frac{1}{r} \delta(\Delta\eta - r) && (6.45)
\end{aligned}$$

where I transformed $\vec{k}$ into polar coordinates and used the identity

$$\frac{1}{\pi} \int_{-\infty}^\infty dx \sin(ax) \sin(bx) = \delta(a - b) - \delta(a + b) \tag{6.46}$$

and $r \equiv |\Delta\vec{x}|$. We finally find the full Green function as

$$\begin{aligned}
G(x, x') =& \frac{\delta(\Delta\eta - r)}{\eta\eta' H_0^2 r} && (6.47)\\
=& \frac{\delta(\Delta\eta - r)}{\eta\eta' H_0^2 \Delta\eta} && (6.48)
\end{aligned}$$

As in flat spacetime, this is proportional to a delta function, so we have no dispersion in this case.

6.3.1 Transforming to cosmological time

In cosmological time, the line element is given as

$$ds^2 = -dt^2 + a^2 \left(dX^2 + dY^2 + dZ^2 \right) \tag{6.49}$$

Comparing with the line element in conformal time, we see that

$$\begin{aligned}
dt =& a d\eta = H_0 \eta d\eta \Leftrightarrow && (6.50)\\
\int_0^t dt =& \int_0^\eta H_0 \eta d\eta \Leftrightarrow && (6.51)\\
t =& \frac{1}{2} H_0 \eta^2 && (6.52)
\end{aligned}$$

and so $a = \sqrt{2H_0 t}$. Using this and the property $\delta(ax) = \frac{1}{a}\delta(x)$ of the delta function, we find

$$G(x, x') = \frac{\delta\left(\sqrt{\frac{2t}{H_0}} - \sqrt{\frac{2t'}{H_0}} - r)\right)}{2\sqrt{tt'}H_0\left(\sqrt{\frac{2t}{H_0}} - \sqrt{\frac{2t'}{H_0}}\right)} \tag{6.53}$$

$$= \frac{\delta\left(\sqrt{t} - \sqrt{t}' - \sqrt{\frac{H_0}{2}}r\right)}{4\sqrt{tt'}\left(\sqrt{t} - \sqrt{t'}\right)} \tag{6.54}$$

This can not be expressed using only Δt and r because the Hubble constant isn't constant, making it theoretically possible to determine the time since the beginning of the universe from the form of the Green function.

6.4 Dust dominated universe, $\omega = 0$

In this case $\alpha = 2$, so Bessel functions in the solution are $J_{\frac{3}{2}}(x)$ and $J_{-\frac{3}{2}}(x)$. The relation

$$J_{a-1}(x) + J_{a+1}(x) = \frac{2a}{x}J_a(x) \tag{6.55}$$

makes it possible to express $J_{\pm\frac{3}{2}}(x)$ using the simple $J_{\pm\frac{1}{2}}(x)$:

$$J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) \tag{6.56}$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{1}{x}\sin x - \cos x\right) \tag{6.57}$$

$$J_{-\frac{3}{2}}(x) = \frac{1}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x) \tag{6.58}$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{1}{x}\cos x - \sin x\right) \tag{6.59}$$

which, inserted into equation (6.37) and absorbing the factor $\sqrt{\frac{2}{\pi}}$ into the arbitrary factors A and B, gives

$$\widehat{g} = A\left[\frac{1}{k\eta}\sin(k\Delta\eta) - \cos(k\Delta\eta)\right] + B\left[\frac{1}{k\eta}\cos(k\Delta\eta) - \sin(k\Delta\eta)\right] \tag{6.60}$$

I have here again replaced η with $\Delta\eta$ in the arguments of sin and cos, as this is transparent to the differential equation. The boundary condition

$\widehat{g}(\eta = \eta') = 0$ gives $B = Ak\eta'$, and $\dot{\widehat{g}}(\eta = \eta') = 4\pi$ gives

$$4\pi = \frac{A}{\eta'} - \frac{B}{k\eta'^2} - kB \Leftrightarrow \tag{6.61}$$

$$A = -\frac{4\pi}{k^2\eta'} \tag{6.62}$$

$$B = -\frac{4\pi}{k} \tag{6.63}$$

The full expression for $\widehat{g}$ is thereby

$$\widehat{g} = \frac{4\pi}{k}\left(\frac{1}{k\eta'}\cos(k\Delta\eta) - \frac{1}{k^2\eta\eta'}\sin(k\Delta\eta) + \sin(k\Delta\eta) - \frac{1}{k\eta}\cos(k\Delta\eta)\right) \tag{6.64}$$

$$= \frac{4\pi}{k}\left(\left[1 - \frac{1}{k^2\eta\eta'}\right]\sin(k\Delta\eta) + \frac{\Delta\eta}{k\eta\eta'}\cos(k\Delta\eta)\right) \tag{6.65}$$

To find the reduced green function, we Fourier transform again:

$$g = (2\pi)^{-3}\theta(\Delta\eta)\int d^3\vec{k}e^{i\vec{x}\cdot\vec{k}}\widehat{g} \tag{6.66}$$

$$= \frac{\theta(\Delta\eta)}{\pi}\int_0^\infty dk\int_0^\pi d\theta_k e^{irk\cos\theta_k}\sin\theta_k\left[k\left(1 + \frac{1}{\eta\eta' k^2}\right)\sin(k\Delta\eta) - \frac{\Delta\eta\cos(k\Delta\eta)}{\eta\eta'}\right] \tag{6.67}$$

$$= \frac{2\theta(\Delta\eta)}{r\pi}\int_0^\infty dk\Bigg[\sin(rk)\sin(k\Delta\eta) + \frac{1}{\eta\eta'}\left(k^{-2}\sin(k\Delta\eta) - \Delta\eta k^{-1}\cos(k\Delta\eta)\right)\sin(kr)\Bigg] \tag{6.68}$$

We recognize the first of these terms as the delta integral $\frac{2}{\pi}\int_0^\infty \sin(ax)\sin(bx) = \delta(a-b) + \delta(a+b)$. Because of the step function, and because r is positive, only the first delta function survives. The parenthesis in the second term can be written as a derivative, and so we have

$$g = \frac{\delta(\Delta\eta - r)}{r} - \frac{2\theta(\Delta\eta)}{r\pi\eta\eta'}\underbrace{\int_0^\infty dk\left(\frac{\sin(k\Delta\eta)}{k}\right)'\sin(kr)}_{F} \tag{6.69}$$

$$F = \underbrace{\left[\frac{\sin(k\Delta\eta)}{k}\sin(kr)\right]_0^\infty}_{0} - r\int_0^\infty dk k^{-1}\sin(k\Delta\eta)\cos(kr) \tag{6.70}$$

$$= -\frac{r\pi}{2}\left(\theta(\Delta\eta - r) + \theta(-\Delta\eta - r)\right) \Rightarrow \tag{6.71}$$

$$g = \frac{\delta(\Delta\eta - r)}{\Delta\eta} + \frac{\theta(\Delta\eta - r)}{\eta\eta'} \tag{6.72}$$

The full Green function is then

$$G(x,x') = \frac{16}{H_0^4\eta^2\eta'^2}\left(\frac{\delta(\Delta\eta - r)}{\Delta\eta} + \frac{\theta(\Delta\eta - r)}{\eta\eta'}\right) \tag{6.73}$$

This time the delta function is joined by a step function, so the Green function is not zero inside the light cone: it has a tail.

6.4.1 Transforming to cosmological time

The relationship between cosmological time and conformal time is given by

$$dt = ad\eta = \frac{1}{4}H_0^2\eta^2 d\eta \Leftrightarrow \tag{6.74}$$

$$\int_o^t dt = \frac{1}{4}H_0^2\int_0^\eta \eta^2 d\eta \Leftrightarrow \tag{6.75}$$

$$t = \frac{1}{12}H_0^2\eta^3 \Leftrightarrow \eta = \sqrt[3]{\frac{12t}{H_0^2}} \tag{6.76}$$

Inserting this into the Green function, we find

$$G(x,x') = \frac{16}{H_0^4\left(\frac{12}{H_0^2}\right)^{\frac{4}{3}}\sqrt[3]{t^2t'^2}}\left(\frac{\delta(\sqrt[3]{\frac{12t}{H_0^2}} - \sqrt[3]{\frac{12t'}{H_0^2}} - r)}{\sqrt[3]{\frac{12t}{H_0^2}} - \sqrt[3]{\frac{12t'}{H_0^2}}} + \frac{\theta(\sqrt[3]{\frac{12t}{H_0^2}} - \sqrt[3]{\frac{12t'}{H_0^2}} - r)}{\left(\frac{12}{H_0^2}\right)^{\frac{2}{3}}\sqrt[3]{tt'}}\right) \tag{6.77}$$

$$= \frac{1}{9}\left(\frac{\delta\left(\sqrt[3]{t} - \sqrt[3]{t'} - \sqrt[3]{\frac{H_0^2}{12}r}\right)}{\sqrt[3]{t^2t'^2}\left(\sqrt[3]{t} - \sqrt[3]{t'}\right)} + \frac{\theta\left(\sqrt[3]{t} - \sqrt[3]{t'} - \sqrt[3]{\frac{H_0^2}{12}r}\right)}{tt'}\right) \tag{6.78}$$

which is again seen to depend on the values of t and t', and not only on their difference. As in the case for a radiation dominated universe, this comes about because the universe does not look the same at every instance; its relative expansion gradually slows down.

6.5 Lorentz-invariant vacuum energy dominated (de Sitter) universe, $\omega = -1$

Since we now have $\omega < -\frac{1}{3}$, we make the variable change $\epsilon = \eta + \frac{\alpha}{H_0} = \eta - H_0^{-1}$, since $\alpha = \frac{2}{1+3\omega} = -1$. We would now normally calculate $J_{-\frac{3}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$, find initial conditions, Fourier transform, and so on, but closer inspection of the differential equation, (equation (6.32)), shows us that only the product $\alpha(\alpha-1)$ matters. In this case this is 2, which is the same value one gets for $\omega = 0 \Rightarrow \alpha = 2$, the dust case we computed in the last section. We therefore immediately know that

$$g = \frac{\delta(\Delta\epsilon - r)}{\Delta\epsilon} + \frac{\theta(\Delta\epsilon - r)}{\epsilon\epsilon'} \tag{6.79}$$

$$= \frac{\delta(\Delta\eta - r)}{\Delta\eta} + \frac{\theta(\Delta\eta - r)}{\left(\eta - \frac{1}{H_0}\right)\left(\eta' - \frac{1}{H_0}\right)} \tag{6.80}$$

Since $a = \frac{1}{1-H_0\eta}$, we find

$$G(x,x') = \frac{\left(\eta - H_0^{-1}\right)\left(\eta' - H_0^{-1}\right)}{\Delta\epsilon}\delta(\Delta\epsilon - r) + H_0^2\theta(\Delta\epsilon - r) \tag{6.81}$$

6.5.1 Transforming to cosmological time

$$dt = ad\eta = \frac{d\eta}{1 - H_0\eta} \Leftrightarrow \tag{6.82}$$

$$t = -\frac{1}{H_0}\ln(1 - H_0\eta) \tag{6.83}$$

$$\eta = \frac{1}{H_0}\left(1 - e^{-H_0 t}\right) \tag{6.84}$$

Inserting this into the Green function, we find

$$G(x,x') = \overbrace{H_0\frac{e^{-H_0t}e^{-H_0t'}}{e^{-H_0t'} - e^{-H_0t}}\delta\left(\frac{1}{H_0}\left[e^{-H_0t'} - e^{-H_0t}\right] - r\right)}^{J} + \underbrace{H_0^2\theta\left(\frac{1}{H_0}\left[e^{-H_0t'} - e^{-H_0t}\right] - r\right)}_{K} \tag{6.85}$$

$$J = \frac{H_0^2\delta\left(-e^{-H_0t} + e^{-H_0t'} + H_0r\right)}{e^{H_0t} - e^{H_0t'}} \tag{6.86}$$

$$= \frac{H_0^2\delta\left(1 - e^{-H_0\Delta t} - H_0re^{H_0t'}\right)}{e^{H_0\Delta t} - 1} \tag{6.87}$$

We now use the property $\delta(f(x) - y) = \frac{1}{f'(x)}\delta(x - f^{-1}(y))$ to isolate the Δt part of the argument of the delta function. In this case we have $f(x) = e^{-H_0x}$, so $f'(x) = -H_0e^{-H_0x}$ and $f^{-1}(x) = -\frac{1}{H_0}\ln x$. Thus

$$J = \frac{H_0\delta\left(\Delta t + \frac{1}{H_0}\ln\left[1 - H_0re^{H_0t'}\right]\right)}{1 - e^{-H_0\Delta t}} \tag{6.88}$$

This manipulation has not changed when the argument of the delta function goes from negative to positive. Since the step function in part K shares the same argument, and the step function only depends on when this change takes place, we can make the same change in the argument of the step function as we did with the delta function above.

$$K = H_0^2 \theta \left(\Delta t + \frac{1}{H_0} \ln \left[1 - H_0 r e^{H_0 t'} \right] \right) \tag{6.89}$$

All in all

$$G(x, x') = \frac{H_0 \delta \left(\Delta t + \frac{1}{H_0} \ln \left[1 - H_0 r e^{H_0 t'} \right] \right)}{1 - e^{-H_0 \Delta t}} + H_0^2 \theta \left(\Delta t + \frac{1}{H_0} \ln \left[1 - H_0 r e^{H_0 t'} \right] \right) \tag{6.90}$$

This may appear to depend on absolute time, like in radiation and dust dominated universes, due to the t'-dependence of the logarithm. However, in conformal time we had $\delta(\Delta\eta - r)$, which corresponds to a light-like separation between x and x' when the arguments of the δ and step functions change from negative to positive. Therefore the temporal and spatial distance between x and x' must be equal then, and thus it must be possible to write

$$G = \frac{H_0 \delta(\Delta t - L)}{1 - e^{-H_0 \Delta t}} + H_0^2 \theta(\Delta t - L) \tag{6.91}$$

where L is the spatial distance between x and x' as measured along a geodesic connecting them. This, as opposed to the coordinate difference, is measurable, and so no experiment would measure a dependence on absolute time for the Green function. This is not the case for the dust and radiation dominated universes, as their t-dependence also occurs outside the argument of the delta and step functions.

It is interesting to note that the tail of the Green function is constant inside the light cone. Since the Green function is the potential caused by a source existing at a given point in spacetime, and a world line consists of a continuous series of such points, the potential around a source will gradually increase, without limit. But since only the gradient of the potential has a physical meaning, this is not a problem.

6.6 Physical relevance

These 3 cases are representative for different phases in the history of the universe, which started out as radiation dominated, then became dust dominated, and is now believed to be LIVE-dominated. Though we only found the scalar Green function, the electromagnetic Green function will be qualitatively similar, which means that we should expect some dispersion of light as it traverses the universe.

Though we have seen how the Green function can develop a tail due to curvature in this chapter, it was only for the scalar wave equation, and not the electromagnetic wave equation, which is more relevant physically. This was due to the difficulty introduced by the mixing effect of the wave operator $\Box$ in curved space-time. In the next chapter, I will use numerical methods to find the electromagnetic Green function in another case where the wave operator introduces mixing. Here we will also see how complicated $\Box$ can be when expressed in coordinate form.

Chapter 7

The tail term in the Schwarzschild spacetime

Until now, we have examined the various effects that can result in a tail term from various angles, one angle at a time, and have avoided the most obvious case where a tail term appears in the equation of motion for a charged particle: the Schwarzschild spacetime. I have put this off until now, not only because it is best to understand the individual effects that make up the tail term before one looks at a case which combines them, but also because this case is dramatically more complicated than the previous cases we have considered. In certain circumstances, this problem can be considered analytically. DeWitt and DeWitt [6] have considered the general problem of falling charges in the linearized theory, and Smith and Will [14] have analyzed the special case of a charge at rest at a constant distance from a black hole. But in general, the self force is too complicated to handle analytically in this case. I will therefore take a numerical approach here.

7.1 The electromagnetic wave equation

As before, a general form of the electromagnetic wave equation is

$$A^{\mu;\nu}{}_{\nu} - R^{\mu}{}_{\nu}A^{\nu} = -4\pi j^{\mu} \tag{7.1}$$

which can be derived from Maxwell's equations with Lorenz gauge. But for complicated cases, this equation is not the most efficient way of arriving at a useful form of the wave equation. It is actually easiest to start from Maxwell's equations themselves.

In form language, Maxwell's four equations reduce to the single equation

$$\underline{d} * \underline{F} = * \mu_0 \underline{j} \tag{7.2}$$

where $\underline{F}$ and $\underline{j}$ are forms corresponding to the electromagnetic field tensor and four-current respectively. In form language, the field potential is related to the field itself by $\underline{dA} = \underline{F}$. Using this, and the identity $**\underline{dT} = s(-1)^{p(n-p)}\underline{T}$, where $\underline{T}$ is some form of order p, where s is the sign of the metric determinant (-1 in this case), and n is the number of dimensions, we get

$$*\underline{d} * \underline{dA} = \mu_0 \underline{j} \tag{7.3}$$

Working in coordinate basis, we have $\underline{A} = A_\mu \partial x^\mu$, and

$$\begin{aligned}\underline{dA} =&(A_{r,t} - A_{t,r})\underline{d}t \wedge \underline{d}r + (A_{\theta,t} - A_{t,\theta})\underline{d}t \wedge \underline{d}\theta \\ &+(A_{\phi,t} - A_{t,\phi})\underline{d}t \wedge \underline{d}\phi + (A_{\theta,x} - A_{x,\theta})\underline{d}r \wedge \underline{d}\theta \\ &+(A_{\phi,x} - A_{x,\phi})\underline{d}r \wedge \underline{d}\phi + (A_{\phi,\theta} - A_{\theta,\phi})\underline{d}\theta \wedge \underline{d}\phi\end{aligned} \tag{7.4}$$

The hodge star is given by

$$\begin{aligned}*\underline{T} =&\langle \underline{\varepsilon}, \underline{T}\rangle \\ =&\frac{\sqrt{g}}{n!(n-p)!}\epsilon_{\mu_1\cdots\mu_p\nu_1\cdots\nu_{n-p}}T^{\mu_1\cdots\mu_p}\underline{\omega}^{\nu_1} \wedge \cdots \wedge \underline{\omega}^{\nu_{n-p}} \\ =&\sqrt{g}\epsilon_{|\mu_1\cdots\mu_p||\nu_1\cdots\nu_{n-p}|}T^{|\mu_1\cdots\mu_p|}\underline{\omega}^{|\nu_1} \wedge \cdots \wedge \underline{\omega}^{\nu_{n-p}|}\end{aligned} \tag{7.5}$$

were $\epsilon_{\mu_1\cdots\mu_n} = s\epsilon^{\mu_1\cdots\mu_n}$, the latter which is the totally antisymmetric Levi-Chevita symbol, and g is the absolute value of the determinant of the metric. Since this is a linear operation, we need only calculate the effect it has on our basis forms. $\underline{a} = \underline{d}t \wedge \underline{d}r$ has tensor components $a_{tr} = -a_{rt} = 1$, with corresponding contravariant components of $a^{tr} = -a^{rt} = -a_{tr}$, where I have used $g_{\mu\nu} = \mathrm{diag}(-\beta, \beta^{-1}, r^2, r^2\sin\theta)$, where $\beta = 1 - \frac{R}{r}$. With this, we find

$$*(\underline{d}t \wedge \underline{d}r) = -\sqrt{g}\epsilon^{tr\theta\phi}(-1)\underline{d}\theta \wedge \underline{d}\phi = \sqrt{g}\underline{d}\theta \wedge \underline{d}\phi \tag{7.6}$$

