Particles in Quantum Field Theory and Non-inertial Reference Frames

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Thanks, hugs and all that

No man is an island

Completing a thesis is not something one does very often (thankfully). It is also not done in a vacuum, even when vacuum is a main subject of the thesis. I would therefore like to thank all the people who have contributed to and enriched both this thesis and my life for the past nearly two and a half years. First of all thank you to my supervisor Jon Magne Leinaas, for suggesting a topic that, despite more than a fair share of depressions and frustrations along the way, has given me a well of new understanding about the foundations of quantum field theory and about theoretical physics in general. Thank you especially for your patience when the thesis work ground to a halt for a long time before taking a very different direction than originally intended. Thank you to all the people at the Department of Physics here in Oslo who have contributed with valuable discussions. Particular thanks to Joakim Bergli for both unending and unendingly stimulating conversations about the fundamentals of physical theories which have focused my thoughts and given me fascinating new insights. Thank you also to my girlfriend and my family for putting up with my incessant business in these years, though I am not sure I can promise that that will change very much. Thank you finally to all the industrious and lovable students at the department who make its social scene truly brilliant and unique (and a special thanks to Bjørn Hallvard Samset who got me pulled into it even before I started studying here). Thank you for having given me opportunities to grow and develop myself further and in more ways than I could have imagined when I began my studies five years ago. I dedicate this work to you, and only hope that I have been able to contribute something lasting in return. Thank you also for ensuring that this thesis was not finished sooner. ☺

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Abstract

In this thesis I discuss various topics relating to the definition of particles and vacuum states in quantum field theory in general, and apply it to non-inertial reference frames in Minkowski spacetime. The particle concept in quantum field theory is shown to be rather ambiguous and subjective.

I discuss generally what particles are and how they should be defined in quantum field theories. I then discuss what ambiguities are inherent in such a definition and in particular what ambiguities there are for observers in different stationary non-inertial reference frames in Minkowski spacetime. I use this to gain a broader perspective on the Unruh effect, the effect by which an accelerated observer will view the vacuum state of an inertial reference frame as being filled by a thermal ensemble of particles. I conclude that the effect actually depends on how the solutions of the field equation are joined across the event horizon that is present in hyperbolicly accelerated reference frames, and that the effect is really more subjective than is commonly assumed in the literature. Finally, I investigate the behaviour of a model particle detector. I conclude that it does not necessarily reflect the spectrum of particles that is present in the reference frame of the detector, because the excitation spectrum of the detector may be distorted by particle states with negative energy, which are present in many non-inertial reference frames. The results of detection experiments will therefore not generally agree with the usual definition of particles in quantum field theory.

Along the way I provide explicit calculations of all possible Killing vector fields and stationary trajectories in Minkowski spacetime, discussions of what the corresponding reference frames look like, as well as solutions of the Klein-Gordon equation in coordinates appropriate to the different reference frames.

Contents

Introduction — QFT and spurious particles

Is it or is it not — that is the question

There are certain things even physicists usually take for granted. One of these things is that a physical object either exists or it does not. Nevertheless, quantum physics can blur not only the momentum and position of an object, but make even its existence a matter of subjective judgement.

This was the astounding conclusion reached by William G. Unruh in a seminal paper ([1]) that appeared in 1976. Unruh was investigating the then recent result by Stephen Hawking that quantum field theory could allow black holes to emit particles with a thermal energy distribution. During his investigation, he discovered that an analogous effect also existed for accelerated reference frames in flat spacetime. An observer travelling along a straight trajectory with constant proper acceleration through a perfect vacuum in flat spacetime should in fact observe a thermal bath of particles pervading the vacuum. The temperature of this bath would be proportional to the acceleration, and Unruh was also able to show that a model particle detector travelling along such a trajectory would respond and become excited in just the way one would expect from a thermal bath of particles. Unruh's results showed that even for a spacetime as seemingly familiar as flat, Minkowski spacetime, the concept of a vacuum was not at all well-defined in quantum field theory. Whether space was empty or not and what kind of particles it contained would depend on the motion of the observer. The effect was tiny and the vacuum would mostly be quite chilly — a whopping 2.5×10^{20} m/s^2 of acceleration would be needed to produce just 1 K. But it would still be there. In effect, the question "Is there anything there?" would be answered differently by different observers, even without any black holes or other mysterious entities around.

Unruh's original paper has since prompted a steady stream of papers and a continuing discussion on how to interpret and generalize the phenomenon that he discovered, now referred to as the "Unruh effect". However, the effect and especially generalizations of it still seem to be marked by some confusion, and a comprehensive treatment of the fundamentals

of the effect and a motivation for all the assumptions one makes in deriving it, does not seem to exist.

This thesis was originally intended to be part of a discussion started by John S. Bell and Jon Magne Leinaas about whether electrons orbiting in storage rings could be used to study the Unruh effect (see [2] and [3]). After all, electrons in a circular orbit will experience a constant centripetal acceleration which one could imagine would give rise to a vacuum temperature in the same way as linear acceleration. And electrons in storage rings would be a lot easier to study than linearly accelerating particles that would zip off into the sunset long before any thermal effects could be measured. However, electrons in a storage ring would be experiencing a constant rotation of their reference frame in addition to the acceleration, and it was known that this causes some serious complications. A detector that does not follow a perfectly straight trajectory would respond differently to the vacuum than to a thermal bath, although with a bit of good-will it could be made to look approximately thermal. Bell and Leinaas therefore proposed that the excitations could be interpreted using an effective temperature, and that they were caused by a thermal Unruh effect with some additional complications due to the rotation.

Bell and Leinaas' conclusions were however flatly contradicted by results obtained by just looking at the states of the quantum field without involving any detectors. In fact, adding a little rotation to the acceleration should produce no change at all in the Unruh effect or the vacuum temperature, whereas adding enough rotation to produce a closed orbit, such as that of an electron in a storage ring, would remove it altogether (see the paper [4] by John R. Letaw and Jonathan D. Pfautsch). As work with the thesis progressed (or rather stagnated) it became frustratingly clear that just calculating detector responses would not paint a very clear picture of what was going on. A better understanding of the underlying quantum field theoretical formalism was needed.

In the literature however, there was not much comprehensive material to be found on how to define vacua and particle states in quantum field theory in a general way, and there even seemed to be some conflicting definitions. Clearly the method that Letaw and Pfautsch had used did not agree with e.g. defining particles as being things detected by particle detectors. More frustratingly, there was virtually nothing on *why* particle states and vacua should be constructed the way they were. Nobody seemed to have bothered to write anything about what a particle in a quantum field theory really is or why it should be what we usually consider it to be. The claims that some or other vacuum contained such and such particles with respect to an observer moving in this or that fashion were therefore correspondingly difficult to make real sense of.

The goal of this thesis therefore, are simply:

- to give a (hopefully) complete and consistent account of what particles are in terms of quantum field theories and why, as well as what ambiguities are inherent in these definitions
- to investigate how this definition is interpreted by observers in different reference frames in Minkowski spacetime, and how the Unruh effect fits into this
- to investigate the behaviour of a simple model particle detector in these reference frames and to give a full account of why it does not necessarily detect the kinds of

particles it should according to our definitions.

Chapters 1, 2 and parts of chapter 3 will be devoted to the first point, the rest of chapter 3 as well as chapters 4, 5 and 6 will discuss various aspects of the second point, and finally the detector issue will be discussed and resolved in chapter 7.

The approaches that I have taken in this thesis are technically rather simple. Although I do start out by treating general fields, I restrict the larger part of the thesis to real scalar fields, i.e. Klein-Gordon fields in order to make the discussion as transparent as possible. I have strived to make the discussion as thorough as possible, taking pains to explain and state my motivation every step along the way, since I felt that precisely this was frustratingly absent in much of the literature. I am not sure whether I have succeeded or not, but it has at least resulted in a rather voluminous thesis. My apologies if the reader feels that the number of words used is rather larger than he or she would have preferred.

Some of the results and conclusions in this thesis may be entirely new, although I suspect a lot of them may be known to the select few who have taken pains to develop a good understanding of what the Unruh effect is all about and what the foundations of quantum field theory really tell us about particles. Nevertheless, many of these conclusions seem to be either completely absent from the literature, or one has to carefully mine selected papers for clues that hint at them. Because there was very little comprehensible and complete material in the literature about the topics I discuss, I have had to arrive at many of the conclusions and develop many approaches (especially in the first two chapters) independently, although it seems a bit surprising if they are not "out there" already.

The results should mostly not be shocking or seem difficult to anticipate with some hindsight. Still, there may be some surprises that at least I did not expect beforehand. In particular, it will turn out that not only is the definition of vacua and particles a matter of subjective judgement. How subjective the judgement is, is in fact also rather subjective! May the reader not be bored...

Conventions

Before we start, a word about the conventions used in this thesis:

- I use "God-given" units in which $\hbar$ and c are dimensionless and equal to 1, unless otherwise stated.
- Boldface symbols, e.g. $\mathbf{P}$, denote three-dimensional vectors. Symbols with arrows over them, e.g. $\vec{P}$, denote four-vectors.
- In the beginning of the thesis I denote quantum mechanical operators by placing a hat over them, e.g. $\hat{H}$, to avoid confusing them with classical quantities. From chapter 3 onwards though, when there is little danger of confusion, I will stop using the hats.

Finally, I apologize for any blatant crimes against good typography committed in this thesis. It just goes to show that it pays to study the finer points of LaTeX layout before writing too much...

Chapter 1

The particle concept

A main topic in this thesis will be to investigate how particles, or particle-like quantum states, arise in quantum field theories, and how these particles behave and change when viewed from different kinds of reference frames in Minkowski spacetime. Before we can do that however, we first need to have a clear picture of what a particle is (or what we want it to be), and what kind of properties a particle should have. To that end, I will first look into the idea of a particle in classical mechanics, the theory in which the concept first appeared. As we shall see throughout this thesis, it is also the only kind of theory where the concept really has a clear and unambiguous meaning. After having developed some intuition for what a particle is classically, we will then see how the particle concept is adapted when we quantize the classical theory of single particles. In the next chapter we will then use all this to see how we can recognize particle-like states when we quantize a classical field. Note that in this and the following chapter, I have allowed for the particles and the fields or wave-functions that represent them to be of any type and have an arbitrary number of indices (to denote tensor components, spin, isospin, etc.). However, for the rest of the thesis I deal exclusively with real scalar (i.e. Klein-Gordon) particles, so if all the indices in the first two chapters become too messy to keep tabs on, they can be safely ignored. In some places I have also not been quite consistent in writing out the indices explicitly, and just let their presence be implicit.

1.1 The classical particle - what for?

In classical mechanics, the concept of a particle seems fairly intuitive and obvious. A (point) particle is usually thought of simply as a piece of matter having no extension (i.e. being point shaped), moving around in space according to Newton's laws of motion and having some intrinsic properties, such as mass and electric charge. Aside from any philosophical qualms we might have about squeezing a finite mass into a mathematical, unextended point, this pretty much sums it up. This kind of particle concept is of course intuitive to anyone who has played around with grains of sand and who has the imagination to picture them becoming infinitely small. But why is this concept useful, and how do we define it somewhat more stringently?

Classical mechanics, like any other part of physics, owes its success to the ability to

idealize systems by just the right amount. This is done by introducing concepts which capture precisely the properties and degrees of freedom of a system that interest us and which determine its behaviour (or the interesting part of its behaviour anyway), while discarding everything else that might muddy the picture. Since classical mechanics is basically a theory of how matter moves and behaves, an obvious concept to introduce is that of an idealized unit of matter. This is the role served by the particle.

The word particle stems from the Latin *particula*, meaning small part or little piece. Particles hence are small, idealized pieces of matter, possessing only the properties and degrees of freedom needed for them to be able to build up larger bodies of matter and endow these bodies with the properties that are relevant to the theory. In classical mechanics the relevant degrees of freedom for a particle are usually position and some form of momentum (linear, angular or otherwise, depending on which is more convenient for the system in question), while the relevant intrinsic properties are usually mass and sometimes electric charge (although one could imagine others, for example intrinsic spin, isospin, colour charge and other properties introduced in elementary particle physics). When several particles build up an extended body, their relative positions and momenta then give rise to degrees of freedom of the extended body which individual particles do not posses, such as shape and angular momentum, while their intrinsic properties can be added up in some appropriate way to give the total mass, total charge and so on of the extended body.

One may of course ask, why are the quantities listed as examples above better than any other arbitrary set of properties and degrees of freedom we might cook up? What sort of quantities is it prudent to use when describing a particle? One tentative answer is the following: if we want to find a set of quantities to describe some system, it makes sense to choose them so that as many of them as possible are conserved, i.e. constant in time. This way, our description of the system will not change with time unless the system interacts with some external system capable of altering its otherwise conserved quantities. Therefore properties used to describe a particle should be conserved quantities as long as the particle is isolated. And if it interacts with another particle, then the sum of the value of this property for each of the two particles should be conserved. This is usually satisfied by the quantities we commonly use to label particles, such as mass, electric charge, linear and/or angular momentum, etc. Also, by using time-independent quantities to describe a particle, we can assure ourselves that it has a measure of stability and constancy to it. If its defining properties could change at a whim, we could not be quite sure that the particle would even go on *being* a particle the next moment. Clearly, we do not want a particle to be a sort of “now you see it, now you don’t” phenomenon, unless there is some interaction (such as annihilation with an anti-particle) causing it to behave that way. Of course, for a completely arbitrary physical system, we cannot be sure that such conserved quantities always exist, or that it makes sense to talk about well-behaved particles with this kind of conservative and predictable behaviour. For most realistic systems (or at least for most systems that we physicists care to work with), certain symmetries insure that there are certain conserved quantities that we can use to characterize the system and the particles contained in it. We will now delve a bit deeper into the nature and origins of such physical conservatism.

1.2 Symmetries and conserved quantities in classical mechanics

For a classical particle, the precise origin of conserved quantities are not really all that important. All we have to know is that they exist and what they are, and then we just use them to describe and label our particles and simplify calculations. In field theories however, the way in which these conserved quantities arise will be a crucial guide to how to quantize the theories and how to recognize the appearance of particles in the resulting quantum field theory. Therefore, if nature has chosen to be conservationist, we need to understand why, at least at some level.

In classical mechanics, some particle properties are conserved by definition. In non-relativistic mechanics for example, mass and electric charge are just defined, unchangeable properties of everlasting, indestructible point particles. Other more dynamic quantities, such as mechanical energy and linear and angular momentum, may however still be or not be conserved. But in corresponding relativistic theories, particles may be created and annihilated, so that not even conservation of charge and mass (which relativisticly is just another form of energy) or similar intrinsic particle properties can be taken for granted. Even in e.g. the classical field theory of electromagnetism, the conservation of electric charge is not automatic but requires the famous addition of the displacement current term to Ampere's law, which allowed Maxwell to predict the existence of electromagnetic waves. So, even though we can often allow ourselves to take conservation of many quantities for granted in non-relativistic classical point-particle mechanics, we should still take a look at where it comes from in order to more easily generalize the particle concept to relativistic and quantum theories.

Conserved quantities are usually conserved because there is some continuous symmetry in the theory that gives rise to their conservation, as is demonstrated by Noether's symmetry theorem. By "continuous symmetry" we mean some transformation T_λ which transforms the dynamical variables of the system in a continuous manner as the parameter λ is changed, but still leaves the physical behaviour of the system unchanged (if we have a physical system with rotational symmetry, T_λ could e.g. be a rotation of the system, and the parameter λ could be the angle of rotation). If we have such a continuous symmetry transformation, then there exists some quantity Q_T corresponding to this transformation which is conserved, i.e. $\frac{d}{dt}Q_T = 0$.

The most frequently used method for showing this, is by using the Lagrangian formalism, in which the physical system is parametrized by a set of dynamical variables $\{q^i\}$ (typically the coordinates of the particles in a classical mechanical particle theory) and their time derivatives $\{\dot{q}^i\}$, and where one postulates that the time evolution followed by the system is the one which minimizes some functional $S[q^1(t),\, q^2(t),\, \ldots,\, q^N(t)]$, called the *action*. One almost always assumes that the action can be written as $S[q_i(t)] = \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt$, where $L(q_i(t), \dot{q}_i(t))$ is called the *Lagrangian* and is a function (not a functional) of the q_i and their time derivatives. A symmetry transformation is then one which leaves the action invariant (this will leave the physics of the problem unchanged, since the physics is determined by the action), and it is possible to use this to show that for every continuous symmetry transformation there is a corresponding conserved quantity.

Here I will follow a different approach, using instead the Hamiltonian formalism. As we shall see, this formalism has the advantage that the correspondence between generators of continuous transformations and their associated conserved quantities is more explicit. This will be especially useful when using conserved quantities to quantize field theories, as we will be doing in the next chapter.

In the Hamiltonian formalism, the physical system is parametrized by a set of coordinates q^i and a set of canonical conjugate momenta p_i (which, given a Lagrangian, can be related to the time derivatives of the q^i by the formula $p_i = \partial L/\partial \dot{q}^i$). The time evolution of the system is then governed by a function $H(q^i, p_i)$, the Hamiltonian, through Hamilton's equations:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = \frac{\partial H}{\partial q^i} \tag{1.1}$$

The variables q^i and p_i can be thought of as coordinates of a so-called phase space, and the state of a system can then be represented as a point in this phase space. The observable quantities of the system are functions of the q^i and p_i, i.e. they are functions defined on the phase space of the system. But, as we shall see, any such function can also act as a generator of a continuous transformation acting on the phase space.

To see this, it is useful to define the so-called *Poisson bracket*. If we have two quantities A and B which are both functions $A(q^i, p_i)$ and $B(q^i, p_i)$ on the phase space of the system, we define the Poisson bracket $\{A,\ B\}$ as follows:

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q^i}\frac{\partial A}{\partial p_i} \right) \tag{1.2}$$

If we have some function T defined on the phase space, we can use this function to generate an infinitesimal transformation as follows:

$$q^i \longrightarrow q^i + \epsilon \frac{\partial T}{\partial p_i} \qquad\qquad p_i \longrightarrow p_i - \epsilon \frac{\partial T}{\partial q^i} \tag{1.3}$$

where ϵ is an infinitesimal quantity parametrizing the transformation. Defining the transformation in this way, we can then write the infinitesimal change δA of any quantity A under the transformation in the following convenient way:

$$\delta A = \sum_i \left(\frac{\partial A}{\partial q^i}\delta q^i + \frac{\partial A}{\partial p_i}\delta p_i \right) = \sum_i \epsilon \left(\frac{\partial A}{\partial q^i}\frac{\partial T}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial T}{\partial q^i} \right) = \epsilon \{A, T\} \tag{1.4}$$

Here we see a direct correspondence between physical quantities and generators of continuous transformations: Any physical quantity is a function defined on phase space, and will hence generate a continuous transformation. Conversely, a generator of a continuous transformation of the kind given in eq. 1.4 is a function on phase space, and can therefore be interpreted as some kind of physical quantity. We can also see directly that if (and only if) A *commutes* with T, in the sense that $\{A, T\} = 0$, then the physical quantity A will be invariant under the transformation generated by T, and vice versa.

This finally brings us back to symmetry transformations and conserved quantities. A symmetry transformation is one that does not alter the behaviour of the system, i.e. which does not change the form of Hamilton's equations. This will be the case if it leaves the Hamiltonian invariant, that is if $\{H, T\} = 0$, where T is the generator of the transformation. But this will also mean that T is invariant under the transformation generated by H. And the transformation generated by H is precisely time translations:

$$\{T, H\} = \sum_i \left(\frac{\partial T}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial T}{\partial p_i}\frac{\partial H}{\partial q^i} \right) = \sum_i \left(\frac{\partial T}{\partial q^i}\frac{\mathrm{d}q^i}{\mathrm{d}t} + \frac{\partial T}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} \right) = \frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\partial T}{\partial t} \quad (1.5)$$

where $\frac{\partial T}{\partial t}$ is any explicit time dependence that T might exhibit in addition to its time dependence through p_i and q_i. Hence if $\{T, H\} = 0$ and $\frac{\partial T}{\partial t} = 0$, i.e. if T generates a symmetry transformation and does not have any explicit time dependence, T will be a conserved quantity.

1.3 A brief account of quantum mechanics

In quantum mechanics we need to take into account the experimental fact that not all properties of a particle (such as position and momentum) can be simultaneously measured or even defined with arbitrary precision, and that a particle can be in a superposition of several different physical states at the same time. I of course assume that the reader is familiar with standard quantum mechanics. But in order to prepare the way for certain kinds of arguments that I will use later on to intuitively define what a particle is in quantum field theory, and to tie the quantum mechanical particle together with the classical one in a straight-forward way, I will give a brief introduction to quantum mechanical formalism from a slightly different perspective than is usually given in textbooks. In the discussion in this chapter, I will denote all quantum mechanical linear operators by placing hats above them in order to distinguish between a classical observable itself (e.g. A) and its corresponding quantum mechanical operator ($\hat{A}$). Later on though, I will drop these hats in order to economize on the notation.

In quantum mechanics a physical system is described by a complex vector space of the various physical states which the system can occupy, each of which may have a definite value for a given physical quantity or be a superposition of states with different definite values for that quantity. A physical quantity possessed by the system is represented as a linear operator that acts on the vector space of states, namely the operator which generates the same transformation on the space of states as the corresponding classical quantity would have generated on classical phase space (in a sense to be more precisely defined in a moment). This is to say, if $|\psi\rangle$ is the quantum mechanical state of a system, $T(q^i, p_i)$ is a physical quantity and generator of a classical phase space transformation, and $\hat{T}$ is the corresponding quantum mechanical linear operator, then $-i\epsilon\hat{T}\,|\psi\rangle$ will be the infinitesimal change in the state $|\psi\rangle$ under an infinitesimal T-transformation (the reason for inserting the $-i$ in this definition of the generator will become clear in a moment). Under a finite T-transformation with parameter λ, the state $|\psi\rangle$ will then become $e^{-i\lambda\hat{T}}\,|\psi\rangle$. Note by the way

that the parameters ϵ and λ will have some dimensionality depending on the dimensionality of T. From eq. 1.4 we see in fact that ϵ (and thus also λ) must have a dimensionality which is the inverse of T, times length times momentum. This means that even if we decide on what units to measure T in, this does not necessarily fully define the units to be used for measuring ϵ and λ. We must also decide on a unit for length times momentum. This is exactly the role played by $\hbar$ in quantum mechanics; $\hbar$ sets the scale for measuring quantities of length times momentum. In this thesis I use $\hbar \equiv 1$ dimensionlessly, but if one does not use this convention of "God-given" units, then you need to replace ϵ and λ by $\frac{\epsilon}{\hbar}$ and $\frac{\lambda}{\hbar}$ in the exponential factors above in order to make the exponents dimensionless.

We further make the definition that if a system is in a state described by the state vector $|\psi\rangle$ and we make a measurement which amounts to asking the question "is the system in state $|\phi\rangle$?", then the probability of the answer being "yes" will be $|\langle\phi|\psi\rangle|^2$, where $\langle\phi|\psi\rangle$ is the inner product between $|\psi\rangle$ and $|\phi\rangle$. In so doing we also assume that the state $|\psi\rangle$ is normalized, i.e. $\langle\psi|\psi\rangle = 1$, so that the probability of finding the system in state $|\psi\rangle$ when it actually *is* in the state $|\psi\rangle$ will be 1. Since the normalization of the state vectors should not change under a canonical phase space transformation, the transformation generated by $\hat{T}$ will need to be unitary with respect to the inner product between the state vectors. This means that $(e^{-i\hat{T}})^\dagger = (e^{-i\hat{T}})^{-1}$, i.e. $e^{i\hat{T}^\dagger} = e^{i\hat{T}}$, so we must have $\hat{T}^\dagger = \hat{T}$, i.e. the operator of any physical observable must be hermitian. We will return to the question of how to define the inner product in a moment.

Finally we assume that the states with a definite value for the physical observable T are the states which are left physically unchanged by the transformation generated by $\hat{T}$. By physically unchanged, I mean that the state vector before and after the transformation are parallel, i.e. one is equal to a complex number times the other, since multiplying the state vector with a complex number with modulus 1 does not change the physical result of any measurements (the complex number must have modulus 1 since we have demanded that the transformation generated by $\hat{T}$ be unitary). These vectors are thus the the eigenvectors of $\hat{T}$, and we further assume that the measured value of the observable T is given by the corresponding eigenvalue. Since $\hat{T}$ is hermitian, the eigenvalues are all real, as they should be for a physical observable quantity. This furthermore implies that the T-transformation $e^{-i\lambda\hat{T}}$ only changes the eigenstates by a complex phase factor (determined by the eigenvalue), so that they remain the same physical states, as it should be, and the measured value for T in these states is simply the magnitude of the corresponding phase shift (divided by λ). This desired hermiticity of operators corresponding to physical observables was the reason for introducing the factor $-i$ in the definition of the generator.

From these assumptions, we get that if we have a system in the state $|\psi\rangle$ and we try to measure the value of the observable quantity A whose q.m. operator $\hat{A}$ has eigenvalues $\{a_i\}$ and eigenvectors $|a_i\rangle$, then we get the result a_i with probability $P_i = |\langle a_i|\psi\rangle|^2$. This means that the mean or expectation value of the measurement will be:

$$\langle A\rangle = \sum_i P_i a_i = \sum_i \langle\psi|a_i\rangle \langle a_i|\psi\rangle a_i = \sum_i \langle\psi|a_i\rangle \langle a_i|\hat{A}|\psi\rangle = \langle\psi|\hat{A}|\psi\rangle \qquad (1.6)$$

(where we have used the fact that since $\hat{A}$ is hermitian, its eigenvectors span the whole vector space, which implies that $\sum_i |a_i\rangle\langle a_i| = \mathbf{1}$, when we take the states $\{|a_i\rangle\}$ to be

orthonormal.) We could by the way also have done things the other way around; by postulating eq. 1.6 we could have concluded that the eigenstates of $\hat{A}$ would have been the only states with a definite value for A (since one can show that these would be the only ones for which the variance, $\langle A^2\rangle - \langle A\rangle^2 = \langle\psi|\,\hat{A}^2\,|\psi\rangle - \langle\psi|\,\hat{A}\,|\psi\rangle^2$, would be zero), and then worked our way backwards through the eigenstate expansion in eq. 1.6 to conclude that the probability of measuring a_i would have to be $\langle\psi|a_i\rangle\langle a_i|\psi\rangle = |\,\langle a_i|\psi\rangle\,|^2$.

Using this mean value we can give more precise meaning to the statement that the operator $\hat{T}$ should generate the same transformation on the vector space of q.m. states as the classical quantity T does on classical phase space. By this we will mean that if classically T generates an infinitesimal transformation under which $A \to A + \epsilon G$ for some physical quantity G (i.e. $\{A, T\} = G$, then $\hat{T}$ should generate an infinitesimal transformation under which the mean value of $\hat{A}$ transforms likewise, i.e.

$$\langle A\rangle = \langle\psi|\,\hat{A}\,|\psi\rangle \to \langle A\rangle + \epsilon\,\langle G\rangle = \langle\psi|\,\hat{A}\,|\psi\rangle + \epsilon\,\langle\psi|\,\hat{G}\,|\psi\rangle \tag{1.7}$$

This allows us to find the commutation relations between the operators. Under an infinitesimal T-transformation the quantum mechanical state vectors transform as $|\psi\rangle \to (1 - i\epsilon\hat{T})\,|\psi\rangle$. This gives us the following transformation relation for the mean value $\langle A\rangle$:

$$\langle A\rangle = \langle\psi|\,\hat{A}\,|\psi\rangle \to \langle\psi|\,(1 + i\epsilon\hat{T})\hat{A}(1 - i\epsilon\hat{T})\,|\psi\rangle = \langle\psi|\,\hat{A}\,|\psi\rangle - i\epsilon\,\langle\psi|\,\left[\hat{A}, \hat{T}\right]\,|\psi\rangle \tag{1.8}$$

By comparing eqs. 1.7 and 1.8 we then see that $[\hat{A}, \hat{T}] = i\hat{G}$. In other words, we get the quantum mechanical commutators from the classical Poisson brackets by making the substitution $[\hat{A}, \hat{T}] \leftrightarrow i\{A, T\}$, which e.g. gives us the fundamental commutation relation

$$\{q^i,\, p_j\} = \delta^i_j \qquad \Longrightarrow \qquad [q^i,\, p_j] = i\,\delta^i_j \tag{1.9}$$

Note that it may not be possible to apply the correspondence $[\hat{A}, \hat{T}] \leftrightarrow i\{A, T\}$ to all operators $\hat{A}$ and $\hat{T}$. $\hat{A}$ and $\hat{T}$ are both supposed to be functions of $\hat{q}^i$ and $\hat{p}_i$ in the same way that A and T are functions of q^i and p_i. However, the ordering of $\hat{q}^i$ and $\hat{p}_i$ is important since they do not commute, whereas the ordering of q^i and p_i when writing out A and T does not matter. For such quantities, there may be several different operators that correspond to the same classical quantity, so that the correspondence between the Poisson bracket and the quantum mechanical commutators cannot hold for all of them. In these cases, we have to express the operators as functions of other operators that do obey uniquely defined commutation relations (1.9 is always a sure bet) and use these to work out the commutators of the more ambiguous operators.

1.4 Q.M. for particles

After this perhaps somewhat lengthy introduction to familiar quantum mechanics, we are ready to tackle the question of what is meant by a quantum mechanical particle. I will do this quite thoroughly, even though it might not seem absolutely necessary. In particular, I will pay much attention to defining an inner product between the state vectors of a particle,

since the definition of this inner product will turn out to be very significant when it comes to discovering particle-like states in the quantum theory of fields in the next chapter. The definition of a suitable inner product at the end of this section is a bit tedious, and its conclusion is not especially surprising, but it is still instructive to see to what extent the familiar inner product for quantum mechanical wave functions has to be postulated and to what extent it *must* be the way it is. Also, to make the discussion as general as possible, I will allow for any kind of particles carrying any kind of indices (scalar, vector, spinor or whatever), although I will not make full use of this generality later on.

If we want to construct a quantum theory for a single particle, we can basically just take our standard classical notion of what a particle is and quantize it with the procedure mentioned above. If a particle is described classically by some coordinates q^i and momenta p_i and dynamical variables which are functions of the q^i and p_i, just simply convert them all into operators and impose the commutation relation in eq. 1.9. If you use a set of conserved quantities to label the particle, these quantities will still be conserved in the quantum theory, since if some quantity $A(q^i, p_i)$ is conserved for the classical particle, then $\{A, H\} = 0$, so in the quantum theory, $\hat{A}$ will commute with the Hamiltonian operator $\hat{H}$. If we use the Heisenberg picture, where the operators carry the time dependence and the state vectors are independent of time, this means that $\frac{d}{dt}\hat{A} = -i[A, H] = 0$, so $\hat{A}$ does not change with time. If we instead use the Schrödinger picture, where the state vectors carry all the time dependence, then if the state $|\psi(t = 0)\rangle$ starts out at time $t = 0$ as an eigenstate $|a\rangle$ of $\hat{A}$ with eigenvalue a, then at a later time t' we will have

$$\begin{gathered} |\psi(t')\rangle = e^{-i\hat{H}t}\,|\psi(0)\rangle = e^{-i\hat{H}t}\,|a\rangle \\ \Downarrow \\ \hat{A}\,|\psi(t)\rangle = \hat{A}e^{-i\hat{H}t}\,|a\rangle = e^{-i\hat{H}t}\hat{A}\,|a\rangle = a\,|\psi(t)\rangle \end{gathered} \tag{1.10}$$

Either way, we see that the value of A does not change with time, hence A is conserved for the quantum particle also and can be used to label the quantum particle just as well as the classical particle.

There is however one very important difference. In quantum mechanics we can use the same quantities as in classical mechanics to characterize the particle, but we may not be able to use them all at the same time. We said that a state with a definite value for some quantity A is an eigenstate for the corresponding operator $\hat{A}$. But two operators $\hat{A}$ and $\hat{B}$ can only have a shared set of eigenstates if they commute. Therefore we can only use a set of quantities whose operators all commute to label our particle. In quantum mechanics we therefore need to be more careful about how we choose the quantities we use to describe a particle, and the behaviour and nature of the particle depends on our choice. This is demonstrated e.g. by the (in)famous *particle-wave duality*: a particle such as the electron or the photon may behave as a particle (with a fairly sharply defined position, but not momentum) in certain contexts, such as absorption or emission by atoms or when hitting a photographic plate, but as waves in other contexts (with a sharply-defined momentum but not position), e.g. when passing through a crystal lattice or the slits of a diffraction grating. In the former case, the most important variable for characterizing the behaviour of the particle is position, in the latter it is momentum or wavelength.

The most usual quantities used to label a particle in quantum mechanics are quantities such as momenta, angular momenta and energy itself, since the operators of these quantities frequently commute with the Hamiltonian operator and are conserved. These operators in turn do *not* usually commute with the position operators $\hat{q}^i$. Nevertheless, one of the hallmarks of a classical particle is that it is localized to a mathematical point, i.e. to a well defined position. Also most interactions between a particle and its surroundings (such as our measurement apparatuses) are local. Hence we would still like to talk about the whereabouts of a quantum mechanical particle, even though a well-defined position is incompatible with the other variables we usually use to characterize the particle. Furthermore, if we want to talk about quantum *fields*, we obviously need to bring spatial positions into the picture, since a field naturally is some quantity that varies with position in space (and time).

To do explicit calculations and to define an inner product between state vectors, we also need to work in a specific representation of the operator algebra and the state vector space. The obvious choice if we want to talk about spatial positions is then to represent the state vectors by functions of position, $\psi(\mathbf{q})$ (scalar, vector functions, or other kinds of functions with various kinds of indices, depending on what sort of particle the state should describe), and the operators by differential and multiplicative operators acting on these functions. Since the momentum operator $\hat{p}_i$ generates translations in the coordinate q^i, a natural choice of operator to represent $\hat{p}_i$ in this representation is $-i\frac{\partial}{\partial q^i}$ (or the covariant derivative $-i\nabla_i$ for non-scalar functions). The commutation relation 1.9 is then satisfied if we simply let $\hat{q}^i$ act on $\psi(\mathbf{q})$ by multiplying it by q^i, i.e. $\hat{q}^i\,|\psi\rangle$ is represented by the function $q^i\psi(\mathbf{q})$. Since the eigenstates of $\hat{q}^i$ are states with a sharp, definite value for the particle position, this definition of $\hat{q}^i$ as being represented by multiplication by q^i also makes sense if we wish to interpret $\psi(\mathbf{q})$ as some kind of probability density for finding the particle at the point $\mathbf{q}$. The eigenfunctions of the multiplicative operator q^i are functions which only have support for a single value of the coordinate q^i, so that if we interpret the magnitude of the wave-function at a point q^i as indicating the probability density of finding the particle at that point, then the eigenfunctions of $\hat{q}^i$ are precisely those that are sharply peaked around one point, as they should be (but as we shall see, this interpretation is not as straight-forward in the general case as in the usual non-relativistic Schrödinger theory).

1.5 Inner products for quantum particles

We can now start to grapple with the question of defining the inner product between the state vectors, which as I mentioned is important to how we will later define particle states in quantum field theories. We will define the inner product between the state vectors by defining an inner product between the position-dependent functions that represent them. For the moment, we will work in the Schroödinger picture, so that the functions will depend on time as well as spatial coordinates. The method that I will use here is based on a discussion in the book [5] by Takahashi, although I have elaborated on it a bit.

Let $\langle f, h\rangle$ denote the inner product between two functions f and g. It must satisfy the following demands:

- At least the two following axioms of a complex inner product must be satisfied:

1. Symmetry: $\langle f, g\rangle = \langle g, f\rangle^*$
2. Linearity: $\langle f, ag + bh\rangle = a\langle f, g\rangle + b\langle f, h\rangle$ for complex numbers a and b.

Normally, an inner product must also satisfy the axiom of positivity, i.e. the property that $\langle f, f\rangle \geq 0$, with equality if and only if f is identically zero. We will however find that we may have to loosen up a bit on this requirement,

- Operators corresponding to observable quantities must be hermitian with respect to the inner product. This is satisfied if and only if the operators $\hat{q}^i$ and $\hat{p}_i$ are all hermitian, i.e. $\langle f\ ,\ q^i g\rangle = \langle q^i f\ ,\ g\rangle$ and $\langle f\ ,\ -i\nabla_i\, g\rangle = \langle -i\nabla_i\, f\ ,\ g\rangle$ (since any symmetrized polynomial of hermitian operators is itself hermitian. We implicitly assume that when operators are written out as polynomials of $\hat{q}^2$ and $\hat{p}^2$, they are symmetrized in order to satisfy this requirement).

For any set of sufficiently well behaved functions, linearity implies that $\langle f, g\rangle$ can be written as $\int \mathcal{D}(f, g)dV$, where $\mathcal{D}$ is a linear function of f and g and/or their derivatives (to any order in general), and the integral is taken over all of space. The symmetry requirement can then be fulfilled by letting $\mathcal{D}(f, g)$ have the form $f^{I\,*}\, D_{IJ} g^J + g^I\, D^*_{IJ} f^{J\,*}$, where D is some linear combination of differential operators (possibly both space and time derivatives), which can possibly act both to the left and to the right. The indices I and J used here collectively represent all indices (vector, spinor, gauge, ...) the functions may be carrying, and if such indices are present, then D_{IJ} represents the components of a corresponding matrix of differential operators. Repeated indices are implicitly summed over.

If the operator $\hat{p}_i = -i\nabla_i$ is to be hermitian, we must have:

$$\begin{aligned}
\langle f, -i\nabla_i\, g\rangle &= \int \left(f^{I\,*}\, D_{IJ}(-i\nabla_i\, g^J) + (-i\nabla_i\, g^I)\, D^*_{IJ} f^{J\,*}\right) dV \\
&= \int \left[(-i\nabla_i\, f^I)^*\, D_{IJ} g^J + f^{I\,*}(i\nabla_i\, D_{IJ}) g^J + g^I\, (i\nabla_i\, D^*_{IJ}) f^{J\,*}\right. \\
&\qquad \left. + g^I\, D^*_{IJ}(-i\nabla_i f^J)^*\right] dV \\
&\stackrel{!}{=} \langle -i\nabla_i f, g\rangle = \int \left[(-i\nabla_i\, f^I)^*\, D_{IJ} g^J + g^I D^*_{IJ}(-i\nabla_i\, f^J)^*\right] dV
\end{aligned} \tag{1.11}$$

(where we have used partial integration and assumed that the functions vanish at infinity, so that the border terms are zero) For this equality to hold in general, we should have $\nabla_i\, D = 0$. This will be the case if D is either position-independent or depends on position only by being a function of the metric tensor $g_{\mu\nu}$, assuming that the covariant derivative is defined using a metric connection, so that the covariant derivative of the metric is zero. There are of course cases in which eq. 1.11 is satisfied without having $\nabla_i\, D = 0$, but in most theories it will anyway be desirable to have an inner product which is determined by the metric, since the inner product will then be expressed in explicitly general relativistically covariant terms.

The demand that $\hat{q}^i$ be hermitian puts further restrictions on D:

$$\begin{aligned}\langle f, q^i g\rangle &= \int \left(f^{I\,*} D_{IJ}(q^i g^J) + q^i g^I D^*_{IJ} f^{J\,*}\right) dV \\ &= \int \left(q^i (f^{I\,*} D_{IJ} g^J + g^I\, D^*_{IJ} f^{J\,*}) + f^{I\,*}\,[D_{IJ}, q^i](g^J)\right) dV \\ &\stackrel{!}{=} \langle q^i f, g\rangle \;=\; \int \left(q^i f^{I\,*} D_{IJ} g^J + g^I\, D^*_{IJ} q^i f^{J\,*}\right) dV \\ &= \int \left(q^i (f^{I\,*} D_{IJ} g^J + g^I\, D_{IJ} f^{J\,*}) + g^I\,[D^*_{IJ}, q^i](f^{J\,*})\right) dV\end{aligned} \tag{1.12}$$

We thus get the restriction

$$f^{I\,*}\,[D_{IJ}, q^i](g^J) = g^I\,[D^*_{IJ}, q^i](f^{J\,*}) \tag{1.13}$$

for any functions f and g that are legal wave functions, or at least that the integral of the two sides be equal.

Despite all these restrictions, we are still left with a huge range of differential operators D that all comply, especially if we demand only that the integral of the two sides in eq. 1.13 be equal. However, the inner product is linked to probabilities. More precisely, if the system is in the state represented by $\psi(\mathbf{q})$ we want to interpret the integrand of the inner product $\langle\psi, \psi\rangle$ as the probability density $\rho(\mathbf{q})$ of finding the particle at the point $\mathbf{q}$ in a precise position measurement. This follows from the fact that

$$|\langle\delta(\mathbf{q} - \mathbf{q}_0), \psi(\mathbf{q})\rangle|^2 = \langle\psi(\mathbf{q}), \delta(\mathbf{q} - \mathbf{q}_0)\rangle\,\langle\delta(\mathbf{q} - \mathbf{q}_0), \psi(\mathbf{q})\rangle \tag{1.14}$$

is by definition equal to the probability density of finding the particle at $\mathbf{q}_0$, since $\delta(\mathbf{q} - \mathbf{q}_0)$ is the eigenfunciton of $\hat{\mathbf{q}}$. This must be equal to the integrand of $\langle\psi, \psi\rangle$, since

$$\langle\psi, \psi\rangle = \langle\psi|\psi\rangle = \sum_{\mathbf{q}_0} \langle\psi|\mathbf{q}_0\rangle\,\langle\mathbf{q}_0|\psi\rangle \equiv \int d^3\mathbf{q}_0\,\langle\psi(\mathbf{q}), \delta(\mathbf{q} - \mathbf{q}_0)\rangle\,\langle\delta(\mathbf{q} - \mathbf{q}_0, \psi(\mathbf{q})\rangle \tag{1.15}$$

Since probability is conserved (the Hamiltonian operator is required to be hermitian, so that time-evolution is a unitary transformation), ρ should obey a continuity equation:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0 \tag{1.16}$$

for some "probability current" density $\mathbf{j}$, or, in explicitly covariant relativistic notation:

$$\nabla_\mu j^\mu = 0 \tag{1.17}$$

where j^μ is the 4-vector $(\rho, \mathbf{j})$. This equation should follow in a natural and direct way from the behaviour of the wave functions. The behaviour of the wave functions is determined by using the fact that the Hamiltonian operator generates time translations, which gives us the Schrödinger equation:

$$i\frac{\partial}{\partial t}\,|\psi\rangle = \hat{H}\,|\psi\rangle \tag{1.18}$$

We can use either this equation directly or some modified version of it to get a linear differential equation, from now on referred to as the "field equation", which determines the time evolution of the wave functions (by using eq. 1.18 directly we can e.g. get the non-relativistic Schrödinger equation for a non-relativistic particle or the Dirac equation for a relativistic spinor particle, or if we have a relativistic scalar particle, we can "square" 1.18 to get the Klein-Gordon equation).

If we now write the field equation in the form $\Lambda_{IJ}\psi^J(\mathbf{q},t) = 0$, where Λ is the appropriate linear differential operator, then we can multiply this by ψ^* from the left to get $\psi^I(\mathbf{q},t)^*\,\Lambda_{IJ}\psi^J(\mathbf{q},t) = 0$. We can then "symmetrize" this equation by adding to it its complex conjugate and get

$$\psi^{I\,*}\Lambda_{IJ}\psi^J + \psi^I(\Lambda_{IJ}\psi^J)^* = 0 \tag{1.19}$$

We can then hope that we will be able to rewrite this differential equation as

$$\frac{\partial}{\partial t}\left(\psi^{I\,*}\Gamma^t_{IJ}\psi^J + \psi^I\Gamma^{t\,*}_{IJ}\psi^{J\,*}\right) + \nabla\cdot\left(\psi^{I\,*}\mathbf{\Gamma}_{IJ}\psi^J + \psi^I\mathbf{\Gamma}^*_{IJ}\psi^{J\,*}\right) = 0 \tag{1.20}$$

for some differential operators Γ^t and $\mathbf{\Gamma}$ (which may act both to the left and to the right, and where $\mathbf{\Gamma}$ is a vector-valued), or as

$$\nabla_\mu\left(\psi^{I\,*}\Gamma^\mu_{IJ}\psi^J + \psi^I(\Gamma^\mu_{IJ})^*\psi^{J\,*}\right) = 0 \tag{1.21}$$

in relativistically covariant notation, where $\Gamma^\mu = (\Gamma^t;\mathbf{\Gamma})$. If we can achieve this, then Γ^t, i.e. Γ^0, would be an excellent candidate to use for the differential operator D.

To get further, expand the differential operator Λ and write the field equation as

$$\Lambda_{IJ}\psi^J = \sum_{n=0}^{N}\Lambda^{\mu_1\mu_2\ldots\mu_n}_{n\;IJ}\nabla_{\mu_0}\nabla_{\mu_1}\cdots\nabla_{\mu_n}\psi^J = 0 \tag{1.22}$$

where N is the order of the differential equation, and where summation over repeated indices is implied. Note by the way that although I am using relativistically covariant notation, the following analysis will be equally valid for a non-relativistic theory if we just consider the tensor quantities to be matrices that are not necessarily relativistically covariant. In this thesis however, we will not concern ourselves with non-relativistic problems.

Now write the differential operator $\Gamma^\mu = (\Gamma_t,\mathbf{\Gamma})$ as

$$\Gamma^\mu_{IJ} = \sum_{n=0}^{N}\sum_{i=0}^{n}\Gamma^{\mu_1\ldots\mu_i,\,\mu,\,\mu_{i+1}\ldots\mu_n}_{IJ}\overleftarrow{\nabla}_{\mu_1}\ldots\overleftarrow{\nabla}_{\mu_i}\nabla_{\mu_{i+1}}\ldots\nabla_{\mu_n} \tag{1.23}$$

where $\overleftarrow{\nabla}$ denotes (covariant) derivation acting to the left. Note that unless we work in flat spacetime, the order of the derivatives is important, since covariant derivatives will not in general commute. Eq. 1.21 now becomes

$$\sum_{n=0}^{N}\sum_{i=0}^{n}[\Gamma_{IJ}^{\mu_1\dots\mu_i\,,\,\mu\,,\,\mu_{i+1}\dots\mu_n}\Big(\psi^{I\,*}\overleftarrow{\nabla}_{\mu_1}\dots\overleftarrow{\nabla}_{\mu_i}\overleftarrow{\nabla}_{\mu}\,\nabla_{\mu_{i+1}}\dots\nabla_{\mu_n}\psi^{J} + \psi^{I\,*}\overleftarrow{\nabla}_{\mu_1}\dots\overleftarrow{\nabla}_{\mu_i}\,\nabla_{\mu}\nabla_{\mu_{i+1}}\dots\nabla_{\mu_n}\psi^{J}\Big) + (\Gamma_{IJ}^{\mu_1\dots\mu_i\,,\,\mu\,,\,\mu_{i+1}\dots\mu_n})^{*}\Big(\psi^{I}\overleftarrow{\nabla}_{\mu_1}\dots\overleftarrow{\nabla}_{\mu_i}\overleftarrow{\nabla}_{\mu}\,\nabla_{\mu_{i+1}}\dots\nabla_{\mu_n}\psi^{J\,*} + \psi^{I}\overleftarrow{\nabla}_{\mu_1}\dots\overleftarrow{\nabla}_{\mu_i}\,\nabla_{\mu}\nabla_{\mu_{i+1}}\dots\nabla_{\mu_n}\psi^{J\,*}\Big)] \quad (1.24)$$

We see that in order for this to reproduce eq. 1.19, we should have $\Gamma_{IJ}^{-\,,\,\mu\,,\,\mu_1\dots\mu_n\,,} = \frac{1}{2}\Lambda_{(n+1)\,IJ}^{\mu\mu_1\dots\mu_n}$ and $\Gamma_{IJ}^{\mu_1\dots\mu_n\,,\,\mu\,,\,-} = \frac{1}{2}(\Lambda_{(n+1)\,JI}^{\mu\mu_n\dots\mu_1})^{*}$ in order that the terms differentiating only one of the functions come out right (with the dash indicating that there are no indices in this location). Furthermore, if we have in general that $\Gamma_{IJ}^{\mu_1\dots\mu_{j-1}\,,\,\mu_j\,,\,\mu_{j+1}\dots\mu_n} = -\Gamma_{IJ}^{\mu_1\dots\mu_j\,,\,\mu_{j+1}\,,\,\mu_{j+2}\dots\mu_n}$, then all the terms that differentiate both to the left and to the right will cancel out, and we are almost left with eq. 1.19 as desired. The only thing missing are the terms that do not involve any differentiation at all. However, we are free to multiply Λ with some arbitrary c-number without altering the field equations, so we can make the term of Λ that involves no differentiation purely imaginary. The non-differentiating terms will cancel due to the addition of the complex conjugate and drop out of eq. 1.19.

Note that the previous assumptions are only consistent if $\Lambda_{(n+1)\,IJ}^{\mu\mu_1\dots\mu_n} = (-1)^n(\Lambda_{(n+1)\,JI}^{\mu_n\dots\mu_1\mu})^{*}$. This almost makes sense, since the Hamiltonian should only be a function of hermitian operators, i.e. of the $\hat{q}^i$ and $\hat{p}_i$. Otherwise, the Hamiltonian will not be hermitian, time evolution will not be unitary and there will be no reason why probabilities should be conserved and no reason that our construction of a conserved inner product should be possible. Since $\hat{p}^i = -i\nabla_i$, all derivatives in the field equation should appear together with one factor of $-i$, so that Λ_n when complex conjugated should switch sign for n odd and remain the same for n even. However, this appears to be the opposite of what the above discussion requires; there Λ_n should reverse sign for even n and stay the same for odd n. This suggests that we simply multiply Λ by i in the above relations, and everything should be in order. We then get:

$$\Gamma_{IJ}^{-\,,\,\mu\,,\,\mu_1\dots\mu_n} = \frac{i}{2}\Lambda_{(n+1)\,IJ}^{\mu\mu_1\dots\mu_n} \quad (1.25)$$

$$\Gamma_{IJ}^{\mu_1\dots\mu_n\,,\,\mu\,,\,-} = -\frac{i}{2}(\Lambda_{(n+1)\,JI}^{\mu_n\dots\mu_1\mu})^{*} \quad (1.26)$$

where the requirement on Λ now is that

$$\Lambda_{(n+1)\,IJ}^{\mu\mu_1\dots\mu_n} = (-1)^{n+1}(\Lambda_{(n+1)\,JI}^{\mu_n\dots\mu_1\mu})^{*} \quad (1.27)$$

and the other relations between the components of Γ^{μ} still hold.

To satisfy eq. 1.11, Λ must also not be explicitly dependent on position other than through the metric tensor (except possibly for Λ_0 since that term drops out of the inner product). But this also makes some sense, since in a theory that depends on position in

space in an arbitrary way, there is no reason why spatial translations should be unitary transformations, and hence $\hat{p}_i$ a hermitian operator, as we have required it to be.

Our inner product then finally becomes:

$$\langle f, g \rangle = \int \left(f^{I\,*} \Gamma^0_{IJ} g^J + g^I (\Gamma^0_{IJ})^* f^{J\,*} \right) dV \tag{1.28}$$

where Γ^0 is defined by eq. 1.23 and 1.26. If we have a relativistic theory and want to make the whole inner product explicitly covariant, we can write the whole integral in an explicitly covariant form by not taking the integral of the "density" defined by Γ^0 over all of "space" as defined by one particular reference frame, but rather take the integral of the "current" defined by Γ^μ over a spacelike hypersurface Σ, as follows:

$$\langle f, g \rangle = \int_\Sigma \left(f^{I\,*} \Gamma^\mu_{IJ} g^J + g^I (\Gamma^\mu_{IJ})^* f^{J\,*} \right) d\Sigma_\mu \tag{1.29}$$

(For the more stringently mathematical minded: this notation means, in $N+1$ dimensions, take the vector $(f^{I\,*} \Gamma^\mu_{IJ} g^J + g^I (\Gamma^\mu_{IJ})^* f^{J\,*}) \mathbf{e}_\mu$, feed it to the volume form $\mathbf{d}V = \sqrt{-g}\, \mathbf{d}x^0 \wedge \ldots \wedge \mathbf{d}x^N$ and integrate the resulting N-form over the surface Σ.) By using Stoke's theorem and assuming that the wave functions vanish at spatial infinity, eq. 1.21 then ensures that the resulting inner product is in fact independent of which spacelike hypersurface Σ we choose to use, so that the inner product and the probabilities that it represents are indeed conserved, and time evolution is hence unitary.

As an example of how to implement this inner product, take the Schrödinger equation for a spinless non-relativistic particle with mass m:

$$\left[-\frac{\partial}{\partial t} - \frac{1}{2m} \nabla^2 + V(\mathbf{x}) \right] \psi(\mathbf{x}, t) = 0 \tag{1.30}$$

In this case we have $\Lambda_0 = V(x)$ (note that it may vary with position, but as we earlier noted this does not matter for $\Lambda_{(0)}$), $\Lambda^0_{(1)} = -i$ and $\Lambda^{ij}_2 = -\delta^{ij} \frac{1}{2m}$ for $i, j = 1, 2, 3$, while all other Λ-components are zero (since we are working with a scalar particle, there are no indices). Note that these all satisfy eq. 1.27. By eq. 1.26 we then have $\Gamma^{-,0,-} = \frac{1}{2}$. According to eq. 1.28 the inner product is then

$$\langle f, g \rangle = \int f(\mathbf{x})^* g(\mathbf{x})\, d^3x \tag{1.31}$$

which is the usual inner product of non-relativistic Schrödinger quantum mechanics. In this case we see that the inner product satisfies the axiom of positivity, $\langle f, f \rangle \geq 0$, with $\langle f, f \rangle = 0$ if and only if $f(\mathbf{x}) \equiv 0$. This is of course reassuring, since it means that f can be normalized to $+1$ as it should be. Furthermore, the integrand of $\langle f, f \rangle$, $f(\mathbf{x})^* f(\mathbf{x}) = |f(\mathbf{x})|^2$ is to be interpreted as the probability of finding the particle at the point $\mathbf{x}$ when it is in the state represented by f, and this is fortunately also reassuringly positive.

This does not have to be case though. As a further example, consider a relativistic, spinless particle, which is described by the Klein-Gordon equation:

$$\left[\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right] \psi(\mathbf{x}, t) = 0 \tag{1.32}$$

In this case we see that we have $\Gamma^{-,0,0} = \frac{i}{2}$ and $\Gamma^{0,0,-} = -\frac{i}{2}$, giving us the following inner product:

$$\langle f, g \rangle = i \int \left(f^* \frac{\partial g}{\partial t} - g \frac{\partial f^*}{\partial t} \right) d^3x \tag{1.33}$$

By using the Klein-Gordon equation one can easily check that this inner product is conserved, but it is clearly violates the positivity axiom. Just let f be for instance $f(\mathbf{x}, t) = \frac{1}{2E} e^{iEt}$, and you get that $\langle f, f \rangle = -1$. Or let it be any real-valued function, and the norm $\langle f, f \rangle$ will be zero even when f is not identically zero! This is obviously a problem. This means that we cannot necessarily normalize f to $+1$. And, since the integrand $f(\mathbf{x}, t)^* \frac{\partial g(\mathbf{x},t)}{\partial t} - g(\mathbf{x}, t) \frac{\partial f(\mathbf{x},t)^*}{\partial t}$ is not necessarily positive, we could get a negative probability density for finding the particle at a point $\mathbf{x}$! This is clearly not quite acceptable, so we will need to interpret the inner product as something different than just probabilities. This is not feasible in one-particle quantum mechanics, but in the next chapter we shall see that it has a quite natural interpretation in quantum field theory, linked to creation and annihilation of particles.

