The $\nu = 1$ quantum Hall effect and its one-dimensional representation

by

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Preface

This thesis is devoted to the relationship between the quantum Hall system and its one-dimensional representation. The quantum Hall system, which consists of electrons confined to two dimensions and in a strong transverse magnetic field, has an edge which can be described by a one-dimensional Luttinger model. The Luttinger model has the property that it can be bosonized, that is, it can be rewritten as theory of non-interacting bosons, even if the electrons in the model are interacting. This property makes it possible to exactly calculate its correlation function, which in the asymptotic limit has the form of a power law. The general theory for the quantum Hall edge makes the prediction that the power law exponent will have a universal value determined by the bulk state. Tunneling experiments have verified the Luttinger description of the edge, in the sense of a power law behaviour of the correlation function. However, the experimental values of the exponent deviate from the prescribed universal value. Several numerical works suggest that the discrepancy can be explained by taking into account the electron interaction.

This thesis consists of an introductory part where the general background is introduced. My contribution to the field is found in three papers and a chapter. Two of the papers investigate the relationship between the quantum Hall system and its one-dimensional representation from a microscopical point of view. The effect of the interaction is taken into account. In the third paper I study the effects of interaction induced second Landau level mixing. A chapter is devoted to the discussion of fractional charges in the Luttinger model, and the discussion is exemplified by applying the theory to the quantum Hall model discussed in the papers.

How this thesis is organized

Chapter 1 is an introduction to the quantum Hall effect and the necessary concepts are introduced. Chapter 2 is an introduction to the Luttinger model. The bosonization of the model is developed in great detail. Then, in Chapter

3 we will see how the formalism developed in Chapter 2 can be applied to the edge of the quantum Hall system. The Hartree-Fock method, which has been used in the Paper III, is presented in Chapter 4. Chapter 5 is devoted to ongoing research on fractional excitations in the Luttinger model. We will see how the the quantum Hall model introduced in Paper I and II can be used to exemplify the discussion in the literature and give valuable insight not so easily seen in the one-dimensional case.

In Paper I we show how a $\nu = 1$ quantum Hall system can be mapped onto a one-dimensional representation. In the low energy limit the system takes the form of a Luttinger model, and the Luttinger parameters are calculated. The mapping is explicit in the sense that a microscopic description of the system is used, in contrast to the macroscopic arguments mostly used in the literature. The study is followed up in Paper II where we map the one-dimensional correlation function back to the two-dimensional description. The effect of the interaction on the correlation function and the density profile is discussed. Paper III is a numerical study where I investigate the effect of letting the interaction induce mixing with the second Landau level in a $\nu = 1$ quantum Hall model. It is shown that the asymptotic behaviour of the correlation function is not affected by the interaction. However, the interaction gives rise to oscillations in the density profile.

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Joakim Bergli has always been willing to answer and discuss my questions, I really appreciate that. I'm grateful to Juha Suorsa for reading and giving me valuable feedback on my thesis. I'm happy to have shared office with Eirik Ovrum and Per Øyvind Sollid, thank you both for a nice time at Ø469! I have really enjoyed my years at the Theory Group and I would like to thank both the current and former theoreticians there for making it such a nice place to be.

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List of papers

Paper I: Mats Horsdal and Jon Magne Leinaas,
Explicit mapping between a two-dimensional quantum Hall system and a one-dimensional Luttinger liquid. I. Luttinger parameters,
Phys. Rev. B **76**, 195321 (2007)

Paper II: Mats Horsdal and Jon Magne Leinaas,
Explicit mapping between a two-dimensional quantum Hall system and a one-dimensional Luttinger liquid. II. Correlation functions,
Phys. Rev. B **76**, 195322 (2007)

Paper III: Mats Horsdal,
Effects of interaction induced second Landau level mixing in the $\nu = 1$ quantum Hall effect,
Submitted to Phys. Rev. B

Contents

Part I

Introduction

Chapter 1

The quantum Hall effect

This chapter introduces the quantum Hall effect (QHE), a fascinating effect which takes place in a relatively simple system that we still do not fully understand. Most of my work presented in this thesis is related to some of the mysteries found in this system. The presentation here is mainly based on review articles and Ph.D. theses [1, 2, 3, 4].

1.1 Experimental verification

The QHE is a striking phenomenon that occurs in a two-dimensional electron gas with a strong transverse magnetic field and at low temperatures ($T \leq 4$K). The 2D electron gas is confined to the interface between two semiconductors or between a semiconductor and an insulator. The experimental setup is sketched in Figure 1.1, which shows the Hall bar with the 2D electron gas. An electrical field is applied in the vertical direction and this gives rise to a horizontal current. The longitudinal and transverse resistance, $R_{xx} = \frac{V_H}{I}$ and $R_{xy} = \frac{V_H}{I}$, respectively, can then be measured. The effect, which is now called the integer QHE (IQHE), was first discovered by von Klitzing, Dorda and Pepper in 1980 in a very clean sample [5]. They found that the transverse resistance as a function of the magnetic field exhibits plateaus in certain regions of the magnetic field. In the same regions of B the longitudinal resistance vanishes. The plateaus were given by $R_{xy} = \frac{1}{\nu}\frac{h}{e^2}$, where h is Planck's constant, $-e$ is the electron charge and ν is an integer. Two years later Tsui, Stormer and Gossard [6] performed an experiment in a cleaner sample and discovered the fractional QHE (FQHE); that ν could take fractional values. Data from an experiment which shows the IQHE is shown in Figure 1.2. We can clearly see the plateaus in ρ_{xy} (R_{xy}) and the absence of horizontal resistivity at the plateaus.

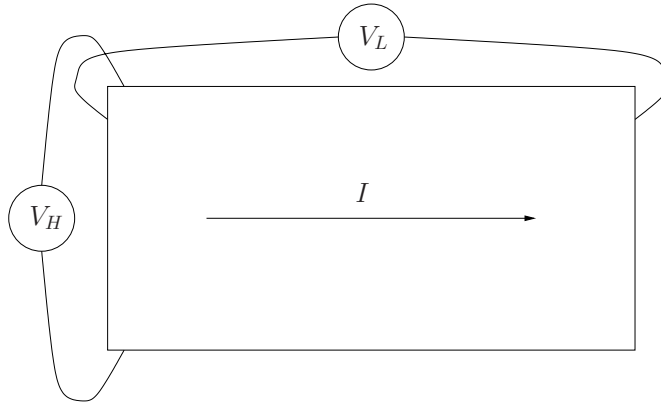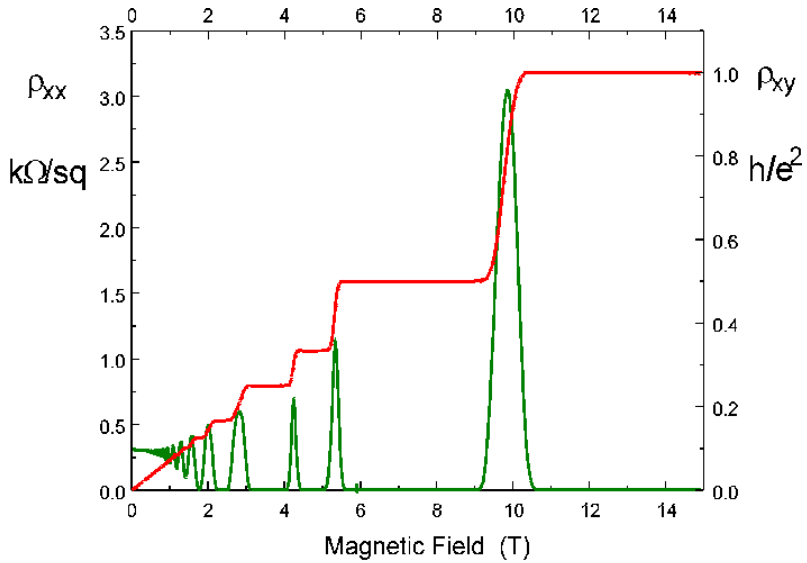Figure 1.1: The geometry of a quantum Hall experiment.

Figure 1.2: Data from an IQHE experiment at 30mK. The resistivities ρ_{xx} and ρ_{xy} are shown as functions of the magnetic field. Courtesy of D.R. Leadley, Warwick University 1997.

The quantization of R_{xy} in experiments is extremely precise and universal, which means that it is not sensitive to the microscopic details of the sample or its size. In fact, the value of h/e^2 can be determined to a precision of at least 1 part in 10^7. The reason for this universality lies in the two dimensional nature of the system. To see this we can consider the resistivity tensor, $\rho_{\alpha\beta}$, which is a local quantity. The resistivity and its inverse, the conductance, $\sigma_{\alpha\beta}$, are related to the current density, $\mathbf{j}$, and electric field, $\mathbf{E}$, as

$$E_\alpha = \rho_{\alpha\beta} j_\beta\,, \quad j_\alpha = \sigma_{\alpha\beta} E_\beta\,.$$

In two dimensions resistivity and resistance have the same dimension. This means that the relationship between them can depend only on the shape and not on the size of the sample. For a rectangular geometry in the case where $\rho_{xx} = 0$, which is the case on a plateau, it turns out that $R_{xx} = \rho_{xx}$ and $R_{xy} = \rho_{xy}$. This explains why such an amazing precision can be achieved in the experiments; the results are not sensitive to the fine details of the sample and the measured quantity is actually the resistivity, which is a local quantity.

1.1.1 The classical Hall conductance

Classically, a plateau in the transverse resistivity is not what we would expect. By assuming Lorentz invariance only, we will now show that the classical resistivity varies linearly with the magnetic field.

Assume that we have electrons confined to the xy-plane and in a transverse magnetic field $\mathbf{B}$. Let us perturb the system with a homogeneous electric field $\mathbf{E}$ in the xy-plane. The electric field is assumed to be weak in the sense that $\frac{E}{cB} \ll 1$. If we assume that there is no preferred reference frame, we can make a Lorentz transformation to a reference frame S' moving with velocity $\mathbf{v} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$. The electric and magnetic field in the system S' is given by the transformation properties of electric and magnetic fields,

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \frac{\gamma - 1}{v^2}(\mathbf{E} \cdot \mathbf{v})\mathbf{v}$$

$$\mathbf{B}' = \gamma\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) - \frac{\gamma - 1}{v^2}(\mathbf{B} \cdot \mathbf{v})\mathbf{v}\,,$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$, as usual. Using these transformations it is easy to show that $\mathbf{B}' \approx \mathbf{B}$, to first order in v/c, and $\mathbf{E}' = 0$. Since the electric field vanishes in S', there cannot be a current in this reference frame, $\mathbf{j}' = 0$. From

the transformation properties for charge density, ρ, and current density,

$$\mathbf{j} = \mathbf{j}' + (\gamma - 1)\frac{\mathbf{v} \cdot \mathbf{j}'}{v^2}\mathbf{v} + \gamma\rho'\mathbf{v}$$
$$\rho = \gamma\left(\rho' + \frac{\mathbf{v} \cdot \mathbf{j}'}{c^2}\right),$$

one finds that $\mathbf{j} = \rho\mathbf{v}$ in the lab system. By writing $\rho = -en$, where n is the electron density in the plane, one sees that $\mathbf{j} = \frac{ne}{B}\hat{\mathbf{z}} \times \mathbf{E}$. This expression can be rewritten in terms of the conductivity tensor as $j_\alpha = \sigma_{\alpha\beta}E_\beta$, where the conductivity tensor can be written as

$$\sigma = \nu\frac{e^2}{h}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.1}$$

where we have defined $\nu = \frac{nh}{eB}$. The resistivity tensor is the inverse of the conductivity tensor,

$$\rho = \frac{1}{\nu}\frac{h}{e^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We see that ρ_{xy} varies linearly in B, in contradiction to what is found in the experiments in a strong magnetic field and at low temperature.

Let us try to interpret the quantity $\nu = \frac{nh}{eB}$ that was introduced in (1.1). From the definition of the magnetic flux quantum $\Phi_0 = \frac{h}{e}$, we see that ν is the ratio of the number of electrons to the number of magnetic flux quanta in the system. A plateau would appear if this ratio could be kept constant under variation of the magnetic field.

Our derivation is general, but it relies on the assumption that there is no preferred reference frame (and that the particle number is constant, which is the case in most experiments). This means that the Lorentz invariance must be broken for a plateau to appear. The quantum Hall effect takes place in very clean samples, but from the argument given here it cannot be completely clean, because that would make the system Lorentz invariant, and the plateaus would disappear. Therefore a small concentration of impurities is essential for the effect to take place.

1.2   Electrons in a magnetic field

To get an understanding of the QHE it is good idea to start with the description of just a single particle in a magnetic field. As we have already discussed, the two dimensionality is very important for the realization of

these phenomena. We therefore write down the Lagrangian for an electron in two dimensions in a perpendicular magnetic field with vector potential $\mathbf{A}$,

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\dot{\mathbf{r}} \cdot \mathbf{A} \tag{1.2}$$

From the Lagrangian the Hamiltonian is easily derived:

$$H = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}\right)^2$$

We will make a choice of gauge, namely the Landau gauge or the linear gauge, $\mathbf{A}_x = -yB$ and $\mathbf{A}_y = 0$. Other gauge choices are also possible, e.g. the circular gauge, which is widely used. Also we assume that $eB > 0$. The Hamiltonian can then be written as

$$H = \frac{1}{2m}\left(p_x + yeB\right)^2 + \frac{1}{2m}p_y^2. \tag{1.3}$$

We see that this Hamiltonian is translationally invariant in the x-direction, which means that the x-momentum can be quantized as $p_x = \hbar k$. By substituting this into (1.3) a new length scale introduces itself; the magnetic length, $\ell_B = \sqrt{\frac{\hbar}{eB}}$. Another quantity also appears naturally; the cyclotron frequency, $\omega_c = \frac{eB}{m}$, which we recognize as the angular frequency of a particle of mass m and charge e undergoing circular motion in a magnetic field B. The Hamiltonian then takes the form

$$H = \frac{1}{2m}p_y^2 + \frac{1}{2}m\omega_c^2\left(y + \ell_B^2 k\right)^2,$$

which we recognize as a one-dimensional harmonic oscillator shifted away from the origin. The eigenstates of this Hamiltonian are given by $\psi_n(y + \ell_B^2 k)$ where ψ_n is the eigenstate of a one-dimensional harmonic oscillator of quantum number n. If we assume that the system is periodic in the x-direction with period L, then $k = \frac{2\pi}{L}m$, where $m \in \mathbb{Z}$, and the two dimensional wavefunction is given by:

$$\psi_{kn}(x, y) = L^{-1/2}e^{ikx}\psi_n(y + \ell_B^2 k). \tag{1.4}$$

We see that this state corresponds to a plane wave in the x-direction and a harmonic oscillator state in the y-direction centered at $y_k = -\ell_B^2 k$. The correspondence between the y-coordinate and the wave vector is shown i Figure 1.3, which shows a geometric representation of these states. By acting with (1.3) on this state we see that it has energy $E_{kn} = \hbar\omega_c\left(n + \frac{1}{2}\right)$, i.e. it is completely degenerate in the quantum number k. This means that if the
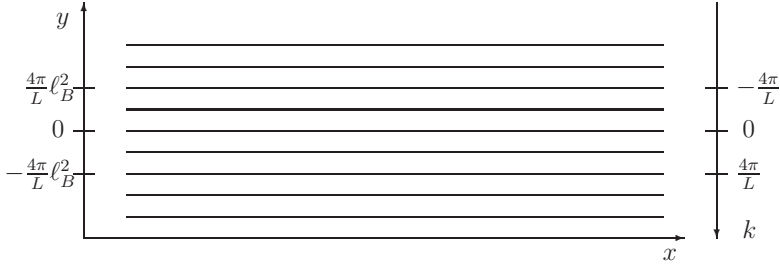
Figure 1.3: A geometrical representation of the eigenvectors (1.4). The states correspond to plane waves with wave vector k in the x-direction, and harmonic oscillator states centered at $y_k = -\ell_B^2 k$ in the y-direction. The lines in the figure show where the states are centered in the y-direction. We clearly see the correspondence between the y-coordinate and the quantum number k.

system has infinite extension in the y-direction, there is an infinite number of states with the same energy. The different harmonic oscillator states is called Landau levels, where the $n = 0$ is called the lowest Landau level (LLL), $n = 1$ the second Landau level and so forth.