The calculation proceeds in an identical manner for the others, and all together, we have

$$\begin{aligned}*(\underline{d}t \wedge \underline{d}r) =&r^2\sin\theta\underline{d}\theta \wedge \underline{d}\phi & *(\underline{d}t \wedge \underline{d}\theta) =& -\beta^{-1}\sin\theta\underline{d}r \wedge \underline{d}\phi \\ *(\underline{d}t \wedge \underline{d}\phi) =&(\beta\sin\theta)^{-1}\underline{d}r \wedge \underline{d}\theta & *(\underline{d}r \wedge \underline{d}\theta) =& -\beta\sin\theta\underline{d}t \wedge \underline{d}\phi \\ *(\underline{d}r \wedge \underline{d}\phi) =&\beta\sin^{-1}\theta\underline{d}t \wedge \underline{d}\theta & *(\underline{d}\theta \wedge \underline{d}\phi) =& -(r^2\sin\theta)^{-1}\underline{d}t \wedge \underline{d}r\end{aligned} \tag{7.7}$$

Thus

$$\begin{aligned}*\underline{dA} =&r^2\sin\theta(A_{r,t} - A_{t,r})\underline{d}\theta \wedge \underline{d}\phi - \beta^{-1}\sin\theta(A_{\theta,t} - A_{t,\theta})\underline{d}r \wedge \underline{d}\phi \\ &+(\beta\sin\theta)^{-1}(A_{\phi,t} - A_{t,\phi})\underline{d}r \wedge \underline{d}\theta - \beta\sin\theta(A_{\theta,r} - A_{r,\theta})\underline{d}t \wedge \underline{d}\phi \\ &+\beta\sin^{-1}\theta(A_{\phi,r} - A_{r,\phi})\underline{d}t \wedge \underline{d}\theta - (r^2\sin\theta)^{-1}(A_{\phi,\theta} - A_{\theta,\phi})\underline{d}t \wedge \underline{d}r\end{aligned} \tag{7.8}$$

Taking the exterior derivative and using the hodge star again is a straightforward but tedious operation, which results in

$$
\begin{aligned}
*\underline{d} * \underline{dA} =& \bigg\{ \frac{2\beta}{r}(A_{r,t} - A_{t,r}) + \beta(A_{r,tr} - A_{t,rr}) + \frac{1}{r^2}\Big[\cot\theta(A_{\theta,t} - A_{t,\theta}) \\
&+ (A_{\theta,t\theta} - A_{t,\theta\theta})\Big] + \frac{1}{r^2\sin^2\theta}(A_{\phi,t\phi} - A_{t,\phi\phi}) \bigg\} \underline{dt} \\
&+ \bigg\{ \frac{1}{\beta}(A_{r,tt} - A_{t,rt}) + \frac{1}{r^2}\Big[\cot\theta(A_{\theta,r} - A_{r,\theta}) + (A_{\theta,r\theta} - A_{r,\theta\theta})\Big] \\
&+ \frac{1}{r^2\sin^2\theta}(A_{\phi,r\phi} - A_{r,\phi\phi}) \bigg\} \underline{dr} \\
&+ \bigg\{ \frac{1}{\beta}(A_{\theta,tt} - A_{t,\theta t}) - \Big[\frac{R}{r^2}(A_{\theta,r} - A_{r,\theta}) + \beta(A_{\theta,rr} - A_{r,\theta r})\Big] \\
&+ \frac{1}{r^2}(A_{\phi,\theta\phi} - A_{\theta,\phi\phi}) \bigg\} \underline{d\theta} \\
&+ \bigg\{ \frac{1}{\beta}(A_{\phi,tt} - A_{t,\phi t}) - \frac{R}{r^2}(A_{\phi,r} - A_{r,\phi}) - \beta(A_{\phi,rr} - A_{r,\phi r}) \\
&- \frac{1}{r^2}\Big[-\cot\theta(A_{\phi,\theta} - A_{\theta,\phi}) + (A_{\phi,\theta\theta} - A_{\theta,\phi\theta})\Big] \bigg\} \underline{d\phi} \qquad (7.9)
\end{aligned}
$$

The mixed derivatives are unexpected, and inspection of equation (7.1) shows that we should not end up with mixed derivatives. It must therefore be possible to eliminate them. From the coefficient of $\underline{dt}$ we have the mixed derivatives

$$
\begin{aligned}
\text{mixed}_t =& \beta A_{r,tr} + \frac{1}{r^2}A_{\theta,t\theta} + \frac{1}{r^2\sin^2\theta}A_{\phi,t\phi} \\
=& \left(\beta A_{r,r} + \frac{1}{r^2}A_{\theta,\theta} + \frac{1}{r^2}A_{\phi,\phi} \right)_{,t} \\
=& \left(g^{\mu\nu}A_{\mu,\nu} + \frac{1}{\beta}A_{t,t} \right)_{,\phi} \\
=& \left(A^{\mu}{}_{;\mu} + g^{\mu\nu}\Gamma^{\rho}{}_{\mu\nu}A_{\rho} + \frac{1}{\beta}A_{t,t} \right)_{,t} \qquad (7.10)
\end{aligned}
$$

As mentioned, equation (7.1) was derived with the assumption of Lorenz gauge, i.e. $A^{\mu}{}_{;\mu} = 0$. Using this, and the Christoffel symbols

$$
\begin{array}{llllll}
\Gamma^t_{tr} = & \frac{R}{2r^2\left(1-\frac{R}{r}\right)} & \Gamma^r_{tt} = & \frac{R}{2r^2}\left(1 - \frac{R}{r}\right) & \Gamma^r_{rr} = & \frac{-R}{2r^2\left(1-\frac{R}{r}\right)} \\
\Gamma^r_{\theta\theta} = & -r\left(1 - \frac{R}{r}\right) & \Gamma^r_{\phi\phi} = & -r\sin^2\theta\left(1 - \frac{R}{r}\right) & \Gamma^\theta_{r\theta} = & \frac{1}{r} \\
\Gamma^\theta_{\phi\phi} = & -\sin\theta\cos\theta & \Gamma^\phi_{r\phi} = & \frac{1}{r} & \Gamma^\phi_{\theta\phi} = & \cot\theta
\end{array}
\qquad (7.11)
$$

one arrives at

$$\text{mixed}_t = \frac{1}{\beta}A_{t,tt} - \frac{\beta+1}{r}A_{r,t} - \frac{\cot\theta}{r^2}A_{\theta,t} \tag{7.12}$$

The mixed derivatives of the other components of $*\underline{d}*\underline{dA}$ are somewhat more complicated. As a representative, I will show the treatment of mixed_r in full.

$$\begin{aligned}
\text{mixed}_r =& -\frac{1}{\beta}A_{t,rt} + \frac{1}{r^2}A_{\theta,r\theta} + \frac{1}{r^2\sin^2\theta}A_{\phi,r\phi} \\
=& \Bigg(\underbrace{-\frac{1}{\beta}A_{t,rt} + \frac{1}{r^2}A_{\theta,r\theta} + \frac{1}{r^2\sin^2\theta}A_{\phi,r\phi}}_{a}\Bigg)_{,r} \\
& - \frac{R}{r^2\beta^2}A_{t,t} + \frac{2}{r^3}A_{\theta,\theta} + \frac{2}{r^3\sin^2\theta}A_{\phi,\phi} \\
a =& g^{\mu\nu}A_{\mu,\nu} - \beta A_{r,r} = g^{\mu\nu}\Gamma^{\rho}{}_{\mu\nu} - \beta A_{r,r} \\
=& -\frac{\beta+1}{r}A_r - \frac{\cot\theta}{r^2}A_\theta - \beta A_{r,r} \Rightarrow \qquad (7.13) \\
\text{mixed}_r =& -\frac{\beta+1}{r}A_{r,r} - \frac{\cot\theta}{r^2}A_{\theta,r} - \beta A_{r,rr} + \frac{2\beta}{r^2}A_r + \frac{2\cot\theta}{r^3}A_\theta \\
& - \frac{R}{r^2}A_{r,r} - \frac{R}{r^2\beta^2}A_{t,t} + \frac{2}{r^3}A_{\theta,\theta} + \frac{2}{r^2\sin^2\theta}A_{\phi,\phi} \qquad (7.14)
\end{aligned}$$

The others proceed in a similar manner, and we get

$$\begin{aligned}
\text{mixed}_\theta =& -\frac{1}{r^2}A_{\theta,\theta\theta} - \frac{\cot\theta}{r^2}A_{\theta,\theta} + \frac{2\cot\theta}{r^2\sin^2\theta}A_{\phi,\phi} - \frac{\beta+1}{r}A_{r,\theta} + \frac{1}{r^2\sin^2\theta}A_\theta \\
\text{mixed}_\phi =& -\frac{1}{r^2\sin^2\theta}A_{\phi,\phi\phi} - \frac{\beta+1}{r}A_{r,\phi} - \frac{\cot\theta}{r^2}A_{\theta,\phi} \qquad (7.15)
\end{aligned}$$

Reinserting this into equation (7.9), we find

$$\begin{aligned}
\mu_0 j_t =& \left(-g^{\mu\mu}\partial_\mu^2 - \frac{2\beta}{r}\partial_r - \frac{\cot\theta}{r^2}\partial_\theta\right)A_t - \frac{R}{r^2}\partial_t A_r \\
\mu_0 j_r =& \left(-g^{\mu\mu}\partial_\mu^2 - \frac{2}{r}\partial_r - \frac{\cot\theta}{r^2}\partial_\theta + \frac{2\beta}{r^2}\right)A_r \\
& - \frac{R}{r^2\beta^2}\partial_t A_t + \frac{2}{r^3}\partial_\theta A_\theta + \frac{2}{r^3\sin^2\theta}\partial_\phi A_\phi + \frac{2\cot\theta}{r^3}A_\theta \\
\mu_0 j_\theta =& \left(-g^{\mu\mu}\partial_\mu^2 - \frac{R}{r^2}\partial_r - \frac{\cot\theta}{r^2}\partial_\theta + \frac{1}{r^2\sin^2\theta}\right)A_\theta \\
& - \frac{2\beta}{r}\partial_\theta A_r + \frac{2\cot\theta}{r^2\sin^2\theta}\partial_\phi A_\phi \\
\mu_0 j_\phi =& \left(-g^{\mu\mu}\partial_\mu^2 - \frac{R}{r^2}\partial_r + \frac{\cot\theta}{r^2}\partial_\theta\right)A_\phi - \frac{2\beta}{r}\partial_\phi A_r - \frac{2\cot\theta}{r^2}\partial_\phi A_\theta
\end{aligned} \tag{7.16}$$

where $g^{\mu\mu}\partial_\mu^2 = -\frac{1}{\beta}\partial_t^2 + \beta\partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2\sin^2\theta}\partial_\phi^2$. This is starting to resemble the familiar wave equation.

To find the contravariant version of this equation, we use the metric to raise the indices of j and A. Since the equation involves derivatives of A, this operation will introduce derivatives of the metric, and as the metric does not depend on t and ϕ, we only need to consider r and θ:

$$\begin{aligned}
A_{t,r} =& -\beta A^t, r - \frac{R}{r^2}A^t \\
A_{t,rr} =& -\beta A^t{}_{,rr} - \frac{2R}{r^2}A^t{}_{,r} + \frac{2R}{r^3}A^t \\
A_{r,r} =& \frac{1}{\beta}A^r{}_{,r} - \frac{R}{r^2\beta^2}A^r \\
A_{r,rr} =& \frac{1}{\beta}A^r{}_{,rr} - \frac{2R}{r^2\beta^2}A^r{}_{,r} + \frac{2R}{r^3\beta^3}A^r \\
A_{\theta,r} =& r^2A^\theta{}_{,r} + 2rA^\theta \\
A_{\theta,rr} =& r^2A^\theta{}_{,rr} + 4rA^\theta{}_{,r} + 2A^\theta \\
& \text{and identical with an extra factor of } \sin^2\theta \text{ for } A_{\phi,r} \text{ and } A_{\phi,rr} \\
A_{\phi,\theta} =& r^2\left(\sin^2\theta A^\phi{}_{,\theta} + 2\sin\theta\cos\theta A^\phi\right) \\
A_{\phi,\theta\theta} =& r^2\left(\sin^2\theta A^\phi{}_{,\theta\theta} + 4\sin\theta\cos\theta A^\phi{}_{,\theta} + 2(\cos^2\theta - \sin^2\theta)A^\phi\right)
\end{aligned} \tag{7.17}$$

We can finally find the contravariant components of the wave equation:

$$-\mu_0 j^\mu = \left(g^{\mu\mu}\partial_\mu^2 + \frac{\cot\theta}{r^2}\partial_\theta\right)A^\mu + M^\mu{}_\nu A^\nu \tag{7.18}$$

$$M^\mu{}_\nu = \begin{pmatrix} \frac{2}{r}\partial_r & -\frac{R}{\beta^2r^2}\partial_t & 0 & 0 \\ -\frac{R}{r^2}\partial_t & \frac{2\beta}{r}\left[\partial_r - \frac{1}{r}\right] & -\frac{2\beta}{r}\left[\partial_\theta + \cot\theta\right] & -\frac{2\beta}{r}\partial_\phi \\ 0 & \frac{2}{r^3}\partial_\theta & \frac{1+3\beta}{r}\partial_r - \frac{\cot^2\theta-1}{r^2} & -\frac{2\cot\theta}{r^3}\partial_\phi \\ 0 & \frac{2}{r^3\sin^2\theta}\partial_\phi & \frac{2\cot\theta}{r^2\sin^2\theta} & \frac{1+3\beta}{r}\partial_r + \frac{2\cot\theta}{r^2}\partial_\theta \end{pmatrix} \tag{7.19}$$

To get the same formula in cgs units, we multiply j^μ by a factor of $4\pi\epsilon_0$. Since $\epsilon_0\mu_0 = c^{-2}$, and we use units where $c = 1$, this is equivalent to replacing μ_0 with 4π. The vector wave equation in the Schwarzschild space-time is a set of four coupled second order partial differential equations, and from their shape, it should be obvious that an analytical solution is very hard to find. Depending on what one wishes to compute, though, some simplifications are possible. The easiest case is the electrostatic one: A particle is held at some fixed position in our coordinate system. This will assure π and t symmetry, reducing the number of equations to two. One will not find the Green function that way, as that per definition is time-dependent (it has a δ shaped source in both time and space), but will instead find its integral, the potential. This has been done by [14], who found the self force to be repulsive in this case. Since we are interested in examining the tail term, we will need to find the Green function, and must therefore forsake t-symmetry. However, we still have ϕ, symmetry, as we always can choose our coordinate system such that the emission happens at one of its poles.

7.2 Which version of the equation to simulate

We are mainly interested in finding the Green function, because the dispersive part of the tail term exposed in it. The differential equation corresponding to the Green function can be obtained from equation (7.18) by replacing j^μ with $g^\mu{}_{\mu'}\delta(\mathbf{x}, \mathbf{x}')$, making the equation one of tensor rank 2 instead of 1. To avoid this, we will solve the equation as it is, but take j^μ to be delta shaped in time and approximately delta shaped (i.e. Gaussian) in space, and consider the case of a stationary source, meaning j^μ pointing in the t direction only, but the simulation also supports other directions.

In order to separate this limited Green function from the full Green function $G^\mu{}_{\mu'}$ and the full potential A^μ, I will label the quantity discussed here G^μ. It is like A^μ in that the charge and velocity of the source has been taken into account, and like $G^\mu{}_{\mu'}$ in that the source is delta-shaped.

7.3 Numerical method

The most straightforward way of solving a set of differential equations like this, is to rewrite them as a set of first-order equations, divide space-time into a regular grid, compute the spatial derivatives at regular values of t, and use these to compute how the values at each point in the grid should change when going to the next step. This is called the finite difference method, since we replace the continuous spacetime with a discrete, regular grid.

A series of pitfalls is associated with solving this set of equations numerically.

7.3.1 Delta functions

The source, $j^\mu(\mathbf{x}) = \delta(\mathbf{x}, \mathbf{x}')qu^\mu$ involves an infinitely sharp and tall spike, which is not representable numerically. Introducing as sharp a spike as the grid can represent - one a single grid square wide - will not do either, as abrupt changes in field value will give inaccurate numerical derivatives, which manifest as extra short-wavelength waves appearing in the solution.

Instead, I approximate delta functions using a Gaussian distribution with integral 1:

$$\delta(x) \sim \frac{1}{\epsilon\sqrt{\pi}} e^{-\left(\frac{x}{\epsilon}\right)^2} \tag{7.20}$$

Replacing a delta function with a Gaussian has the effect of blurring the solution, replacing sharp wave fronts with more diffuse ones. For optimal results, the Gaussian must be as narrow as possible while avoiding the extra waves, and an ϵ of the order of a few grid squares works well. Gaussians will only replace delta functions in space, not in time. The reason for this, is that having a Gaussian in time would mean that the source has time to move during the emission, ruining the ϕ-symmetry. We therefore use

$$\delta(\mathbf{x}, \mathbf{x}') = \frac{1}{\epsilon^3\sqrt{\pi}^3} \mathbf{e}^{-\left(\frac{|\Delta\vec{x}|}{\epsilon}\right)^2} \tag{7.21}$$

Equation (7.20) as an approximation to the delta function is derived in a Cartesian coordinate system in flat space-time. In our case, it must therefore be employed in a local orthonormal coordinate system around the source, or, if the Gaussian would extend outside the area where these coordinates are usable, one must use the more general Fermi normal coordinates. This can be achieved by replacing $|\Delta\vec{x}|$ with the spatial distance from $\vec{x}$ to $\vec{x}'$, and since we assume that the Gaussian is narrow, we can calculate this by using Pythagoras: Purely radial and angular distances are precomputed in the program, and we make the approximation $\mathrm{dist}(\vec{x}, \vec{x}')^2 \approx \mathrm{dist}_r(\vec{x}, \vec{x}')^2 + \mathrm{dist}_\theta(\vec{x}, \vec{x}')^2$.

7.3.2 Grid shape

A polar coordinate system is natural for this problem because of the ϕ-symmetry, and because the metric takes on a nice shape in this system. Our grid could therefore be a 2-dimensional polar grid, with coordinates r and θ. t is missing from the grid because we will solve the equation time step by time step, and only need to keep one slice in memory at a time, and ϕ is missing because of symmetry.

Such a grid would work, but there are problems associated with it. A grid with points at regular intervals of r would have increasing spatial distance between each point as one approaches the horizon, and for this reason, and because time moves slower near the horizon, a wave approaching the it would be compressed, taking up fewer and fewer grid points as it moves. But as we have discussed already, sharp changes in field value over a few grid points make the numerical derivatives inaccurate. It is therefore better to decrease the spacing between grid points near the horizon.

Naming the grid radius variable μ, I calculate $r(\mu)$ by sending a photon inwards from some point r_0, recording its position at regular intervals of t. This ensures that even in the worst case - a light-like wave traveling radially inwards - the wavelength in terms of number of cells spanned will not decrease.

In the θ direction things are a bit simpler, but we still have the $\cot\theta$ factor to worry about. This factor changes rapidly near the coordinate poles, so one could again fear inaccuracies here. To handle this, I have added support for increasing resolution around the poles, though this turned out to be unnecessary.

A final consideration concerning the grid shape, is that a finer grid usually requires finer time steps too, to avoid instability.