Chapter 2

Particles in quantum fields

In the previous chapter we dealt with theories where we started out by assuming the existence of particles and where the positions and velocities of the particles were usually the dynamic variables. These were usually functions of a time parameter, so that the theories were defined in terms of the positions of individual particles at each point in time. In this chapter we will start out with field theories, in which the dynamical variables are fields, defined on some N-dimensional space-time manifold, so that the classical theories are defined in terms of the value of a field at each point in space-time. We will then go on to define quantum field theory by carefully examining how these fields can be quantized. As we do so, we will see how quantities reminiscent of particles in the way defined in the previous chapter naturally arise from conserved quantities in the fields. However, as we will investigate further in the next chapter, the nature of these particles may depend on which conserved quantities and which symmetries we choose to use to quantize the theory.

2.1 Classical fields

Before we start quantizing them and before we start worrying about particles, let us take a brief look at what fields are in classical physics. In this section we will recapitulate basic classical field theoretical formalism, and then spend a section investigating the particulars of linear field equations, which have a special significance physically. After that we will move on to quantize the fields.

Classical field theories are theories in which the degrees of freedom are not just the position of one or more particles as a function of time, but rather the values of a whole field, as a function of position in space *and* time (as such, the theory of a classical particle can be viewed as just a field theory in $0+1$-dimensional space-time, the spatial coordinates themselves being the field).

One usually derives the behaviour of the field by demanding that the action functional $S[\phi]$ have an extremum when the field ϕ has an actual physical configuration. The action functional can also usually be written as the time integral of a Lagrangian, $L[\phi]$ (where L now is a functional of the field ϕ and its first order time derivatives $\dot{\phi}$ for a fixed point in time), and the Lagrangian can usually further be written as the space integral of a Lagrangian *density* $\mathcal{L}(\phi, \phi_{;\mu})$, which is a function of the field ϕ and its first order (covariant)

derivatives $\phi_{;\mu}$ at a single point in space time. In $N+1$-dimensional space time, the action functional then becomes:

$$S[\phi] = \int \mathcal{L}(\phi, \phi_{;\mu})\, d^{N+1}x \tag{2.1}$$

By writing the action in this form, using a Lagrangian density that depends only on the field and its derivatives at a single point in spacetime, we ensure that the theory is local, i.e. that the field at one point is directly affected only by the values of the field at infinitesimal distances from the first point, and not by whatever antics the field might be performing at some other point far away. This is clearly a desirable property for any physical theory that seeks to describe the universe that we are familiar with. Things on Earth do not care what happens at some instant in the Andromeda galaxy unless electromagnetic radiation or something else has traversed the distance in between to inform them of it. Secondly, by making the Lagrangian density a function of the value and the *first* order derivatives of the field, we are saying that the value and first order derivatives of the field are the only relevant variables in the theory. Higher order derivatives do not affect the theory explicitly, but only in the way that they affect the values and first order derivatives of the field. This implies that the field will be described by a differential equation of not more than second order (as can be seen from eq. 2.2). This seems to be a property of all theories that describe our universe, although it could perhaps conceivably be otherwise.

By minimizing (or at least extremizing) eq. 2.1, we get the Euler-Lagrange equation which determines the behaviour of the field:

$$\nabla_\mu \frac{\partial \mathcal{L}}{\partial \phi^I_{;\mu}} - \frac{\partial \mathcal{L}}{\partial \phi^I} = 0 \tag{2.2}$$

where I as in the previous chapter represents any kind of indices the field may be carrying.

The Hamiltonian formalism can be derived by defining a conjugate momentum Π to the field:

$$\Pi_I(x) = \frac{\delta L[\phi, \dot{\phi}]}{\delta \dot{\phi}^I(x)} = \frac{\partial \mathcal{L}(\phi(x))}{\partial \phi^I_{;0}} \tag{2.3}$$

The Hamiltonian $H[\phi, \Pi]$ (now a functional of ϕ and Π at some point in time) is defined in the usual way by:

$$H[\phi, \Pi] = \int \dot{\phi}^I(x)\Pi_I(x)\, d^N x \; - \; L \tag{2.4}$$

(with implicit summation over the repeated indices I) and is now a functional of the field and its conjugate momentum field. The integration is taken over a spacelike hypersurface with constant x^0. This can also be made into a local quantity by defining a Hamiltonian density $\mathcal{H}(\phi, \Pi)$:

$$\mathcal{H}(\phi, \Pi) = \phi^I(x)\Pi_I(x) - \mathcal{L} \tag{2.5}$$

so that $H[\phi, \Pi] = \int \mathcal{H}(\phi, \Pi)\, d^N x$, where $\mathcal{H}$ is a function of the field and its conjugate momentum at a single point. The time-evolution of the field and the conjugate momentum is then given by Hamilton's equations, either in their functional derivative form:

$$\dot{\phi}^I(x) = \frac{\delta H[\phi, \Pi]}{\delta \Pi_I(x)} \qquad \dot{\Pi}_I(x) = -\frac{\delta H[\phi, \Pi]}{\delta \phi^I(x)} \tag{2.6}$$

or in a local form, as a normal set of differential equations:

$$\dot{\phi^I}(x) = \frac{\partial \mathcal{H}(\phi(x), \Pi(x))}{\partial \Pi_I} \qquad \dot{\Pi_I}(x) = -\frac{\partial \mathcal{H}(\phi(x), \Pi(x))}{\partial \phi^I} \tag{2.7}$$

Note that although they are equivalent, the Lagrangian and Hamiltonian formalisms outlined here are quite different in nature. The Lagrangian formalism is explicitly relativistically covariant in form (provided of course that the Lagrangian density is). Although of course the definition of the Lagrangian *functional* $L[\phi, \dot{\phi}]$ involves choosing some particular spacelike hypersurface to integrate over, the choice of this hypersurface is insignificant as long as the theory can be expressed locally by a Lagrangian *density* $\mathcal{L}(\phi(x), \dot{\phi}(x))$, since the action can then be written as an integral of this density over all of spacetime, and the Euler-Lagrange equations 2.2 assume an explicitly covariant form. The Hamiltonian formalism on the other hand is expressed through the canonical conjugate momentum $\Pi(x)$, and defining this quantity entails explicitly choosing a time coordinate. We also explicitly choose spacelike constant-x^0 hypersurfaces in order to define the Hamiltonian functional $H[\phi, \Pi]$. Although we can use the locally defined Hamiltonian density $\mathcal{H}(\pi(x), \Pi(x))$ to get around this explicit dependence on foliating spacetime into spacelike hypersurfaces, we still need to explicitly choose the x^0-coordinate to define $\Pi(x)$. And the spacelike hypersurfaces are also important to the whole spirit of the formalism. In the Hamiltonian formalism, rather than view the field as a field over all of spacetime to start with, you view it as a field living on separate spacelike hypersurfaces, and the Hamiltonian functional then plays the role of generating transitions from one spacelike hypersurface to the next. This interpretation is especially crucial when quantizing the field and defining field operators, as the operators will then be defined on the spacelike hypersurfaces and have their time-evolution, i.e. how operators defined on one spacelike hypersurface act when transported along to the next hypersurface, governed by the Hamiltonian operator.

Finally, we note that just as in the Hamiltonian formalism for particle theories, we can define a Poisson bracket between functionals of $\phi(\mathbf{x})$ and $\Pi(\mathbf{x})$. For any functionals A and B on the phase space of the field, we define:

$$\{A, B\} = \sum_I \int d^3x \, \left(\frac{\delta A}{\delta \phi^I(\mathbf{x})} \frac{\delta B}{\delta \Pi_I(\mathbf{x})} - \frac{\delta A}{\delta \Pi_I(\mathbf{x})} \frac{\delta B}{\delta \phi^I(\mathbf{x})} \right) \tag{2.8}$$

Hence we get the same correspondence between physical quantities (function(al)s on phase space) and generators of phase space transformations as we had for particle theories.

2.2 Free fields, inner products and prelude to quantization

Although the field equation 2.2 can only be of second order or less, it could in principle be quite messy, with lots of non-linear terms. If however it is linear, it has a special property; any linear superposition of solutions to the field equation is itself a solution. This means that if you have one solution of the field equation, then you can take any other solution of the field equation, multiply it by any real or complex number and add it to the original solution, and the result is a function that satisfies the field equation. In a sense the field is oblivious to what values it already has, and new solutions can be added to it independently of how it looked before. If on the other hand the field equation is messy and does contain non-linear terms, then in general we cannot do this. If we start out with one solution of the field equation and want to add another function to it, then what functions we are allowed to add and still get a result that solves the field equation depends on what function we started out with. Solutions can now *not* be added independently of one another.

Physically we can interpret this as meaning that a linear field equation describes a field where different contributions to the field are independent of each other, that they are not interacting. A non-linear field equation on the other hand describes a field where different contributions to the field do depend on each other and interfere with each other. A field described by a linear field equation can therefore be aptly called a *free* field, whereas a field described by a non-linear field equation can be called an *interacting* field.

In this chapter we will want to study free fields, that do not interact with themselves (or with other fields that might exist in the theory, i.e. we want each field to appear alone in its own separate linear field equation, so that we can effectively treat them as belonging to separate theories, with their own separate Lagrangians). Our goal is to see if we can find particle-like entities in the fields when they are quantized. As we shall see, this requires that we can define an inner product between solutions of the field equation. Defining an inner product requires by definition that the solutions of the field equation must form a vector space. This means that a linear combination of solutions must itself be a solution, i.e. the field equations need to be linear.

In a way, this also makes sense. To have something that we can recognize as a particle in the sense that it resembles what we defined as being a particle in chapter 1, we need to be able to isolate it from other particles or other kinds of entities that might be lurking in the field. But this is impossible if the field interacts strongly with itself, as the presence of a particle will then interact with the rest of the field close to it and may cause all sorts of other disturbances to pop in the field around it at all times. If the interaction is weak, we can of course use a perturbation expansion around a free field linear solution, where each term in the expansion is expressed in terms of solutions of linear, free field equations, and identify the terms of increasing order in the expansion as an increasing number of "particles" interacting with each other. But to derive the existence of particles in the first place, we will concentrate on truly free fields, governed by linear field equations from the start.

A linear field equation of second order or less (one that can be derived from a normal Lagrangian) for a field ϕ can be written in the following form:

$$\Lambda\phi(x) = \left(\Lambda_{2\,IJ}^{\mu\nu}\nabla_\mu\nabla_\nu + \Lambda_{1\,IJ}^{\mu}\nabla_\mu + \Lambda_{0\,IJ}\right)\phi^J(x) = 0 \tag{2.9}$$

In line with our requirements for the inner product between quantum mechanical wave functions derived in eq. 1.11, we also demand that Λ_1 and Λ_2 only depend on position through the metric tensor. We can then try to find a Lagrangian for the field equation 2.9.

A Lagrangian density for the field equation 2.9 must have the form:

$$\mathcal{L}(\phi,\phi_{;\mu}) = \frac{1}{2}\Lambda_{2\,IJ}^{\mu\nu}\nabla_\mu\phi^I\nabla_\nu\phi^J \;+\; \frac{1}{2}\Lambda_{1\,IJ}^{\mu}(\nabla_\mu\phi^I)\,\phi^J \;-\; \frac{1}{2}\Lambda_{0\,IJ}\phi^I\phi^J \tag{2.10}$$

But if we plug this Lagrangian density into the Euler-Lagrange equation and compare it with eq. 2.9, we see that this requires that the following three relations hold:

$$\Lambda_{2\,IJ}^{\mu\nu} = \Lambda_{2\,JI}^{\nu\mu} \qquad \Lambda_{1\,IJ}^{\mu} = -\Lambda_{1\,JI}^{\mu} \qquad \Lambda_{0\,IJ} = \Lambda_{0\,JI} \tag{2.11}$$

Otherwise the field equation will not be derivable from a Lagrangian. Furthermore, we see that the first term in eq. 2.10 is symmetric under interchange of μ and ν. Because of this we may as well require that $\Lambda_{IJ}^{\mu\nu}$ also be symmetric under this interchange, i.e.

$$\Lambda_{IJ}^{\mu\nu} = \Lambda_{IJ}^{\nu\mu} \tag{2.12}$$

We can now define an inner product between solutions of the field equation in the same way as we did for quantum mechanical wave functions in section 1.5. According to this, the inner product between to functions f and g satisfying the field equation, taken over some spacelike hypersurface Σ, will be

$$\begin{aligned}\langle f,g\rangle = \frac{i}{2}\int_\Sigma \Big[& f^{I\,*}\Lambda_{2\,IJ}^{\mu\nu}\nabla_\nu g^J - f^{I\,*}\overleftarrow{\nabla_\nu}\,(\Lambda_{2\,JI}^{\nu\mu})^* g^J - g^I(\Lambda_{2\,IJ}^{\mu\nu})^*\nabla_\nu f^{J\,*} \\ & + g^I\overleftarrow{\nabla_\nu}\,\Lambda_{2\,JI}^{\nu\mu} f^{J\,*} + f^{I\,*}\Lambda_{1\,IJ}^{\mu} g^J - g^I(\Lambda_{1\,IJ}^{\mu} f^J)^*\Big]\, d\Sigma_\mu\end{aligned} \tag{2.13}$$

If we assume that Λ is kind enough to satisfy eqs. 1.27, 2.11 and 2.12, use partial integration on the second and fourth terms of eq. 2.13 and neglect the boundary terms, we can rewrite this as

$$\langle f,g\rangle = i\int_\Sigma \left(f^{I\,*}\Lambda_{2\,IJ}^{\mu\nu}\nabla_\nu g^J - g^I\Lambda_{2\,IJ}^{\mu\nu}\nabla_\nu f^{J\,*} + \frac{1}{2}f^{I\,*}\Lambda_{1\,IJ}^{\mu} g^J - \frac{1}{2}g^I\Lambda_{1\,IJ}^{\mu} f^{J\,*}\right) d\Sigma_\mu \tag{2.14}$$

Comparing this to eq. 2.10, we see that this can actually be written as

$$\langle f,g\rangle = i\int_\Sigma \left(f^{I\,*}\left.\frac{\partial\mathcal{L}}{\partial\phi_{;\mu}^I}\right|_{\phi=g} - g^I\left.\frac{\partial\mathcal{L}}{\partial\phi_{;\mu}^I}\right|_{\phi=f^*}\right) d\Sigma_\mu \tag{2.15}$$

If we now choose Σ to be a plane of simultaneity in some reference frame and a coordinate basis adapted to this plane, so that all spacelike basis vectors lie in the plane, then the inner product becomes

$$\begin{aligned}\langle f, g\rangle &= i\int \left(f^{I\,*}\left.\frac{\partial\mathcal{L}}{\partial\phi^I_{;0}}\right|_{\phi=g} - g^I\left.\frac{\partial\mathcal{L}}{\partial\phi^I_{;0}}\right|_{\phi=f^*}\right) d^N x \\ &= i\int \left(f^{I\,*}\,\Pi_I|_{\phi=g} - g^I\,\Pi_I|_{\phi=f^*}\right) d^N x\end{aligned} \tag{2.16}$$

where $\Pi|_{\phi=g}$ and $\Pi|_{\phi=f}$ are the values of the conjugate momentum when the field is in the configuration of f and g respectively. This expression will be very useful for deriving the existence of particle-like states in the quantized fields.

2.3 Quantizing free fields — Voilà, particles!

We will now see how a classical field $\phi(x)$ can be quantized in a straight-forward manner using the canonical quantization prescription presented in section 1.3. We will see however, that we have more freedom in choosing exactly how to carry out this prescription when quantizing fields than when quantizing single particles.

Assume then that we have a field $\phi(x)$. The degrees of freedom are now the value of the field at each *point* in spacetime. To start with, space and time are on an equal footing as far as the field is concerned (assuming that we are working with relativistic theories), but in order to quantize the theory, we need to divide the spacetime into equal-time spacelike hypersurfaces, as described at the end of section 2.1. We then get one operator $\hat{\phi}(\mathbf{x})$ for every point $\mathbf{x}$ on an equal-time hypersurface. How the operators evolve from one hypersurface to the next, depends on what picture we choose, since we can "distribute" the time dependence freely between the operators and the state vectors. In this thesis, I will always (unless otherwise noted) use the Heisenberg picture, in which state vectors are constant and the time-evolution of the field operators, i.e. how the field operators are transported from one equal-time surface to the next, is governed by the Hamiltonian operator $\hat{H}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\phi}(\mathbf{x}) = -i\left[\hat{\phi}(\mathbf{x}), \hat{H}\right] \tag{2.17}$$

The field operators now depend on the time parameter $t \equiv x^0$, so that I will henceforth write it conveniently as $\hat{\phi}(x) = \hat{\phi}(\mathbf{x}, t)$. But it is important to remember that $\hat{\phi}(x)$ does not represent one operator, nor does it really represent one separate operator for each point x in spacetime. Rather, it represents one operator $\hat{\phi}(\mathbf{x})$ for each point $\mathbf{x}$ on a spacelike hypersurface, which is evolving with a time-parameter $t \equiv x^0$ as it passes from hypersurface to hypersurface. This picture is unproblematic as long as it actually possible to divide the whole spacetime into disjoint equal-time hypersurfaces. There may however exist spacetimes where this is not possible. In such cases, the Hamiltonian, operator-based formalism of quantum mechanics becomes somewhat awkward, and if it is at all usable, we will at least have to live with singularities in the places where equal-time hypersurfaces intersect or are otherwise ill-defined. Other quantization schemes such as path-integral quantization may provide a more suitable formalism in such spacetimes, but I will not discuss this any further here.

The operators in a quantum field theory act on a complex Hilbert space of *functionals* on each equal-time hypersurface. The algebra of these operators on any equal-time hypersurface is determined by imposing the equal-time commutation relation

$$[\hat{\phi}^I(\mathbf{x}), \hat{\Pi}_J(\mathbf{x}')] = i\hbar\,\delta(\mathbf{x}-\mathbf{x}')\delta^I{}_J \tag{2.18}$$

for bosonic fields and the anticommutation relation

$$\{\hat{\phi}^I(\mathbf{x}), \hat{\Pi}_J(\mathbf{x}')\} = i\hbar\,\delta(\mathbf{x}-\mathbf{x}')\delta^I{}_J \tag{2.19}$$

for fermionic fields. In this thesis I will be only be concerned with bosonic fields, except for a brief section at the end of this chapter. So unless otherwise noted, all field operators will be bosonic. To simplify the discussion, I will in the following also assume that each component of ϕ^I is a real-valued field, so that each component of the field operator becomes hermitian when quantized. This does not really limit the applicability of the results since any complex valued field can be written as a two-component real field, with one component being the real part and the other the imaginary part of the field.

Although for a real field it is in principle possible to observe the actual value of the field at individual points in space, it is rather awkward to work in a basis of states where the field has a sharp definite value, i.e. eigenstates of the individual field operators $\hat{\phi}(x)$, and the value of the field in individual points is certainly not usually a conserved quantity. Therefore, instead of using the individual field operators $\hat{\phi}(x)$ directly, it might be more convenient to combine these to form a set of operators that is more closely linked with quantities that we do want to work with.

Suppose now that K is some observable that we are interested in. As an observable, it can be written as a functional of ϕ and Π. K will generate a transformation on the (classical) phase space of the field and all functions on the phase space by way of the Poisson bracket, as defined by eq. 2.8. In particular, K generates an infinitesimal transformation on the field itself:

$$\delta\phi^I(\mathbf{x},t) = \epsilon \sum_J \int d^3x' \left(\frac{\delta\phi^I(\mathbf{x},t)}{\delta\phi^J(\mathbf{x}')}\frac{\delta K}{\delta\Pi_J(\mathbf{x}')} - \frac{\delta\phi^I(\mathbf{x},t)}{\delta\Pi_J(\mathbf{x}')}\frac{\delta K}{\delta\phi_J(\mathbf{x})} \right) = \epsilon\,\frac{\delta K}{\delta\Pi_I(\mathbf{x})} \tag{2.20}$$

since by definition of the functional derivative, $\frac{\delta\phi^I(\mathbf{x})}{\delta\phi^J(\mathbf{x}')} = \delta(\mathbf{x}-\mathbf{x}')\delta^I{}_J$, while $\phi(\mathbf{x})$ and $\Pi(\mathbf{x}')$ are independent variables so that $\frac{\delta\phi^I(\mathbf{x})}{\delta\Pi_J(\mathbf{x}')} = 0$. ϵ is the infinitesimal parameter of the transformation and the integration is taken over the hypersurface $x^0 = t$.

A particular set of solutions of the field equation are now of interest, namely those functions $f_i(x)$ which are *eigenfunctions* of the transformation, that is those functions for which $\delta f_i(x) = -i\epsilon\kappa_i f_i(x)$, where i is an index to label the eigenfunctions and the κ_i are the *eigenvalues* of the transformation. This statement is supposed to mean that if the field ϕ is in the configuration of f_i, then the infinitesimal change in the field given by eq. 2.20 will be $\delta\phi(x)|_{\phi=f_i} = -i\epsilon\kappa_i\,\phi(x)|_{\phi=f_i}$ (the reason for inserting these factors of $-i$ will become clear in a moment). I will refer to these functions interchangeably as eigenfunctions, mode functions or wave functions throughout the thesis, for reasons that will become clear along the way.

Since we are seeking to find something particle-like in the quantum fields, it would be natural to try and make a connection between these eigenfunctions and the wave-functions of particles in single-particle quantum mechanics. It would then also be natural to connect the transformation that K generates on the phase space of the field to a transformation generated by the operator of a corresponding observable acting on quantum mechanical wave-functions. In that case, since quantum mechanical operators generate *linear* transformations on wave-functions, we should restrict K to be a quantity that generates *linear* transformations on the field. By this I mean that $\frac{\delta K}{\delta \Pi(\mathbf{x})}$ should be a linear function of $\phi(\mathbf{x})$ and $\Pi(\mathbf{x})$.

Note that $\frac{\delta K}{\delta \Pi(\mathbf{x})}$ may depend explicitly on the point $\mathbf{x}$. It can also depend on the spatial derivatives of $\phi(\mathbf{x})$, which may be included explicitly in the definition of the functional $K[\phi, \Pi]$ (it can just not depend explicitly on the time derivative of $\phi(x)$, since this must be included through the conjugate momentum Π). From the formula 2.10 for permissible free-field Lagrangians, we see that Π must be proportional either to the first order time-derivative of the field or to the field itself. Therefore, if $\frac{\delta K}{\delta \Pi(\mathbf{x})}$ is a linear function of $\phi(\mathbf{x})$ and $\Pi(\mathbf{x})$, this means that it can be written as a linear function of the field $\phi(x)$ and its first-order, spatial and temporal derivatives, as well as the position x. Most familiar quantities used in physics will satisfy this requirement. E.g. energy, momentum and angular momentum generate time translations, space-translations and spatial rotations respectively, and the infinitesimal change $\delta\phi(x)$ can be expressed as a linear function of $\phi(x)$ and its first-order derivatives (as well as the spatial position $\mathbf{x}$ in the case of spatial rotations).

If the transformation generated by K is linear in this sense, the generator of the transformation of the field can therefore actually be represented as a linear differential and multiplicative operator acting on the field. I will denote this operator by D_K. This means that our eigenfunctions f_i satisfy the equation $iD_K f_i(x) = \kappa_i f_i(x)$. If furthermore iD_K is a Hermitian operator with resepct to some inner product over each spacelike hypersurface, then the eigenfunctions will form a complete set for all functions defined on each hypersurface, and all of the eigenfunctions with distinct eigenvalues will be orthogonal to each other with resepct to that inner product. The inner product 2.16 is an obvious candidate if we want to make a connection with quantum mechanical wave-functions, and we will henceforth assume that iD_K is Hermitian with respect to it (again, this will be the case for commonly used physical quantities). We see that this operator looks a lot like the Hermitian transformation-generating operators acting on quantum mechanical wave functions that we introduced in section 1.3 to represent observables. Making this resemblence more direct is the reason why I have introduced all these factors of $-i$ here.

In general, iD_K can have the following form:

$$iD_K = A(x) + B^\mu(x)\, i\frac{\partial}{\partial x^\mu} \tag{2.21}$$

If iD_K is to be hermitian with respect to 2.16, both $A(x)$ and $B(x)$ must be real-valued functions. However, I will require that $A(x) = 0$ for the following reasons: If $A(x)$ and $B(x)$ are both non-zero, then the complex conjugates $f_i^*(x)$ of the eigenfunctions may not be eigenfunctions themselves. But if they are not, it will create complications for our discussions later in this chapter, and in particular the commutation relations between the

annihilation and creation operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ that I will define in a minute, will not work out right. The discussion would then fail to give us the clear cut particles that we are looking for. And if $B(x) = 0$, so that $iD_K = A(x)$ with no $\frac{\partial}{\partial x^\mu}$ involved, the "eigenfunctions" would not be functions at all but distributions (e.g. δ-functions). These are rather ill-defined and awkward to work with. If one insists on using that kind of operator, one should instead work with the Fourier transform of the field rather than the field itself, where multiplying by x^μ becomes derivation with respect to the Fourier-transformed variable instead, and the treatment becomes analogous to what follows here. We therefore assume that $iD_K = B^\mu(x)\, i\frac{\partial}{\partial x^\mu}$.

Since iD_K is assumed to be hermitian, the eigenfunctions f_i form a complete set, so we can expand the field in terms of these eigenfunctions:

$$\phi(\mathbf{x}) = \sum_i a_i f_i(\mathbf{x}) \tag{2.22}$$

So far this expression is only valid on one specific equal-time hypersurface. But we can expand the expression to be valid for all times by defining $f_i(\mathbf{x}, t) \equiv f_i(x)$ to be equal to the eigenfunction $f_i(\mathbf{x})$ at a starting time, but then evolve with time in such a way that it satisfies the field equation. We can then write:

$$\phi(x) = \sum_i a_i f_i(x) \tag{2.23}$$

Note however that the functions $f_i(x)$ will then not necessarily be eigenfunctions of iD_K at all times, only at the starting time. If however $\{K, H\} = 0$, then time-evolution will not interfere with the transformations generated by K and the functions $f_i(x)$ will be eigenfunctions of iD_K at all times.

By expanding the field like this, the degrees of freedom for the field are now the values of the coefficients a_i. They can be found from the values of the field itself by using the orthogonality of the eigenfunctions with respect to the inner product 2.16. They will therefore depend on the configuration $\phi(\mathbf{x})$ and $\Pi(\mathbf{x})$ of the field and the conjugate momentum on the hypersurface being integrated over. They can therefore be viewed as functionals of ϕ and Π. We can then write the change in the field under an infinitesimal K-transformation as:

$$\delta\phi(x) = \epsilon\,\{\phi(x), K\} = \epsilon \sum_i \{a_i, K\}\, f_i(x) \tag{2.24}$$

At the same time, we know that the change must be:

$$\delta\phi(x) = \sum_i a_i\, \delta f_i(x) = -i\epsilon \sum_i \kappa_i a_i\, f_i(x) \tag{2.25}$$

From this we see that we must in fact have:

$$\{a_i, K\} = -i\kappa_i a_i \tag{2.26}$$

Now we can do the same thing with the *quantized* field operator $\hat{\phi}(x)$ and expand it in eigenfunctions $f_i(x)$. The degrees of freedom when expanding the classical field were the

values of the coefficients a_i, so for the quantized field, these coefficients become operators $\hat{a}_i$:

$$\hat{\phi}(x) = \sum_i \hat{a}_i f_i(x) \tag{2.27}$$

The quantity K now also becomes a Hermitian operator acting on the Hilbert space of field states, denoted by $\hat{K}$. Note that the use of the term "Hermitian" here means Hermitian not with respect to the inner product 2.16, which is an inner product on the space of functions on spacetime and which has nothing to do with the field operator $\hat{K}$, but rather with respect to the inner product between field *states* or the *functionals* representing these states. I have not yet defined such an inner product, and I will not do so either, just assume that it exists. As was noted in section 1.3, the commutator $[\hat{a}_i, \hat{K}]$ must be given by simply letting the classical quantities in the Poisson bracket eq. 2.26 become operators and multiply the expression by i. This will ensure that the expectation value of $\delta\hat{\phi}$ will be equal to $\delta\phi$ under the transformation generated by $\hat{K}$, so that $\hat{K}$ in a sense generates the same transformation on the Hilbert space of the quantum field as K does on the phase space of the classical field. We need to assume that K does not have a form that makes ordering of the field operators ambiguous when constructing $\hat{K}$, so that we can be sure the correspondence between Poisson bracket and commutator holds. We then get:

$$[\hat{a}_i, \hat{K}] = \kappa_i \hat{a}_i \tag{2.28}$$

Note that even though we have assumed that ϕ is a real field and that the field equation is real, the eigenfunctions f_i can and usually will be complex. Even though they are supposed to satisfy a real field equation, there is nothing suspect about this, as long as they can be added up to give a real result, and as long as the sum 2.22 adds up to be hermitian. Allowing complex solutions is after all a common method for solving real differential equations. But since ϕ is supposed to be a hermitian operator, we must have the following relation:

$$\hat{\phi}(x) = \hat{\phi}(x)^\dagger = \sum_i \hat{a}_i^\dagger f_i(x)^* \tag{2.29}$$

The complex conjugated functions f_i^* must also be solutions of the field equation, since the field equation is real. They also have to be eigenfunctions of iD_K, since we assumed that $iD_K = B^\mu(x)\, i\frac{\partial}{\partial x^\mu}$, so that

$$iD_K f_i^*(x) = B^\mu(x)\, i\frac{\partial f_i^*(x)}{\partial x^\mu} = -\left(B^\mu(x)\, i\frac{\partial f_i(x)}{\partial x^\mu}\right)^* = -\kappa_i f_i^*(x) \tag{2.30}$$

So f_i^* is also an eigenfunction of iD_K with eigenvalue $-\kappa_i$. The expansion 2.29 will then itself be an expansion in eigenfunctions and their associated operators. We can then conclude that if for some i and j we have $f_j = f_i^*$, then we will have $a_j = a_i^\dagger$. This relation will come in handy later. The commutator between $\hat{a}_i^\dagger$ and $\hat{K}$ can be found easily by adjoining eq. 2.28 and using the fact that $\hat{K}$ is Hermitian so $\hat{K}^\dagger = \hat{K}$:

$$[\hat{a}_i^\dagger, \hat{K}] = \left([\hat{K}, \hat{a}_i^\dagger]\right)^\dagger = -\kappa_i \hat{a}_i^\dagger \tag{2.31}$$

So can these commutation relations tell us what the operators a_i really are? Yes, at least in part, if we look at what happens if we let $\hat{a}_i$ act on some eigenstate $|K\rangle$ of the field operator $\hat{K}$ (K here denotes the eigenvalue of $\hat{K}$ for this state). If we apply $\hat{K}$ to the resulting state, we get the following:

$$\hat{K}a_i|K\rangle = \left(\hat{a}_i\hat{K} - [\hat{a}_i, \hat{K}]\right)|K\rangle = (\hat{a}_i K - \hat{a}_i\kappa_i)|K\rangle = (K - \kappa_i)\,\hat{a}_i|K\rangle \tag{2.32}$$

Thus we see that $\hat{a}_i|K\rangle$ is another eigenstate of $\hat{K}$, with eigenvalue $K-\kappa_i$, i.e. $\hat{a}_i|K\rangle \propto |K-\kappa_i\rangle$. In other words, the effect of the operator $\hat{a}_i$ is to lower the value of the observable K by κ_i.

In the same way, we can show that the adjoint $\hat{a}_i^\dagger$ of $\hat{a}_i$ has the opposite effect. Letting $\hat{a}_i^\dagger$ act on $|K\rangle$ we get:

$$\hat{K}\hat{a}_i^\dagger|K\rangle = \left(\hat{a}_i^\dagger\hat{K} - [\hat{a}_i^\dagger, \hat{K}]\right)|K\rangle = (K + \kappa_i)\,\hat{a}_i^\dagger|K\rangle \tag{2.33}$$

Thus we get that $\hat{a}_i^\dagger|K\rangle \propto |K+\kappa_i\rangle$, so that $\hat{a}_i^\dagger$ increases the field observable K by κ_i.

We now have a pair of operators which "create" or "annihilate" a certain amount of the quantity K in the field. If now K is a quantity which we want to use to characterize particles, it would be tempting to interpret these operators as creating and annihilating *actual* particles! Using $\hat{a}_i^\dagger$ on some state could then cause a particle with $K = \kappa_i$ to appear in that state, while using $\hat{a}_i$ could cause the same particle to vanish.

If we wish to adopt this kind of interpretation however, there are several questions that need to be resolved. Some examples of potentially troublesome issues are the following:

1. If the quantity K can be both positive and negative, how can we be sure that $\hat{a}_i$ is different from $\hat{a}_j^\dagger$ if $\kappa_j = -\kappa_i$ (which it will need to be, since creating a particle with $K = -\kappa_i$ is not necessarily the same as annihilating a particle with $K = \kappa_i$)?

2. If $\kappa_i + \kappa_j = \kappa_k$, are the products $\hat{a}_i^\dagger\hat{a}_j^\dagger$ and $\hat{a}_j^\dagger\hat{a}_i^\dagger$ different from $a_k^\dagger$, i.e. can one differentiate between a single particle with $K = \kappa_k$ and two particles whose K-values add up to κ_k?

3. In a free field theory, where particles do not interact, will $\hat{a}_i$ and $\hat{a}_i^\dagger$ commute with $\hat{a}_j$ and $\hat{a}_j^\dagger$ for boson fields and anticommute for fermion fields when $i \neq j$ (which they should since particles of kind i and particles of kind j would not interact and thus be created and annihilated independently of one another)?

4. The $\hat{a}_i$ and the $\hat{a}_i^\dagger$ would only increase or decrease the number of particles in the field, so it is not obvious that one can meaningfully define a well-defined *total* number of particles in any given state. Is this possible, and can one then define a vacuum state, i.e. a state with no particles in it? And if so, what should happen when $\hat{a}_i$ acts on a state with no particles of kind i in it?

The answer to these questions depends on what the commutation relations between the different $\hat{a}_i$ and $\hat{a}_i^\dagger$ operators. We can find these relations by using the orthogonality between

the eigenfunctions f_i and the expansions 2.27 and 2.29 to obtain the following expressions for $\hat{a}_i$ and $\hat{a}_i^\dagger$:

$$\hat{a}_i = \frac{\langle f_i, \hat{\phi}\rangle}{\langle f_i, f_i\rangle} \qquad \hat{a}_i^\dagger = \frac{\langle f_i^*, \hat{\phi}\rangle}{\langle f_i^*, f_i^*\rangle} \tag{2.34}$$

If we use a coordinate basis adapted to the spacelike hypersurface Σ which we integrate over when calculating the scalar product, so that all spacelike basis vectors are tangent to Σ, we get the following expression for the scalar product between a function f and the field operator $\hat{\phi}$:

$$\langle f, \phi\rangle = i\int_\Sigma \left(f^{I\,*}\,\frac{\partial\mathcal{L}(\hat{\phi})}{\partial\phi^I_{;\mu}} - \hat{\phi}^I\,\frac{\partial\mathcal{L}(f^*)}{\partial\phi^I_{;\mu}}\right) d\Sigma_\mu = i\int \left(f^{I\,*}\,\hat{\Pi}_I - \hat{\phi}^I\,\frac{\partial\mathcal{L}(f^*)}{\partial\phi^I_{;0}}\right) d^{N-1}x \tag{2.35}$$

where N is the number of spacetime dimensions, and where $\mathcal{L}(\phi)$ means to take ϕ and substitute it into the expression for $\mathcal{L}$. Using the equal time commutation relation $[\phi(\mathbf{x}), \Pi(\mathbf{y})] = i\delta^{N-1}(\mathbf{x}-\mathbf{y})$, we then get:

$$\begin{aligned}
[\hat{a}_i, \hat{a}_j] &= \frac{[\langle f_i, \hat{\phi}\rangle,\, \langle f_j, \hat{\phi}\rangle]}{\langle f_i, f_i\rangle\,\langle f_j, f_j\rangle} \\
&= \frac{1}{\langle f_i, f_i\rangle\,\langle f_j, f_j\rangle}\left[i\int \left(f_i^{I\,*}(x)\,\hat{\Pi}_I(x) - \hat{\phi}^I(x)\,\frac{\partial\mathcal{L}(f_i^*(x))}{\partial\phi^I_{;0}}\right) d^{N-1}x\,,\right. \\
&\qquad\qquad \left. i\int \left(f_j^{J\,*}(y)\,\hat{\Pi}_J(y) - \hat{\phi}^J(y)\,\frac{\partial\mathcal{L}(f_j^*(y))}{\partial\phi^J_{;0}}\right) d^{N-1}y\right] \\
&= \frac{1}{\langle f_i, f_i\rangle\,\langle f_j, f_j\rangle}\int\int \left(f_i^{I\,*}(x)\frac{\partial\mathcal{L}(f_j^*(y))}{\partial\phi^J_{,\mu}}\left[\hat{\Pi}_I(x), \hat{\phi}^J(y)\right] + \right. \\
&\qquad\qquad \left. \frac{\partial\mathcal{L}(f_i^*(x))}{\partial\phi^I_{,\mu}} f_j^{J\,*}(y)\left[\hat{\phi}(x)^I, \hat{\Pi}_J(y)\right]\right) d^{N-1}x\, d^{N-1}y \\
&= \frac{1}{\langle f_i, f_i\rangle\,\langle f_j, f_j\rangle}\int \left(-i f_i^{I\,*}(x)\frac{\partial\mathcal{L}(f_j^*(x))}{\partial\phi^I_{,\mu}} + i f_j^{I\,*}(x)\frac{\partial\mathcal{L}(f_i^*(x))}{\partial\phi^I_{,\mu}}\right) d^{N-1}x \\
&= \frac{\langle f_j,\, f_i^*\rangle}{\langle f_i, f_i\rangle\,\langle f_j, f_j\rangle}
\end{aligned} \tag{2.36}$$

If $f_j \neq f_i^*$, this will be zero, whereas if $f_j = f_i^*$, we will get $1/\langle f_i, f_i\rangle$. From the discussion earlier in this section we know that if $f_j = f_i^*$, then we must have $\hat{a}_j = \hat{a}_i^\dagger$. This means that we can write the field operator as

$$\phi(x) = \sum_i \left(\hat{a}_i f_i(x) + \hat{a}_i^\dagger f_i^*(x)\right) \tag{2.37}$$

where the sum now is only over modes f_i with positive norm $\langle f_i, f_i\rangle$. We then get the following commutation relations:

$$[\hat{a}_i,\, \hat{a}_j^\dagger] \;=\; \frac{1}{\langle f_i, f_i\rangle}\,\delta_{ij} \tag{2.38}$$

$$[\hat{a}_i,\, \hat{a}_j] \;=\; [\hat{a}_i^\dagger,\, \hat{a}_j^\dagger] \;=\; 0 \qquad \text{if} \qquad f_j \neq f_i^* \tag{2.39}$$

This commutation relation provides most of the clues we need to illuminate the nature of the $\hat{a}_i$-operators. First of all we note that since $\hat{a}_j = \hat{a}_i^\dagger$ and $\langle f_j, f_j\rangle = -\langle f_i, f_i\rangle$ whenever $f_j = f_i^*$, we can split the $\hat{a}_i$ into well defined creation and annihilation operators based on whether the norm $\langle f_i, f_i\rangle$ of the associated mode is positive or negative (if modes with norm 0 exist, we ignore them for the time being). If a mode f_i has positive norm, we thus choose to write its associated operator as $\hat{a}_i$, an annihilation operator. If a mode has negative norm, we choose to write the mode itself as f_i^*, the complex conjugate of a mode with positive norm, and its associated operator as $\hat{a}_i^\dagger$, a creation operator. Why we make this particular choice, will become apparent shortly. For convenience we also choose to normalize the positive norm modes so that $\langle f_i, f_i\rangle = 1$ (which implies $\langle f_i^*, f_i^*\rangle = -1$). We then get the following familiar looking commutation relations:

$$[\hat{a}_i,\, \hat{a}_j^\dagger] \;=\; \delta_{ij} \qquad\qquad [\hat{a}_i,\, \hat{a}_j] \;=\; [\hat{a}_i^\dagger,\, \hat{a}_j^\dagger] \;=\; 0 \tag{2.40}$$

This procedure obviously breaks down for eigenfunctions with norm equal to zero. However, the eigenfunctions f_i look tantalizingly like wave-functions for a single-particle quantum mechanical theory, being eigenfunctions of the linear operator iD_K with their eigenvalues κ_i being the same as the K-value of the particle that is supposedly created or annihilated by $\hat{a}_i^\dagger$ or $\hat{a}_i$. We wish to preserve this correspondence, and since quantum mechanical wave-functions are not allowed to have zero norm, we will not allow eigenfunctions with zero norm. We are not being quite consistent here since we nevertheless allow functions with negative norm. But these functions obviously have a role to play in the second-quantized multi-particle/field theory by defining the difference between creation and annihilation operators. Zero-norm functions may also have some physical significance, and if we want to, we can produce them by making linear combinations of positive- and negative-norm eigenfunctions and their corresponding operators *after* we have deduced the commutation relations between them. I will not go into this any further in this thesis.

We thus see that for a free field, for which the inner product we have used and hence the above commutation relations always can be used, separate field modes do not interfere with each other, since their creation and annihilation operators commute. Also, even if $\kappa_i + \kappa_j$ is equal to κ_k, $\hat{a}_i^\dagger\hat{a}_j^\dagger$ is not the same operator as $\hat{a}_k^\dagger$, since the former commutes with a_k, whereas the latter does not. Thus creating *two* particles with K-values adding up to κ_k will not be the same as creating *one* particle with $K = \kappa_k$.

The commutation relations also gives a natural definition of absolute particle numbers in a state. To see this, we will show that the operator

$$\hat{\mathcal{N}}_i = \hat{a}_i^\dagger\hat{a}_i \tag{2.41}$$

gives a natural definition of the number of particles of type i in the field. This can be shown in the same way as it is usually done for the quantum mechanical harmonic oscillator:

First we note that $\hat{\mathcal{N}}_i$ is equal to its own adjoint, so it is Hermitian. Therefore it has real eigenvalues, which I will denote $n_{i,j}$ (the subscripts denote the jth eigenstate of $\hat{\mathcal{N}}_i$), and a complete set of eigenstates $|n_{i,j}\rangle$. As in eqs. 2.32 and 2.33, the commutation relations 2.40 give us that

$$\begin{aligned} [\hat{a}_i, \hat{\mathcal{N}}_i] = \hat{a}_i \qquad &\Longrightarrow \qquad \hat{a}_i |n_{i,j}\rangle \propto |n_{i,j} - 1\rangle \\ [\hat{a}_i^\dagger, \hat{\mathcal{N}}_i] = -\hat{a}_i^\dagger \qquad &\Longrightarrow \qquad \hat{a}_i^\dagger |n_{i,j}\rangle \propto |n_{i,j} + 1\rangle \end{aligned} \tag{2.42}$$

$\hat{a}_i^\dagger$ and $\hat{a}_i$ thus act as creation and annihilation operators for the quantity $\mathcal{N}_i$ as well as for K, so it might be possible to interpret the eigenvalues of $\hat{\mathcal{N}}_i$ as particle numbers.

We then investigate the norm of the state $\hat{a}_i |n_{i,j}\rangle$, which is:

$$\langle n_{i,j}| \hat{a}_i^\dagger \hat{a}_i |n_{i,j}\rangle = \langle n_{i,j}| \mathcal{N}_i |n_{i,j}\rangle = n_{i,j} \langle n_{i,j}|n_{i,j}\rangle = n_{i,j} \tag{2.43}$$

(We let the eigenstates $|n_{i,j}\rangle$ be normalized to 1.) This means that if field states are not to have negative norm, $n_{i,j}$ will have to be non-negative. However, we know that we can use $\hat{a}_i$ to lower $n_{i,j}$ by integer steps as many times as we want. The only thing that can halt this descent and keep $n_{i,j}$ from becoming negative, is if we eventually get $n_{i,j} = 0$, so that if we apply $\hat{a}_i$ once more, the state vanishes. This requires that $n_{i,j}$ always be integer.

So we have an operator $\mathcal{N}_i$ which has integer non-negative eigenvalues, for which $\hat{a}_i$ and $\hat{a}_i^\dagger$ act as annihilation and creation operators. One final observation should hopefully let us convice ourselves that it would be natural to interpret this operator as giving the number of particles with $K = \kappa_i$ in the field. Define the operator $\hat{K}'$ as follows:

$$\hat{K}' = \sum_i \kappa_i \hat{a}_i^\dagger \hat{a}_i = \sum_i \kappa_i \hat{\mathcal{N}}_i \tag{2.44}$$

Using the commutation relations 2.40 one can check that $\hat{K}'$ has the same commutation relations with all of the $\hat{a}_i$ and $\hat{a}_i^\dagger$ as does $\hat{K}$. If we assume that all operators in the theory can be written as a function of the field operator and its conjugate momentum operator, then all operators can be written as polynomials or expanded as Taylor series of creation and annihilation operators. $\hat{K}$ and $\hat{K}'$ must then satisfy the same commutation relations with *any* operator. This furthermore means that the operator $\hat{K} - \hat{K}'$ commutes with *all* operators. It is however possible to show that if all operators can be expressed in terms of $\hat{a}_i$ and $\hat{a}_i^\dagger$ operators, then any operator which commutes with all other operators must be a c-number (just write any polynomial of such creation and annihilation operators. If it contains any terms that include $\hat{a}_i^\dagger$, it will not commute with $\hat{a}_i$, and if it contains any terms with $\hat{a}_i$, it cannot commute with $\hat{a}_i^\dagger$. Doing this for all possible indices i, we see that if the polynomial contains any operators at all other than just c-numbers, there will always be some $\hat{a}_i$ or $\hat{a}_i^\dagger$ that does not commute with it). Thus we have that $\hat{K} = \hat{K}' + C$ for some complex number C (which actually must be a real number, since $\hat{K}$ and $\hat{K}'$ are both Hermitian), i.e.

$$\hat{K} = \sum_i \kappa_i \hat{\mathcal{N}}_i + C \tag{2.45}$$

This is precisely what we want; if we interpret $\hat{\mathcal{N}}_i$ as the operator giving the number of particles of type i, then K becomes simply κ_i times the number of particles in mode i, summed over all i, exactly as it should be. We also see that there must exist a state $|0\rangle$ which is an eigenvalue of all the $\hat{\mathcal{N}}_i$ with eigenvalue $n_i = 0$, i.e. a state which is annihilated by all the $\hat{a}_i$. This we can interpret as the "empty" or vacuum state, the state with no particles in it. The number C in eq. 2.45 can then be interpreted as a vacuum value of K.

Now, by adding together all the different number operators, we can also form a *total* number operator $\hat{\mathcal{N}}$:

$$\hat{\mathcal{N}} \equiv \sum_i \hat{\mathcal{N}}_i = \sum_i \hat{a}_i^\dagger \hat{a}_i \tag{2.46}$$

The eigenvalues of this operator gives us the total number of particles in the field with respect to our definitions. Note that this operator is much more general than the individual $\hat{\mathcal{N}}_i$-operators. The latter are tied explicitly to the eigenstates of $\hat{K}$. If the field is to be in an eigenstate of $\hat{\mathcal{N}}_i$, it has to be in an eigenstate of $\hat{K}$. These number operators are therefore not likely to be compatible with those defined using an arbitrary other observable $\hat{K}'$. $\hat{\mathcal{N}}$ on the other hand might still be. The field may be in an eigenstate of $\hat{\mathcal{N}}$ without being in an eigenstate of $\hat{K}$ or any of the $\hat{\mathcal{N}}_i$'s. This just means that the field contains a definite number of particles, only these particles do not all have a definite value for K. They might have definite values for some other observable K' instead, so that the field will be in an eigenstate of $\hat{K}'$. This will be discussed further in the next chapter.

Note that here I have used just one observable, K, to define the particles. Usually the value of only one observable is not enough to determine the state of a field or a particle (even if a particle has e.g. a known energy, it could still zip off in any direction, circle around in a loop or perform somersaults). We will usually use a set of commuting operators and their eigenvalues to label the state of a field an its particles and make the states uniquely determined. We then expand the field in a set of eigenfunctions that are common to all these observables. The operators associated with these eigenfunctions will then have the same commutation relations with all of the observables as in eq. 2.28, and following the same line of reasoning as for the observable K, we can conclude that they will create or annihilate one particle with the corresponding eigenvalues for each of the commuting observables.

2.4 A note on fermions

Before we leave this chapter, I will just say a few very brief words about what happens in the case of a fermionic field. In this case, God has decided that we must use the fundamental *anti*commutator relation 2.19 between the field operator and the conjugate momentum rather than the commutator relation 2.18. This means that the commutator relations 2.40 between the creation and annihilation operators become anticommutators instead:

$$\{\hat{a}_i,\, \hat{a}_j^\dagger\} = \delta_{ij} \qquad \{\hat{a}_i,\, \hat{a}_j\} = \{\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\} = 0 \tag{2.47}$$

The commutation relations with $\hat{K}$ are as before, so that $\hat{a}_i$ acts as an annihilation operator

for K, and $\hat{a}_i^\dagger$ acts as a creation operator. Now define the operator $\hat{\mathcal{N}}_i$ to be $\hat{\mathcal{N}}_i = \hat{a}_i^\dagger \hat{a}_i$, just as in eq. 2.41. By using the formula

$$[\hat{A}, \hat{B}] = \{\hat{A}, \hat{B}\} - 2\hat{B}\hat{A} \tag{2.48}$$

and after a little computation, we find that the commutation relations of $\hat{a}_i$ and $\hat{a}_i^\dagger$ with $\hat{\mathcal{N}}_i$ are the same as for bosonic fields (as in eq. 2.42). This means that $\hat{a}_i$ still lowers the eigenvalue $n_{i,j}$ of $\hat{\mathcal{N}}_i$ by 1, while $\hat{a}_i^\dagger$ raises it by 1. The expression 2.45 for $\hat{K}$ also still holds true, so it would be natural to interpret $\hat{\mathcal{N}}_i$ as a number operator for the mode i.

We also still have that the norm of $\hat{a}_i \ |n_{i,j}\rangle$ is $n_{i,j}$ (as in eq. 2.43). But now, because of the anticummutation relations 2.47, we have

$$\begin{aligned} \{\hat{a}_i, \hat{a}_i\} = 2\hat{a}_i^2 = 0 \quad &\Longrightarrow \quad \hat{a}_i^2 = 0 \\ \{\hat{a}_i^\dagger, \hat{a}_i^\dagger\} = 2(\hat{a}_i^\dagger)^2 = 0 \quad &\Longrightarrow \quad (\hat{a}_i^\dagger)^2 = 0 \end{aligned} \tag{2.49}$$

This implies that

$$\langle n_{i,j}| \, \hat{a}_i^\dagger \hat{\mathcal{N}}_i \hat{a}_i \, |n_{i,j}\rangle = \langle n_{i,j}| \, (\hat{a}_i^\dagger)^2 \hat{a}_i^2 \, |n_{i,j}\rangle = 0 \tag{2.50}$$

But this is also equal to

$$\langle n_{i,j}| \, \hat{a}_i^\dagger \hat{\mathcal{N}}_i \hat{a}_i \, |n_{i,j}\rangle = n_{i,j} \ \langle n_{i,j} - 1| \, \hat{\mathcal{N}}_i \, |n_{i,j} - 1\rangle = n_{i,j} \, (n_{i,j} - 1) \tag{2.51}$$

which implies that $n_{i,j}$ must be either 0 or 1. A mode therefore cannot accomodate more than a single particle, as we would expect for fermions.

$\hat{a}_i$ obviously annihilates states where $n_{i,j} = 0$. If we now define the total number operator $\hat{\mathcal{N}} = \sum_i \hat{\mathcal{N}}_i$ as before, we therefore get that the state where $\mathcal{N} = 0$ is annihilated by all the $\hat{a}_i$ as before, so it makes sense to define this as the vacuum state. However, by inspection we also find that $\hat{a}_i^\dagger$ annihilates any state for which $n_{i,j} = 1$. If we wish, we could therefore define $\hat{\mathcal{N}}_i' = \hat{a}_i \hat{a}_i^\dagger$ and use this as a number operator instead of $\hat{\mathcal{N}}_i$. This number operator would then have eigenvalues 1 or 0 as before, but the roles of both the 0- and 1-state and the creation and annihilation operators would be reversed. What was previously the occupied state will now be the vacuum, and the vacuum will now be the occupied state. $\hat{a}_i^\dagger$ will now act as an annihilation operator for $\hat{\mathcal{N}}_i'$, while $\hat{a}_i$ acts as a creation operator, since the commutation relations turn out to be the reverse of eq. 2.42. The form 2.45 for $\hat{K}$ furthermore will become

$$\hat{K} = \sum_i \kappa_i \hat{a}_i^\dagger \hat{a}_i + C = \sum_i \kappa_i \left(1 - \hat{a}_i \hat{a}_i^\dagger\right) + C = -\sum_i \kappa_i \hat{\mathcal{N}}_i' + C' \tag{2.52}$$

where we have absorbed the terms $\sum_i \kappa_i$ into a new vacuum value C' for K.

For fermions we therefore have two options when it comes to defining the vacuum and the particle states. We can either choose the vacuum to be the state that is annihilated by all the $\hat{a}_i$, in which case the particle states will be those that we defined first. Or, we can choose the vacuum state to be the one that is annihilated by all the $\hat{a}_i^\dagger$. The particles will then be a kind of "holes", that are simply defined as the absence of a particle according to the first definition, and with opposite signs for their K-values.

Chapter 3

Particular ambiguities

In the previous chapter we saw how particle-like objects arise in field theories when they are quantized. Defining and deriving the existence of these objects was fairly straight-forward. However, the procedure completely ignored the question of uniqueness; will the field contain the same number of particles and will these particles have the same characteristics regardless of how we carry out the quantization? For instance, we relied heavily on a single observable, K, to derive the existence of particle-like states for the quantum field. As part of the Hamiltonian formalism, we also chose a specific reference frame to work in, represented by the choice of a time-coordinate and a family of equal-time spacelike hypersurfaces to integrate all of our integrals over. Will our particles look the same and will there be equally many of them in any given field state no matter what observable or what reference frame we choose to use? In this chapter I will try to answer these questions and find an interesting mix of "yes" and "no" answers that will form the basis for the rest of the investigations of this thesis.

3.1 Particles - what kind and how many?

We made several assumptions in deriving the existence of particles in quantized fields which a priori might seem rather arbitrary. But they were mostly motivated by and necessary to the goal of finding quantum states that would satisfy our requirements for what states with definite numbers of particles in them should look like. Our choice of the observable K used to define the particles however, was completely arbitrary, any Hermitian field operator could have played the part. But we know that at some level it *must* matter what operator we choose. States with a definite number of particles with a given value κ_i of K, i.e. eigenstates of the number operator $\hat{\mathcal{N}}_i$, are eigenstates of $\hat{K}$. So if the operators $\hat{K}$ and $\hat{K}'$ of two observables do not have a common complete set of eigenstates, then a state with a definite number of particles with a particular value of K cannot also in general be a state with a definite number of particles with a particular value of K'. In other words, if $\hat{K}$ and $\hat{K}'$ do not commute, then they do not share the same set of number operators $\hat{\mathcal{N}}_i$, so in this sense at least, the definition of particles and particle numbers with respect to K will not be equivalent to that with respect to K'. But even if $\hat{K}$ and $\hat{K}'$ do not commute and their individual number operators $\hat{\mathcal{N}}_i$ are not equivalent, there is still the question of whether

or not their total number operators $\hat{\mathcal{N}}$ are the same. And if they are not, there is also the possibility that they may not even share the same vacuum state, so that a state which is completely void of particles with respect to K could be teeming with particles if we define them using K' instead.