   We also note that if the magnetic field is very strong, the electron will be confined to the LLL. This means that the electron state is given by a single quantum number only, which in this case is the wavevector k. The system is therefore effectively one-dimensional. This fact has been used in Paper I and II, where we map a quantum Hall model onto its one-dimensional representation.

   This one-dimensionality can also be seen in another way. We know that the energy is degenerate within each Landau level, which means that all electrons have the same kinetic energy. This suggests that the kinetic term can be ignored. If we rewrite the Lagrangian (1.2) in the Landau gauge and ignore the kinetic term, the Lagrangian takes the form $L = -\hbar \ell_B^{-2} \dot{x} y$, where the definition of the magnetic length has been used. The momentum canonical to x is then given by $p_x = -\hbar \ell_B^{-2} y$. If we naively quantize this theory by $[x, p_x] = i\hbar$, we find that

$$[x, y] = -i\ell_B^2 \,.$$

The x and y variable no longer commute! The commutator resembles the commutator between x and p for a one-dimensional system. This means that a state vector, $\psi(x, y)$, can be viewed as wave function defined on phase

space instead of the configuration space, just like a coherent state. We then know that there exists an alternative formulation where the wavefunctions only depend on one variable, e.g. $\psi(x)$.

1.2.1 Number of states in a Landau level

In the real world the Hall bar is of course not infinite in size, this means that there is a finite number of of electron orbitals in each Landau level. Let us see if we can count them:

First, let us assume that the system has a length L in the x-direction, and a width W in the y-direction. By placing the x-axis along the center of the bar, the two edges will have positions $y_{\text{edge}} = \pm W/2$. The two highest momentum states will be characterized by $k = \pm k_F$, the Fermi momenta. From the correspondence between the k and y-space, we see that $y_{\text{edge}} = \pm \ell_B^2 k_F$, which gives $k_F = W/2\ell_B^2$. The number of states is given by $N = \sum_{k=-k_F}^{k_F} 1$, which in the large L limit can be written;

$$N = \frac{L}{2\pi} \int_{-k_F}^{k_F} dk = \frac{L}{2\pi} 2k_F = \frac{WL}{2\pi\ell_B^2} = WLB\frac{e}{2\pi\hbar},$$

where the definition of the magnetic length have been used in the last line. We recognize WLB as the total magnetic flux through the sample and $\frac{2\pi\hbar}{e} = \frac{h}{e}$ as the magnetic flux quantum. This means that the number of states in a Landau level is given by

$$N = \frac{\Phi}{\Phi_0} = N_{\Phi_0} \,.$$

That is, the number of states is equal to the number of magnetic flux quanta penetrating the bar; there is on state per magnetic flux quantum.

We now see that the quantity $\nu = \frac{N_e}{N_{\Phi_0}}$, that was introduced in (1.1), can also be interpreted as the ratio of the number of electrons to the number of available states. We will call this ratio the filling fraction. The filling fraction tells us how many of the of the available states that are actually occupied: $\nu = 1$ means that the whole LLL is filled, $\nu = 2$ means that both the LLL and the second Landau level is filled, $\nu = \frac{1}{3}$ means that only one third of the LLL is filled. We see that when a Landau level is completely filled there is an excitation gap to all excited states.

1.2.2 The integer quantum Hall effect

From the previous discussion it is not clear how the IQHE plateaus can appear; when the magnetic field is changed slightly the filling fraction changes
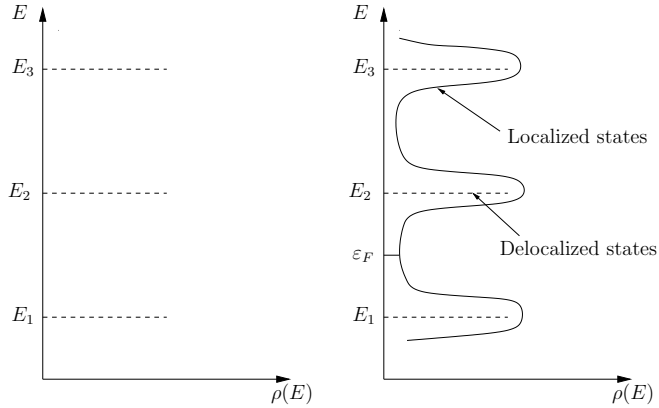
Figure 1.4: The left figure shows the density of states of a system without impurities. We see that the Landau levels corresponds to delta functions. Impurities will give rise to a broadening of the Landau levels as shown in the figure to the right. Only the original Landau levels corresponds to delocalized states, while the states that originates from the impurities are localized.

and no plateau is possible. This is where the impurities come in [7, 8]. The left figure in Figure 1.4 shows the density of states of a clean sample; the Landau levels can clearly be seen as deltafunctions situated at E_1, E_2 etc. Impurities in the sample will give rise to a broadening of the Landau levels as shown in the right figure. The states that originate from the impurities are localized. This means that electrons that occupy these states do not contribute to the transport properties of the system. Only the delocalized states, which correspond to the original Landau levels, contribute to the transport.

We can now understand how a plateau can appear: Assume that the Fermi level, ε_F, lies in a mobility gap as shown in Figure 1.4. Assume that the Fermi level is increased, the transport properties will not change because only the number of occupied localized states increases. But as soon as the Fermi level crosses one of the Landau levels (localized at E_2 in this example), we will get a transition where more electrons contribute to the transport, and we have a new plateau.

In an experimental setting the magnetic field is varied, and from the picture presented here we see that the localized states act as electron reservoirs. As the magnetic field is increased available states open up in the Landau levels, but since there exist occupied localized states with higher energies,

these states provide electrons for the available delocalized states.

The FQHE can be explained in a similar manner, but for this explanation to work we see that we need a mobility gap. How can there be a gap when a Landau level is only partially filled? From what we have seen here the ground state will even be degenerate, since all single particle orbitals in a Landau level have the same energy. The solution to this mystery is the electron interaction, which we have ignored until now.

1.3 The fractional quantum Hall effect

For $\nu = \frac{1}{m}$, m odd, all particle orbitals are confined to the lowest Landau level. Since these orbitals are degenerate, the total energy will effectively be given by the repulsive Coulomb interaction, $\frac{e^4}{4\pi\epsilon r}$.

When discussing these states it is more convenient to work in the symmetric gauge, $\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}$, which preserves rotational symmetry about the origin. This means that the single particle orbitals will also be eigenstates of the angular momentum operator. It can be shown that the energy eigenstates in the LLL are given by

$$\varphi_m(z) = \frac{1}{\ell_B \sqrt{2\pi 2^m m!}} z^m e^{-|z|^2/4},$$

where the dimensionless complex number $z = (x + iy)/\ell_B$ represents the position of the particle in the xy-plane and $m \geq 0$ is an integer. The angular momentum of this state is $\hbar m$. We also see that the peak of $\varphi_m(z)$ is located at a radius of $R = \sqrt{2m}\ell_B$.

The ground state of a $\nu = 1$ system of N electrons is a Slater determinant with all $m < N$ orbitals occupied. This corresponds to a circular shaped quantum Hall droplet of radius $\sqrt{2(N-1)}\ell_B$. It can be shown that this Slater determinant can be written as (up to a normalization factor)

$$\Psi_1(\{z\}) = \prod_{i<j}^{N}(z_i - z_j)e^{-\frac{1}{4}\sum_{j=1}^{N}|z_j|^2},$$

where $\{z\} = (z_1, z_2, .., z_N)$.

Laughlin [9] suggested that the ground state for the $\nu = \frac{1}{m}$, m odd, FQHE could be written as

$$\Psi_{1/m}(\{z\}) = \prod_{i<j}^{N}(z_i - z_j)^m e^{-\frac{1}{4}\sum_{j=1}^{N}|z_j|^2}. \tag{1.5}$$

Let us discuss the form of this wavefunction. Since m is odd it is clearly antisymmetric, which is required by the Pauli principle. The electrons are effectively kept apart, since the wavefunction vanishes quickly as $z_i \rightarrow z_j$ due to the factor $(z_i - z_j)^m$. The wavefunction therefore includes the effects of the interaction. The highest order of z that appears in (1.5) is $m(N-1)$, which gives a droplet radius of $R = \sqrt{2\pi m(N-1)}\ell_B$. This droplet contains $\frac{\pi R^2 B}{\Phi_0} = m(N-1)$ flux quanta, by the use of the definition for the magnetic length and Φ_0. In the thermodynamic limit this gives a filling factor of $\nu = \frac{1}{m}$.

For the Laughlin state it can be shown that there is a finite gap to excitations, and that the elementary excitations are quasi-holes and quasi-particles with fractional charge $\frac{1}{m}$ [9, 10, 1]. The fact that the excitations have fractional charge means that they also have fractional statistics [2, 11]. Anyons, particles with fractional statistics, were first discussed by Leinaas and Myrheim [12]. The same mechanism that gave rise to the plateaus in the IQHE can now be used to explain the FQHE plateaus: As the magnetic field is increased (decreased) quasi-holes (-particles) will be created, but, as in the IQHE, these holes (particles) will be trapped by the impurities, and will therefore not contribute to the charge transport. The ratio of the number of conducting states to the number of flux quanta is therefore unchanged, and we have a plateau. The value of the Hall conductance at a plateau follows from a gauge invariance argument that we will not go into here [7, 9]. It can be shown that the Laughlin state is incompressible [9]. That the ground state can be viewed as an incompressible quantum Hall fluid will be used when we discuss edge excitations in Chapter 3.

But does the Laughlin state have anything to do with the real world? The Laughlin state is not the ground state of a Hamiltonian with Coulomb interaction. The exact ground state of such a Hamiltonian can be calculated numerically for a limited number of particles. It turns out that the overlap between the exact ground state and the Laughlin state is very large. Even more, the overlap between the Laughlin state and the exact ground states with other repulsive interactions is also very large [9]. This shows that the interaction type is not very important for the effect to take place, and that the Laughlin state is a very good approximative wavefunction that captures the essential physics.

1.3.1 Composite fermions

Lauglins wavefunction is a good approximation to the $\nu = \frac{1}{m}$ ground state. But what about other fractions? Jain have proposed wavefunctions that easily captures the essential physics of the $\nu = \frac{n}{2mn+1}$ states, where $m \geq 0$

and $n > 0$ are integers [13, 14]. Jain's idea is that the $\nu = \frac{n}{2mn+1}$ FQHE can be thought of as an IQHE of socalled composite fermions (CF), which can be viewed as electrons with an even number of flux quanta attached.

The Lauglin state (1.5) can be written as $\Psi_{1/(2m+1)} = D^m \Psi_1$, where Ψ_1 is the $\nu = 1$ ground state, m is an integer and the Jastrow factor is given by

$$D^m \equiv \prod_{j<k}^{N} (z_j - z_k)^{2m}.$$

Let us fix all z_i except z_1. If z_1 is brought in a loop around any other electron it can be shown that D^m will contribute with a phase $4m\pi$. Since the Aharanov-Bohm phase acquired when an electron is brought in a loop around a unit flux quanta is 2π, Jain suggests that we can think of D^m as effectively attaching $2m$ flux quanta to each electron. We can then view the $\nu = \frac{1}{2m+1}$ state as the $\nu = 1$ ground state of non-interacting CFs, that is electrons with $2m$ flux quanta attached. Jain then generalizes this picture and looks at the state $\Psi_\nu = D^m \Psi_n$, where Ψ_n, n integer, is the $\nu = n$ ground state. This state will correspond to the ground state of n completely filled CF Landau levels. But which electronic filling fraction does this state correspond to? The electrons in the system will see a different magnetic field than the CFs. From a mean field point of view an electron will see both the magnetic field arising from all the flux quanta attached to the CFs, as well as the magnetic background field in the CF system, B^*. The total magnetic flux in the electron system is therefore given by

$$\Phi = \Phi^* + 2mN\Phi_0 \, ,$$

where Φ and Φ^* is the total magnetic flux penetrating the electron droplet and the CF droplet, respectively. From the definition of the filling fraction we see that the electron filling fraction is given by

$$\nu = \frac{n}{2mn+1} \, .$$

At first sight the wavefunction $D^m \Psi_n$ doesn't seem like a good candidate for a ground state in the $\nu = \frac{n}{2mn+1}$ case, since it obviously includes mixing with higher Landau levels. However, the Jastrow factor is very efficient at pushing the electrons down to the LLL [14]. To get a state completely in the LLL, Jain's suggestion for the $\nu = \frac{n}{2mn+1}$ state is therefore

$$\Psi_\nu(\{z\}) = \mathcal{P} \prod_{j<k}^{N} (z_j - z_k)^{2m} \Psi_n(\{z\}). \tag{1.6}$$

where $\mathcal{P}$ is an operator that projects the state onto the LLL. We see that the gap to excited states is easily accounted for since the only possible excitations involve the excitation of one or more CFs from the completely filled Landau levels to the non-occupied ones. The Jastrow factor effectively keeps the CFs apart and compensates for the CF interaction; as z_j approaches z_k we see that $\Psi_\nu^{\text{CF}}(\{z\})$ goes to zero at least as fast as $(z_j - z_k)^{2m}$. Therefore the CF-interaction can usually be ignored, and the CFs can be viewed as non-interacting particles.

We see that the CF picture reduces the complicated problem of interacting electrons in a partially filled LLL to the more or less trivial picture of non-interacting CFs in completely filled Landau levels. However, this picture is mainly based on arguments and is difficult to verify directly. But one can compare the ground states constructed with the CF approach with exact ground states: The exact ground states of $\nu = \frac{n}{2mn+1}$ systems can be calculated numerically for a limited number of electrons. It can be shown that the overlap between the exact ground state and the CF constructed wavefunction is very large, which suggests that the Jain states are good approximations to the real ground states. In Ref. [14] numerical results involving eight particles is presented. We also note that the Laughlin series emerges as a special case of the Jain series; it can be translated into a filled CF LLL only.

Chapter 2

The Luttinger model and bosonization

The bosonization technique has been used extensively in my work. In this chapter we will see how this technique makes it possible to study the low energy limit of a one-dimensional system. The presentation given here is mainly based on the original paper by Haldane [15], but some ideas are taken from Ref. [16].