7.3.3 Boundary conditions

We are interested in solving the wave equation in the Schwarzschild spacetime, which goes asymptotically to Minkowski spacetime in the distance. We therefore expect waves to continue expanding indefinitely after leaving the vicinity of the black hole. However, there are limits on how far the grid can extend. Though is is possible to create a grid that extends to infinity while still only having a finite number of cells, this would result in some cells being very large, leading to inaccurate simulation there. The typical symptoms of this are artificial dispersion and reflection.

Instead of using such a grid, I will be using the one described in the last section, and arbitrarily end it at some distance from the center. Ideally, the boundary here would have so-called non-reflective boundary conditions, which would let a wave "pass through" as if the grid continued on the other side. However, implementing this is only easy in 1 dimension, and will be beyond our scope. Instead, we will place the boundary far away from the point of emission, and end the simulation before a signal has time to reach the boundary, be reflected, and reach the area of interest again. This places an effective limit on the duration of the simulation.

Of course, the limit of large r is not the only place we need a boundary. Due to our choice of coordinates, the grid must stop near the horizon, and the same considerations will apply here.

The final boundary to consider, is the θ boundary. We are using a 3-dimensional polar coordinate system, and are assuming rotational symmetry around the pole, so our coordinate system is effectively only a 2-dimensional polar system, with the difference that the angular coordinate θ only runs from 0 to π. Where the other half of a normal 2-dimensional polar system would have been ϕ-symmetry leaves us with a mirror image of the first half. This means that we must have mirror boundary conditions at the θ-boundaries. If the grid starts at g_0 and ends at g_n in the θ-direction, this is done by mapping $g_{-i} = g_i$, $g_{n+i} = g_{n-i}$. This is used in the numerical derivatives at the boundaries.

7.3.4 Derivatives

I will use the common 2nd order accuracy numerical derivatives.

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \tag{7.22}$$

$$f''(x) \approx \frac{f(x+h) - f(x) + f(x-h)}{h^2} \tag{7.23}$$

This is complicated somewhat by my uneven grid intervals. By using the chain rule repeatedly, we have $f' = \frac{\partial f}{\partial x} = \frac{\partial y}{\partial x}\frac{\partial f}{\partial y} = \frac{\dot{f}}{\dot{x}}$ and $f'' = \frac{1}{\dot{x}^2}\ddot{f} - \frac{\ddot{x}}{\dot{x}^3}\dot{f}$, where a prime indicates derivative with respect to the old variable x, while a dot indicates derivative with respect to the new variable y. By using the second order approximation on each of the derivatives in these expressions, we get second order approximations for f' and f''.

7.3.5 Time evolution

Equation (7.18) can be rewritten as

$$\begin{aligned}\ddot{\underline{A}} =& 4\pi\beta\underline{j} + \beta\left(g^{ii}\partial_i^2 + r^{-2}\cot\theta\partial_\theta + \mathbf{M}\right)\underline{A} \\ =& f(\underline{A}, \dot{\underline{A}})\end{aligned} \tag{7.24}$$

where f is an operator with no time derivatives. We can now regard $\underline{A}$ and $\underline{B} \equiv \dot{\underline{A}}$ as separate variables, and get the set of equations

$$\dot{\underline{B}} = f(\underline{A}, \underline{B}) \qquad \dot{\underline{A}} = \underline{B} \tag{7.25}$$

These equations can be solved by the simple iteration

$$\underline{B}(t+\Delta t) = \underline{B}(t) + f(\underline{A}(t), \underline{B}(t))\Delta t \qquad \underline{A}(t+\Delta t) = \underline{A}(t) + \underline{B}(t)\Delta t \tag{7.26}$$

which is called the forward Euler method, and is the simplest but usually least accurate method.

There is also the more sophisticated Runge-Kutta method:

$$\begin{aligned}
\underline{B_1} =& f(\underline{A}, \underline{B}) & \underline{A_1} =& \underline{B} \\
\underline{B_2} =& f(\underline{A} + \frac{1}{2}\underline{A_1}\Delta t, \underline{B} + \frac{1}{2}\underline{B_1}\Delta t) & \underline{A_2} =& \underline{B} + \frac{1}{2}\underline{A_1}\Delta t \\
\underline{B_3} =& f(\underline{A} + \frac{1}{2}\underline{A_2}\Delta t, \underline{B} + \frac{1}{2}\underline{B_2}\Delta t) & \underline{A_3} =& \underline{B} + \frac{1}{2}\underline{A_2}\Delta t \\
\underline{B_4} =& f(\underline{A} + \underline{A_3}\Delta t, \underline{B} + \underline{B_3}\Delta t) & \underline{A_4} =& \underline{B} + \underline{A_3}\Delta t \\
\underline{B}(t+\Delta t) =& \underline{B}(t) + \frac{\Delta t}{6}(B_1 + 2B_2 + 2B_3 + B_4) \\
\underline{A}(t+\Delta t) =& \underline{A}(t) + \frac{\Delta t}{6}(A_1 + 2A_2 + 2A_3 + A_4)
\end{aligned} \tag{7.27}$$

Each step is here much slower, but this is usually more than compensated by the greater accuracy and stability of this method, allowing one to use larger time steps.

7.3.6 The force

Though the particle only acts as a source at the very beginning of the simulation, when it is a one of the poles of the system, the program keeps track of where the particle is at later times too, in order to calculate the self force. By integrating the force the Green function imparts on the particle over all time in the simulation, and dividing by the duration of the emission (really infinitesimal, but finite here), we find the self force. The force corresponding to the Green function is found by calculating the corresponding field $F_{\mu\nu} = G_{\nu,\mu} - G_{\mu,\nu}$, and then using the Lorentz force $f^\mu = F^{\mu\nu} j^\nu$.

Again, j is really delta shaped in space, but is replaced by a Gaussian. However, this gives it a spatial extent, meaning that the Lorentz force must be integrated over this area, and to do this we need the spatial volume element $dV = \frac{r}{\sqrt{1-\frac{R}{r}}} dr d\theta$. The ϕ part is missing from this expression. This is because the grid does not contain ϕ and we thus do not need to replace a delta function in this direction with a Gaussian. The integral therefore only goes over r and θ, giving the volume element above.

7.4 The light cone

In addition to solving the wave equation, the program also calculates the shape of the light cone for the emission event. This is done by emitting a set of light-like test particles in all directions from the source, and updating their position in step with the wave equation. Their motion is dictated by the geodetic equation

$$\frac{Du^\mu}{D\tau} = 0 \Leftrightarrow \tag{7.28}$$

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta \tag{7.29}$$

This equation is, as it stands, not suitable for a numerical implementation, as the proper time τ is constant for a light-like particle. To get a more suitable equation, we introduce the 3-speed, $v^\mu \equiv \frac{u^\mu}{u^t}$. We can now write the equation as

$$\frac{du^\mu}{d\tau} = \frac{\partial t}{\partial \tau}\frac{du^\mu}{dt} = u^t \frac{d}{dt}(u^t v^\mu) = u^{t2}\frac{dv^\mu}{dt} + v^\mu \frac{du^t}{d\tau} = -\Gamma^\mu{}_{\alpha\beta} u^{t2} v^\alpha v^\beta \Leftrightarrow$$

$$\frac{dv^\mu}{dt} = \left(\Gamma^t{}_{\alpha\beta} v^\mu - \Gamma^\mu{}_{\alpha\beta}\right) v^\alpha v^\beta \tag{7.30}$$

Because of the symmetry, the particles can stay in the $r\theta$ plane, and since the 3-speed has $v^t = 1$ per definition, we have

$$\begin{aligned}
\frac{dv^r}{dt} =& 2\Gamma^t{}_{tr} {v^r}^2 - \Gamma^r{}_{tt} - \Gamma^r{}_{rr} {v^r}^2 - \Gamma^r{}_{\theta\theta} {v^\theta}^2 \\
=& \frac{R}{2r^2\beta}(3{v^r}^2 - \beta^2) + r\beta {v^\theta}^2 \qquad (7.31) \\
\frac{dv^\theta}{dt} =& 2\Gamma^t{}_{tr} v^\theta v^r - -2\Gamma^\theta{}_{r\theta} v^r v^\theta \\
=& -\frac{2R}{r^2\beta}(r - \frac{3}{2}R) v^r v^\theta \qquad (7.32)
\end{aligned}$$

The forward Euler method is then used to solve these nonlinear differential equations.

7.5 Simulation units

To solve a differential equation numerically, it is necessary to convert it to dimensionless form. This is done by introducing standard values of the various dimensioned quantities the equation involves, and measuring everything in multiples of these, effectively introducing a new unit system tailored for the specific problem. In this case, I choose some arbitrary r_0 as the unit in the r direction, giving the dimensionless radial coordinate $\overline{r} = \frac{r}{r_0}$, where an overline here is used to indicate the dimensionless variables. Since we use $c = 1$, time is also measured in units of length, and so we can just as well use $\overline{t} = \frac{t}{r_0}$. We now choose the unit of G such that they make the dimensionless equation as simple as possible. The original equation is

$$\Box G^\mu = -4\pi j^\mu = -4\pi q \delta(\mathbf{x}, \mathbf{x}') u^\mu = -4\pi {r_0^{-4}} \delta(\overline{\mathbf{x}}, \overline{\mathbf{x}}') u^\mu \qquad (7.33)$$

by inspecting equation (7.18), we see that $[\Box G^\mu] = [G^\mu] r_0^{-2}$, where the notation $[G]$ means the units of G. To get rid of the units in the equation, then, we introduce $\overline{G}^\mu = \frac{G^\mu}{G_0^\mu}$ such that $[G_0^\mu] r_0^{-2} = 4\pi q [u^\mu] r_0^{-4} \Leftrightarrow [G_0^\mu] = 4\pi q [u^\mu] r_0^{-2}$. The four-velocity has units $[u^{t|r}] = 1$ and $[u^{\theta|\phi}] = r_0^{-1}$, which gives $[G_0^{t|r}] = 4\pi q r_0^{-2}$, $[G_0^{\theta|\phi}] = 4\pi q r_0^{-3}$. This results in the dimensionless equation

$$\overline{\Box G}^\mu = -\delta(\overline{\mathbf{x}}, \overline{\mathbf{x}}') \overline{u}^\mu \qquad (7.34)$$

which is the one we will solve.

7.6 Simulation output and results

The simulator plots various components of the potential and field in the $r\theta$ plane. In the plot, shades of red indicate a positive amplitude, and shades of blue indicate a negative one. Brighter color means a larger absolute value of the amplitude. When in logarithmic mode, white and black are separated by a factor of 1000. The fields are normalized by default, the normalization variable displayed as "Amp" in the upper left corner of the plot. On top of this plot, the part of the light cone contained in the current slice through space-time appears as a thin white line, and the position of the emitting particle, had it still been there, is plotted as a white dot.

7.6.1 Setup

Here the particle is held at fixed r and $\theta = \pi$, giving it a velocity of $u^\mu = (1 - \frac{R}{r}, 0, 0, 0)$. In Minkowski spacetime, this would have resulted in only G^t having a nonzero value. In our case, there is a mixing between the components, but we expect the time component to dominate.

To get accurate results, we use a fairly high resolution in both space and time, choosing 400 points in the θ direction, 1500 points in the r direction, and an r initial r step size at the position of the particle of 0.5. The particle position and black hole radius are at their default values of 30 and 10, respectively. Due to the high spatial resolution, it is necessary to use a time step of 0.03. Finally, we want logarithmic plotting, self force measurements and display of the light cone. The command line options specifying this is

```
log 1 lc 1 force 1 dt0 0.25 dt 0.03 nmu 1500
```

7.6.2 Evolution of the field

Figure 7.1, 7.2 and 7.3 show G^t, G^r and G^θ respectively ($G^\phi = 0$ due to symmetry). In all of these, there is not much dispersion at first, but after the wave fronts have crossed on the other side of the black hole, it becomes more an more apparent that the field is nonzero outside the wave front. This is more apparent near the black hole, which is to be expected, as the curvature is highest there. This dispersion is the "tail" of the Green function, and is what gives the "tail term" its name.

Note that the field starts by moving away from the point of emission, but that parts of it curves around the black hole, and intercepts it again. If the simulation were to continue, this would repeat endlessly, with rapidly diminishing amplitude. This effect is equivalent to what I called the topological tail term in chapter 3. Note that the "amplitude" displayed on the figures is only the maximum amplitude occurring somewhere in the figure, and thus does not necessarily reflect the average amplitude of the wave front.

More physically relevant than the potential, is the field. Again we have 3 relevant components: F^{tr}, corresponding to E^r, $F^{t\theta}$, corresponding to E^θ, and $F^{r\theta}$, corresponding to B^ϕ. These are displayed in figure 7.4, 7.5 and 7.6.

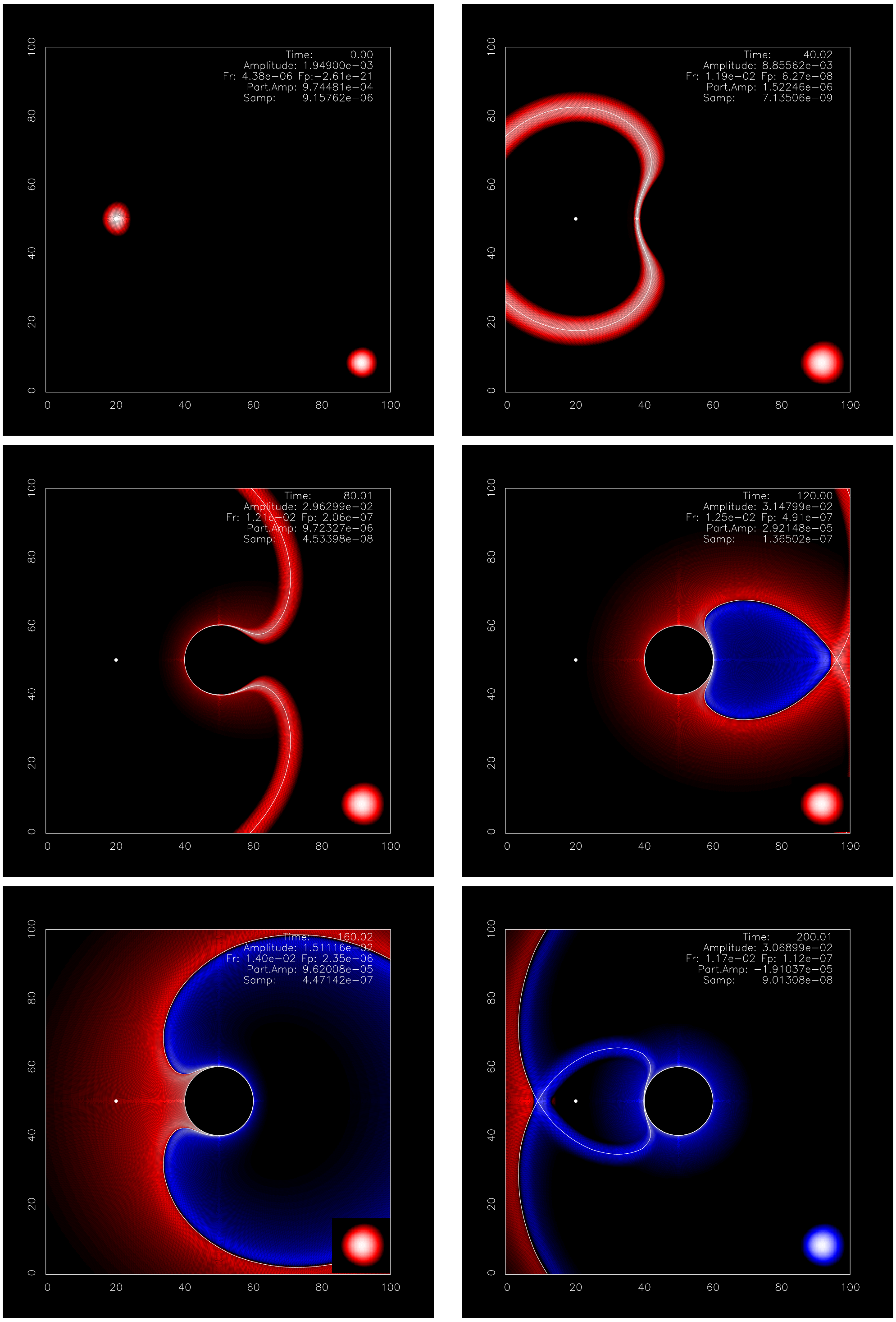

Figure 7.1: Logarithmic plot of G^t for selected times.

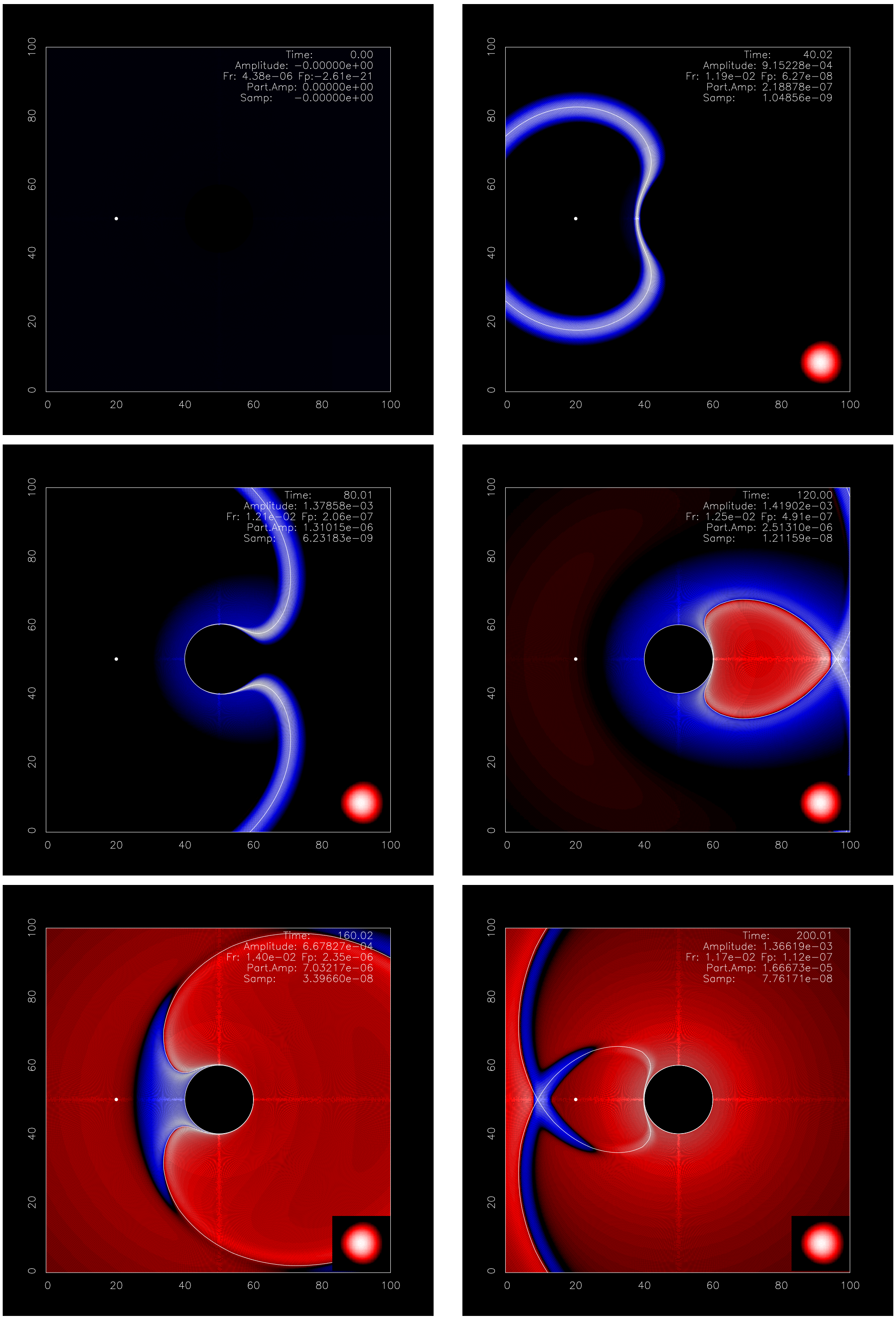

Figure 7.2: Logarithmic plot of G^r for selected times.