Let us therefore investigate the relationship between particle number operators defined with respect to different observables a little further. Note that from now on, I will drop the hats on the operators to economize on the notation, unless there is danger of confusion. Suppose that we have two observables F and G (I will call them this from here on rather than K and K' in order to have a clearer notation). Expand the field operator in eigenfunctions for each of them:

$$\phi(x) = \sum_i \Big(a_i f_i(x) + a_i^\dagger f_i^*(x) \Big) = \sum_I \Big(b_I g_i(x) + b_I^\dagger g_I^*(x) \Big) \tag{3.1}$$

where the f_i and g_I are positive norm eigenfunctions, F and G respectively, and a_i, $a_i^\dagger$, b_i and $b_i^\dagger$ are the corresponding operators. I use capitalized indices on the functions g_I and their operators to indicate that the indices on these functions are not the same as those on the eigenfunctions f_i of F. Since the positive norm eigenfunctions f_i of F and g_I of G, and their negative norm counterparts f_i^* and g_I^* together should form a complete set, we can expand them both in terms of each other:

$$g_I = \sum_j \left(\alpha_{Ij} f_j + \beta_{Ij} f_j^* \right) \qquad g_I^* = \sum_j \left(\alpha_{Ij}^* f_j^* + \beta_{Ij}^* f_j \right) \tag{3.2}$$

and

$$f_j = \sum_I \left(\alpha'_{jI}\, g_I + \beta'_{jI}\, g_I^* \right) \qquad f_j^* = \sum_I \left(\alpha'^*_{jI}\, g_I^* + \beta'^*_{jI}\, g_I \right) \tag{3.3}$$

Assume that the positive norm eigenfunctions f_j and g_I are all normalized to $+1$ and hence the negative norm eigenfunctions f_j^* and g_I^* all to -1 (if the eigenfunction indices can take a continuum of values, we will need to normalize them to δ-functions instead, but this is a rather straight-forward generalization that I will not treat in detail). We then get:

$$\begin{gathered} \alpha_{Ij} = \langle f_j, g_I \rangle \qquad \beta_{Ij} = -\langle f_j^*, g_I \rangle \\ \alpha'_{jI} = \langle g_I, f_j \rangle = \alpha_{Ij}^* \qquad \beta'_{jI} = -\langle g_I^*, f_j \rangle = -\langle f_j, g_I^* \rangle^* = \langle f_j^*, g_I \rangle = -\beta_{Ij} \end{gathered} \tag{3.4}$$

where we have used the property of the inner product 2.16 that $\langle f, g \rangle = -\langle f^*, g^* \rangle^*$.

This expansion allows us to express the field operator as follows:

$$\begin{aligned} \phi(x) &= \sum_j \Big(a_j f_j(x) + a_j^\dagger f_j(x)^* \Big) \\ &= \sum_{Ij} \Big[a_j \left(\alpha_{Ij}^* g_I(x) - \beta_{Ij} g_I(x)^* \right) + a_j^\dagger \left(\alpha_{Ij} g_I(x)^* - \beta_{Ij}^* g_I(x) \right) \Big] \\ &\stackrel{!}{=} \sum_I \Big(b_I g_I(x) + b_I^\dagger g_I(x)^* \Big) \end{aligned} \tag{3.5}$$

For this last equality to hold, we must have the following relationship between the creation and annihilation operators:

$$b_I = \sum_j \left(\alpha^*_{Ij} a_j - \beta^*_{Ij} a_j^\dagger\right) \qquad b_I^\dagger = \sum_j \left(\alpha_{Ij} a_j^\dagger - \beta_{Ij} a_j\right) \tag{3.6}$$

and in the same way we obtain:

$$a_j = \sum_I \left(\alpha_{Ij} b_I + \beta^*_{Ij} b_I^\dagger\right) \qquad a_j^\dagger = \sum_I \left(\alpha^*_{Ij} b_I^\dagger + \beta_{Ij} b_I\right) \tag{3.7}$$

This kind of transformation from one set of creation and annihilation operators to another is known as a Bogoliubov transformation, and the α- and β-coefficients are known as Bogoliubov coefficients.

For later reference, we also list some relations satisfied by the Bogoliubov coefficients. The normalization $\langle f_i, f_j\rangle = \delta_{ij}$ straight away gives us:

$$\sum_K \left(\alpha_{Ki}\alpha^*_{Kj} - \beta^*_{Ki}\beta_{Kj}\right) = \delta_{ij} \tag{3.8}$$

and the orthogonality $\langle f_i, f_j^*\rangle = 0$ gives us:

$$\sum_K \left(\beta^*_{Ki}\alpha_{Kj} - \alpha_{Ki}\beta^*_{Kj}\right) = 0 \tag{3.9}$$

The normalization $\langle g_I, g_J\rangle = \delta_{IJ}$ gives us:

$$\sum_k \left(\alpha^*_{Jk}\alpha_{Jk} - \beta^*_{Ik}\beta_{Jk}\right) = \delta_{IJ} \tag{3.10}$$

and finally the orthogonality $\langle g_I, g_J^*\rangle = 0$ gives us:

$$\sum_k \left(\alpha_{Ik}\beta_{Jk} - \beta_{Ik}\alpha_{Jk}\right) = 0 \tag{3.11}$$

From these algebraic exercises, we see that creating or annihilating a particle with a sharp F-value will be equivalent to creating or annihilating a particle with a sharp G-value if and only if all the β_{Ij} are zero and exactly one of the α_{Ij} is non-zero for each j (more of them could be non-zero if there are degeneracies). But we also see that if only all the β_{Ij} are zero, even if more than one of the α_{ij} are non-zero, creating or annihilating a particle with respect to F will still be like creating of annihilating one particle with respect to G, even if that particle may not have a sharp value of both F and G. More precisely all the annihilation operators b_I of G can in this case be written as a combination of annihilation operators a_i of F only, without involving the creation operators $a_i^\dagger$. Any state that is annihilated by all of the a_i, i.e. a vacuum state of F, will then also be annihilated by all the the b_I and thus be a vacuum state of G as well. So, If all the β_{Ij} are zero, that is if the expansion of the positive norm eigenfunctions of one observable does not contain any negative norm eigenfunctions of the other, then the two observables will agree on the definition of the vacuum state.

Now if all the β_{Ij} vanish so that F and G agree on what the vacuum is and both agree on creating or annihilating one particle is, then presumably they should be able to agree on

the total number of particles in any given state. That is, their total number operators, which I will denote $\mathcal{N}_F$ and $\mathcal{N}_G$, should be the same. Let us now see if this holds true. We have:

$$\begin{aligned} \mathcal{N}_F &\equiv \sum_j \mathcal{N}_{F,j} = \sum_j a_j^\dagger a_j \\ \mathcal{N}_G &\equiv \sum_I \mathcal{N}_{G,I} = \sum_I b_I^\dagger b_I \end{aligned} \tag{3.12}$$

Expanding the creation and annihilation operators of F gives us:

$$\begin{aligned} \mathcal{N}_F &= \sum_j a_j^\dagger a_j = \sum_j \sum_I \left(\alpha_{Ij}^* b_I^\dagger + \beta_{Ij} b_I\right) \sum_K \left(\alpha_{Kj} b_K + \beta_{Kj}^* b_K^\dagger\right) \\ &= \sum_{IjK} \left(\alpha_{Ij}^* \alpha_{Kj}\, b_I^\dagger b_K + \beta_{Kj} \beta_{Kj}^*\, b_I b_K^\dagger + \beta_{Ij} \alpha_{Kj}\, b_I b_I + \alpha_{Ij}^* \beta_{Kj}^*\, b_I^\dagger b_K^\dagger\right) \end{aligned} \tag{3.13}$$

As expected this is not equal to $\mathcal{N}_G$ if any $\beta_{Ij} \neq 0$. However, if all $\beta_{Ij} = 0$, then all terms but the first vanish, and we can use eq. 3.10 to obtain:

$$\mathcal{N}_F = \sum_{IjK} \alpha_{Ij}^* \alpha_{Kj} b_I^\dagger b_K = \sum_{IK} \delta_{IK} b_I^\dagger b_K = \sum_I b_I^\dagger b_I = \mathcal{N}_G \tag{3.14}$$

So, if all β_{Ij} vanish, i.e. there is no overlap between positive norm eigenfunctions of F and negative norm eigenfunctions of G or vice versa, then F and G will agree on what states contain a definite number of particles and the number of particles in those states.

Bogoliubov coefficients can thus help elucidate the relationship between particle states defined with respect to different observables and the ambiguities that arise from the choice of such an observable. But even if we have chosen one single observable to use for defining particles, all ambiguities are not necessarily laid to rest. The creation and annihilation operators, and with them the spectrum of particle states, are only uniquely defined as long as the eigenstates of the observable are not degenerate, i.e. as long as no two eigenstates have the same eigenvalue. If we do have degeneracies, then there will be more than one way to define the set of eigenfunctions f_i of an observable F. Suppose that two positive norm eigenfunctions f_{i_1} and f_{i_2} have the same eigenvalue κ_i, are orthogonal to each other and are normalized to $+1$. Then e.g. the two functions $f_{I_1} = a f_{i_1} + b f_{i_2}$ and $f_{I_2} = b f_{i_1} - a f_{i_2}$ will also both have the same eigenvalue κ_i. If we choose a and b such that $a^2 + b^2 = 1$ (a convenient way often employed is to choose $a = \cos\theta$ and $b = \sin\theta$ for some appropriate mixing angle θ), the functions will be normalized to $+1$ and will be orthogonal to each other, so that we may as well use f_{I_1} and f_{I_2} in the set of positive norm eigenfunctions instead of f_{i_1} and f_{i_2}. In general we can apply a Bogoliubov transformation with $\alpha_{Ij} \neq 0$ for any eigenfunctions f_I and f_j that have the same eigenvalue.

As long as we only mix *positive* norm eigenfunctions in this way, there is no problem. This just amounts to saying that if you e.g. have a two-dimensional harmonic oscillator potential, then creating a particle oscillating at an angle to the x- and y-axes in a certain energy level is the same as creating one particle oscillating along the x-axis and one oscillating along the y-axis in the same energy level and then superposing the two states

with an appropriate relative coefficient. This makes perfect physical sense. But if we have a *negative* norm eigenfunction f_j^* with the same eigenvalue κ_i as f_i (i.e. $\kappa_j = -\kappa_i$, so that the eigenvalue of f_j^* is $-\kappa_j = \kappa_i$), then we can wreak a bit more havoc (according to the previously introduced convention, we write f_j^* as the complex conjugate of a positive norm eigenfunction f_j). We can define two new eigenfunctions $g_K = cf_i + df_j^*$ and $g_L^* = df_i + cf_j^*$ for some numbers c and d, which will take to be real (why we write g_L^* as a complex conjugate will become apparent in a moment). Assuming that f_i and f_j^* are orthogonal and normalized to 1 and -1 respectively, the new eigenfunctions will have the norms $\langle g_K, g_K\rangle = c^2 - d^2$ and $\langle g_L^*, g_L^*\rangle = d^2 - c^2$, and the inner product between them will be $\langle g_K, g_L^*\rangle = 0$. If we now choose c and d such that $c^2 - d^2 = 1$, which can in general be done by choosing $c = \cosh\chi$ and $d = \sinh\chi$ for some real number χ, then g_K will have norm 1 and g_L^* will have norm -1. We then have two new orthogonal eigenfunctions with eigenvalue κ_i, one with norm 1 and one with norm -1, as before. But if we wish, we can view this as a Bogoliubov transformation and express the annihilation and creation operators b_K and $b_L^\dagger$ associated with g_K and $g_L^\dagger$ by the annihilation and creation operators a_i and $a_j^\dagger$ associated with f_i and f_j^*. In that case we have the following Bogoliubov coefficients:

$$\begin{aligned} \alpha_{Ki} &= c = \cosh\chi \qquad & \alpha^*_{Lj} &= \alpha_{Lj} = c = \cosh\chi \\ \beta_{Kj} &= d = \sinh\chi \qquad & \beta^*_{Li} &= \beta_{Li} = d = \sinh\chi \end{aligned} \tag{3.15}$$

From eq. 3.6 the new annihilation and creation operators are then:

$$b_K = \cosh\chi\, a_i - \sinh\chi\, a_j^\dagger \qquad b_L^\dagger = \cosh\chi\, a_j^\dagger - \sinh\chi\, a_i \tag{3.16}$$

With these definitions we still preserve the right commutation relations:

$$\begin{aligned} [b_K, b_K^\dagger] &= -[b_L^\dagger, b_L] = \cosh^2\chi - \sinh^2\chi = 1 \\ [b_K, b_L] &= [b_K^\dagger, b_L^\dagger] = [b_K^\dagger, b_L] = [b_K, b_L^\dagger] = 0 \end{aligned} \tag{3.17}$$

Thus we have a new set of annihilation and creation operators that satisfy all the usual commutation relations, but e.g. creating a particle with $b_L^\dagger$ is now a superposition of creating one with $a_j^\dagger$ and *annihilating* one with a_i. If we have positive and negative norm eigenfunctions sharing the exact same eigenvalue, there will therefore be an ambiguity in what is creation and what is annihilation of particles.

Since the eigenvalue of a negative-norm function f_j^* is $-\kappa_j$ for the kinds of operators that we have in section 2.3 (of the form $B^\mu(x)\, i\frac{\partial}{\partial x^\mu}$), if f_j^* has the same eigenvalue as some positive-norm eigenfunction f_i, it means that $\kappa_j = -\kappa_i$. This means that the cause of the ambiguity is that we cannot tell the difference between creating a particle with eigenvalue $-\kappa_i$ and annihilating one with eigenvalue $+\kappa_i$. So the concerns about this that I raised on page 31, turned out to be justified. If there are eigenvalues that have perfectly symmetric "partner" eigenvalues with the same magnitude but opposite sign, we will therefore have an unavoidable ambiguity in our definition of particles. It is important to note though, that if we use several commuting observables to describe our particles, then the functions f_i and f_j^* must have degenerate eigenvalues for every one of them, not only one. Otherwise, the eigenfunctions g_K and g_L^* might no longer be eigenfunctions for all of the observables.

3.2 Reference frames and "natural" observables

After all this general talk about defining particles and arbitrary observables, it is time to be a bit more specific. Although most observables are equal, for the purpose of defining particles in quantum field theories some observables are more equal than others. For any observer, one natural observable to use would be the one that generates time translations for that observer, i.e. the Hamiltonian. The Hamiltonian corresponds to energy as defined with respect to the observer, so this will ensure that a particle has a well defined energy, which is desirable in most cases.

What "time translation" means, of course depends on what the spacetime looks like and how the observer moves in that spacetime. An observer could of course be moving in any way he, she or it (in case we have a little green Martian scientist with no gender) desires. But for the sake of defining particles, we would prefer to disallow using an operator that generates just any kind of observer trajectory. More specifically, we would prefer to use a trajectory which is such that the Hamiltonian is not explicitly dependent on the proper time of the observer. If we defined particles with respect to an explicitly time dependent Hamiltonian, then the Hamiltonian at one point in time might not commute with itself at another point in time. This would imply that the number operator would be explicitly time dependent and a state with a well defined particle number at one time might not contain a well defined number of particles any more at a later time. The vacuum state would probably also not be constant, so that particles could spontaneously appear in the vacuum, even in the absence of any interactions or sources of energy (with a time dependent Hamiltonian, energy with respect to that Hamiltonian would obviously also not be conserved). There is of course nothing fundamentally wrong about this, but this is not the kind of behaviour we would like particles in general and the vacuum in particular to indulge in.

If the Hamiltonian is to be independent of time, i.e. stationary, the observer with respect to which the Hamiltonian is defined must experience the surroundings as being stationary as he or she coasts through spacetime along his or her world line. This means in particular that the metric tensor must be constant in the observer's reference frame, i.e. its Lie derivative along the lines of constant spatial position as defined by the observer's reference frame must be zero. This further means that the tangent vectors of these lines, i.e. the timelike basis vectors in the natural coordinate system of the observer, must form a Killing vector field. The kind of Hamiltonian that we want to use for defining particles is thus one which generates translations along a Killing vector field (all these rather loose references to concepts such as "reference frames", "natural coordinate systems" and the like will be treated more intimately in the next chapter).

This thesis will primarily be concerned with Minkowski spacetime, that is flat $3 + 1$-dimensional spacetime. Minkowski spacetime is blessed by being rather poor in ambiguities as far as particles are concerned. Most reasonable quantum field theories in Minkowski spacetime can be cast in an explicitly Lorentz-covariant form, which means that the field and its particles should have the same properties in all inertial reference frames. In particular, as we shall see, the vacuum state is Lorentz invariant, so that if space is empty as viewed from one inertial reference frame, then it is empty in all other inertial reference frames as well. However, it turns out that there are also other, non-inertial frames of reference which can serve just as well for defining particles and a vacuum states. And these

particle and vacuum states are not necessarily equivalent to the ones defined with respect to inertial frames, so that even in Minkowski spacetime there may be room for ambiguities in the definition of particles and the vacuum.

Let us first investigate the observables linked to time translation in inertial frames in Minkowski spacetime. The operator that generates time translations on the Hilbert space in any particular inertial frame is the Hamiltonian operator H in that reference frame (equivalent to the differential operator $i\frac{\partial}{\partial t}$ in the coordinate representation for that reference frame). The vectors generating time translation in an arbitrary inertial frame (the vectors $\frac{\partial}{\partial t}$ in that frame) form a Killing vector field since the Minkowski metric does not depend on any inertial time coordinate, so the Hamiltonian of an inertial frame is obviously acceptable for defining particles and their energies.

The generator of time translation in some other inertial frame moving at constant velocity relative to the first one, involves a spatial translation along the direction of motion in the original frame. Spatial translations of the Hilbert space are generated by the momentum operators P_i, where $i = 1, 2, 3$ in $3+1$-dimensional spacetime. The generator for time translations with respect to another inertial frame is thus equal to the linear combination $\gamma(H + \boldsymbol{\beta} \cdot \mathbf{P})$, as one can find by using a standard Lorentz transformation ($\boldsymbol{\beta} = \mathbf{v}/c$ where $\mathbf{v}$ is the velocity vector of the second inertial system relative to the original one, $\gamma = (1 - \mathbf{v}^2/c^2)$ and $\mathbf{P} = (P_1, P_2, P_3)$). The *vectors* generating time translations on the spacetime in this inertial frame are $\gamma(\frac{\partial}{\partial t} + \beta^i \frac{\partial}{\partial x^i})$ (sum over i from 1 to 3), which is of course a Killing vector field since the vectors $\frac{\partial}{\partial x^i}$ generating spatial translations are Killing vectors for the Minkowski metric. Furthermore, since the operators P_i generating spatial translations commute with the old Hamiltonian, one may choose to define the particle states of the new Hamiltonian to be the same as those defined with respect to the old one, only with different values for the energies and momenta of the particles. Defining particles by using the Hamiltonian and the momentum operators, so that particles are entities with definite energies and momenta (plane waves) is also the most usual way to define particles in Minkowski spacetime. This is why e.g. the quantized Klein-Gordon field with mass m is most usually written in the form $\phi(x) \sim \int (a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x}) d^3p$, with $p = (E, \mathbf{p})$ and $E = \sqrt{\mathbf{p}^2 + m^2}$. The Klein-Gordon field is then expanded in simultaneous eigenfunctions of the Hamiltonian and momentum operators, so that the operators $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ are precisely annihilation and creation operators for particles with momentum $\mathbf{p}$ and energy $\sqrt{\mathbf{p}^2 + m^2}$.

However, the vector fields generated by linear combinations of spatial and timelike translations (or in short by the vector fields $\frac{\partial}{\partial x^\mu}$) are not the only Killing vector fields in Minkowski spacetime. Since the metric is invariant under all Lorentz transformations, any vector field that generates a general Lorentz transformation is a Killing vector field. Such a vector field is a linear combination of the vectors fields generating spatial and temporal translations, spatial rotations and boosts (which can be thought of as 4D rotations involving one spatial axis and the time axis). The vector fields generating these transformations are listed in table 3.1 along with the operators generating the corresponding transformations on the Hilbert space. Coordinates used for the vector fields and the coordinate-representation operators are standard Cartesian Minkowski coordinates, with conventional summation over repeated indices, although one specific coordinate system is used so that repeated indices need not occupy different positions vertically.

Transformation	**Vector field**	**Hilbert space op.**	**Op. in coordinate rep.**
Time translation	$\frac{\partial}{\partial t}$	H	$i\frac{\partial}{\partial t}$
Spatial translation	$\frac{\partial}{\partial x^i}$	P_i	$i\frac{\partial}{\partial x^i}$
Spatial rotation	$\frac{1}{2}\epsilon_{ijk}\left(x^j\frac{\partial}{\partial x^k}-x^k\frac{\partial}{\partial x^j}\right)$	$J_i=\frac{1}{2}\epsilon_i{}^{jk}J_{jk}$	$\frac{1}{2}i\epsilon_{ijk}\left(x^j\frac{\partial}{\partial x^k}-x^k\frac{\partial}{\partial x^j}\right)$
Boost	$x^i\frac{\partial}{\partial t}+t\frac{\partial}{\partial x^i}$	$B_i=J_{0i}$	$ix^i\frac{\partial}{\partial t}+it\frac{\partial}{\partial x^i}$
General translation	$\frac{\partial}{\partial x^\mu}$	P_μ	$i\frac{\partial}{\partial x^\mu}$
General rotation	$x_\mu\frac{\partial}{\partial x^\nu}-x_\nu\frac{\partial}{\partial x^\mu}$	$J_{\mu\nu}$	$i\left(x_\mu\frac{\partial}{\partial x^\nu}-x_\nu\frac{\partial}{\partial x^\mu}\right)$

Table 3.1: Generators of general Lorentz transformations (using $\epsilon^{xyz}=+1$ for Levi-Civita pseudotensor)

I should also make a note on the sign on the Hilbert space momentum operators. They are listed as $+i\frac{\partial}{\partial x^i}$, whereas in conventional quantum mechanics one usually writes them as $-i\frac{\partial}{\partial x^i}$. This is because in the usual quantum mechanics, one usually wants a generator of spatial translations to be the an operator for the momentum *vector*, which is a contravariant quantity. The generator in this case should therefore really be written as $-i\frac{\partial}{\partial x_i}$. With our sign convention for the metric $(+1,-1,-1,-1)$, this becomes $+i\frac{\partial}{\partial x^i}$ when we raise the i-index. The generators given here represent the momentum-*co*vector, a covariant quantity, which is really more appropriate in a relativistic setting. Still, when we come to calculating explicit eigenfunctions in chapter 5, I will be more conventional and treat the momentum as a contravariant vector quantity. Another reason for using the convention $P^i=-i\frac{\partial}{\partial x^i}$ in conventional quantum mechanics, is that it should satisfy the fundamental commutation relation $[x^i,P^j]=i\delta^{ij}$, which requires that we use $-i$ rather than $+i$. The reason for this is that the the transformation generated by $P^i=-i\frac{\partial}{\partial x^i}$ with this commutator is a different kind of transformation than that generated by $H=i\frac{\partial}{\partial t}$ or the $P_i=+i\frac{\partial}{\partial x^i}$ that we use here. The infinitesimal transformation $|\psi\rangle\rightarrow(1-\epsilon iP_i)|\psi\rangle$ is a transformation under which we so to speak leave the coordinate system in place, but move the whole physical system a distance ϵ in the x^i-direction, so that $\psi(x^i)$ after the transformation is equal to what $\psi(x^i-\epsilon)$ was before the transformation. The transformations generated by $H=i\frac{\partial}{\partial t}$ and $P_i=i\frac{\partial}{\partial x^i}$ simply lets the physical system evolve with increasing t or x^i so that $\psi(t,x^i)\rightarrow\psi(t+\epsilon_t,x^i+\epsilon^i)$ under the transformation, which gives us the minus sign of P_i relative to P^i. The same analyses also apply to the rotation generator J_i. The sign convention usually used in quantum mechanics is the opposite here, since one usually views the angular momentum as a pseudo*vector* J^i, whereas here it is given as a pseudo*covector* J_i. The same remarks regarding what sort of transformations the two operators generate, also still apply.

For later reference we will also need the commutation relations between the generators. They can easily be calculated using the coordinate representation, and are found to be:

$$\begin{aligned}[H,P_i] &= 0\\ [H,J_i] &= 0\\ [H,B_i] &= iP_i\\ [P_i,P_j] &= 0\end{aligned}$$

$$\begin{aligned}
[P_i, J_j] &= -i\epsilon_{ijk}P_k \\
[P_i, B_j] &= i\delta_{ij}H \\
[J_i, J_j] &= i\epsilon_{ijk}J_k \\
[J_i, B_j] &= i\epsilon_{ijk}B_k \\
[B_i, B_j] &= -i\epsilon_{ijk}J_k
\end{aligned} \tag{3.18}$$

For our purposes then, which is to define particle states, an acceptable kind of time translation is any which is generated by a linear combination of the vector fields listed in table 3.1, and an acceptable Hamiltonian operator is any which is expressed as the same linear combination of the corresponding operators. As I will show in the next chapter, there are no more generators of Killing vectors fields in Minkowski spacetime than those listed here, so that linear combinations of those are also the *only* acceptable generators of stationary time-translations. The Hamiltonian H_{RF} (RF for reference frame) of any stationary but not necessarily inertial reference frame in Minkowski spacetime can therefore be expressed as follows:

$$H_{\mathrm{RF}} = \gamma\left(H + \beta^i P_i + \omega^i J_i + a^i B_i\right) \tag{3.19}$$

Such a frame is in general both moving, rotating and accelerating relative to the inertial frame with respect to which the operators H, P_i, J_i and B_i are defined. The β^i can thus be interpreted as velocity parameters, the a^i as proper acceleration and the ω^i as proper angular velocity of the new reference frame (i.e. the acceleration and angular velocity experienced by an observer for which the new reference frame is the rest frame). Note that for the remainder of this thesis, operators subscripted with RF will be defined relative to some stationary but not necessarily inertial reference frame. Non-subscripted Lorentz generators on the other hand, such as those on the right hand side of eq. 3.19, will always be defined relative to some specific *inertial* frame (with a specific origin, to make the "spacetime rotation" operators J_i and B_i well defined).

For later use, it is worth noting that eq. 3.19 can be written in an explicitly Lorentz-covariant way as follows:

$$H_{\mathsf{RF}} = \gamma\left(\beta^\mu P_\mu + \frac{1}{2}\omega^{\mu\nu}J_{\mu\nu}\right) \tag{3.20}$$

where $\beta^\mu = (1; \beta^1, \beta^2, \beta^3)$, $P^\mu = (H; P_1, P_2, P_3)$, and $\omega^{\mu\nu}$ and $J_{\mu\nu}$ can be written in matrix form as

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & -a^1 & -a^2 & -a^3 \\ a^1 & 0 & -\omega^3 & \omega^2 \\ a^2 & \omega^3 & 0 & -\omega^1 \\ a^3 & -\omega^2 & \omega^1 & 0 \end{pmatrix} \qquad J^\mu{}_\nu = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & J_3 & -J_2 \\ -B_2 & -J_3 & 0 & J_1 \\ -B_3 & J_2 & -J_1 & 0 \end{pmatrix} \tag{3.21}$$

so that the vectors $\mathbf{a}$ and $\boldsymbol{\omega}$ used previously are defined by $a^i = \omega^{0i}$ and $\omega^i = \frac{1}{2}\epsilon^{ijk}\omega_{jk}$.

$J_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}$ transforms as a tensor under Lorentz-transformation, so $\omega^{\mu\nu}$ must do so too. This means that the following two quantities are invariant when we change from one inertial reference frame to another:

$$\omega^{\mu\nu}\omega_{\mu\nu} = |\boldsymbol{\omega}|^2 - |\mathbf{a}|^2 \qquad \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\omega^{\mu\nu}\omega^{\rho\sigma} = \mathbf{a}\cdot\boldsymbol{\omega} \tag{3.22}$$

These invariants will come in handy in the next chapter.

If we now let all the a^i and ω^i be equal to zero, so that H_{RF} is composed of only H and P_i, then the frame of H_{RF} is just moving with constant velocity with respect to the original inertial frame, and is therefore itself an inertial frame. As already mentioned, since $[H, P_i] = 0$ for all i, H and H_{RF} can be arranged to share the same particle states and the same vacuum. Since the $\beta^i P_i$-term can be always be removed by a single Lorentz transformation in the appropriate direction, we will choose to set $\beta^i = 0$ from the start. This simply means that we start with a different original inertial frame with respect to which we define the operators H, J_i and B_i. This is the frame which at some point is the instantaneous rest frame of an observer located at the origin in the reference frame of H_{RF}.

By defining H_{RF} relative to an instantaneous rest frame of the observer, it is easy to give a physical interpretation to the parameters a^i and ω^i. The terms $a^i B_i + \omega^i J_i$ in the expression for H_{RF} tell us that we get from one instantaneous rest frame of the observer to one an infinitesimal proper time interval $d\tau$ later by boosting the frame by a velocity of $a^i d\tau$ in the x^i-direction and rotating it through an angle of $\omega^i d\tau$ around the x^i-axis. This means that if we fix a coordinate axis cross to the observer, then we can say that the observer is accelerating with a proper acceleration of a^i along the x^i-axis and constantly turning his/her direction of motion and acceleration with an angular velocity of ω^i around the x^i-axis. The parameters a^i and ω^i thus directly measure the acceleration and rotation experienced by the observer in his/her non-inertial frame of reference. Note that defining the Hamiltonian in this way, relative to the instantaneous rest frame of an observer, is crucially different from the approach taken in many other earlier works, notably J. R. Letaw and J. D. Pfautsch ([4]). They instead start out with a reference frame where the observer is not at rest but where the observer's motion has a simple form. I will follow this approach much of the way when it comes to doing practical computations, but when we want to interpret the results, it will often be important to keep contact with the instantaneous rest frame of the observer that I have been talking about here.

Here I claimed that a^i and ω^i represent the proper acceleration and rotation experienced by the observer in his/her own non-inertial reference frame. However, one might object that the operators B_i and J_i in the expression for H_{RF} were defined relative to a fixed, inertial reference frame, which by setting $\boldsymbol{\beta} = 0$ we made an instantaneous rest frame of the observer at one point in time. But only at one point. Once the observer has been transported away from that original instantaneous rest frame, how can we be sure that such a direct interpretation of the terms $\omega^i J_i + a^i B_i$ still holds? A simple, qualitative answer is that the observer is moving along a Killing vector field and experiences a stationary situation, so that the proper acceleration and rotation felt in the original instantaneous rest frame, must be the same as that experienced at all times along the observer's trajectory. This can also be shown explicitly as follows: Suppose that we have two inertial frames, 1 and 2. Let U be

the unitary operator that performs the transformation that takes us from frame 1 to frame 2. Now let the (hermitian) operator L_1 be some Lorentz generator or linear combination of Lorentz generators defined with respect to frame 1. We then want to know what kind of Lorentz generator L_1 looks like when viewed from frame 2, call that generator L_2. The effect of applying L_1 must be the same as first applying U^{-1}, so that frame 2 is transformed into frame 1, then applying L_2, which is the infinitesimal Lorentz transformation that an observer in frame 2 is supposed to see being applied, and finally transforming everything back to the right reference frame by applying U. Thus we get:

$$L_1 = U L_2 U^{-1} \tag{3.23}$$

To apply this to the question of interpreting H_{RF}, let frame 1 be the original rest frame of the observer, with respect to which the Lorentz generators in eq. 3.19 are defined, and let frame 2 be the instantaneous rest frame of the observer after some time interval Δt has passed relative to frame 1. Since H_{RF} all the time is the generator of infinitesimal time translations for the observer, the Schrödinger equation gives us $i\frac{\partial}{\partial t}U(t) = H_{\mathsf{RF}}$, so that $U(\Delta t) = e^{-iH_{\mathsf{RF}}\Delta t}$ is the operator taking us from frame 1 to frame 2. We then get that the observer in his/her instantaneous reference frame sees the action of H_{RF} as $U(\Delta t)H_{\mathsf{RF}}U(\Delta t)^{-1} = e^{-iH_{\mathsf{RF}}\Delta t}H_{\mathsf{RF}}e^{iH_{\mathsf{RF}}\Delta t} = e^{-iH_{\mathsf{RF}}\Delta t}(H + \omega^i J_i + a^i B_i)e^{iH_{\mathsf{RF}}\Delta t}$. But since H_{RF} obviously commutes with itself and any polynomial of itself, this is just equal to H_{RF}. The time translation changes nothing. Therefore the situation is exactly the same in frame 2 as it was in frame 1; the observer experiences a proper acceleration a^i along the x^i-axis and a rotation with angular velocity ω^i around the x^i-axis.

Finally, to do anything more constructive with H_{RF} in general terms, we need to see what kind of world line it generates for our observer for different values of the parameters a^i and ω^i, and what the corresponding instantaneous rest frames look like. This is the topic of the next chapter, along with a more rigorous discussion of what I really mean by the concept of a reference frame that I have been throwing around in this section without any real definition.

To conclude with, we see that there are several "natural" ways to define particles, even if we restrict ourselves merely to particles defined using "acceptable" Hamiltonians in plain ordinary Minkowski spacetime. In chapter 5 we will investigate the different kinds of particles that arise from choosing different values for the parameters in eq. 3.19 before we go on in chapter 6 to discuss what any discrepancies with our usual, familiar inertial particles and vacuum might mean.

Chapter 4

Minkowski spacetime through stationary eyes

In the previous chapter we argued that particle states of a quantum field should preferably be defined using the Hamiltonian that generates proper time translations on Hilbert space relative to some observer, along with a suitable set of observables that commute with the Hamiltonian. It was also argued that this observer should follow a so-called stationary trajectory, i.e. that the vector field $\frac{\partial}{\partial \tau}$ (where τ is the proper time parameter of the observer) should be a Killing vector field. In this chapter we will discuss in more detail what this means, and calculate what stationary trajectories exist in Minkowski spacetime, as well as what the reference frame of an observer following such a trajectory would look like. Although Minkowski spacetime looks rather boring seen from an inertial frame, it will turn out that exotic structures such as event horizons and static limits appear when seen from some non-inertial frames, and that Minkowski spacetime can even be made to look like the limit of a spacetime close to the surface of an infinitely large black hole. I will first start by discussing what the term "reference frame" really means, since this is a term I will use a lot, and have already used quite frequently without really saying much about what is meant by it.

4.1 What is a reference frame?

An observer in any spacetime is represented by a timelike world-line $x^\mu(\tau)$ running through that spacetime, where x^μ denotes a point in spacetime and is a function of some parameter τ. Any physical observer is assumed to be massive and to respect the rules of causality so that his or her world-line has to be timelike, i.e. if $\vec{v}$ is a tangent vector of the world-line, then $\vec{v} \cdot \vec{v} \equiv g_{\mu\nu} v^\mu v^\nu < 0$. We will let the parameter τ be equal to the proper time of the observer, i.e. time as measured by a clock that the observer is carrying along (this means by definition that $\dot{x}^\mu \dot{x}_\mu \equiv g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1$).

The observer is obviously not just interested in his or her own world line and the proper time associated with that, but will want to be able to describe events that happen anywhere in the spacetime. To do this, the observer needs to define a set of coordinates for the part of the spacetime that he or she is interested in observing (in the case of Minkowski

spacetime, it is possible to define coordinates that cover all of the spacetime. However, in general spacetime manifolds, this may not be possible. The observer will then have to content him- or herself with *local* coordinates that cover only some neighbourhood U around the observer. It will usually be possible to make this neighbourhood cover all parts of the spacetime that the observer can actually observe, i.e. every point that can be joined to the observer's world-line by a timelike or null curve).

These coordinates can of course be defined in any way that the observer sees fit as long as they form a coordinatization of the relevant region of spacetime (technically speaking, the coordinates must be a diffeomorphism from the relevant region of spacetime onto an open subset of $\mathbf{R}^4$), but some coordinates are more "physical" than others. For instance, the observer will feel that time passes as he or she moves along the world-line, and observe time pass everywhere else in the observable part of spacetime. Furthermore, the observer will feel that locally, in a small neighbourhood around his or her world-line, there is a set of natural three-dimensional spacelike hypersurfaces that are perpendicular to his or her world-line and defines what is "space" at that moment in the eyes of the observer.

A natural set of coordinates would thus be one which close to the observer's world-line uses a time coordinate whose unit vectors coincide with the tangent vectors of world-line, and is constant in the directions that the observer perceives as being space, as well as three spatial coordinates that coordinatize some spacelike hypersurface that coincides with the observer's local notion of "space" and are constant along the world-line. A coordinate system that is locally adapted to the world-line of an observer in this way and then extended to cover the whole observable part of spacetime is what I will mean by a reference frame. Note however the use of the indefinite article "a" here. In several cases throughout this thesis, I will talk about "the observer" and his or her reference frame, but often I will describe the motion of the observer in coordinate systems that are not adapted to that particular observer (in which the observer is e.g. travelling in a straight line or along a circular path), but I may still refer to those coordinate systems as "reference frames". By calling them that, I imagine the first observer being observed in the reference frame of some *other* observer who is not at rest with respect to the first one.

Defining reference frames by spacelike hyperplanes and a temporal coordinate defined locally around the observer's world-line was straight-forward. However, extending these spacelike hypersurfaces and the spatial coordinates outwards from the world-line to cover the whole spacetime, and defining the time coordinate in other regions than along the world-line, is not a trivial matter. There is no unique way of doing this, since there is no unique way of defining which events simultaneously or before or after other events if they are separated by spacelike distances. Our only real constraint is that the curves of constant spatial coordinates must be timelike in an open neighbourhood around the world-line (but they may become spacelike beyond some boundary, which we shall see is natural e.g. if we are using a rotating coordinate system), and that the hypersurfaces of constant time coordinate (which are coordinatized by the spatial coordinates) are correspondingly spacelike in a suitable neighbourhood of the world-line. In the following, we will denote the time coordinate by τ, since on the world line of the observer it is taken to agree with the observer's proper time. One might also wish to require that the curves of constant spatial coordinates be perpendicular to the hypersurface of constant time coordinate everywhere, and not just at the world-line of the observer, but this would rule out important classes of coordinate systems

(e.g. in rotating cylindrical coordinates $(t; r, \theta, z)$ in Minkowski spacetime, the metric has an off-diagonal term $g_{r\theta} = r^2\omega$, so that the equal-time surfaces in this rather natural coordinate system are not perpendicular to $\frac{\partial}{\partial t}$, and indeed one can show that no smooth surfaces exist that are everywhere perpendicular to $\frac{\partial}{\partial t}$).

In our case though, we are only interested in observers that follow a stationary trajectory and that perceive their situation as being stationary. If our definition of the observers' reference frame is to reflect this stationarity, then this constrains our choice of time coordinate for the whole of the spacetime. The unit time-vectors $\frac{\partial}{\partial \tau}$ will have to form a Killing-vector field, leaving the metric invariant under the transformation it generates. What this means and a more thorough investigation of the implications is the topic of the next section.

But these constraints only apply to the time coordinate, and only gives us the lines of constant spatial position as the integral lines of the unit time-vectors, not which points along each line corresponds to which τ-value. The spatial hypersurfaces of constant time are not pinned down. If we have chosen an equal-time hypersurface for some specific time, e.g. $\tau = 0$, then all the other hypersurfaces will be uniquely determined simply by letting the points on this surface be moved along the integral lines of $\frac{\partial}{\partial \tau}$. But the choice of this initial surface is still completely arbitrary, as long as it is smooth and has the right behaviour locally around the world-line of the observer. Still, when we are working in Minkowski spacetime, we are in a rather privileged situation, since for every point along the observer's world-line there is a uniquely defined *inertial* frame with respect to which the observer is instantaneously at rest. And inertial frames are privileged in the sense that they have a "natural" kind of spacelike hypersurfaces defined for them, namely those given by normal Cartesian Minkowski coordinates $(t; x, y, z)$, which are everywhere orthogonal to the vector field $\frac{\partial}{\partial t}$. This choice of hypersurfaces for the inertial reference frame is also distinguished by the fact that the inertial observer will define the spatial distance along the spacelike hypersurfaces to an event as being the time it takes a light signal from that event to propagate to the observer, divided by the speed of light, so that this definition of spacelike hypersurfaces is in a sense more physical than other hypersurfaces. For a non-inertial observer moving through Minkowski spacetime it would therefore be natural to define each spacelike hypersurface of his or her reference frame at each point in time as being the spacelike hypersurface of the instantaneous rest frame of the observer at that time. This is in practice also what I will do with all the reference-frames used in this thesis, but I stress that it is not necessary to make this choice, it is just convenient and has a reassuringly physical feel to it.

4.2 Constructing stationary reference frames

After having defined a suitable reference frame for an observer, time evolution through some time interval $\Delta\tau$ as perceived by the observer now consists in the transformation that takes each point on a surface of constant time coordinate τ to the one with time coordinate $\tau + \Delta\tau$. This transformation can be viewed as a continuous family of mappings $f_{\Delta\tau} : \mathcal{M} \mapsto \mathcal{M}$ of the spacetime $\mathcal{M}$ onto itself, parametrized by $\Delta\tau$. These mappings generate a continuous family of transformations on the phase-space of any physical system in the spacetime, and the Hamiltonian operator H_{RF} of the previous chapter is the generator of the

corresponding family of transformations on the Hilbert space of any quantum system in the spacetime. As mentioned in the previous chapter, if H_{RF} is going to be used to define rock solid stable particles, then H_{RF} needs to be independent of τ. The observer must perceive the reference frame as being constant, in the sense that no constant-time hypersurface looks different to the observer than any other. This means in particular that the metric must be invariant under time translation. This can be formulated more precisely by saying that the mapping $f_{\Delta\tau}$ must be an isometry.

Let us digress for a moment to define more clearly what this means. The discussion here will be rather brief, so the more interested reader who is not familiar with this sort of material can read a more thorough definition of the concepts in Wald's book [7]. The introductory chapters of Thedore Frankel's book ([8]) is also highly recommended reading for anyone who is interested in a thorough and fairly mathematical introduction that is still very pedagogical and rich in physical applications.

If we have a continuous family of mappings $\psi_{\Delta\tau} : \mathcal{M} \mapsto \mathcal{M}$, then we can define the vector field $\vec{v}_\psi$ of *generators* of $f_{\Delta\tau}$ by saying that $\vec{v}_\psi$ used on a differentiable function $f : \mathcal{M} \mapsto \mathcal{M}$ (where we now view the vector $\vec{v}_\psi$ as a differentiable operator, namely $\vec{v}_\psi = v^\mu_\psi \frac{\partial}{\partial x^\mu}$) is equal to the derivative of $f \circ \psi_{\Delta\tau}$, i.e.

$$\vec{v}_\psi(f(x)) = \frac{\mathrm{d}}{\mathrm{d}\Delta\tau} f(\psi_{\Delta\tau}(x))\bigg|_{\Delta\tau=0} \tag{4.1}$$

In our case, $\vec{v}_\psi$ will be simply the time derivative with respect to proper time, i.e. unit vectors in the τ-direction. Once we have defined the mapping $\psi_{\Delta\tau}$ on the spacetime itself we can also define a corresponding mapping on the space of vectors on $\mathcal{M}$, namely the so-called *push-forward* $\psi^*_{\Delta\tau}$ (also known as the *differential*). This can be defined by saying that the vector $\vec{u}_x$ based at the point x is mapped to a vector $\psi^*_{\Delta\tau}\vec{u}_x$ based at $\psi_{\Delta\tau}(x)$ defined by requiring that it give the following result when applied to a function f:

$$\left(\psi^*_{\Delta\tau}\vec{u}_x\right)\left(f(x)\right) = \vec{u}_x\left(f(\psi_{\Delta\tau}(x))\right) \tag{4.2}$$

i.e. the push-forward of $\vec{u}_x$ used on f is simply $\vec{u}_x$ used on the composite function $f \circ \psi_{\Delta\tau}$. If we now define $y = \psi_{\Delta\tau}(x)$ and use the definition of the push-forward, we can find the push-forward of a vector $\vec{u}$ by noting that for any differentiable function $f : \mathcal{M} \mapsto \mathcal{M}$, we have:

$$\psi^*_{\Delta\tau}\vec{u}\left(f(y)\right) = \left(\psi^*_{\Delta\tau}\vec{u}\right)^\mu \frac{\partial f}{\partial y^\mu} \tag{4.3}$$

But at the same time we have

$$\psi^*_{\Delta\tau}\vec{u}\left(f(y)\right) \equiv \vec{u}\left(f(y(x))\right) = v^\nu \frac{\partial f(y(x))}{\partial x^\nu} = v^\nu \frac{\partial y^\mu}{\partial x^\nu}\frac{\partial f(y)}{\partial y^\mu} \tag{4.4}$$

so that we get:

$$\left(\psi^*_{\Delta\tau}\vec{u}\right)^\mu = v^\nu \frac{\partial y^\mu}{\partial x^\nu} \tag{4.5}$$

With this definition, we can further define a mapping on covariant tensors, called the *pull-back* $(\psi_{\Delta\tau})_*$ (so named because it "pulls back" the covariant tensors from $\psi_{\Delta\tau}(x)$ to

x, i.e. it works in the opposite direction of $\psi_{\Delta\tau}$ and $\psi^*_{\Delta\tau}$). The pull-back of a covariant tensor $\mathbf{T}$ of rank n based at the point $\psi_{\Delta\tau}(x)$ is defined by requiring that it act as follows on vectors based at the point x:

$$\left((\psi_{\Delta\tau})_*\mathbf{T}\right)(\vec{v}_1,\ldots,\vec{v}_n) = \mathbf{T}(\psi^*_{\Delta\tau}\vec{v}_1,\ldots,\psi^*_{\Delta\tau}\vec{v}_n) \tag{4.6}$$

Using the same kind of technique as for deriving eq. 4.5, we get that the components of the pull-back of $\mathbf{T}$ are given by:

$$\left((\psi_{\Delta\tau})_*\mathbf{T}\right)_{\mu_1\mu_2\ldots\mu_n} = T_{\nu_1\nu_2\ldots\nu_n}\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\frac{\partial y^{\nu_2}}{\partial x^{\mu_2}}\cdots\frac{\partial y^{\nu_n}}{\partial x^{\mu_n}} \tag{4.7}$$

Finally, we can define the concept of the *Lie derivative* $\mathcal{L}_{\vec{v}_\psi}\mathbf{T}$ of the covariant tensor $\mathbf{T}$ of rank n with respect to the transformation-generating vector field $\vec{v}_\psi$ as follows:

$$\mathcal{L}_{\vec{v}_\psi}\mathbf{T} \equiv \lim_{\Delta\tau\to 0}\frac{(\psi_{(-\Delta\tau)})_*\mathbf{T} - \mathbf{T}}{\Delta\tau} \tag{4.8}$$

When $\psi_{\Delta\tau}$ is a family of diffeomorphism, i.e. smooth bijective mappings with a smooth inverse (which $\psi_{\Delta\tau}$ indeed is in our case), the definition can be extended to tensors of any rank, also mixed ranks (see the references given at the beginning of this section), but we will only need to use the Lie derivative on covariant tensors so I won't bother with that generalization.

With this machinery in place, we can finally define what it means that $\psi_{\Delta\tau}$ must be an isometry. An isometry is now defined as a mapping for which the pull-back $\psi_{\Delta\tau *}g$ of the metric at some point x is equal to the metric itself at that point, i.e. that the Lie derivative with respect to the generating vector field $\vec{v}_\psi$ is zero, $\mathcal{L}_{\vec{v}_\psi}g = 0$. The vector field $\vec{v}_f$ that generates such an isometry, is called a *Killing vector field*.

Our criterion for whether the Hamiltonian of a particular reference frame can be used to define particles, is now that the vector field that generates time translations in this reference frame must be a Killing vector field, and the world line of the observer must be an integral line of this Killing vector field which is timelike everywhere. If the observer follows a world-line which is not an integral line of any Killing vector field, then the observer is not following a stationary trajectory, and will perceive his or her reference frame as changing with time, so that it allows no unchanging definition of particle states.

Let us now finally deduce what kind of relations a Killing vector field must satisfy. Look at a mapping $\psi_{\Delta\tau}$ generated by the vector field $\vec{v}_\psi$. Let $\Delta\tau = \epsilon$ be an infinitesimal parameter. The mapping ψ_ϵ is then an infinitesimal transformation, and if we once again let $y = \psi_{\Delta\tau}(x)$, then we have $y^\mu \simeq x^\mu + \epsilon v^\mu_\psi$. Using eq. 4.7 we then get the following for the pull-back of the metric (we now drop the subscript ψ on $\vec{v}$ for notational simplicity):

$$\begin{aligned}\left((\psi_\epsilon)_*g\right)_{\mu\nu}(x) &= g_{\rho\sigma}(x+\epsilon\vec{v})\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} = \left(g_{\rho\sigma}+\epsilon v^\lambda\frac{\partial g_{\rho\sigma}}{\partial x^\lambda}\right)\left(\delta^\rho_{\ \mu}+\epsilon\frac{\partial v^\rho}{\partial x^\mu}\right)\left(\delta^\sigma_{\ \nu}+\epsilon\frac{\partial v^\sigma}{\partial x^\nu}\right)\\ &\simeq g_{\mu\nu}+\epsilon g_{\mu\nu,\lambda}\,v^\lambda+\epsilon g_{\rho\nu}v^\rho_{\ ,\mu}+\epsilon g_{\mu\sigma}v^\sigma_{\ ,\nu}\\ &= g_{\mu\nu}(x)+\epsilon\left(g_{\mu\nu,\lambda}\,v^\lambda+v_{\mu,\nu}+v_{\nu,\mu}\right)\end{aligned} \tag{4.9}$$

where the metric is to be evaluated at the point x when no explicit argument is given. If $\psi_{\Delta\tau}$ is an isometry and $\vec{v}$ is a Killing vector field, this must be equal to $g_{\mu\nu}(x)$, so a necessary and sufficient condition for $\vec{v}$ to be a Killing vector field is:

$$g_{\mu\nu,\lambda}\, v^{\lambda} + v_{\mu,\nu} + v_{\nu,\mu} = 0 \tag{4.10}$$

We are working in flat Minkowski spacetime and we will use Cartesian coordinates, so that the metric is constant. This equation then becomes:

$$v_{\mu,\nu} + v_{\nu,\mu} = 0 \tag{4.11}$$

Solving this coupled set of differential equations is not all too difficult. First note that for $\mu = \nu$, we get $v_{\mu,\mu} = 0$, so that v_μ (and hence all of its derivatives) must be independent of x^μ for any μ. Since $v_{\nu,\mu\mu} = -v_{\mu,\nu\mu} = -v_{\mu,\mu\nu} = 0$, this also means that if we take any covariant component of $\vec{v}$ and derivate it more than once with respect to the same coordinate, the result must be zero. This means that it must be possible to write each component v_μ as a polynomial in which no term contains more than one power of each coordinate, and where of course the coordinate x^μ itself does not appear at all. But we can then show that each term must in fact contain only a single coordinate. To prove this, suppose first that this were not the case. v_μ can then be expressed by the following Nth-order polynomial (we could have $N \to \infty$, but that does not matter):

$$v_\mu = \sum_{n=0}^{N} \frac{1}{n!} c_{\mu\nu_1\ldots\nu_n} x^{\nu_1} \cdots x^{\nu_n} \tag{4.12}$$

Because the ordering of the x^{ν_i} in each term does not matter, we can take all the coefficients $c_{\mu\nu_1\ldots\nu_n}$ to be symmetric under interchange of any of the indices after the first one. Eq. 4.11 also requires that $c_{\mu\nu_1\ldots\nu_n}$ must be antisymmetric under interchange of the first index with any of the other ones. For any of the coefficients with $n > 1$ we therefore have:

$$c_{\mu\nu_1\ldots\nu_n} = -c_{\nu_1\mu\ldots\nu_n} \tag{4.13}$$

but by the same token we also have

$$c_{\mu\nu_1\ldots\nu_n} = -c_{\nu_n\nu_1\ldots\mu} = c_{\nu_1\nu_n\ldots\mu} = c_{\nu_1\mu\ldots\nu_n} = -c_{\mu\nu_1\ldots\nu_n} \tag{4.14}$$

which means that $c_{\mu\nu_1\ldots\nu_n} = 0$ for all $n > 1$. Hence v_μ contains only constant terms and terms proportional to *one* power of *one* coordinate. $v_{\mu,\nu}$ must therefore be a constant for any μ and ν.

Now that we know that all the $v_{\mu,\nu}$ are constants, it is of course very easy to find solutions to eq. 4.11. The general solution can be written compactly as:

$$v_\mu = \omega_{\mu\nu} x^\nu + \beta_\mu \tag{4.15}$$

where $\omega_{\mu\nu}$ is an arbitrary antisymmetric tensor and β_μ are arbitrary constants. Raising the index μ to get the true components of the Killing vector and writing out the full vector, we get:

$$\vec{v} = \frac{1}{2}\omega^{\mu\nu}\left(g_{\nu\lambda}x^{\lambda}\frac{\partial}{\partial x^{\mu}} - g_{\mu\lambda}x^{\lambda}\frac{\partial}{\partial x^{\nu}}\right) + \beta^{\mu}\frac{\partial}{\partial x^{\mu}} \tag{4.16}$$

If μ and ν in the first term are both spatial indices, i and j, and we are in $3+1$-dimensional Minkowski spacetime with inertial Cartesian coordinates, then the terms in the parentheses become

$$x^{i}\frac{\partial}{\partial x^{j}} - x^{j}\frac{\partial}{\partial x^{i}} \tag{4.17}$$

which we immediately recognize as the vector field generating rotation around the axis perpendicular to the x^i- and x^j-axes. Correspondingly, if either μ or ν in the first term is 0 and the other a spatial index i, then the vector field is

$$x^{i}\frac{\partial}{\partial t} + t\frac{\partial}{\partial x^{i}} \tag{4.18}$$

which we recognize as the vector field generating boosts along the x^i-axis (where $t = x^0$). The second term in eq. 4.16 is merely a translation along the direction of $\vec{\beta}$. This means that eq. 4.16 is quite analogous to eq. 3.20, and the parameters $\omega_{\mu\nu}$ and β^{μ} are precisely the same as those used in section 3.2. We now define the vector field $\vec{B}_i$ generating boosts in the x^i-direction as

$$\vec{B}_i = x^{i}\frac{\partial}{\partial t} + t\frac{\partial}{\partial x^{i}} \tag{4.19}$$

and the vector field $\vec{J}_i$ generating rotations around the x^i-axis as

$$\vec{J}_i = \frac{1}{2}\epsilon_{ijk}\left(x^{j}\frac{\partial}{\partial x^{k}} - x^{k}\frac{\partial}{\partial x^{j}}\right) \tag{4.20}$$

and the vector field $\vec{P}_{\mu}$ generating translations in the x^{μ}-direction as simply $\vec{P}_{\mu} = \frac{\partial}{\partial x^{\mu}}$ (note that there is a mismatch in the placement of indexes 'upstairs' or 'downstairs' in the above equations, since we are now explicitly using a specific coordinate system, namely Cartesian coordinates). We can then write a general Killing vector field in Minkowski spacetime as:

$$\vec{v} = \beta^{\mu}\vec{P}_{\mu} + \omega^{i}\vec{J}_i + a^{i}\vec{B}_i \tag{4.21}$$

where the spatial vectors $\mathbf{a}$ and $\boldsymbol{\omega}$ are now as defined in chapter 3.