2.1 The low energy limit

Let us start with the quadratic Hamiltonian for a non-interacting and non-relativistic one-dimensional system:

$$H_0 = \int_{-L/2}^{L/2} d\xi \, \psi^\dagger(\xi) \frac{(-i\hbar\partial_\xi)^2}{2m} \psi(\xi).$$

If we assume periodic boundary conditions of period L, the field operator can be written as $\psi(\xi) = L^{-1/2} \sum_k e^{ik\xi} c_k$, where c_k is a fermion annihilation operator and $k = \frac{2\pi}{L}n$, $n \in \mathbb{Z}$. In terms of this expression the Hamiltonian takes the form

$$H_0 = \frac{\hbar^2}{2m} \sum_k k^2 c_k^\dagger c_k.$$

The dispersion is quadratic in the wavenumber k. Figure 2.1 shows the particle energy as a function of k. The ground state consists of all states between the Fermi-points, $\pm k_F$, occupied. We define the Fermi point to lie in between the highest occupied level and the lowest non-occupied level in the ground state of the system. In terms of the number of particles N_0,
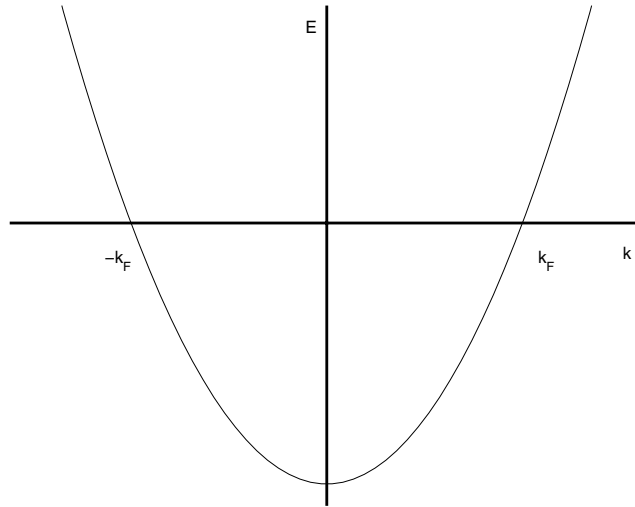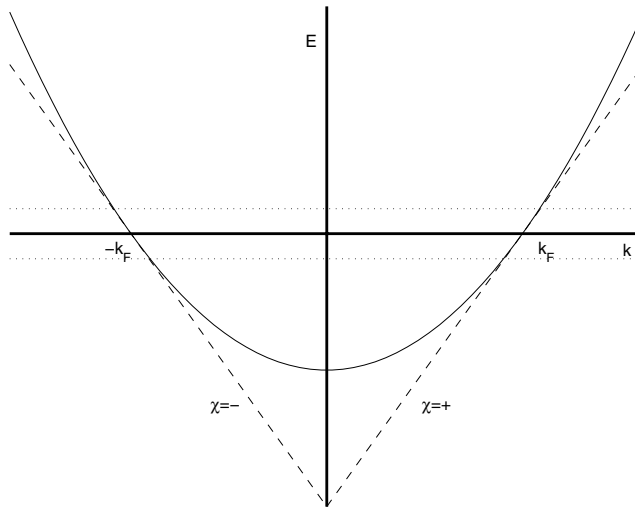
Figure 2.1: The quadratic dispersion.

Figure 2.2: The quadratic dispersion linearized at the Fermi points. The dotted lines show the low energy region.

which is assumed to be odd, the Fermi point is given by $k_F = \frac{\pi}{L} N_0$. At low energies only excitations close to the Fermi-points are possible. To a good approximation we can therefore expand the dispersion at the Fermi points and keep only the lowest (i.e. first) order contribution. Figure 2.2 shows the dispersion linearized at the Fermi points. To perform such a linearization formally it is a good idea to introduce chiral annihilation operators defined by

$$
c_{+,k} = c_k, \quad k \geq 0
$$
$$
c_{-,k} = c_k, \quad k < 0.
$$

In terms of these operators the Hamiltonian takes the form

$$
H_0 = \frac{\hbar^2}{2m} \sum_{\chi,k} k^2 \theta(\chi k) c_{\chi,k}^\dagger c_{\chi,k},
$$

where the chiral variable takes the values $\chi = \pm$, and $\theta(x)$ is the step function. By expanding k^2 around $\pm k_F$ we find that

$$
H_0 = \frac{\hbar^2 k_F^2}{2m} \sum_{\chi,k} \theta(\chi k) \left\{ \left(\frac{\chi k - k_F}{k_F} \right)^2 + 2 \left(\frac{\chi k - k_F}{k_F} \right) + 1 \right\} c_{\chi,k}^\dagger c_{\chi,k}.
$$

In the low energy sector the first term can be ignored compared to the second term. The last term is proportional to the total number of particles. If we assume that the number of particles is constant, this term is uninteresting and can be ignored as well. We are then left with only the second term. If we measure the the energy relative to the ground state (i.e. all states between the Fermi points occupied), the Hamiltonian can be written as

$$
H_0 = v_F \hbar \sum_{\chi,k} \theta(\chi k)(\chi k - k_F) \left\{ c_{\chi,k}^\dagger c_{\chi,k} - \theta(k_F - \chi k) \right\}, \quad (2.1)
$$

where the Fermi velocity is defined as $v_F = \frac{\hbar k_F}{m}$. The left branch includes states with wavenumbers that range from zero to minus infinity, while the right branch includes states with wave numbers that range from zero to plus infinity. However, there is no reason not to expand the branches to include all possible wavenumbers: The low energy sector will not be affected by such an extension and the Hamiltonian (2.1) will still be well defined since it is defined relative to the ground state. Relative to this new ground state, $|F\rangle$, we see that

$$
c_{\chi,k}|F\rangle = 0 \quad \text{if} \quad \chi k > k_F
$$
$$
c_{\chi,k}^\dagger|F\rangle = 0 \quad \text{if} \quad \chi k < k_F.
$$

In terms of this ground state we define the normal ordering of operators by placing the operator that annihilates the ground state to the right;

$$: c^\dagger_{\chi,k} c_{\chi,k} : = \begin{cases} c^\dagger_{\chi,k} c_{\chi,k} & \text{if} \quad \chi k > k_F \\ -c_{\chi,k} c^\dagger_{\chi,k} & \text{if} \quad \chi k < k_F \end{cases}$$
$$= c^\dagger_{\chi k} c_{\chi k} - \theta(k_F - \chi k).$$

The last line follows from the anti-commutation relation of the fermion operators

$$\{c_{\chi,k}, c^\dagger_{\chi',k'}\} = \delta_{\chi,\chi'} \delta_{k,k'} \,. \tag{2.2}$$

The low energy Hamiltonian can then be compactly written as

$$H_0 = v_F \hbar \sum_{\chi,k} (\chi k - k_F) : c^\dagger_{\chi,k} c_{\chi,k} : \,. \tag{2.3}$$

2.2   The density operator and its properties

The chiral creation and annihilation operators motivate the definition of a chiral field operator,

$$\psi_\chi(\xi) = L^{-1/2} \sum_k e^{ik\xi} c_{\chi,k} \,, \tag{2.4}$$

and a chiral density operator

$$\rho_\chi(\xi) =: \psi^\dagger_\chi(\xi) \psi_\chi(\xi) :$$
$$= L^{-1} \sum_q e^{iq\xi} \rho_{\chi,q} \,, \tag{2.5}$$

where the Fourier transform is given by

$$\rho_{\chi,q} = \begin{cases} \sum_k c^\dagger_{\chi,k} c_{\chi,k+q} & \text{if} \quad q \neq 0 \\ N_\chi & \text{if} \quad q = 0 \,. \end{cases}$$

The number operator N_χ counts the number of χ-particles relative to the ground state, and is given by

$$N_\chi = \sum_k : c^\dagger_{\chi,k} c_{\chi,k} : \,.$$

We easily see that $\rho^\dagger_{\chi,q} = \rho_{\chi,-q}$. The density operator plays a central role in the bosonization technique since, as we are about to see, its commutation relations suggest the definition of bosonic creation and annihilation operators.

We begin by calculating the commutation relation between the density operator and the chiral field operator

$$[\rho_{\chi,q}, \psi_\chi^\dagger(\xi)] = L^{-1/2} \sum_{k,k'} e^{ik'\xi} [: c_{\chi,q}^\dagger c_{\chi,q} :, c_{\chi',q'}^\dagger]$$

$$= \delta_{\chi,\chi'} e^{-iq\xi} \psi_{\chi'}^\dagger(\xi) \,. \tag{2.6}$$

The last line follows by the use of the the anticommutation relation (2.2) and the definition of the field operator.

The commutation relation between the density operators is more complicated. By repeated use of the fermion anticommutation relations it is straightforward to show that

$$[\rho_{\chi,q}, \rho_{\chi',-q'}] = \delta_{\chi,\chi'} \sum_{k,k'} \left\{ \delta_{k+q,k'} c_{\chi,k}^\dagger c_{\chi,k'-q'} - \delta_{k,k'-q'} c_{\chi,k'}^\dagger c_{\chi,k+q} \right\} .$$

The commutator, as it stands, is not a well defined operator since it is not normal ordered. To make the commutator well defined we can introduce a cutoff on the k (and k') variable, $-m \le k \le m$. If we assume that $q > 0$ and $q' > 0$ the primed sum can be performed,

$$[\rho_{\chi,q}, \rho_{\chi',-q'}] = \delta_{\chi,\chi'} \left\{ \sum_{k=-m}^{m-q} c_{\chi,k}^\dagger c_{\chi,k+q-q'} - \sum_{k=-m}^{m-q'} c_{\chi,k+q'}^\dagger c_{\chi,k+q} \right\} .$$

By a change of variables in the last expression this reduces to

$$[\rho_{\chi,q}, \rho_{\chi',-q'}] = \delta_{\chi,\chi'} \left\{ \sum_{k=-m}^{-m+q'-2\pi/L} c_{\chi,k}^\dagger c_{\chi,k+q-q'} - \sum_{k=m-q+2\pi/L}^{m} c_{\chi,k}^\dagger c_{\chi,k+q-q'} \right\}$$

Let us consider the $\chi = +$ case first: The second term vanishes when acting on a low energy state since the annihilation operator will act on non-occupied states on the positive branch. The second term will act deep down in the Fermi sea and will also vanish if $q \ne q'$ since the creation operator will act on occupied states. However, if $q = q'$ we see that the first term merely counts the number of states from $-m$ to $-m + q' - 2\pi/L$. Now to the $\chi = -$ case: The first term vanishes when acting on a low energy state since the annihilation operator acts on non-occupied states. The second term also vanishes if $q \ne q'$ since the creation operator will act on occupied states in the Fermi sea. If $q = q'$ the second term will count the number of states between m and $m - q + 2\pi/L$. In both cases we see that the commutator reduces to

$$[\rho_{\chi,q}, \rho_{\chi',-q'}] = \delta_{\chi,\chi'} \delta_{q,q'} \chi \frac{Lq}{2\pi} \,. \tag{2.7}$$

This algebra is a form of the of Kac-Moody algebra. It can be shown that this relation also holds for the other cases ($q, q' < 0$ etc.) [17].

Let us now consider the the commutator between the density operator and the Hamiltonian (2.3). It is easily shown that

$$[H_0, \rho_{\chi,q}] = \sum_k v_F \hbar \left\{ (\chi k - k_F) c^\dagger_{\chi,k} c_{\chi,k+q} - (\chi(k+q) - k_F) c^\dagger_{\chi,k} c_{\chi,k+q} \right\}.$$

For the $q = 0$ case we see that this reduces to

$$[H_0, N_\chi] = 0, \qquad (2.8)$$

and for the $q \neq 0$ we find

$$[H_0, \rho_{\chi,q}] = -v_F \hbar \chi q \rho_{\chi,q}. \qquad (2.9)$$

2.2.1   Bosonic operators and bosonic states

The commutator between the density operators, (2.7), resembles the commutator between bosonic operators. This motivates the definition of bosonic creation and annihilation operators

$$a_q = \sqrt{\frac{2\pi}{L|q|}} \sum_\chi \theta(\chi q) \rho_{\chi,q}, \quad a_q^\dagger = \sqrt{\frac{2\pi}{L|q|}} \sum_\chi \theta(\chi q) \rho_{\chi,-q}, \quad q \neq 0. \quad (2.10)$$

From (2.7) it is easily shown that

$$[a_q, a_{q'}^\dagger] = \delta_{q,q'}, \quad [a_q, a_{q'}] = [a_q^\dagger, a_{q'}^\dagger] = 0. \qquad (2.11)$$

The commutator of the Hamiltonian and the density operator, (2.9), gives

$$[H_0, a_q] = -v_F \hbar |q| a_q, \quad [H_0, a_q^\dagger] = v_F \hbar |q| a_q^\dagger. \qquad (2.12)$$

We see that a_q and $a_q^\dagger$ act as lowering and raising operators for the energy, respectively. We will see in the following that this property makes it possible to define bosonic states for the system.

Let us define $|N_+, N_-\rangle$ as the state constructed by adding N_χ electrons to the lowest available orbitals on the χ-branch in the ground state (or by removing $|N_\chi|$ electrons from the highest occupied orbitals on the χ-branch in the ground state if $N_\chi < 0$). We see that this state is an eigenstate of the the operator $: c^\dagger_{\chi,k} c_{\chi,k} :$ with eigenvalue $\left\{ \theta(\frac{2\pi}{L} N_\chi + k_F - \chi k) - \theta(k_F - \chi k) \right\}$. From this it follows that the state is also an eigenstate of the Hamiltonian with energy

$$\hbar v_F \frac{\pi}{L} \sum_\chi N_\chi^2.$$

It is easily shown that the state is annihilated by a_q, which means that the state is a bosonic ground state. Let us define excited bosonic states by

$$|N_+, N_-, \{n_q\}\rangle = \prod_{q\neq 0} \frac{(a_q^\dagger)^{n_q}}{\sqrt{n_q!}} |N_+, N_-\rangle, \qquad (2.13)$$

where $\{n_q\}$ is a set of non-negative integers. It can be shown that these states are normalized and orthogonal. Since the $a_q^\dagger$'s are raising operators for the energy, we see that (2.13) is an eigenvector of the Hamiltonian with eigenvalue

$$\sum_{k\neq 0} \hbar v_F |q| n_q + \hbar v_F \frac{\pi}{L} \sum_\chi N_\chi^2. \qquad (2.14)$$

We have shown that some eigenstates of H can be represented as bosonic states, but it is not obvious that all eigenstates can be written in this representation. The question is therefore, is the set of states $\{|N_+, N_-, \{n_q\}\rangle\}$ a complete set? A fermionic state can be written as $|\{m_{+,k}\}, \{m_{-,k}\}\rangle$, where $m_{\chi,k}$ is the occupation number of the k-orbital on the χ-branch. Since we are dealing with fermions $m_{\chi,k}$ is either zero or one. It can be shown that for a given energy the number of fermionic and bosonic states are the same [17]. This implies that the bosonic set of states is complete, and every state has a bosonic representation. The bosonic number operator is given by $a_q^\dagger a_q$, and the Hamiltonian can therefore be rewritten as

$$H_0 = \sum_{k\neq 0} \hbar v_F |q| a_q^\dagger a_q + \hbar v_F \frac{\pi}{L} \sum_\chi N_\chi^2. \qquad (2.15)$$

2.3 Bosonization of the field operator

The Hamiltonian has been written in a bosonized form, we will now show how the field operator can be bosonized as well. The idea is as follows: We define an operator $O_\chi(\xi)$ which is proportional to $\psi_\chi^\dagger(\xi)$. The operator is constructed so that it commutes with the bosonic operators. This property makes it possible to identify $O_\chi(\xi)$ by its action on the basis states. The expression for $O_\chi(\xi)$ can then be inverted and an expression for the field operator is found.