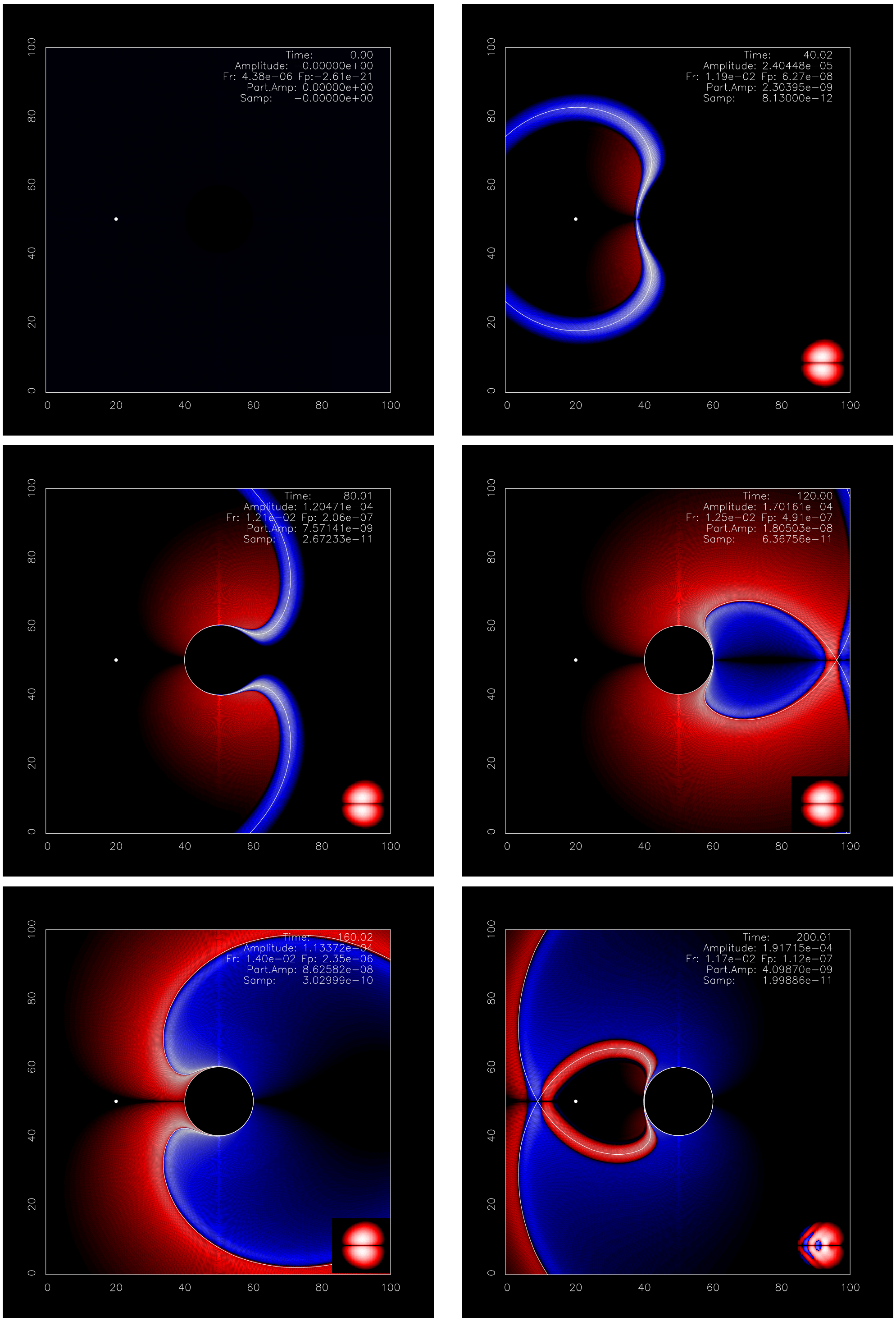

Figure 7.3: Logarithmic plot of G^{θ} for selected times.

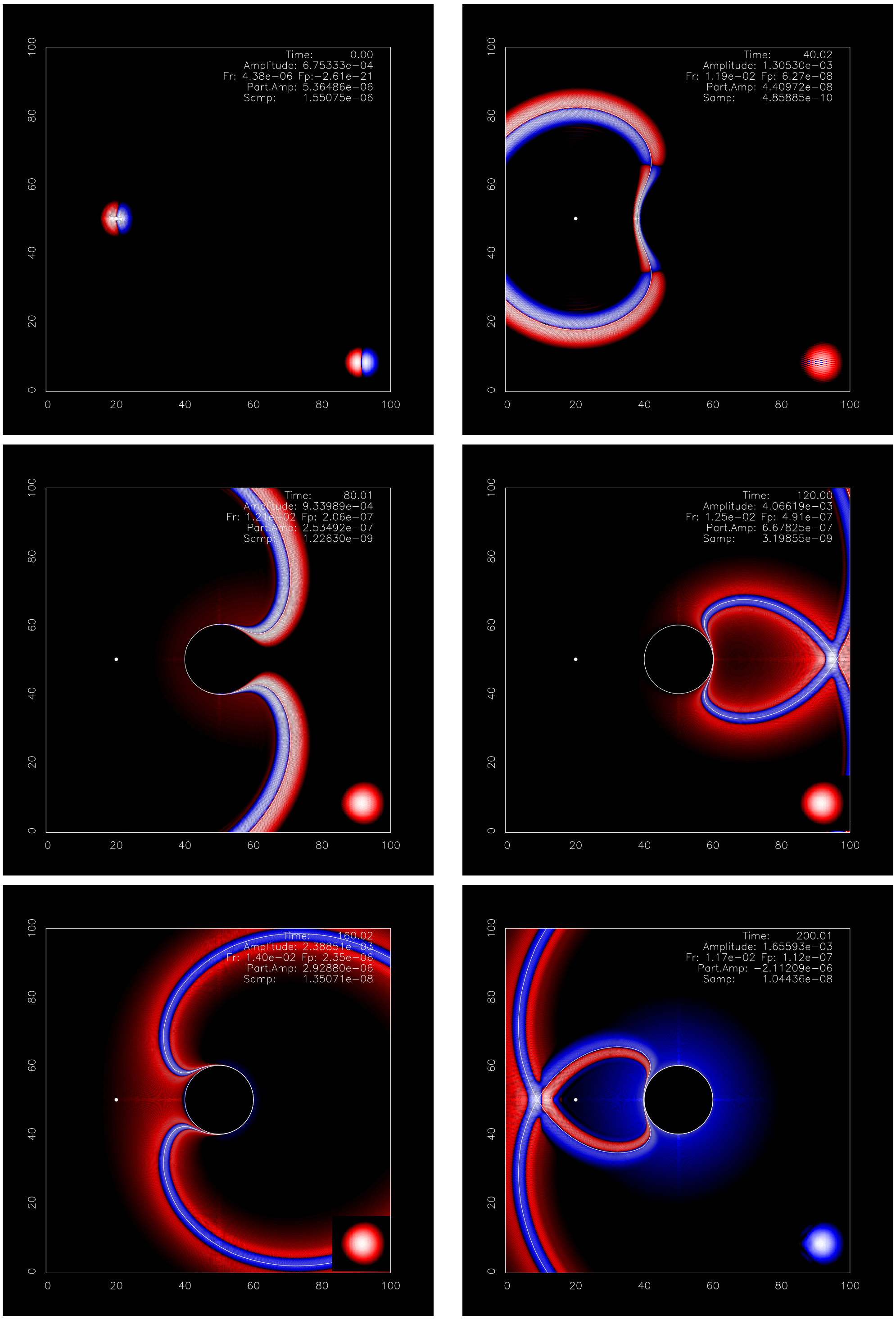

Figure 7.4: Logarithmic plot of F^{tr} corresponding to the Green function, for selected times.

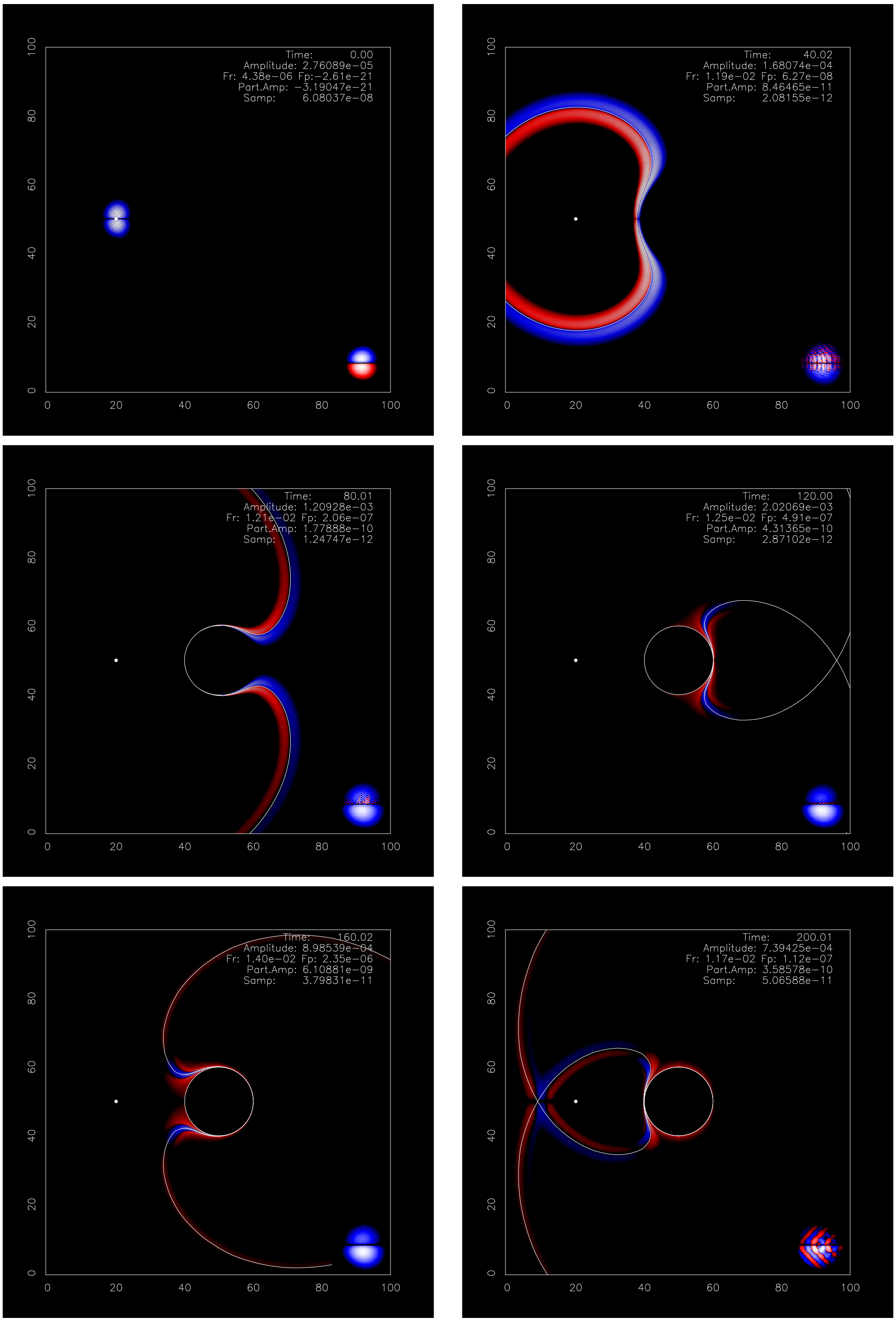

Figure 7.5: Logarithmic plot of $F^{t\theta}$ corresponding to the Green function, for selected times.

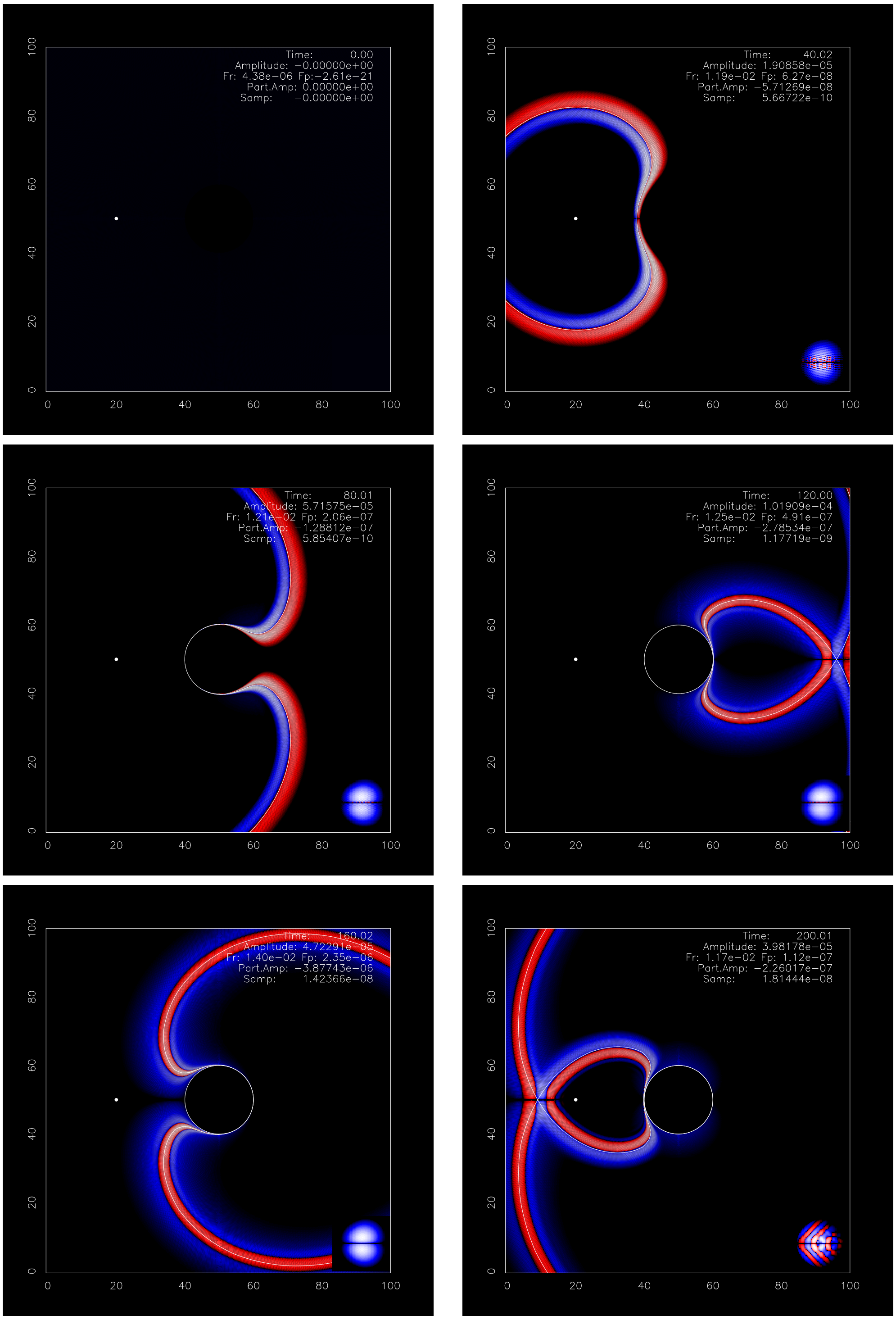

Figure 7.6: Logarithmic plot of $F^{r\theta}$ corresponding to the Green function, for selected times.

Figure 7.4 and 7.5 show that the appearance of the source creates an electric field pointing away from it. However, the following disappearance causes the opposite reaction. In total, we get an expanding wave front consisting of an electric field that first points away from, and then towards the particle. This is thus a longitudinal wave instead of a transversal wave, as a normal electromagnetic wave is.

The magnetic field points out of the simulation plane, which might seem odd when considering our symmetry requirements, but it is actually necessary in order to get a force that does not violate these requirements. It causes a particle moving in the positive θ direction to feel a force in the positive r direction in the red areas, and a particle moving in the positive r direction to feel a force in the negative θ direction in the red areas. This means that $F^{r\theta}$ causes a clockwise rotation in the red areas in the upper half-plane, and opposite in the lower half-plane.

7.6.3 Correspondence with the light cone

The light cone is displayed as a thin white curve superimposed on the field. These are, we recall, computed in totally different ways: The field is found by solving the wave equation, while the light cone is found by solving the geodetic equation. How they match up is a test of the consistency of the simulation, and we see that they they do match well, and stay matched through the simulation.

7.6.4 The self-force

The cumulative force, or total transfered momentum, in the r direction is plotted in figure 7.7. We see that the initial appearance and disappearance of the particle causes a net increase in the particle's momentum, which is then almost constant until the wave front approaches it again. It then changes rapidly, ending up at almost the same level as it was, but we can see a falling trend at the end of the diagram.

The momentum transfer shows no sign of converging towards a final value, which means that the time limit of $t = 200$ is insufficient. The next time the wave front approaches the particle will be at $t = 400$, but since the size of the grid also must be increased to avoid reflection, simulating twice the period takes 4 times as long. The result of this is shown in figure 7.8. The beginning downward trend at the end of the last figure turns out to be the beginning of a runaway decrease. This is clearly at odds with the expectation that the main part of the field should lie near the light cone, and that the field should get weaker with time.

It seems like the simulation is unstable. It gives reasonable results for small values of t, but errors accumulate and eventually come to dominate. Using it to calculate the whole integral of the tail term is therefore

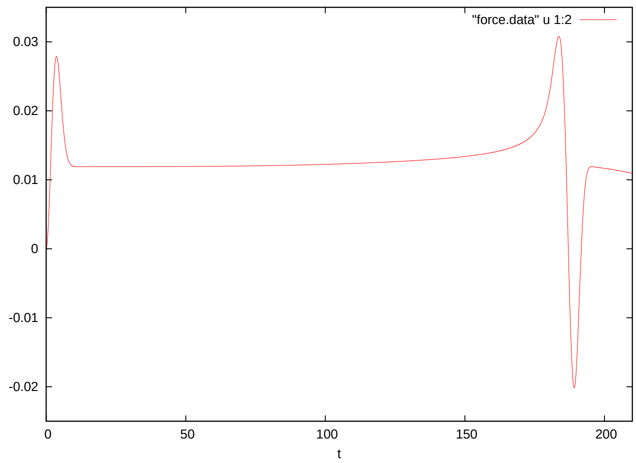

Figure 7.7: The total momentum increase for a particle that is held at rest at fixed coordinates, caused by an emission at $t = 0$, as a function of time.