We have thus shown that any Killing vector field in Minkowski spacetime is a superposition of vector fields generating translations, spatial rotations and boosts, as was suggested in chapter 3. A stationary observer must follow a timelike integral line of some such field, and the whole field can then be used as unit τ-vectors and the corresponding integral lines as lines of constant spatial coordinates relative to the observer. If a Killing vector field is to serve this purpose, the parameters β^{μ}, a^i and ω^i obviously have to be such that the Killing vector field has some integral lines that are timelike everywhere and that the observer can travel along, so that the observer does not have to break the light-speed limit to keep up with it. In the following discussion we will guarantee that this is the case by always choosing $\beta^0 = 1$ and $\beta^i = 0$ for $i = 1, 2, 3$, as we did in the previous chapter. The integral

line through the origin is then always timelike, and will be used as the world-line of the observer. This simply corresponds to letting the observer be at rest at the onset, which we must always be able to do by choosing an appropriate reference frame to start with. This choice therefore does not cause any loss of generality. In the following sections we will describe what the reference frame of the observer looks like for different sets of values of $\mathbf{a}$ and $\boldsymbol{\omega}$. We will find that the observer follows qualitatively different trajectories and experiences the reference frame as qualitatively different depending on whether or not $\mathbf{a}$ and $\boldsymbol{\omega}$ are perpendicular, and if they are, on whether or not $a \equiv |\mathbf{a}|$ is greater than, equal to or less than $\omega \equiv |\boldsymbol{\omega}|$ (even though the trajectory does of course change in a continuous manner between the cases). We will therefore treat these cases separately.

4.3 $\mathbf{a} \perp \boldsymbol{\omega}$ and $a < \omega$: Observer in a circular orbit

If $\mathbf{a}$ and $\boldsymbol{\omega}$ are perpendicular, it makes sense to choose our coordinate axes so that $\mathbf{a}$ and $\boldsymbol{\omega}$ both point along an axis. Let $\mathbf{a}$ point along the x-axis and $\boldsymbol{\omega}$ along the z-axis. The Killing vector field $\vec{v} \equiv \frac{\partial}{\partial \tau}$ then becomes:

$$\vec{v} = \vec{P}_t + \omega \vec{J}_z + a \vec{B}_x = \frac{\partial}{\partial t} + \omega \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) + a \left(x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x} \right) \tag{4.22}$$

We could now of course find the world-line of the observer and all the other curves of constant spatial position relative to the observer's reference frame simply by integrating up the integral lines of the Killing vector field $\vec{v}$, but it would be a little cumbersome to do this directly in the inertial system in which the observer starts at rest. Instead, we can simplify things by performing a Lorentz transformation to view the path of the observer from a different inertial system, which we will denote by primed coordinates. The simplification is the following: according to eq. 3.22, $\mathbf{a} \cdot \boldsymbol{\omega}$ is invariant under Lorentz transformations, and in our case $\mathbf{a} \cdot \boldsymbol{\omega} = 0$. Thus it might be possible to perform a Lorentz transformation so that either $\mathbf{a}$ or $\boldsymbol{\omega}$ vanishes. Since $a^2 - \omega^2$ is also an invariant, which in our case is negative since $a < \omega$, only $\mathbf{a}$ can possibly be made to vanish. Seen from this inertial system, $\vec{v}$ will look like a vector field generating rotations around some axis (presumably different from the location of the observer), and the observer will hence seem to follow a circular orbit around this axis.

The transformation that accomplishes this, can be found as follows: rewrite eq. 4.22 as

$$\vec{v} = \frac{\partial}{\partial t} + x \left(\omega \frac{\partial}{\partial y} + a \frac{\partial}{\partial t} \right) - (\omega y - a t) \frac{\partial}{\partial x} \tag{4.23}$$

In order to make this look like something that is boost-free, we want the terms involving t or $\frac{\partial}{\partial t}$ (except the first one, of course) to disappear. We see that this can be accomplished if we define the transformed coordinates y' and t' by

$$\begin{aligned} y &= \frac{1}{\sqrt{1-\frac{a^2}{\omega^2}}}\left(y' + \frac{a}{\omega}t'\right) \equiv \frac{\omega}{\rho}\left(y' + \frac{a}{\omega}t'\right) \\ t &= \frac{1}{\sqrt{1-\frac{a^2}{\omega^2}}}\left(t' + \frac{a}{\omega}y'\right) \equiv \frac{\omega}{\rho}\left(t' + \frac{a}{\omega}y'\right) \end{aligned} \tag{4.24}$$

(where $\rho \equiv \sqrt{\omega^2 - a^2}$), which is a boost by $\frac{a}{\omega}$ along the positive y-axis, with the inverse transformation given by

$$y' = \frac{\omega}{\rho}\left(y - \frac{a}{\omega}t\right) \qquad\qquad t' = \frac{\omega}{\rho}\left(t - \frac{a}{\omega}y\right) \tag{4.25}$$

x and z are at first taken to transform trivially, $x' = x$ and $z' = z$, but as we shall see, it will be convenient to redefine x' later. The vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial y}$ then transform as:

$$\begin{aligned} \frac{\partial}{\partial y} &= \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} + \frac{\partial t'}{\partial y}\frac{\partial}{\partial t'} = \frac{\omega}{\rho}\left(\frac{\partial}{\partial y'} - \frac{a}{\omega}\frac{\partial}{\partial t'}\right) \\ \frac{\partial}{\partial t} &= \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} - \frac{\partial y'}{\partial t}\frac{\partial}{\partial y'} = \frac{\omega}{\rho}\left(\frac{\partial}{\partial t'} - \frac{a}{\omega}\frac{\partial}{\partial y'}\right) \end{aligned} \tag{4.26}$$

so that $\vec{v}$ becomes

$$\begin{aligned} \vec{v} &= \frac{\omega}{\rho}\left(\frac{\partial}{\partial t'} - \frac{a}{\omega}\frac{\partial}{\partial y'}\right) + x'\frac{\omega}{\rho}\left(\omega - \frac{a^2}{\omega}\right)\frac{\partial}{\partial y'} - \frac{\omega}{\rho}\left(\omega - \frac{a^2}{\omega}\right)y'\frac{\partial}{\partial x'} \\ &= \frac{\omega}{\rho}\frac{\partial}{\partial t'} - \frac{a}{\rho}\frac{\partial}{\partial y'} + \rho\left(x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'}\right) \end{aligned} \tag{4.27}$$

This is almost a vector field generating translations in the t'-direction and rotations around the z'-axis. However, the term $\frac{a}{\rho}\frac{\partial}{\partial y'}$ shows that the rotation is not around the z'-axis, but some other axis parallel to it. This is because we chose the origin to be at the starting point of the observer, which is not on the rotational axis. We can easily remedy this by letting $x' \to x' + \frac{a}{\rho^2}$, which will place the origin at the rotational axis. $\vec{v}$ then finally becomes:

$$\vec{v} = \frac{\omega}{\rho}\frac{\partial}{\partial t'} + \rho\left(x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'}\right) = \frac{\omega}{\rho}\vec{P}_{t'} + \rho\vec{J}_{z'} \tag{4.28}$$

In order to integrate this vector field, it is convenient to introduce cylindrical coordinates $(t'; r, \theta, z')$:

$$x' = r\cos\theta \qquad y' = r\cos\theta \tag{4.29}$$

i.e.

$$r = \sqrt{x'^2 + y'^2} \qquad \theta = \arctan(\frac{y'}{x'}) \tag{4.30}$$

while t' and z' remain unchanged. We then get:

$$\begin{aligned}\frac{\partial}{\partial x'} &= \frac{\partial r}{\partial x'}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x'}\frac{\partial}{\partial \theta} = \cos\theta\,\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\,\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y'} &= \frac{\partial r}{\partial y'}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y'}\frac{\partial}{\partial \theta} = \sin\theta\,\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\,\frac{\partial}{\partial \theta}\end{aligned} \tag{4.31}$$

so that $\vec{v}$ now becomes:

$$\vec{v} \equiv \frac{\partial}{\partial \tau} = \frac{\omega}{\rho}\frac{\partial}{\partial t'} + \rho\frac{\partial}{\partial \theta} \tag{4.32}$$

It is now very easy to integrate this vector field in these coordinates. Simply compare eq. 4.32 to the following formula:

$$\frac{\partial}{\partial \tau} = \frac{\partial t'}{\partial \tau}\frac{\partial}{\partial t'} + \frac{\partial r}{\partial \tau}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial \tau}\frac{\partial}{\partial \theta} + \frac{\partial z}{\partial \tau}\frac{\partial}{\partial z} \tag{4.33}$$

We then immediately obtain the following parametrization for the integral lines of $\vec{v} = \frac{\partial}{\partial \tau}$:

$$\begin{aligned} t'(\tau) &= t'(0) + \frac{\omega}{\rho}\tau \\ r(\tau) &= r(0) = const. \\ \theta(\tau) &= \theta(0) + \rho\tau \\ z'(\tau) &= z'(0) = const. \end{aligned} \tag{4.34}$$

or, in Cartesian coordinates:

$$\begin{aligned} t'(\tau) &= t'(0) + \frac{\omega}{\rho}\tau \\ x'(\tau) &= r(\tau)\cos(\theta(\tau)) = r(0)\cos(\theta(0) + \rho\tau) \\ &= r(0)\left[\cos(\theta(0))\cos(\rho\tau) - \sin(\theta(0))\sin(\rho\tau)\right] \\ &= x'(0)\cos(\rho\tau) - y'(0)\sin(\rho\tau) \\ y'(\tau) &= r(\tau)\sin(\theta(\tau)) = r(0)\sin(\theta(0) + \rho\tau) \\ &= y'(0)\cos(\rho\tau) + x'(0)\sin(\rho\tau) \\ z'(\tau) &= z'(0) \end{aligned} \tag{4.35}$$

We can also express this in terms of the original unprimed coordinates (the coordinates relative to the inertial system where the observer starts from rest at the origin) by substituting $t' = \frac{\omega}{\rho}t + \frac{a}{\rho}y$, $x' = x - \frac{a}{\rho^2}$, $y' = \frac{\omega}{\rho}y + \frac{a}{\rho}t$ and $z' = z$. The result is rather messy but nevertheless of interest. If we choose take $\tau = 0$ and $t(0) = 0$, then in unprimed coordinates we get the point $(x, y, z) = (x(0), y(0), z(0))$ on the spacelike hypersurface that makes up the observer's instantaneous rest frame at $\tau = t = 0$. If we now let $x(0)$, $y(0)$ and $z(0)$ remain constant and let τ run, then we get the time evolution of this point along the integral lines of $\frac{\partial}{\partial \tau}$. This means that by expressing eq. 4.35 in unprimed coordinates with a certain value of $(x(0), y(0), z(0))$ and τ, we actually get the location of the corresponding point (x, y, z) of the instantaneous rest frame of the observer at proper time τ, given in standard Minkowski

coordinates in that frame. So let us go through the trouble of calculating this after all. Note that we set $t(0) = 0$:

$$\begin{aligned} t(\tau) &= \frac{\omega}{\rho}\left(t'(\tau) + \frac{a}{\omega}y'(\tau)\right) = \frac{\omega}{\rho}\left(t'(0) + \frac{\omega}{\rho}\tau\right) + \frac{a}{\rho}\left(y'(0)\cos(\rho\tau) + x'(0)\sin(\rho\tau)\right) \\ &= \frac{\omega^2}{\rho^2}\tau + \frac{\omega^2}{\rho^2}\left(t(0) - \frac{a}{\omega}y(0)\right) + \frac{a\omega}{\rho^2}\cos(\rho\tau)\left(y(0) - \frac{a}{\omega}t(0)\right) \\ &\quad + \frac{a}{\rho}\sin(\rho\tau)\left(x(0) - \frac{a}{\rho^2}\right) \\ &= \frac{\omega^2}{\rho^2}\tau + \frac{a\omega}{\rho^2}\left(\cos(\rho\tau) - 1\right)y(0) + \frac{a}{\rho}\sin(\rho\tau)\left(x(0) - \frac{a}{\rho^2}\right) \end{aligned} \tag{4.36}$$

$$\begin{aligned} x(\tau) &= x'(\tau) + \frac{a}{\rho^2} = x'(0)\cos(\rho\tau) - y'(0)\sin(\rho\tau) + \frac{a}{\rho^2} \\ &= x(0)\cos(\rho\tau) + \frac{a}{\rho^2} - \frac{a}{\rho^2}\cos(\rho\tau) - \frac{\omega}{\rho}\left(y(0) - \frac{a}{\omega}t(0)\right)\sin(\rho\tau) \\ &= \left(x(0) - \frac{a}{\rho^2}\right)\cos(\rho\tau) - \frac{\omega}{\rho}\sin(\rho\tau)y(0) + \frac{a}{\rho^2} \end{aligned} \tag{4.37}$$

$$\begin{aligned} y(\tau) &= \frac{\omega}{\rho}\left(y'(\tau) + \frac{a}{\omega}t'(\tau)\right) \\ &= \frac{\omega}{\rho}y'(0)\cos(\rho\tau) + \frac{\omega}{\rho}x'(0)\sin(\rho\tau) + \frac{a}{\rho}t'(0) + \frac{a\omega}{\rho^2}\tau \\ &= \frac{\omega^2}{\rho^2}y(0)\cos(\rho\tau) + \frac{\omega}{\rho}\sin(\rho\tau)\left(x(0) - \frac{a}{\rho^2}\right) - \frac{a^2}{\rho^2}y(0) + \frac{a\omega}{\rho^2}\tau \\ &= \frac{\omega^2}{\rho^2}y(0)\left(\cos(\rho\tau) - \frac{a^2}{\omega^2}\right) + \frac{\omega}{\rho}\left(x(0) - \frac{a}{\rho^2}\right)\sin(\rho\tau) + \frac{a\omega}{\rho^2}\tau \end{aligned} \tag{4.38}$$

$$z(\tau) = z(0) \tag{4.39}$$

Finally we note that the world-line of the observer is the one that starts with $x_{obs}(0) = y_{obs}(0) = z_{obs}(0) = 0$, so that the world-line is parametrized by:

$$\begin{aligned} t_{obs}(\tau) &= \frac{\omega^2}{\rho^2}\tau - \frac{a^2}{\rho^3}\sin(\rho\tau) \\ x_{obs}(\tau) &= \frac{a}{\rho^2}\left(1 - \cos(\rho\tau)\right) \\ y_{obs}(\tau) &= \frac{a\omega}{\rho^2}\tau - \frac{a}{\rho^2}\sin(\rho\tau) \\ z_{obs}(\tau) &= 0 \end{aligned} \tag{4.40}$$

which after a little investigation one can actually find out is a cycloid. But a much clearer picture of what the trajectory looks like, can be had by expressing it in the cylindrical coordinates of the primed reference frame by inserting $t'_{obs}(0) = 0$, $r_{obs}(0) = \frac{a}{\rho^2}$, $\theta_{obs}(0) = 0$ and $z'_{obs}(0) = 0$ into eq. 4.34:

$$\begin{aligned} t'_{obs}(\tau) &= \frac{\omega}{\rho}\tau \\ r_{obs}(\tau) &= \frac{a}{\rho^2} \\ \theta_{obs}(\tau) &= \rho\tau \\ z'_{obs}(\tau) &= 0 \end{aligned} \tag{4.41}$$

For later reference I also list the result in Cartesian primed coordinates:

$$\begin{aligned} t'_{obs}(\tau) &= \frac{\omega}{\rho}\tau \\ x'_{obs}(\tau) &= -\frac{a}{\rho^2}\cos(\rho\tau) \\ y'_{obs}(\tau) &= -\frac{a}{\rho^2}\sin(\rho\tau) \\ z'_{obs}(\tau) &= 0 \end{aligned} \tag{4.42}$$

This is clearly a circular orbit with radius $\frac{a}{\rho^2}$ and angular velocity $\rho \cdot \frac{\rho}{\omega} = \frac{\rho^2}{\omega} = \omega\left(1 - \frac{a^2}{\omega^2}\right)$ with respect to the time coordinate t'. The factor $\frac{\omega}{\rho}$ in the expression for t'_{obs} can be readily interpreted as time dilatation due to the orbital speed of the observer. The fact that $\theta_{obs}(\tau)$ is equal to $\rho\tau = \omega\sqrt{1 - \frac{a^2}{\omega^2}}$ rather than simply $\omega\tau$ is due to the curious phenomenon known as Thomas precession, which has the effect that the observer actually experiences *more* than one rotation of his or her own reference frame each time he or she completes one orbit in the primed reference frame.

Hence we have seen that when $\mathbf{a} \perp \boldsymbol{\omega}$ and $a < \omega$, the trajectory of the observer is simply a circular orbit when seen in an appropriate inertial frame. The reference frame of the observer does not display much interesting behaviour in itself, not even when we view a quantum field in this frame (as we shall do in the next chapter), but there is one interesting feature to note. The proper-time translation generating Killing vector field $\frac{\partial}{\partial\tau}$ is not timelike everywhere, but becomes spacelike beyond a certain boundary. To see this, we use that the line element or metric in cylindrical coordinates is given by:

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = dt'^2 - dr^2 - r^2 d\theta^2 - dz'^2 \tag{4.43}$$

The Killing vector field $\vec{v} = \frac{\partial}{\partial\tau}$ is given by eq. 4.32, and the norm is:

$$v^2 \equiv g_{\mu\nu}v^\mu v^\nu = \frac{\omega^2}{\rho^2} - r^2\rho^2 \tag{4.44}$$

We see that if $r > \frac{\omega}{\rho^2}$, then $v^2 < 0$, so that $\vec{v}$ becomes spacelike. This means that it is impossible for anything to follow an integral line of $\frac{\partial}{\partial\tau}$ beyond $r = \frac{\omega}{\rho^2}$, i.e. it is impossible for anything behind this boundary to stay spatially at rest with respect to the observer's reference frame (fortunately, since the observer is located at $r = \frac{a}{\rho^2}$ and $a < \omega$, we do not run into trouble with the timelikeness of the observer's own world-line). This boundary is

therefore called a *static limit*. In chapter 5 we shall see that this limit implies the existence of particle states with negative energy in a quantized field, and in chapter 7 this will turn out to be important for the behaviour of detectors following a circular trajectory.

4.4 $\mathbf{a} \perp \boldsymbol{\omega}$ and $a > \omega$: A linearly accelerating observer (with a twist)

In the case when still $\mathbf{a} \perp \boldsymbol{\omega}$ but $a > \omega$, we can still choose $\mathbf{a}$ to lie along the x-axis and $\boldsymbol{\omega}$ to lie along the z-axis as in the previous section, and the Killing vector field $\vec{v}$ is still given by eq. 4.22. This time however, $a > \omega$, so $a^2 - \omega^2 > 0$. This means that we cannot make the acceleration term vanish, but we may try to make the rotation term vanish instead. To accomplish this, we must now make the terms involving y and $\frac{\partial}{\partial y}$ disappear from eq. 4.23 rather than those involving t and $\frac{\partial}{\partial t}$. The procedure for doing this is almost exactly the same as in the previous section, but this time boost by $\frac{\omega}{a}$ in the positive y-direction rather than $\frac{a}{\omega}$ (this is possible now because $\omega < a$ and $\frac{\omega}{a} < 1$, so that the boost is less than the speed of light):

$$\begin{aligned} y &= \frac{1}{\sqrt{1-\frac{\omega^2}{a^2}}}\left(y' + \frac{\omega}{a}t'\right) \equiv \frac{a}{\sigma}\left(y' + \frac{\omega}{a}t'\right) \\ t &= \frac{1}{\sqrt{1-\frac{\omega^2}{a^2}}}\left(t' + \frac{\omega}{a}y'\right) \equiv \frac{a}{\sigma}\left(t' + \frac{\omega}{a}y'\right) \end{aligned} \tag{4.45}$$

where now I have defined $\sigma = \sqrt{a^2 - \omega^2}$. The inverse transformation is now

$$y' = \frac{a}{\sigma}\left(y - \frac{\omega}{a}t\right) \qquad t' = \frac{a}{\omega}\left(t - \frac{\omega}{a}y\right) \tag{4.46}$$

and the vectors $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial t}$ transform as

$$\begin{aligned} \frac{\partial}{\partial y} &= \frac{a}{\sigma}\left(\frac{\partial}{\partial y'} - \frac{\omega}{a}\frac{\partial}{\partial t'}\right) \\ \frac{\partial}{\partial t} &= \frac{a}{\sigma}\left(\frac{\partial}{\partial t'} - \frac{\omega}{a}\frac{\partial}{\partial y'}\right) \end{aligned} \tag{4.47}$$

We still let $x' = x$ and $z' = z$ for now, but from our experience from the previous section, we expect that we may need to redefine x' later. The redefinition will however have a slightly different motivation than that for $a < \omega$.

With this transformation, the Killing vector field $\vec{v}$ becomes:

$$\begin{aligned} \vec{v} &= \frac{a}{\sigma}\left(\frac{\partial}{\partial t'} - \frac{\omega}{a}\frac{\partial}{\partial y'}\right) + x'\frac{a}{\sigma}\left(a - \frac{\omega^2}{a}\right)\frac{\partial}{\partial t'} + \frac{a}{\sigma}\left(a - \frac{\omega^2}{a}\right)t'\frac{\partial}{\partial x'} \\ &= \frac{a}{\sigma}\frac{\partial}{\partial t'} - \frac{\omega}{\sigma}\frac{\partial}{\partial y'} + \sigma\left(x'\frac{\partial}{\partial t'} + t'\frac{\partial}{\partial x'}\right) \end{aligned} \tag{4.48}$$

This is almost the vector field that generates a trajectory which is continuously boosted in the x'-direction, i.e. a system with constant linear proper acceleration along the x'-axis. However, we must also contend with the term $-\frac{\omega}{\sigma}\frac{\partial}{\partial y'}$, which we cannot get rid of unless $\omega = 0$. This shows that in addition to the acceleration, the system is also moving at a constant velocity (with respect to the proper time, not necessarily with respect to the time coordinate t') $\frac{\omega}{\sigma}$ in the negative y'-direction. However, we can simplify the expression for $\vec{v}$ a bit by letting $x' \to x' - \frac{a}{\sigma^2}$, so that $\vec{v}$ becomes:

$$\vec{v} = \sigma\left(x'\frac{\partial}{\partial t'} + t'\frac{\partial}{\partial x'}\right) - \frac{\omega}{\sigma}\frac{\partial}{\partial y'} = \sigma\vec{B}_{x'} - \frac{\omega}{\sigma}\vec{P}_{y'} \tag{4.49}$$

With this redefinition of x', the term $\frac{a}{\sigma}\frac{\partial}{\partial t'}$ is absorbed into the boost-part of the Killing vector field (this puts the observer at $x' = \frac{a}{\sigma^2}$ rather than at the origin, so that the boosting now generates the translation of the observer in the t'-direction as well as the acceleration of the observer's reference frame).

To simplify the expression for $\vec{v} = \frac{\partial}{\partial\tau}$ further before integrating it, we introduce so-called hyperbolic cylindrical coordinates (suitable for systems where there is "rotation" involving the time-axis, i.e. boosting):

$$x' = \xi\cosh\chi \qquad\qquad t' = \xi\sinh\chi \tag{4.50}$$

i.e.

$$\xi = \pm\sqrt{x'^2 - t'^2} \qquad\qquad \chi = \operatorname{artanh}\frac{t'}{x'} \tag{4.51}$$

where the plus sign applies for $x' > 0$ and the minus sign for $x' < 0$. We then get:

$$\begin{aligned}\frac{\partial}{\partial x'} &= \pm\cosh\chi\frac{\partial}{\partial\xi} - \frac{1}{\xi}\sinh\chi\frac{\partial}{\partial\chi}\\ \frac{\partial}{\partial t'} &= \mp\sinh\chi\frac{\partial}{\partial\xi} + \frac{1}{\xi}\cosh\chi\frac{\partial}{\partial\chi}\end{aligned} \tag{4.52}$$

which gives the following expression for $\vec{v}$:

$$\vec{v} \equiv \frac{\partial}{\partial\tau} = \sigma\frac{\partial}{\partial\chi} - \frac{\omega}{\sigma}\frac{\partial}{\partial y'} \tag{4.53}$$

This finally can be integrated to yield the following parametrization of integral lines for $\frac{\partial}{\partial\tau}$:

$$\begin{aligned}\chi(\tau) &= \chi(0) + \sigma\tau\\ \xi(\tau) &= \xi(0) = const.\\ y'(\tau) &= y'(0) - \frac{\omega}{\sigma}\tau\\ z'(\tau) &= z'(0) = const.\end{aligned} \tag{4.54}$$

or, in Cartesian coordinates:

$$\begin{aligned}
t'(\tau) &= \xi(\tau)\sinh(\chi(\tau)) = \xi(0)\sinh(\chi(0)+\sigma\tau) \\
&= \xi(0)\left[\sinh(\chi(0))\cosh(\sigma\tau)+\cosh(\chi(0))\sinh(\sigma\tau)\right] \\
&= t'(0)\cosh(\sigma\tau)+x'(0)\sinh(\sigma\tau) \\
x'(\tau) &= \xi(0)\cosh(\chi(0)+\sigma\tau) = \xi(0)\left[\cosh(\chi(0))\cosh(\sigma\tau)+\sinh(\chi(0))\sinh(\sigma\tau)\right] \\
&= x'(0)\cosh(\sigma\tau)+t'(0)\sinh(\sigma\tau) \\
y'(\tau) &= y'(0)-\frac{\omega}{\sigma}\tau \\
z'(\tau) &= z'(0)
\end{aligned} \tag{4.55}$$

In the original unprimed coordinates this assumes the fairly ugly form (once again we set $t(0)=0$):

$$\begin{aligned}
t(\tau) &= \frac{a}{\sigma}\left(t'(\tau)+\frac{\omega}{a}y'(\tau)\right) = \frac{a}{\sigma}t'(0)\cosh(\sigma\tau)+\frac{a}{\sigma}x'(0)\sinh(\sigma\tau)+\frac{\omega}{\sigma}y'(0)-\frac{\omega^2}{\sigma^2}\tau \\
&= -\frac{a\omega}{\sigma^2}y(0)\cosh(\sigma\tau)+\frac{a}{\sigma}\left(x(0)+\frac{a}{\sigma^2}\right)\sinh(\sigma\tau)+\frac{a\omega}{\sigma^2}y(0)-\frac{\omega^2}{\sigma^2}\tau \\
&= \frac{a}{\sigma}\left(x(0)+\frac{a}{\sigma^2}\right)\sinh(\sigma\tau)+\frac{a\omega}{\sigma^2}y(0)\left(1-\cosh(\sigma\tau)\right)-\frac{\omega^2}{\sigma^2}\tau \\
x(\tau) &= x'(\tau)-\frac{a}{\sigma^2} = x'(0)\cosh(\sigma\tau)+t'(0)\sinh(\sigma\tau)-\frac{a}{\sigma^2} \\
&= \left(x(0)+\frac{a}{\sigma^2}\right)\cosh(\sigma\tau)-\frac{\omega}{\sigma}y(0)\sinh(\sigma\tau)-\frac{a}{\sigma^2} \\
y(\tau) &= \frac{a}{\sigma}\left(y'(\tau)+\frac{\omega}{a}t'(\tau)\right) = \frac{a}{\sigma}y'(0)-\frac{a\omega}{\sigma^2}\tau+\frac{\omega}{\sigma}t'(0)\cosh(\sigma\tau)+\frac{\omega}{\sigma}x'(0)\sinh(\sigma\tau) \\
&= \frac{a^2}{\sigma^2}y(0)-\frac{\omega^2}{\sigma^2}y(0)\cosh(\sigma\tau)+\frac{\omega}{\sigma}\left(x(0)+\frac{a}{\sigma^2}\right)\sinh(\sigma\tau)-\frac{a\omega}{\sigma^2}\tau \\
&= \frac{a^2}{\sigma^2}y(0)\left(1-\frac{\omega^2}{a^2}\cosh(\sigma\tau)\right)+\frac{\omega}{\sigma}\left(x(0)+\frac{a}{\sigma^2}\right)\sinh(\sigma\tau)-\frac{a\omega}{\sigma^2}\tau \\
z(\tau) &= z(0)
\end{aligned} \tag{4.56}$$

The world line of the observer is given by $x_{obs}(0) = y_{obs}(0) = z_{obs}(0) = 0$, so as τ evolves, the following world-line is traced out:

$$\begin{aligned}
t_{obs}(\tau) &= \frac{a^2}{\sigma^3}\sinh(\sigma\tau)-\frac{\omega^2}{\sigma^2}\tau \\
x_{obs}(\tau) &= \frac{a}{\sigma^2}\left(\cosh(\sigma\tau)-1\right) \\
y_{obs}(\tau) &= \frac{a\omega}{\sigma^3}\sinh(\sigma\tau)-\frac{a\omega}{\sigma^2}\tau \\
z_{obs}(\tau) &= 0
\end{aligned} \tag{4.57}$$

In hyperbolic cylindrical coordinates of the primed frame, we get a slightly more convenient expression by inserting $\chi(0)=0$, $\xi(0)=\frac{a}{\sigma^2}$, $y'(0)=0$ and $z'(0)=0$ into eq. 4.54:

$$\begin{aligned} \chi_{obs}(\tau) &= \sigma\tau \\ \xi_{obs}(\tau) &= \frac{a}{\sigma^2} \\ y'_{obs}(\tau) &= -\frac{\omega}{\sigma}\tau \\ z'(\tau) &= 0 \end{aligned} \tag{4.58}$$

For later reference I also note that in primed Cartesian coordinates, this becomes:

$$\begin{aligned} t'_{obs}(\tau) &= \frac{a}{\sigma^2}\sinh(\sigma\tau) \\ x'_{obs}(\tau) &= \frac{a}{\sigma^2}\cosh(\sigma\tau) \\ y'_{obs}(\tau) &= -\frac{\omega}{\sigma}\tau \\ z'_{obs}(\tau) &= 0 \end{aligned} \tag{4.59}$$

This is almost the so-called hyperbolic trajectory of a body which is accelerated with constant proper acceleration a along the x'-axis, but corrected with a time dilatation factor of $\frac{\sigma}{a}$ (corresponding to the boosting of the primed coordinates relative to the original coordinates), so that the acceleration along the x'-axis seen in the primed reference frame is not the proper acceleration a that the observer him- or herself experiences, but $\sigma = \sqrt{a^2 - \omega^2}$ instead. In addition, the observer has a velocity of $\frac{\mathrm{d}y'}{\mathrm{d}\tau} = -\frac{\omega}{\sigma}$ in the y'-direction. Note that this is *not* a constant velocity with respect to the time coordinate t' or t, but it is constant with respect to the observer's proper time τ, so that the observer will still experience a stationary situation in his or her own reference frame.

In this case, the observer has a rather more interesting reference frame than for $a < \omega$. In figure 4.1 I have plotted the projection of the integral lines of $\frac{\partial}{\partial\tau}$ (i.e. the lines of constant spatial position in the observer's reference frame) onto the $x't'$-plane. The lines running from left to right are curves of constant ξ, which are equal to the sections of the observer's planes of simultaneity with the $x't'$-plane. The thick line is the projection of the observer's world line (the full world-line simultaneously travels in the y'-direction, or "into the paper", and the planes of simultaneity will also be tilted in the y'-direction). A distinctive feature is that the $\frac{\partial}{\partial\tau}$ integral lines and planes of simultaneity are concentrated in the two regions marked R and L (for Left and Right). We see that the hyperbolic cylindrical coordinates cover only these regions, not the regions marked U and D (for Up and Down), unless one allows ξ and χ to take complex values. I will comment on this shortly.

Another feature which does not immediately show up on the figure, is the fact that there is a static limit in the observer's reference frame, i.e. a limit beyond which nothing can remain spatially at rest with with respect to the observer's reference frame. The line element in hyperbolic cylindrical coordinates is:

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = \xi^2 d\chi^2 - d\xi^2 - dy'^2 - dz'^2 \tag{4.60}$$

From this, we see that the norm of the Killing vector field $\vec{v}$ in eq. 4.53 is:

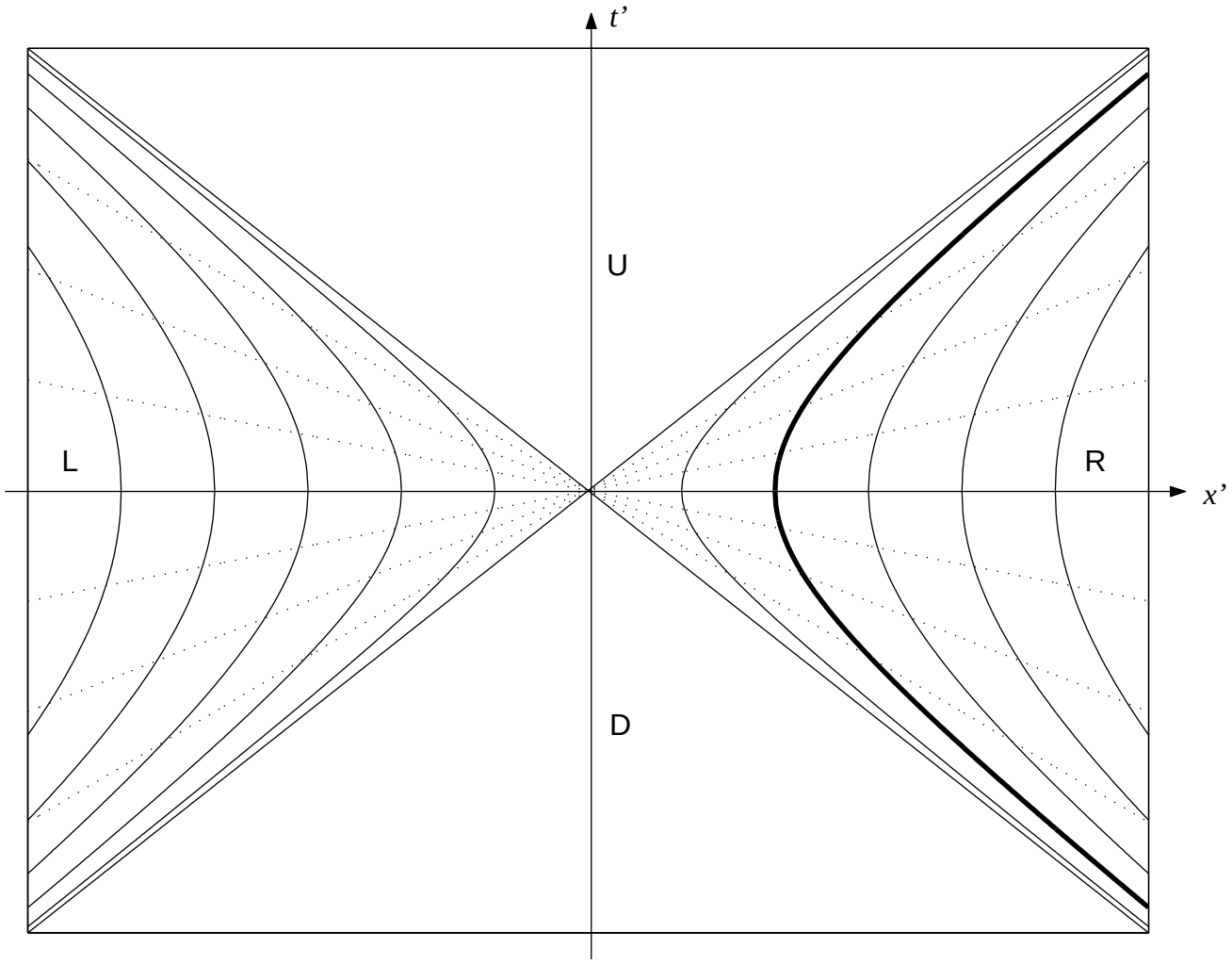

Figure 4.1: Section of hyperbolicly accelerated system

$$v^2 = g_{\mu\nu} v^\mu v^\nu = \sigma^2 \xi^2 - \frac{\omega^2}{\sigma^2} \tag{4.61}$$

For $|\xi| < \frac{\omega}{\sigma^2}$, this is negative. $\vec{v}$ becomes spacelike, so that nothing can travel along the integral lines of $\vec{v}$ beyond this boundary. Hence we have a static limit at $\xi = \pm\frac{\omega}{\sigma^2}$, just as we did for $r = \frac{\omega}{\rho^2}$ in the case of $a < \omega$ (in this case, $a > \omega$, so the location of the observer at $\xi = \frac{a}{\sigma^2}$ is on the right side of the static limit). In this case though, we have something more. There is not just a static limit, beyond which nothing can be spatially at rest with respect to the observer's reference frame, but it turns out that there is also a *horizon*, beyond which all events are causally disconnected from the observer, so that no information about events behind the horizon can ever reach the observer and vice versa. This can be clearly seen in figure 4.1. The regions L and R are causally disconnected, since nothing can pass from one region to the other without following a path with a slope of more than $45°$ relative to the t'-axis, which implies that there is no timelike or null curve along which light signals or massive objects can travel causally between the two regions. This can also be seen clearly

from the line element 4.60. At $\xi = 0$, the term $\xi^2 d\chi^2$, the only positive term in the line element, vanishes, so any non-zero vector based at $\xi = 0$ will have negative norm and be spacelike. Hence there is no way for anything to travel past $\xi = 0$ along a timelike or null curve.

The plane $\xi = 0$ hence acts as a horizon, blocking all flow of information or anything else from R to L (corresponding to $\xi > 0$ and $\xi < 0$ respectively) and vice versa. Since the world line of the observer is in region R, on the right side of the horizon, there is no way that the observer learn anything about or affect events taking place in region L, on the left side of or "behind" the horizon. Note however that it is possible for signals from events in region D to reach the observer, and the observer can him- or herself send a signal to events in region U (although signals cannot travel the other way), even though those regions are outside the hyperbolic cylindrical coordinate system and outside the region covered by integral lines of $\frac{\partial}{\partial\tau}$. This curiously enough means that the observer can learn about and influence events that do not take place within a finite time into the future or into the past as seen by the observer (since there exists no value of τ corresponding to these events). The way signals coming from or travelling to such events looks to the observer, is that signals coming from D seem to start out arbitrarily close to the horizon infinitely far in the past, before rising and arriving at the observer at some finite τ, whereas signals going to U seem to asymptotically approach the horizon in the infinite future (to be more specific, one can calculate that the signals will approach the limit $\xi = 0$ exponentially as $\tau \to -\infty$ and $\tau \to +\infty$ respectively).

But now that we have found one horizon, we may ask if there could be other horizons as well. One could imagine that even within the region R there might be areas that are not connected with the observer by any causal path. This obviously does not seem to be the case from looking at figure 4.1, but that figure is only the projection of the world-line onto the $x't'$-plane. When the y'-dimension is included, we cannot infer just from looking at the projection that no parts from region R will disconnect from the observer. Fortunately though, this turns out not to be the case. A causal path linking any point in space in R (i.e. any integral curve of $\frac{\partial}{\partial\tau}$) to the observer can be found using hyperbolic cylindrical coordinates as follows: Any point in R has $\xi > 0$. From the line element 4.60, we then see that by pointing a vector suitably in the $\chi\xi$-plane, we can go straight to the plane $\xi = \frac{a}{\sigma^2}$ (in which the observer is located) along a timelike line, even though it may be slow going at first if we start with ξ very close to zero. Once we are somewhere in the $\xi = \frac{a}{\sigma^2}$ plane, we can go to $z' = 0$ along a straight, timelike curve. Our ξ- and z'-coordinates are now lined up with those of the observer. All we have to do now is to catch up with the observer, who is moving in the negative y'-direction with a speed of $\frac{\mathrm{d}y'}{\mathrm{d}\tau} = -\frac{\omega}{\sigma}$. But according to the line element at $\xi = \frac{a}{\sigma^2}$ it is possible to travel straight in the y'-direction with a speed of up to $|\frac{\mathrm{d}y'}{\mathrm{d}\tau}| = |\sigma\frac{\mathrm{d}y'}{\mathrm{d}\chi}| = \frac{a}{\sigma}$, which is greater than $\frac{\omega}{\sigma}$, so we will always be able to catch up with the observer eventually. Following the same reasoning in reverse, we can also construct a causal path leading from the observer to any spatial point in R, so all of R is causally connected to the observer.

Before we move on, let us take a moment to compare the results for $a < \omega$ and $a > \omega$. The situations seem quite different. When seen from the respective primed reference frames, the world-line with $a < \omega$ is a closed, circular trajectory, whereas the world-line for

$a > \omega$ is a hyperbolicly accelerated trajectory bending a little in the y'-direction and is very non-closed indeed. These trajectories can obviously not be continuously deformed into one another as either $a \to \omega$ or $\omega \to a$ from below. The description of the observer's reference frame in the two cases also looks rather different. When $a > \omega$ the reference frame has an event horizon as well as a static limit that is stretched out parallel to the horizon. When $a < \omega$ on the other hand, there is no event horizon and the static limit encircles the observer (although it is further away from the observer in the direction of $\mathbf{a}$ and closer in the opposite direction). But the observer obviously should not experience any abrupt change when a and ω pass each other, but should feel it as just a continuous change of the parameters in his or her trajectory. Indeed, if we compare eq. 4.39 to eq. 4.56 and eq. 4.40 to eq. 4.57 we see that in the unprimed reference frame, both the time evolution of the coordinates and the world-line of the observer him- or herself do in fact deform continuously into one another as we cross the line $a = \omega$. This makes sense, since the unprimed reference frame is the one from which the observer starts at rest. The primed reference frames on the other hand are obtained from the unprimed ones by a boost of $\frac{a}{\omega}$ if $a < \omega$ and $\frac{\omega}{a}$ if $a > \omega$. As a and ω approaches each other, this boosting velocity approaches the speed of light, and this causes the discontinuity in the primed reference frames at $a = \omega$.

Despite the (dis)appearance of the horizon and the peculiar behaviour of the static limit when we cross the $a = \omega$ border, the observer also perceives these features as changing in a continuous manner. For $a > \omega$ the horizon is located at $\xi = 0$ and the observer at $\xi = \frac{a}{\sigma^2}$, so the distance to the horizon is $\frac{a}{\sigma^2} = \frac{a}{a^2-\omega^2}$. What happens as $\omega \to a$ from below or vice versa is that the horizon disappears off into infinity and vanishes at the point where $a = \omega$. Similarly, when $a < \omega$ the observer is surrounded by the static limit. The static limit is at $r = \frac{\omega}{\rho^2}$ and the observer is at $r = \frac{a}{\rho^2}$. At the closest, the distance between the observer and the static limit is $\frac{\omega}{\rho^2} - \frac{a}{\rho^2} = \frac{\omega-a}{\omega^2-a^2} = \frac{1}{\omega+a}$, which is always finite. But the distance to the furthest point of the static limit, on the opposite side of the rotational axis, is $\frac{a}{\rho^2} + \frac{\omega}{\rho^2} = \frac{a+\omega}{\omega^2-a^2} = \frac{1}{\omega-a}$, which goes to infinity when $a \to \omega$ from below or vice versa. Since the static limit always forms a cylinder around the rotational axis, this means that as a approaches ω and the diameter of the cylinder goes to infinity, the part of it that is still close to the observer becomes increasingly flat. If we let a approach and then pass ω, what happens is then that both the rotational axis and the far end of the static limit fly off to infinity. When $a = \omega$, they both vanish, and the closest part of the static limit closer, a distance $\frac{1}{a+\omega}$ away from the observer, becomes completely planar. As a increases further, the static limit has become the plane static limit at $\xi = \frac{\omega}{\sigma^2}$ (which is still a distance $\frac{1}{a+\omega}$ away from the observer, so its position changes continuously) that we are supposed to have for $a > \omega$, and at the same time an event horizon appears at infinity and rushes towards the observer from the direction opposite to that in which the rotational axis and the far end of the static limit vanished. As the ratio $\frac{a}{\omega}$ increases further, the horizon comes ever closer, and as $\frac{a}{\omega} \to \infty$, it and the static limit merge altogether. All of this proceeds in a completely smooth and continuous manner.

4.5 $\mathbf{a} \perp \boldsymbol{\omega}$ and $a = \omega$: A borderline case

If $a = \omega$, then both $\mathbf{a} \cdot \boldsymbol{\omega} = 0$ and $a^2 - \omega^2 = 0$. It will then not be possible to make either the boost term or the rotational term in the Killing vector field $\vec{v}$ disappear without making them *both* disappear. That however can obviously not be done unless $\mathbf{a}$ and $\boldsymbol{\omega}$ were both zero to start with; $\mathbf{a}$ and $\boldsymbol{\omega}$ determine the components of the tensor $\omega^{\mu\nu}$ according to eq. 3.21. If they could be made to be zero in one frame, then all the components of $\omega^{\mu\nu}$ would be zero in that frame. But since $\omega^{\mu\nu}$ is a tensor, its components and hence $\mathbf{a}$ and $\boldsymbol{\omega}$ would have to be zero in *all* inertial frames, which would not be the case if they were not zero to start with.

This means that we cannot find any "primed" reference frame of the kind that we used when a was genuinely either less than or greater than ω. However, finding the integral lines of the Killing vector field $\frac{\partial}{\partial \tau}$ is very easy in the unprimed coordinates since we have already found them in the two previous cases. In unprimed coordinates, the integral lines can be found simply by taking the limit $a = \omega$ in either eq. 4.39 or in eq. 4.56. By using the fact that $\sinh(x) \to x + \frac{1}{6}x^3$ and $\cosh(x) - 1 \to \frac{1}{2}x^2$ when $x \to 0$, we get the following limit:

$$\begin{aligned}
t(\tau) &= (1 + ax(0))\,\tau - \frac{1}{2}a\omega y(0)\tau^2 + \frac{1}{6}a^2\tau^3 \\
&= (1 + ax(0))\,\tau - \frac{1}{2}a^2 y(0)\tau^2 + \frac{1}{6}a^2\tau^3 \\
x(\tau) &= x(0) - \omega y(0)\tau + \frac{1}{2}a\tau^2 = x(0) - a y(0)\tau + \frac{1}{2}a\tau^2 \\
y(\tau) &= \left(1 - \frac{1}{2}\omega^2\right) y(0)\tau^2 + \omega x(0)\tau + \frac{1}{6}a\omega\tau^3 \\
&= \left(1 - \frac{1}{2}a^2\right) y(0)\tau^2 + a x(0)\tau + \frac{1}{6}a^2\tau^3 \\
z(\tau) &= z(0)
\end{aligned} \tag{4.62}$$

In the unprimed coordinates, the observer starts out at the origin, $t_{obs}(0) = x_{obs}(0) = y_{obs}(0) = z_{obs}(0) = 0$, so the world-line of the observer is:

$$\begin{aligned}
t_{obs}(\tau) &= \tau + \frac{1}{6}a^2\tau^3 \\
x_{obs}(\tau) &= \frac{1}{2}a\tau^2 \\
y_{obs}(\tau) &= \frac{1}{6}a^2\tau^3 \\
z_{obs}(\tau) &= 0
\end{aligned} \tag{4.63}$$

How the observer experiences this reference frame can also be inferred by looking at the limit $a = \omega$ for the two previous cases. The horizon for the case $a > \omega$ is located at a distance $\frac{a}{a^2 - \omega^2}$ from the observer. This goes off to infinity in the limit $a = \omega$, so that there is no horizon in this reference frame. The static limit for $a > \omega$ on the other hand is located at

$\xi = \frac{\omega}{a^2-\omega^2}$, so that the distance between it and the observer is $\frac{1}{a+\omega}$. There will therefore be a static limit in the reference frame located a distance $\frac{1}{a+\omega} = \frac{1}{2a}$ away from the observer in the opposite direction of $\mathbf{a}$. As noted at the end of the previous section, this also agrees with the limit taken from $a < \omega$. The reference frame where $\mathbf{a} \perp \boldsymbol{\omega}$ and $a = \omega$ is therefore in every aspect simply the borderline between $a > \omega$ and $a < \omega$ and is determined by taking the limit $\omega \to a$ from either side.

4.6 $\mathbf{a} \not\perp \boldsymbol{\omega}$: An accelerated observer on a helical trajectory

If we no longer have $\mathbf{a} \perp \boldsymbol{\omega}$, we have that $\mathbf{a} \cdot \boldsymbol{\omega} \neq 0$, so it will not be possible to make either the rotational term or the boost term of the Killing vector field $\vec{v}$ vanish by viewing it from an appropriate inertial frame. However, it might be possible to choose a frame which is such that the boosting takes place along the direction of the rotational axis, i.e. a frame where $\mathbf{a} \| \boldsymbol{\omega}$.

Since $\mathbf{a}$ and $\boldsymbol{\omega}$ are not perpendicular, this time we cannot choose them to lie along two different coordinate axes. Instead, choose $\boldsymbol{\omega}$ to point along the positive x-axis and $\mathbf{a}$ to lie in the xy-plane. The Killing vector field $\vec{v}$ is then given by:

$$\vec{v} = \vec{P}_t + \omega \vec{J}_x + a_x \vec{B}_x + a_y \vec{B}_y =$$
$$\frac{\partial}{\partial t} + \omega \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) + a_x \left(x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x} \right) + a_y \left(y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y} \right) \tag{4.64}$$

To change the relative directions of rotation and boosting in $\vec{v}$, we obviously need to perform a boost in some direction, the question is which one. The only direction that really distinguishes itself when we have two linearly independent vectors $\mathbf{a}$ and $\boldsymbol{\omega}$ in a three-dimensional space, is the direction perpendicular to both of them, in this case the z-direction.

We therefore boost the system by a velocity β to transform to primed coordinates $(t'; x', y', z')$ and see what happens to $\vec{v}$:

$$\begin{aligned} z &= \frac{1}{\sqrt{1-\beta^2}} \left(z' + \beta t' \right) \equiv \gamma \left(z' + \beta t' \right) \\ t &= \gamma \left(t' + \beta z' \right) \end{aligned} \tag{4.65}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. The inverse transformation is

$$z' = \gamma \left(z - \beta t \right) \qquad\qquad t' = \gamma \left(t - \beta z \right) \tag{4.66}$$

and the vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial z}$ transform as

$$\frac{\partial}{\partial z} = \gamma \left(\frac{\partial}{\partial z'} - \beta \frac{\partial}{\partial t'} \right) \qquad \frac{\partial}{\partial t} = \gamma \left(\frac{\partial}{\partial t'} - \beta \frac{\partial}{\partial z'} \right) \tag{4.67}$$

while we let $y' = y$ and $x' = x$ for now. Inserting this into eq. 4.64 gives us:

$$\begin{aligned}\vec{v} = \gamma\left(\frac{\partial}{\partial t'} - \beta\frac{\partial}{\partial z'}\right) + \gamma\omega\left(y'\frac{\partial}{\partial z'} - z'\frac{\partial}{\partial y'} - \beta y'\frac{\partial}{\partial t'} - \beta t'\frac{\partial}{\partial y'}\right) \\ + \gamma a_x\left(x'\frac{\partial}{\partial t'} + t'\frac{\partial}{\partial x'} - \beta x'\frac{\partial}{\partial z'} + \beta z'\frac{\partial}{\partial x'}\right) \\ + \gamma a_y\left(y'\frac{\partial}{\partial t'} + t'\frac{\partial}{\partial y'} - \beta y'\frac{\partial}{\partial z'} + \beta z'\frac{\partial}{\partial y'}\right) \quad (4.68)\end{aligned}$$

Reorganizing a little, we get:

$$\begin{aligned}\vec{v} &= \gamma\frac{\partial}{\partial t'} - \beta\gamma\frac{\partial}{\partial z'} + \gamma\left(\omega - \beta a_y\right)\left(y'\frac{\partial}{\partial z'} - z'\frac{\partial}{\partial y'}\right) + \gamma\beta a_x\left(z'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial z'}\right) \\ &\quad + \gamma a_x\left(x'\frac{\partial}{\partial t'} + t'\frac{\partial}{\partial x'}\right) + \gamma\left(a_y - \beta\omega\right)\left(y'\frac{\partial}{\partial t'} + t'\frac{\partial}{\partial y'}\right) \quad (4.69) \\ &\equiv \gamma\left(\vec{P}_{t'} - \beta\vec{P}_{z'} + \omega'_x\vec{J}_{x'} + \omega'_y\vec{J}_{y'} + a'_x\vec{B}_{x'} + a'_y\vec{B}_{y'}\right)\end{aligned}$$

where now

$$\omega'_x = \omega - \beta a_y \qquad \omega'_y = \beta a_x \qquad a'_x = a_x \qquad a'_y = a_y - \beta\omega \quad (4.70)$$

We want the boost direction and the axis of rotation to be parallel in the primed system, so we demand the following:

$$\frac{\omega'_y}{\omega'_x} \stackrel{!}{=} \frac{a'_y}{a'_x} \qquad \Longrightarrow \qquad \frac{\beta a_x}{\omega - \beta a_y} \stackrel{!}{=} \frac{a_y - \beta\omega}{a_x}$$

This gives us the following quadratic equation for β:

$$\begin{aligned}&\left(a_y - \beta\omega\right)\left(\omega - \beta a_y\right) - \beta a_x^2 = 0 \\ \Longrightarrow \qquad &\beta^2 a_y\omega - \beta\left(\omega^2 + a_x^2 + a_y^2\right) + a_y\omega = 0 \quad (4.71)\end{aligned}$$

Use now that $a_x^2 + a_y^2 = a^2$, and we get the following standard solutions:

$$\beta = \frac{\omega^2 + a^2 \pm \sqrt{(\omega^2 + a^2)^2 - 4a_y^2\omega^2}}{2a_y\omega} = \frac{\omega^2 + a^2}{2a_y\omega} \pm \sqrt{\left(\frac{\omega^2 + a^2}{2a_y\omega}\right)^2 - 1} \quad (4.72)$$

β must however be less than 1 for a boost to be physically possible. Since $\omega^2 + a^2 \geq 2a\omega > 2a_y\omega$, we must therefore use the solution with the minus sign in eq. 4.72. If we do so, a boost will always be possible, which can be seen as follows: If we derivate the expression for β (with the minus sign) with respect to a_y for a and ω fixed, we get:

$$\frac{\mathrm{d}\beta}{\mathrm{d}a_y} = -\frac{\omega^2 + a^2}{2a_y^2\omega} + \frac{\omega^2 + a^2}{2a_y^2\omega}\frac{\omega^2 + a^2}{2a_y\omega}\left[\left(\frac{\omega^2 + a^2}{2a_y\omega}\right)^2 - 1\right]^{-\frac{1}{2}}$$

$$= \frac{\omega^2+a^2}{2a_y^2\omega}\left\{\frac{\omega^2+a^2}{2a_y\omega}\left[\left(\frac{\omega^2+a^2}{2a_y\omega}\right)^2-1\right]^{-\frac{1}{2}}-1\right\} > 0 \tag{4.73}$$

This means that β is an increasing function of a_y. β is therefore bounded above by its limit value when $a_y \to a$ (since a_y is necessarily less than a). Hence β is less than the value we get by inserting $a_y = a$ into eq. 4.72, which is:

$$\frac{\omega^2+a^2}{2a\omega}-\sqrt{\left(\frac{\omega^2+a^2}{2a\omega}\right)^2-1}=\frac{\omega^2+a^2}{2a\omega}-\sqrt{\left(\frac{\omega^2-a^2}{2a\omega}\right)^2}=\frac{\omega^2+a^2-|\omega^2-a^2|}{2a\omega} \tag{4.74}$$

If now $a \leq \omega$, this becomes equal to $\frac{a}{\omega} \leq 1$ and conversely, if $a \geq \omega$, then it becomes $\frac{\omega}{a} \leq 1$. So in any case we find that $\beta < 1$ when we choose the minus sign in eq. 4.72.