The operator is defined as

$$O_\chi(\xi) = \psi_\chi^\dagger(\xi) e^{i\Lambda_\chi(\xi)} e^{i\Lambda_\chi^\dagger(\xi)}, \qquad (2.16)$$

where

$$\Lambda_\chi(\xi) = -i \sum_{q\neq 0} \sqrt{\frac{L}{2\pi|q|}} \theta(\chi q) e^{iq\xi} a_q. \qquad (2.17)$$

By using the commutation relation between the density and the field operator, (2.6), and the fact the $[f, e^g] = [f, g]e^g$ if $[f, [f, q]] = 0$, it can indeed be shown that the operator $O_\chi(\xi)$ commutes with both a_q and $a_q^\dagger$. We will now identify the the operator $O_\chi(\xi)$. It is easily seen that $O_\chi(\xi)|N_+, N_-\rangle$ has no overlap to excited states,

$$\langle N_+, N_-, \{n_q\}|O_\chi(\xi)|N_+, N_-\rangle = \langle N_+, N_-|\prod_{q\neq 0} \frac{a_q^{n_q}}{\sqrt{n_q!}}O_\chi(\xi)|N_+, N_-\rangle = 0.$$

The last equality follows by commuting the a_q operators past $O_\chi(\xi)$, the operators will then annihilate the ket to the right.

Let us now consider the effect of $O_\chi(\xi)$ on states with no bosonic excitations

$$\langle N_\chi', N_{-\chi}'|\psi_\chi^\dagger(\xi)|N_\chi, N_{-\chi}\rangle$$
$$= \langle N_\chi', N_{-\chi}'|O_\chi(\xi)e^{-i\Lambda_\chi^\dagger(\xi)}e^{-i\Lambda_\chi(\xi)}|N_\chi, N_{-\chi}\rangle$$
$$= \langle N_\chi', N_{-\chi}'|O_\chi(\xi)|N_\chi, N_{-\chi}\rangle$$

The last line follows by commuting the first exponential to the left. It is easily shown that

$$\langle N_\chi', N_{-\chi}'|\psi_\chi^\dagger(\xi)|N_\chi, N_{-\chi}\rangle = \eta L^{-1/2}\delta_{N_\chi', N_\chi+1}\delta_{N_{-\chi}, N_{-\chi}'}e^{-i\chi(k_F - \frac{\pi}{L} + (N_\chi+1)\frac{2\pi}{L})\xi},$$

where $\eta = \pm 1$ depends on how the state $|N_\chi, N_\chi\rangle$ is constructed. It then follows that

$$O_\chi(\xi)|N_\chi, N_{-\chi}\rangle = \eta L^{-1/2}e^{-i\chi(k_F - \frac{\pi}{L})}e^{-i\frac{2\pi}{L}\chi N_\chi \xi}|N_\chi + 1, N_{-\chi}\rangle,$$

where N_χ in the exponential is the number operator. Both $O_\chi(\xi)$ and N_χ commute with $a_q^\dagger$, which means that the effect of $O_\chi(\xi)$ on a general bosonic state is given by

$$O_\chi(\xi)|N_\chi, N_{-\chi}, \{n_q\}\rangle = \eta L^{-1/2}e^{-i\chi(k_F - \frac{\pi}{L})}e^{-i\frac{2\pi}{L}\chi N_\chi \xi}|N_\chi + 1, N_{-\chi}, \{n_q\}\rangle.$$

This means that $O_\chi(\xi)$ can be written as

$$O_\chi(\xi) = L^{-1/2}e^{-i\chi(k_F - \frac{\pi}{L})}e^{-i\frac{2\pi}{L}\chi N_\chi \xi}U_\chi,$$

where U_χ is a ladder operator that increases the fermion number on the χ-branch by one. If we assume that the state $|N_+, N_-\rangle$ is constructed by adding

the N_+ particles to the positive branch before N_- particles are added to the negative branch, we see that U_χ can be written as

$$U_+ = \sum_{N_\chi, N_{-\chi}, \{n_q\}} (-1)^{N_-} |N_+ + 1, N_-, \{n_q\}\rangle\langle N_+, N_-, \{n_q\}|$$

$$U_- = \sum_{N_\chi, N_{-\chi}, \{n_q\}} |N_+, N_- + 1, \{n_q\}\rangle\langle N_+, N_-, \{n_q\}|.$$

The factor $(-1)^{N_-}$ gives the phase acquired when a creation operator for the positive branch is commuted past the creation operators for the negative branch. This factor ensures that U_+ and U_- anti-commute. Since the bosonic states are orthonormal it follows that U_χ is unitary and that

$$[N_\chi, U_{\chi'}] = \delta_{\chi,\chi'} U_{\chi'}. \tag{2.18}$$

By inverting the expression for $O_\chi(\xi)$, we have found an expression for the field operator

$$\psi_\chi^\dagger(\xi) = L^{-1/L} e^{-i\chi k_F \xi} e^{i\phi_\xi^\dagger(\xi)} U_\chi e^{i\phi_\xi(\xi)}, \tag{2.19}$$

where

$$\phi_\chi(\xi) = -\frac{\pi}{L}\chi N_\chi \xi + i \sum_{q \neq 0} \sqrt{\frac{2\pi}{L|q|}} \theta(\chi q) e^{iq\xi} a_q. \tag{2.20}$$

We have now bosonized both the Hamiltonian and the field operator in the case of a non-interacting theory. In the next section we will see what happens when interactions are introduced.

2.4 The interacting model

Let us introduce an interaction term in the Hamiltonian,

$$H_I = \frac{\pi}{L} \sum_{\chi,k} \{V_{1,q} \rho_{\chi,q} \rho_{\chi,-q} + V_{2,q} \rho_{\chi,q} \rho_{-\chi,-q}\}. \tag{2.21}$$

At first sight this term looks a bit odd; first we note that the interaction between particles of the same chirality, $V_{1,q}$, is different from the interaction between particles of opposite chiralities, $V_{2,q}$. We also note that the density operators in the interaction term are already normal ordered. At the moment we will just accept this interaction term, and not be bothered about how it might arise.

From the definition of the boson operators we see that the density operator can be written as

$$\rho_{\chi,q} = \delta_{q,0} N_\chi + \sqrt{\frac{L|q|}{2\pi}} \left\{ \theta(\chi q) a_q + \theta(-\chi q) a_{-q}^\dagger \right\}. \tag{2.22}$$

By using this expression for the density operator, the full Hamiltonian, $H = H_0 + H_I$, can be written as

$$H = \frac{1}{2} \sum_{q \neq 0} (\hbar v_F + V_{1,q})|q| \left\{ a_q a_q^\dagger + a_q^\dagger a_q \right\} + \frac{1}{2} \sum_{q \neq 0} V_{2,q}|q| \left\{ a_q a_{-q} + a_{-q}^\dagger a_q^\dagger \right\}$$

$$+ \frac{\pi}{L} \left\{ (\hbar v_F + V_{1,0}) \sum_\chi N_\chi^2 + V_{2,0} \sum_\chi N_\chi N_{-\chi} \right\} - \frac{1}{2} \sum_{q \neq 0} \hbar v_F |q|. \tag{2.23}$$

By defining an operator for the total number of particles, N, and the total current, J, by

$$N = N_0 + \sum_\chi N_\chi, \qquad J = \sum_\chi \chi N_\chi, \tag{2.24}$$

the $q = 0$ contribution to H can be compactly written as

$$\frac{\pi}{L} \left\{ (\hbar v_F + V_{1,0}) \sum_\chi N_\chi^2 + V_{2,0} \sum_\chi N_\chi N_{-\chi} \right\} = \frac{\pi}{2L} \left\{ v_N (N - N_0)^2 + v_J J^2 \right\},$$

where we have defined the velocities

$$v_N = \hbar v_F + V_{1,0} + V_{2,0} \quad v_J = \hbar v_F + V_{1,0} - V_{2,0}. \tag{2.25}$$

The $q \neq 0$ contribution can be considerably simplified by defining new bosonic operators by a Bogoliubov transformation,

$$b_q = \cosh \zeta_q a_q - \sinh \zeta_q a_{-q}^\dagger, \tag{2.26}$$

where the parameter ζ_q is assumed to depend on $|q|$ only. It is easily shown that the new operator obeys bosonic statistics, $[b_q, b_{q'}^\dagger] = \delta_{q,q'}$. By expressing a_q in terms of b_q (and $b_{-q}^\dagger$), the Hamiltonian can be cast in a non-interacting form

$$H = \sum_{q \neq 0} \hbar \omega_q b_q^\dagger b_q + \frac{\pi}{2L} \left\{ v_N (N - N_0)^2 + v_J J^2 \right\} + \frac{1}{2} \sum_{q \neq 0} \hbar (\omega_g - v_F |q|), \tag{2.27}$$

if we require that

$$2(\hbar v_F + V_{1,q}) \cosh \zeta_q \sinh \zeta_q + V_{2,q}(\cosh^2 \zeta_q + \sinh^2 \zeta_q) = 0.$$

This requirement determines the ζ_q parameter,

$$\tanh(2\zeta_q) = -\frac{V_{2,q}}{\hbar v_F + V_{1,q}}. \tag{2.28}$$

The frequency ω_q is given by

$$\omega_q = |q|\sqrt{(v_F + \hbar^{-1}V_{1,q})^2 - (\hbar^{-1}V_{2,q})^2}\,,$$

which shows that the system is gapless, just like the non-interacting system, since arbitrarily small excitations are possible. We have seen that the redefinition of bosonic operators made it possible to transform the interacting Hamiltonian into a Hamiltonian of free bosons. In the next subsection we will see that the interacting ground state is related to the non-interacting one by a unitary transformation.

2.4.1   The interacting ground state

By using the formula

$$e^{-B}Ae^{B} = \sum_{n=0}^{\infty} \frac{1}{n!}[A,B]_n = A + [A,B] + \frac{1}{2!}[[A,B],B] + \dots\,,$$

where $[A,B]_{n+1} = [[A,B]_n, B]$ and $[A,B]_0 \equiv A$, one can show that the definition of the interacting boson operator, (2.26), can be related to the non-interacting one by a unitary transformation

$$b_q = Sa_q S^\dagger\,, \tag{2.29}$$

where

$$S = e^{\frac{1}{2}\sum_{q\neq 0}\zeta_q(a_q^\dagger a_{-q}^\dagger - a_q a_{-q})}. \tag{2.30}$$

This transformation also makes it possible to relate the non-interacting ground state to the interacting one. We see from the interacting Hamiltonian (2.27) that the ground state has to be annihilated by b_q. From $a_q|\mathrm{F}\rangle = 0$, where $|\mathrm{F}\rangle$ is the non-interacting ground state, and the unitary transformation of a_q, we see that $b_q S|\mathrm{F}\rangle = 0$. This means that the interacting ground state is related to the non-interacting one by

$$|\mathrm{IGS}\rangle = S|\mathrm{F}\rangle\,.$$

2.4.2   The interacting field operator

It will be helpful to express the field operator in terms of the interacting boson operators. From (2.26) we see that the bosonic field in the expression for the field operator (2.19) can be written as

$$\phi_\chi(\xi) = -\frac{\pi}{L}\chi N_\chi\xi + i\sum_{q\neq 0}\sqrt{\frac{2\pi}{L|q|}}\theta(\chi q)e^{iq\xi}\left(\cosh\zeta_q b_q + \sinh\zeta_q b^\dagger_{-q}\right).$$

The field operator can be normal ordered with respect to the boson operators by using the formula

$$e^A e^B = e^B e^A e^{[A,B]}$$
$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}, \qquad (2.31)$$

which hold if $[A,[A,B]] = [B,[A,B]] = 0$. The field operator then takes the form

$$\psi^\dagger_\chi(\xi) = L^{-1/2}e^{-\sum_{q\neq 0}\frac{2\pi}{L|q|}\sinh^2\zeta_q}e^{-i\chi k_F\xi}e^{i\varphi^\dagger_\chi(\xi)}U_\chi e^{i\varphi_\chi(\xi)}, \qquad (2.32)$$

where the bosonic field is given by

$$\varphi_\chi(\xi) = -\frac{\pi}{L}\chi N_\chi\xi$$
$$+ i\sum_{q\neq 0}\sqrt{\frac{2\pi}{L|q|}}\left\{\theta(\chi q)\cosh\zeta_q - \theta(-\chi q)\sinh\zeta_q\right\}e^{iq\xi}b_q. \qquad (2.33)$$

The action of all the operators in (2.32) on the interacting ground state is known, and in the next section we will see how this makes it possible to calculate the correlation function exactly.

2.5   The correlation function

We want to calculate the correlation function $\langle\psi^\dagger_\chi(\xi)\psi_{\chi'}(\xi')\rangle$. The only operator that can change the fermion number on a branch is the U_χ operator, for this reason the expectation value will vanish unless $\chi = \chi'$. The quantity of interest is therefore

$$C_\chi(\xi - \xi') = \langle\psi^\dagger_\chi(\xi)\psi_\chi(\xi')\rangle$$
$$= L^{-1}e^{-2\sum_{q\neq 0}\frac{2\pi}{L|q|}\sinh^2\zeta_q}e^{-i\chi k_F(\xi-\xi')}$$
$$\times \langle F|S^\dagger e^{i\varphi^\dagger_\chi(\xi)}U_\chi e^{i\varphi_\chi(\xi)}e^{-i\varphi^\dagger_\chi(\xi')}U^\dagger_\chi e^{-i\varphi_\chi(\xi')}S|F\rangle.$$

The last line can be simplified by noting that $\varphi_\chi(\xi)S|F\rangle = 0$. By using $[N_\chi, U_\chi] = 1$, we can commute U_χ to the right and $U_\chi^\dagger$ to the left. These operators will then cancel due to the unitarity of U_χ. The expectation value in the last line is then given by

$$e^{i\chi\frac{\pi}{L}(\xi-\xi')}e^{[\varphi_\chi(\xi),\varphi_\chi^\dagger(\xi')]} ,$$

where the commutator follows from (2.31). The commutator is easily shown to give

$$\sum_{q>0}\frac{2\pi}{Lq}e^{i\chi q(\xi-\xi')} + \sum_{q>0}\frac{2\pi}{Lq}\sinh^2\zeta_q\left\{e^{i\chi q(\xi-\xi')} + e^{-i\chi q(\xi-\xi')}\right\} .$$

To calculate the exponential of this expression, let us first consider the function $S(x) = e^{\sum_{n=1}^{\infty} n^{-1}e^{inx}}$. To make this function well behaved for all x, it is a good idea to introduce a small imaginary part, $x \to x + i\varepsilon$, where $\varepsilon > 0$. The derivative of S is given by

$$\frac{dS(x)}{dx} = S(x)i\sum_{n=1}^{\infty}e^{in(x+i\varepsilon)}$$

$$= S(x)i\frac{e^{i(x+i\varepsilon)}}{1 - e^{i(x+i\varepsilon)}} ,$$

where the last line follows by performing the sum in the second line. By integrating this expression with suitable boundary conditions, one gets

$$e^{\sum_{q>0}\frac{2\pi}{Lq}e^{i\chi q\xi}} = \left(1 - e^{i(\chi\frac{2\pi}{L}+i\varepsilon)}\right)^{-1} . \tag{2.34}$$

Since the correlation function only depends on $\xi - \xi'$ one can write the full expression for the correlation function as

$$C_\chi(\xi) = \frac{i}{2L}e^{-i\chi k_F\xi}e^{-A(\xi)}\frac{1}{\sin\left[\frac{\pi}{L}(\chi\xi + i\varepsilon)\right]} ,$$

where

$$A(\xi) = 4\sum_{q>0}\frac{2\pi}{Lq}\sinh^2\zeta_q\sin^2(q\xi/2) .$$

We see that the interaction contribution sits in the function $A(\xi)$. It can be shown that in the large ξ (and large L) limit, the correlation function takes the asymptotic form

$$C_\chi(\xi) \sim e^{-i\chi k_F\xi}|\xi|^{-\gamma},$$

where $\gamma = \cosh(2\zeta_0)$. From the definition of ζ_q we find that

$$\gamma = \sqrt{\frac{(\hbar v_F + V_{1,0})^2}{(\hbar v_F + V_{1,0})^2 - V_{2,0}^2}}. \tag{2.35}$$

We see that an interaction between particles of opposite chirality will give rise to a renormalization of the exponent away from its noninteracting value $\gamma = 1$.