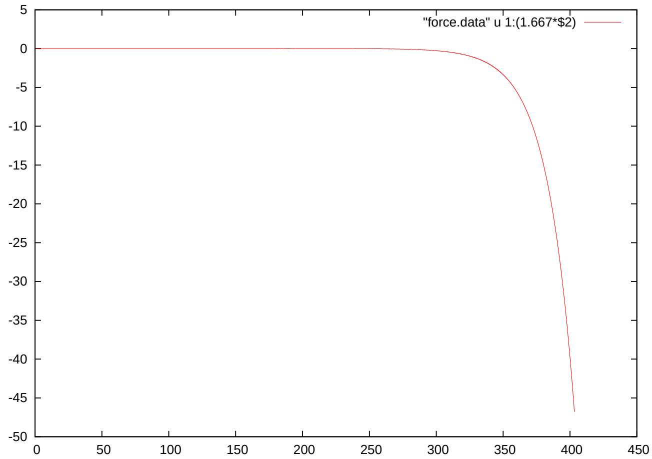

Figure 7.8: The total momentum increase for a particle that is held at rest at fixed coordinates, caused by an emission at $t = 0$, as a function of time.

impossible. However, it should still serve as a qualitative illustration of the tail term.

7.7 Green functions and energy, and the physicality of Green functions

An obvious test of the simulation, is to see if energy-momentum is preserved in it. The local preservation of energy-momentum is given by the continuity equation $T^{\mu\nu}{}_{;\nu} = 0$, so if four-momentum is preserved in the simulation, a plot of the divergence of the stress-energy tensor should give zero values, or at least very low values, everywhere. The special relativistic version of this equation is $T^{\mu\nu}{}_{,\nu} = \frac{\partial E}{\partial t} + \nabla \cdot \vec{p} = 0$, which simply states that any change in energy at a given point must be compensated for by a corresponding in- or outflow of momentum. Put another way: Energy can only disappear by going somewhere else. Figure 7.9 shows the t, r and θ components of the four-momentum at selected times. We see that the energy (the t component) is nonzero at the front and back of the wave, and that the other components, corresponding to the 3-momentum, are much smaller than this. So we have energy moving outwards from where the particle appeared and disappeared, but only a small amount of momentum doing the same, and not pointing in the direction the energy is moving. This is at odds with the continuity equation, and the lower right part of figure 7.9 shows the value of $T^{\mu\nu}{}_{;\nu}$ at the same times as the other figures.

There are two important points to note about this figure. Firstly, we see that the continuity violation is of the same order as the energy itself, so this is not just some small numerical deviation, and secondly, it looks very much like the time derivative of the energy field itself.

To investigate this continuity violation more closely, let us, for a moment, return to the case of flat spacetime. From equation (3.5) we know that a static charge sets up the field

$$A_\mu = -\frac{q}{r}\delta^t_\mu \tag{7.35}$$

Since the field only moves on the light cone, a time-varying charge is introduced simply by replacing q with the value q had when the part of the field we are considering was emitted. If the charge varies as $qf(t)$, then we get a field of

$$A_\mu = -\frac{q}{r}f(t-r)\delta^t_\mu \tag{7.36}$$

which could also have been easily found by using the Green function. What, then, is the point of looking at a variable charge, when charge is known to be a conserved quantity? Recall that a Green function is the

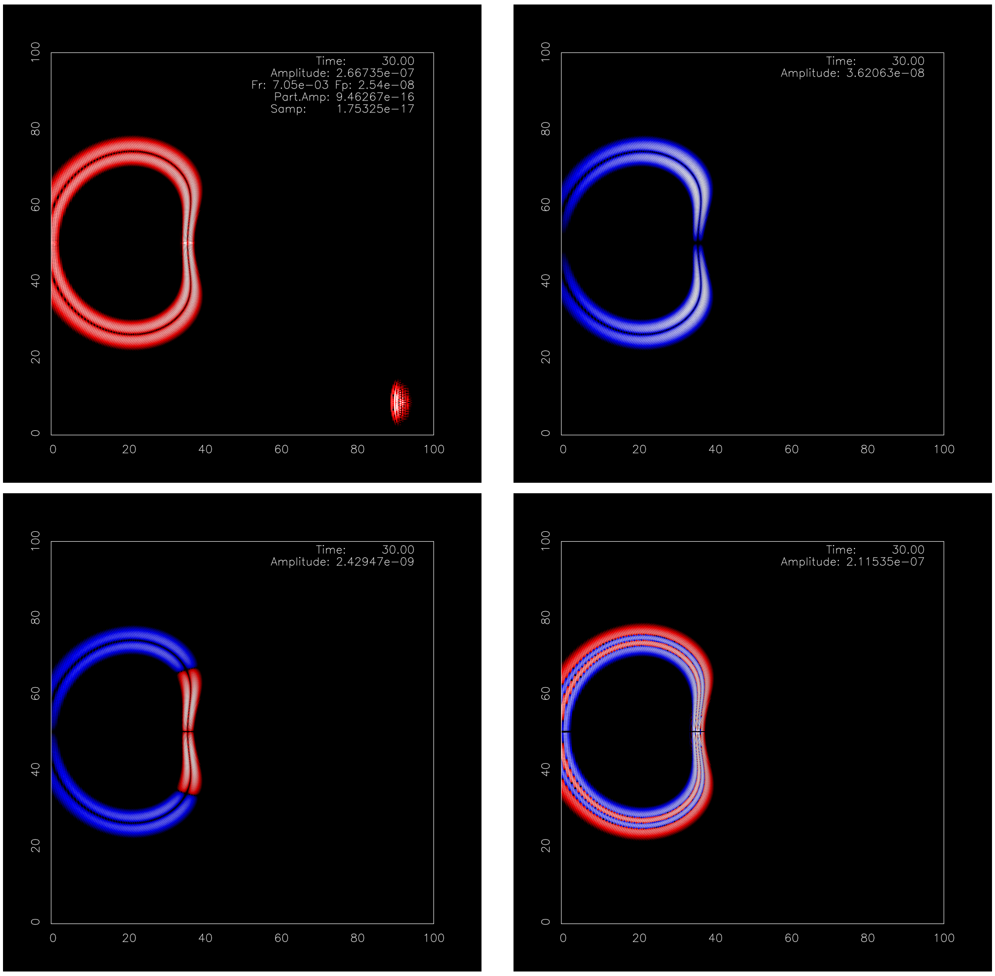

Figure 7.9: From upper left, to upper right, lower left and lower right are the t, r and θ components of the four-momentum and the t-component of the divergence of the stress energy tensor, respectively.

field potential generated by a charge that appears and disappears, so a Green function involves, by its very definition, this unphysical process. The function f is introduced as a generalization, and setting $f(t) = \delta(t)$ will reproduce the Green function.

With this form for A, we find the electromagnetic field to be

$$\begin{aligned} F_{tr} =& - A_{t,r} = \frac{q}{r^2}\left(f(t-r) + f'(t-r)r\right) \\ F_{tr} =& F_{t\theta} = F_{t\phi} = F_{ij} = 0 \end{aligned} \tag{7.37}$$

so we have a purely electric field, pointing in the radial direction and propagating outwards. The energy-momentum content of this purely electric wave can be found by using the stress-energy tensor $p^\mu = T^{t\mu} = \frac{1}{4\pi}\left(F^{t\rho}F^\mu{}_\rho - \frac{1}{4}g^{t\mu}F^{\rho\sigma}F_{\rho\sigma}\right)$, and we readily find

$$\begin{aligned} p^\mu =& \frac{1}{8\pi}F_{tr}^2\delta_t^\mu \\ =& \frac{q^2}{8\pi r^4}\left(f(t-r) + f'(t-r)r\right)^2\delta_t^\mu \end{aligned} \tag{7.38}$$

Since $(f(t-r) + f'(t-r)r \neq 0$ in general, this electric wave is carrying energy, but no 3-momentum. The t-component of the continuity equation takes the form

$$\begin{aligned} 0 =& T^{t\nu}{}_{;\nu} = T^{t\nu}{}_{,\nu} + \Gamma^t{}_{\rho\nu}T^{\rho\nu} + \Gamma^\nu{}_{\rho\nu}T^{t\rho} \\ =& T^{tt}{}_{,t} = \frac{\partial p^t}{\partial t} \end{aligned} \tag{7.39}$$

which is similar to the numerical result in the Schwarzschild spacetime, and in conflict with equation (7.38), which in general shows p^t to be neither zero nor static in general, and thus has a nonzero time-derivative.

We have then, it seems, a conundrum. Each event on the world line of a stationary particle sets up a purely electrical wave, carrying energy away to infinity while violating the continuity equation not only at the point of emission, where this could be expected (since a particle is appearing and disappearing there), but continuing to violate it during its propagation. It is clear that the Green function corresponds to an unphysical solution of the Maxwell equations. This is not a problem per se, as the Green function only is a building block for the full solution. However, since each of them seems to be carrying energy away, it seems as if a particle at rest in Minkowski space-time should radiate!

The solution to this problem lies in observing that while the superposition principle applies to the field (due to the linearity of the wave equation), the energy is a quadratic function of the field, and so the superposition principle does not apply to this. Thus, while the total field is built up of the sum of the Green function for each point on the world line, the same does not apply for the energy. Hence, the Green function is ill suited for analysis of the energy balance.

7.8 Summary

We have seen how the seemingly simple operator $\Box$ takes on a very complicated form when operating on a vector in Schwarzschild coordinates, and how this turns what was four independent differential equations in Minkowski coordinates into coupled ones here. Such equations are only solved numerically, and we have seen the results of such a numerical solution. This solution exhibited both the topological and tail parts of the tail term, i.e. it involved the light cone reencountering the particle, and is also nonzero off the light cone. It thus serves to sum up the different parts of the light cone we examined isolatedly so far.

We have also examined the odd properties of a Green function for the electromagnetic wave equation.

- While normal electromagnetic waves are transverse, the Green function can be purely longitudinal, meaning that the field points in the same direction as the motion.
- While a normal electromagnetic wave must have both magnetic and electric components, the Green function can be purely electrical.
- It may carry energy, but have no 3-momentum, thus violating the continuity equation.
- It may radiate away energy to infinity, even if it corresponds to a particle at fixed coordinates in an inertial coordinate system in flat spacetime, where the particle clearly should not radiate.

These properties follow from the violation of charge preservation which is inherent in the definition of the Green function: it is the potential caused by a particle only existing at one point in spacetime. However, adding together all the points that make out a particle's world line removes the violation of charge conservation, and the corresponding adding together of Green functions must also eliminate the strange behavior listed above, through destructive interference.

We did not succeed in using the simulation to measure the actual self-force, as it turned out that the numerical implementation used here is unstable. This is unfortunate; not only because this removes the possibility of obtaining numbers detailing the strength of this force, but also because it casts doubt on the qualitative correctness of the shape and propagation of the field. This is an obvious opportunity for further work in the future.

Chapter 8

Conclusion

The generalized Lorenz-Abraham-Dirac equation is superficially similar to the normal LAD equation, but the final extra term, the “tail term”, changes the equation from being local, to depending on the entire past history of the particle, and thus being explicitly nonlocal. We have seen that two independent mechanisms can give rise to this term.

The first mechanism is called the “tail” of the Green function, and is what gives name to the “tail term”. We usually take it for granted that the electromagnetic wave equation only has solutions that move at the speed of light, and thus are zero everywhere but on the light cone. We say that there is no dispersion of light in vacuum. This was shown to be an exception, rather than the norm, in chapter 5.

To have zero potential inside the light cone, one needs the potential to fall off by exactly the same amount after the wave front has passed, as it increased when the wave front arrived, and this balance is disturbed by introducing curvature.

Perhaps even more surprising, is that this also will happen in Minkowski spacetime if the number of spatial dimensions is 1 or even. In chapter 4, we examined the case of four spatial dimensions and one time dimension, and there the Green function not only developed a tail, it even diverged for $\sigma = 0$, and is in this sense more important here than for the 3+1-dimensional cases considered.

The tail of the Green function is not inherently connected to curvature. It is an inherent property of the wave equation, and the proper question is not “why does light disperse in curved spacetime” but rather “why does light only move at the speed c in the special case of 3 spatial dimensions, 1 time dimension, and flat spacetime”, as the first of these is the norm, while the last is an exception, albeit a very common one.

However, the tail of the Green function is only part of what causes the tail term. When a particle’s field moves slower than the speed of light, it is obvious that the particle might catch up to it, and interact with it. This is what one usually thinks of as the cause of the tail term. For example, DeWitt and Brehme [5] divides the Green function into the part u, which is on the light cone, and v, which is the tail of the Green function, and defines the tail term only in terms of v. This only works inside the local convex neighborhood of an event, where two points can't both have a light-like separation and a time-like separation at once. This breaks down outside the LCN, and a particle can therefore cross a light cone centered on a previous position on its world line. This was examined in chapter 3 and 7.

Despite all its complexities, the generalized LAD equation is still a simplified version of the Lorentz force equation $f^\mu = F^\mu{}_\nu j^\nu$. In chapter 4, we saw that the GLAD equation breaks down in 4+1 dimensions due to its use of the Hadamard ansatz, which asserts that the tail part of the Green function is nondivergent. We therefore went through the process of solving the wave equation and using this solution with the Lorentz force equation to derive the equation of motion. This is the same sort of derivation that resulted in the GLAD equation, but it is much less general.

Finally, we have examined the properties of the Green function as a solution of the wave equation. The retarded Green function, which is the Green function used here, embodies cause and effect for the field. Specifically, it answers the question "what effect does the existence of current at this point in spacetime have on the field potential". This makes it useful to calculate the Green function even in cases where it is a complication, rather than a simplification. However, as we saw in chapter 7, the Green function is not a physical solution of Maxwell's equation, and therefore exhibits several odd properties. One should therefore take care about what one reads out of the Green function directly. To recover a physical solution, it is always necessary to perform an integration of the Green function over some source field, and many of the properties of the Green function are lost due to destructive interference in this process.

8.1 Deficiencies and possibilities for further work

Two topics stand out as unsatisfactory in this work. Firstly, the equation of motion for a hyperbolically accelerated charge in 4+1 dimensions resulted in an infinite self force. It would be interesting to examine if this only is due to the use of point particles, or if it persists even with extended charges.

Secondly, the numerical solution of the electromagnetic wave equation in chapter 7 turned out to be unstable. I had hoped to be able to compare the measured self force with Smith and Will [14]'s results for a particle suspended over a black hole, and with those for a particle in orbit around a black hole[2]. An improved simulation, perhaps with a coordinate system better suited to task, could shed better light on the Green term and self force for a particle near a black hole.

Additionally, it would be interesting to consider the question of radiation and energy-momentum conservation in more detail. As mentioned in the introduction, the Larmor formula is in need of generalization, and one must expect a generalized version to be as complicated as the GLAD equation. We also found the field to act on the horizon with a nonzero force in the 3+1 and 4+1-dimensional Rindler cases. This force needs an explanation, and needs a corresponding, oppositely directed force.

There is also the possibility of looking at the self force for extended bodies, or for other forces than electromagnetism, curvature- and topology-induced self-interaction of quantum particles, gravitational self-interaction, and so on. This field of research has by no means been completed during the almost 50 years since the GLAD equation was derived.

Chapter 9

Appendix

9.1 Relations for coordinate systems

This section contains various relations, mostly Christoffel symbols, that I have found useful when dealing with various coordinate systems.

9.1.1 Polar

Metric

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

Christoffel symbols

$$t\colon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad r\colon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \theta\colon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \phi\colon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{llllllll} \Gamma^r_{\theta\theta} = & -r & \Gamma^r_{\phi\phi} = & -r\sin^2\theta & \Gamma^\theta_{r\theta} = & \frac{1}{r} & \Gamma^\theta_{\phi\phi} = & -\cos\theta\sin\theta \\ \Gamma^\phi_{r\phi} = & \frac{1}{r} & \Gamma^\phi_{\theta\phi} = & \cot\theta & & & & \end{array}$$

Laplace

$$\Delta\phi = \left(g^{\mu\mu}\partial_\mu^2 + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\right)\phi$$

$$\Delta\vec{A} = \left(g^{\mu\mu}\partial_\mu^2 + \frac{2}{r}\partial_r + \frac{\cot\theta}{r^2}\partial_\theta\right)\vec{A} + \begin{pmatrix} 0 \\ -\frac{2}{r}\left(\frac{1}{r}A^r + \partial_\theta A^\theta + \partial_\phi A^\phi + \cot\theta A^\theta\right) \\ \left[\frac{2}{r}\partial_r - \frac{\cot^2\theta - 1}{r^2}\right]A^\theta + \frac{2}{r^3}\partial_\theta A^r - \frac{2\cot\theta}{r^3}\partial_\phi A^\phi \\ \left[\frac{2}{r}\partial_r + \frac{2\cot\theta}{r^2}\partial_\theta\right]A^\phi + \frac{2}{r^2\sin^2\theta}\left(\frac{1}{r}\partial_\phi A^r + \cot\theta\partial_\phi A^\phi\right) \end{pmatrix}$$

where

$$g^{\mu\mu}\partial_\mu^2 = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

9.1.2 Rotating polar

Metric

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \omega^2 r^2\sin^2\theta\right) & 0 & 0 & -\omega r^2\sin^2\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ -\omega r^2\sin^2\theta & 0 & 0 & r^2\sin^2\theta \end{pmatrix} g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & -\omega \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ -\omega & 0 & 0 & \frac{1}{r^2\sin^2\theta} - \omega^2 \end{pmatrix}$$

Christoffel symbols

$$t\!: \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad r\!: \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \theta\!: \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \phi\!: \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{llllllll} \Gamma^r_{tt} = & -r\omega^2\sin^2\theta & \Gamma^r_{t\phi} = & \omega r\sin^2\theta & \Gamma^r_{\theta\theta} = & -r & \Gamma^r_{\phi\phi} = & -r\sin^2\theta \\ \Gamma^\theta_{tt} = & -\omega^2\sin\theta\cos\theta & \Gamma^\theta_{t\phi} = & \omega\sin\theta\cos\theta & \Gamma^\theta_{r\theta} = & \frac{1}{r} & \Gamma^\theta_{\phi\phi} = & -\sin\theta\cos\theta \\ \Gamma^\phi_{tr} = & -\frac{\omega}{r} & \Gamma^\phi_{t\theta} = & -\omega\cot\theta & \Gamma^\phi_{r\phi} = & \frac{1}{r} & \Gamma^\phi_{\theta\phi} = & \cot\theta \end{array}$$

9.1.3 Schwartzchild

Metric

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{R}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{R}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} g^{\mu\nu} = \begin{pmatrix} \frac{-1}{1-\frac{R}{r}} & 0 & 0 & 0 \\ 0 & 1 - \frac{R}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

Christoffel symbols

$$t\colon \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; r\colon \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; \theta\colon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; \phi\colon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{array}{llllllll} \Gamma^t_{tr} = & \frac{R}{2r^2\left(1-\frac{R}{r}\right)} & \Gamma^r_{tt} = & \frac{R}{2r^2}\left(1-\frac{R}{r}\right) & \Gamma^r_{rr} = & \frac{-R}{2r^2\left(1-\frac{R}{r}\right)} & \Gamma^r_{\theta\theta} = & -r\left(1-\frac{R}{r}\right) \\ \Gamma^r_{\phi\phi} = & -r\sin^2\theta\left(1-\frac{R}{r}\right) & \Gamma^\theta_{r\theta} = & \frac{1}{r} & \Gamma^\theta_{\phi\phi} = & -\sin\theta\cos\theta & \Gamma^\phi_{r\phi} = & \frac{1}{r} \\ \Gamma^\phi_{\theta\phi} = & \cot\theta & & & & & & \end{array}$$