Now that we have finally showed that we can make the boost direction and the rotational axis line up in parallel, it makes sense to choose a new coordinate system, $(t''; x'', y'', z'')$ which is rotated in the $x'y'$-plane so that the boost direction and rotational axis both lie along the x''-axis (but for now we let $t'' = t'$ and $z'' = z'$). Define also the boost and acceleration parameters a'' and ω'' with respect to the double primed frame as follows:

$$a'' = \sqrt{a_x'^2 + a_y'^2} = \sqrt{a^2+\beta^2\omega^2-2\beta a_y\omega} \tag{4.75}$$

$$\omega'' = \sqrt{\omega_x'^2 + \omega_y'^2} = \sqrt{\omega^2+\beta^2 a^2-2\beta\omega a_y} \tag{4.76}$$

In these coordinates and with these parameters, we finally get a fairly convenient form for $\vec{v}$:

$$\vec{v} = \gamma\left[\frac{\partial}{\partial t''}-\beta\frac{\partial}{\partial z''}+a''\left(x''\frac{\partial}{\partial t''}+t''\frac{\partial}{\partial x''}\right)+\omega''\left(y''\frac{\partial}{\partial z''}-z''\frac{\partial}{\partial y''}\right)\right] \tag{4.77}$$

As usual though, we can make this form even more convenient if we redefine y'' and shift the origin by letting $y'' \to y'' + \beta/\omega''$. We then rid ourselves of the term $-\gamma\beta\frac{\partial}{\partial z''}$. Furthermore, we also let $x'' \to x'' - \frac{1}{a''}$ to incorporate the time translation into the boosting, so that $\vec{v}$ finally becomes:

$$\begin{aligned}\vec{v} \equiv \frac{\partial}{\partial\tau} &= a''\left(x''\frac{\partial}{\partial t''}+t''\frac{\partial}{\partial x''}\right)+\omega''\left(y''\frac{\partial}{\partial z''}-z''\frac{\partial}{\partial y''}\right)\\ &= a''\vec{B}_{x''}+\omega''\vec{J}_{x''}\end{aligned} \tag{4.78}$$

This time, $\vec{v}$ contains both a boost and a rotational part, with the plane of rotation (the $y''z''$-plane) being perpendicular to the boost direction (the x''-direction). This suggests that, in order to calculate explicitly the integral lines of $\frac{\partial}{\partial\tau}$, we should introduce hyperbolic cylindrical coordinates ξ and χ in place of x'' and t'', and ordinary cylindrical coordinates r and θ in place of y'' and z'':

$$\begin{aligned} x'' &= \xi \cosh \chi & t'' &= \xi \sinh \chi \\ y'' &= r \cos \theta & z'' &= r \sin \theta \end{aligned} \tag{4.79}$$

or

$$\begin{aligned} \xi &= \pm\sqrt{x''^2 - t''^2} & \chi &= \operatorname{artanh} \frac{t''}{x''} \\ r &= \sqrt{y''^2 + z''^2} & \theta &= \arctan \frac{z''}{y''} \end{aligned} \tag{4.80}$$

$\vec{v}$ then becomes:

$$\vec{v} \equiv \frac{\partial}{\partial \tau} = a'' \frac{\partial}{\partial \chi} + \omega'' \frac{\partial}{\partial \theta} \tag{4.81}$$

and we get the following integral lines for $\frac{\partial}{\partial \tau}$:

$$\begin{aligned} \chi(\tau) &= \chi(0) + a''\tau \\ \xi(\tau) &= \xi(0) = const. \\ \theta(\tau) &= \theta(0) + \omega''\tau \\ r(\tau) &= r(0) = const. \end{aligned} \tag{4.82}$$

which in the Cartesian double primed coordinates becomes:

$$\begin{aligned} t''(\tau) &= \xi(\tau)\sinh(\chi(\tau)) = \xi(0)\sinh\left(\chi(0) + a''\tau\right) \\ &= \xi(0)\left[\sinh(\chi(0))\cosh(a''\tau) + \sinh(a''\tau)\cosh(\chi(0))\right] \\ &= t''(0)\cosh(a''\tau) + x''(0)\sinh(a''\tau) \\ x''(\tau) &= \xi(\tau)\cosh(\chi(\tau)) = \xi(0)\cosh\left(\chi(0) + a''\tau\right) \\ &= \xi(0)\left[\cosh(\chi(0))\cosh(a''\tau) + \sinh(\chi(0))\sinh(a''\tau)\right] \\ &= x''(0)\cosh(a''\tau) + t''\sinh(a''\tau) \\ y''(\tau) &= r(\tau)\cos(\theta(\tau)) = r(0)\cos\left(\theta(0) + \omega''\tau\right) \\ &= r(0)\left[\cos(\theta(0))\cos(\omega''\tau) - \sin(\theta(0))\sin(\omega''\tau)\right] \\ &= y''(0)\cos(\omega''\tau) - z''(0)\sin(\omega''\tau) \\ z''(0) &= r(\tau)\sin(\theta(\tau)) = r(0)\sin\left(\theta(0) + \omega''\tau\right) \\ &= r(0)\left[\sin(\theta(0))\cos(\omega''\tau) + \cos(\theta(0))\sin(\omega''\tau)\right] \\ &= z''(0)\cos(\omega''\tau) + y''(0)\sin(\omega''\tau) \end{aligned} \tag{4.83}$$

This time, calculating the corresponding relations for the unprimed coordinates becomes a *very* messy job indeed. Since I will not be using the result later in this thesis either, I skip this calculation altogether. Instead, I just simply give the world line of the observer

in the hyperbolic cylindrical coordinates of the doubly primed reference frame. Since the observer to start with is located at the origin of the unprimed reference frame and we later moved the origin of the $x''y''$-plane a distance $-\frac{1}{a''}$ in the x''-direction and a distance $\frac{\beta}{\omega''}$ in the y''-direction, eq. 4.80 gives us that the world-lined of the observer is characterized by $\chi_{obs}(0) = 0$, $\xi_{obs}(0) = \frac{1}{a''}$, $r_{obs}(0) = \frac{\beta}{\omega''}$ and $\theta_{obs}(0) = 0$. The world-line is therefore:

$$\begin{aligned} \chi_{obs}(\tau) &= a''\tau \\ \xi_{obs}(\tau) &= \frac{1}{a''} \\ r_{obs}(\tau) &= \frac{\beta}{\omega''} \\ \theta_{obs}(\tau) &= \omega''\tau \end{aligned} \tag{4.84}$$

We see that this is a trajectory which performs circular orbits around the x''-axis with constant angular velocity ω'' (with respect to τ, the angular velocity with respect to t'' is not constant because the acceleration causes the time dilatation factor to change continuously), while simultaneously being accelerated with constant proper acceleration (hyperbolic acceleration) in the x''-direction, so that the resulting trajectory is a kind of accelerating helix (one could perhaps call it a "hyperbolic helicoid" if one is in desperate need of a fancy name to put on it).

Since the observer is following a trajectory with constant proper acceleration in the x''-direction, there is obviously an event horizon at $\xi = 0$, for the same reasons that I mentioned in the case with $\mathbf{a} \perp \boldsymbol{\omega}$ and $a > \omega$. The region at $\xi < 0$ is causally disconnected from the observer. But the observer stays in a bounded region around the x''-axis, so his or her motion in the y''- and z''-direction cannot cause the appearance of any further horizons; the whole region with $\xi > 0$ is clearly causally connected to the observer, so that the observer will always be able to learn of and transmit information to any event that happens on that side of the horizon.

But since we have circular motion, there should obviously also be some static limit in this reference frame. The line element in combined cylindrical and hyperbolic cylindrical coordinates is:

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = \xi^2 d\chi^2 - d\xi^2 - dr^2 - r^2 d\theta^2 \tag{4.85}$$

According to eq. 4.81, the norm of $\vec{v}$ is then:

$$v^2 \equiv g_{\mu\nu}v^\mu v^\nu = \xi^2 a''^2 - r^2\omega''^2 \tag{4.86}$$

If now $r > \frac{a''}{\omega''}\xi$, v^2 will be negative, and $\vec{v}$ will be spacelike, so that nothing can stay spatially at rest with respect to the integral lines of $\frac{\partial}{\partial\tau}$ in this limit. So we have a static limit at the surface $r = \frac{a''}{\omega''}\xi$, which is a cone around the ξ-axis with its apex at the origin.

Chapter 5

Particles and vacua in Minkowski spacetime

In this chapter we will investigate what the particle and vacuum states look like in the different stationary reference frames in Minkowski spacetime, and how they relate to each other. As mentioned in section 3.2, we will define particles with respect to the Hamiltonian H_{RF}, which generates translations in proper time τ with respect to the observer of that reference frame. Initially, we will express the Hamiltonian in terms of Lorentz transformation generators from the inertial rest frame of the observer (i.e. as in eq. 3.19 with no P_i-term). However, we will not let ourselves be completely bound to this starting point frame of reference, since as we saw in the last chapter, we can achieve a lot of simplification by viewing things from a different reference frame which is never at rest with respect to the first one. We will see that the aforementioned ambiguities inherent in defining particles will all come to haunt us. In fact, we will find that depending on how liberal or strict you want to be in placing restrictions on how to define the particle states, the vacuum and particle states of the different stationary reference frames in Minkowski spacetime can be either uniquely defined, divided into just two distinct classes, or be almost completely arbitrary. The second possibility seems to be the only one which has been discussed in the literature that I am familiar with.

5.1 Comparing particles in different reference frames

Before we start looking at specific cases, let us see what we can say in general about the relationship between particles defined with respect to the Hamiltonians of different stationary reference frames. If we have two stationary reference frames 1 and 2 with Hamiltonians $H^{(1)}_{\mathsf{RF}}$ and $H^{(2)}_{\mathsf{RF}}$, then we are guaranteed that we can find a common set of particle states with a definite number of particles and definite energies if the two Hamiltonians commute, $[H^{(1)}_{\mathsf{RF}}, H^{(2)}_{\mathsf{RF}}] = 0$. Let us see when this is the case. Write the Hamiltonians on the form of eq. 3.19 and express both of the Hamiltonians through Lorentz generators defined with respect to the same inertial reference frame. We then get:

$$
\begin{aligned}
[H^{(1)}_{\mathsf{RF}}, H^{(2)}_{\mathsf{RF}}] &= [H + \beta_1^i P_i + \omega_1^i J_i + a_1^i B_i \,,\; H + \beta_2^i P_i + \omega_2^i J_i + a_2^i B_i] \\
&= (a_2^i - a_1^i)[H, B_i] \;+\; (\beta_1^i \omega_2^j - \beta_2^i \omega_1^j)[P_i, J_j] \;+\; (\beta_1^i a_2^j - \beta_2^i a_1^j)[P_i, B_J] \\
&\quad +\; (\omega_1^i a_2^j - \omega_2^i a_1^j)[J_i, B_j] \;+\; \omega_1^i \omega_2^j [J_i, J_j] \;+\; a_1^i a_2^j [B_i, B_j] \\
&= (a_2^i - a_1^i)\, i P_i - (\beta_1^i \omega_2^j - \beta_2^i \omega_1^j)\epsilon_{ijk}\, i P_k + (\beta_1^i a_2^j - \beta_2^i a_1^j)\delta_{ij}\, i H \\
&\quad +\; (\omega_1^i a_2^j - \omega_2^i a_1^j)\epsilon_{ijk}\, i B_k \;+ (\omega_1^i \omega_2^j - a_1^i a_2^j)\epsilon_{ijk}\, i J_k
\end{aligned}
\tag{5.1}
$$

If the Hamiltonians are to commute, all the terms in this expression have to vanish. For this to happen, we must have $\boldsymbol{\beta}_1 \cdot \mathbf{a}_2 = \boldsymbol{\beta}_2 \cdot \mathbf{a}_1$, $\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 = \mathbf{a}_1 \times \mathbf{a}_2$, $\mathbf{a}_1 \times \boldsymbol{\omega}_2 = \mathbf{a}_2 \times \boldsymbol{\omega}_1$ and $\boldsymbol{\beta}_1 \times \boldsymbol{\omega}_2 - \boldsymbol{\beta}_2 \times \boldsymbol{\omega}_1 = \mathbf{a}_2 - \mathbf{a}_1$. There are obviously non-trivial transitions for which these relations all hold, but they are rather singular. This result therefore suggest that there could be a bewildering number of distinct classes of trajectories between which the Hamiltonians do not commute and for which one therefore cannot construct any common set of particle states with definite numbers of particles with definite energies, maybe not even a common vacuum state. One could fear that different stationary observers would have very many different opinions on what the particle states looked like and what empty space would be. In particular, we see that if $\mathbf{a}_1 = \boldsymbol{\omega}_1 = \boldsymbol{\beta}_1 = 0$ so that trajectory 1 is just standing still in the original inertial frame, then $H^{(2)}_{\mathsf{RF}}$ commutes with $H^{(1)}_{\mathsf{RF}} = H$ only if $\mathbf{a}_2 = 0$. Just a tiny bit of acceleration therefore breaks the commutativity, so that it would seem that our usual inertially defined particles have a very limited range of applicability indeed.

Of course, we have not here discussed the Bogoliubov coefficients between the eigenfunctions of the different Hamiltonians. If the Bogoliubov β-coefficients are zero for most transitions (which it fortunately turns out that they are, as we shall see later in this chapter), the observers might still agree on what states contain a definite number of particles, how many particles they contain and what the vacuum looks like. But as long as the Hamiltonians do not commute, they will still not be able to agree on whether these particles have well-defined energies, i.e. are in an eigenstate of the respective Hamiltonians. The particles that they observe would therefore look qualitatively quite different, so that we would still have many sets of arguably different-looking particle-states.

However, there is some redundancy in the parameters of eq. 5.1, namely in the choice of the parameters $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$. We could hope that just as we were able to reduce the bewildering number of different stationary trajectories down to just four qualitatively different kinds of motion in the last chapter, we might also be able to reduce the number of qualitatively different sets of particle states by applying an appropriate set of Lorentz transformations to the reference frames. I will devote the rest of this section to demonstrating that this is indeed the case. The rest of the chapter will be used to calculate explicit particle eigenfunctions in the case of a scalar (Klein-Gordon) field and finding the Bogoliubov coefficients for transitions between them in order to determine what inequivalent vacuum and particle states we have.

Simplifying the number of distinct kinds of particle states in fact turns out to be exactly the same as the reduction of the number of qualitatively distinct stationary trajectories that we found in the last chapter. In the following subsections I will show this for each of the four kinds of stationary trajectory that we identified.

Frames with $\mathbf{a} \perp \boldsymbol{\omega}$, $a < \omega$

We start out with the normal general form of the stationary Hamiltonian H_{RF} of the observer's reference frame, where I now take the Lorentz generators to be defined with respect to the inertial frame in which the observer starts at rest, so that $\boldsymbol{\beta} = 0$. Since $\mathbf{a}$ is perpendicular to $\boldsymbol{\omega}$, I choose $\mathbf{a}$ to lie along the x-axis and $\boldsymbol{\omega}$ along the z-axis as in section 4.3. Remember that with these definitions, the Killing vector field $\frac{\partial}{\partial\tau}$ generating proper time translation for the observer had the following form (eq. 4.22):

$$\frac{\partial}{\partial\tau} = \vec{P}_t + \omega\vec{J}_z + a\vec{B}_x = \frac{\partial}{\partial t} + \omega\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + a\left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right) \tag{5.2}$$

The form of the Hamiltonian stands in one-to-one correspondence to this expression, since it too generates proper time translations for the observer:

$$H_{\mathsf{RF}} = H + \omega J_z + aB_x \tag{5.3}$$

Because of the presence of the B_x-term, this does not commute with the inertial Hamiltonian H, at least not the inertial Hamiltonian of the original inertial frame. But as we found in section 4.3, by boosting with a velocity $\frac{a}{\omega}$ along the y-axis into another inertial frame, denoted by primed coordinates $(t'; x', y', z')$, and moving the origin a distance $\frac{a}{\rho^2}$ in the x-direction, the Killing vector field $\frac{\partial}{\partial\tau}$ could be given the following form (eq. 4.28):

$$\frac{\partial}{\partial\tau} = \vec{v} = \frac{\omega}{\rho}\frac{\partial}{\partial t'} + \rho\left(x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'}\right) = \frac{\omega}{\rho}\vec{P}_{t'} + \rho\vec{J}_{z'} \tag{5.4}$$

where $\rho = \sqrt{\omega^2 - a^2}$. The Hamiltonian H_{RF} then correspondingly has the form:

$$H_{\mathsf{RF}} = \frac{\omega}{\rho}H' + \rho J_{z'} = \frac{\omega}{\rho}\left(H' + \frac{\rho^2}{\omega}J_{z'}\right) \tag{5.5}$$

From this expression we see that H_{RF} actually commutes with the new inertial Hamiltonian H', since H' commutes with $J_{z'}$ (and obviously with itself). The particle states defined using H_{RF} can therefore be made to coincide with particle states defined using the inertial Hamiltonian H' by defining it to be the set of particle states that are common eigenstates of both H' and $J_{z'}$, and thereby also of H_{RF}. Since these states are in fact eigenstates of H_{RF} no matter what the value of a and ω, as long as $a < \omega$, this particle spectrum is actually a common particle spectrum for *all* trajectories with $\mathbf{a} \perp \boldsymbol{\omega}$ and $a < \omega$, which gives us the kind of reduction of distinct particle spectra that we hoped for. Note however that observers on different trajectories will experience this common set of definite energy particle states somewhat differently. The energy that they observe for each state will be scaled by the time-dilatation factor $\frac{\omega}{\rho}$ in eq. 5.5 and also shifted by the term $\frac{\rho^2}{\omega}J_{z'}$.

All such trajectories therefore share a common particle spectrum among themselves, and this spectrum is an inertial Minkowski particle spectrum. However, it is important to note that this is *not* the usual set of plane-wave, definite-momentum particles, which are simultaneous eigenstates of H' and $\mathbf{P}'$. Instead the set consists eigenstates for H' and $J_{z'}$, i.e. states with definite angular momentum around the z'-axis. This is the reason why we have a common particle spectrum for H_{RF} and H' but not for H_{RF} and H even though H'

and H commute. Whereas the conventional plane-wave, definite momentum eigenstates are simultaneous eigenstates for all inertial Hamiltonian operators, states with definite angular momentum are not.

Frames with $\mathbf{a} \perp \boldsymbol{\omega}$, $a > \omega$

When $\mathbf{a} \perp \boldsymbol{\omega}$ and $a > \omega$, we can still start out with the same form of the proper-time translation generating vector field $\frac{\partial}{\partial\tau}$ and the Hamiltonian H_{RF} as in eqs. 5.2 and 5.3. But in this case, we found in section 4.4 that we could not remove the B_x-term, but we could remove the J_z-term instead, by boosting with a velocity of $\frac{\omega}{a}$ in the y-direction and moving the origin a distance $\frac{a}{\sigma^2}$ in the $-x$-direction. In the new inertial frame, denoted by primed coordinates $(t'; x', y', z')$, $\frac{\partial}{\partial\tau}$ had the following form (eq. 4.49):

$$\vec{v} = \sigma\left(x'\frac{\partial}{\partial t'} + t'\frac{\partial}{\partial x'}\right) - \frac{\omega}{\sigma}\frac{\partial}{\partial y'} = \sigma\vec{B}_{x'} - \frac{\omega}{\sigma}\vec{P}_{y'} \tag{5.6}$$

The Hamiltonian H_{RF} then has the form:

$$H_{\mathsf{RF}} = \sigma B_{x'} - \frac{\omega}{\sigma}P_{y'} = \sigma\left(B_{x'} - \frac{\omega}{\sigma^2}P_{y'}\right) \tag{5.7}$$

From this form of the Hamiltonian, we see immediately that H_{RF} does *not* commute with any inertial Hamiltonian because of the term involving $B_{x'}$ (and since we know from section 4.4 that H_{RF} must always contain a non-vanishing boost-term, it's no good trying to boost ourselves into any other inertial frame either). When $a > \omega$, the particle spectrum therefore truly looks different from any inertial one. Later in this chapter we shall also find that at least if we use the most "naive" eigenfunctions of H_{RF} for a Klein-Gordon field, the β-coefficients of the Bogoliubov transformation between those eigenfunctions and the plane-wave eigenfunctions of an inertial Hamiltonian are non-zero. The particle spectra therefore seem to disagree not only on which states have definite energy but even which states contain a definite number of particles and what the vacuum state is.

Still, eq. 5.7 gives us a notable simplification in the number of distinct particle spectra. Since $B_{x'}$ and $P_{y'}$ commute, we can choose the particles eigenstates of H_{RF} to be those that are common eigenstates for both $B_{x'}$ and $P_{y'}$. These states are eigenstates of H_{RF} for any values of a and ω as long as $a > \omega$, and therefore give us a particle spectrum that is common for all stationary reference frames with $\mathbf{a} \perp \boldsymbol{\omega}$ and $a > \omega$. Note however that just as for the case $a < \omega$, each observer will experience a scaling of the energy of each state by the time-dilatation factor σ and a shift due to the term $-\frac{\omega}{\sigma^2}P_{y'}$.

Frames with $\mathbf{a} \perp \boldsymbol{\omega}$, $a = \omega$

When $\mathbf{a} \perp \boldsymbol{\omega}$ and $a = \omega$, then as in section 4.5 we can still put $\mathbf{a}$ along the x-axis and $\boldsymbol{\omega}$ along the z-axis, so that the vector field $\frac{\partial}{\partial\tau}$ gets the form

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial t} + a\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + a\left(x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}\right) = \vec{P}_t + a\vec{J}_z + a\vec{B}_x \tag{5.8}$$

so that the Hamiltonian has the form

$$H_{\mathsf{RF}} = H + aJ_z + aB_x \tag{5.9}$$

As was noted in section 4.5, this form cannot be simplified any further. We cannot get rid of either the J_z-term or the B_x-term. Since B_x does not commute with H and since J_z does not commute with B_x, we cannot make H_{RF} commute with any inertial Hamiltonian or any "accelerating" Hamiltonian of the form of eq. 5.7. The particle spectrum seen by an observer following this kind of trajectory is therefore distinct from the two earlier cases. If one thinks a bit, this may seem a little strange. I already commented in the last chapter that the observer does not experience any discontinuous jump in the appearance of his or her reference frame when the trajectory passes from $a < \omega$ to $a > \omega$. Therefore it does not seem quite right that the particle spectra should be divided into clear-cut distinct spectra for $a < \omega$, $a > \omega$ and even $a = \omega$.

The origin of this paradox can once again be found in the fact that we have chosen to define our particle states using the "primed" inertial frames, which involve a boost relative to the original unprimed inertial frame (the original instantaneous rest frame of the observer) that approaches the speed of light as a and ω approach each other. If we had instead defined them using the unprimed frames, then the transition between the particle states would have been continuous, but we would then have had to contend with a lot more distinct sets of definite-energy particle states. I will comment a little more on this later in the chapter when we calculate explicit eigenfunctions of the different frames.

Note also that H_{RF} from eq. 5.9 for one value of a does not commute with H_{RF} for another value of a, so there is in fact no common set of definite energy particle states for the case $a = \omega$. Each value of a has its own distinct set of states. We will not concern ourselves too much with the details of the $a = \omega$ trajectories in this thesis, concentrating instead on comparing and interpreting the cases $a > \omega$ and $a < \omega$, but I will mention a bit more about it later in this chapter.

Frames with $\mathbf{a} \not\perp \boldsymbol{\omega}$

As we saw in section 4.6, when $\mathbf{a} \not\perp \boldsymbol{\omega}$ we can perform an appropriate boost, translation and rotation on the original inertial instantaneous rest frame of the observer to obtain what we in that section denoted by a "double primed" reference frame, in which the vector field $\frac{\partial}{\partial\tau}$ has the following form:

$$\frac{\partial}{\partial\tau} = a''\left(x''\frac{\partial}{\partial t''} + t''\frac{\partial}{\partial x''}\right) + \omega''\left(y''\frac{\partial}{\partial z''} - z''\frac{\partial}{\partial y''}\right) = a''\vec{B}_{x''} + \omega''\vec{J}_{x''} \tag{5.10}$$

In this inertial frame, H_{RF} correspondingly has the form

$$H_{\mathsf{RF}} = a''B_{x''} + \omega''J_{x''} \tag{5.11}$$

As we noted in section 4.6, we cannot make the boost term vanish in any way, so H_{RF} cannot be made to commute with any inertial Hamiltonian. Furthermore, unless $\omega = 0$, it cannot be made to commute with a Hamiltonian of the form $\sigma\left(B_x + \frac{\omega}{\sigma^2}P_y\right)$ that we had for the case $\mathbf{a} \perp \boldsymbol{\omega}$, $a > \omega$ since a translation operator such as P_y could not commute with

both $B_{x''}$ and $J_{x''}$ (a momentum operator commutes with an angular momentum operator only when they are parallel, whereas it commutes with a boost operator only when they are orthogonal, so it cannot commute with the sum of a boost and angular momentum operator that are parallel to each other).

But, since $B_{x''}$ and $J_{x''}$ commute, we can find a single set of definite energy particle states that is common to all reference frames following a trajectory with $\mathbf{a} \not\perp \boldsymbol{\omega}$. Simply choose the set that is a common set of eigenstates for both $B_{x''}$ and $J_{x''}$.

So after having examined all the different classes of stationary trajectories, we have been able to identify just three distinct sets of definite energy particle states, along with a continuous family of states for the case $\mathbf{a} \perp \boldsymbol{\omega}$ and $a = 0$, which can serve as the particle spectra of a stationary observer. We now proceed to calculate the explicit particle eigenfunctions for a Klein-Gordon field in the different kinds of stationary reference frames. We shall then use these to calculate Bogoliubov coefficients and thus answer the question of which of these sets that do not agree at all on which states have definite particle numbers, and which sets merely disagree on which particle states that have definite energies.

5.2 Eigenfunctions when $\mathbf{a} \perp \boldsymbol{\omega}$, $a < \omega$

In the following sections I will calculate eigenfunctions and eigenvalues of H_{RF} in the four different kinds of stationary reference frames, to get an understanding for what kind of particles the observer sees in each frame and what their energy spectra are. I will calculate the full eigenfunctions only for the case of a scalar Klein-Gordon field, although for this first case, $\mathbf{a} \perp \boldsymbol{\omega}$ with $a < \omega$, I will give a few hints along the way as to what would need to be done to perform the calculations for a general field, just to illustrate the procedure.

In the last section, we saw that in a stationary reference frame, seen from an appropriate inertial reference frame, the Hamiltonian H_{RF} of the observer had the following form:

$$H_{\mathsf{RF}} = \frac{\omega}{\rho}\left(H' + \frac{\rho^2}{\omega} J_{z'}\right) \tag{5.12}$$

The mode functions for a Klein-Gordon field are scalar functions on Minkowski spacetime. When acting on such functions, the Lorentz generators H' and $J_{z'}$ are represented by the following differential operators:

$$H' = i\frac{\partial}{\partial t'} \qquad J_{z'} = x' P_{y'} - y' P_{x'} = i\left(x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'}\right) \tag{5.13}$$

so that H_{RF} is represented by

$$H_{\mathsf{RF}} = \frac{\omega}{\rho}\left[i\frac{\partial}{\partial t'} + \frac{\rho^2}{\omega} i\left(x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'}\right)\right] \tag{5.14}$$

A note on conventions is in order here. The factor of i added to the differential operators above was mandated by the definitions we made about generators of transformations

acting on the phase space of particles back in section 1.3 and used to derive particles from quantum field theoretical formalism in section 2.3. However, you might react to the sign convention of $J_{z'}$. In most textbooks you will find both it and most other operators involving a spatial derivative defined with a factor of $-i$ rather than $+i$ in front. The reason for this is the remark I made in section 3.2 about the momentum operators as they were listed in table 3.1. The eigenvalues of the angular momentum operators J_i are actually pseudo*covector* quantities. However, the physical angular momentum quantities that we usually use are pseudo*vector* quantities, and with the signature that we use for the metric, these quantities have opposite signs. Therefore, in order that the eigenvalues of the angular momentum operators should be the pseudovectors that we are used to work with and not their negatives, I will use the operators $J^i = -J_i$ instead of J_i. The placement of indices on various quantities may in general become a bit willy-nilly because of this, but as long as the signs work out correctly, it does not really matter. H_{RF} therefore becomes $H_{\mathsf{RF}} = \frac{\omega}{\rho}\left(H' - \frac{\rho^2}{\omega}J^{z'}\right)$

Before we go on, note also that the operators H' and $J^{z'}$ generate time translations and rotations respectively only for purely scalar functions, i.e. mode functions for fields that have no kinds of vector, spinor, gauge or other kinds of indices attached to them. Although I will not pursue this line further, we could easily generalize to fields with arbitrary indices by writing the operators in the following form:

$$H' = i\left(\frac{\partial}{\partial t'} + \mathsf{H}^{IJ}\right) \qquad J^{z'} = -i\left(x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'} + \left(\mathsf{J}^{z'}\right)^{IJ}\right) \tag{5.15}$$

where H^{ij} and $\left(\mathsf{J}^{z'}\right)^{IJ}$ are matrices that act to generate time translations and spatial rotations on the indices of the field. We could then calculate the (indexed) eigenfunctions of these operators instead of just scalar ones.

One final note before we get to work: It may seem a bit odd that I represented H' by $i\frac{\partial}{\partial t}$. After all, I stated in section 2.3 that the operators that act on the quantum field should be viewed as operators defined and acting on functions on a specific equal-time spacelike hypersurface. The eigenfunctions of the observable/operator used to define particle modes were to be viewed as functions on such an equal-time hypersurface, with their time dependence given afterwards by demanding that they satisfy the field equation. Strictly speaking, the Hamiltonian H' should therefore really be written not as $i\frac{\partial}{\partial t'}$, but as the *spatial* differential operator that according to the field equation would have the same effect on solutions of the field equation as $i\frac{\partial}{\partial t'}$. However, this obviously has to be equivalent to the following procedure: Let the mode functions be functions of both space and time to begin with. Demand that they be eigenfunctions of whatever operators they are supposed to be eigenfunctions of, and allow yourself to write these operators in whatever form you care to. Since this procedure originally only determined the behaviour of the eigenfunctions on a spacelike hypersurface, it will not make the eigenfunctions completely determined if they are viewed as functions on all of spacetime. Therefore, it should be possible to demand that they satisfy the field equation, and this will completely determine their behaviour on all of spacetime. This will be a much more convenient way of finding the eigenfunctions, so I keep H' in the form of $i\frac{\partial}{\partial t'}$ rather than writing it as a function of spatial differential operators.

After all these notes, it's time to get down to business and calculate the eigenfunctions. For this purpose, we switch to cylindrical coordinates $(t'; r, \theta, z')$ as we did in section 4.3. $J^{z'}$ then simply has the form

$$J^{z'} = -i\frac{\partial}{\partial\theta} \tag{5.16}$$

As noted in the previous section, we want the eigenfunctions f of H_{RF} to be eigenfunctions of both H' and $J^{z'}$:

$$i\frac{\partial}{\partial t'}f_{Em} = Ef_{Em} \qquad\qquad -i\frac{\partial}{\partial\theta}f_{Em} = mf_{Em} \tag{5.17}$$

where E and m are the eigenvalues of H' and $J^{z'}$ respectively and we index the eigenfunction with these eigenvalues. Note that these equations say nothing about how f_{Em} depends on z'. Since H' and $J^{z'}$ both commute with $P^{z'} = -i\frac{\partial}{\partial z'}$, we can demand that f be an eigenfunction of it too:

$$-i\frac{\partial}{\partial z'}f_{Emk_{z'}} = k_{z'}f_{Emk_{z'}} \tag{5.18}$$

where $k_{z'}$ is the eigenvalue of $P^{z'}$. [1]

Satisfying these very simple equations is easy. The solution is:

$$f_{Emk_{z'}}(t', r, \theta, z') = F_{Emk_{z'}}(r)e^{-iEt'+im\theta+ik_{z'}z'} \tag{5.19}$$

for some function $F_{Emk_{z'}}(r)$ of r. The energy $\epsilon_{Emk_{z'}}$ of this mode as defined in the stationary reference frame is equal to its eigenvalue under H_{RF}:

$$\begin{aligned} H_{\mathsf{RF}}f_{Emk_{z'}} &= i\frac{\omega}{\rho}\left(\frac{\partial}{\partial t'} + \frac{\rho^2}{\omega}\frac{\partial}{\partial\theta}\right)f_{Emk_{z'}}(t', r, \theta, z') \\ &= \frac{\omega}{\rho}\left(E - \frac{\rho^2}{\omega}m\right)f_{Emk_{z'}}(t', r, \theta, z') \end{aligned} \tag{5.20}$$

so that $\epsilon_{Emk_{z'}} = \frac{\omega}{\rho}\left(E - \frac{\rho^2}{\omega}m\right)$. Note the time dilatation factor and the shift in the energy that we have commented on earlier.

Before the eigenfunction is fully determined though, we need to specify the function $F_{Emk_{z'}}(r)$. This is done by demanding that $f_{Emk_{z'}}$ satisfy the field equation. The field equation is found from the Lagrangian density, which in the case of a Klein-Gordon field with mass M (which may be zero) has the following form:

$$\mathcal{L}_{KG} = \frac{1}{2}g^{\mu\nu}\,\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}M^2\phi^2 \tag{5.21}$$

The covariant derivative ∇_μ used on scalar fields like this is of course just equal to the partial derivative, ∂_μ. However, the Euler-Lagrange field equation 2.2 involves taking the

[1] $k_{z'}$ is a vector quantity, and should strictly speaking be denoted by $k^{z'}$. However, since it will make the notation simpler later on, I will keep the index downstairs.

covariant derivative of a covariant derivative, i.e. taking the covariant derivative of a non-scalar quantity, so this is not equal to the plain partial derivative. Rather, the field equation for the Klein-Gordon field, the Klein-Gordon equation, is:

$$\nabla_\mu \left(g^{\mu\nu} \nabla_\nu \phi \right) + M^2 \phi = g^{\mu\nu} \left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} - \Gamma^\alpha_{\ \mu\nu} \right) \phi + M^2 \phi = 0 \tag{5.22}$$

where $\Gamma^\alpha_{\ \mu\nu}$ is a connection symbol for the tensor basis chosen, which in a coordinate basis is given by:

$$\Gamma^\alpha_{\ \mu\nu} = \frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda} \right) \tag{5.23}$$

In cylindrical coordinates, the metric is given by:

$$\begin{aligned} & g_{\mu\nu} dx^\mu dx^\nu = dt'^2 - dr^2 - r^2 d\theta^2 - dz'^2 \\ \Longrightarrow \qquad & g_{t't'} = 1 \qquad g_{rr} = -1 \qquad g_{\theta\theta} = -r^2 \qquad g_{z'z'} = -1 \\ \Longrightarrow \qquad & g^{t't'} = 1 \qquad g^{rr} = -1 \qquad g^{\theta\theta} = -\frac{1}{r^2} \qquad g^{z'z'} = -1 \end{aligned} \tag{5.24}$$

According to eq. 5.23, the non-vanishing Christoffel symbols are then:

$$\Gamma^r_{\ \theta\theta} = -r \qquad\qquad \Gamma^\theta_{\ r\theta} = \Gamma^\theta_{\ \theta r} = -\frac{1}{r} \tag{5.25}$$

The field equation then becomes:

$$\left[\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\partial^2}{\partial z'^2} + M^2 \right] \phi = 0 \tag{5.26}$$

If we insert into this $f_{Emk_{z'}}(t', r, \theta, z') = F_{Emk_{z'}}(r) e^{-iEt' + im_i \theta_i k_{z'} z'}$, we get the following equation for $F_{Emk_{z'}}(r)$:

$$-\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m_i^2}{r^2} + E^2 - k_{z'}^2 - M^2 \right] F_{Emk_{z'}}(r) = 0 \tag{5.27}$$

People a working knowledge of the semi-finer points of ordinary second-order differential equation should recognize this as a Bessel equation. If we demand that the function be well-defined everywhere and do not diverge as $r \to 0$, then the solution must be

$$F_{Emk_{z'}}(r) = A_{Emk_{z'}} J_m(Q_{Ek_{z'}} r) \qquad Q_{Ek'_z} = \sqrt{E^2 - k_{z'}^2 - M^2} \tag{5.28}$$

where J_m is a Bessel function of the first kind and order m, and $A_{Emk_{z'}}$ is some normalization constant that we do not really need to determine, at least not yet. Hence, the complete eigenfunctions of H_{RF} are the following:

$$f_{Emk_{z'}}(t', r, \theta, z') = A_{Emk_{z'}} e^{-iEt' + im\theta + ik_{z'} z'} J_m(Q_{Ek_{z'}} r) \tag{5.29}$$

The question now is what values are allowed for the parameters E, $k_{z'}$ and m? Note that the Bessel function can be written as a series $J_m(x) = x^m \sum_{n=0}^{\infty} c_{2n} x^{2n}$, where the coefficients c_k obey the following recursion relation:

$$c_k = \frac{-1}{(k+m)^2 - m^2} c_{k-2} \tag{5.30}$$

From this relation we see that the coefficients c_k asymptotically behave approximately as the coefficients of a sine or cosine function as $k \to \infty$. Actually the difference will cause the coefficients to diminish more rapidly, and ensure that $J_m(x) \to 0$ as $x \to \infty$. This means that if $Q_{Emk_{z'}}$ is real, $f_{Emk_{z'}}$ will converge towards zero as $r \to \infty$, so that it is normalizable and acceptable as a wave function. If on the other hand $Q_{Emk_{z'}}$ is imaginary, then $J_m(Q_{Emk_{z'}} r)$ will behave more like a hyperbolic sine or cosine, which means that $f_{Emk_{z'}}$ will diverge rapidly to infinity as $r \to \infty$. This is not acceptable for a wave function, so we must demand that $Q_{Emk_{z'}}$ be real. For E this gives $E^2 > k_{z'}^2 + M^2$, as one might have expected. As long as they satisfy this constraint though, E and $k_{z'}$ may freely take any value from $-\infty$ to $+\infty$ (although $|E|$ can of course never be smaller than M). Note also that since θ is a periodical coordinate with period 2π, we must have $f_{Emk_{z'}}(t', r, \theta, z') = f_{Emk_{z'}}(t', r, \theta + 2\pi, z')$, which implies that m must be an integer number.

With these specifications, we have defined all the eigenfunctions of H_{RF} and their corresponding eigenvalues. The next thing we need to find out, is which of them have positive and which have negative norm. According to eq. 1.33, the inner product between two solutions $f_{Emk_{z'}}$ and $f_{E'm'k'_{z'}}$ of the Klein-Gordon equation is:

$$\langle f_{Emkz'}, f_{E'm'k'_{z'}} \rangle = i \int \left(f^*_{Emk_{z'}} \frac{\partial f_{E'm'k'_{z'}}}{\partial t} - \frac{\partial f^*_{Emkz'}}{\partial t} f_{E'm'k'_{z'}} \right) d^3x \tag{5.31}$$

To keep the following calculations from sprawling across too many lines, I will drop the subscripts on the eigenfunctions $f_{Emk_{z'}}$ and $f_{E'm'k'_{z'}}$ and on the parameters $Q_{Emk_{z'}}$ and $Q_{E'm'k'_{z'}}$ and just denote them by f, f', Q and Q' instead. The same thing applies to the normalization constants $A_{Emk_{z'}}$ of eq. 5.29. Using cylindrical coordinates, the inner product integral becomes:

$$\begin{aligned} &A^* A' \,(E' + E)\, e^{-i(E'-E)t'} \int_{-\infty}^{\infty} dz'\, e^{i(k'_{z'} - k_{z'})z'} \int_0^{2\pi} d\theta\, e^{i(m'-m)\theta} \int_0^{\infty} dr\, r J_m(Qr)^* J_{m'}(Q'r) \\ &= A^* A' (E' + E)\, e^{-i(E'-E)t'}\, 2\pi\, \delta(k'_{z'} - k_{z'})\, 2\pi\, \delta_{m',m} \int_0^{\infty} dr\, r J_m(Qr)^* J_{m'}(Q'r) \end{aligned} \tag{5.32}$$

The last integral is a bit tricky to calculate, so the calculation is left to appendix A. The result is:

$$\int_0^{\infty} dr\, r J_m(Qr) J_{m'}(Q'r) = \frac{\delta(Q - Q')}{\pi^2 Q} \tag{5.33}$$

This means that we only get a non-vanishing inner product if $Q = Q'$, $k_{z'} = k'_{z'}$ and $m = m'$. Since the mass M is constant, these three equalities together imply $E = E'$, as one would expect. The inner product then finally becomes:

$$\begin{aligned}\langle f_{Emk_{z'}}, f_{E'm'k'_{z'}} \rangle &= A^*_{Emkz'} A_{E'm'k'_{z'}} \frac{8E}{Q_{Emk_{z'}}} \delta_{m,m'} \delta(k_{z'} - k'_{z'}) \delta(Q_{Emk_{z'}} - Q_{E'm'k'_{z'}}) \\ &= |A_{Emk_{z'}}|^2 \frac{8E}{|E|} \delta_{m,m'} \delta(k_{z'} - k'_{z'}) \delta(E - E')\end{aligned} \tag{5.34}$$

From this we see that the norm of an eigenfunction f_i of course diverges, but this is just due to the fact that we are dealing with a free particle in an infinite volume, so that it must be normalized to a product of δ-functions rather than a finite number. What is important to note now, is that $f_{Emk_{z'}}$ has positive norm if E is positive, and negative norm if E is negative. This is what we would expect; a free particle eigenfunction with positive norm has positive energy, an eigenfunction with negative norm has negative energy. But this is if we look at the energy E defined in the primed *inertial* frame. For the observer him/herself on the rotating trajectory the story is quite different, since proper time vector of the observer is mixed up with the vectors generating rotation around the z'-axis and points in a different direction than the unit t'-vector. The energy $\epsilon_{Emk_{z'}}$ that the observer sees, is not E but given by eq. 5.20, namely $\epsilon_{Emk_{z'}} = \frac{\omega}{\rho}\left(E - \frac{\rho^2}{\omega} m\right)$. m may take *any* integer value, but the eigenfunctions have positive norm whenever E is positive, regardless of the value of m.

Since m can be arbitrarily large, independent of the value of E, this means that particle states with sufficiently large values of m will have negative energy $\epsilon_{Emkz'}$ as measured by the observer in the stationary reference frame. But since the eigenvalue of both H', $J_{z'}$ and hence H_{RF} are zero in the vacuum state, the observer will observe the existence of particle states with a *lower* energy than the vacuum. As long as we have a free or isolated quantum field that does not interact with anything, this is OK, but if we couple the field to some sort of detector, the vacuum becomes unstable. We will look further into how detectors respond to the different vacua in the next two chapters.

As we suggested in the previous section, the eigenfunctions $f_{Emk_{z'}}$ calculated here are not the set of plane wave eigenfunctions that are usually used to describe the particle states for a Klein-Gordon field in inertial reference frames. They are therefore not necessarily eigenfunctions with respect to any arbitrary inertial Hamiltonian defined in another inertial frame than H'. One may therefore legitimately ask whether the particle states and the vacuum defined by these eigenfunctions really correspond to the kind of particles that we are used to observing in Minkowski spacetime. Since they are eigenfunctions of H', we know that they are *a* set of particle states and *a* vacuum of the inertial Hamiltonian H', but will other inertial Hamiltonians agree on this? Equivalently, is the total number operator of H' and H_{RF} with respect to this set of eigenfunctions equal to the "usual" common total number operator of all inertial Hamiltonians? When we use the definite-energy, definite-momentum plane wave eigenfunctions $f_{\mathbf{k}'}(t', \mathbf{x}') = \frac{1}{\sqrt{(2\pi)^3 2E_{\mathbf{k}'}}} e^{-iE_{\mathbf{k}'}t' + i\mathbf{k}'\cdot\mathbf{x}'}$ (where $E_{\mathbf{k}'} = \pm\sqrt{\mathbf{k}'^2 + M^2}$, with the plus sign for positive norm functions and the minus sign for nega-

tive norm functions), we know that this is always the case. Unlike the eigenfunctions $f_{Emk_{z'}}$ we have used here, these are known to be eigenfunctions of *all* inertial Hamiltonians.

According to the discussion in chapter 3, the total number operator with respect to the eigenfunctions $f_{Emk_{z'}}$ will be the same as the normal total number operator for inertial Hamiltonians if all the Bogoliubov β-coefficients $\beta(E, m, k_{z'}; \mathbf{k}')$ between the $f_{Emk_{z'}}$ and the plane wave eigenfunctions $f_{\mathbf{k}'}$ are all. Let us now calculate these (in the following calculations I drop the subscripts on $Q_{Emk_{z'}}$ since there are no other Qs around for it to be confused with):

$$\begin{aligned}\beta(E, m, k_{z'}; \mathbf{k}') &= -\langle f^*_{\mathbf{k}'}, f_{Emk_{z'}} \rangle \\ &\sim (E_{\mathbf{k}'} - E)\, e^{-i(E_{\mathbf{k}'}+E)t'} \int e^{i\mathbf{k}'\cdot\mathbf{x}'} e^{ik_{z'}z + im\theta} J_m(Qr) d^3x'\end{aligned} \tag{5.35}$$

where $f_{Emk_{z'}}$ and $f_{\mathbf{k}'}$ are both positive norm functions, so that E and $E_{\mathbf{k}'}$ are both positive. Let us now express the whole integral in cylindrical coordinates. Since the system is cylindrically symmetric around the z'-axis, we can choose the projection of $\mathbf{k}'$ onto the xy-plane to point in any direction we please. Choose it to point along the y'-axis. If we then denote the length of this projection by $k'_{xy} = \sqrt{k'^2_{x'} + k'^2_{y'}}$, $\beta(E, m, k_{z'}; \mathbf{k}')$ becomes:

$$(E_{\mathbf{k}'} - E)\, e^{-i(E_{\mathbf{k}'}+E)t'} \int_{-\infty}^{\infty} dz\, e^{i(k'_{z'}+k'_{z'})z'} \int_0^{\infty} dr\, r \int_0^{2\pi} d\theta\, e^{-im\theta - ik'_{xy} r \sin\theta} J_m(Qr) \tag{5.36}$$

The integral over θ is in fact an integral representation of a Bessel function (see [6]):

$$J_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-im\theta + ix\sin\theta} d\theta = \frac{1}{2\pi}(-1)^m \int_0^{2\pi} e^{-im\theta - ix\sin\theta} d\theta \tag{5.37}$$

(where in the last step we simply substituted $\theta \to \theta + \pi$) The integral 5.36 then becomes:

$$\begin{aligned}&(-1)^m \frac{1}{2\pi} (E_{\mathbf{k}'} - E)\, e^{-i(E'_{\mathbf{k}}+E)t'} \delta(k'_{z'} + k_{z'}) \int_0^{\infty} dr\, r J_m(k'_{xy} r) J_m(Qr) \\ &= (-1)^m \frac{1}{2\pi} (E_{\mathbf{k}'} - E)\, e^{-i(E_{\mathbf{k}'}+E)t'} \delta(k'_{z'} + k_{z'}) \frac{1}{\pi^2 Q} \delta(Q - k'_{xy})\end{aligned} \tag{5.38}$$

Hence $\beta(E, m, k_{z'}; \mathbf{k}') = 0$ unless $k'_{z'} = k_{z'}$ and $Q = k'_{xy} = \sqrt{k'^2_{x'} + k'^2_{y'}}$. But since $E = \sqrt{Q^2 + k^2_{z'} + M^2}$ and $E_{\mathbf{k}'} = \sqrt{k'^2_{x'} + k'^2_{y'} + k'^2_{z'} + M^2}$, this means that we must have $E = E_{\mathbf{k}'}$. The factor $E_{\mathbf{k}'} - E$ is then equal to zero, and $\beta(E, m, k_{z'}; \mathbf{k}')$ vanishes. All the Bogoliubov β-coefficients therefore vanish, and the number operators for the two sets of eigenfunctions are identical. In other words, particle states defined relative to H_{RF} are in fact the same thing as particle states defined in an inertial frame.

5.3 Eigenfunctions when a $\perp \boldsymbol{\omega}$, $a > \omega$

Now we will calculate what the particle eigenfunctions of H_{RF} look like when $a > \omega$. In this case the observer is no longer following a circular trajectory seen from any reference frame. Instead, we saw in section 4.4 that it is possible to find a frame with respect to which the observer seems to be accelerating with constant proper acceleration in one direction while moving at a constant proper speed in a direction perpendicular to the direction of acceleration. In this frame, H_{RF} had the form (eq. 5.7):

$$H_{\mathsf{RF}} = \sigma \left(B_{x'} - \frac{\omega}{\sigma^2} P_{y'} \right) \tag{5.39}$$

In order that the eigenvalue of the momentum operator correspond to the a vector momentum rather than a covector momentum with opposite sign, I will replace $P_{y'}$ by $P^{y'} = -P_{y'}$ in this expression. With this convention, we get the following differential operators when $B_{x'}$ and $P^{y'}$ act on scalar functions:

$$B_{x'} = i \left(x' \frac{\partial}{\partial t'} + t' \frac{\partial}{\partial x'} \right) \qquad\qquad P^{y'} = -i \frac{\partial}{\partial y'} \tag{5.40}$$

If we use the hyperbolic cylindrical coordinates $(\chi; \xi, y', z')$ that we introduced in section 4.4, then $B^{x'}$ takes the simple form:

$$B_{x'} = i \frac{\partial}{\partial \chi} \tag{5.41}$$

H_{RF} as a differential operator acting on scalar functions is therefore equal to

$$H_{\mathsf{RF}} = \sigma \left(B_{x'} + \frac{\omega}{\sigma^2} P^{y'} \right) = \sigma i \left(\frac{\partial}{\partial \chi} - \frac{\omega}{\sigma^2} \frac{\partial}{\partial y'} \right) \tag{5.42}$$

We can now start calculating the eigenfunctions of H_{RF} in the case of a scalar Klein-Gordon field. The calculations follow the same lines as in the previous section. We want the eigenfunctions of H_{RF} to be simultaneous eigenfunctions of $B_{x'}$ and $P^{y'}$. Since $P^{z'}$ commutes with both $B_{x'}$ and $P^{y'}$, we can let the eigenfunctions be eigenfunctions of $P^{z'}$ as well, just as for $a < \omega$. We then have the following eigenvalue equations

$$i \frac{\partial}{\partial \chi} g_{k_{y'} k_{z'} \Omega} = \Omega g_{k_{y'} k_{z'} \Omega} \qquad -i \frac{\partial}{\partial y'} g_{k_{y'} k_{z'} \Omega} = k_{y'} g_{k_{y'} k_{z'} \Omega} \qquad -i \frac{\partial}{\partial z'} g_{k_{y'} k_{z'} \Omega} = k_{z'} g_{k_{y'} k_{z'} \Omega} \tag{5.43}$$

where Ω, $k_{y'}$ and $k_{z'}$ are the eigenvalues of $B_{x'}$, $P^{y'}$ and $P^{z'}$ respectively. The solution is easy to find:

$$g_{k_{y'} k_{z'} \Omega}(\chi, \xi, y', z') = G_{k_{y'} k_{z'} \Omega}(\xi) e^{-i\Omega\chi + i k_{y'} y' + i k_{z'} z'} \tag{5.44}$$

To find the function $G_{k_{y'} k_{z'} \Omega}(\xi)$ we need to find the Klein-Gordon equation in hyperbolic cylindrical coordinates. The metric in these coordinates is:

$$
\begin{aligned}
& g_{\mu\nu}dx^\mu dx^\nu = \xi^2 d\chi^2 - d\xi^2 - dy'^2 - dz'^2 \\
\Longrightarrow \qquad & g_{\chi\chi} = \xi^2 \qquad g_{\xi\xi} = -1 \qquad g_{y'y'} = -1 \qquad g_{z'z'} = -1 \\
\Longrightarrow \qquad & g^{\chi\chi} = \frac{1}{\xi^2} \qquad g^{\xi\xi} = -1 \qquad g^{y'y'} = -1 \qquad g^{z'z'} - 1
\end{aligned}
\tag{5.45}
$$

Eq. 5.23 then gives us the following non-vanishing Christoffel symbols:

$$
\Gamma^{\xi}_{\chi\chi} = \xi \qquad \Gamma^{\chi}_{\xi\chi} = \Gamma^{\chi}_{\chi\xi} = \frac{1}{\xi}
\tag{5.46}
$$

The field equation eq. 5.22 is then:

$$
\left[\frac{1}{\xi^2}\frac{\partial^2}{\partial\chi^2} - \frac{\partial^2}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial}{\partial\xi} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + M^2\right]\phi(\chi,\xi,y,z) = 0
\tag{5.47}
$$

Inserting $g_{k_{y'}k_{z'}\Omega}(\chi,\xi,y',z') = G_{k_{y'}k_{z'}\Omega}(\xi)e^{-i\Omega\chi+ik_{y'}y'+ik_{z'}z'}$ gives:

$$
\begin{aligned}
& -\left[\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\Omega^2}{\xi^2} - k_{y'}^2 - k_{z'}^2 - M^2\right] G_{k_{y'}k_{z'}\Omega}(\xi) = 0 \\
\Longrightarrow \qquad & \left[\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\Omega^2}{\xi^2} - \Xi^2_{k_{y'}k_{z'}}\right] G_{k_{y'}k_{z'}\Omega}(\xi) = 0
\end{aligned}
\tag{5.48}
$$

where we have defined $\Xi_{k_{y'}k_{z'}} = \sqrt{k_{y'}^2 + k_{z'}^2 + M^2}$. If we demand that the solution do not diverge to infinity in the limit $\xi \to \infty$, the solution to this equation is a so-called *modified* Bessel function, or a Bessel function the second kind of the second kind, of imaginary order $i\Omega$, denoted $K_{i\Omega}(\Xi_{k_{y'}k_{z'}}\,\xi)$ (see [16] for a definition). However, $K_{i\Omega}(\Xi_{k_{y'}k_{z'}})$ does *not* necessarily behave well when $\xi \to -\infty$. For $\xi < 0$ it may take complex values, and though its real part does not diverge when $\xi \to -\infty$, its imaginary part may do so. I will comment more on this shortly, but at least for $\xi > 0$, the eigenfunctions $g_{k_{y'}k_{z'}\Omega}$ can be written as

$$
g_{k_{y'}k_{z'}\Omega}(\chi,\xi,y,z) = A_{k_{y'}k_{z'}}e^{-i\Omega\chi+ik_{y'}y'+ik_{z'}z'}K_{i\Omega}(\Xi_{k_{y'}k_{z'}}\,\xi)
\tag{5.49}
$$

In this case, we have no conditions on the parameters Ω, $k_{y'}$ and $k_{z'}$ so they can all take any value from $-\infty$ to ∞, independently of one another.