2.6   Coordinate space formulation

In this section we will reformulate the bosonized Hamiltonian in terms of fields $\Theta(\xi)$ and $\Phi(\xi)$ instead of the Fourier modes. In the presentation given here we will be using a notation similar to the one used in Ref. [18], since that article will be discussed more closely in Chapter 5.

We start by defining the fields

$$\Theta(\xi) = \Theta_0 - \frac{1}{2\sqrt{\pi}} \sum_\chi \left\{ \phi_\chi(\xi) + \phi_\chi^\dagger(\xi) \right\} \tag{2.36}$$

$$\Phi(\xi) = \Phi_0 + \frac{1}{2\sqrt{\pi}} \sum_\chi \chi \left\{ \phi_\chi(\xi) + \phi_\chi^\dagger(\xi) \right\}, \tag{2.37}$$

where $\phi_\chi(\xi)$ is given by (2.20), and the operators Θ_0 and Φ_0 will be discussed later. It is easily shown that the derivatives and of the fields are related to the charge and current density,

$$\rho(\xi) - \rho_0 = -\frac{1}{\sqrt{\pi}} \partial_\xi \Phi(\xi), \qquad j(\xi) = \frac{1}{\sqrt{\pi}} \partial_\xi \Theta(\xi),$$

where the charge and current is given by

$$\rho(\xi) = \rho_0 + \sum_\chi \rho_\chi(\xi), \qquad j(\xi) = \sum_\chi \chi \rho_\chi(\xi),$$

and the ground state density by $\rho_0 = \frac{N_0}{L}$.

In the low-energy limit we can expand the interaction to zeroth order in the expression for the interacting Hamiltonian (2.23). If we define $u = \sqrt{v_N v_J}$ and $K = \sqrt{\frac{v_J}{v_N}}$, we see that the Hamiltonian can be expressed in terms of the fields (2.36) and (2.37) as

$$H = \frac{u}{2} \int_{-L/2}^{L/2} d\xi \left\{ K^{-1} \left(\partial_\xi \Phi(\xi) \right)^2 + K \left(\partial_\xi \Phi(\xi) \right)^2 \right\} - \frac{1}{2} \sum_{q \neq 0} \hbar v_F |q|. \tag{2.38}$$

This is the form used in Ref. [18].

It is also useful to express the field operator, (2.19), in terms of the Θ and Φ fields. We easily see from the definition of these fields that the field operator can be formulated as

$$\psi_\chi(\xi) = L^{-1/2} e^{i\chi k_F \xi} e^{-i\phi_\chi^\dagger(\xi)} U_\chi^{-1} e^{-i\phi_\chi(\xi)}$$
$$\sim e^{i\chi k_F \xi} e^{i\sqrt{\pi}(\Theta(\xi) - \chi \Phi(\xi))}. \tag{2.39}$$

The U_χ operator can be expressed in terms of the exponential of the Θ_0 and Φ_0 operators. Since this is rather technical, as discussed in Ref. [19], we will not go into the details here.

Chapter 3

Edge states in the quantum Hall effect

In this chapter we will see how the Luttinger model can be applied to the edge of a quantum Hall system. As we saw in the previous chapter the Luttinger model exhibits gapless bosonic excitations. We have learned that the excitations of the quantum Hall system are gapped, so how can the Luttinger model be applied to such a system? It turns out that even if the bulk of the system is gapped, the edge of the system exhibits gapless modes [8].

The presentation here is mainly based on Wen's hydrodynamical approach [20], but I have tried to make the theory more rigorous.

3.1 Hydrodynamical description

Let us consider the edge of a two-dimensional incompressible quantum Hall fluid of density $n = \frac{\nu}{2\pi\ell_B^2}$, as shown in Figure 3.1. The fluid is subject to a transverse magnetic field and a confining electric field in the negative y-direction (to be in accordance with the notations used in Chapter 1 and Papers I-III, the charge of the fluid will be assumed to be positive). The edge profile is given by the profile function $h(x)$, which defines a one-dimensional density $\rho(x) = nh(x)$.

We will now try to find a Lagrangian for such a system. The Lagrangian for a particle of charge q in a magnetic field and an electric potential ϕ is given by

$$L = \frac{1}{2}m\mathbf{v}^2 + q\mathbf{v} \cdot \mathbf{A} - q\phi,$$

where $\mathbf{A}$ is the magnetic vector potential. We argued in Chapter 1 that for
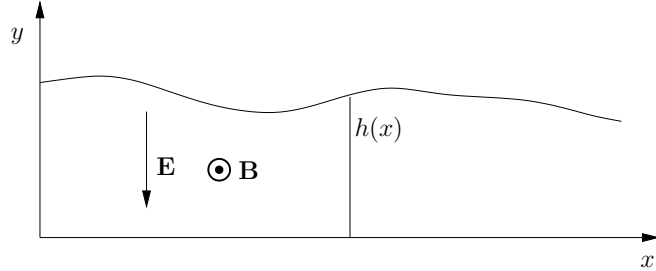
Figure 3.1: The edge of an incompressible quantum Hall fluid.

a system restricted to the LLL, the kinetic term can be ignored. We will therefore consider the Lagrangian $L = L_A + L_\phi$, where $L_A = q\mathbf{v} \cdot \mathbf{A}$ and $L_\phi = -q\phi$.

Let us start by evaluating L_A. We will be working in the Landau gauge, $\mathbf{A} = (-yB, 0, 0)$. By summing over all volume elements in the liquid we see that L_A can be written as

$$L_A = -neB \int_0^L dx \int_0^{h(x)} dy\, v_x y,$$

where L is the length of the system. If we make the approximation that the velocity component, v_x, is only dependent on x, the y-integral can be performed

$$L_A = -\pi\hbar\nu^{-1} \int_0^L dx\, v_x \rho^2(x),$$

where the definition of n and $\rho(x)$ have been used. The velocity can be found from the expression for the one-dimensional current density, $j(x) = \rho(x)v_x$. To find an expression for the current density we can integrate the one-dimensional continuity equation, $\partial_x j = \dot\rho$. We then find that the velocity is given by

$$v_x = -\frac{1}{\rho(x)} \int_0^x dx' \dot\rho(x') + \frac{j(0)}{\rho(x)}.$$

If we assume periodic boundary conditions the density can be expressed in terms of Fourier components as

$$\rho(x) = L^{-1} \sum_q e^{iqx} \rho_q\,. \tag{3.1}$$

The following expression for L_A can then be found

$$L_A = \frac{\pi\hbar}{\nu L} \sum_{q\neq 0} \frac{1}{iq} \dot{\rho}_q \rho_{-q} - \frac{\pi\hbar}{\nu L} Q \sum_{q\neq 0} \frac{1}{iq} \dot{\rho}_q - \frac{\pi\hbar}{\nu} Q j(0), \qquad (3.2)$$

where Q is the $q = 0$ Fourier transform of $\rho(x)$. Q corresponds to the total number of electrons in the edge region, $y > 0$, and is assumed to be constant.

The electric potential is expressed in terms of the electric field as $\phi(x) = Ey$. L_ϕ can then be expressed as

$$L_\phi = -ne \int_0^L dx \int_0^{h(x)} dy\, Ey$$

$$= -\frac{eB}{2n} v \int_0^L dx \rho^2(x),$$

where we have defined the velocity parameter $v = E/B$. Using (3.1), we find that

$$L_\phi = -\frac{\pi\hbar v}{\nu L} \sum_{q\neq 0} \rho_q \rho_{-q} - \frac{\pi\hbar v}{\nu L} Q^2. \qquad (3.3)$$

The last terms in (3.2) and (3.3) correspond to a boundary term and a constant term. If we ignore these terms we find that the Lagrangian is given by

$$L = \frac{\pi\hbar}{\nu L} \sum_{q\neq 0} \frac{1}{iq} \dot{\rho}_q \left(\rho_{-q} - Q \right) - \frac{\pi\hbar v}{\nu L} \sum_{q\neq 0} \rho_q \rho_{-q}. \qquad (3.4)$$

The canonical momentum is given by

$$\Pi_q = \frac{\partial L}{\partial \dot{\rho}_q} = \frac{\pi\hbar}{\nu L} \frac{1}{iq} \left(\rho_{-q} - Q \right). \qquad (3.5)$$

It is then easily seen that the equations of motion, $\dot{\Pi}_q = \frac{\partial L}{\partial \rho_q}$, is given by

$$\dot{\rho}_q = ivq\rho_q.$$

The Fourier transform of this equation is $\left(\partial_t - v\partial_x \right) \rho(x,t) = 0$, with solutions of the form $\rho(x,t) = \rho(x+vt)$. The solutions are chiral and move to the left with velocity $v = \frac{E}{B}$. This is also what we would expect: A charged particle in a crossed electric and magnetic field will drift with a velocity $\mathbf{v} = \frac{\mathbf{E}\times\mathbf{B}}{\mathbf{B}^2}$. A height profile $h(x)$ in the charged liquid should therefore move to the left with velocity v.

From the Lagrangian, (3.4), and the canonical momenta, (3.5), the Hamiltonian is easily found

$$H = \frac{\pi\hbar v}{\nu L} \sum_{q\neq 0} \rho_q \rho_{-q}. \qquad (3.6)$$

The Hamiltonian is equal to the electrostatic contribution and we notice that it does not depend on the canonical momenta, the reason for this is that the Lagrangian is only first order in the velocities $\dot{\rho}_q$.

3.1.1   Quantization

To canonically quantize this theory is not straightforward because the Hamiltonian formulation of this theory is not well defined as it stands. The reason for this is that the phase space is constrained since the coordinates and canonical momenta can be identified with each other. The physical states are therefore constrained to lie on a surface in phase space. As we see from (3.5) the surface is defined by $\phi_q = 0$, where $\phi_q = \Pi_q - \frac{\pi\hbar}{\nu L}\frac{1}{iq}\left(\rho_{-q} - Q\right)$. The Hamiltonian formulation can be generalized to deal with such constrained systems [21, 22]. The Poisson brackets are generalized to Dirac brackets, which enable quantization. We will not go into the details here, but it can be shown that the commutator between the density operators is given by

$$[\rho_q, \rho_{q'}] = -\nu\frac{Lq}{2\pi}\delta_{q,-q'} \, . \tag{3.7}$$

The commutator between the Hamiltonian, (3.6), and the density operator then easily follows

$$[H, \rho_q] = \hbar v q \rho_q \, . \tag{3.8}$$

The electron operator creates a particle of unit charge, which means that it has to obey $\left[\rho(x'), \psi^\dagger(x)\right] = \delta(x' - x)\psi^\dagger(x)$. The Fourier transform of this commutator is

$$\left[\rho_q, \psi^\dagger(x)\right] = e^{-iqx}\psi^\dagger(x). \tag{3.9}$$

We recognize (3.7), (3.8) and (3.9) as the corresponding commutators for the negative chirality density operator in the Luttinger model, (2.7), (2.9) and (2.6), respectively, except that the Kac-Moody algebra found here has been modified with a factor ν. The theory can therefore be bosonized, but the factor ν in (3.7) gives rise to a modified exponent in the asymptotic electron correlation function

$$\left\langle \psi^\dagger(x)\psi(x')\right\rangle \sim \frac{1}{(x - x')^{1/\nu}} \, .$$

The commutator (3.9) only ensures that the electron creates a unity charge, the requirement that the electron operator has to be fermionic, i.e. it has to anticommute with itself, constrains the filling fraction to be $\nu = 1/m$, with m odd [20], i.e. the Laughlin series. However, the edge theory can be generalized, by a Chern-Simons description, to account for other fractions as well [20], which gives rise to other forms of the correlator exponents. Since the fluid model gives the results needed here, we will not go into that formalism.

3.2 Experimental and numerical results

The derivation in the previous section predicts that the edge of a quantum Hall system behaves as a Luttinger liquid. The electron correlation function will therefore have a power-law behaviour and the exponent takes the universal value of ν^{-1} (in the case of the Laughlin series).

The correlation function can be related to the current-voltage characteristics for electron tunneling into the edge, $I \sim V^{\alpha}$, where α is the correlation function exponent. Several experiments have been performed to test this prediction [23, 24, 25, 26], and the experiments do confirm a Luttinger liquid behaviour in the sense that the current-voltage characteristics behave as a power-law. However, the experimentally measured exponent does not agree with the universal prediction. A clear observation of a plateau behaviour for the exponent is also missing. However, a weak plateau like structure for α close to the ρ_{xy} plateau in the $\nu = 1/3$ case is reported in Ref. [25].

So far no widely accepted theoretical explanation for the experimental results exists, but several numerical works suggest that the discrepancy between theory and experiment can be ascribed to the effects of the electron interaction [27, 28, 29]. Of special interest to Paper III is Ref. [27], where Mandal and Jain (MJ) suggest that mixing with the second CF Landau level can lead to renormalization of the correlation function exponent for the $\nu = 1/3$ case. As we saw in Chapter 1 the $\nu = 1/3$ FQHE can be seen as a $\nu = 1$ IQHE of CFs. The question then arises: Can a similar renormalization effect be seen in the $\nu = 1$ IQHE if one allows the interaction to cause mixing with the second electronic Landau level? This is the motivation behind Paper III. In that paper I allow for mixing with the second electronic Landau level in a fashion similar to the approach in MJ's work. The single particle orbitals, that in the non-interacting case is given by (1.4) with $n = 0$, is allowed to be in a superposition of a LLL and a second Landau level state

$$\varphi_k(x,y) = C_0(k)\psi_{k0}(x,y) + C_1(k)\psi_{k1}(x,y)\,, \qquad (3.10)$$

where $C_0(k)$ and $C_1(k)$ are mixing coefficients. Normalization requires that $|C_0(k)|^2 + |C_1(k)|^2 = 1$. The interaction, which has a Gaussian shape, will give rise to mixing coefficients different from the non-interacting case, $C_0(k) = 1$ and $C_1(k) = 0$. Since we work in the $\nu = 1$ case, the many-particle ground state of the system is a Slater determinant of the single-particle states (3.10). Since the ground state is given as a Slater determinant it can be calculated numerically within the Hartree-Fock approximation. In Paper III knowledge of the Hartree-Fock approximation is assumed, for that reason an introduction to that topic is given in Chapter 4. As opposed to the $\nu = 1/3$

case I find no renormalization of the correlation function exponent. However, the interaction is shown to give rise to oscillations in the density profile.

3.3 Explicit mapping

The hydrodynamic model is based on very general arguments, and it is not derived from a microscopical point of view. The motivation for the work in Paper I and II is to study a model where the relationship between the two-dimensional quantum Hall system and the one-dimensional Luttinger model can be made precise. One of the advantages of this approach compared to the effective edge description is that interactions can be included at the microscopic level and can be dealt with in a natural way. The fact that the quantum Hall system has two edges that can interact with each other is also naturally described in this approach. The disadvantage is that we have to consider the case of integer filling since this case has a ground state that can easily be dealt with.