Laplace

$$\Delta\phi = \left(g^{\mu\mu}\partial^2_\mu + \left[\frac{2}{r} - \frac{R}{r^2}\right]\partial_r + \frac{\cot\theta}{r^2}\partial_\theta\right)\phi$$

$$\Delta\vec{A} = \left(g^{\mu\mu}\partial^2_\mu + \frac{\cot\theta}{r^2}\partial_\theta\right)\vec{A} + \begin{pmatrix} \frac{2}{r}\partial_r A^t - \frac{R}{(r-R)^2}\partial_t A^r \\ \frac{2}{r}\left(1-\frac{R}{r}\right)\left(\partial_r A^r - \frac{1}{r}A^r - \partial_\theta A^\theta - \partial_\phi A^\phi - \cot\theta A^\theta\right) - \frac{R}{r^2}\partial_t A^t \\ \left[\frac{1}{r}\left(4+\frac{3R}{r}\right)\partial_r - \frac{\cot^2\theta-1}{r^2}\right]A^\theta + \frac{2}{r^3}\partial_\theta A^r - \frac{2\cot\theta}{r^3}\partial_\phi A^\phi \\ \left[\frac{1}{r}\left(4+\frac{3R}{r}\right)\partial_r + \frac{2\cot\theta}{r^2}\partial_\theta\right]A^\phi + \frac{2}{r^2\sin^2\theta}\left(\frac{1}{r}\partial_\phi A^r + \cot\theta\partial_\phi A^\phi\right) \end{pmatrix}$$

where

$$g^{\mu\mu}\partial^2_\mu = -\frac{1}{1-\frac{R}{r}}\partial^2_t + \left(1+\frac{R}{r}\right)\partial^2_r + \frac{1}{r^2}\partial^2_\theta + \frac{1}{r^2\sin^2\theta}\partial^2_\phi$$

9.1.4 Rotating Schwartzchild

Metric

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{R}{r}\right) + \omega^2 r^2\sin^2\theta & 0 & 0 & -\omega r^2\sin^2\theta \\ 0 & \frac{1}{1-\frac{R}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ -\omega r^2\sin^2\theta & 0 & 0 & r^2\sin^2\theta \end{pmatrix} g^{\mu\nu} = \begin{pmatrix} \frac{-1}{1-\frac{R}{r}} & 0 & 0 & \frac{-\omega}{1-\frac{R}{r}} \\ 0 & 1-\frac{R}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ \frac{-\omega}{1-\frac{R}{r}} & 0 & 0 & \frac{1}{r^2\sin^2\theta} - \frac{\omega^2}{1-\frac{R}{r}} \end{pmatrix}$$

Christoffel symbols

$$t\colon \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; r\colon \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \; \theta\colon \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \; \phi\colon \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$
\begin{array}{llllll}
\Gamma^t_{tr} = & \frac{R}{2r^2\left(1-\frac{R}{r}\right)} & \Gamma^r_{tt} = & \left(1-\frac{R}{r}\right)\left(\frac{R}{2r^2}-r\omega^2\sin^2\theta\right) & \Gamma^r_{t\phi} = & \left(1-\frac{R}{r}\right)\omega r\sin^2\theta \\
\Gamma^r_{rr} = & \frac{-R}{2r^2\left(1-\frac{R}{r}\right)} & \Gamma^r_{\theta\theta} = & -r\left(1-\frac{R}{r}\right) & \Gamma^r_{\phi\phi} = & -r\sin^2\theta\left(1-\frac{R}{r}\right) \\
\Gamma^\theta_{tt} = & -\omega^2\sin\theta\cos\theta & \Gamma^\theta_{t\phi} = & \omega\sin\theta\cos\theta & \Gamma^\theta_{r\theta} = & \frac{1}{r} \\
\Gamma^\theta_{\phi\phi} = & -\sin\theta\cos\theta & \Gamma^\phi_{tr} = & -\frac{\omega}{r}\left(1-\frac{R}{r}\right) & \Gamma^\phi_{t\theta} = & -\omega\cot\theta \\
\Gamma^\phi_{r\phi} = & \frac{1}{r} & \Gamma^\phi_{\theta\phi} = & \cot\theta & &
\end{array}
$$

9.1.5 Rindler

Metric

$$
g_{\mu\nu} = \begin{pmatrix} -g^2x^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} \frac{-1}{g^2x^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Christoffel symbols

$$
t\colon \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; x\colon \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; y\colon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; z\colon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\Gamma^t_{tx} = \; \frac{1}{x} \quad \Gamma^x_{tt} = \; g^2x
$$

9.1.6 Polar Rindler

Metric

$$
g_{\mu\nu} = \begin{pmatrix} -g^2x^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} \frac{-1}{g^2x^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{r^2} \end{pmatrix}
$$

Christoffel symbols

$$
t\colon \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; x\colon \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; r\colon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; \theta\colon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

$$
\begin{array}{llll}
\Gamma^t_{tx} = & \frac{1}{x} & \Gamma^x_{tt} = & g^2x \\
\Gamma^r_{\theta\theta} & = -r & \Gamma^\theta_{r\theta} & = \frac{1}{r}
\end{array}
$$

9.2 Source code

I here present the source code of the program used in chapter 7. Note that the coordinate called "phi" here corresponds not to the ϕ coordinate, but to θ, for historical reasons.