Equation 5.48 is symmetric under the interchange $\xi \to -\xi$. Therefore, the functions $G_{k_{y'}k_{z'}\Omega}(-\xi)$ must form a complete set of eigenfunction solutions when $\xi < 0$. We could therefore simply extend the eigenfunctions $g_{k_{y'}k_{z'}\Omega}(\chi,\xi,y,z)$ into the region $\xi < 0$ by defining them to be $A_{k_{y'}k_{z'}}e^{-i\Omega\chi+ik_{y'}y'+ik_{z'}z'}K_{i\Omega}(-\Xi_{k_{y'}k_{z'}}\,\xi)$ for $\xi < 0$. But this turns out not to be very appropriate if we wish to use these functions as eigenfunctions, since they will actually have norm 0 if defined in this way. Note however that $K_{i\Omega}(\Xi_{k_{y'}k_{z'}}\xi)$ is not well-behaved as $\xi \to 0$. It is bounded, but oscillates wildly. This is to be expected, since the field equation, the coordinates and the metric in these coordinates are all singular at the point $\xi = 0$. Since the field equation and the solutions are singular at $\xi = 0$, we do not have to follow the prescription above to define the eigenfunctions for $\xi < 0$. We could in fact choose solutions completely independently of one another on either side of $\xi = 0$; If

we have specified a solution on any open neighbourhood on one side of $\xi = 0$, the field equation tells us what the solution has to look like on all of that side. But since the field equation breaks down at $\xi = 0$, it cannot tell us what the solution has to be on the other side, so we can choose it freely from all possible solutions. This corresponds to the fact that for an observer following the path generated by H_{RF} there is a horizon at $\xi = 0$. The two regions on either side of the horizon are not causally connected, and an observer observing the field on his or her side cannot say anything about what the field looks like on the other side.

Of course, if we choose eigenfunctions $g_{k_{y'}k_{z'}\Omega}$ with different eigenvalues Ω, $k_{y'}$ and $k_{z'}$ on either side of the horizon, then the field will not have the same values for these observables on both sides. However, since an observer on one side of the horizon cannot know anything about the values of these observables on the other side, there is no apparent reason why they should have to be the same as far as the observer is concerned. He or she will not be able to tell the difference anyway.

On the other hand, if we choose somewhat less singular coordinates and allow *complex* values for them, then the solutions need to be analytic at every point except for an isolated singularity that will persist at the origin (they must be analytic in order to be differentiable with respect to a complex coordinate, and if they are not differentiable, they can obviously not be solutions of a differential equation). Since we can circumvent the origin in a complex coordinate-plane, the choice of field configuration on one side of the horizon *will* uniquely define the field on the other side if allow this sort of analytic extension of $g_{k_{y'}k_{z'}}$. We will see in the next chapter that this has great significance for the behaviour of the field across the horizon and for the interpretation of the relationship between particles defined in the stationary accelerated reference frame and in an inertial frame. For now though we will treat the field separately on each side of the horizon, by demanding that our eigenfunctions have support on only one side of the horizon and are identically zero on the other. We make the following definition:

$$
\begin{aligned}
{}^{R}g_{k_{y'}k_{z'}\Omega}(\chi,\xi,y',z') &= \begin{cases} A_{k_{y'}k_{z'}} e^{-i\Omega\chi + ik_{y'}y' + ik_{z'}z'} K_{i\Omega}(\Xi_{k_{y'}k_{z'}}\,\xi) & \xi > 0 \\ 0 & \xi < 0 \end{cases} \\
{}^{L}g_{k_{y'}k_{z'}\Omega}(\tau,\xi,y,z) &= \begin{cases} 0 & \xi > 0 \\ A_{k_{y'}k_{z'}} e^{-i\Omega\chi + ik_{y'}y' + ik_{z'}z'} K_{i\Omega}(-\Xi_{k_{y'}k_{z'}}\,\xi) & \xi < 0 \end{cases}
\end{aligned}
\tag{5.50}
$$

Let us calculate the norm of these functions to find out which ones we should use as the positive norm eigenfunctions. Before we start however, I would like to warn that the rest of this section involves some rather hairy integrals that have been calculated in a direct but equally hairy way. I have also not bothered to calculate any normalization constants that are not needed. A more complete calculation of the normalization constants of the modified Bessel functions, as well as those of the Bogoliubov coefficients to be calculated later in this chapter, can be found in chapter 2 of the paper [9] by S. Takagi. The method of calculation there is perhaps a bit more elegant, but more roundabout.

Now let's get down to work. First we need to calculate what the surface-form $d\Sigma_\mu$ and the canonical momentum in the inner product 2.15 looks like in hyperbolic cylindrical coordinates. We choose for our surface of integration a surface with constant χ, and choose

$\chi = 0$. On this surface we have $dx' = d\xi$ and $dt' = \xi d\chi$, so that the Minkowski space volume form $dV = dt' \wedge dx' \wedge dy' \wedge dz'$ in hyperbolic cylindrical coordinates on this surface becomes $dV = \xi d\chi \wedge d\xi \wedge dy' \wedge dz'$. The surface form $d\Sigma$ is then

$$d\Sigma = \xi d\xi \wedge dy' \wedge dz' \tag{5.51}$$

The canonical momentum in these coordinates is $\Pi = \frac{\partial \mathcal{L}}{\partial \phi_{,\chi}}$. From the form 5.21 of the Klein-Gordon Lagrangian, we get that

$$\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} = g^{\mu\nu}\phi_{,\nu} \qquad \Longrightarrow \qquad \Pi = \frac{\partial \mathcal{L}}{\partial \phi_{,\chi}} = \frac{1}{\xi^2}\frac{\partial \phi}{\partial \chi} \tag{5.52}$$

The inner product between two eigenfunctions ${}^R g_{k_{y'} k_{z'}} \Omega$ and ${}^L g_{k'_{y'} k'_{z'} \Omega'}$ on the right side of the horizon then becomes (once again I drop the subscripts on the g's, Ξ's and A's and just distinguish them by primes in order to abbreviate the notation):

$$\begin{aligned}
\langle {}^R g, {}^R g' \rangle &= i \int \frac{1}{\xi} \left({}^R g^* \frac{\partial}{\partial \chi} {}^R g' - (\frac{\partial}{\partial \chi} {}^R g^*) {}^R g' \right) d\xi \, dy' \, dz' \\
&= A^* A' \left(\Omega' + \Omega\right) e^{-i(\Omega' - \Omega)\chi} \int_{-\infty}^{\infty} dz' \, e^{i(k'_{z'} - k_{z'})} \int_{-\infty}^{\infty} dy' \, e^{i(k'_{y'} - k_{y'})} \\
&\qquad \int_0^{\infty} d\xi \, \frac{1}{\xi} K_{i\Omega}(\Xi\xi) K_{i\Omega'}(\Xi'\xi) \\
&= A^* A' \left(\Omega' + \Omega\right) e^{i(\Omega' - \Omega)\chi} 4\pi^2 \delta(k'_{z'} - k_{z'}) \delta(k'_{y'} - k_{y'}) \int_0^{\infty} d\xi \, \frac{1}{\xi} K_{i\Omega}(\Xi\xi) K_{i\Omega'}(\Xi'\xi) \\
&\sim \delta(k'_{z'} - k_{z'}) \delta(k'_{y'} - k_{y'}) \delta(\Omega - \Omega')
\end{aligned} \tag{5.53}$$

The last transition was calculated in section A.2 of appendix A. So we see that modes with different eigenvalues are orthogonal. I did not calculate the proportionality constant here, but when the inner product is non-vanishing, we must have $\Omega = \Omega'$. We then have $A^* A' = |A|^2 > 0$ and $\Omega' + \Omega = 2\Omega$. Moreover, the integrand of the ξ-integral is genuinely positive, so that the sign of the norm will be the same as the sign of Ω. So we see that, as we might have expected, that modes with a positive value for the "Rindler energy" Ω on the right side of the horizon have positive norm, whereas modes with negative Ω have negative norm (note: the name "Rindler energy" is just a convenient name for Ω, but Ω is not really an energy, it is in fact a dimensionless quantity). All particles defined with respect to H_{RF} will therefore have positive Ω-values on the right side of the horizon. The energy of a particle in the mode $g_{k_{y'} k_{z'} \Omega}$ observed by the observer according to H_{RF} is however not proportional to Ω, but to $\epsilon_{k_{y'} k_{z'} \Omega} = \sigma\left(\Omega + \frac{\omega}{\sigma^2} k_{y'}\right)$. $k_{y'}$ can have any value, including arbitrarily negative ones, so when $\omega \neq 0$, the observer may observe particles with negative energy. We will return to this point later.

First we note that if we take the inner product between two eigenfunctions ${}^L g_{k_{y'} k_{z'} \Omega}$ and ${}^L g_{k_{y'} k_{z'} \Omega}$ on the *left* side of the horizon, the integral becomes exactly the same as eq. 5.53, except that the integral over ξ now runs from $-\infty$ to 0 instead of from 0 to $+\infty$. We can turn this into an integral from 0 to $+\infty$ by substituting $\xi \to -\xi$ and reversing the direction

of integration, thereby introducing two minus signs that cancel. But the factor $\frac{1}{\xi}$ now reads $-\frac{1}{\xi}$ instead, so that we get precisely

$$\left\langle {}^{L}g_{k_{y'}k_{z'}\Omega}\,,\,{}^{L}g_{k'_{y'}k'_{z'}\Omega'}\right\rangle = -\left\langle {}^{R}g_{k_{y'}k_{z'}\Omega}\,,\,{}^{R}g_{k'_{y'}k'_{z'}\Omega'}\right\rangle \tag{5.54}$$

This means that positive norm modes on the left side of the horizon have *negative* values for Ω, whereas negative norm modes have *positive* values, exactly the opposite to the case on the right side of the horizon. Therefore, from now on I will denote all positive-norm eigenfunctions by $g_{k_{y'}k_{z'}\Omega}$, and whether they exist on the left or on the right side of the horizon will be determined by the sign of Ω, i.e.:

$$g_{k_{y'}k_{z'}\Omega}(\chi,\xi,y',z') = \begin{cases} A_{k_{y'}k_{z'}}e^{-i\Omega\chi+ik_{y'}y'+ik_{z'}z'}K_{i\Omega}(\Xi_{k_{y'}k_{z'}}\,\xi) & \Omega>0,\ \xi>0 \\ 0 & \Omega>0,\ \xi<0 \\ 0 & \Omega<0,\ \xi>0 \\ A_{k_{y'}k_{z'}}e^{-i\Omega\chi+ik_{y'}y'+ik_{z'}z'}K_{i\Omega}(-\Xi_{k_{y'}k_{z'}}\,\xi) & \Omega<0,\ \xi<0 \end{cases} \tag{5.55}$$

Since the particle spectrum of H_{RF} includes particle states with negative energy even when $\omega = 0$ and because we have a horizon and singular behaviour of the eigenfunctions at $\xi = 0$, we should not expect the number operator and the vacuum defined with respect to H_{RF} in this case to be equivalent to that defined in any inertial frame. We will now show that they are indeed inequivalent. To show this, we calculate the Bogoliubov β-coefficients for transforming from the plane wave eigenfunctions $f_{\mathbf{k}'}(t',\mathbf{x}') = \frac{1}{\sqrt{(2\pi)^3 2E_{\mathbf{k}'}}}e^{-iE_{\mathbf{k}'}t'+i\mathbf{k}'\cdot\mathbf{x}'}$ of an inertial Hamiltonian to the eigenfunctions $g_{k_{y'}k_{z'}\Omega}$ of H_{RF}. In calculating the inner product between $g_{k_{y'}k_{z'}\Omega}$ and $f_{\mathbf{k}'}$, we will use hyperbolic cylindrical coordinates and express $f_{\mathbf{k}'}(t',x',y',z')$ as a function of χ and ξ along with y' and z':

$$\beta(k_{y'},k_{z'},\Omega;\,\mathbf{k}') = -\langle f^*_{\mathbf{k}'},g_{k_{y'}k_{z'}\Omega}\rangle = -i\int_\Sigma \frac{1}{\xi}\left(f_{\mathbf{k}'}\frac{\partial g_{k_{y'}k_{z'}\Omega}}{\partial\chi} - g_{k_{y'}k_{z'}\Omega}\frac{\partial f_{\mathbf{k}'}}{\partial\chi}\right)d\xi\,dy'\,dz' \tag{5.56}$$

where Σ is a surface of constant χ. We have:

$$\frac{\partial f_{\mathbf{k}'}}{\partial\chi} = \left(\frac{\partial t'}{\partial\chi}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial\chi}\frac{\partial}{\partial x'}\right)f_{\mathbf{k}'} =$$

$$B_{\mathbf{k}'}\xi\,(-iE_{\mathbf{k}'}\cosh\chi + ik'_{x'}\sinh\chi)\,e^{-iE_{\mathbf{k}'}\xi\sinh\chi+ik'_{x'}\xi\cosh\chi-ik'_{y'}y'-ik'_{z'}z'} \tag{5.57}$$

(where we denote the normalization constant $\frac{1}{\sqrt{(2\pi)^3 2E_{\mathbf{k}'}}}$ of $f_{\mathbf{k}'}$ by $B_{\mathbf{k}'}$) Since the integral 5.56 does not depend on what constant-χ surface Σ we choose for evaluating it, we can choose χ to have any constant value we want. To simplify the calculations, we will choose $\chi = 0$. Inserting the expression 5.55 for $g_{k_{y'}k_{z'}\Omega}$ into 5.56 and setting $\chi = 0$, we get

$$\beta(k_{y'},k_{z'},\Omega;\,\mathbf{k}') = -A_{k_{y'}k_{z'}\Omega}B_{\mathbf{k}'}\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\int d\xi\frac{1}{\xi}\left(\Omega-\xi E_{\mathbf{k}'}\right)e^{i(k_{y'}+k'_{y'})y'+i(k_{z'}+k'_{z'})z'}$$
$$e^{ik'_{x'}\xi}K_{i\Omega}(\pm\Xi_{k_{y'}k_{z'}}\,\xi)$$
$$= -A_{k_{y'}k_{z'}\Omega}B_{\mathbf{k}'}4\pi^2\delta\big(k_{y'}+k'_{y'}\big)\delta(k_{z'}+k'_{z'})\int d\xi\,\left(\frac{\Omega}{\xi}-E_{\mathbf{k}'}\right)$$
$$e^{ik'_{x'}\xi}K_{i\Omega}(\pm\Xi_{k_{y'}k_{z'}}\,\xi) \tag{5.58}$$

The plus-sign is for $\Omega > 0$ (i.e. for functions defined on the right side of the horizon) and the minus-sign is for $\Omega < 0$ (functions defined on the left side of the horizon). For $\Omega > 0$, the ξ-integration runs from $\xi = 0$ to $\xi = \infty$, whereas for $\Omega < 0$ it runs from $\xi = -\infty$ to $\xi = 0$. To get the same limits for both cases we can reverse the order of integration and substitute $\xi \to -\xi$ when $\Omega < 0$. The term $\frac{\Omega}{\xi}$ will then switch sign, which means that it becomes $-\frac{\Omega}{\xi}$. The integral in eq. 5.58 then becomes:

$$\int_0^{\infty}d\xi\,\left(\pm\frac{\Omega}{\xi}-E_{\mathbf{k}'}\right)e^{\pm ik'_{x'}\xi}K_{i\Omega}(\Xi_{k_{y'}k_{z'}}\,\xi) \tag{5.59}$$

This last integral is solved in appendix A. The final result is (eq. A.22):

$$\beta(k_{y'},k_{z'},\Omega;\,\mathbf{k}') = \pm4\pi^2A_{k_{y'}k_{z'}\Omega}B_{\mathbf{k}'}\delta\big(k_{y'}+k'_{y'}\big)\delta(k_{z'}+k'_{z'})\frac{\pi\,e^{\mp\frac{\pi\Omega}{2}}}{\sinh(\pi\Omega)}\left(\frac{E_{\mathbf{k}'}-k'_{x'}}{E_{\mathbf{k}'}+k'_{x'}}\right)^{\frac{i\Omega}{2}}$$
$$= 4\pi^2A_{k_{y'}k_{z'}\Omega}B_{\mathbf{k}'}\delta\big(k_{y'}+k'_{y'}\big)\delta(k_{z'}+k'_{z'})\frac{\pi\,e^{-\frac{\pi|\Omega|}{2}}}{\sinh(\pi|\Omega|)}\left(\frac{E_{\mathbf{k}'}-k'_{x'}}{E_{\mathbf{k}'}+k'_{x'}}\right)^{\frac{i\Omega}{2}} \tag{5.60}$$

So $\beta\big(k_{y'}k_{z'}\Omega;\,E_{\mathbf{k}'}$ does not vanish. This means that the total number operator defined with respect to H_{RF} by the modes $g_{k_{y'}k_{z'}\Omega}$ is *not* the same as the usual number operator defined with respect to plane wave modes of an inertial Hamiltonian. What is meant by the number of particles in the field is thus not the same in the two cases, and in particular the vacuum defined in one case will seem to contain a superposition of different numbers of particles as defined in the other case. It therefore appears that an accelerated observer travelling through what an inertial observer would call a vacuum, will actually claim to observe particles in the supposed vacuum! This effect is what is usually referred to as the *Unruh effect*, and was first presented by W. G. Unruh in the paper [1] in 1976. We will see in the next chapter however, that this statement needs to be taken with a pinch of salt, as what an observer will actually see is by no means uniquely defined without making more assumptions.

In fact, the Unruh effect as presented here is simply an artifact of how we have chosen to define our eigenfunctions, namely as functions that have support only on one side of the horizon and vanish on the other side. This definition leaves room for a redefinition. In particular, there is a degeneracy between positive norm eigenfunctions on one side of the horizon and *negative* norm eigenfunctions with the same eigenvalues for Rindler energy

and momenta on the other side of the horizon. As stated in chapter 3, this allows us to redefine the eigenfunctions in a way that mixes creation and annihilation operators, so that the number operator of H_{RF} is actually not uniquely defined. It will turn out that by mixing modes on both sides of the horizon in this way, we can produce a set of eigenfunctions whose Bogoliubov β-coefficients relative to the inertial plane-wave modes do vanish, so that it is not at all clear whether an accelerated observer should observe any particles in the inertial vacuum state or not. Nevertheless, these non-inertially defined particles *might* be perceived as real. I will refer to particles defined using the functions $g_{k_{y'}k_{z'}\Omega}$ as *Rindler particles* and the functions themselves as *Rindler modes* due to the fact that Minkowski space seen from a hyperbolicly accelerating reference frame is often called Rindler space. The vacuum defined relative to these particles will be referred to as the *Rindler vacuum* (often also called the Fulling or Rindler-Fulling vacuum), as opposed to the vacuum defined by the inertial plane wave modes $f_{\mathbf{k}'}$, which I will refer to as the *Minkowski vacuum*.

I will now also calculate the Bogoliubov α-coefficients, since we will be needing them in the next chapter. We have:

$$\begin{aligned}\alpha(k_{y'}k_{z'}\Omega;\,\mathbf{k}') &= \langle f_{\mathbf{k}'}, g_{k_{y'}k_{z'}\Omega}\rangle = i\int \frac{1}{\xi}\left(f^*_{\mathbf{k}'}\frac{\partial g_{k_{y'}k_{z'}\Omega}}{\partial\chi} - g_{k_{y'}k_{z'}\Omega}\frac{\partial f^*_{\mathbf{k}'}}{\partial\chi}\right)d\xi\,dy'\,dz' \\ &= -A_{k_{y'}k_{z'}\Omega}B_{\mathbf{k}'}\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\int d\xi\frac{1}{\xi}\,(-\Omega-\xi E_{\mathbf{k}'}) \\ &\qquad e^{i(k_{y'}-k'_{y'})y'+i(k_{z'}-k'_{z'})z'}e^{-ik'_{x'}\xi}K_{i\Omega}(\pm\Xi_{k_{y'}k_{z'}}\,\xi) \qquad (5.61)\end{aligned}$$

This integral is almost the same as eq. 5.58, except that the terms involving $\mathbf{k}'$ in the exponents change sign, and all terms and factors involving Ω also change sign. The result is then found simply by making these substitutions in eq. 5.60:

$$\begin{aligned}\alpha(k_{y'},k_{z'},\Omega;\,\mathbf{k}') &= \pm 4\pi^2 A_{k_{y'}k_{z'}\Omega}B_{\mathbf{k}'}\delta\big(k_{y'}-k'_{y'}\big)\delta(k_{z'}-k'_{z'})\frac{\pi\,e^{\pm\frac{\pi\Omega}{2}}}{\sinh(\pi\Omega)}\left(\frac{E_{\mathbf{k}'}+k'_{x'}}{E_{\mathbf{k}'}-k'_{x'}}\right)^{-\frac{i\Omega}{2}} \\ &= 4\pi^2 A_{k_{y'}k_{z'}\Omega}B_{\mathbf{k}'}\delta\big(k_{y'}-k'_{y'}\big)\delta(k_{z'}-k'_{z'})\frac{\pi\,e^{\frac{\pi|\Omega|}{2}}}{\sinh(\pi|\Omega|)}\left(\frac{E_{\mathbf{k}'}-k'_{x'}}{E_{\mathbf{k}'}+k'_{x'}}\right)^{\frac{i\Omega}{2}} \\ &= e^{\pi|\Omega|}\beta(-k_{y'},-k_{z'},\Omega;\,\mathbf{k}')\end{aligned} \qquad (5.62)$$

Looking at these relations, there seems to be something amiss. The Bogoliubov coefficient $\beta(k_{y'},k_{z'},\Omega;\,\mathbf{k}')$ is apparently completely independent of any parameters of the observer's trajectory, since neither a nor ω appears anywhere in eq. 5.60. This means that it is non-zero even in the limit $a \to \omega$, and even $a \to 0$ regardless of ω. This seems to imply that the observer would see a full-blown spectrum of Rindler particles in the Minkowski vacuum even when the acceleration is close to zero, and the particle spectrum will certainly not deform continuously into the vacuum state as $a \to \omega$. The reason for this is once again that we defined the modes $g_{k_{y'}k_{z'}\Omega}$ to be common eigenfunctions for *all* stationary observers with $\mathbf{a} \perp \boldsymbol{\omega}$ and $a > \omega$. This means first of all that getting from the original "unprimed"

instantaneous rest frame of the observer to the "primed" frame that we have used to define the mode functions, we need to perform a boost which approaches the speed of light when $a \to \omega$, so that the transition from $a > \omega$ to $a < \omega$ is discontinuous. But in addition to this, we also used the "Rindler energy" Ω to label the modes. This parameter is a dimensionless quantity which is common to all stationary observers with $a > \omega$. However, it is not a very physical parameter as regarded by any one observer. A more physical parameter to use for labeling the modes would be the energy $\epsilon_{k_{y'}k_{z'}\Omega}$ of the mode defined by the observer's Hamiltonian H_{RF}, which is

$$\epsilon_{k_{y'}k_{z'}\Omega} = \sigma\left(\Omega + \frac{\omega}{\sigma^2}k_{y'}\right) \tag{5.63}$$

When $a \to \omega$ or $a \to 0$ so that $\sigma \to 0$, the relationship between $\epsilon_{k_{y'}k_{z'}\Omega}$ and Ω becomes degenerate, so that the modes when labeled by Ω will not behave as they "should" in these limits. We could therefore instead use the following set of modes:

$$\breve{g}_{k_{y'}k_{z'}\epsilon}(\chi, \xi, y', z') = g_{k_{y'}k_{z'}\Omega} \qquad \text{where} \qquad \Omega = \frac{1}{\sigma}\left(\epsilon + \frac{\omega}{\sigma}k_{y'}\right) \tag{5.64}$$

Note that this is more than just a simple reparametrization, it is actually a redefinition of the entire set of mode functions. The transition between the g- and the $\breve{g}$-modes depends explicitly on the value of a and ω and is therefore not the same for every stationary observer. The modes $\breve{g}$ are explicitly tied to one observer. If we now want the Bogoliubov β-coefficient $\beta(k_{y'}, k_{z'}, \epsilon;\ E_{\mathbf{k}'})$ between these modes and the Minkowski plane wave modes, we can get them simply by inserting $\Omega = \frac{1}{\sigma}\left(\epsilon + \frac{\omega}{\sigma}k_{y'}\right)$ into eq. 5.60. This gives us:

$$\beta(k_{y'}, k_{z'}, \epsilon;\ E_{\mathbf{k}'}) \sim \frac{e^{-\frac{\pi|\epsilon + \frac{\omega}{\sigma}k_{y'}|}{2\sigma}}}{\sinh(\pi\frac{|\epsilon - \frac{\omega}{\sigma}k_{y'}|}{\sigma})} \tag{5.65}$$

This tends to zero as $\sigma \to 0$. This means that as $a \to \omega$ or $a \to 0$, the modes $\breve{g}_{k_{y'}k_{z'}\epsilon}$ become equivalent to the Minkowski plane-wave modes in terms of defining particle states and the vacuum state. An observer that labels his or her particles using their energy ϵ as he or she perceives it through H_{RF} instead of the abstract parameter Ω will therefore see the particles that are present in the Minkowski vacuum vanish and give way to a complete vacuum as the trajectory approaches $a = 0$ or the limit $a = \omega$.

5.4 Particles when a $\perp$ $\boldsymbol{\omega}$, $a = \omega$

The case $a = \omega$ is a rather singular case, and forms a simple but non-trivial boundary between $a < \omega$ and $a > \omega$. Unfortunately, it is a bit of a hassle to find the eigenfunctions of H_{RF} in this case. One obviously cannot take the limit of either $f_{Emk_{z'}}$ or $g_{k_{y'}k_{z'}\Omega}$ as $\omega \to a$ or vice versa, since these sets of mode functions do not behave in a continuous way across the $a = \omega$ border. We could perform a Lorentz transformation on them to get back to the unprimed coordinates of the observer's original instantaneous rest frame, label them using the energy ϵ as defined by H_{RF} rather than E or Ω and hope that we could then take the limit $\omega \to a$ or vice versa. However, this is rather messy. Calculating the eigenfunctions from scratch when $a = \omega$ is also a bit of work. Since I am not going to need the functions

and will not be treating this case any more in this thesis, I will therefore not calculate them. Those who are interested in an explicit solution can look at the article [4] by Letaw and Pfautsch, where they have been calculated (or at least listed) explicitly.

We just note that since the Bogoliubov β-coefficients between the eigenfunctions of H_{RF} and the Minkowski plane-wave modes are all zero when $a < \omega$, and since they all tend to zero for the case $a > \omega$ when $a \to \omega$ if we use the modes $\breve{g}_{k_{y'}k_{z'}\epsilon}$ that I defined in the last section, we expect that the β-coefficients will be zero when $a = \omega$ also. This is indeed confirmed in [4]. Therefore, for a stationary reference frame with $\mathbf{a} \perp \boldsymbol{\omega}$ and $a = \omega$, the states of definite particle number are the same as in an inertial frame, and the vacuum state is just the ordinary Minkowski vacuum.

5.5 Eigenfunctions when $\mathbf{a} \not\perp \boldsymbol{\omega}$

Finally we we will calculate particle eigenfunctions in the case where $\mathbf{a}$ and $\boldsymbol{\omega}$ are not perpendicular to each other. In this case we found that there exists a frame that we denoted with doubly primed coordinates in which the observer follows an accelerated motion along the x''-axis and simultaneously orbits the x''-axis in a helical motion. In this frame H_{RF} had the form (eq. 5.11):

$$H_{\mathsf{RF}} = a'' B_{x''} + \omega'' J_{x''} \tag{5.66}$$

When represented by differential operators that act on the scalar Klein-Gordon field and using the combined hyperbolic cylindrical and ordinary cylindrical coordinates $(\chi; \xi, r, \theta)$ that we introduced in section 4.6, the operators $B_{x''}$ and $J^{x''}$ (with $J^{x''}$ once again in pseudovector rather than pseudocovector form) can be written as

$$B_{x''} = i\frac{\partial}{\partial \chi} \qquad J^{x''} = -i\frac{\partial}{\partial \theta} \tag{5.67}$$

As a differential operator H_{RF} is therefore equal to

$$H_{\mathsf{RF}} = a'' B_{x''} - \omega'' J_{x''} = ia''\frac{\partial}{\partial \chi} + i\omega''\frac{\partial}{\partial \theta} \tag{5.68}$$

Now we want to find eigenfunctions of H_{RF} that are eigenfunctions of both $B_{x''}$ and $J^{x''}$:

$$i\frac{\partial}{\partial \chi} h_{m\Omega} = \Omega h_{m\Omega} \qquad -i\frac{\partial}{\partial \theta} h_{m\Omega} = m h_{m\Omega} \tag{5.69}$$

The solution is simple:

$$h_{m\Omega}(\chi, \xi, r, \theta) = H_{m\Omega}(\xi, r) e^{-i\Omega\chi + im\theta} \tag{5.70}$$

where $H_{m\Omega}(\xi, r)$ is some function of ξ and r that needs to be determined by the Klein-Gordon equation. The energy $\epsilon_{m\Omega}$ defined by H_{RF} for these functions is:

$$\begin{aligned} & H_{\mathsf{RF}} h_{m\Omega} = i\left(a''\frac{\partial}{\partial\chi} + \omega''\frac{\partial}{\partial\theta}\right) h_{m\Omega} = \left(a''\Omega - \omega'' m\right) h_{m\Omega} \\ \Longrightarrow \qquad & \epsilon_{m\Omega} = a''\Omega - \omega'' m \end{aligned} \tag{5.71}$$

Now let us derive the Klein-Gordon equation in the combined hyperbolic and ordinary cylindrical coordinates. The metric in these coordinates is:

$$\begin{aligned} & g_{\mu\nu}dx^\mu dx^\nu = \xi^2 d\chi^2 - d\xi^2 - dr^2 - r^2 d\theta^2 \\ \Longrightarrow \qquad & g_{\chi\chi} = \xi^2 \qquad g_{\xi\xi} = -1 \qquad g_{rr} = -1 \qquad g_{\theta\theta} = -r^2 \\ \Longrightarrow \qquad & g^{\chi\chi} = \frac{1}{\xi^2} \qquad g^{\xi\xi} = -1 \qquad g^{rr} = -1 \qquad g^{\theta\theta} = -\frac{1}{r^2} \end{aligned} \tag{5.72}$$

By combining our results from eqs. 5.25 and 5.46 we then get the following non-vanishing Christoffel symbols:

$$\Gamma^{\xi}_{\ \chi\chi} = \xi \qquad \Gamma^{\chi}_{\ \xi\chi} = \Gamma^{\chi}_{\ \chi\xi} = \frac{1}{\xi} \qquad \Gamma^{r}_{\ \theta\theta} = -r \qquad \Gamma^{\theta}_{\ r\theta} = \Gamma^{\theta}_{\ \theta r} = -\frac{1}{r} \tag{5.73}$$

The Klein-Gordon equation then becomes:

$$\left[\frac{1}{\xi^2}\frac{\partial^2}{\partial\chi^2} - \frac{\partial^2}{\partial\xi^2} - \frac{1}{\xi}\frac{\partial}{\partial\xi} - \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + M^2\right] h_{m\Omega}(\chi,\xi,r,\theta) = 0 \tag{5.74}$$

which yields the following equation for $H_{m\Omega}(\xi, r)$:

$$-\left[\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{\Omega^2}{\xi^2} - \frac{m^2}{r^2} - M^2\right] H_{m\Omega}(\xi, r) = 0 \tag{5.75}$$

If we separate this equation and use the solutions of eqs. 5.27 and 5.48 we get:

$$h_{m\Omega Q}(\chi,\xi,r,\theta) = C_{m\Omega Q} e^{-i\Omega\chi + im\theta} J_m(Qr) K_\Omega(\pm\Xi_Q\,\xi) \tag{5.76}$$

where the plus-sign is for $\xi > 0$ and the minus-sign for $\xi < 0$, and where $\Xi_Q = \sqrt{Q^2 + M^2}$. Q is here actually an independent parameter (which is why I suddenly included it in the subscript of $h_{m\Omega Q}$). It may take any value from 0 to $+\infty$, but can not be negative since the Bessel functions $J_m(x)$ diverge to infinity when $x \to -\infty$. Ω may take any value from $-\infty$ to ∞, whereas m may take any integer value between $-\infty$ and ∞.

Just as for $\mathbf{a} \perp \boldsymbol{\omega}$, $a > \omega$ the solutions are not well behaved at $\xi = 0$ where we have a horizon, and just as before we can therefore split the solutions arbitrarily into eigenfunctions defined on either side of the horizon or on the other. As before, we choose the eigenfunctions to have support only on one side of the horizon or on the other.

Turning to the inner product between two eigenfunctions $h_{m\Omega Q}$ and $h_{m'\Omega' Q'}$ defined on the right side of the horizon ($\xi > 0$), we get the following by combining the results from

the cases $a > \omega$ and $a < \omega$ (dropping the subscripts on the C's and Ξ's and distinguishing them by primes instead):

$$\langle h_{m\Omega Q}\ ,\ h_{m'\Omega'Q'}\rangle =$$

$$C^*C'\,(\Omega'+\Omega)\int_0^{2\pi} d\theta\, e^{i(m'-m)\theta}\int_0^\infty d\xi\,\frac{1}{\xi}K_{i\Omega'}(\Xi'\,\xi)K_{i\Omega}(\Xi\,\xi)\int_0^\infty dr\, r J_{m'}(Q'\,r)J_m(Q\,r) \tag{5.77}$$

The solutions of the last two integrals are already given in eqs. 5.33 and 5.53, so we get:

$$\langle h_{m\Omega Q}\ ,\ h_{m'\Omega'Q'}\rangle \sim |C|^*\,\frac{2\Omega}{Q}\delta_{m,m'}\delta(Q'-Q)\delta(\Omega-\Omega') \tag{5.78}$$

Since we assume that $Q > 0$, we once again obtain that positive norm eigenfunctions on the right side of the horizon have positive values for Ω. However, the energy as defined by H_{RF} is equal to $\epsilon_{m\Omega} = a''\Omega + \omega'' m$, and since m has neither an upper nor a lower bound, this means that $\epsilon_{m\Omega}$ can be negative, with the same implications as for $a > \omega$ (the only difference is that m only has integer values, whereas $k_{y'}$ can assume a continuum of values).

Just as before we also get that the inner product between two eigenfunctions defined on the left side of the horizon ($\xi < 0$) will be the negative of the corresponding functions on the right side, so that positive norm eigenfunctions on the left side of the horizon have *negative* values for Ω and vice versa. As in the case of $a > \omega$ we can therefore define

$$h_{m\Omega Q}(\chi,\xi,r,\theta) \;=\; \begin{cases} C_{m\Omega Q}\,e^{-i\Omega\chi+m\theta}K_{i\Omega}(\Xi_Q\,\xi)J_m(Qr) & \Omega>0,\ \xi>0 \\ 0 & \Omega>0,\ \xi<0 \\ 0 & \Omega<0,\ \xi>0 \\ C_{m\Omega Q}\,e^{-i\Omega\chi+im\theta}K_{i\Omega}(-\Xi_Q\,\xi)J_m(Qr) & \Omega<0,\ \xi<0 \end{cases} \tag{5.79}$$

Finally we turn to the number operator. Let now $g_{k'_{y'}k'_{z'}\Omega'}$ to be an eigenfunction for the case where $\mathbf{a} \perp \boldsymbol{\omega}$ and $a > \omega$, i.e. a "Rindler" eigenfunction:

$$\begin{aligned} g_{k'_{y'}k'_{z'}\Omega'} &= A_{k'_{y'}k'_{z'}\Omega'}e^{-i\Omega'\chi+ik'_{y'}y'+ik'_{z'}z'}K_{i\Omega'}(\Xi_{k'_{y'}k'_{z'}}\xi) \\ &= A_{k'_{y'}k'_{z'}\Omega'}e^{-i\Omega'\chi+ik'_{y'}r\cos\theta+ik'_{z'}r\sin\theta}K_{i\Omega'}(\Xi_{k_{y'}k_{z'}\Omega'}\,\xi) \end{aligned} \tag{5.80}$$

We now calculate the Bogoliubov β-coefficients between the modes $g_{k'_{y'}k'_{z'}\Omega'}$ and the modes $h_{m\Omega Q}$ (I will drop the subscripts on the functions and on the normalization constants for brevity since they are denoted by different letters). To simplify the discussion, I first assume that both Ω and Ω' are positive, so that both functions have support on the right side of the horizon:

$$\begin{aligned}\beta(m,\Omega,Q;\,k_{y'},k_{z'},\Omega) &= -\langle g'^{*},h\rangle \\ &= -A'C\,(\Omega-\Omega')\int_0^{2\pi}d\theta\,\int_0^{\infty}dr\,re^{im\theta}e^{ik'_{y'}r\cos\theta+ik'_{z'}r\sin\theta}J_m(Qr) \\ &\qquad \int_0^{\infty}d\xi\,K_{i\Omega'}(\Xi_{k'_{y'}k'_{z'}}\xi)K_{i\Omega'}(\Xi_Q\xi)\end{aligned} \tag{5.81}$$

We now make use of the following identity:

$$k'_{y'}\cos\theta + k'_{z'}\sin\theta = k\sin(\theta+\alpha) \tag{5.82}$$

where $k=\sqrt{k'^2_{y'}+k'^2_{z'}}$ and $\tan\alpha=\frac{k'_{z'}}{k'_{y'}}$. Inserting this into eq. 5.81 gives us:

$$\begin{aligned}&-A'C\,(\Omega-\Omega')\int_0^{2\pi}d\theta\,\int_0^{\infty}dr\,re^{im(\theta+\alpha-\alpha)}e^{ikr\sin(\theta-\alpha)}J_m(Qr) \\ &\qquad\qquad \int_0^{\infty}d\xi\,K_{i\Omega'}(\Xi_{k'_{y'}k'_{z'}}\,\xi)K_{i\Omega}(\Xi_Q\,\xi)\end{aligned} \tag{5.83}$$

Using eq. 5.37 this becomes:

$$\begin{aligned}&-A'C\,(\Omega-\Omega')\,(-1)^m\,2\pi e^{im\alpha}\int_0^{\infty}dr\,rJ_m(kr)J_m(Qr)\int_0^{\infty}d\xi\,K_{i\Omega'}(\Xi_{k'_{y'}k'_{z'}}\,\xi)K_{i\Omega}(\Xi_Q\,\xi) \\ &= -A'C\,(\Omega-\Omega')\,(-1)^m\,\frac{2}{\pi Q}\delta(k-Q)e^{im\alpha}\int_0^{\infty}d\xi\,K_{\Omega'}(\Xi_{k_{y'}k_{z'}}\,\xi)K_{i\Omega'}(\Xi_Q\,\xi)\end{aligned} \tag{5.84}$$

Since $\Xi_{k_{y'}k_{z'}}=\sqrt{k'^2_{y'}+k'^2_{z'}+M^2}=\sqrt{k^2+M^2}$ and $\Xi_Q=\sqrt{Q^2+M^2}$ the δ-function $\delta(k-Q)$ implies that $\beta(m,\Omega,Q;\,k_{y'},k_{z'},\Omega)$ will vanish unless $\Xi_{k'_{y'}k'_{z'}}=\Xi_Q$. But if this is the case then the final integral over ξ gives us:

$$\beta(m,\Omega,Q;\,k_{y'},k_{z'},\Omega)\sim(\Omega-\Omega')\,\frac{2}{\pi Q}\delta(k-Q)\delta(\Omega-\Omega') \tag{5.85}$$

so if $\Omega\neq\Omega'$ the δ-function vanishes, but if $\Omega=\Omega'$, the prefactor $(\Omega-\Omega')$ vanishes, so in either case we get $\beta(m,\Omega,Q;\,k_{y'},k_{z'},\Omega)=0$. The β-coefficient when Ω and Ω' are both negative, so that both functions have their support on the left side of the horizon can be calculated in the same way and will yield the same result. If Ω and Ω' have differing signs, then the β-coefficient obviously vanishes since the functions then are never non-zero at the same time anywhere along the ξ-axis. This means that the number operator defined using the modes $h_{m\Omega Q}$ is equivalent to the number operator of the Rindler modes $g_{k'_{y'}k'_{z'}\Omega}$, so particles defined with respect to these modes are equivalent to each other. This also means that particles defined with respect to the modes $h_{m\Omega Q}$ are *not* equivalent to particles defined using inertial plane-wave modes. However, the same caveats as for $\mathbf{a}\perp\boldsymbol{\omega}$, $a>\omega$ still apply when it comes to the question of whether an observer whose motion is generated by H_{RF} will actually observe any particles when travelling through the inertial Minkowski vacuum.

Chapter 6

Making sense of inequivalent vacua

In the last chapter we saw that a for an observer following a stationary world-line characterized by a constant proper acceleration $\mathbf{a}$ and constant proper angular velocity $\boldsymbol{\omega}$, the number operator defined by the observer's Hamiltonian H_{RF} when $a > \omega$ or when $\mathbf{a}$ and $\boldsymbol{\omega}$ are not perpendicular to each other, appears not to be equivalent to the number operator defined by the Hamiltonian of an inertial observer. In this chapter we will examine the consequences of this further, and see that the inequivalence can be given a thermodynamic interpretation. We will also see that it is actually not quite straight-forward to say whether or not H_{RF} in these cases *really* defines an inequivalent set of particle states and an inequivalent vacuum. Furthermore we will investigate under what circumstances inequivalent sets of particle states may exist and explore the relationship between inequivalent number operators and the existence of event horizons. Finally, we will discuss whether we can treat any particular vacuum and set of particle states in Minkowski spacetime as more fundamental than the other. Much of the discussion here was motivated by the article [4] by Letaw and Pfautsch. In that paper they examine the particle eigenfunctions of stationary reference frames on a case by case basis, as I have done in the last two chapters, and conclude that there are just two inequivalent vacuum states (and hence two inequivalent definitions of the particle number operator) for stationary reference frames in Minkowski spacetime, namely the inertial Minkowski vacuum and the Rindler-Fulling vacuum. However, they did not consider the possibility of mixing solutions from either side of the horizon. In this chapter I will show that when this is allowed, the conclusion becomes quite different, and the number of distinct vacua actually becomes largely a matter of taste.

6.1 Rindler particles and the inertial vacuum

In section 5.3 we calculated the Bogoliubov coefficients for the transformation from particle eigenfunctions $g_{k_{y'}k_{z'}\Omega}$ defined relative to the Hamiltonian H_{RF} of a stationary accelerated observer with $\mathbf{a} \perp \boldsymbol{\omega}$ and $a > \omega$, and plane wave eigenfunctions $f_{\mathbf{k}'}$ defined relative to an inertial Hamiltonian. Now we want to see how we can use these to calculate what the inertial vacuum looks like in terms of the "Rindler" particle states of the accelerated observer. We will stick to the case where $\mathbf{a} \perp \boldsymbol{\omega}$ and $a > \omega$, i.e. the case where seen from a suitable reference frame e have constant linear acceleration along the ξ-axis superposed on

a constant velocity along the negative y'-axis. The calculations when $\mathbf{a}$ is not perpendicular to $\boldsymbol{\omega}$ would be analogous step by step if one only replaces the velocity along the y'-axis with rotation around the x''-axis, so we will not treat it separately.

Let us first see in general how we can relate a vacuum state $|0\rangle$ defined using one set of annihilation and creation operators a_i and $a_i^\dagger$, to another vacuum state $|0'\rangle$ defined using another set of operators b_I and $b_I^\dagger$. $|0\rangle$ will be a superposition of particle states with respect to $(b_I, b_I^\dagger)$, so it can be written as

$$|0\rangle = F(\mathbf{b}^\dagger)\,|0'\rangle \tag{6.1}$$

where $F(\mathbf{b}^\dagger)$ is a polynomial or a power series of the creation operators $b_I^\dagger$. $|0\rangle$ is annihilated by all the annihilation operators a_i, so we have

$$a_i\,|0\rangle = a_i F(\mathbf{b}^\dagger)\,|0'\rangle = 0 \tag{6.2}$$

According to eq. 3.7 this can be written as:

$$\sum_J \left(\alpha_{Ji} b_J + \beta^*_{Ji} b_J^\dagger\right) F\,|0'\rangle = 0 \tag{6.3}$$

But since by definition $b_J\,|0'\rangle = 0$, we have that $b_J F\,|0\rangle_R = (b_J F - F b_J)\,|0'\rangle = [b_J, F]\,|0'\rangle$, so

$$\sum_J \left(\alpha_{Ji}\,[b_J, F] + \beta^*_{Ji} b_J^\dagger F\right)|0'\rangle = 0 \tag{6.4}$$

Now note that because of the commutation relation $[b_I, b_J] = \delta_{IJ}$, we have

$$\begin{aligned}\left[b_J\,,\,(b_1^\dagger)^{n_1}(b_2^\dagger)^{n_2}\cdots(b_J^\dagger)^{n_J}\cdots\right] &= n_J(b_1^\dagger)^{n_1}(b_2^\dagger)^{n_2}\cdots(b_J^\dagger)^{n_J-1}\cdots \\ &= \frac{\partial}{\partial b_J^\dagger}\left((b_1^\dagger)^{n_1}(b_2^\dagger)^{n_2}\cdots(b_J^\dagger)^{n_J}\cdots\right)\end{aligned} \tag{6.5}$$

where the partial derivation symbolically means derivating the expression as if the $b_K^\dagger$s were c-numbers (but we must of course keep their ordering). Since F is a sum of such products of creation operators, we see that eq. 6.4 will be satisfied if

$$\sum_J \alpha_{Ji}\frac{\partial F}{\partial b_J^\dagger} = -\sum_J \beta^*_{Ji} b_J^\dagger F \tag{6.6}$$

Now define the "inverse" α^{-1}_{iI} by:

$$\sum_i \alpha^{-1}_{iI}\alpha_{Ji} \;=\; \delta_{IJ} \tag{6.7}$$

We then get:

$$\frac{\partial F}{\partial b_I^\dagger} = -\sum_{Ji} \beta^*_{Ji}(\alpha^{-1}_{iI}) b_J^\dagger F \tag{6.8}$$

If we define $V_{IJ} = \sum_i (\alpha_{iI}^{-1})\beta_{Ji}^*$, this equation has the solution:

$$F \propto \exp\left(-\sum_{IJ} b_J^\dagger V_{IJ} b_I^\dagger\right) \tag{6.9}$$

This outlines the general procedure of how to calculate how two inequivalent vacuum states are related. In our specific case however, the states are labeled by continuous quantities, $\mathbf{k}' = (k'_{x'}, k'_{y'}, k'_{z'})$ for the inertial plane wave functions $f_{\mathbf{k}'}$, and $k_{y'}$, $k_{z'}$ and Ω for the Rindler mode functions $g_{k_{y'}k_{z'}\Omega}$ respectively, so the sums should be converted into integrals and the normalizations done using δ-functions. Since in this chapter I will be working almost exclusively in the "primed" reference frame defined in section 4.4 and rarely if ever refer to the original unprimed instantaneous rest frame of the observer, I will drop all the primed on the coordinates y' and z'. They will always refer to the primed reference frame unless otherwise noted. Eq. 6.9 then becomes:

$$F \propto \exp\left(-\int dk_{1y}\, dk_{1z}\, d\Omega_1\, dk_{2y}\, dp_{2z}\, d\Omega_2\, b^\dagger_{k_{y_1}k_{z_1}\Omega_1} V(k_{1y}, k_{1z}, \Omega_1; k_{2y}, k_{2z}, \Omega_2) b^\dagger_{k_{2y}k_{2z}\Omega_2}\right) \tag{6.10}$$

where now

$$V(k_{1y}, k_{1z}, \Omega_1; k_{2y}, k_{2z}, \Omega_2) = \int d^3k'\, \beta(k_{1y}, k_{1z}, \Omega_1; \mathbf{k}')^*\, \alpha^{-1}(\mathbf{k}';\, k_{2y}, k_{2z}, \Omega_2) \tag{6.11}$$

and where the inverse $\alpha^{-1}(\mathbf{k}';\, k_{2y}, k_{2z}, \Omega_2)$ is defined by

$$\int d^3k' \alpha(k_{1y}, k_{1z}, \Omega_1;\, \mathbf{k}')\, \alpha^{-1}(\mathbf{k}';\, k_{2y}, k_{2z}, \Omega_2) = \delta(k_{1y} - k_{2y})\delta(k_{1z} - k_{2z})\delta(\Omega_1 - \Omega_2) \tag{6.12}$$

Now from eq. 5.60 and 5.62 we can see that $\alpha(k_y, k_z, \Omega;\, \mathbf{k}')^* = \alpha(k_y, k_z, -\Omega;\, \mathbf{k}')$ and $\beta(k_y, k_z, \Omega;\, \mathbf{k}') = e^{-\pi|\Omega|}\, \alpha(-k_y, -k_z, \Omega;\, \mathbf{k}')$. Using this on eq. 6.11 we get the following:

$$\begin{aligned} V(k_{1y}, k_{1z}, \Omega_1; k_{2y}, k_{2z}, \Omega_2) &= \int d^3k'\, e^{-\pi|\Omega_2|} \alpha(-k_{1y}, -k_{1z}, -\Omega_1; \mathbf{k}') \alpha^{-1}(\mathbf{k}';\, k_{2y}, k_{2z}, \Omega_2) \\ &= e^{-\pi|\Omega_2|} \delta(k_{1y} + k_{2y})\, \delta(k_{1z} + k_{2z})\, \delta(\Omega_1 + \Omega_2) \end{aligned} \tag{6.13}$$

Inserting this into eq 6.11 finally gives us the following expression for the inertial Minkowski vacuum $|0\rangle_M$ in terms of the Rindler vacuum $|0\rangle_R$:

$$
\begin{aligned}
|0\rangle_M &= C \exp\left(- \int dk_y\, dk_z\, d\Omega\, e^{-\pi|\Omega|} b^\dagger_{k_y k_z \Omega} b^\dagger_{(-k_y)(-k_z)(-\Omega)} \right) |0\rangle_R \\
&= C \prod_{k_y,k_z,\Omega} \exp\left(-e^{-\pi|\Omega|} b^\dagger_{k_y k_z \Omega} b^\dagger_{(-k_y)(-k_z)(-\Omega)} \right) |0\rangle_R \\
&= C \prod_{k_y,k_z,\Omega} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \left(e^{-\pi|\Omega|} b^\dagger_{k_y k_z \Omega} b^\dagger_{(-k_y)(-k_z)(-\Omega)} \right)^N |0\rangle_R \\
&= C \prod_{k_y,k_z,\Omega} \sum_{N=0}^{\infty} (-1)^N e^{-N\pi|\Omega|} \, |N, k_y, k_z, \Omega\rangle \otimes |N, -k_y, -k_z, -\Omega\rangle
\end{aligned}
\tag{6.14}
$$

where C is some proportionality constant, and where $|N, k_y, k_z, \Omega\rangle \otimes |N, -k_y, -k_z, -\Omega\rangle$ denotes a state with N particles with momenta and energy k_y, k_z, Ω on the right side of the horizon and N particles with momenta and energy $-k_y, -k_z, -\Omega$ on the left side of the horizon if $\Omega > 0$, or vice versa if $\Omega < 0$ (I have used the fact that $(b^\dagger_{k_y k_z \Omega})^N |0\rangle_R = \sqrt{N!}\,|N, k_y, k_z, \Omega\rangle$).

So we see that the Minkowski vacuum is a superposition of states with any number of pairs of Rindler particles, each pair consisting of one particle on the right side of the horizon (with $\Omega > 0$) and another on the left side ($\Omega < 0$) with exact opposite values for the momenta, and each weighted with a factor of $e^{-\pi|\Omega|}$.

6.2 Rindler particles and thermodynamics

The factor $e^{-\pi|\Omega|}$ in eq. 6.14 looks tantalizingly like something that could be related to a Boltzmann factor in statistical mechanics. We shall now see that it can indeed be interpreted in this way.

Suppose that we are in the position of an observer cruising along through the vacuum in Minkowski spacetime on a stationary trajectory with $a > \omega$ and $\mathbf{a} \perp \boldsymbol{\omega}$. We define our particles as we did in section 5.3, i.e. we use Rindler particles, and then ask the question "How many Rindler particle pairs are there in mode pair number i", by which we mean a pair of modes, one on each side of the horizon, characterized by momenta $\pm k_{iy}$, $\pm k_{iz}$ and Rindler energy $\pm\Omega_i$ (plus for the right side of the horizon, minus for the left side). Let us denote the probability for finding N_i particles in this mode by P_{N_i}, and to avoid all those pesky absolute value marks, we label each pair by the energy and momenta of the state on the right side of the horizon, so that $\Omega_i > 0$. From eq. 6.14, using standard quantum mechanical axioms, we immediately see that

$$
P_{N_i} \propto |e^{-N_i \pi \Omega_i}|^2 = e^{-N_i\, 2\pi\Omega_i} \tag{6.15}
$$

Since the total probability of finding any number of particles in mode i must be 1, we get:

$$
\sum_{N_i=0}^{\infty} P_{N_i} \stackrel{!}{=} 1
$$

$$\sum_{N_i=0}^{\infty} e^{-N_i\, 2\pi\Omega_i} = \frac{1}{1 - e^{2\pi\Omega_i}}$$

$$\Longrightarrow \qquad P_{N_i} = \frac{e^{-N_i\, 2\pi\Omega_i}}{1 - e^{-2\pi\Omega_i}} \tag{6.16}$$

This is very reminiscent of the Bose-Einstein distribution for an energy level with energy Ω_i in an ideal gas of bosons at temperature $kT = \frac{1}{2\pi}$, so it would seem that we are getting somewhere. However, the energy of mode i as defined by H_{RF} is not Ω (which is not an energy at all, but a dimensionless parameter) but $\epsilon_i = \sigma\left(\Omega_i + \frac{\omega}{\sigma^2}k_{i_y}\right)$ (where once again $\sigma = \sqrt{a^2 - \omega^2}$). In other words, we have

$$\Omega_i = \frac{1}{\sigma}\left(\epsilon_i - \frac{\omega}{\sigma}k_{i_y}\right) \tag{6.17}$$

This finally allows us to write P_{N_i} in the following form; If we define $kT = \frac{\sigma}{2\pi}$ and $v = \frac{\omega}{a}$, then we get

$$P_{N_i} = \frac{e^{-N_i\frac{\epsilon_i - v\frac{a}{\sigma}k_{i_y}}{kT}}}{1 - e^{-\frac{\epsilon_i - v\frac{a}{\sigma}k_{i_y}}{kT}}} \tag{6.18}$$

(the reason for defining v in this way will become clear in a moment) This is almost precisely the Bose-Einstein distribution for an energy level i with energy ϵ_i in an ideal boson gas at temperature $kT = \frac{\sigma}{2\pi} = \frac{\sqrt{a^2-\omega^2}}{2\pi}$ and zero chemical potential, except for the extra term $-\frac{v\frac{a}{\sigma}k_{i_y}}{kT}$ in the exponential. But this term can be readily explained in thermodynamic terms. It simply accounts for the fact that the observer is *moving* relative to the centre-of-mass rest frame of the gas (which in this case is the primed inertial frame as defined in section 4.4), as we can qualitatively explain as follows:

An ideal gas viewed from its center-of-mass rest frame obeys the following thermodynamic identity:

$$dE = T\, dS \tag{6.19}$$

where E is the total energy of the gas, S is the total entropy and T is the temperature, and where we have neglected the effect of the volume of the gas (the "vacuum gas" of Rindler particles in the Minkowski vacuum obviously has infinite volume, so there are no volume changes to be considered in this case). The energy E in this case is all internal energy E_{int}, due to energy in internal degrees of freedom in the gas, such as thermal motion, rotation and vibration of the gas molecules. So we have $E = E_{int}$, or $E = E_{int} + M$ in the relativistic case, where M is the total rest mass of the gas. A change dE_{int} in the internal is reflected in a change dS in the entropy. By using the Shannon formula $S = \sum_i -kP_i ln(P_i)$ for the entropy and maximizing the entropy for a given average total energy E along with Bose-Einstein statistics, we get the usual Bose-Einstein distribution $P_{N_i} = e^{-N_i\epsilon_i/kT}/(1 - e^{-\epsilon_i/kT})$ for the probability of having N_i particles in an energy level with energy ϵ_i. If however we observe the gas from a reference frame in which the gas as a whole is moving at a velocity $\mathbf{v}$, then we must add to this the kinetic energy of the gas as a whole. This means that we get

$$E = E_{int} + \frac{1}{2}Mv^2 = E_{int} + \frac{P^2}{2m} \tag{6.20}$$

for a non-relativistic gas, or

$$E = E_{int} + \sqrt{P^2 + M^2} \tag{6.21}$$

for a relativistic gas, where $\mathbf{P}$ is the total momentum of the gas. In both the relativistic and non-relativistic case, one can calculate that the thermodynamic identity becomes:

$$dE = T\,dS + \mathbf{v}\cdot d\mathbf{P} \qquad \Longrightarrow \qquad T\,dS = dE - \mathbf{v}\cdot d\mathbf{P} \tag{6.22}$$

(where for the relativistic case, we differentiate E with respect to $\mathbf{P}$ and use the identity $\mathbf{v} = \mathbf{P}/E$). Maximizing the entropy as before, we get the following probability P_{N_i} for having N_i particles in a mode with energy ϵ_i and momentum $\mathbf{k}_i$:

$$P_{N_i} = \frac{e^{\frac{\epsilon_i - \mathbf{v}\cdot\mathbf{k}_i}{kT}N_i}}{1 - e^{\frac{\epsilon_i - \mathbf{v}\cdot\mathbf{k}_i}{kT}}} \tag{6.23}$$

which has almost the same form as eq. 6.18, with $\mathbf{v}$ equal to $\frac{\omega}{a}$ along the y-axis. This value of $\mathbf{v}$ of course makes sense, since the observer actually *is* moving at a velocity $\frac{\omega}{a}$ in the primed inertial frame as described in section 4.4. The reason for the prefactor $\frac{a}{\sigma}$ in eq. 6.18 is that the momentum $\mathbf{k}_i$ is defined relative to the rest frame of the gas, which is the primed reference frame in hyperbolic coordinates. But the observer is moving along the y'-axis relative to this frame, and therefore experiences a time dilatation factor of $\frac{a}{\sigma}$ which scales momentum (whereas ϵ_i is defined in the observer's own rest frame, and so does not have any time-dilatation factor in front of it)

The final twist to note before we can give a thermodynamic interpretation to how an accelerating observer will experience the Minkowski vacuum, is that the expression 6.14 is not really a thermal distribution of states, so the Minkowski vacuum is not really made up of a thermal ensemble of Rindler particles. Rather, 6.14 describes a coherent superposition of particle-*pair* states, and each state has total energy and momentum equal to zero, since each state contains one particle on one side of the horizon and one on the other side with exactly opposite energies and momenta. However, the observer is not causally connected to the left side of the horizon, and can never actually see the particles there. The observer therefore only sees one half of each particle pair. If we simply ignore what particles exist in states on the left side of the horizon and average out of over all possible particle numbers in these states, then 6.14 *is* indeed a thermal distribution. Since we then only care about the particles on the right side of the horizon, these particles do not seem to have any "partners" that cancel out their energies and momenta. For this reason we can say that the stationary accelerated observer travelling through the Minkowski vacuum actually sees this vacuum as being a thermal ideal gas of bosons with temperature $kT = \frac{\sigma}{2\pi}$ moving at velocity $v = \frac{\omega}{a}$ in the y-direction.