   We will now discuss some of the ideas and results of Paper I and II. In Chapter 1 we pointed out that in the LLL the quantum Hall system is effectively one-dimensional, due to the fact the wavefunction (1.4) is given by one quantum number only. Let us exploit this fact. A single particle state in the LLL can be expanded in a basis consisting of the states (1.4),

$$\psi(x,y) = L^{-1/2} \sum_k c_k e^{ikx} \psi_0(y + \ell_B^2 k) \,.$$

However, since the state is determined by the c_k parameters alone it is equivalent to a one-dimensional wave function

$$\psi(\xi) = L^{-1/2} \sum_k c_k e^{ik\xi} \,.$$

By using the expression for the harmonic oscillator ground state, ψ_0, one can find a mapping between the 1D and 2D representations

$$\psi(\xi) = \int dx dy f(x - \xi, y) \psi(x, y) \,,$$

$$\psi(x, y) = \int d\xi f(x - \xi, y) \psi(\xi) \,,$$

where

$$f(x, y) = \frac{1}{\sqrt{2\pi\sqrt{\pi}\ell_B^3}} e^{-x(x+2iy)/2\ell_B^2} \,. \tag{3.11}$$

The transition function f does not only make it possible to map wavefunctions between one and two dimensions, and vice versa, also matrix elements of arbitrary operators can be mapped in this way. This means that any operator defined in the two-dimensional model has a corresponding one in the one-dimensional representation.

To model a proper Hall bar there has to be a confining potential in the system. This we will take to be a harmonic potential in the y-direction, $\frac{1}{2}m\omega^2 y^2$. This confining potential can be absorbed in the Hamiltonian (1.3) by a redefinition of the parameters. This means that the eigenstates will have the same form and the mapping just described still holds, except for a redefinition of some parameters. The confining potential will lead to a particle drift in the system in a similar way to the $\mathbf{E} \times \mathbf{B}$ drift in the hydrodynamic case, except that the current is not restricted to the edges only, it will extend through the whole system and reaches its maximum at the edges. Another effect of the interaction is that it lifts the degeneracy in the LLL, the energy of the single particle states is now given by

$$\frac{\hbar^2}{2M}k^2 + \frac{1}{2}\hbar\bar{\omega}_c \,,$$

where M is a mass parameter and $\bar{\omega}_c$ is a redefined cyclotron frequency. The ground state of this system can be represented by Figure 1.3 with all the lowest momentum states occupied, this will give rise to two well defined edges in terms of the Fermi momenta, $y_{\text{edge}} = \pm\ell_B^2 k_F$. The particles in the upper half plane will all drift toward the left, while the particles in the lower half plane drift to the right.

An interaction between the particles can be introduced, and we have chosen an interaction with a Gaussian form, $V(x,y) = V_0 e^{-\alpha^2(x^2+y^2)}$. In addition to giving easy control of both the interaction strength, V_0, and interaction length, $1/\alpha$, it can be explicitly mapped to the one-dimensional representation due to the fact that the the transition function, (3.11), has a Gaussian shape. This gives rise to a one-dimensional Gaussian interaction that is no longer a local density interaction. The one-dimensional picture now emerges: We have a well defined non-interacting groundstate defined by all momentum states $|k| < k_F$ occupied. The dispersion is quadratic in the wavenumber k and the interaction is non-local in the densities. By taking the low-energy limit, just as we did when we discussed the Luttinger model in Chapter 2, the dispersion can be linearized at the Fermi points. By assuming that excitations close to the Fermi points are allowed the momentum representation of the interaction can also be given a low energy form, where the interaction splits into two parts: One part corresponds to the interaction between particles with same chirality, $V_1(q)$, and the other part to inter-

actions between particles of opposite chirality, $V_2(q) \approx e^{-\alpha^2 W^2} V_1(q)$, where $W = 2\ell_B^2 k_F$ is the width of the quantum Hall bar. The 1D representation can now be bosonized with the technique developed in Chapter 2. As we know, this leads to a correlation function which in the asymptotic limit has the shape of a power law, where the exponent is renormalized by the interaction. However, only the interaction between opposite chiralities will lead to renormalization, as seen from (2.35).

We see that our approach has no need for effective arguments, since it is based on an exact mapping from the two-dimensional to the one-dimensional representation. We also see that this mapping is not restricted to the edges only, as in the hydrodynamical approach, in fact, the whole quantum Hall system is mapped onto its one-dimensional representation.

The one-dimensional representation has now been described, but in real life the quantum Hall system is two-dimensional. What is needed is to map the expression for the one-dimensional correlation function back to the 2D case. This is the topic of Paper II, where we show how the 2D correlation function, $\langle \psi^\dagger(x,y)\psi(x',y') \rangle$, can be derived. The expression for the correlation function close to the edge is shown to approach a power law in the asymptotic limit with the exponent equal to the exponent in the one-dimensional representation. Since we have an explicit expression for the 2D correlation function the electron density profile is given simply by putting $x = x'$ and $y = y'$. What we find is that even if the interaction between the edges, i.e. the different chiralities, can lead to substantial renormalization of the exponent, the effect of the interaction on the density profile is minimal compared to the non-interacting case. Even more, the deviation from the non-interacting density is only found close to the edges of the system.

Chapter 4

The Hartree-Fock approximation

In Paper III, the Hartree-Fock (HF) algorithm was used to calculate the ground state of a quantum Hall system where the single particle states have contributions from the second Landau level. In this chapter I will derive the HF equation and show how it can be used. The review I present here is mainly based on Refs. [30, 31].

4.1   The Hartree-Fock equation

The HF approximation is based on the idea that an interacting many-particle system can be approximately described by a system of non-interacting particles, where the particles move in an effective potential set up by all the other particles.

The idea is therefore that the Hamiltonian of the N particles (in the x-representation)

$$H = \sum_{i=1}^{N} h_i + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}} V(\mathbf{r}_i - \mathbf{r}_j),$$

where the first term consists of single particle operators (kinetic energy and background potential, if it exists) and the second term is the interaction term, can be replaced by a Hamiltonian of single particle operators,

$$H^{\mathrm{HF}} = \sum_{i=1}^{N} h_i^{\mathrm{HF}}. \tag{4.1}$$

The ground state of (4.1) will then be a Slater determinant of single particle

orbitals,

$$\Phi(\{\mathbf{r}\}) = \frac{1}{\sqrt{N!}} \det(\varphi_i(\mathbf{r}_j)), \tag{4.2}$$

where the single particle orbitals, $\varphi_i(\mathbf{r})$, are normalized, $1 = \int d\mathbf{r}\varphi_i^*(\mathbf{r})\varphi_i(\mathbf{r})$. The ground state orbitals are chosen so that they minimize the energy expectation value of the full Hamiltonian, $\langle\Phi|H|\Phi\rangle$. We will see that this requirement will lead to single particle eigenvalue equations that determine the single particle operators h_i^{HF} in (4.1). Since the the single particle orbitals have to be normalized, we have to minimize

$$\mathcal{H} = \langle\Phi|H|\Phi\rangle - \sum_{i=1}^{N} \lambda_i \left\{ \int d\mathbf{w}\varphi_i^*(\mathbf{w})\varphi_i(\mathbf{w}) - 1 \right\}, \tag{4.3}$$

where the Lagrange multipliers $\{\lambda_i\}$ arise from the constraint that $\{\varphi_i\}$ have to be normalized. The energy expectation value can be written as

$$\langle\Phi|H|\Phi\rangle = \sum_{i=1} \int d\mathbf{w}\varphi_i^*(\mathbf{w})h_i\varphi_i(\mathbf{w})$$

$$+ \frac{1}{2} \sum_{i,j=1}^{N} \left\{ \int d\mathbf{w}d\mathbf{w}'\varphi_i^*(\mathbf{w})\varphi_j^*(\mathbf{w}')V(\mathbf{w}-\mathbf{w}')\varphi_i(\mathbf{w})\varphi_j(\mathbf{w}') \right.$$

$$\left. - \int d\mathbf{w}d\mathbf{w}'\varphi_i^*(\mathbf{w})\varphi_j^*(\mathbf{w}')V(\mathbf{w}-\mathbf{w}')\varphi_j(\mathbf{w})\varphi_i(\mathbf{w}') \right\}. \tag{4.4}$$

The variation of $\mathcal{H}$ with respect to $\varphi_k^*(\mathbf{r})$ is then given by

$$\frac{\delta\mathcal{H}}{\delta\varphi_k^*(\mathbf{r})} = h_k\varphi_k(\mathbf{r}) + \sum_{i=1}^{N} \int d\mathbf{w}\varphi_i^*(\mathbf{w})V(\mathbf{r}-\mathbf{w})\varphi_i(\mathbf{w})\varphi_k(\mathbf{r})$$

$$- \sum_{i=1}^{N} \int d\mathbf{w}\varphi_i^*(\mathbf{w})V(\mathbf{r}-\mathbf{w})\varphi_i(\mathbf{r})\varphi_k(\mathbf{w}) - \lambda_k\varphi_k(\mathbf{r}).$$

For the Φ to minimize the energy, the variation of $\mathcal{H}$ has to vanish, $\frac{\delta\mathcal{H}}{\delta\varphi_k^*(\mathbf{r})} = 0$, and this leads to the HF equation,

$$\left\{ h + \sum_{i=1}^{N} (I_i - K_i) \right\} \varphi_k(\mathbf{r}) = \lambda_k\varphi_k(\mathbf{r}), \tag{4.5}$$

where the operator I_i is defined by

$$I_i = \int d\mathbf{w}\varphi_i^*(\mathbf{w})V(\mathbf{r}-\mathbf{w})\varphi_i(\mathbf{w}) \tag{4.6}$$

and the exchange operator by

$$K_i\psi(\mathbf{r}) = \int d\mathbf{w}\,\varphi_i^*(\mathbf{w})V(\mathbf{r}-\mathbf{w})\psi(\mathbf{w})\varphi_i(\mathbf{r}). \qquad (4.7)$$

It is easily shown that the operators on the LHS of (4.5) are Hermitian,

$$\langle f|I_i|g\rangle = \int d\mathbf{r}d\mathbf{w}\,f^*(\mathbf{r})\varphi_i^*(\mathbf{w})V(\mathbf{r}-\mathbf{w})g(\mathbf{r})\varphi_i(\mathbf{w})$$

$$= \left\{\int d\mathbf{r}d\mathbf{w}\,g^*(\mathbf{r})\varphi_i^*(\mathbf{w})V(\mathbf{r}-\mathbf{w})f(\mathbf{r})\varphi_i(\mathbf{w})\right\}^* = \langle g|I_i|f\rangle^*, \quad (4.8)$$

and similarly for K_i (h is of course Hermitian). We see that the problem of finding the single particle orbitals has been reduced to finding the solution to an eigenvalue equation, (4.5), where all the eigenvalues are real (since the operator on the LHS is Hermitian). On the other hand, the operators I_i and K_i themselves depends on the single particle orbitals. The way forward is a self consistent approach:

1. Make an initial approximation for $\varphi_i(\mathbf{r})$ for all $i \in \{1,..,N\}$.

2. Calculate I_i and K_i for all $i \in \{1,..,N\}$.

3. Solve the HF equation (4.5).

4. Select the N orbitals with the lowest eigenvalue λ_i.

The procedure is then repeated from Step 2 with the new orbitals. This cycle is iterated until the orbitals found are equal to the orbitals found in the previous iteration.

4.1.1 The Hartree-Fock energy

From (4.5) we would expect the HF energy to be $\sum_{i=1}^N \lambda_i$, but we will see that this is not the case. By multiplying (4.5) with $\int d\mathbf{r}\varphi_k^*(\mathbf{r})$ and using the fact that the single particle states are orthonormal (the orthogonality follows from the fact that the operator on the LHS of (4.5) is Hermitian), we find that

$$\int d\mathbf{w}\,\varphi_i^*(\mathbf{r})h\varphi_i(\mathbf{r}) \equiv \langle\varphi_i|h|\varphi_i\rangle$$

$$= \lambda_i - \sum_{j=1}^N \left\{\langle\varphi_i\varphi_j|V|\varphi_i\varphi_j\rangle - \langle\varphi_i\varphi_j|V|\varphi_j\varphi_i\rangle\right\}, \qquad (4.9)$$

where the matrix element in the last line is defined by

$$\langle \varphi_i \varphi_j | V | \varphi_k \varphi_l \rangle = \int d\mathbf{r} d\mathbf{w} \varphi_i^*(\mathbf{r}) \varphi_j^*(\mathbf{w}) V(\mathbf{r} - \mathbf{w}) \varphi_k(\mathbf{r}) \varphi_l(\mathbf{w})$$

By inserting this expression in (4.4) we find that the HF energy is given by

$$E^{\text{HF}} = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i,j=1}^{N} \left\{ \langle \varphi_i \varphi_j | V | \varphi_i \varphi_j \rangle - \langle \varphi_i \varphi_j | V | \varphi_j \varphi_i \rangle \right\}, \tag{4.10}$$

where we indeed see that there is a correction to the naive guess that $E^{\text{HF}} = \sum_{i=1}^{N} \lambda_i$.

4.2   Stability of the Hartree-Fock solution

The variational principle used to derive the HF equation only guarantees that the solution is stationary with respect to variation of the orbitals, and not necessarily that the solution is a minimum. A necessary condition for this is that the energy of an "excited state" is higher than E^{HF}. Let us consider the state $|\Phi_{ph}\rangle$, constructed by replacing one of the ground state orbitals φ_h, $h \leq N$ with one of the discarded orbitals, φ_p, $p > N$. From (4.4) we see that the energy of this state is given by

$$E_{ph} = \langle \Phi_{ph} | H | \Phi_{ph} \rangle = \sum_{\substack{i=1 \\ i \neq h}}^{N,p} \langle \varphi_i | h | \varphi_i \rangle + \frac{1}{2} \sum_{\substack{i,j=1 \\ i,j \neq h}}^{N,p} \left\{ \langle \varphi_i \varphi_j | V | \varphi_i \varphi_j \rangle - \langle \varphi_i \varphi_j | V | \varphi_j \varphi_i \rangle \right\}$$

$$= E^{\text{HF}} + \langle \varphi_p | h | \varphi_p \rangle + \sum_{j=1}^{N} \left\{ \langle \varphi_p \varphi_j | V | \varphi_p \varphi_j \rangle - \langle \varphi_p \varphi_j | V | \varphi_j \varphi_p \rangle \right\}$$

$$- \langle \varphi_h | h | \varphi_h \rangle - \sum_{j=1}^{N} \left\{ \langle \varphi_h \varphi_j | V | \varphi_h \varphi_j \rangle - \langle \varphi_p \varphi_j | V | \varphi_j \varphi_p \rangle \right\}$$

$$- \left\{ \langle \varphi_p \varphi_h | V | \varphi_p \varphi_h \rangle - \langle \varphi_p \varphi_h | V | \varphi_h \varphi_p \rangle \right\}$$

For the last equality the expression for the HF energy, (4.10), have been used. From (4.9) it follows that

$$E_{ph} = E^{\text{HF}} + \lambda_p - \lambda_h - \left\{ \langle \varphi_p \varphi_h | V | \varphi_p \varphi_h \rangle - \langle \varphi_p \varphi_h | V | \varphi_h \varphi_p \rangle \right\}.$$

By defining $\Gamma_{ph} = E_{ph} - E^{\text{HF}}$, we see that the necessary condition for E^{HF} to be a minimum is that

$$\Gamma_{ph} > 0, \quad \text{for all} \begin{cases} h \leq N \\ p > N \end{cases} \tag{4.11}$$

In Paper 3 this check have been performed on the HF solutions.