The important functions here are "main", which contains the initialization and main loop, and "ddcommon" and "ddextra", which implement the wave equation.

```
#include <cstdio>
#include <cstdlib>
#include <cmath>
#include <cstring>
#include <cpgplot.h>
#include <curses.h>
#include <list>

using namespace std;

// Full electromagnetic version

// utility functions
int max(int a, int b) { return a > b ? a : b; }
int min(int a, int b) { return a < b ? a : b; }
int mod(int a, int b) { int c = a % b; return c < 0 ? c+b :
    c; }
double modf(double a, double b)
{
  double c = fmod(a,b);
  return c < 0 ? c+b : c;
}
double cat(double x, double a)
{
  double tmp = sin(x);
  tmp += tmp < 0 ? -a : a;
  return cos(x)/tmp;
}

const int cnum = 100;
const int maxneg = cnum/2, maxpos = cnum-1;
const double pi = 4*atan(1.0);

int screen, psfile;

// pgplot initialization. Initializes both the screen and a
    file.
void init(double maxrR, double wsize, double R, char *
   filepg)
{
```

```
  double nx = maxrR*R;
  double ny = nx;
  psfile = cpgopen(filepg);
  screen = cpgopen("/xserv");
  cpgask(0);
  cpgpap(wsize, ny/nx);
  cpgenv(-0.5,nx,-0.5,ny,1,1);
  for(int i = 0; i <= maxneg; i++)
    cpgshls(i, 0, double(i)/maxneg, 1);
  for(int i = maxneg+1; i <= maxpos; i++)
    cpgshls(i, 120, double(i-maxneg-1)/(maxpos-maxneg-1),
       1);
  cpgslct(psfile);
  cpgask(0);
  cpgpap(wsize, ny/nx);
  for(int i = 0; i <= maxneg; i++)
    cpgshls(i, 0, double(i)/maxneg, 1);
  for(int i = maxneg+1; i <= maxpos; i++)
    cpgshls(i, 120, double(i-maxneg-1)/(maxpos-maxneg-1),
       1);
  cpgslct(screen);
}

struct particle { double xr, xp, vr, vp; };
typedef list<particle> Plist;

// This class is basically a 2D matrix of doubles.
class field
{
public:
  field(int xdim = 0, int ydim = 0):nx(xdim),ny(ydim),n(
     ydim*xdim),data(new double[xdim*ydim]) {}
  field(const field & f):nx(f.nx),ny(f.ny),n(f.n),data(new
     double[f.nx*f.ny])
  {
    for(int i = 0; i < nx*ny; i++) data[i] = f.data[i];
  }
  ~field() { delete[] data; }
  double & operator()(int x, int y) { return data[x+nx*y];
     }
  double operator()(int x, int y) const { return data[x+nx*
     y]; }
  double & operator[](int i) { return data[i]; }
  double operator[](int i) const { return data[i]; }
  void operator=(const field & f)
  {
    if(f.n != n)
    {
      delete [] data;
```

```
      n = f.n; nx = f.nx; ny = f.ny;
      data = new double[n];
    }
    for(int i = 0; i < f.n && i < n; i++)
      data[i] = f[i];
  }
  int mudim() const { return nx; }
  int phidim() const { return ny; }
  int dim() const { return n; }
private:
  int nx, ny, n;
  double * data;
};

// Handle cyclic coordinates
double fieldcyc(const field & f, int mu, int phi)
{
  if(mu < 0 || mu >= f.mudim()) return 0;
  phi = mod(phi,2*f.phidim());
  if(phi >= f.phidim()) phi = 2*f.phidim()-2-phi;
  return f(mu, phi);
}

// cuts out a small section of the full field.
field cutfield(const field & f, int mu, int phi, int s)
{
  field res(s,s);
  for(int i = 0; i < s; i++)
  for(int j = 0; j < s; j++)
    res(i,j) = fieldcyc(f, mu+i-(s-1)/2,phi+j-(s-1)/2);
  return res;
}

// Normalization
double maxamp(const field & f, int from = 0)
{
  double minv = 1.0/0.0, maxv = -1.0/0.0;
  for(int i = from; i < f.mudim(); i++)
  for(int j = 0; j < f.phidim(); j++)
  {
    double tmp = f(i,j);
    if(minv > tmp) minv = tmp;
    if(maxv < tmp) maxv = tmp;
  }
  return maxv > -minv ? maxv : -minv;
}

// The main simulator class. It is somewhat messy.
class Simulator
```

```
{
public:
  Simulator():buflen(100) { buffer = new char[buflen]; }
  ~Simulator() { delete[] buffer; }
  // Reads command line arguments, and uses default
  // values for those that are not given.
  void cmd(int argn, char ** args)
  {
    printlog = false; muprint = false; calcE = false;
    wsize = 10; nmu = 800; nphi = 400; R = 10;
    maxrR = 10; dt = 0.05; dt0 = 0.5; r0 = 30; ri = 2;
    component = 0; lcprint = false; runge = false;
    skip = 0, interval = 10, count = 0, Fa = 0, Fb = 1;
    orbit = false; force = false; damprad = 0; maxdamp = 1;
    cum = true; printtime = true;
    polar = 0; foo = 1e-3; deltalim = 1e-5;
    mode = 0; maxtime = -1; every = -1; filepg = "plot.ps/
        cps";
    for(char ** i = args+1, ** j = args+2; *i != 0 && *j !=
         0; i += 2, j += 2)
    {
      if(!strcmp(*i, "nmu")) nmu = atoi(*j);
      else if(!strcmp(*i, "nphi")) nphi = atoi(*j);
      else if(!strcmp(*i, "wsize")) wsize = atof(*j);
      else if(!strcmp(*i, "R")) R = atof(*j);
      else if(!strcmp(*i, "maxrR")) maxrR = atof(*j);
      else if(!strcmp(*i, "dt")) dt = atof(*j);
      else if(!strcmp(*i, "dt0")) dt0 = atof(*j);
      else if(!strcmp(*i, "ri")) ri = atof(*j);
      else if(!strcmp(*i, "r0")) r0 = atof(*j);
      else if(!strcmp(*i, "polar")) polar = atof(*j);
      else if(!strcmp(*i, "maxcot")) foo = atof(*j);
      else if(!strcmp(*i, "deltalim")) deltalim = atof(*j);
      else if(!strcmp(*i, "component")) component = atoi(*j
         );
      else if(!strcmp(*i, "interval")) interval = atoi(*j);
      else if(!strcmp(*i, "Fa")) Fa = atoi(*j);
      else if(!strcmp(*i, "Fb")) Fb = atoi(*j);
      else if(!strcmp(*i, "lc")) lcprint = atoi(*j);
      else if(!strcmp(*i, "skip")) skip = atoi(*j);
      else if(!strcmp(*i, "orbit")) orbit = atoi(*j);
      else if(!strcmp(*i, "force")) force = atoi(*j);
      else if(!strcmp(*i, "damprad")) damprad = atoi(*j);
      else if(!strcmp(*i, "maxdamp")) maxdamp = atoi(*j);
      else if(!strcmp(*i, "cum")) cum = atoi(*j);
      else if(!strcmp(*i, "E")) calcE = atoi(*j);
      else if(!strcmp(*i, "runge")) runge = atoi(*j);
      else if(!strcmp(*i, "maxtime")) maxtime = atof(*j);
      else if(!strcmp(*i, "every")) every = atof(*j);
```

```
    else if(!strcmp(*i, "mode")) mode = atoi(*j);
    else if(!strcmp(*i, "log")) printlog = atoi(*j);
    else if(!strcmp(*i, "filepg")) filepg = *j;
    else if(!strcmp(*i, "pgfile")) filepg = *j;
  }
}
// Fixes phi values that are off limits.
int fixphi(int phi)
{
  phi = mod(phi, 2*nphi-2);
  if(phi >= nphi) return 2*nphi-2-phi;
  else return phi;
}
double fixphif(double phi)
{
  phi = modf(phi, 2*pi);
  if(phi > pi) return 2*pi-phi;
  else return phi;
}
// The following functions represent numerical
// second and first derivatives. This ones
// marked with an extra underscore are used
// at boundaries.
double dd_r(const field & f, int mu, int phi, double dr)
{
  return (f(mu+1,phi)-2*f(mu,phi)+f(mu-1,phi)
    + (-mu2r[mu+1]+2*mu2r[mu]-mu2r[mu-1])
    * (f(mu+1,phi)-f(mu-1,phi))/2/dr)/dr/dr;
}
double dd_p(const field & f, int mu, int phi, double dp)
{
  return (f(mu,phi+1)-2*f(mu,phi)+f(mu,phi-1)
    + (-nu2phi[phi+1]+2*nu2phi[phi]-nu2phi[phi-1])
    * (f(mu,phi+1)-f(mu,phi-1))/2/dp)/dp/dp;
}
double dd_p_(const field & f, int mu, int phiu, int phim,
     int phid, double dp)
{
  return (f(mu,phiu)-2*f(mu,phim)+f(mu,phid)
    + (-dnu2phi[phiu]+dnu2phi[phid])
    * (f(mu,phiu)-f(mu,phid))/4)/dp/dp;
}
double d_r(const field & f, int mu, int phi, double dr)
{
  return (f(mu+1,phi)-f(mu-1,phi))/2/dr;
}
double d_p(const field & f, int mu, int phi, double dp)
{
  return (f(mu,phi+1)-f(mu,phi-1))/2/dp;
```

```
}
double d_p_(const field & f, int mu, int phiu,int phim,
    int phid, double dp)
{
  return (f(mu,phiu)-f(mu,phid))/2/dp;
}
// given how other things work, it is hard to make this
    one pretty
// it derives component a of the field with respect to
    the b coordinate
void curl_(const field * f, const field * g, field & cf,
    int a, int b, int sign, int dmu = -1, int dphi = -1)
{
  // offset of local mu and phi from global values
  // phi not really used
  // The mu value sent in is that of the center of the
      field
  // but we need the beginning
  if(dmu == -1) dmu = 0;
  else dmu -= (cf.mudim()-1)/2;

  int p = cf.phidim()-1;
  switch(b)
  {
  case 0:
    for(int i = 0; i < cf.mudim(); i++)
    for(int j = 0; j < cf.phidim(); j++)
      cf(i,j) += g[a](i,j)*sign;
    break;
  case 1:
    for(int i = 1; i < cf.mudim()-1; i++)
    {
      double dr = (mu2r[i+dmu+1]-mu2r[i+dmu-1])/2;
      for(int j = 0; j < cf.phidim(); j++)
        cf(i,j) += d_r(f[a], i, j, dr)*sign;
    }
    break;
  case 2:
    for(int i = 0; i < cf.mudim(); i++)
    {
      // yet another ugly hack. only apply boundaries
      // when we have the whole field. oh well.
      // There are now many places in the code that
      // assume that a full field is nmu*nphi big.
      if(cf.phidim() == nphi)
      {
        cf(i,0) += 0*d_p_(f[a], i, 1, 0, p, dnu2phi[0])*
            sign;
        cf(i,p) += 0*d_p_(f[a], i, 0, p, p-1, dnu2phi[p])
```

```
          *sign;
      }
      for(int j = 1; j < cf.phidim()-1; j++)
      {
        cf(i,j) += d_p(f[a], i, j, dnu2phi[j])*sign;
      }
    }
    break;
  }
}
// the need to raise and lower, and to make copies
   because of this,
// makes this function slow and clumsy
void curl(const field * f, const field * g, field & cf,
   int a, int b)
{
  field A[] = { f[0],f[1],f[2] }, At[] = { g[0],g[1],g[2]
      };
  for(int i = 0; i < 3; i++)
  {
    raise_or_lower(A[i],A[i],i,0);
    raise_or_lower(At[i],At[i],i,0);
  }
  for(int i = 0; i < cf.dim(); i++) cf[i] = 0;
  curl_(A,At,cf,b,a,1);
  curl_(A,At,cf,a,b,-1);
  raise_or_lower(cf,cf,a,1);
  raise_or_lower(cf,cf,b,1);
}
// computes the curl in a square with sides 2*size+1
   centered on
// mu,phi. The returned area has a border of 1 on all
   sides, so
// coordinates need an offset of 1+size-mu etc.
field curl_area(const field * f, const field * g, int a,
   int b, int mu, int phi, int size)
{
  int ext = 3+2*size;
  field cf(ext,ext);
  field A[3], At[3];
  for(int which = 0; which < 3; which++)
  {
    A[which] = cutfield(f[which],mu,phi,ext);
    At[which] = cutfield(g[which],mu,phi,ext);
  }
  for(int i = 0; i < 3; i++)
  {
    raise_or_lower(A[i],A[i],i,0,mu,phi);
    raise_or_lower(At[i],At[i],i,0,mu,phi);
```

```
  }
  for(int i = 0; i < cf.dim(); i++) cf[i] = 0;
  curl_(A,At,cf,b,a,1,mu,phi);
  curl_(A,At,cf,a,b,-1,mu,phi);
  raise_or_lower(cf,cf,a,1,mu,phi);
  raise_or_lower(cf,cf,b,1,mu,phi);
  return cf;
}
// input and output may be the same object
void raise_or_lower(const field & f, field & uf, int dim,
    int up, int dmu = -1, int dphi = -1)
{
  if(dmu == -1) dmu = (f.mudim()-1)/2;
  if(dphi == -1) dphi = (f.phidim()-1)/2;
  for(int i = 0; i < f.mudim(); i++)
  {
    double r = mu2r[i-(f.mudim()-1)/2+dmu];
    double a;
    switch(dim)
    {
    case 0: a = -1/(1-R/r); break;
    case 1: a = (1-R/r); break;
    case 2: a = 1/r/r;
    }
    if(!up) a = 1/a;

    for(int j = 0; j < f.phidim(); j++)
      uf(i,j) = f(i,j)*a;
  }
}

// overwrites ddf with the common parts of the spacetime
// wave equation result
// Works best for even phidim, and needs phi ~ -phi.
// Point emission may only happen at phi = 0 or phi = pi
// so gradual emission during motion is out of the
    picture.
void ddcommon(const field * fs, field * ddfs)
{
  for(int which = 0; which < 3; which++)
  {
    const field & f = fs[which];
    field & ddf = ddfs[which];

    // border
    for(int i = 1; i < f.mudim()-1; i++)
    {
      double r = mu2r[i];
      double beta = 1-R/r;
```

```
      double dr = (mu2r[i+1]-mu2r[i-1])/2;
      int p = f.phidim()-1;
      double dp = dnu2phi[0];
      ddf(i,0) = beta*(beta*dd_r(f,i,0,dr)
         + dd_p_(f,i,1,0,1,dp)/r/r
         + 0/foo/r/r*d_p_(f, i, 1, 0, 1, dp)); // 1/foo
            from cos/sin
      double phi = nu2phi[p];
      double ct = cat(phi,foo);
      dp = dnu2phi[p];
      ddf(i,p) = beta*(beta*dd_r(f,i,p,dr)
         + dd_p_(f,i,p-1,p,p-1,dp)/r/r
         + 0*ct/r/r*d_p_(f,i,p-1,p,p-1,dp));
    }
    for(int j = 0; j < f.phidim(); j++)
    {
      ddf(0,j) = 0; ddf(f.mudim()-1,j) = 0;
    }
    // interior
    for(int i = 1; i < f.mudim()-1; i++)
    {
      double r = mu2r[i];
      double beta = 1-R/r;
      for(int j = 1; j < f.phidim()-1; j++)
      {
        double dr = (mu2r[i+1]-mu2r[i-1])/2;
        double phi = nu2phi[j], dp = dnu2phi[j];
        ddf(i,j) = beta*(beta*dd_r(f, i, j, dr)
         + dd_p(f,i,j,dp)/r/r
         + cat(phi,foo)/r/r*d_p(f,i,j,dp));
      }
    }
  }
}
void ddextra(const field * f, const field * g, field *
   ddf)
{
  int phidim = f[0].phidim(), mudim = f[0].mudim();

  int p = phidim-1;
  for(int i = 1; i < mudim-1; i++)
  {
    double r = mu2r[i];
    double beta = 1-R/r;
    double dr = (mu2r[i+1]-mu2r[i-1])/2;
    double ct = 1/foo;
    double dp = dnu2phi[0];
    ddf[0](i,0) += 2*beta/r*d_r(f[0],i,0,dr) - R/beta/r/r
       *g[1](i,0);
```

```
    ddf[1](i,0) += beta*(2*beta/r*(d_r(f[1],i,0,dr)-1/r*f
        [1](i,0)
      -0*d_p_(f[2],i,1,0,1,dp)-ct*f[2](i,0))-R/r/r*g[0](i
          ,0));
    ddf[2](i,0) += beta*((1+3*beta)/r*d_r(f[2],i,0,dr)
      - (ct*ct-1)/r/r*f[2](i,0) + 0*2/r/r/r*d_p_(f[1],i
          ,1,0,1,dp));
    double phi = nu2phi[p];
    dp = dnu2phi[p];
    ct = cat(phi,foo);
    ddf[0](i,p) += 2*beta/r*d_r(f[0],i,p,dr) - R/beta/r/r
        *g[1](i,p);
    ddf[1](i,p) += beta*(2*beta/r*(d_r(f[1],i,p,dr)-1/r*f
        [1](i,p)
      -0*d_p_(f[2],i,p-1,p,p-1,dp)-ct*f[2](i,p))-R/r/r*g
          [0](i,p));
    ddf[2](i,p) += beta*((1+3*beta)/r*d_r(f[2],i,p,dr)
      - (ct*ct-1)/r/r*f[2](i,p) + 0*2/r/r/r*d_p_(f[1],p
          -1,p,p-1,p,dp));
  }
  // interior
  for(int i = 1; i < mudim-1; i++)
  {
    double r = mu2r[i];
    double beta = 1-R/r;
    for(int j = 1; j < phidim-1; j++)
    {
      double dr = (mu2r[i+1]-mu2r[i-1])/2;
      double phi = nu2phi[j], dp = dnu2phi[j];
      double ct = cat(phi,foo);
      ddf[0](i,j) += 2*beta/r*d_r(f[0],i,j,dr) - R/beta/r
          /r*g[1](i,j);
      ddf[1](i,j) += beta*(2*beta/r*(d_r(f[1],i,j,dr)-1/r
          *f[1](i,j)
        -d_p(f[2],i,j,dp)-ct*f[2](i,j))-R/r/r*g[0](i,j));
      ddf[2](i,j) += beta*((1+3*beta)/r*d_r(f[2],i,j,dr)
        - (ct*ct-1)/r/r*f[2](i,j) + 2/r/r/r*d_p(f[1],i,j,
            dp));
    }
  }
}
double field_energy(const field * f, const field * g)
{
  field Ftr(nmu,nphi), Ftp(nmu,nphi), Frp(nmu,nphi);
  curl(f, g, Ftr, 0, 1);
  curl(f, g, Ftp, 0, 2);
  curl(f, g, Frp, 1, 2);
  double E = 0;
  for(int i = 1; i < nmu-1; i++)
```

```
  {
    double r = mu2r[i];
    double beta = 1-R/r;
    double dr = (mu2r[i+1]-mu2r[i-1])/2;
    for(int j = 0; j < nphi; j++)
    {
      double dp = dnu2phi[j];
      E += (Ftr(i,j)*Ftr(i,j)/beta+
        Ftp(i,j)*Ftp(i,j)*r*r+
        Frp(i,j)*Frp(i,j)*r*r/beta/beta)/8*
        r*r*fabs(sin(nu2phi[j]))/sqrt(beta)*dp*dr;
    }
  }
  return E;
}
void calc_momentum(const field * f, const field * g)
{
  field Ftr(nmu,nphi), Ftp(nmu,nphi), Frp(nmu,nphi);
  curl(f, g, Ftr, 0, 1);
  curl(f, g, Ftp, 0, 2);
  curl(f, g, Frp, 1, 2);
  for(int i = 1; i < nmu-1; i++)
  {
    double r = mu2r[i];
    double beta = 1-R/r;
    double dr = (mu2r[i+1]-mu2r[i-1])/2;
    for(int j = 0; j < nphi; j++)
    {
      double dp = dnu2phi[j];
      momentum[1](i,j) = r*r/4/pi*Frp(i,j)*Ftp(i,j);
      momentum[2](i,j) = -1.0/4/pi/beta*Ftr(i,j)*Frp(i,j)
          ;
      double E = (Ftr(i,j)*Ftr(i,j)/beta+
        Ftp(i,j)*Ftp(i,j)*r*r+
        Frp(i,j)*Frp(i,j)*r*r/beta/beta)/8/pi;
      if(j > 0 && j < nphi-1)
      {
        divergence(i,j) = (E-momentum[0](i,j))/dt
          + d_r(momentum[1],i,j,dr)
          + d_p(momentum[2],i,j,dp)
          + (R/r/r/beta+2/r)*momentum[1](i,j)
          + cat(nu2phi[j],foo)*momentum[2](i,j);
      }
      momentum[0](i,j) = E;
    }
  }
}

void printsmall(const field & f, double x1, double x2,
```

```
    double y1, double y2)
{
  cpgbbuf();
  float xt[] = { x1-0.5,x2+0.5,x2+0.5, x1-0.5 },
    yt[] = {y1-0.5,y1-0.5,y2+0.5,y2+0.5};
  cpgsci(0); cpgpoly(4, xt, yt);
  double div = norm ? maxamp(f, 0) : 1;
  double dx = (x2-x1)/f.mudim(), dy = (y2-y1)/f.phidim();
  for(int i = 0; i < f.mudim(); i++)
  {
    xt[0] = x1+i*dx; xt[1] = x1+(i+1)*dx;
    xt[2] = xt[1]; xt[3] = xt[0];
    for(int j = 0; j < f.phidim(); j++)
    {
      yt[0] = y1+j*dy; yt[1] = yt[0];
      yt[2] = y1+(j+1)*dy; yt[3] = yt[2];
      double value;
      if(printlog)
      {
        int sign = 1;
        double V = f(i,j)/div;
        if(V < 0) { sign = -1; V = -V; }
        value = (log(V)/log(10)+logfact)/logfact;
        if(value < 0) value = 0;
        if(value > 1) value = 1;
        value *= sign;
      } else value = f(i,j)/div;
      if(value < 0) cpgsci(int(-value*maxneg));
      else cpgsci(maxneg+1 + int(value*(maxpos-maxneg-1))
          );
      cpgpoly(4, xt, yt);
    }
  }
  cpgebuf();
}

void printfield(const field & f,const Plist & light_cone,
    bool do_norm)
{
  cpgbbuf();
  double r1 = R, r2;
  double nr = maxrR*R/2;
  if(do_norm) div = maxamp(f, muprint ? 0 : mumin);

  // clear screen
  //float xt[] = { 0,2*nr,2*nr, 0 },yt[] = {0,0,2*nr,2*nr
      };
  //cpgsci(0); cpgpoly(4, xt, yt);
  cpgeras();
```

```
if(muprint) r1 = 0;

double cp[f.phidim()+1], sp[f.phidim()+1];
if(!muprint)
{
  cp[0] = cos(nu2phi[0]); cp[f.phidim()] = cos(nu2phi[f
      .phidim()-1]);
  sp[0] = sin(nu2phi[0]); sp[f.phidim()] = sin(nu2phi[f
      .phidim()-1]);
  for(int j = 1; j < f.phidim(); j++)
  {
    cp[j] = cos((nu2phi[j-1]+nu2phi[j])/2);
    sp[j] = sin((nu2phi[j-1]+nu2phi[j])/2);
  }
}
else
{
  cp[0] = 1; cp[f.phidim()] = -1;
  sp[0] = 0; sp[f.phidim()] = 0;
  for(int j = 1; j < f.phidim(); j++)
  {
    double alpha = pi*(j-0.5)/(f.phidim()-1);
    cp[j] = cos(alpha);
    sp[j] = sin(alpha);
  }
}

for(int i = 0; i < murR; i++)
{
  if(muprint) r2 = (double)i/murR*nr;
  else r2 = mu2r[i];
  if(muprint || (r2-r1 > nr/300 && r1 < 2*nr))
  {
    for(int j = 0; j < f.phidim(); j++)
    {
      double value;
      if(printlog)
      {
        int sign = 1;
        double V = f(i,j)/div;
        if(V < 0) { sign = -1; V = -V; }
        value = (log(V)/log(10)+logfact)/logfact;
        if(value < 0) value = 0;
        if(value > 1) value = 1;
        value *= sign;
      } else value = f(i,j)/div;
      if(value < 0) cpgsci(int(-value*maxneg));
      else cpgsci(maxneg+1 + int(value*(maxpos-maxneg
```

```
        -1)));
      float xpos[] = { nr+r1*cp[j], nr+r2*cp[j], nr+r2*
          cp[j+1], nr+r1*cp[j+1] },
           ypos[] = { nr+r1*sp[j], nr+r2*sp[j], nr+r2*sp
               [j+1], nr+r1*sp[j+1] };
      cpgpoly(4, xpos, ypos);
      for(int k = 0; k < 4; k++) ypos[k] = 2*nr-ypos[k
          ];
      cpgpoly(4, xpos, ypos);
    }
    r1 = r2;
  }
}

if(lcprint)
{
  cpgsci(maxpos);
  if(light_cone.size())
  {
    Plist::const_iterator i = light_cone.begin();
    if(!muprint)
    {
      cpgmove(nr+i->xr*cos(i->xp), nr+i->xr*sin(i->xp))
          ;
      while(++i != light_cone.end())
        cpgdraw(nr+i->xr*cos(i->xp), nr+i->xr*sin(i->xp
            ));
    }
    else
    {
      double r = r2mu(i->xr)*nr/murR;
      cpgmove(nr+r*cos(i->xp), nr+r*sin(i->xp));
      while(++i != light_cone.end())
      {
        r = r2mu(i->xr)*nr/murR;
        cpgdraw(nr+r*cos(i->xp), nr+r*sin(i->xp));
      }
    }
  }
  if(muprint)
  {
    double r = r2mu(source_p.xr)*nr/murR;
    cpgcirc(nr+r*cos(source_p.xp),nr+r*sin(source_p.xp)
        , 0.5);
  }
  else cpgcirc(nr+source_p.xr*cos(source_p.xp),nr+
      source_p.xr*
    sin(source_p.xp), 0.5);
}
```

```
  int at = -1;
  cpgsci(maxpos);
  //snprintf(buffer, buflen, "Particles: %10d",
      light_cone.size());
  //cpgmtxt("T", at--, 0.95, 1.0, buffer);
  if(printtime)
  {
    snprintf(buffer, buflen, "Time: %10.2lf", dt*count);
    cpgmtxt("T", at--, 0.95, 1.0, buffer);
  }
  snprintf(buffer, buflen, "Amplitude: %10.5le", div);
  cpgmtxt("T", at--, 0.95, 1.0, buffer);
  if(force)
  {
    snprintf(buffer, buflen, "Fr:%9.2le Fp:%9.2le", fr,
        fp);
    cpgmtxt("T", at--, 0.95, 1.0, buffer);
    snprintf(buffer, buflen, "Part.Amp: %10.5le",
        part_amp);
    cpgmtxt("T", at--, 0.95, 1.0, buffer);
    snprintf(buffer, buflen, "Samp: %15.5le", maxamp(
        small));
    cpgmtxt("T", at--, 0.95, 1.0, buffer);
  }
  if(calcE)
  {
    snprintf(buffer, buflen, "Energy: %10.2le", E);
    cpgmtxt("T", at--, 0.95, 1.0, buffer);
  }
  int L = (int)(R*maxrR);
  printsmall(small, L, 5*L/6, L/6, 0);
  cpgsci(maxpos);
  cpgbox("BCIN", 0.0, 0, "BCIN", 0.0, 0);
  cpgebuf();
}

double ducalc(double r, double u, double R)
{
  double beta = 1-R/r;
  return -R/2/r/r*(beta-3*u*u/beta);
}
void runge_step_r(double & r, double & u, double dt,
   double R)
{
  double du1 = ducalc(r,u,R), dr1 = u;
  double du2 = ducalc(r+dt*dr1/2,u+dt*du1/2,R), dr2 = u+
     dt*du1/2;
  double du3 = ducalc(r+dt*dr2/2,u+dt*du2/2,R), dr3 = u+
```

```
    dt*du2/2;
  double du4 = ducalc(r+dt*dr3,u+dt*du3,R), dr4 = u+dt*
    du3;
  r += dt*(dr1+2*dr2+2*dr3+dr4)/6;
  u += dt*(du1+2*du2+2*du3+du4)/6;
}

void makeMu2r(int nmu, double r0, double R, double dt)
{
  mu2r = new double[nmu];
  dists = new double[nmu];
  double beta = 1-R/r0;
  double r = r0, u = -sqrt(beta);
  double d = 0;
  mu2r[nmu/2] = r;
  dists[nmu/2] = d;
  fprintf(stderr, "%lf\n", dt);
  for(int i = nmu/2-1; i >= 0; i--)
  {
    runge_step_r(r,u,dt,R);
    mu2r[i] = r;
    // dt*u = dr because of the step size here
    // the spatial distance between two points
    // is dr/sqrt(beta)
    d += dt*u/sqrt(beta);
    dists[i] = d;
  }
  r = r0;
  beta = 1-R/r0;
  d = 0; u = sqrt(beta);
  for(int i = nmu/2+1; i < nmu; i++)
  {
    runge_step_r(r,u,dt,R);
    mu2r[i] = r;
    d += dt*u/sqrt(beta);
    dists[i] = d;
  }

  fprintf(stderr, "%10d%10.4lf\n\r", 0, mu2r[0]);
  fprintf(stderr, "%10d%10.4lf\n\r", nmu-1, mu2r[nmu-1]);
}

// make a phi grid with extra resolution at the poles,
// paying extra attention to symmetry. a modifies how
// much resolution is increased,
void makeNu2phi(int nphi, double a)
{
  nu2phi = new double[nphi];
  dnu2phi = new double[nphi];
```

```
  double delta = atan2(1.0,a);
  int m = nphi-1;
  for(int nu = 0; nu <= m; nu++)
  {
    double x = 2.0*nu/m - 1;
    nu2phi[nu] = atan2(1.0,-x*a)+delta*x;
  }
  dnu2phi[0] = nu2phi[1];
  dnu2phi[m] = nu2phi[1];
  for(int nu = 1; nu < nphi-1; nu++)
    dnu2phi[nu] = (nu2phi[nu+1]-nu2phi[nu-1])/2;
}

// computes the *radial* distance from a grid point to a
    particle
// does this by first finding the clost grid point, and
    then
// adding the distance from the point to this
double r_distance(int mu, int phi, const particle & p)
{
  int pmu = (int)r2mu(p.xr);
  double dist = dists[mu]-dists[pmu];
  return dist + (mu2r[pmu]-p.xr)/sqrt(1-R/p.xr);
}

Plist LC_init(double r, double phi, int num)
{
  Plist P;
  particle p = { r, phi, 0, 0 };
  double rfact = 1-R/r, phifact = sqrt(1-R/r)/r;
  for(int i = 0; i < num; i++)
  {
    double angle = 2*pi/num*i;
    p.vr = cos(angle)*rfact;
    p.vp = sin(angle)*phifact;
    P.push_back(p);
  }
  return P;
}

void LC_prop(Plist & P)
{
  for(Plist::iterator i = P.begin(); i != P.end(); i++)
    prop_particle(*i);
}
void prop_particle(particle & p)
{
  double r = p.xr;
  double beta = 1-R/r;
```

```
  double dvr = R/2/r/r*(3*p.vr*p.vr/beta-beta)
    + r*beta*p.vp*p.vp;
  double dvp = -2*(r-1.5*R)/r/r/beta*p.vr*p.vp;
  p.vr += dt*dvr;
  p.vp += dt*dvp;
  p.xr += dt*p.vr;
  p.xp += dt*p.vp;
}

// makes a "delta function" (gaussian) at the position of
    p
// Now uses dists to get better, though still approximate
    ,
// spatial distances
// Should have norm 1
void makedelta(field & help, const particle & p, int dmu
    = -1, int dphi = -1, double rmod = 1)
{
  double norm = 0;
  if(dmu == -1) dmu = (help.mudim()-1)/2;
  if(dphi == -1) dphi = (help.phidim()-1)/2;
  double rm = ri*rmod;
  for(int i = 0; i < help.mudim(); i++)
  {
    int mu = i-(help.mudim()-1)/2+dmu;
    if(mu < 0 || mu >= nmu) continue;
    double r = mu2r[mu];
    for(int j = 0; j < help.phidim(); j++)
    {
      int phi = fixphi(j-(help.phidim()-1)/2+dphi);
      // cyclic coordinate
      double dp = fabs(nu2phi[phi]-fixphif(p.xp));
      double rad2 = (pow(r_distance(mu,phi,p),2)+
        pow(r*dp,2));
      help(i,j) = exp(-rad2/(rm*rm))/rm/rm/pi-deltalim;
      if(help(i,j) < 0) help(i,j) = 0;
    }
  }
}

void makesource(field * targ, const particle & p, double
    gamma)
{
  // the following sets up the source, using gaussians in
      place
  // of delta functions.
  double betai = 1-R/p.xr;
  field help(nmu,nphi);
  makedelta(help, p);
```

```
  for(int i = 0; i < targ[0].dim(); i++)
  {
    targ[0][i] = gamma*help[i];
    targ[1][i] = gamma*help[i]*p.vr;
    targ[2][i] = gamma*help[i]*p.vp;
  }
}

// particles are assumed to be close, and this function
   keeps
// them close
void LC_more(Plist & P, double tol)
{
  Plist::iterator a = P.begin(), b = P.begin(), c = P.
      begin();
  b++; c++; c++;
  // this function uses its own maxrR, to avoid confusion
  // when zooming
  double maxrR = 3*r0/R;
  while(c != P.end())
  {
    double beta = 1-R/a->xr;
    if(a->xr < R*maxrR)
    {
      if(beta*(a->xr - b->xr)*(a->xr-b->xr)
        + a->xr*a->xr*(a->xp-b->xp)*(a->xp-b->xp) > tol)
      {
        particle p = { (a->xr+b->xr)/2,
          (a->xp+b->xp)/2, (a->vr+b->vr)/2,
          (a->vp+b->vp)/2 };
        double r = p.xr;
        beta = 1-R/r;
        a = P.insert(b, p);
      }
      else { a++; b++; c++; }
    }
    else
    {
      if(fabs(a->xp - c->xp) < pi/50)
      {
        b = P.erase(b); c++;
      }
      else { a++; b++; c++; }
    }
  }
}

// this is a slow operation, but necessary to show the
```

```
    light cone
// in mu mode
double r2mu(double r)
{
  int a = 0, b = nmu-1, old;
  if(r < mu2r[a]) return a;
  if(r > mu2r[b]) return b;
  while(b - a > 1)
  {
    old = a;
    a = (a+b)/2;
    if(mu2r[a] > r) { b = a; a = old; }
  }
  return a + (b-a)*(r-mu2r[a])/(mu2r[b]-mu2r[a]);
}
double phi2nu(double phi)
{
  int a = 0, b = nphi-1, old;
  if(phi < nu2phi[a]) return a;
  if(phi > nu2phi[b]) return b;
  while(b - a > 1)
  {
    old = a;
    a = (a+b)/2;
    if(nu2phi[a] > phi) { b = a; a = old; }
  }
  return a + (b-a)*(phi-nu2phi[a])/(nu2phi[b]-nu2phi[a]);
}
void calcmurR()
{
  int i;
  for(i = 0; i < nmu && mu2r[i] < maxrR*R; i++); murR = i
      ;
  for(i = 0; i < nmu && mu2r[i]-R < maxrR*0.05; i++);
     mumin = i;
}

// returns the integral of the product of two fields
// The two fields may cover only sections of the global
// field, and to support this, the extra position
    arguments
// support this, and give the coordinates of the field
// centers. This is needed to compute the volume elements
    .
// FUDGE: when fudge is true, any values of the field
// coming from the lower half-plane have their sign
// changed.
double inner_product(const field & a, const field & b,
    int mu, int phi, bool fudge = false)
```

```
{
  int modifier = 1;
  double res = 0;
  for(int i = 0; i < a.mudim(); i++)
  {
    int I = i-(a.mudim()-1)/2+mu;
    if(I <= 0 || I >= nmu) continue;
    double dr = (mu2r[I+1]-mu2r[I-1])/2;
    double velem = mu2r[I]/sqrt(1-R/mu2r[I])*dr;
    for(int j = 0; j < a.phidim(); j++)
    {
      int J = mod(j-(a.phidim()-1)/2+phi,2*phi);
      if(J >= nphi)
      {
        if(fudge) modifier=-1;
        J = 2*nphi-2-J;
      }
      else if(fudge && (J == 0 || J == nphi-1)) modifier
          = 0;
      res += a(i,j)*b(i,j)*velem*dnu2phi[J]*modifier;
    }
  }
  return res;
}

void direct_product(field & a, const field & b, int mu,
    int phi, bool fudge = false)
{
  int modifier = 1;
  double res = 0;
  for(int i = 0; i < a.mudim(); i++)
  {
    int I = i-(a.mudim()-1)/2+mu;
    if(I <= 0 || I >= nmu) continue;
    double dr = (mu2r[I+1]-mu2r[I-1])/2;
    double velem = mu2r[I]/sqrt(1-R/mu2r[I])*dr;
    for(int j = 0; j < a.phidim(); j++)
    {
      int J = mod(j-(a.phidim()-1)/2+phi,2*phi);
      if(J >= nphi)
      {
        if(fudge) modifier=-1;
        J = 2*nphi-2-J;
      }
      else if(fudge && (J == 0 || J == nphi-1)) modifier
          = 0;
      a(i,j) = a(i,j)*b(i,j)*velem*dnu2phi[J]*modifier;
    }
  }
```

```
}

void runge_kutta(field * f, field * ft)
{
  double t = count*dt;
  Ft(f,ft,fa,t);
  Ftt(f,ft,fta,t);

  for(int which = 0; which < 3; which++)
  for(int i = 0; i < f[which].dim(); i++)
  {
    fa[which][i] *= dt; fta[which][i] *= dt;
    fc[which][i] = fa[which][i]/6;
    ftc[which][i] = fta[which][i]/6;
    fa[which][i] = f[which][i] + fa[which][i]/2;
    fta[which][i] = ft[which][i] + fta[which][i]/2;
  }

  Ft(fa,fta,fb,t+dt/2);
  Ftt(fa,fta,ftb,t+dt/2);

  for(int which = 0; which < 3; which++)
  for(int i = 0; i < f[which].dim(); i++)
  {
    fb[which][i] *= dt; ftb[which][i] *= dt;
    fc[which][i] += fb[which][i]/3;
    ftc[which][i] += ftb[which][i]/3;
    fb[which][i] = f[which][i] + fb[which][i]/2;
    ftb[which][i] = ft[which][i] + ftb[which][i]/2;
  }

  Ft(fb,ftb,fa,t+dt/2);
  Ftt(fb,ftb,fta,t+dt/2);

  for(int which = 0; which < 3; which++)
  for(int i = 0; i < f[which].dim(); i++)
  {
    fa[which][i] *= dt; fta[which][i] *= dt;
    fc[which][i] += fa[which][i]/3;
    ftc[which][i] += fta[which][i]/3;
    fa[which][i] = f[which][i] + fa[which][i];
    fta[which][i] = ft[which][i] + fta[which][i];
  }

  Ft(fa,fta,fb,t+dt);
  Ftt(fa,fta,ftb,t+dt);
  for(int which = 0; which < 3; which++)
  for(int i = 0; i < f[which].dim(); i++)
  {
```

```
    f[which][i] += fc[which][i] + dt*fb[which][i]/6;
    ft[which][i] += ftc[which][i] + dt*ftb[which][i]/6;
  }
}

void Ftt(const field * f, const field * g, field *h,
    double t)
{
  ddcommon(f,h);
  ddextra(f,g,h);
  double offset = 4*sqrt(betai)*ri;
  // gradual emission turned out to break symmetry,
  // and so we emit everything the first time
  if(t < dt)
  {
    makesource(scratch, source_p, gamma);
    //double fact = exp(-betai*(t-offset)
    //   *(t-offset)/ri/ri)/ri/sqrt(pi);
    //   // beta factors cancel
    double fact = 1/dt;
    for(int i = 0; i < f[0].dim(); i++)
    for(int which = 0; which < 3; which++)
      h[which][i] += scratch[which][i]*fact;
  }
}
void Ft(const field * f, const field * g, field * h,
    double t)
{
  for(int which = 0; which < 3; which++)
  for(int i = 0; i < h[which].dim(); i++)
    h[which][i] = g[which][i];
}
void psmooth(field & f, double fact, int maxlat)
{
  if(maxlat < 1) return;
  if(maxlat > f.phidim()/2) maxlat = f.phidim()/2;
  int p = f.phidim()-1;
  for(int mu = 0; mu < f.mudim(); mu++)
  {
    double a = f(mu,1), b;
    for(int phi = 0; phi < maxlat; phi++)
    {
      b = f(mu,phi);
      f(mu,phi) = (f(mu,phi)+fact*(a+f(mu,phi+1)))/(1+2*
          fact);
      a = b;
    }
    a = f(mu,p-1);
    for(int phi = p; p-phi < maxlat; phi--)
```

```
    {
      b = f(mu,phi);
      f(mu,phi) = (f(mu,phi)+fact*(a+f(mu,phi+1)))/(1+2*
          fact);
      a = b;
    }
  }
}

int main(int argn, char ** args)
{
  bool nodel = true;
  cmd(argn, args);
  mumin = 10, mumax = nmu-10;
  makeMu2r(nmu, r0, R, dt0);
  makeNu2phi(nphi, polar);
  calcmurR();
  norm = true;
  forcefile = fopen("force.data", "w");

  //for(int i = 1; i < nmu-1; i++) printf("%le\n", (mu2r[
      i+1]-mu2r[i-1])/2/dt0); exit(1);

  WINDOW * win = initscr();
  nodelay(win, nodel);
  init(maxrR,wsize,R, filepg);

  // variables used to dump image to file
  double firstpage = true, printpage = false;

  logfact = 3;

  field f[] = { field(nmu, nphi), field(nmu, nphi), field
      (nmu, nphi) },
         g[] = { field(nmu, nphi), field(nmu, nphi), field
             (nmu, nphi) },
       h[] = { field(nmu, nphi), field(nmu, nphi), field(
           nmu, nphi) },
       help(nmu,nphi);
  // these are work fields used for runge kutta +
      momentum
  for(int which = 0; which < 3; which++)
  {
    scratch[which] = field(nmu,nphi);
    if(runge)
    {
      fa[which] = field(nmu,nphi);
      fb[which] = field(nmu,nphi);
      fc[which] = field(nmu,nphi);
```

```
      fta[which] = field(nmu,nphi);
      ftb[which] = field(nmu,nphi);
      ftc[which] = field(nmu,nphi);
    }
    momentum[which] = field(nmu,nphi);
  }
  divergence = field(nmu,nphi);
  // start point has r = r0
  // the following sets up the source, using gaussians in
       place
  // of delta functions. if orbit, the source particle is
       moving in a
  // circular orbit with radius r0 when the emission
       occurs
  betai = 1-R/r0;

  source_p.xr = r0; source_p.xp = pi;
  gamma = pow(1-1.5*R/r0,-2);
  source_p.vr = 0; source_p.vp = sqrt(R/2/r0)/r0;
  if(!orbit) {
    gamma = sqrt(betai);
    source_p.vr = 0;
    source_p.vp = 0;
  }

  // Light cone will be initialized half way through the
  // source production
  Plist light_cone;

  // display At by default (perhaps Ftr would be better?)
  field * disp = f;
  if(mode == 2) disp = momentum;

  // initialize force to zero
  fr = 0; fp = 0;

  printfield(f[0], light_cone, norm);
  double lasttime = 0;
  while(maxtime < 0 || count*dt <= maxtime)
  {
    int ci; char c;
    // very ugly, but I don't care, since I'm near
         finished anyway
    bleh:
    ci = getch(); c;
    if(ci == ERR) goto work;
    c = (char)ci;
    switch(c)
    {
```

```
case 'l': printlog ^= 1; break;
case 'f': disp = f; mode = 0; break;
case 'g': disp = g; mode = 0; break;
case 'h': disp = h; mode = 0; break;
case 'e': disp = momentum; mode = 2; break;
case 'D': mode = 3; break;
case 't': component = 0; break;
case 'r': component = 1; break;
case 'p': component = 2; break;
case '1': Fa = 0; Fb = 1; mode = 1; break;
case '2': Fa = 0; Fb = 2; mode = 1; break;
case '3': Fa = 1; Fb = 2; mode = 1; break;
case 'n': norm ^= 1; break;
case 'm': muprint ^= 1; break;
case '+': skip += 10; break;
case '*': skip += 100; break;
case '0': skip = 0; break;
case 'i': interval++; break;
case 'j': interval--; if(interval <= 0) interval = 1;
    break;
case '9': logfact += 1; break;
case '8': logfact -= 1; break;
case 'd': printpage = true; break;
case ' ': nodel ^= true; nodelay(win,nodel); break;
case '>':
  maxrR /= 1.2;
  cpgsci(maxpos);
  cpgenv(-0.5,maxrR*R,-0.5,maxrR*R,1,1);
  calcmurR();
  break;
case '<':
  maxrR *= 1.2;
  cpgsci(maxpos);
  cpgenv(-0.5,maxrR*R,-0.5,maxrR*R,1,1);
  calcmurR();
  break;
case 'c': lcprint ^= 1; break;
case 'q': goto end;
}
if(mode == 0 || mode == 2) printfield(disp[component
   ], light_cone, norm);
else if(mode == 1)
{
  curl(f, g, help, Fa, Fb);
  printfield(help, light_cone, norm);
}
goto bleh;
work:
// update physics
```

```
if(runge) runge_kutta(f,g);
else
{
  Ftt(f,g,h,count*dt);
  for(int which = 0; which < 3; which++)
  {
    for(int i = 0; i < f[which].dim(); i++)
      g[which][i] += h[which][i]*dt;
    for(int i = 0; i < f[which].dim(); i++)
      f[which][i] += g[which][i]*dt;
  }
}
if(mode == 3) calc_momentum(f,g);
// no gradual emission
//int offset = (int)(4*sqrt(betai)*ri/dt);
int offset = 0;
// damping. as usual this only seems to make things
   worse, so
// it is turned off by default
for(int which = 0; which < 3; which++)
for(int j = 0; j < f[which].phidim(); j++)
{
  for(int i = 0; i < damprad; i++)
  {
    f[which](i,j) *= 1-maxdamp*dt*(damprad-i)/damprad
       ;
    f[which](f[which].mudim()-1-i,j) *=
      1-maxdamp*dt*(damprad-i)/damprad;
  }
}
if(count == offset)
  light_cone = LC_init(source_p.xr, source_p.xp, 200)
     ;
else if(count > offset)
{
  LC_prop(light_cone);
  LC_more(light_cone, R*R/400);
}
if(orbit) prop_particle(source_p);

// accumulate force. slow and inefficient, but pah!
// It creates a distribution representing the
   particle
// in an area around it, computes the field
   components
// in that area, and collects the force
if(force)
{
  if(!cum) { fp = 0; fr = 0; }
```

```
    double sfact = min(r0*dnu2phi[nphi-1],(dists[nmu/2+1]-dists[nmu/2-1])/2);
    int size = (int)ceil(3*ri/sfact);
    int ext = 3+2*size;
    int sm = (int)r2mu(source_p.xr),
      sp = (int)phi2nu(fixphif(source_p.xp));
    double h = R*maxrR/5;
    field pdist(ext, ext);
    makedelta(pdist, source_p, sm, sp);
    field c;
    // get amplitude at particle. will use
    // the actually computed Fs instead of the displayed
    // ones, for simplicity
    if(mode == 0 || mode == 2)
    {
      small = cutfield(disp[component],sm,sp,ext);
      part_amp = inner_product(pdist,small,sm,sp);
    }
    c = curl_area(f,g, 1, 0, sm,sp,size);
    raise_or_lower(c,c,0,0, sm, sp);
    double tmp = inner_product(pdist, c, sm, sp);
    fr += gamma*tmp;
    if(mode==1 && Fa == 0 && Fb == 1) { small = c; part_amp = tmp; }
    c = curl_area(f,g,1, 2,sm,sp,size);
    raise_or_lower(c,c,2,0, sm, sp);
    tmp = inner_product(pdist,c,sm,sp);
    fr += gamma*source_p.vp*tmp;
    if(mode==1 && Fa == 1 && Fb == 2) { small = c; part_amp = tmp; }
    c = curl_area(f,g,2, 1,sm,sp,size);
    raise_or_lower(c,c,1,0, sm, sp);
    tmp = inner_product(pdist,c,sm,sp,true);
    fp += gamma*source_p.vr*tmp;
    c = curl_area(f,g,2, 0,sm,sp,size);
    raise_or_lower(c,c,0,0, sm, sp);
    tmp = inner_product(pdist,c,sm,sp,true);
    fp += gamma*tmp;
    if(mode==1 && Fa == 0 && Fb == 2) { small = c; part_amp = tmp; }
    direct_product(small, pdist, sm, sp, false);
    fprintf(forcefile, "%17.7le%15.7le\n", count*dt, fr);
  }

  // display stuff
  bool thistime = false;
  if(every > 0 && count*dt >= lasttime)
```

```
    {
      thistime = true;
      lasttime += every;
      printpage = true;
    }
    if(thistime || (skip <= 0 && !(count%interval)))
    {
      if(mode == 2) calc_momentum(f,g);
      if(calcE)
      {
        E = field_energy(f,g);
        printf("%15.7lf %15.7le\n", count*dt, E);
      }
      if(printpage) // redirect to page
      {
        cpgslct(psfile);
        if(firstpage) firstpage = false;
        else cpgpage();
        cpgsci(maxpos);
        cpgenv(-0.5,maxrR*R,-0.5,maxrR*R,1,1);
      }
      if(mode == 0||mode==2) printfield(disp[component],
          light_cone, norm);
      else if(mode == 1)
      {
        curl(f, g, help, Fa, Fb);
        printfield(help, light_cone, norm);
      }
      else printfield(divergence, light_cone, norm);
      if(printpage) // redirect back
      {
        cpgslct(screen);
        printpage = false;
      }
    }
    else skip--;
    fprintf(stderr, ".");
    count++;
  }
  end:
  fclose(forcefile);
  endwin();
  cpgend();
  return 0;
}
double *mu2r, *dists, *nu2phi, *dnu2phi;
double murR;
int nmu, nphi;
double wsize, R, r0;
```

```
  double maxrR, dt, dt0, ri, betai, gamma;
  double logfact, maxdamp, E, polar, foo, deltalim, div;
  double maxtime, every;
  bool muprint, printlog, norm, lcprint, orbit, force, cum,
       calcE;
  bool printtime, runge;
  int mumin, mumax, component, skip, count, Fa, Fb, damprad
     ;
  int interval, mode;
  int buflen;
  particle source_p;
  double fr, fp;
  char * buffer;
  field scratch[3], fa[3], fb[3], fc[3], fta[3], ftb[3],
     ftc[3], small,
    momentum[3], divergence;
  double part_amp;
  char * filepg;
  FILE * forcefile;
};

int main(int argn, char ** args)
{
  Simulator sim;
  sim.main(argn, args);

  return 0;
}
```