Note that it is crucial for this thermal analogy that we use the Rindler eigenfunctions $g_{k_{y'}k_{z'}\Omega}$ which have support on only one side of the horizon and vanish on the other. If we let the eigenfunctions behave differently across the horizon, the Bogoliubov coefficients

will look differently and the thermal effects may be distorted or vanish. This is connected with the point mentioned in section 3.1 that if we have degeneracies between eigenfunctions with positive and negative norm, then one cannot uniquely define what is meant by creation and annihilation of particles with respect to the observable linked to those eigenfunctions. In the next section we shall address this point and show that the "vacuum temperature" can be made to disappear if we allow ourselves to mix states on either side of the horizon.

6.3 Mixing across the horizon

In section 5.3 we chose our eigenfunctions so that positive norm eigenfunctions with $\Omega > 0$ were identically zero on the left side of the horizon, while those with $\Omega < 0$ vanished on the right side of the horizon. In this way we effectively split the particle spectrum into one set of particles existing only on the right side of the horizon and one set existing only on the left side. This made sense because the horizon acts as a natural boundary in spacetime for a stationary accelerated observer. No information about what goes on behind the horizon can ever reach the observer, so what happens to particle modes on the left side of the horizon should not affect the observer. Similarly, no information about the observer can ever cross to the left side of the horizon, so whatever the observer might do to the particle modes in his or her part of the spacetime, should not affect particle modes on the left side of the horizon.

But we know that if we have degeneracies in the system, in the sense that some of the eigenfunctions share the same eigenvalues for all the observables that we have used to define them, we are free to define new sets of eigenfunctions by making linear combinations of these degenerate eigenfunctions, since the resulting combination will also be an eigenfunction of the same observables with the same eigenvalues. In the case of the eigenfunctions $g_{k_y k_z \Omega}$ of H_{RF} in the Rindler case, it is clear that we cannot mix positive norm eigenfunctions on across the horizon; all positive norm eigenfunctions with support on the right side of horizon have $\Omega > 0$ and all those with support on the left side have $\Omega < 0$. There is therefore no degeneracy between positive norm eigenfunctions on different sides of the horizon. However, if we allow ourselves to mix positive and *negative* norm eigenfunctions, it is a different story. For each positive norm eigenfunction $g_{k_y k_z \Omega}$ on one side of the horizon, there is a corresponding positive norm eigenfunction $g_{(-k_y)(-k_z)(-\Omega)}$ with exactly opposite values for Ω and the momenta on the other side of the horizon. Its complex conjugate $g^*_{(-k_y)(-k_z)(-\Omega)}$ however, is a negative norm eigenfunction with the *same* Ω- and momentum-eigenvalues as $g_{k_y k_z \Omega}$. Therefore we may make linear combinations of the two to get a new set of eigenfunctions.

This degeneracy does not only allow us to make sets of particle states that cross over the horizon. Since we are also mixing positive norm functions with negative norm ones, we are mixing creation on one side of the horizon with annihilation on the other side; If we define a new eigenfunction $\tilde{g}_{k_y k_z \Omega}$ by

$$\tilde{g}_{k_y k_z \Omega} = A g_{k_y k_z \Omega} + B g_{(-k_y)(-k_z)(-\Omega)} \tag{6.24}$$

then we know from section 3.1 that the corresponding annihilation and creation operators $\tilde{b}_{k_y k_z \Omega}$ and $\tilde{b}^\dagger_{k_y k_z \Omega}$ will be given by:

$$\begin{aligned}\tilde{b}_{k_y k_z \Omega} &= A^* \, b_{k_y k_z \Omega} - B^* \, b^\dagger_{(-k_y)(-k_z)(-\Omega)} \\ \tilde{b}^\dagger_{k_y k_z \Omega} &= A \, b^\dagger_{k_y k_z \Omega} - B \, b_{(-k_y)(-k_z)(-\Omega)}\end{aligned} \tag{6.25}$$

so we see that with respect to the "old" particles, creating a particle using $\tilde{b}^\dagger_{k_y k_z \Omega}$ is a superposition of creating a particle on $g_{k_y k_z \Omega}$'s side of the horizon and annihilating one on $g^*_{(-k_y)(-k_z)(-\Omega)}$'s side, and vice versa when annihilating a particle using $\tilde{b}_{k_y k_z \Omega}$.

I will now show explicitly that with this kind of mixing, we can define ourselves a new set of Rindler particle eigenfunctions that are equivalent to the plane wave inertial eigenfunctions, i.e. a set for which the β-coefficients of the Bogoliubov-transformation from the set of inertial plane-wave eigenfunctions are all zero. Denote the new set of Rindler eigenfunctions by $\tilde{g}_{k_y k_z \Omega}$, and let the inertial plane wave eigenfunctions as before be denoted by $f_{\mathbf{k}'}$. The Bogoliubov coefficients $\tilde{\alpha}$ and $\tilde{\beta}$ for the transformation from $f_{\mathbf{k}'}$ to $\tilde{g}_{k_y k_z \Omega}$ are then given by:

$$\begin{aligned}\tilde{\alpha}(k_y, k_z, \Omega; \mathbf{k}') &= \langle f_{\mathbf{k}'} \, , \, \tilde{g}_{k_y k_z \Omega} \rangle = A \, \langle f_{\mathbf{k}'} \, , \, g_{k_y k_z \Omega} \rangle + B \, \langle f_{\mathbf{k}'} \, , \, g^*_{(-k_y)(-k_z)(-\Omega)} \rangle \\ &= A \, \alpha(k_y, k_z, \Omega; \mathbf{k}') - B \, \beta(-k_y, -k_z, -\Omega; \mathbf{k}')^* \\ &= \left(A - e^{-\pi |\Omega|} B \right) \alpha(k_y, k_z, \Omega; \mathbf{k}')\end{aligned} \tag{6.26}$$

and

$$\begin{aligned}\tilde{\beta}(k_y, k_z, \Omega; \mathbf{k}') &= \langle f^*_{\mathbf{k}'} \, , \, \tilde{g}_{k_y k_z \Omega} \rangle = A \, \langle f^*_{\mathbf{k}'} \, , \, g_{k_y k_z \Omega} \rangle + B \, \langle f^*_{\mathbf{k}'} \, , \, g^*_{(-k_y)(-k_z)(-\Omega)} \rangle \\ &= A \, \beta(k_y, k_z, \Omega; \mathbf{k}') - B \, \alpha(-k_y, -k_z, -\Omega; \mathbf{k}')^* \\ &= \left(A - e^{\pi |\Omega|} B \right) \beta(k_y, k_z, \Omega; \mathbf{k}')\end{aligned} \tag{6.27}$$

We see now that by choosing $B = e^{-\pi |\Omega|} A$, we get $\tilde{\beta}(k_y, k_z, \Omega; \mathbf{k}') = 0$. We then need to choose $A = (1 - e^{-2\pi |\Omega|})^{-1/2}$ to get the proper normalization. Thus if we let the functions

$$\tilde{g}_{k_y k_z \Omega} = \left(1 - e^{-2\pi |\Omega|} \right)^{-\frac{1}{2}} \left(g_{k_y k_z \Omega} + e^{-\pi |\Omega|} g^*_{(-k_y)(-k_z)(-\Omega)} \right) \tag{6.28}$$

define a new set of modes, we get a set of modes that are both eigenfunctions of the Rindler observables we have used *and* have the same number operator as the inertial plane wave modes. Thus we see clearly that the inequivalence of the Rindler and Minkowski vacua is actually an artifact of how we join (or do not join) the eigenfunctions on either side of the horizon.

By using the observation that Rindler modes can be mixed to produce a set of particle modes with the same number operator as the inertial plane wave modes, we could also have derived the relation 6.14 in a simpler way. Since the number operator of the modes $\tilde{g}_{k_y k_z \Omega}$ is equivalent to that of the inertial modes $f_{\mathbf{k}'}$, the annihilation operators $\tilde{b}_{k_y k_z \Omega}$ must be a linear combinations of the inertial plane-wave annihilation operators $a_{\mathbf{k}'}$ alone, and not of the creation operators $a^\dagger_{\mathbf{k}'}$. Hence $\tilde{b}_{k_y k_z \Omega}$ must annihilate the Minkowski vacuum:

$$\tilde{b}_{k_y k_z \Omega} \, |0\rangle_M = 0$$

$$\Longrightarrow \qquad \left(1 - e^{-\pi|\Omega|}\right)^{-\frac{1}{2}} \left(b_{k_y k_z} - e^{-\pi|\Omega|} b^\dagger_{(-k_y)(-k_z)(-\Omega)}\right) |0\rangle_M = 0 \tag{6.29}$$

Inserting $|0\rangle_M = F\,|0\rangle_R$ into this equation and proceeding as before, we obtain:

$$\frac{\partial F}{\partial b^\dagger_{k_y k_z}} = e^{-\pi|\Omega|} b^\dagger_{(-k_y)(-k_z)(-\Omega)} F \tag{6.30}$$

which once again gives us eq. 6.14. This may not seem like too much of a simplification, but note that we have now actually derived this expression without making explicit use of the Bogoliubov coefficients α and β, only the fact that the linear combination $\tilde{b}_{k_y k_z \Omega} \propto b_{k_y k_z} - e^{-\pi|\Omega|} b^\dagger_{(-k_y)(-k_z)(-\Omega)}$ annihilates the Minkowski vacuum. As we shall see in the next section, this fact can be derived without calculating any Bogoliubov coefficients at all, and the way in which this is done, turns out to be quite instructive.

First a final few notes about the statistical mechanics of Rindler particles in the Minkowski vacuum. In statistical mechanics a quantum system is described by a density matrix ρ:

$$\rho = \sum_i p_i \,|\psi_i\rangle \langle\psi_i| \tag{6.31}$$

where p_i is the probability that the system is in state $|\psi_i\rangle$ (a probability that is due to ignorance of what state the system was actually prepared in, *not* quantum uncertainties). In our case, the field is in the Minkowski vacuum state, a pure state, so the density matrix is simply:

$$\rho = |0\rangle_M \,{}_M\langle 0| \tag{6.32}$$

But the state is only pure as long as we choose to remember the correlation between states on the right side and on the left side of the horizon, i.e. that if we observe N_i particles in mode number i with momenta and Rindler energy k_{i_y}, k_{i_z} and Ω_i on the right side of the horizon, then there will be exactly N_i corresponding particles with momenta and energy $-k_{i_y}$, $-k_{i_z}$ and $-\Omega_i$ on the left side of the horizon. If we choose to forget about this correlation and ignore the all the states on the left side of the equation, then we get a new density matrix ρ' by tracing ρ over all the states on the left side of the horizon:

$$\begin{aligned}
\rho' &= \prod_{j,\Omega_j<0} \sum_{N_j} \left\langle N_j, k_{y_j}, k_{z_j}, \Omega_j \right| \rho \left| N_j, k_{y_j}, k_{z_j}, \Omega_j \right\rangle \\
&= \prod_{j,\Omega_j<0} \sum_{N_j} \left\langle N_j, -k_{y_j}, -k_{z_j}, \Omega_j | 0 \right\rangle_M \,{}_M\!\left\langle 0 | N_j, -k_{y_j}, -k_{z_j}, \Omega_j \right\rangle \\
&\propto \prod_{j,\Omega_j>0} \sum_{N_j} e^{-N_j \frac{2\pi\Omega_j}{|a_0|}} \left| N_j, k_{y_j}, k_{z_j}, \Omega_j \right\rangle \left\langle N_j, k_{y_j}, k_{z_j}, \Omega_j \right| \\
&= \prod_{j,\Omega_j>0} \sum_{N_j} e^{-N_j \frac{2\pi\,|\epsilon_i + \frac{\omega}{\sigma} k_{iy}|}{\sigma}} \left| N_j, k_{y_j}, k_{z_j}, \Omega_j \right\rangle \left\langle N_j, k_{y_j}, k_{z_j}, \Omega_j \right|
\end{aligned} \tag{6.33}$$

which is precisely a thermal density matrix with temperature $kT = \frac{\sigma}{2\pi}$ over the states with $\Omega_j > 0$ for a gas moving with velocity $\frac{\omega}{a}$ in the y-direction relative to the observer. This

is no longer the density matrix of a pure state, but of a mixed state, due to the fact that we have chosen to ignore information about what goes on behind the horizon. Thus we see explicitly once again that the thermal nature of the Minkowski vacuum seen in a stationary accelerating reference frame is due to the way in which we use the horizon to separate particle modes and then choose to ignore the modes on the left side of the horizon.

We have considered two special cases when it comes to choosing how positive norm Rindler eigenfunctions on one side of the horizon join up with negative norm eigenfunctions with the same Ω-, k_y- and k_z-value on the other side. The first case is when we choose that they do not join up at all, that we use the eigenfunctions $g_{k_y k_z \Omega}$ which all vanish on one side of the horizon. In this case we get the distribution 6.14 of Rindler particles in the Minkowski vacuum. The other case is when we choose that they join up to form the eigenfunctions $\tilde{g}_{k_y k_z \Omega}$ as given in eq. 6.28. In that case we get a set of eigenfunctions whose number operator is identical to that of the inertial plane wave eigenfunctions, so the Minkowski vacuum looks completely empty with respect to the particle states defined by these eigenfunctions. But suppose that we choose some other way of joining the eigenfunctions to define a set of particle modes. In principle there are no restrictions on how we do this, since the field equation says nothing about how the eigenfunctions behave at the horizon. We could therefore do it in any way we want, and by doing so we can get pretty much any kind of distribution of particles we want in the Minkowski vacuum. By choosing the modes appropriately we can make any factor appear in eq. 6.29 instead of $e^{-\pi|\Omega|}$. This factor will then appear in eq. 6.14 instead of $e^{-\pi|\Omega|}$ and give us any desired kind of distribution. Apparently we therefore have not just one or two but infinitely many distinct vacuum states and sets of particle states in Minkowski spacetime!

6.4 Horizons and inequivalent particle modes

In the last section we saw that for a stationary reference frame with $\mathbf{a} \perp \boldsymbol{\omega}$ there exist an unending number of distinct sets of particle eigenfunctions that define inequivalent sets of particle states apart from the functions $g_{k_y k_z \Omega}$ and $\tilde{g}_{k_y k_z \Omega}$ that we have defined so far. So what if anything really distinguishes the modes $g_{k_y k_z \Omega}$ and $\tilde{g}_{k_y k_z \Omega}$ physically? From the observer's point of view, the modes $g_{k_y k_z \Omega}$ are obviously distinguished by the fact that they do not extend across the horizon. Since everything behind the horizon is causally disconnected from and can neither affect nor be affected by the observer, this is a natural, physical way of splitting the modes, and ought to be the modes best suited to describe what the observer experiences. The physical significance of the modes $\tilde{g}_{k_y k_z \Omega}$ on the other hand is not quite so obvious, except for the fact that they are apparently equivalent to the inertial plane-wave modes in some way. In this section we shall look at the physical significance of these modes and study more generally the connection between event horizons and the presence of thermal/statistical distributions of particles with respect to one set of particle modes in the vacuum state defined by another kind of particle modes.

The significance of different ways of combining the positive norm eigenfunction $g_{k_y k_z \Omega}$ on one side of the horizon with the negative norm eigenfunction $g^*_{(-k_y)(-k_z)(-\Omega)}$ on the other side of the horizon can be probed further by examining the behaviour of these two functions near the horizon. If we first take the limit of these functions as $x \to -t$ (in which both ξ

and τ are ill-defined) and then see what happens as we try to cross the singularity at the origin (where $(x,t)=(0,0)$, or $\xi=0$), we will make some interesting discoveries. I follow here the method used by Unruh in his article [1].

A difficulty in continuing the Rindler eigenfunctions across the horizon, is that the coordinates ξ and τ are singular at the horizon, and there do not even exist values of these coordinates that correspond to the border surfaces $x=t$ and $x=-t$. We therefore need to define some more convenient coordinates with which to express the eigenfunctions. Let us define the so-called null coordinates u and v by $u=t-x$ and $v=t+x$ ("null" because the vectors ∂_u and ∂_v are future-directed null vectors). In these coordinates the inertial plane-wave modes $f_{\mathbf{k}'}\sim e^{-iE_{\mathbf{k}'}t+i\mathbf{k}'\cdot\mathbf{x}}$ are given by:

$$f_{\mathbf{k}'}\sim e^{-iE_{\mathbf{k}'}t+i\mathbf{k}'\cdot\mathbf{x}}=e^{-\frac{i}{2}E_{\mathbf{k}'}(u+v)+\frac{i}{2}k'_x(v-u)+i(k'_y y+k'_z z)}=e^{-\frac{i}{2}(E_{\mathbf{k}'}+k'_x)u-\frac{i}{2}(E_{\mathbf{k}'}-k'_x)v+i(k'_y y+k'_z z)} \tag{6.34}$$

If we restrict these functions to the hypersurface $x=-t$ so that $v=0$ is constant (this is the 45° downwards slanting line in figure 4.1) and keep y and z constant (e.g. equal to 0), we may view $f_{\mathbf{k}'}$ as a function of u alone:

$$f_{\mathbf{k}'}(u)\sim e^{-\frac{i}{2}(E_{\mathbf{k}'}+k'_x)u} \tag{6.35}$$

For positive norm inertial modes, we have $E_{\mathbf{k}'}=+\sqrt{\mathbf{k}^2+M^2}>|k'_x|$, whereas we get a negative norm mode if we let $E_{\mathbf{k}'}=-\sqrt{\mathbf{k}^2+M^2}<-|k'_x|$. This means that if we now make an extension of $f_{\mathbf{k}'}$ by allowing u to be a complex variable, then all positive norm plane-wave modes will be analytic and bounded in the lower half complex u-plane, i.e. for $\mathrm{Im}(u)<0$. The negative norm plane wave modes on the other hand, will not be bounded on the lower half u-plane, and will in fact all diverge to infinity as $\mathrm{Im}(u)\to-\infty$, but will instead be analytic and bounded in the *upper* half of the complex u-plane. This observation will be useful for deciding whether or not the decomposition of a function into plane wave eigenfunctions contains only positive norm plane wave eigenfunctions or not; Any function that is analytic and bounded on the lower half of the complex u-plane must be a combination of only positive norm plane-wave functions, since any admixture of negative-norm functions would cause divergences or singularities on the lower half of the u-plane.

Now look at what happens to the Rindler eigenfunction $g_{k_y k_z\Omega}$ for $\xi>0$, $\Omega>0$ as we approach the limit $x=-t$. As I said, the Rindler coordinates (ξ,τ) are not well-defined there, so we need to express $g_{k_y k_z\Omega}$ in terms of the Cartesian coordinates (x,t) instead. We will assume that we are close to the origin $(x,t)=(0,0)$ or $\xi=0$. As $\xi\to0$, the Rindler modes have the asymptotic form (see remarks in section A.2):

$$\begin{aligned} g_{k_y k_z\Omega}(\chi,\xi,y,z)&\simeq\left(D_\Omega\xi^{i\Omega}+D^*_\Omega\xi^{-i\Omega}\right)e^{-i\Omega\chi}e^{i(k_y y+k_z z)}\\ &=\left(D_\Omega e^{i\Omega\ln\xi}+D^*_\Omega e^{-i\Omega\ln\xi}\right)e^{-i\Omega\chi}\,e^{i(k_y y+k_z z)}\end{aligned} \tag{6.36}$$

where D_Ω is a normalization factor. We do not need to know the value of this factor, only the fact that since the modified Bessel functions $K_{i\Omega}$ that give the ξ-dependence of

$g_{k_y k_z \Omega}(\chi, \xi, y, z)$ are defined in such a way that $K_{i\Omega} = K_{-i\Omega}$ for any real Ω, we must have $D_{-\Omega} = D^*_\Omega$. Inserting $\xi = \sqrt{x^2 - t^2}$ and $\chi = \operatorname{artanh} \frac{t}{x} = \frac{1}{2} \ln \frac{x+t}{x-t}$, we get:

$$\begin{aligned} g_{k_y k_z \Omega}(t, x, y, z) &\simeq \left(D_\Omega e^{i\Omega \frac{1}{2} \ln(x^2 - t^2)} + D^*_\Omega e^{-i\Omega \frac{1}{2} \ln(x^2 - t^2)} \right) e^{-i\Omega \frac{1}{2} \ln \frac{x+t}{x-t}} e^{i(k_y y + k_z z)} \\ &= \left(D_\Omega e^{i\Omega \ln(x-t)} + D^*_\Omega e^{-i\Omega \ln(x+t)} \right) e^{i(k_y y + k_z z)} \\ &= \left(D_\Omega e^{i\Omega \ln(-u)} + D^*_\Omega e^{-i\Omega \ln v} \right) e^{i(k_y y + k_z z)} \end{aligned} \tag{6.37}$$

For the corresponding negative norm mode $g_{(-k_y)(-k_z)(-\Omega)}$ on the left side of the horizon, at $\xi < 0$, the asymptotic behaviour is:

$$g^*_{(-k_y)(-k_z)(-\Omega)}(\chi, \xi, y, z) \simeq \left(D^*_{-\Omega} (-\xi)^{i\Omega} + D_{-\Omega} (-\xi)^{-i\Omega} \right) e^{-i\Omega\chi} e^{i(k_y y + k_z z)} \tag{6.38}$$

This time inserting $\xi = -\sqrt{x^2 - t^2}$ and $\chi = \operatorname{artanh} \frac{t}{x} = \frac{1}{2} \ln \frac{x+t}{x-t}$, using the fact that $D_{-\Omega} = D^*_\Omega$ and following the same procedure as previously, we get (keep in mind this time that $x + t$ and $x - t$ are negative, so that $\ln(x+t)^2 = 2\ln(-x-t)$, *not* $2\ln(x+t)$, and similarly for $\ln(x-t)^2$):

$$g^*_{(-k_y)(-k_z)(-\Omega)} \simeq \left(D_\Omega e^{i\Omega \ln u} + D^*_\Omega e^{-i\Omega \ln(-v)} \right) e^{i(k_y y + k_z z)} \tag{6.39}$$

This is the general form of the function when u and v are both small, i.e. when we are close to the origin. Let us now take the limit $x \to -t$ to get onto the surface $x = -t$. In this limit-taking, u approaches some definite value (which we have assumed to be small, i.e. we are close to the origin), and $v \to 0$. This means the u-dependent terms of $g_{k_y k_z \Omega}$ and $g^*_{(-k_y)(-k_z)(-\Omega)}$ approach some well-defined value, whereas the v-dependent terms oscillate wildly. This oscillation has a constant amplitude so that the v-dependent parts do not converge to any specific value, but this is only because we are using eigenfunctions with a completely sharp value of Ω. If we instead construct wave packets, peaked sharply but with non-zero width around a specific Ω-value, the contributions from the different frequencies will cancel out, and the v-dependent terms tend to zero (or, if you don't like constructing wave packets, pretend that the v-dependent terms have some unknown but constant value on the surface $x = -t$, since v is constant on the entire surface. That will do the same trick, since all that matters for the following discussion is that $g_{k_y k_z \Omega}$ and $g^*_{(-k_y)(-k_z)(-\Omega)}$ only depend on u on this surface). We can then discard the v-dependent terms in this limit. We will also suppress the y- and z-dependence, since only the u-dependence is relevant to the following discussion. We can then write:

$$g_{k_y k_z \Omega}(u) = D_\Omega e^{i\Omega \ln(-u)} \qquad g_{(-k_y)(-k_z)(-\Omega)}(u)^* = D_\Omega e^{-i\Omega \ln u} \tag{6.40}$$

where the first function is defined for $u < 0$ (the right side of the horizon) and the second for $u > 0$ (the left side of the horizon). As we have mentioned many times already, these functions are Rindler eigenfunctions with the same eigenvalue Ω, and can therefore be combined to yield new eigenfunctions. However, unlike ξ and χ, the coordinate u that we are now using, is *not* singular at the origin or anywhere else, and nor is the field equation expressed in terms of this coordinate. This means that we are no longer free to combine

the two functions on each side of the origin in any way we want. Rather, they should be combined in such a way that the resulting function becomes an analytic function of u when u is viewed as a complex variable. If is this were not the case, then we would have to introduce some border across which the field equation were not valid. But Minkowski spacetime contains no such natural borders, so doing this would have to be an artificial construct not supported by the physics of Minkowski spacetime. However, we have still demanded that the combination be an eigenfunction of $\frac{\partial}{\partial\chi} = -u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$, which is equal to $-u\frac{\partial}{\partial u} = \frac{\partial}{\partial \ln|u|}$ at the $v = 0$ surface, so we must expect a singularity at $u = 0$ and possibly a branch cut extending in some direction out from the origin, but otherwise the function should be analytic.

With these constraints, we can combine the functions in the following way: first denote the combined function by $\hat{g}_{k_y k_z \Omega}(u)$ and define it as:

$$\hat{g}_{k_y k_z \Omega}(u) = g_{k_y k_z \Omega}(u) + C g^*_{(-k_y)(-k_z)(-\Omega)}(u) \tag{6.41}$$

where $g_{k_y k_z \Omega}(u)$ by definition has support for $\mathrm{Re}\ u < 0$ and disappears for $\mathrm{Re}\ u > 0$, and vice versa for $g^*_{(-k_y)(-k_z)(-\Omega)}(u)$. C is now some appropriate constant which we need to choose so that $\hat{g}_{k_y k_z \Omega}(u)$ actually becomes an analytic function, with only the sort of singularities that we have listed above.

The expressions for $g_{k_y k_z \Omega}(u)$ and $g^*_{(-k_y)(-k_z)(-\Omega)}(u)$ in eq. 6.40 involve $\ln u$ and $\ln(-u)$, so in order to match them up, we first need to decide where to put the branch cut for the function $\ln z$. First let us lay it along the positive imaginary axis. In that case, we have that if $\mathrm{Re}\ u < 0$, then $\ln(-u) = \ln u + i\pi$, so that

$$g_{k_y k_z \Omega}(u) = D_\Omega e^{i\Omega \ln(-u)} = D_\Omega e^{i\Omega \ln u} e^{-\pi\Omega} \tag{6.42}$$

So, we see that if choose $C = e^{-\pi\Omega}$, then $g_{k_y k_z \Omega}(u)$ and $g^*_{(-k_y)(-k_z)(-\Omega)}(u)$ match up perfectly. So when we choose to have the branch of the logarithm function on the positive imaginary axis, we get

$$\hat{g}_{k_y k_z \Omega}(u) = g_{k_y k_z \Omega}(u) + e^{-\pi\Omega} g^*_{(-k_y)(-k_z)(-\Omega)}(u) \tag{6.43}$$

This is precisely the combination of $g_{k_y k_z \Omega}$ and $g_{(-k_y)(-k_z)(-\Omega)}$ we had in the function $\tilde{g}_{k_y k_z \Omega}$ that we defined in section 6.3! Note now that since we chose to put the branch cut of the logarithm in the upper half of the complex u-plane, $\hat{g}_{k_y k_z \Omega}(u)$ is in fact bounded and analytic on the entire lower half of the u-plane. As we found from eq. 6.35, this means that $\hat{g}_{k_y k_z \Omega}(u)$ must be a linear combination of only positive norm inertial plane wave mode functions! ($\hat{g}_{k_y k_z \Omega}(u)$ is of course bounded on the upper half of the complex u-plane as well, but because of the branch cut it is not analytic there, and hence we cannot use its behaviour there to draw any conclusions about what sort of plane wave modes it consists of.) This analysis has of course so far been only for the surface $x = -t$, but by specifying the value of a function on this surface, we have a boundary condition that fixes the value of the function everywhere in Minkowski spacetime. This means that the function $g_{k_y k_z \Omega}(\chi, \xi, y, z) + e^{-\pi\Omega} g^*_{(-k_y)(-k_z)(-\Omega)}(\chi, \xi, y, z)$ must in fact be a linear combination of positive norm plane wave functions on the whole of Minkowski spacetime. This is an alternative way of proving that the function $\tilde{g}_{k_y k_z \Omega}(\chi, \xi, y, z)$ consists only of positive norm

inertial plane wave functions, which leads directly to eq. 6.29, which further gives us the expression 6.14 for how the Minkowski vacuum is related to the Rindler particle states, without making any direct use of Bogoliubov transformations, and thus avoiding the more direct but fairly tedious calculation of the Bogoliubov coefficients.

If we had instead chosen the branch cut of the logarithm function to be in the lower half of the complex u-plane, e.g. along the negative imaginary axis, we would have obtained $C = e^{+\frac{\pi\Omega}{|a_0|}}$. Using this factor to combine $g_{k_y k_z \Omega}$ and $g_{(-k_y)(-k_z)(-\Omega)}$ would have given us something proportional to the function $\tilde{g}_{(-k_y)(-k_z)(-\Omega)}$ instead. Using the fact that this function would now have been bounded and analytic on the entire *upper* half of the complex u-plane, we could have concluded that it is a combination of only *negative* norm plane wave functions, which is of course entirely consistent with what we derived above. Instead of using the hypersurface $t = -x$ where $v = 0$, we could also have used the hypersurface $t = +x$, where $u = 0$, and viewed everything as functions of v instead of u. Using the same procedures, the conclusions in this case would have been exactly the same.

In doing all this, we have not only proved in a more circuitous way that combining functions on either side of the horizon into the functions $\tilde{g}_{k_y k_z \Omega}$ gives us a set of particle modes that are equivalent to the inertial plane wave modes. We have also demonstrated that this is in some sense the "correct" way to join up the eigenfunctions on each side of the horizon. Since there are no physical singularities anywhere in Minkowski spacetime, and since the field equation is not singular anywhere apart from coordinate singularities in the coordinates that we have used to write it out, the solutions should be "as analytic as possible", except that there may be singularities associated with requiring that they be eigenfunctions of some operator (e.g. $\frac{\partial}{\partial\chi}$) that might be singular in some places. The combination of Rindler modes that form the functions $\tilde{g}_{k_y k_z \Omega}$ is exactly the kind of joining that makes them as analytic as they can get. The only "excuse" that we have for splitting up the mode functions in any other fashion, as the functions $g_{k_y k_z \Omega}$ or otherwise, is that precisely because of the presence of the horizon, no stationary observer on one side of the horizon can tell how the modes behave across the horizon anyway.

To sum up, we could conclude that for any stationary reference frame in Minkowski spacetime, the only "true" kind of particles and the only "true" number operator is that defined by an inertial Hamiltonian, and the Minkowski vacuum is the only "true" vacuum: All maximally analytic sets of mode functions define the same number operator. Any non-equivalent number operators are simply an effect of splitting up modes in an arbitrary, non-analytic manner. Indeed, as we saw in the previous section, with such splitting we can construct any number of number operators and vacua that are not equivalent to the inertial ones, not just the Rindler vacuum and its corresponding number operator. But they are all based on modes that are less analytic than the structure of Minkowski spacetime would warrant, and in that sense "artificial".

However, in reference frames where there is a horizon, an observer cannot causally affect or be affected by what goes on behind the horizon, so from an observer's point of view it can still be seen as natural to actually divide the mode functions as we did with the set $g_{k_y k_z \Omega}$, with functions that only have support on one side of the horizon, and then average out over all possible states on the far side of the horizon to get a sort of thermally mixed state in front of the horizon, even if this procedure is based on mode functions that

are not maximally analytic. In this case we do get an inequivalent number operator due to the non-analytic splitting of the mode functions. And then, because of the averaging, we make the Minkowski vacuum look like a thermal state with a certain temperature. As we shall see in the next chapter, a naively constructed particle detector travelling along a stationary trajectory in the Minkowski vacuum will in fact seem to respond to these kinds of particles when there is a horizon present, since the detector cannot detect the part of the modes that lie beyond the horizon, and also cannot affect that part of the modes. However, we get some additional complications when there exist modes with negative energy that are not confined behind the horizon, i.e. when $\omega \neq 0$. And even when $\omega = 0$ it is in fact possible to interpret the detector response without making use of hot Rindler particles flying around in the vacuum.

Note that these remarks about the presence of a horizon and analytic versus non-analytic particle mode functions also hold in general spacetimes, not just Minkowski spacetime. When there is an event horizon present, we can argue that it is natural to choose modes that have support only on one side of the horizon. It also seems natural that a particle detector should respond to particles defined using that kind of modes and averaging over all states behind the horizon, regardless of the analyticity of these modes. If on the other hand there is no horizon anywhere, then it would be natural to demand that the modes have to be maximally analytic, and this might rule out certain ambiguities about how to define particle states in the spacetime. However, in spacetimes other than Minkowski spacetime, it may be possible to find maximally analytic sets of particle mode functions that do not all give rise to equivalent number operators, so that we may have inequivalent number operators and vacua without having to involve horizons and more or less artificial mode splittings. A horizon of some sort is nevertheless necessary in order to construct a thermal density matrix out of the vacuum as in eq. 6.33. The non-pure thermal nature of this density matrix arises precisely because our observer is not causally connected to the events that take place behind the horizon so that we can trace out the density matrix over all states there. Without this tracing, the density matrix is just that of a pure state, so that there are no thermal or statistical mechanical effects.

Chapter 7

Detecting particles — or then again?

In the last chapter we found that the Minkowski vacuum is a superposition of different Rindler particle states, and found that if we ignore what happens on the far side of the horizon and average out over field configurations there, the Minkowski vacuum looks like a thermal ensemble of Rindler particles at temperature $kT = \sigma/2\pi \equiv \frac{\sqrt{a^2-\omega^2}}{2\pi}$. Therefore an observer travelling along a stationary trajectory with either $\mathbf{a}$ not perpendicular to $\boldsymbol{\omega}$ or with $\mathbf{a} \perp \boldsymbol{\omega}$ and $a > \omega$ should see the Minkowski vacuum as actually containing a thermal distribution of particles. However, what we have looked at, is really just what kind of states the Minkowski vacuum is comprised of in terms of eigenstates of the Hamiltonian H_{RF} that generates proper time translations with respect to the observer. What an observer will actually "see" with his or her own eyes, is not necessarily the same thing.

In this chapter we will try to address the question of what an observer really sees when travelling through the Minkowski vacuum along the various kinds of stationary trajectories. Obviously, the eyes of an observer may not be the best measuring devices to use in observing a quantum field. Instead, we will look at how a model particle detector travelling alongside the observer responds to the Minkowski vacuum state surrounding it. It has been noted in many publications that a model particle detector that is travelling with linear constant proper acceleration (i.e. following a stationary orbit with $\boldsymbol{\omega} = 0$) will indeed register a thermal distribution of particles with the temperature associated with the "standard" Unruh effect. This was first noted already by Unruh ([1]). It has also been frequently noted that if rotation is thrown into the mix ($\boldsymbol{\omega} \neq 0$, whether $a > \omega$ or not), the spectrum detected by the detector will have a non-thermal character and be non-vanishing even when $a < \omega$ (see e.g. [10]). However, few articles go into much detail about what really causes this discrepancy. Trying to offer an explanation for why the detector acts as it does in simple physical terms is therefore the main topic for this final chapter of the thesis. It will turn out that the detector actually *does* detect a thermal distribution of Rindler particles when following the appropriate kinds of trajectories, even when $\omega \neq 0$, but that it also detects *more* in the vacuum than just lukewarm gases of Rindler particles.

7.1 Model particle detector

In order to model the detection of particles in a quantum field $\phi(x)$, we obviously need to devise some sort of quantum system coupled to the field that can act as a model particle detector. Since the purpose of this detector is only to detect particles in the quantum field, we will only be interested in how it interacts with the field, and not concern ourselves with the details of its inner workings or how it behaves in the absence of any field. Indeed, we would like our results to be as independent of the specifics of the detector as possible. The kind of model detector that I will construct in this section is a so-called monopole detector, also known as a de Witt-detector, that is commonly employed in the literature, with numerous variations. The specific detector that I construct here is modelled on the one used in Birrell and Davies' book [11], although I have added most of the physical reasoning as to why this is a viable way to construct a model particle detector myself.

So, let us first assume that the detector is some quantum system that can be described by a quantum field $\Delta(x)$ (this obviously is an assumption one should always be able to get away with). Since we are not really interested in the detector itself, we won't bother to specify any free Lagrangian or Hamiltonian for it, just say that we denote the free Hamiltonian of the detector (i.e. the Hamiltonian when it is not yet coupled to the field $\phi(x)$) by H_{RF}^{Δ}, and the free Hamiltonian of the field $\phi(x)$ by H_{RF}^{ϕ}. The total free Hamiltonian is then $H_{\mathsf{RF}} = H_{\mathsf{RF}}^{\Delta} + H_{\mathsf{RF}}^{\phi}$. The coupling between Δ and ϕ we denote by an interaction Hamiltonian density $\mathcal{V}$. We will assume that $\mathcal{V}$ has the form

$$\mathcal{V}(x) = \mathcal{M}(x)\phi(x) \tag{7.1}$$

where $\mathcal{M}(x)$ is some function of the detector field operator $\Delta(x)$. This ought to be a realistic model for any interaction Hamiltonian. Practically every type of particle detector that we have ever really built, basically boils down to this type of interaction, namely that between the electron and the photon field, which has the form $\bar{\psi}\gamma^{\mu}\psi A_{\mu}$ (where ψ is the electron four-component spinor field, A_{μ} is the photon 4-vector field, γ^{μ} are the 4×4 Dirac matrices and $\bar{\psi} = \psi^{\dagger}\gamma^{0}$).

Now, we want the detector in a sense to act as the eyes of the observer, so we want the detector to follow the exact same trajectory as the observer. This requires that the detector (or at least its interaction with the field) be localized to precisely that trajectory. In order to achieve this, we need $\mathcal{V}(x)$ to vanish everywhere except when x is on the trajectory of the observer. Therefore, $\mathcal{M}(x)$ must have the form

$$\mathcal{M}(x') = g(\tau)m(\tau)\,\delta\left((x' - x(\tau)\right) \tag{7.2}$$

where τ is the proper time of the observer, $x(\tau)$ denotes the position of the observer/detector at proper time τ and $m(\tau)$ is some operator acting on the state space of the detector. $g(\tau)$ is a coupling constant, whose proper time dependence we will comment on later.

This kind of localization of a quantum mechanical system is of course impossible due to the uncertainty principle. However, idealized detectors always have a tinge of classicality about them anyway, so we will try not to let this bother us to much. We can assume that the mass of the detector is very large, practically infinite, compared with the energies involved, so that we can with some justification ignore any small quantum uncertainties about its

position as well as any recoils from the interaction with the quantum field. The detector will still capture the physics which is of interest to us here. More realistic treatments (at least with a somewhat liberal interpretation of 'realistic') with less completely localized detectors can be found e.g. in Unruh's original article [1], the article [13] by Bell, Hughes and Leinaas, or the article [14] by B. Reznik which deals with the first order corrections due to quantum uncertainties in the trajectory of the detector and recoil from particle detection. The

We assume that the detector possesses a complete set of stationary energy eigenstates $\left|\epsilon_i^\Delta\right\rangle$, i.e. states that are eigenstates of H_{RF}^Δ, and the corresponding eigenvalues ϵ_i^Δ are the possible energies that the detector may have with respect to H_{RF}. Furthermore, the detector should reflect only the proper time dependence of the quantum field ϕ that it measures and not introduce any explicit time dependence of its own, so we assume that $m(\tau)$ depends on time only through the Hamiltonian, i.e.

$$\frac{d}{d\tau}m(\tau) = -i\left[m(\tau),\, H\right] \tag{7.3}$$

Now let us see what happens to the detector when it is put in the quantum field and allowed to interact with it. We have the following Hamiltonian H for the combined system of the quantum field ϕ and the detector:

$$H = H_{\mathsf{RF}} + \int d^3x\, \mathcal{V}(x) = H_{\mathsf{RF}} + g(\tau)\, m(\tau)\, \phi\left(x(\tau)\right) \equiv H_{\mathsf{RF}} + V\left(\tau\right) \tag{7.4}$$

We want to calculate the probability that a detector will be excited as it travels through the quantum field. We assume that the detector starts out in some free energy eigenstate $\left|\epsilon_i^\Delta\right\rangle$, which is an eigenstate of H_{RF}^Δ at $\tau = \tau_i$, and that the field starts out in some state $\left|\phi_i\right\rangle$ and denote their joint state by $\left|\epsilon_i^\Delta,\, \phi_i\right\rangle$. We then want to calculate the probability $P_{i\to f}$ that the detector ends up in the energy eigenstate ϵ_f^Δ at $\tau = \tau_f$ while the field ends up in some state $\left|\phi_f\right\rangle$, written jointly as $\left|\epsilon_f^\Delta,\, \phi_f\right\rangle$. If the detector ends up in this state, we will at first take this to mean it has detected a particle with energy $\epsilon_f^\Delta - \epsilon_i^\Delta$ in the field (although this will turn out to not always be a good interpretation).

Until now we have used the Heisenberg picture, in which the operators evolve with proper time as in eq. 7.3 whereas the state vectors remain constant. To calculate transition probabilities between free energy eigenstates, it will be more convenient to work with the interaction picture. In this picture the operators evolve as the non-interacting operators do in the Heisenberg picture, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}\tau}A = -i\left[A, H_0\right] \tag{7.5}$$

for any operator A, where H_0 is the non-interacting Hamiltonian, which in our case is H_{RF}. This kind of behaviour is convenient, since it means that H_{RF} will be constant, and therefore it will have the same eigenstates at all τ. The time evolution of the states $\left|\Psi\right\rangle$ can then be found by noting that since the quantity $\left\langle\Phi\right| A \left|\Psi\right\rangle$ for any states $\left|\Psi\right\rangle$ and $\left|\Phi\right\rangle$ must be the same independently of which picture we use, we must have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left\langle\Phi\right| A \left|\Psi\right\rangle\right) = -i\left\langle\Phi\right|\left[A, H\right]\left|\Psi\right\rangle = -i\left\langle\Phi\right|\left[A, H_{\mathsf{RF}}\right]\left|\Psi\right\rangle - i\left\langle\Phi\right|\left[A, V\right]\left|\Psi\right\rangle \tag{7.6}$$

in any picture. If we are now in the interaction picture, where the time dependence of A is given by eq. 7.5, the time dependence of the states must be given by

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\left|\Psi\right\rangle = V\left|\Psi\right\rangle \tag{7.7}$$

This equation has the following solution:

$$\left|\Psi(\tau)\right\rangle = \left(1 + (-i)\int_{\tau_0}^{\tau} d\tau_1\, V(\tau_1) + (-i)^2 \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2\, V(\tau_1)V(\tau_2) + \ldots\right)\left|\Psi(\tau_0)\right\rangle \tag{7.8}$$

In the higher order integrals we are free to interchange the order of integration, as long as we keep the ordering of the operators and are careful to rearrange the integration limits appropriately. By writing out all possible permutations of each term in the integral expansion in this way, we can arrange it so that they add up in such a way that all the integrals run from τ_0 to τ, but with a factor of $1/n!$ in front, where n is the number of integrals in the term. We then get:

$$\begin{aligned}\left|\Psi(\tau)\right\rangle &= \sum_{n=0}^{\infty} T\frac{(-i)^n}{n!}\int_{\tau_0}^{\tau} d\tau_1\, V(\tau_1)\cdots\int_{\tau_0}^{\tau} d\tau_n\, V(\tau_n) \\ &= T\exp\left(-i\int_{\tau_0}^{\tau} d\tau'\, V(\tau')\right)\end{aligned} \tag{7.9}$$

where the symbol T signifies that all operator products behind it must be time-ordered, i.e.

$$TV(\tau_1)V(\tau_2) = \begin{cases} V(\tau_1)V(\tau_2)\,, & \tau_1 > \tau_2 \\ V(\tau_2)V(\tau_1)\,, & \tau_1 < \tau_2 \end{cases} \tag{7.10}$$

and correspondingly for products of more than two operators.

Now we can calculate the transition probability for the detector and the field. If we start out in the state $\left|\epsilon_i^{\Delta},\, \phi_i\,(\tau_i)\right\rangle$ at $\tau = \tau_i$, then at $\tau = \tau_f$ the system will be in the state:

$$\begin{aligned}\left|\epsilon_i^{\Delta},\, \phi_i\,(\tau_f)\right\rangle &= T\exp\left(-i\int_{\tau_i}^{\tau_f} d\tau'\, V(\tau')\right)\left|\epsilon_i^{\Delta},\, \phi_i\,(\tau_i)\right\rangle \\ &= T\exp\left(-i\int_{\tau_i}^{\tau_f} g(\tau')\, m(\tau')\, \phi(x(\tau'))\right)\left|\epsilon_i^{\Delta},\, \phi_i\,(\tau_i)\right\rangle\end{aligned} \tag{7.11}$$

Since the detector should detect the field as accurately as possible without disturbing it too much, the coupling must only be a weak perturbation. We therefore assume that the coupling strength $g(\tau)$ is very small, and that it suffices to expand eq. 7.11 to first order in $g(\tau)$:

$$\left|\epsilon_i^{\Delta},\, \phi_i\,(\tau_f)\right\rangle \simeq \left(1 - i\int_{\tau_i}^{\tau_f} d\tau'\, g(\tau')\, m(\tau')\, \phi(x(\tau'))\right)\left|\epsilon_i^{\Delta},\, \phi_i\,(\tau_i)\right\rangle \tag{7.12}$$

The probability amplitude $A_{i\to f}$ to find the system in the state $\left|\epsilon_f^\Delta,\,\phi_f\right\rangle$ at $\tau=\tau_f$ is then equal to:

$$A_{i\to f}=\left\langle\epsilon_f^\Delta,\,\phi_f\right|\left(1-i\int_{\tau_i}^{\tau_f}d\tau'\,g(\tau')\,m(\tau')\,\phi(x(\tau'))\right)\left|\epsilon_i^\Delta,\,\phi_i\right\rangle \tag{7.13}$$

We are only interested in transitions that the detector makes between states with different energies, $\epsilon_i^\Delta\neq\epsilon_f^\Delta$, which means that $\left\langle\epsilon_i^\Delta|\epsilon_f^\Delta\right\rangle=0$ so that we can ignore the first term in eq. 7.13. The probability $P_{i\to f}$ is then given by:

$$\begin{aligned}P_{i\to f}&=\left|A_{i\to f}\right|^2\\&=\left\langle\epsilon_i^\Delta,\,\phi_i\right|\int_{\tau_i}^{\tau_f}d\tau\,g(\tau)^*\,m^\dagger(\tau)\,\phi^\dagger(x(\tau))\left|\epsilon_f^\Delta,\,\phi_f\right\rangle\\&\quad\left\langle\epsilon_f^\Delta,\,\phi_f\right|\int_{\tau_i}^{\tau_f}d\tau'\,g(\tau')\,m(\tau')\,\phi(x(\tau'))\left|\epsilon_i^\Delta,\,\phi_i\right\rangle\end{aligned} \tag{7.14}$$

We are not really interested in the final state of the field ϕ, only that of the detector. Therefore, if we redefine $P_{i\to f}$ to be the probability of the detector making a transition from $\left|\epsilon_i^\Delta\right\rangle$ to $\left|\epsilon_f^\Delta\right\rangle$ regardless of the final configuration of the field (but retaining the condition that the field be initially in the state $|\phi_i\rangle$), we must sum sum over all the final field states $|\phi_f\rangle$ in eq. 7.14. We can then make use of the completeness relation $\sum_n|\phi_n\rangle\langle\phi_n|=1$ to remove the final field states completely from eq. 7.14. Furthermore, since we are in the interaction picture, $m(\tau)$ evolves in time as a non-interacting operator:

$$m(\tau)=e^{iH_{\mathsf{RF}}^\Delta\tau}m(0)e^{-iH_{\mathsf{RF}}^\Delta\tau}\equiv e^{iH_{\mathsf{RF}}^\Delta\tau}m\,e^{-iH_{\mathsf{RF}}^\Delta\tau} \tag{7.15}$$

Inserting this into eq. 7.14 then finally gives us:

$$P_{i\to f}=\int_{\tau_i}^{\tau_f}d\tau\int_{\tau_i}^{\tau_f}d\tau' g(\tau)^*g(\tau')e^{-i(\epsilon_f^\Delta-\epsilon_i^\Delta)(\tau-\tau')}\mathcal{T}(\epsilon_f^\Delta,\epsilon_i^\Delta)\left\langle\phi_i\right|\phi\left(x(\tau)\right)\phi\left(x(\tau')\right)\left|\phi_i\right\rangle \tag{7.16}$$

where $\mathcal{T}(\epsilon_f^\Delta,\epsilon_i^\Delta)=\left|\left\langle\epsilon_f^\Delta\right|m\left|\epsilon_i^\Delta\right\rangle\right|^2$. This function is a measure of the sensitivity of the detector to the transition between the energy levels ϵ_i^Δ and ϵ_f^Δ and depends on the details of how the detector works, whereas the rest of the expression 7.16 does not.

This expression is quite general, using only the form of the interaction Hamiltonian and the weakness of the coupling as assumptions. In our specific case of measuring particles in the Minkowski vacuum with a detector following a stationary trajectory, we can make a few more assumptions. First of all we will get rid of the time dependence of $g(\tau)$. This time dependence allows us to switch the detector on at early times and switch it off again later, so that the initial and final states will be stationary, free energy eigenstates. We assume now that the detector is switched on very early and switched off again very late, and slowly in such a way that this switching will have a completely negligible probability of causing any excitations in the detector, so that for all practical purposes we can assume that $g(\tau)\simeq g$ is constant throughout the measurement process (to be more precise, we let the Fourier transform $\tilde{g}(\omega)$ of $g(\tau)$ be peaked sharply around $\omega=0$, with the width of the peak much,

much smaller than $\epsilon_f^\Delta - \epsilon_i^\Delta$ for all possible energy differences involved). We also assume that g is real. Next, in our case we are only interested in the case where the quantum field starts out in the Minkowski vacuum state, $|\phi_i\rangle = |0\rangle_M$. Therefore we can write

$$\langle\phi_i|\,\phi(x(\tau))\,\phi(x(\tau'))\,|\phi_i\rangle = {}_M\langle 0|\,\phi(x(\tau))\,\phi(x(\tau'))\,|0\rangle_M \equiv G^+(x(\tau), x(\tau')) \tag{7.17}$$

where G^+ is called the positive frequency Wightman function. Furthermore, since the detector is following a stationary trajectory, $G^+(x(\tau), x(\tau'))$ should only depend on the proper time difference $\tau - \tau'$ (this of course needs to be calculated explicitly, but will turn out to be true). We therefore define $G^+(\tau-\tau') = G^+(x(\tau), x(\tau'))$. With these assumptions, eq. 7.16 becomes:

$$P_{i\to f} = g^2\mathcal{T}(\epsilon_f^\Delta, \epsilon_i^\Delta)\int_{\tau_i}^{\tau_f} d\tau \int_{\tau_i}^{\tau_f} d\tau'\, e^{-i(\epsilon_f^\Delta - \epsilon_i^\Delta)(\tau-\tau')}G^+(\tau-\tau') \tag{7.18}$$

Now change variables to $u = \tau + \tau'$ and $s = \tau - \tau'$. We can then let $\tau_i - \tau_f$ and $\tau_f - \tau_i$ be the integration limits for s, which implies that $2\tau_i - s$ and $2\tau_f - s$ will be the integration limits for u, since $u + s = 2\tau$, so that $2\tau_i \le u + s \le 2\tau_f$:

$$\begin{aligned} P_{i\to f} &= g^2\mathcal{T}(\epsilon_f^\Delta, \epsilon_i^\Delta)\int_{\tau_i-\tau_f}^{\tau_f-\tau_i} ds \int_{2\tau_i-s}^{2\tau_f-s} du\, e^{-i(\epsilon_f^\Delta - \epsilon_i^\Delta)s}G^+(s) \\ &= 2\,(\tau_f - \tau_i)\, g^2\mathcal{T}(\epsilon_f^\Delta, \epsilon_i^\Delta)\int_{\tau_i-\tau_f}^{\tau_f-\tau_i} e^{-i(\epsilon_f^\Delta - \epsilon_i^\Delta)s}G^+(s) \end{aligned} \tag{7.19}$$

So, finally, the transition *rate* from $\left|\epsilon_i^\Delta\right\rangle$ to $\left|\epsilon_f^\Delta\right\rangle$ is given by:

$$\Gamma_{i\to f} \equiv \frac{P_{i\to f}}{\tau_f - \tau_i} = 2g^2\mathcal{T}(\epsilon_f^\Delta, \epsilon_i^\Delta)\int_{\tau_i-\tau_f}^{\tau_f-\tau_i} ds\, e^{-i(\epsilon_f^\Delta - \epsilon_i^\Delta)s}G^+(s) \tag{7.20}$$

For the rest of this chapter, I will assume that the detector is either equally sensitive to all energies, or that we can correct for any variations in sensitivity. For simplicity I will therefore assume that $\mathcal{T}(\epsilon_f^\Delta, \epsilon_i^\Delta) = 1$.