4.3 Expansion of single particle state orbitals

Usually a computer is used for HF calculations. It will therefore be useful to convert the integral equation (4.5) into an algebraic equation. This can be done if we expand the single particle orbitals in a basis of a finite number of functions, $\phi_\mu(\mathbf{r})$, so that $\varphi_k(\mathbf{r}) = \sum_\mu C_\mu(k)\phi_\mu(\mathbf{r})$. We will see that the number, Ω, of basis functions has to be larger than the number of particles, N. By multiplying (4.5) with $\int d\mathbf{r}\phi_\nu^*(\mathbf{r})$ and assuming that the basis orbitals are orthonormal we see that the the HF equation can be written as

$$\sum_\mu \left\{ \int d\mathbf{r}\phi_\nu^*(\mathbf{r})h\phi_\mu(\mathbf{r}) + \sum_{i=1}^N \int d\mathbf{r}\phi_\nu^*(\mathbf{r})(I_i - K_i)\phi_\mu(\mathbf{r}) \right\} C_\mu(k) = \lambda_k C_\nu(k),$$

which can be written as

$$\sum_\mu F_{\nu\mu}C_\mu(k) = \lambda_k C_\mu(k). \tag{4.12}$$

The matrix $F_{\nu\mu}$ can be found by rewriting the I_i and K_i operators in terms of the expansion of φ_i, the result is

$$F_{\mu\nu} = \langle\phi_\nu|h|\phi_\mu\rangle$$
$$+ \sum_{i=1}^N \sum_{\sigma\rho} C_\sigma^*(i)C_\rho(i) \left\{ \langle\phi_\nu\phi_\sigma|V|\phi_\mu\phi_\rho\rangle - \langle\phi_\nu\phi_\sigma|V|\phi_\rho\phi_\mu\rangle \right\}. \tag{4.13}$$

When a basis has been chosen, all the matrix elements in (4.13) can be explicitly calculated. The problem has then been reduced to solving the algebraic eigenvalue equation (4.12) for all coefficients $C_\mu(i)$. The algorithm is similar to the previously discussed procedure:

1. Make an initial choice of $\{C_\mu(i)\}$.

2. Calculate the matrix $F_{\mu\nu}$

3. Solve the eigenvalue equations (4.12) for all i

4. Select the N sets of coefficients, $\{C_\mu(i)\}$, with the lowest eigenvalues (because $\Omega > N$, the number of solutions will generally be larger than N).

The whole procedure is then repeated from Step 2, and iterated until the coefficients do not change from one iteration to the next.

The orbital expansion method was used to find the HF ground state in Paper 3.

Part II

Ongoing Research and Summary

Chapter 5

Fractional excitations in Luttinger liquids

In this chapter I will discuss an ongoing study that is related to the discussion of fractional charges in one-dimensional systems. The work presented here is done in collaboration with Jon Magne Leinaas and Hans Hansson.

5.1   Introduction

In the literature it has been suggested that the elementary excitations of one-dimensional systems carry fractional charges [32, 33, 18]. The suggestions are based on two ideas. The first approach, taken by Heinonen and Kohn (HK) [32], is based on the idea from Fermi liquid theory that excitations somehow "become" dressed when the interaction is adiabatically turned on. The second approach, taken by Fisher and Glazman (FG) [33] and Pham, Gabay and Lederer (PGL) [18], is based on the idea that the Luttinger liquid have a natural separation into chiral modes. It is interesting that HK's quasiparticle charge, which is only calculated to lowest order in the interaction, seem to agree with PGL's result. It is an open question whether this agreement also holds to higher order, and that is one of the things that will be discussed in this chapter.

This chapter is organized as follows: I begin by giving a short review of the ideas and results in the three cited papers, and point out similarities and differences between them. In Section 5.2 and 5.3 the ideas by PGL and HK will be applied to the quantum Hall model introduced in Paper I and II. We will see that the two edges of the systems gives an interesting aspect to the discussion. In the Summary I suggest some questions for further study.

5.1.1 Motivation

I will now discuss the three different approaches that, although different, seemingly agree on the values of the charges.

In Ref. [32] HK discuss the propagation of a quasiparticle in a one-dimensional system with Hamiltonian

$$H = \sum_k \varepsilon_k c_k^\dagger c_k + \frac{1}{2} \sum_{k_1,k_2,p} V(p) c_{k_1-p}^\dagger c_{k_2}^\dagger c_{k_2-p} c_{k_1}. \tag{5.1}$$

The wavepacket is constructed as $|\Phi_{k_0}\rangle = \sum_k A_k c_k^\dagger |F\rangle$, where $|F\rangle$ is the non-interacting ground state and $c_k^\dagger$ is the creation operator for a particle with wavenumber k. The wavepacket correspond to the injection of a unit charge, which imposes a normalization on the envelope function, $\sum_k |A_k|^2 = 1$. The interaction is then adiabatically turned on, which means that the state $|\Phi_{k_0}\rangle$ evolves into a quasiparticle wavepacket $|\Psi_{k_0}\rangle$. The expectation value of the density, in the Fourier representation, in this state, $\langle \Psi_{k_0}|\rho(q)|\Psi_{k_0}\rangle$, is calculated to first order perturbation theory in the interaction $V(p)$. It is found that the density has a discontinuity as $q \to 0$, which can be interpreted as a delocalization of some of the charge to infinity. The quasiparticle charge is identified by taking the $q \to 0$ limit of the density, and it is found to be

$$e^* = 1 - \frac{1}{4\pi k_F} \left\{ V(0) - V(2k_F) \right\}. \tag{5.2}$$

The last term, $V(2k_F)$, is a high-momentum term that is absent in the Luttinger model. In the model studied by HK the interaction between particles with the same chirality is equal to the interaction between particles of opposite chirality, i.e. $V_{1,q} = V_{2,q}$. If we ignore the last term in (5.2) and translate the interaction in terms of the formalism used in this thesis, we find that the charge can be written as

$$e^* = 1 - \frac{1}{2k_F} V_{1/2,0}. \tag{5.3}$$

FG study a completely different process than HK, and they also conclude that fractional excitations exist in one dimension. FG work within the formalism of the Luttinger model, and their conclusion is based on what is called chiral separation [18]. If we go back to Eq. (2.38) and make a redefinition of the fields $\Theta_{R/L}(\xi) = \Theta(\xi) \mp K^{-1}\Phi(\xi)$, we see that the Hamiltonian can be cast in the form

$$H = \frac{uK}{4} \int_{-L/2}^{L/2} d\xi \left\{ [\partial_\xi \Theta_R(\xi)]^2 + [\partial_\xi \Theta_L(\xi)]^2 \right\}.$$

It can be shown that these fields satisfy the equations of motion

$$\left(\partial_\xi \pm \frac{1}{u}\partial_t\right)\Theta_{\mathrm{R/L}}(\xi,t) = 0,$$

which means that the fields are chiral, $\Theta_{\mathrm{R/L}}(\xi,t) = \Theta_{\mathrm{R/L}}(x \mp ut)$. FG study the effect of backscattering by an impurity potential in the 1D wire. The dominant $2k_F$ backscattering is given by the operator

$$H_{\mathrm{imp}} \sim \int_{L/2}^{-L/2} \delta(\xi)\left(\psi_+^\dagger(\xi)\psi_-(\xi) + h.c.\right). \tag{5.4}$$

The field operators are given by eq. (2.39), and they can be reformulated in terms of the fields $\Theta_{\mathrm{R/L}}(\xi)$. This means that (5.4) will give rise to two counterpropagating modes, and FG identify the charges of these modes as $\pm K$.

PGL have a very general approach to the existence of fractional charges [18]. They claim that the *fundamental* excitations in the Luttinger liquid are fractional. Their approach gives rise to a multitude of different fractional charges depending of how they are created. We will not go into the details here, but rather point out the general idea, which is based on the injection of particles. The central operator in their approach is an operator that injects Q particles with momentum q and current J,

$$V_{Q,J}(q) = \frac{1}{\sqrt{L}}\int_{-L/2}^{L/2} d\xi e^{i(q-Jk_F)\xi} : e^{-i\sqrt{\pi}(Q\Theta(\xi)-J\Phi(\xi))} :, \tag{5.5}$$

where $: .. :$ means normal ordering with respect to the bosonic operators. The quantum number J is as usual the difference between right- and left moving particles in the non-interacting ground state. Note that for $Q = 1$ and $J = 1$ this operator corresponds to the Fourier transform of $\psi_+^\dagger(\xi)$ in (2.39). In terms of the $\Theta_{\mathrm{R/L}}(\xi)$ operators the injection (5.5) can be split into a right moving mode and a left moving mode, which carry charges $Q_\pm = \frac{1}{2}(Q\pm KJ)$. Hur et. al have proposed and experiment that involves the tunneling into a quantum wire, and they predict charges in agreement with the analysis of PGL [34].

We see that PGL's picture corresponds to FG's conclusion, since a backscattering of a particle is a $Q = 0$, $J = 2$ excitation which will give rise to counterpropagating charges $\pm K$. However, there is not complete agreement with HK's result. We saw that the injection of a unit charge in HK's approach only gave rise to one chiral charge. The injection of a unit charge is a

$Q = 1$ and $J = 1$ excitation which, according to PGL, will give rise to counterpropagating charges $Q_\pm = \frac{1}{2}(1\pm K)$. We see that to first order in the interaction the right moving charge is given by $Q_+ = \frac{1}{2}\left(1 + \sqrt{\frac{v_J}{v_N}}\right) \approx 1 - \frac{1}{2\hbar v_F}V_{2,0}$, which does indeed agree with HK (if we put $\hbar = m = 1$, which is the notation used by HK).

It is interesting that the charge found by HK agree with what PGL predict. However, it is an open question whether this agreement also holds to higher order in the interaction. As we will see, this agreement does not hold when the calculation is made exact. Another way to try to shed some light on the excitations in the Luttinger model is by use of our quantum Hall model described in Paper I and II. In contrast to the Luttinger model, which is only one dimensional, our quantum Hall model is two-dimensional, which means that excitations can not only be characterized by their chirality, also the edge they reside on must be taken into account. We will in this chapter look at two scenarios: First we will discuss the injection of a particle on the upper edge. This particle will be injected into the interacting system, and in that sense it resembles the injection mechanism discussed by PGL. We will see that this approach gives results that agree with the the right and left moving charges in PGL, but the picture is more complex since the charges are distributed on two edges. The second approach is also an injection on the upper edge. However, the injection is into the non-interacting system and the interaction is then adiabatically turned on in a fashion similar HK's approach. In contrast to HK's calculation the powerful machinery of bosonization makes it possible to identity the charge exactly, not only to the lowest order in the interaction. What we find is this charge is *not* equal to PGL's charge, but it does agree with HK's result to first order in the interaction.

5.2   Electron injection on the edge

Before we go into a detailed calculation it is a good idea to get some physical insight into the system in question. We will start by discussing the non-interacting system. We remember from Eq. (2.22) that the density operator could be written as

$$\rho_{\chi,q} = \delta_{q,0}N_\chi + \sqrt{\frac{L|q|}{2\pi}}\left\{\theta(\chi q)a_q + \theta(-\chi q)a_{-q}^\dagger\right\}.$$

This operator is related to the density operator in the x-representation by (2.5). We know that the time dependent non-interacting bosonic operators is given by $a_q(t) = e^{-iv_F|q|t}a_q$. We then see that the expectation value of $\rho_\chi(x)$

in a given state gives the density moving in the χ-direction. For the non-interacting system particles with negative chirality correspond to the upper edge of the 2D system and particles with positive chirality correspond to the lower edge. From now on we will therefore take the parameter χ to mean the upper edge for $\chi = -$ and the lower edge for $\chi = +$.

If we now turn to the interacting system the original chiral operators are not chiral anymore. The easiest way to see this is by considering the non-interacting bosonic operator a_q, which in the non-interacting system is an operator with positive chirality for positive q. By inverting the Bogoliubov transformation (2.26) we find that

$$a_q = \cosh \zeta_q b_q + \sinh \zeta_q b_{-q}^\dagger \,. \tag{5.6}$$

The time dependent interacting operator is given by $b_q(t) = e^{i\omega_q t} b_q$. This means that for $q > 0$ the operator a_q includes both chiralities.

Let us now turn to the operator for the densities on the edges. By use of (5.6) it can be re-expressed as

$$\rho_{\chi,q}(t) = \delta_{q,0} N_\chi + \sqrt{\frac{L|q|}{2\pi}} \cosh \zeta_q \left\{ \theta(\chi q) e^{-i\omega_q t} b_q + \theta(-\chi q) e^{i\omega_q t} b_{-q}^\dagger \right\}$$

$$+ \sqrt{\frac{L|q|}{2\pi}} \sinh \zeta_q \left\{ \theta(-\chi q) e^{-i\omega_q t} b_q + \theta(\chi q) e^{i\omega_q t} b_{-q}^\dagger \right\} \tag{5.7}$$

By taking the expectation value of this operator in a given state we see that we in general can identify left moving excitations both on the upper and lower edge, and right moving excitations on both edges. We see that the left moving excitations on the lower edge is equal to κ times the left moving excitation on the upper edge, where $\kappa = \tanh \zeta_q$, and that the right moving excitation on the upper edge is equal to κ times the right moving excitation on the lower edge. The picture that emerges is that when the interactions are turned on,the chiralities are no longer confined to each edge. The interaction "pushes" some of the left moving chirality down to the lower edge, and some of the right moving chirality up to the upper edge.

Let us exemplify this picture. Assume that a particle is injected into the upper edge in the interacting system, this will give rise to a left propagating charge Q on the upper edge. Some of the left moving charge will be found on the lower edge with value κQ. The total charge on the lower edge has to be conserved, which means that there has be a right moving charge $-\kappa Q$ on the lower edge as well. However, this right moving charge means that there is a right moving charge $-\kappa^2 Q$ on the upper edge as well, see Figure 5.1. Charge conservation on the upper edge means that $Q - \kappa^2 Q = 1$. If we identify the charge by taking the $q \to 0$ limit, we find that $Q = \cosh^2 \zeta_0$.
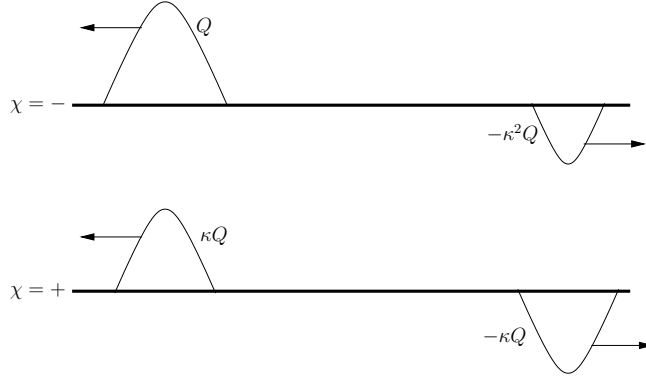
Figure 5.1: The injection of an electron on the upper edge, will give rise to both left moving and right moving excitations on both edges.