Bibliography

[1] Kevin Brown. Math pages. http://www.mathpages.com/home/kmath242/kmath242.htm.

[2] L. M. Burko. Self-force on a particle in orbit around a black hole. *Phys. Rev. Lett.*, 84:4529–4532, 2000.

[3] L. M. Burko and E. Poisson A. I. Harte. Mass loss by a scalar charge in an expanding universe. *Phys. Rev. D*, 65:124006, 2002.

[4] S. Detweiler and B. F. Whiting. Self-force via a green's function decomposition. *Phys. Rev. D*, 67:024025, 2003.

[5] B. S. DeWitt and R. W. Brehme. Radiation damping in a gravitational field. *Annals of Physics*, 9:220, 1960.

[6] Cecile Morette DeWitt and Bryce S. DeWitt. Falling charges. *Physics*, 1:3–20, 1964.

[7] Paul Dirac. Classical theory of radiating electrons. *Proc. R. Soc. London, Ser. A*, 167, 1938.

[8] E. E. Flanagan and R. M. Wald. Does back reaction enforce the averaged null energy condition in semiclassical gravity? *Phys. Rev. D*, 54:6233–6283, 1996.

[9] W. Fugmann and M. Kretzschmar. *Il Nuovo Cimento B*, 106:351, 1991.

[10] D. V. Gal'tsov. Radiation reaction in various dimensions. *Phys. Rev. D*, 66:025016, 2002. arXiv:hep-th/0112110v1.

[11] J. Hadamard. *Lectures on Cauchy's problem in linear partial differential equations*. Yale University Press, 1923.

[12] J. M. Hobbs. A vierbein formalism of radiation damping. *Annals of Physics*, 47:141, 1968.

[13] Eric Poisson. The motion of point particles in curved spacetime. *Living Rev. Relativity*, 7, 2004. URL: http://www.livingreviews.org/lrr-2004-6, cited Sept. 2007.

[14] A. G. Smith and Clifford M. Will. Force on a static charge outside a schwarzschild black hole. *Phys. Rev. D*, 22:1276, 1980.