Let us do this analysis more quantitatively. We will consider a wave packet on the χ-edge, where χ refer to the edges (i.e. the non-interacting chirality), defined by

$$|\Phi_\chi\rangle = \int d\xi \varphi(\xi)\psi_\chi^\dagger(\xi)|\text{IGS}\rangle,$$

where $|\text{IGS}\rangle$ is the interacting ground state. Normalization of this state means that

$$\int d\xi_1 d\xi_2 \varphi^*(\xi_1)\varphi(\xi_2)\langle\text{IGS}|\psi_\chi(\xi_1)\psi_\chi^\dagger(\xi_2)|\text{IGS}\rangle = 1.$$

To be able to identify the different propagating modes, we will study the expectation value of the time dependent density operator in this state. We will therefore consider the quantity

$$D_{\chi',\chi}(q,t;\xi_1,\xi_2) = \langle\text{IGS}|\psi_{\chi'}(\xi_1)\rho_{\chi,q}(t)\psi_{\chi'}^\dagger(\xi_2)|\text{IGS}\rangle, \qquad (5.8)$$

where $\rho_{\chi,q}(t)$ is given by (5.7). We see that $D_{\chi',\chi}$ includes matrix elements of the form

$$\langle\text{IGS}|\psi_{\chi'}(\xi_1)b_q\psi_{\chi'}^\dagger(\xi_2)|\text{IGS}\rangle \quad \text{and} \quad \langle\text{IGS}|\psi_{\chi'}(\xi_1)b_{-q}^\dagger\psi_{\chi'}^\dagger(\xi_2)|\text{IGS}\rangle.$$

Since $b_q|\text{IGS}\rangle = 0$, these matrix elements can be calculated by commuting b_q to the right and $b_{-q}^\dagger$ to the left. From the definition of the the non-interacting

bosonic operators, (2.10) and the commutation relation between the density operator and the field operator, (2.6), we easily see that

$$
[a_q, \psi_\chi^\dagger(\xi)] = \sqrt{\frac{2\pi}{L|q|}} \theta(\chi q) e^{-iq\xi} \psi_\chi^\dagger(\xi)
$$

$$
[a_q^\dagger, \psi_\chi^\dagger(\xi)] = \sqrt{\frac{2\pi}{L|q|}} \theta(\chi q) e^{iq\xi} \psi_\chi^\dagger(\xi) \,. \tag{5.9}
$$

From the Bogoliubov transformation, (2.26), it then follows

$$
[b_q, \psi_\chi^\dagger(\xi)] = \sqrt{\frac{2\pi}{L|q|}} \left\{ \theta(\chi q) \cosh \zeta_q - \theta(-\chi q) \sinh \zeta_q \right\} e^{-iq\xi} \psi_\chi^\dagger(\xi)
$$

$$
[\psi_\chi(\xi), b_q^\dagger] = \sqrt{\frac{2\pi}{L|q|}} \left\{ \theta(\chi q) \cosh \zeta_q - \theta(-\chi q) \sinh \zeta_q \right\} e^{iq\xi} \psi_\chi(\xi) \,.
$$

Using (2.18), we find that

$$
\begin{aligned}
&D_{\chi',\chi}(q, t; \xi_1, \xi_2) \\
&= \delta_{\chi,\chi'} \langle \mathrm{IGS} | \psi_{\chi'}(\xi_1) \psi_{\chi'}^\dagger(\xi_2) | \mathrm{IGS} \rangle \\
&\quad \times \left[\delta_{q,0} + \cosh \zeta_q \left\{ \theta(\chi q) e^{-i(q\xi_2 + \omega_q t)} + \theta(-\chi q) e^{-i(q\xi_1 - \omega_q t)} \right\} \right. \\
&\qquad \left. - \sinh \zeta_q \left\{ \theta(\chi q) e^{-i(q\xi_1 - \omega_q t)} + \theta(-\chi q) e^{-i(q\xi_2 + \omega_q t)} \right\} \right] \\
&\quad + \delta_{\chi,-\chi'} \langle \mathrm{IGS} | \psi_{\chi'}(\xi_1) \psi_{\chi'}^\dagger(\xi_2) | \mathrm{IGS} \rangle \\
&\qquad \times \sinh \zeta_q \cosh \zeta_q \left[\left\{ \theta(\chi q) e^{-i(q\xi_1 - \omega_q t)} + \theta(-\chi q) e^{-i(q\xi_2 + \omega_q t)} \right\} \right. \\
&\qquad\qquad \left. - \left\{ \theta(\chi q) e^{-i(q\xi_2 + \omega_q t)} + \theta(-\chi q) e^{-i(q\xi_1 - \omega_q t)} \right\} \right] \,. \tag{5.10}
\end{aligned}
$$

Let us assume that the state $|\Phi_\chi\rangle$ is constructed by injecting an electron on the upper edge ($\chi = -$) in the point ξ_0, this means that the envelope function goes as $\varphi(\xi) \sim \delta(\xi - \xi_0)$. We identify the localized charge by taking the $q \to 0$ limit. To find the correct chirality we have to remember that the density operator is given by $\rho_\chi(\xi) \sim \sum_q e^{iq\xi} \rho_{\chi,q}$, the correct chirality is therefore determined by the relative sign between ξ and t in the exponentials. The expectation values in (5.10) will not contribute to the charge due to the normalization condition of the state $|\Phi_\chi\rangle$. The charges can then be read out from (5.10) by taking the $q \to 0$ limit. The excitations and their charges are summed up in the table below:

	Left moving	Right moving
Upper edge ($\chi = -$)	$\cosh^2 \zeta_0$	$-\sinh^2 \zeta_0$
Lower edge ($\chi = +$)	$\cosh \zeta_0 \sinh \zeta_0$	$-\cosh \zeta_0 \sinh \zeta_0$

We see that the result do agree with the discussion at the beginning of this section. Let us compare our results with the charges found by PGL. From the definition of ζ_q, (2.28), and v_N and v_J, (2.25), we see that $K = \sqrt{\frac{v_J}{v_N}} = e^{2\zeta_0}$. The total left moving and right moving charges are then found to be

$$Q_- = \cosh^2 \zeta_0 + \cosh \zeta_0 \sinh \zeta_0 = (1 + K)/2$$
$$Q_+ = -\sinh^2 \zeta_0 - \cosh \zeta_0 \sinh \zeta_0 = (1 - K)/2\,.$$

Since the injection of a particle on the upper edge is a $Q = 0$ and $J = -1$ excitation, we see that our result is in full agreement with the charges predicted by PGL, $Q_\pm = \frac{1}{2}(Q \pm KJ)$.

5.3   Adiabatic approach

We will now study the propagation of a wavepacket by a different approach. The wavepacket is introduced on the edge of the *non-interacting* system, and the interaction is then adiabatically switched on. This approach is similar to the idea in HK. The transformation from the non-interacting system to the interacting one is conveyed by the operator S, (2.30). The state is therefore given by

$$|\tilde{\Phi}_\chi\rangle = S \int d\xi \tilde{\varphi}(\xi) \psi_\chi^\dagger(\xi)|\mathrm{F}\rangle.$$

The S operator transforms a non interacting bosonic operator into an interacting one, that means that an operator O will have the same effect in a non interacting system as $SOS^\dagger$ on an interacting system. Since the operator $\psi_\chi^\dagger(\xi)$ creates a particle with chirality χ in the non interacting system, $S\psi_\chi^\dagger(\xi)S^\dagger$ will also create a particle with chirality χ, but in the interacting system. We will therefore expect the state $|\tilde{\Phi}_\chi\rangle$ to contain only modes with chirality χ. Let us assume that we inject a particle on the upper edge, this will give rise to a left moving charge Q on the upper edge. But due to the interaction some of the left moving chirality is found on the lower edge, and, as we see from the previous discussion, carry a charge κQ.

As in the previous section, we will study the time dependent density of the state $|\tilde{\Phi}_\chi\rangle$, which is assumed to be normalized. The quantity of interest is therefore

$$G_{\chi',\chi}(q,t;\xi_1,\xi_2) = \langle \mathrm{F}|\psi_{\chi'}(\xi_1)S^\dagger \rho_{\chi,q}(t)S\psi_{\chi'}^\dagger(\xi_2)|\mathrm{F}\rangle. \qquad (5.11)$$

From (2.29) we see that $S^\dagger b_q S = a_q$, this gives that $S^\dagger \rho_{\chi,q}(t)S$ is given by (5.7) with b_q and $b_{-q}^\dagger$ replaced by a_q and $a_{-q}^\dagger$, respectively. We see that (5.11)

will include matrix elements of the form

$$\langle F|\psi_{\chi'}(\xi_1)a_q\psi^\dagger_{\chi'}(\xi_2)|F\rangle \quad \text{and} \quad \langle F|\psi_{\chi'}(\xi_1)a^\dagger_{-q}\psi^\dagger_{\chi'}(\xi_2)|F\rangle\,.$$

Since $a_q|F\rangle = 0$, these matrix elements can be calculated by commuting a_q to the right, and $a^\dagger_{-q}$ to the left. By use of the commutators (5.9) we easily find that

$$\begin{aligned}
G_{\chi',\chi}&(q,t;\xi_1,\xi_2)\\
&= \delta_{\chi,\chi'}\langle F|\psi_{\chi'}(\xi_1)\psi^\dagger_{\chi'}(\xi_2)|F\rangle\\
&\quad\times \left[\delta_{q,0} + \cosh\zeta_q\left\{\theta(\chi q)e^{-i(q\xi_2+\omega_q t)} + \theta(-\chi q)e^{-i(q\xi_1-\omega_q t)}\right\}\right]\\
&\quad + \delta_{\chi,-\chi'}\langle F|\psi_{\chi'}(\xi_1)\psi^\dagger_{\chi'}(\xi_2)|F\rangle\\
&\quad\times \sinh\zeta_q\left\{\theta(\chi q)e^{-i(q\xi_1-\omega_q t)} + \theta(-\chi q)e^{-i(q\xi_2+\omega_q t)}\right\}\,.
\end{aligned} \tag{5.12}$$

First of all we notice that $G_{\chi',\chi}$ is discontinuous as $q \to 0$, which means that some of the charge has been delocalized. If we assume that we inject an electron on the upper edge we see by an analysis similar to the one preceding Eq. (5.10) that the charges that appear in the system are as given in the table below:

	Left moving	Right moving
Upper edge ($\chi = -$)	$\cosh\zeta_0$	0
Lower edge ($\chi = +$)	$\sinh\zeta_0$	0

We see some clear differences from the results in the previous section: First of all we note that there is no right moving charge, this agrees with the discussion at the beginning of this section. Secondly we note that the left moving charges in this case are different from the charges found previously. On the other hand, the ratio between the charge on the lower edge and the upper edge, $\tanh\zeta_0$, is the same. In the introduction to this chapter we saw that there was agreement between the charges found in PGL and HK, but we have to remember that HK's result is only correct to first order in the interaction. We see that that the bosonization technique made it possible for us to calculate the same quantity exactly.

Let us relate the total left moving charge to the parameter K, we easily see that

$$\tilde{Q}_- = \cosh\zeta_0 + \sinh\zeta_0 = \sqrt{K},$$

whereas the left moving charge found in the previous section was given by $Q_- = (1 + K)/2$. If we assume that the interaction is weak can make the approximation $K \approx 1 - \frac{V_{2,0}}{\hbar v_F}$, it then follows that Q_- and $\tilde{Q}_-$ agree to first order in the interaction, as expected. Note that this agreement holds in the

case where the interaction between particles of the same chirality is different from the interaction between particles of opposite chirality, a case that was not discussed by HK.

5.4   Discussion

Motivated by the discussion of fractional charges in the Luttinger model we have studied the excitations on the edges of the quantum Hall model introduced in Paper I and II. The excitations are created by the injection of a particle on one edge only. We see that the injection gives rise to a rich structure in the sense that the excitations reside on both edges. From our analysis we see that these excitations carry fractional charges. However, the two different approaches gives rise to two different excitations with different charges. The adiabatic turning on of the interaction gives rise to a charge that does not fit into the general approach of PGL. One can therefore ask whether the concept of fractional charges in the Luttinger model is useful, since the charges created are dependent on the way they are created, but further study is required in this field.

Another interesting project for further study is to examine the two-dimensional density when an electron is injected into the two-dimensional quantum Hall system discussed in Paper I and II. This will involve the calculation of four point functions, which can be calculated exactly in one dimension due to the bosonization technique. A mapping, in analogy with the procedure used to calculate the 2D correlation function in Paper II, should in principle give the full 2D description of the density. On the other hand this mapping is much more complicated than the one solved in Paper II.

Summary

In this thesis I have discussed the quantum Hall system and its relation to the one-dimensional Luttinger model. In chapter 1 I gave an introduction to the quantum Hall effect. We saw how the restriction to the lowest Landau level suggested that a one-dimensional representation could be given. Both the integer and the fractional effect was discussed, and an introduction to Jain's composite fermion approach were given. Chapter 2 gave an introduction to the Luttinger model and it was shown how the bosonization technique made it possible to calculate correlation functions exactly. Then, in Chapter 3, the edge states of the quantum Hall effect was discussed. We saw how a general hydrodynamical approach predicted that the edge of the quantum Hall system could be described by a Luttinger liquid with a universal correlation function exponent. However, the value of the exponent do not agree with experiments. Some work have suggested that this discrepancy is due to the electron interaction. It is within this discussion my work is focused. My work is presented in three papers and one chapter and I will now show how they are related to this topic and how they are related to each other.

In Paper I we exploit the fact that there exists an explicit one-to-one mapping between a state in the lowest Landau level and a one-dimensional representation. We study a model of a $\nu = 1$ quantum Hall system with interacting electrons confined to a narrow linear channel by a harmonic potential. The system is mapped to its one-dimensional representation. The mapping is done at the microscopical level which means that there is no need for the qualitative arguments usually employed. In the low energy limit the one-dimensional representation takes the form of a Luttinger liquid with both chiralities present. The Luttinger parameters are calculated and it is shown that they are modified by the interaction. However, the current parameter is not renormalized and we relate that to the non-renormalization of the Hall conductivity due to the interaction. We calculate the 1D correlation function and find that the asymptotic exponent is renormalized in the case where the interaction reaches across the Hall bar.

The work in Paper I is followed up in Paper II, where we derive the

two-dimensional correlation function by applying the explicit mapping to the one-dimensional correlation function. We see that the 2D correlation function naturally splits into a bulk part not modified by the interaction and an edge part that is sensitive to the interaction. From the correlation function we find the electron density. The density is only minimally affected by the interaction, and the effect is only found close to the edges. On the other hand, the interaction can lead to renormalization of the exponent of the asymptotic correlation function. The exponent agrees with the exponent found in the one-dimensional representation.

Paper III is a numerical study motivated by a work by Mandal and Jain, where they suggest that interaction induced second composite Fermion Landau level mixing can lead to renormalization of the exponent in the fractional effect. Since the fractional effect can be seen as an integer effect of composite fermions, I wanted to see whether a similar renormalization effect could be seen in the $\nu = 1$ quantum Hall effect if mixing with the second Landau level was allowed. Each electron orbital in a completely filled lowest Landau level is allowed to mix with the corresponding momentum orbital in the second Landau level. Note that this approach is somehow complementary to the work in Paper I and II since scattering across the Fermi point is forbidden in this paper, while the study in Paper I and II was restricted to the lowest Landau level. The ground state of the interacting system was calculated by the use of the Hartree-Fock algorithm, for that reason an introduction to the algorithm was given in Chapter 4. No renormalization of correlation function exponent is found. However, the interaction gives rise to oscillations in the density profile.

Chapter 5 was devoted to an ongoing study motivated by the discussion of fractional excitations in one-dimensional systems. In particular we saw how the quantum Hall model discussed in Paper I and II could be used to give insight on the propagation of excitations in the Luttinger model. We also saw that different values of the charge could be identified depending on how the excitations are created in the system. However, further study is needed and I believe that our model can be a useful tool in this study.

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