So now we have an expression for the rate at which the detector is excited into the state $\left|\epsilon_f^\Delta\right\rangle$ if it travels along a given stationary trajectory. Still, I have not yet said anything about what this really tells us about the state of the quantum field ϕ and what particles it contains. In order to describe fully the behaviour of the detector and its interaction with the field, we could calculate the function $G^+(s)$ and evaluate explicitly the integral 7.20. We could then hopefully deduce exactly what the particle distribution that the detector has detected is by calculating the corresponding transition rate of a detector following an inertial world-line with the field starting out in an initial state $|\phi_i\rangle$ which is some distribution of Minkowski particle states.

Calculating $G^+(s)$ is no problem. The form of ${}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M$ for a Klein-Gordon field ϕ is found in any standard textbook on quantum field theory, and is fairly straightforward to calculate anyway. However, the integral 7.20 can only be evaluated analytically for a few special stationary trajectories (see [10] for a case-by-case discussion). Also,

whether we calculate it numerically or analytically, it is not quite clear how one should interpret the result. As I mentioned, we could compare it to the transition rate of an inertial detector in a certain background of Minkowski particles. But a stationary non-inertial detector is not supposed to detect Minkowski particles, it is supposed to detect particles as defined using H_{RF} for a stationary non-inertial observer, and it is not at all obvious that the detector should react in the same way to that kind of particles as it would to Minkowski particles with the same energy (after all, even the "energies" in question are defined differently, one as the eigenvalue of an inertial Hamiltonian and the other as the eigenvalue of H_{RF} for a non-inertial observer).

So what then should we calculate and how should we interpret it in order to find out what is going on with the particles in the field ϕ? Actually, we are not necessarily interested in knowing the exact details of how many non-inertial particles there are of what kind in the Minkowski vacuum. What really interests us, is to see whether or not the detector accurately reflects the thermal distribution of Rindler particles in the Minkowski vacuum that we found in the previous chapter. By "accurately reflects" I mean that the probability of finding the detector in a given energy level after the measurement should be equal to what we would expect if the detector were a system thermally coupled to a heat reservoir at the temperature in question. This will be so if the *ratio* of the probability P_1 of finding the detector in the state $\left|\epsilon_1^{\Delta}\right\rangle$ and the probability P_2 of finding it in the state $\left|\epsilon_2^{\Delta}\right\rangle$ is equal to $P_1/P_2 = e^{(\epsilon_2^{\Delta}-\epsilon_1^{\Delta})/kT}$. So how do we calculate this ratio? Note that whatever the state of the quantum field is, if the operator $m(\tau)$ is hermitian (which it had better be, if the interaction Hamiltonian $\mathcal{V}$ is to be Hamiltonian), the probability $P_{1\to 2}$ of going from state 1 to 2 given that the system is already in state 1 is equal to the probability $P_{2\to 1}$ of going the other way if the system is in state 2. If P_1 and P_2 are the probabilities of the system actually being in the states 1 or 2 respectively, then the transition rates must be

$$\Gamma_{1\to 2} \propto P_1 P_{1\to 2} \qquad\qquad \Gamma_{2\to 1} \propto P_2 P_{2\to 1} \tag{7.21}$$

Since $P_{1\to 2} = P_{2\to 1}$, the sought-after probability ratio must be:

$$\frac{P_1}{P_2} = \frac{\Gamma_{1\to 2}}{\Gamma_{2\to 1}} \tag{7.22}$$

which is the expression we need to interpret the results of the detector measurements.

In the following, we will always take the limit $\tau_i \to -\infty$ and $\tau_f \to \infty$ in order to make the integral 7.20 simpler to solve. This also makes sense, since quantum mechanically the energy of a transition is only well defined in the limit where the time interval over which the transition takes place becomes infinitely long, and we also want to make sure that the detector has reached a thermal equilibrium with the field and that all transient behaviour has died down. With all these definitions, assumptions and calculations behind us, we will use the remaining sections of this chapter to discuss and interpret what happens to the detector when it is placed on various kinds of stationary trajectories through the Minkowski vacuum.

7.2 Detecting Rindler particles — or not?

We will now study what happens to a detector like the one that we constructed in the previous section when it travels along a stationary world-line with $\mathbf{a} \perp \boldsymbol{\omega}$ and $a > \omega$ through the Minkowski vacuum of the quantum field ϕ. We assume that the detector has (at least) two distinct energy levels $\left|\epsilon_1^\Delta\right\rangle$ and $\left|\epsilon_2^\Delta\right\rangle$, and calculate the transition rates $\Gamma_{1\to2}$ and $\Gamma_{2\to1}$ between them as outlined in the last section.

Let us begin by calculating the correlation function $G^+(x,x') \equiv {}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M$. This is most easily calculated using normal Cartesian coordinates $(t';x',y',z')$ in the "primed" inertial frame from section 4.4 where the rotational part of H_{RF} has been removed (after boosting, as given by eq. 4.46), and by expanding the field operator in inertial plane wave modes, $f_{\mathbf{k}} = ((2\pi)^3 2E)^{-1/2} e^{-iEt'+i\mathbf{k}\cdot\mathbf{x}'}$. As we shall see though, this much-used approach does not offer quite enough insight in cases where the result is not completely straightforward to interpret. Still, it is convenient for deriving an expression that is practical to use when performing numerical integration in the general case of $\boldsymbol{\omega} \neq 0$, so let us use it to find out what the problem is before we turn to more instructive but computationally awkward methods:

$$\begin{aligned}
G^+(x,x') &\equiv {}_M\langle 0|\,\phi(x)\phi(x')\,|0\rangle_M \\
&= \int d^3k \int d^3k' \frac{1}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}\sqrt{2E_{\mathbf{k}'}}}\, {}_M\langle 0|\left(a_{\mathbf{k}}e^{-iE_{\mathbf{k}}t+i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^\dagger e^{+iE_{\mathbf{k}}t-i\mathbf{k}\cdot\mathbf{x}}\right) \\
&\qquad\qquad \left(a_{\mathbf{k}'}e^{-iE_{\mathbf{k}'}t'+i\mathbf{k}'\cdot\mathbf{x}'} + a_{\mathbf{k}'}^\dagger e^{+iE_{\mathbf{k}'}t'-i\mathbf{k}'\cdot\mathbf{x}'}\right)|0\rangle_M \\
&= \int d^3k \int d^3k' \frac{1}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}\sqrt{2E_{\mathbf{k}'}}}\, {}_M\langle 0|\,a_{\mathbf{k}}a_{\mathbf{k}'}^\dagger\,|0\rangle_M\; e^{-i(E_{\mathbf{k}}t-E_{\mathbf{k}'}t')+i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{x}')} \\
&= \int d^3k \frac{1}{(2\pi)^3\,2E_{\mathbf{k}}} e^{-iE_{\mathbf{k}}(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}
\end{aligned} \tag{7.23}$$

To solve this integral it is helpful to express $\mathbf{k}$ in spherical coordinates. We also assume that the field is massless, so that $k \equiv |\mathbf{k}| = E_{\mathbf{k}}$. To simplify he expression, denote $|\mathbf{x}-\mathbf{x}'|$ by Δr and $t-t'$ by Δt. Using the direction of $\mathbf{x}-\mathbf{x}'$ as the central axis, we get:

$$\begin{aligned}
G^+(x,x') &= 2\pi \int_0^\infty k^2\,dk \int_{-1}^1 d(\cos\theta) \frac{1}{(2\pi)^3 2k} e^{-ik\Delta t+ik\Delta r\cos\theta} \\
&= \frac{1}{8\pi^2}\int_0^\infty dk\, \frac{e^{ik\Delta r}-e^{-ik\Delta r}}{i\Delta r}\, e^{-ik\Delta t} \\
&= \frac{1}{8\pi^2}\left[\frac{1}{\Delta r}\left(\frac{e^{ik\Delta r-ik\Delta t}}{\Delta t-\Delta r} - \frac{e^{-ik\Delta r-ik\Delta t}}{\Delta t+\Delta r}\right)e^{-ik\Delta t}\right]_{k=0}^{k=\infty}
\end{aligned} \tag{7.24}$$

This expression has no well defined limit for $k \to \infty$. To make it well defined, we use a widely used trick[1] in quantum field theory by adding an infinitesimal imaginary term to the

[1] It is really not "just a trick". It can be physically motivated e.g. by the Gell-Mann-Lowe-theorem in which

time difference, $\Delta t \to \Delta t - i\epsilon$, where ϵ is small and positive. The whole expression then vanishes for $k \to +\infty$, and we are left with

$$\begin{aligned} G^+(x,x') &= -\frac{1}{8\pi^2}\frac{1}{\Delta r}\left(\frac{1}{\Delta t - i\epsilon - \Delta r} - \frac{1}{\Delta t - i\epsilon + \Delta r}\right) \\ &= \frac{-1}{4\pi^2\left[(\Delta t - i\epsilon)^2 - (\Delta r)^2\right]} \end{aligned} \tag{7.25}$$

Now we can calculate the function $G^+(\tau,\tau') \equiv G^+(x_{obs}(\tau), x_{obs}(\tau'))$ for the trajectory of the observer. In the coordinates $(t';x',y',z')$ that we are using, the trajectory of the observer is given by eq. 4.59. This gives us:

$$\begin{aligned} \Delta t'_{obs} &= \frac{a}{\sigma^2}\left[\sinh(\sigma\tau) - \sinh(\sigma\tau')\right] \\ \Delta x'_{obs} &= \frac{a}{\sigma^2}\left[\cosh(\sigma\tau) - \cosh(\sigma\tau')\right] \\ \Delta y'_{obs} &= -\frac{\omega}{\sigma}(\tau - \tau') = -\frac{\omega}{\sigma}\Delta\tau \\ \Delta z'_{obs} &= 0 \end{aligned} \tag{7.26}$$

This gives us (dropping the subscript obs for simplicity):

$$\begin{aligned} (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 &= \frac{a^2}{\sigma^4}\left[\sinh^2(\sigma\tau) + \sinh^2(\sigma\tau') - \cosh^2(\sigma\tau) - \cosh^2(\sigma\tau')\right. \\ &\qquad \left. -2\sinh(\sigma\tau)\sinh(\sigma\tau') + 2\cosh(\sigma\tau)\cosh(\sigma\tau')\right] - \frac{\omega^2}{\sigma^2}(\Delta\tau)^2 \\ &= \frac{a^2}{\sigma^4}\left[-2 + 2\cosh(\sigma(\tau-\tau')) - \frac{\omega^2}{a^2}\sigma^2(\Delta\tau)^2\right] = \frac{a^2}{\sigma^4}\left[2\left(\cosh(\sigma\,\Delta\tau) - 1\right) - \frac{\omega^2}{a^2}\sigma^2(\Delta\tau)^2\right] \\ &= \frac{a^2}{\sigma^2}\left[4\sinh^2\left(\frac{\sigma}{2}\Delta\tau\right) - \frac{\omega^2}{a^2}(\sigma\,\Delta\tau)^2\right] \end{aligned} \tag{7.27}$$

where in the last step, I used the identity $\cosh x = 2\sinh^2\left(\frac{x}{2}\right) + 1$. But we must also remember the infinitesimal convergence term $-i\epsilon$ that we added to Δt. Since the precise value of ϵ does not matter, only that it is small and positive, we see by expanding the expression for $\Delta t'_{obs}$ in eq. 7.26 that this will come out right if we replace $\Delta\tau$ by $\Delta\tau - i\epsilon$:

$$(\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 = \frac{4a^2}{\sigma^4}\left[\sinh^2\left(\frac{\sigma}{2}(\Delta\tau - i\epsilon)\right) - \frac{\omega^2}{4a^2}(\sigma(\Delta\tau - i\epsilon))^2\right] \tag{7.28}$$

it picks out the vacuum state as the state with lowest energy. The fact that the vacuum is *not* the minimum-energy state for many stationary trajectories signals that the detector perhaps is not going to agree with our definitions.

so the Wightman function is (replacing $\Delta\tau$ with s for more convenient notation):

$$G^+(s) = -\frac{\sigma^4}{16\pi^2 a^2}\left[\sinh^2\left(\frac{\sigma(s-i\epsilon)}{2}\right) - \frac{\omega^2}{4a^2}\left(\sigma(s-i\epsilon)\right)^2\right]^{-1} \tag{7.29}$$

and the expression for the transition rate $\Gamma_{1\to 2}$ is finally:

$$\Gamma_{1\to 2} = -\frac{\sigma^4 g^2}{8\pi^2 a^2}\int_{-\infty}^{\infty} ds\, \frac{e^{-i(\epsilon_2^\Delta - \epsilon_1^\Delta)s}}{\sinh^2\left(\frac{\sigma(s-i\epsilon)}{2}\right) - \frac{\omega^2}{a^2}\left(\frac{\sigma(s-i\epsilon)}{2}\right)^2} \tag{7.30}$$

If we insert this into eq. 7.20 to obtain the transition rate $\Gamma_{1\to 2}$, we see that the integral we need to perform is in general not very nice. In fact, it can usually only be performed numerically. If we have purely linear acceleration though, that is if $\omega = 0$ (and $\sigma = a$), then the integral can be solved using complex integration and the residue theorem. Note that $\sinh\left(\frac{a}{2}(s + i2\pi/a) - i\epsilon\right) = -\sinh\left(\frac{a(s-i\epsilon)}{2}\right)$. If we integrate from $-\infty$ to $+\infty$ along the real axis and back again along the line from $+\infty + i\frac{2\pi}{a}$ to $-\infty + i\frac{2\pi}{a}$ to form a rectangular contour, we enclose a double pole at $s = i\epsilon$. The integration becomes:

$$\begin{aligned}\left(\int_{-\infty}^{+\infty} + \int_{+\infty+\frac{2\pi i}{a}}^{-\infty+\frac{2\pi i}{a}}\right) ds\, \frac{e^{-i(\epsilon_2^\Delta - \epsilon_1^\Delta)s}}{\sinh^2\left(\frac{a(s-i\epsilon)}{2}\right)} &= \left(1 - e^{\frac{2\pi}{a}(\epsilon_1^\Delta - \epsilon_2^\Delta)}\right)\int_{-\infty}^{\infty} ds\, \frac{e^{-i(\epsilon_2^\Delta - \epsilon_1^\Delta)s}}{\sinh^2\left(\frac{a(s-i\epsilon)}{2}\right)} \\ &= 2\pi i\,\mathrm{Res}\left(s = i\epsilon\right)\end{aligned} \tag{7.31}$$

As I mentioned, the pole at $s = i\epsilon$ is a double pole. To simplify the writing, I take $\epsilon \to 0$ right away so that the pole is at $s = 0$, since the ϵ-term has already played out its role by determining on which side of the integration contour the pole is located. The residue can then be calculated by multiplying the integrand by s^2, derivating once with respect to s and taking the limit $s \to 0$:

$$\mathrm{Res}\left(s = 0\right) = \lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s^2 e^{-i(\epsilon_2^\Delta - \epsilon_1^\Delta)s}}{\sinh^2\left(\frac{as}{2}\right)}\right) = -\frac{4i}{a^2}\left(\epsilon_2^\Delta - \epsilon_1^\Delta\right) \tag{7.32}$$

The desired integral is then:

$$\int_{-\infty}^{\infty} ds\, \frac{e^{-i(\epsilon_2^\Delta - \epsilon_1^\Delta)s}}{\sinh^2\left(\frac{as}{2} - i\epsilon\right)} = -\frac{8\pi}{a^2}\,\frac{\epsilon_2^\Delta - \epsilon_1^\Delta}{e^{\frac{2\pi}{a}(\epsilon_2^\Delta - \epsilon_1^\Delta)} - 1} \tag{7.33}$$

so that the transition rate is:

$$\Gamma_{1\to 2} = \frac{g^2}{\pi}\,\frac{\epsilon_2^\Delta - \epsilon_1^\Delta}{e^{\frac{2\pi}{a}(\epsilon_2^\Delta - \epsilon_1^\Delta)} - 1} \tag{7.34}$$

The transition rate $\Gamma_{2\to 1}$ the other way, from state 2 to state 1, can be found by simply interchanging 1 and 2 in eq. 7.34. The result is:

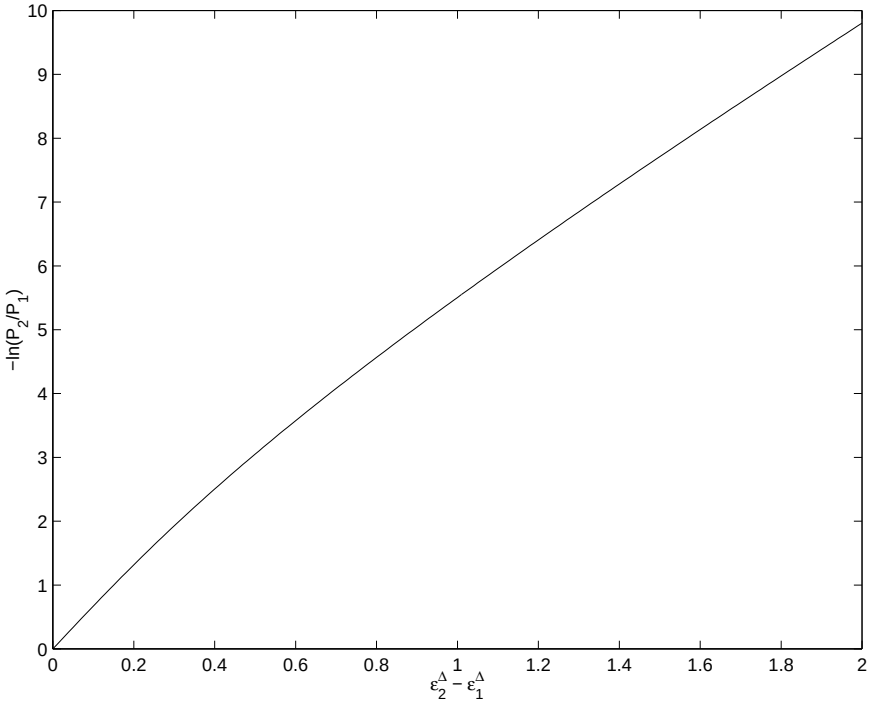

Figure 7.1: Relative populations for $a = 1.0$, $\omega = 0.9$

$$\Gamma_{2\to1} = \frac{g^2}{\pi}\frac{\epsilon_1^\Delta - \epsilon_2^\Delta}{e^{\frac{2\pi}{a}(\epsilon_1^\Delta - \epsilon_2^\Delta)} - 1} = \frac{g^2}{\pi}\frac{\epsilon_2^\Delta - \epsilon_1^\Delta}{1 - e^{-\frac{2\pi}{a}(\epsilon_2^\Delta - \epsilon_1^\Delta)}} = e^{\frac{2\pi}{a}(\epsilon_2^\Delta - \epsilon_1^\Delta)}\Gamma_{1\to2} \tag{7.35}$$

so the probability ratio of finding the detector in state 2 to that of finding it in state 1 after the measuring, is

$$\frac{P_2}{P_1} = \frac{\Gamma_{1\to2}}{\Gamma_{2\to1}} = e^{-\frac{2\pi}{a}(\epsilon_2^\Delta - \epsilon_1^\Delta)} \tag{7.36}$$

This is precisely the Boltzmann factor that characterizes a canonical thermal distribution with temperature $kT = \frac{a}{2\pi}$, just as we would expect.

So if $\omega = 0$, everything is just fine and the detector seems to be kind enough to fulfill our expectations of how it should behave. For $\omega \neq 0$ on the other hand, eq. 7.30 cannot be analytically integrated and the result certainly would not be a nice thermal spectrum. If ω is small compared to a, we could expand the fraction in a Taylor series to first order in $\frac{\omega}{a}$ and presumably get an expression of the form $\frac{P_2}{P_1} = F(\epsilon_2^\Delta - \epsilon_1^\Delta)e^{-\frac{2\pi}{\sigma}(\epsilon_2^\Delta - \epsilon_1^\Delta)}$, where F would be some slowly varying correction factor. But in general, the integral has to be evaluated numerically.

This I have done using a small Matlab program which is listed in appendix B. A sample plot of the logarithm of the relative population $\frac{P_2}{P_1} = \frac{\Gamma_{1\to2}}{\Gamma_{2\to1}}$ as a function of the energy difference $\epsilon_2^\Delta - \epsilon_1^\Delta$ (denoted by E in the plot) is given in figure 7.1. I have here chosen to

set $a = 1$ and $\omega = 0.9$, and E is given in units relative to a. If the excitation spectrum were actually thermal, then $P_2/P_1 = e^{-\frac{2\pi}{\sigma}\left(\epsilon_2^\Delta - \epsilon_1^\Delta\right)}$, so that we should have

$$\ln \frac{P_2}{P_1} = \ln \frac{\Gamma_{1\to 2}}{\Gamma_{2\to 1}} = -\frac{2\pi}{\sigma}\left(\epsilon_2^\Delta - \epsilon_1^\Delta\right) \tag{7.37}$$

In other words, the curve in fig. 7.1 should be a straight line with slope $\frac{2\pi}{\sigma} \approx 14.3$. We see that the curve in fig. 7.1 is in fact fairly straight, so one could perhaps assign some sort of effective temperature to it that varies slowly with the energy difference. However, the curve is not *completely* straight, so the spectrum is obviously not thermal. The slope is also not right at all, it actually seems to be *less* than 2π and certainly nothing like 14.3 (strangely, it seems close to but somewhat less than the slope $\frac{2\pi}{a}$ that we would expect for $\omega = 0$). This suggests that there is indeed something fishy going on when $\omega \neq 0$. This will be well confirmed in the next section when we investigate a detector for which $a < \omega$, which according to our discussions should not detect any particles in the Minkowski vacuum at all.

7.3 Detecting Minkowski particles that should not be there

In the last section we saw that when the detector follows a stationary trajectory with $\boldsymbol{\omega} = 0$, the detector does detect a thermal spectrum of particles in the Minkowski vacuum, just as we would expect. But if $\boldsymbol{\omega} \neq 0$ and perpendicular to $\mathbf{a}$, the spectrum is no longer thermal, even though $a > \omega$. In this section we see what happens when $a < \omega$. We know from chapter 6 that the vacuum in this case is the same as the inertial Minkowski vacuum. The Minkowski vacuum is dead cold as far as a stationary observer is concerned, and there should be no vacuum temperature for the detector to detect. But still we will see that the detector in fact *does* get excited. There is obviously something in the Minkowski vacuum to detect, even though it may not have anything to do with heat and thermal effects.

For $\mathbf{a} \perp \boldsymbol{\omega}$ and $a < \omega$, the trajectory of the observer and the detector is given by eq. 4.42, so that the differences in position between the proper times τ and τ' are:

$$\begin{aligned}
\Delta t'_{obs} &= \frac{\omega}{\rho}(\tau - \tau') = \frac{\omega}{\rho}\Delta\tau \\
\Delta x'_{obs} &= -\frac{a}{\rho^2}\left(\cos(\rho\tau) - \cos(\rho\tau')\right) \\
\Delta y'_{obs} &= -\frac{a}{\rho^2}\left(\sin(\rho\tau) - \sin(\rho\tau')\right) \\
\Delta z'_{obs} &= 0
\end{aligned} \tag{7.38}$$

The squared distance then is (again dropping the subscript obs):

$$\begin{aligned}(\Delta t')^2-(\Delta x')^2-(\Delta y')^2-(\Delta z')^2 &= \frac{\omega^2}{\rho^2}(\Delta\tau)^2-\frac{a^2}{\rho^4}\left[\cos^2(\rho\tau)+\cos^2(\rho\tau')+\sin^2(\rho\tau)\right.\\ &\left.+\sin^2(\rho\tau')-2\cos(\rho\tau)\cos(\rho\tau')-2\sin(\rho\tau)\sin(\rho\tau')\right]\\ &= \frac{\omega^2}{\rho^2}(\Delta\tau)^2-\frac{a^2}{\rho^4}\left[2-2\cos\left(\rho(\tau-\tau')\right)\right]=\frac{\omega^2}{\rho^4}\left[(\rho\,\Delta\tau)^2-2\frac{a^2}{\omega^2}\left(1-\cos(\rho\Delta\tau)\right)\right]\\ &= \frac{4\omega^2}{\rho^4}\left[\left(\frac{\rho\,\Delta\tau}{2}\right)^2-\frac{a^2}{\omega^2}\sin^2\left(\frac{\rho\,\Delta\tau}{2}\right)\right]\end{aligned} \tag{7.39}$$

where in the last transition I used $\cos x = 1-2\sin^2\left(\frac{x}{2}\right)$. As in the previous section, we need to replace $\Delta\tau$ by $\Delta\tau - i\epsilon$ to get the convergence term that we added to Δt when we derived the Wightman function. The Wightman function is then:

$$G^{+}(s) = -\frac{\rho^4}{16\pi^2\omega^2}\left[\left(\frac{\rho\,(s-i\epsilon)}{2}\right)^2-\frac{a^2}{\omega^2}\sin^2\left(\frac{\rho\,(s-i\epsilon)}{2}\right)\right]^{-1} \tag{7.40}$$

The transition rate $\Gamma_{1\to 2}$ is then:

$$\Gamma_{1\to 2} = -\frac{g^2\rho^4}{8\pi^2\omega^2}\int_{-\infty}^{\infty} ds\,\frac{e^{-i(\epsilon_2^{\Delta}-\epsilon_1^{\Delta})s}}{\left(\frac{\rho(s-i\epsilon)}{2}\right)^2-\frac{a^2}{\omega^2}\sin^2\left(\frac{\rho(s-i\epsilon)}{2}\right)} \tag{7.41}$$

Since $\rho \equiv \sqrt{\omega^2-a^2} = i\sqrt{a^2-\omega^2} \equiv i\sigma$ and $\sinh(ix) = i\sin(x)$, we can reassure ourselves by comparing eqs. 7.35 and 7.41 that $\Gamma_{1\to 2}$ does vary continuously with ω and a as we cross over from $a>\omega$ to $a<\omega$. This means that the particle spectrum detected by the detector varies continuously with a and ω, so that it has a well-defined limit as $a\to\omega$. The excitation rate for the case $a=\omega$ should therefore be given simply by using either eq. 7.35 or eq. 7.41 and taking the limit $\sigma\to 0$ or $\rho\to 0$, so I will not treat that case separately.

But let us return to the case at hand. Just as for $a>\omega$ when $\omega\neq 0$, the integral in eq. 7.41 is not solvable analytically, and there is certainly no reason to expect the result to yield a thermal probability distribution for the excitation spectrum of the detector. I used the same Matlab routine as in the previous section to produce a sample plot of the logarithm of the relative populations of levels $\left|\epsilon_2^{\Delta}\right\rangle$ and $\left|\epsilon_1^{\Delta}\right\rangle$ for $a=1$ and $\omega=2$ (actually, to improve numerical convergence, I scaled both a, ω and the energy by a factor of 100 and used these values in the numerical calculation, which simply amounts to using a different set of energy units and should therefore not affect the result). The result is plotted in fig. 7.2. First of all, the curve is not just a flat line at ∞ as it should be if P_2 were zero. Obviously there is something in the Minkowski vacuum that is causing the detector to become excited, even though there should not be any heat bath of Rindler particles in the reference frame of the detector. Moreover, the curve is definitely not straight, so whatever is causing the excitation is not anything purely thermal.

The curve is fairly straight in sections, so if one wants to view the detector excitation as being due to some sort of temperature in the Minkowski vacuum, one could ascribe an

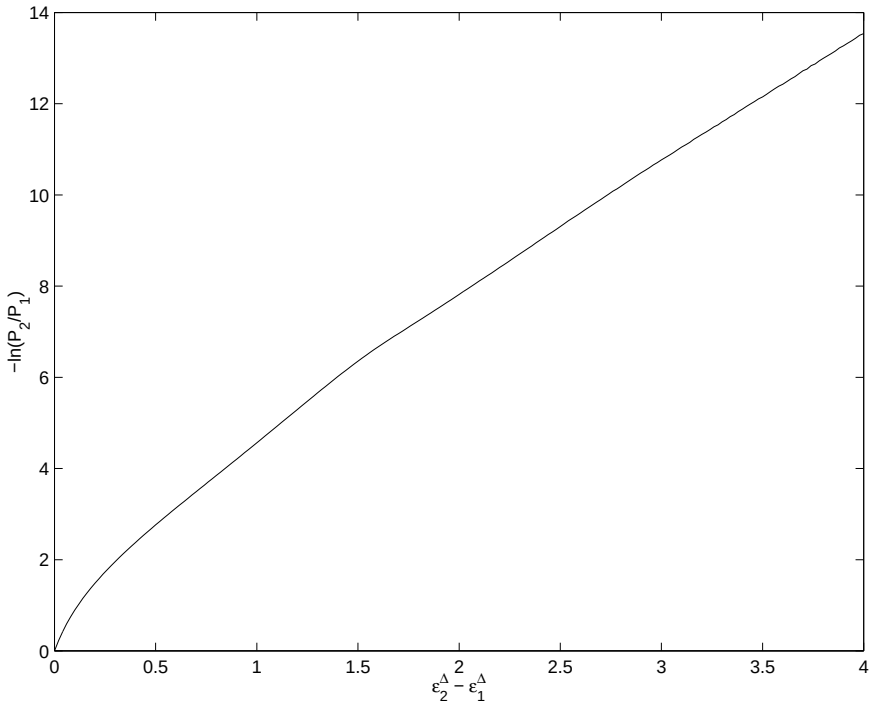

Figure 7.2: Relative populations for $a = 1.0$, $\omega = 2.0$

effective temperature to the excitations. This effective temperature would then not be completely independent of the energy gap $\epsilon_2^\Delta - \epsilon_1^\Delta$, but would vary rather slowly with it when the energy gap is sufficiently large, at least if we are to believe fig. 7.2. This approach was taken e.g. by Bell and Leinaas ([2] and [3]) and was applied to interpret depolarization of electron spins in storage rings. As the ratio $\frac{\omega}{a}$ increases, the curve becomes ever less straight and the bend near the origin becomes increasingly pronounced. But it is nevertheless remarkable that the detector becomes excited in a nearly thermal fashion right up to the point where $a = \omega$ and even well into the region where $\omega > a$. However, according to our definition of what a particle is, there should be no particles at all in the Minkowski vacuum, neither thermally distributed nor otherwise, when viewed from a stationary reference frame with $\mathbf{a} \perp \boldsymbol{\omega}$ and $a < \omega$. So there must be something else in the Minkowski vacuum to detect. In the next section I will try to give a physical interpretation of just what that "something else" actually is.

7.4 Making sense of non-behaving detectors

In the last two sections we saw that when a monopole detector follows a stationary trajectory with $\omega \neq 0$, the excitation spectrum of the detector does not correspond to what we would expect based on the existence or non-existence of a thermal "gas" of Rindler particles in the Minkowski vacuum. Obviously, the detector is doing more than just detecting particles according to our naive definition, but the integrals in eqs. 7.30 and 7.41 unfortu-

nately do not offer many clues to what is really going on. In order to get a better idea, let us do the calculations again from scratch in a different way. The treatment here is rather simple and the interpretations fairly intuitive, but nevertheless they do not seem to be present in any of the literature that I have been able to find.

We first treat the case $a > \omega$. This time we will be concerned with just interpreting the expression we get, be it an expression in closed form or an unfinished integral. So instead of using inertial plane wave modes to expand the field operator, we will use the more "natural" eigenfunctions of H_{RF} in our calculations.

Let us begin by calculating the correlation function $G^+(x,x') \equiv {}_M\langle 0|\, \phi(x)\phi(x')\, |0\rangle_M$. We could expand the field operator $\phi(x)$ using standard Rindler modes $g_{k_y k_z \Omega}$ that are non-zero only on one side of the horizon:

$$\phi(x) = \int dk_y\, dk_z\, d\Omega \left[b_{k_y k_z \Omega} g_{k_y k_z \Omega}(x) + b^\dagger_{k_y k_z \Omega} g^*_{k_y k_z \Omega}(x) \right] \tag{7.42}$$

The problem with this is that the associated annihilation operators $b_{k_y k_z \Omega}$ do not annihilate the Minkowski vacuum $|0\rangle_M$. Instead, we have to use eq. 6.14 and expand $|0\rangle_M$ in terms of Rindler particle states, and evaluating eq. 7.42 becomes rather a hassle. A more convenient choice is to use the "analytically continued" Rindler modes $\tilde{g}_{k_y k_z \Omega}(x)$ that we derived in section 6.3:

$$\phi(x) = \int dk_y\, dk_z\, d\Omega \left[\tilde{b}_{k_y k_z \Omega} \tilde{g}_{k_y k_z \Omega}(x) + \tilde{b}^\dagger_{k_y k_z \Omega} \tilde{g}^*_{k_y k_z \Omega}(x) \right] \tag{7.43}$$

They are also eigenfunctions of H_{RF}, but their associated annihilation operators $\tilde{b}_{k_y k_z \Omega}$ do annihilate the Minkowski vacuum.

Using these modes, the correlation function becomes:

$$\begin{aligned} {}_M\langle 0|\, \phi(x)\phi(x')\, |0\rangle_M = \int dk_y\, dk_z\, d\Omega \int dk'_y\, dk'_z\, d\Omega'\, {}_M\langle 0| \left[\tilde{b}_{k_y k_z \Omega} \tilde{g}_{k_y k_z \Omega}(x) + \tilde{b}^\dagger_{k_y k_z \Omega} \tilde{g}^*_{k_y k_z \Omega}(x) \right] \\ \left[\tilde{b}_{k'_y k'_z \Omega'} \tilde{g}_{k'_y k'_z \Omega'}(x') + \tilde{b}^\dagger_{k'_y k'_z \Omega'} \tilde{g}^*_{k'_y k'_z \Omega'}(x') \right] |0\rangle_M \end{aligned} \tag{7.44}$$

When we multiply out the parentheses and use the fact that $\tilde{b}_{k'_y k'_z \Omega'} |0\rangle_M = 0$ and ${}_M\langle 0|\, \tilde{b}^\dagger_{k_y k_z \Omega} = 0$, we are left with only one term:

$$\begin{aligned} {}_M\langle 0|\, \phi(x)\phi(x')\, |0\rangle_M &= \int dk_y\, dk_z\, d\Omega \int dk'_y\, dk'_z\, d\Omega'\, \tilde{g}_{k_y k_z \Omega}(x) \tilde{g}^*_{k'_y k'_z \Omega'}(x') \\ &\qquad {}_M\langle 0|\, \tilde{b}_{k_y k_z \Omega} \tilde{b}^\dagger_{k'_y k'_z \Omega'} |0\rangle_M \\ &= \int dk_y\, dk_z\, d\Omega \int dk'_y\, dk'_z\, d\Omega'\, \tilde{g}_{k_y k_z \Omega}(x) \tilde{g}^*_{k'_y k'_z \Omega'}(x') \\ &\qquad \delta(k_y - k'_y) \delta(k_z - k'_z)\, \delta(\Omega - \Omega') \\ &= \int dk_y\, dk_z\, d\Omega\, \tilde{g}_{k_y k_z \Omega}(x) \tilde{g}^*_{k_y k_z \Omega}(x') \end{aligned} \tag{7.45}$$

so that for $G^+(x(\tau), x(\tau'))$, we get:

$$G^+\left(x(\tau),x(\tau')\right) = \int dk_y\, dk_z\, d\Omega\, \tilde{g}_{k_y k_z \Omega}\left(x(\tau)\right) \tilde{g}^*_{k_y k_z \Omega}\left(x(\tau')\right) \tag{7.46}$$

$x(\tau)$ is the position of the detector, which in hyperbolic cylindrical coordinates $(\chi;\xi,y',z')$ is given by (cf. eq 4.54):

$$x(\tau) = \left(\sigma\tau\,;\, \frac{a}{\sigma^2}\,,\, -\frac{\omega}{\sigma}\tau\,,\, 0\right) \tag{7.47}$$

for $\sigma \equiv \sqrt{a^2-\omega^2}$. Since the detector is located at $\xi = \frac{a}{\sigma^2}$, i.e. always on the right hand side of the horizon, $\tilde{g}(x(\tau))$ is given by:

$$\begin{aligned}\tilde{g}_{k_y k_z \Omega}(x(\tau)) &= g_{k_y k_z \Omega}(x(\tau) = A_{k_y k_z \Omega}\, e^{-i\Omega\chi}\, e^{ik_y y' + ik_z z'}\, K_{i\Omega}(\Xi\xi) \\ &= A_{k_y k_z \Omega}\, e^{-i\sigma(\Omega + \frac{\omega}{\sigma^2}k_y)\tau}\, K_{i\Omega}\left(\frac{\Xi a}{\sigma^2}\right)\end{aligned} \tag{7.48}$$

when $\Omega > 0$. If $\Omega < 0$, we instead get

$$\begin{aligned}\tilde{g}_{k_y k_z \Omega}(x(\tau)) &= e^{-\pi|\Omega|} g^*_{(-k_y)(-k_z)(-\Omega)}(x(\tau)) = A_{k_y k_z \Omega}\, e^{-\pi|\Omega|} e^{-i\Omega\chi}\, e^{ik_y y' + ik_z z'}\, K_{i\Omega}(\Xi\xi) \\ &= A_{k_y k_z \Omega}\, e^{-\pi|\Omega|} e^{-i\sigma(\Omega + \frac{\omega}{\sigma^2}k_y)\tau}\, K_{i\Omega}\left(\frac{\Xi a}{\sigma^2}\right)\end{aligned} \tag{7.49}$$

so that $G^+(s)$ is:

$$\begin{aligned}G^+(x(\tau),x(\tau')) = \int_{-\infty}^{\infty} dk_y\, dk_z\, d\Omega\ \left|A_{k_y k_z \Omega}\right|^2 \left(\Theta(\Omega) + \Theta(-\Omega)e^{-2\pi|\Omega|}\right) \\ e^{-i\sigma(\Omega + \frac{\omega}{\sigma^2}k_y)(\tau-\tau')} \left|K_{i\Omega}(\Xi\frac{a}{\sigma^2})\right|^2\end{aligned} \tag{7.50}$$

Reassuringly, this once again turns out to be a function only of the proper time difference $\tau - \tau'$, so setting $s \equiv \tau - \tau'$, we finally get the sought-after function $G^+(s)$.

We can now insert this into eq.7.20 to recalculate the transition rate $\Gamma_{1\to 2}$:

$$\begin{aligned}\Gamma_{1\to 2} = 2|g|^2 \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dk_y\, dk_z\, d\Omega\, e^{-i(\epsilon_2^\Delta - \epsilon_1^\Delta)s} \left|A_{k_y k_z \Omega}\right|^2 \left(\Theta(\Omega) + \Theta(-\Omega)e^{-2\pi|\Omega|}\right) \\ e^{-i\sigma(\Omega + \frac{\omega}{\sigma^2}k_y)s} \left|K_{i\Omega}(\Xi\frac{a}{\sigma^2})\right|^2\end{aligned}$$

$$\begin{aligned}= 4\pi|g|^2 \int_{-\infty}^{\infty} dk_y\, dk_z\, d\Omega\, \delta(\epsilon_2^\Delta - \epsilon_1^\Delta + \sigma(\Omega + \frac{\omega}{\sigma^2}k_y)) \left|A_{k_y k_z \Omega}\right|^2 \\ \left(\Theta(\Omega) + \Theta(-\Omega)e^{-2\pi|\Omega|}\right) \left|K_{i\Omega}(\Xi\frac{a}{\sigma^2})\right|^2\end{aligned} \tag{7.51}$$

Recall now that the energy $\epsilon_{k_y k_z \Omega}$ of the field mode $\tilde{g}_{k_y k_z \Omega}$ as defined by the observer, i.e. the eigenvalue of a particular field mode with respect to H_{RF} or $i\frac{\partial}{\partial \tau}$ is $\epsilon_{k_y k_z \Omega} = \sigma(\Omega + \frac{\omega}{\sigma^2} k_y)$. Inserting this into eq. 7.51 we get the following expression:

$$\Gamma_{1\to 2} = 4\pi |g|^2 \int_{-\infty}^{\infty} dk_y\, dk_z\, d\Omega\, \delta(\epsilon_2^\Delta - \epsilon_1^\Delta + \epsilon_{k_y k_z \Omega}) \left| A_{k_y k_z \Omega} \right|^2 \left(\Theta(\Omega) + \Theta(-\Omega) e^{-\frac{2\pi}{\sigma} |\epsilon_{k_y k_z \Omega} - \frac{\omega}{\sigma} k_y|} \right) \left| K_{i\Omega}(\Xi \frac{a}{\sigma^2}) \right|^2 \quad (7.52)$$

In the same way, we calculate the transition rate for a detector following a stationary trajectory with $a < \omega$. In this case we use cylindrical coordinates and the eigenfunctions $f_{m k_z E} = A_{m k_z E} e^{-iEt + im\theta + ik_z z} J_m(Qr)$ from chapter 5. The field operator in these modes is:

$$\phi(x) = \sum_m \int dk_z\, dE \left(a_{m k_z E}\, f_{m k_z E}(x) + a^\dagger_{m k_x E}\, f^*_{m k_z E}(x) \right) \quad (7.53)$$

where the annihilation operators $a_{m k_z E}$ annihilate the Minkowski vacuum. Following the same path as for $a > \omega$, we get the following Wightman function:

$${}_M\langle 0 |\, \phi(x)\phi(x')\, | 0 \rangle_M = \sum_m \int dk_z\, dE\, f_{m k_z E}(x) f^*_{m k_z E}(x') \quad (7.54)$$

The position $x(\tau)$ of the detector in cylindrical coordinates $(t; r, \theta, z)$ is given by:

$$x(\tau) = (\frac{\omega}{\rho}; \frac{a}{\rho^2}, \rho\tau, 0) \quad (7.55)$$

so that the function $G^+(x(\tau), x(\tau'))$ becomes:

$$G^+(x(\tau), x(\tau')) = \sum_m \int dk_z\, dE\, \left| A_{m k_x E} \right|^2 e^{-i\frac{\omega}{\rho} E(\tau - \tau') + im\rho(\tau - \tau')} \left| J_m(Q \frac{a}{\rho^2}) \right|^2 \quad (7.56)$$

which inserted into eq. 7.20 gives the following transition rate:

$$\Gamma_{1\to 2} = 4\pi g^2 \sum_m \int dk_z\, dE\, \delta(\epsilon_2^\Delta - \epsilon_1^\Delta + \frac{\omega}{\rho}(E - \frac{\rho^2}{\omega} m) \left| A_{m k_z E} J_m(Q \frac{a}{\rho^2}) \right|^2 \quad (7.57)$$

Remembering now that the energy $\epsilon_{m k_z E}$ of the field mode $f_{m k_z E}$ as defined by H_{RF} is $\epsilon_{m k_z E} = \frac{\omega}{\rho}(E - \frac{\rho^2}{\omega} m)$, this becomes:

$$\Gamma_{1\to 2} = 4\pi g^2 \sum_m \int dk_z\, dE\, \delta(\epsilon_2^\Delta - \epsilon_1^\Delta + \epsilon_{m k_z E}) \left| A_{m k_z E} J_m(Q \frac{a}{\rho^2}) \right|^2 \quad (7.58)$$

So what can we make of these expressions? Quite a lot, actually. The first thing to note is the δ-function which occurs for both $a > \omega$ and $a < \omega$. This factor ensures energy

conservation with respect to the energy defined by the proper time of the observer/detector, i.e. the energy defined by H_{RF}. This means that the detector is never excited when $\epsilon_2^\Delta - \epsilon_1^\Delta > 0$ if $a = \omega = 0$, i.e. when the detector is in simple inertial motion through the Minkowski vacuum, because all field modes have positive energy with respect to inertial Hamiltonians, and there are no particles in any of the modes to be absorbed. The argument of the δ-function is therefore never zero, and the δ-function always vanishes. This corresponds to what we expect; the Minkowski vacuum is *the* vacuum for an inertial observer, it has zero temperature, so the detector should detect nothing.

But when the motion is no longer inertial, this changes. When $a \neq 0$ but $\omega = 0$, eq. 7.51 still gives us the kind of excitations we expect from the previous discussions: In this case the energy $\epsilon_{k_y k_z \Omega}$ has the same sign as Ω. The term with the first step function, $\Theta(\Omega)$, gives no contribution when $\epsilon_2^\Delta - \epsilon_1^\Delta > 0$, since if $\omega = 0$ the δ-function always vanishes when $\Omega > 0$. The second term on the other hand, which for $\omega = 0$ is equal to $\Theta(-\Omega) e^{-\frac{2\pi|\epsilon_{k_y k_z \Omega}|}{a}}$, does give us excitations, with precisely the thermal weight factor that we expect. It now looks as though the detector is standing still in a gas with temperature $kT = \frac{a}{2\pi}$, being excited by the thermal energy of particles bumping into it. We can even identify the factor $|A_{k_y k_z \Omega} K_{i\Omega}(\Xi \frac{a}{\sigma^2}|^2$ as being the density of the gas at the position of the detector, which will obviously be a factor in determining the excitation rate.

If we now let ω be non-zero, things become a little more complicated. The energy defined with respect to the stationary trajectory of the detector (with respect to $\frac{\partial}{\partial \tau}$ or H_{RF}) is now no longer Ω or E, but rather it is $\epsilon_{k_y k_z \Omega} = \sigma(\Omega + \frac{\omega}{\sigma^2} k_y)$ when $a > \omega$ and $\epsilon_{m k_y E} = \frac{\omega}{\rho}\left(E - \frac{\rho^2}{\omega} m\right)$ when $a < \omega$. Since both k_y and m can take on any arbitrarily large values, both positive and negative, independently of Ω and E respectively, both of these energies can be negative. In the case $a < \omega$, the δ-functions therefore no longer prevent excitations of the detector when $\epsilon_2^\Delta > \epsilon_1^\Delta$, since there may now be modes for which the energy $\epsilon_{m k_z E}$ is equal to $-(\epsilon_2^\Delta - \epsilon_1^\Delta)$, making the argument of the δ-function zero. For the case $a > \omega$, the term $\Theta(\Omega)$ now does give a contribution to the integral, since $\Omega > 0$ no longer implies that the argument of the δ-function is non-zero and positive. The contribution from the term $\Theta(-\Omega) e^{-\frac{2\pi|\epsilon_{k_y k_z \Omega} - \frac{\omega}{\sigma} k_y|}{\sigma}}$ furthermore is complicated by the fact that the value of the exponential now depends on not just the energy, but also on k_y, as well as the fact that Ω and $\epsilon_{k_y k_z \Omega}$ no longer need to have the same sign.

So we can conclude that the Minkowski vacuum is still a vacuum for a detector with $a < \omega$, and it is still a thermal superposition of Rindler particles for a detector with $a > \omega$. But the existence of modes with negative energy modifies the excitation spectrum of the detector since it can now be excited not only by absorbing particles with positive energy, but also by *emitting* particles with negative energy. The shift in energies also modifies the contribution from the thermal Rindler particles, since some modes that previously did not have the right energy to excite the detector now do, while others that did have the right energy no longer do.

The energy shifts can readily be given a physical interpretation. E.g. for the case $a < \omega$, the detector follows a circular trajectory at constant velocity when seen from the appropriate inertial reference frame. The original energy E of the field modes are defined as their energy in this *inertial* frame. But since the detector is moving relative to this frame, the

energy of the modes will be shifted depending on their angular momentum relative to the detector. This is obvious if you think of the mode as a classical particle; if it has angular momentum parallel to and greater than that of the detector, it will seem to be coming at the detector "from behind" and therefore be less energetic relative to the detector than E. Conversely, if it has angular momentum antiparallel to that of the detector, it will seem to be hitting it "head on" and hence be more energetic than E (whereas for particles with angular momentum parallel to but smaller than that of the detector, the detector will be "catching up" with them, so whether their energy relative to the detector is smaller than or greater than E will vary depending on the value of their angular momentum). Similarly, when the detector is travelling at a constant linear velocity in the y-direction, as it appears to be doing in the case $a > \omega$ seen from a hyperbolicly accelerated reference frame, particles (i.e. field modes) will seem more or less energetic depending on whether they are "colliding" with the detector head on or "catching up" with it or "being caught up with" by it, i.e. depending on their k_y-value.

The terms appearing in the δ-functions of eqs. 7.51 and 7.57 correspond precisely to these energy shifts. In the case $a < \omega$, the energy change in the field due to a change dJ in total angular momentum as seen from a reference frame rotating with angular velocity ω, is equal to $\omega\, dJ$. In our case, the change in angular momentum when a field quantum is absorbed, is $dJ = -m$. The angular velocity is actually $\frac{\rho^2}{\omega}$ rather than just ω due to the boost we performed on the original rest frame of the observer in section 4.3, and we also need to multiply the energy with an overall factor of $\frac{\omega}{\rho}$ due to the time dilatation experienced by the detector in its circular orbit. The total change in the energy of the field as defined with respect to H_{RF} is therefore $\Delta\epsilon = -\frac{\omega}{\rho}\left(E - \frac{\rho^2}{\omega}m\right)$, which is exactly what must be balanced by the excitation energy $\epsilon_2^\Delta - \epsilon_1^\Delta$ of the detector for the δ-function in eq. 7.57 to give any contribution.

Correspondingly, the change in energy of a field seen from a frame moving at constant velocity v, is $v\, dP$, where dP is the change in momentum along v. In our case, when $a > \omega$, the change in y-momentum when a field quantum is absorbed is $dP = -k_y$ and the velocity in question is $v = -\frac{\omega}{a}$. Due to the boost we performed on the original rest frame of the observer/detector, the acceleration parameter is actually $\frac{\sigma^2}{a}$ rather than a, and we must multiply the energy by an overall time dilatation factor of $\frac{a}{\sigma}$ due to the motion of the detector. The total change in energy of the field during the absorption or creation of a field quantum as seen by the detector is therefore $\Delta\epsilon = \pm\frac{a}{\sigma}\left(\frac{\sigma^2}{a}\Omega + \frac{\omega}{a}k_y\right) = \pm\sigma\left(\Omega + \frac{\omega}{\sigma^2}k_y\right)$, which is just the term that must compensate $\epsilon_2^\Delta - \epsilon_1^\Delta$ in the δ-function of eq. 7.51. Note that the quantities dJ and dP used here are actually supposed to be infinitesimal, so it might seem that setting them equal to k_y or m is a bit of an abuse. However, we have assumed that the motion of the detector remains constant, so that neither ω nor v ever change. Hence the quantities dJ and dP can be made as large as one wants.

Something about this does not quite make physical sense, though. I first observed an energy shift in the field modes, which gave rise to particles with negative energy, and used this to explain that the detector can use the nature of its trajectory to become excited by creating particles with negative energy. Then I go on to explain this energy shift in terms of particles moving in circular or linear trajectories relative to the observer and colliding with

it or, in the case of creating rather than absorbing a particle, being thrown of it at different energies depending on their motion relative to the detector. But this seems rather counter-intuitive. If for instance you take a couple of tennis balls with you in a car or on a carousel and start throwing them around, certainly they will have different kinetic energies in your reference frame than the energies observed by somebody watching you calmly from the roadside or standing beside the carousel. But the balls will always have positive energy in both frames. No matter which way you throw them, you are not likely to get anything out of it other than aching arms and puzzled looks from any non-physicist bystanders, and you will certainly not gain any energy from it. We will address this paradox in the next and final section of this chapter, along with the potentially puzzling question of where the energy to excite the detector *really* comes from.

For now we just conclude that there are three different factors that can contribute to the excitation spectrum of the detector: Firstly, there is the possible presence of a thermal spectrum of Rindler particles (defined with respect to H_{RF}) in the Minkowski vacuum, which will give a purely thermal contribution to the excitation spectrum with temperature $kT = \frac{a}{2\pi}$ when $a > \omega$, but no contribution (i.e. zero temperature) when $a < \omega$. Secondly, when $\omega \neq 0$ but $a > \omega$, the temperature of the thermal spectrum is changed to $kT = \frac{\sigma}{2\pi}$ due to the time dilatation factor caused by the fact that the detector is moving at a constant velocity in the y-direction. This motion also causes a shift in all the energy levels which modifies the excitation spectrum so that it becomes that of a thermal bath which is *moving* at a constant velocity relative to the detector. Finally, when $\omega \neq 0$ the shift in energy levels due to the constant linear motion of the detector when $a > \omega$ or its constant circular motion when $a < \omega$ gives rise to modes with negative energy, and these modes contribute to the excitation spectrum of the detector since they enable the detector to gain energy by *creating* particles with negative energy.

Thus the non-thermal excitation spectrum of the detector when $\omega \neq 0$ but $a > \omega$ does not reflect any non-thermal behaviour in the quantum field itself. And the fact that the detector becomes excited at all when $a < \omega$ does not mean that there are actually any particles in the vacuum. Rather, it reflects the fact that the detector does not really detect particles at all. What it *really* detects, is the existence of transitions to states of lower energy in the quantum field, making it possible for the detector to extract energy from the field and become excited. Whether this energy comes from "real" particles with positive energy or from the vacuum decaying into a state with lower energy than the vacuum state itself, is irrelevant as far as the detector is concerned. Thus the particle detector detects energy, not particles, since it will detect negative energy particles that are *not* there — it will be excited by negative energy modes with no particles in them to begin with, if the vacuum can give off energy by decaying into them — and it will fail to detect negative energy particles that actually *are* there, since absorbing them would require withdrawing energy from the detector rather than exciting it. Nevertheless, the interaction between the detector and the field is a fairly general one (as I mentioned, any detector we have ever built uses an interaction of this type). This suggests that just looking at what particles are present in the field may be a poor guide to predicting how someone who travels through the field along some trajectory will actually perceive the field.

Before moving on, we can also elaborate a little on this point. We can actually claim that *all* the excitations of the detector in the Minkowski vacuum are due to the presence

of negative energy modes in the field, even those that we first attributed to the presence of a thermal bath of Rindler particles when $a > \omega$. This is once again due to the ambiguity in how to define the behaviour of the mode functions across the horizon when $a > \omega$. If we use the modes $g_{k_y k_z \Omega}$, which are defined to have support only on the right side of the horizon for $\Omega > 0$ and on the left side of the horizon for $\Omega < 0$, we do get that the Minkowski vacuum is filled with a thermal gas of Rindler particles on the right side of the horizon (provided we ignore the correlations with excitations on the left side of the horizon which the detector cannot see). In this case, a certain part of the detector excitations must be naturally interpreted as being due to absorption of these particles. But we can instead e.g. use the modes $\tilde{g}_{k_y k_z \Omega}$, for which the Minkowski vacuum really *is* a vacuum. If we use these modes, there are no particles in the Minkowski vacuum for the detector to absorb. But now, the modes with negative Ω-values, which were confined to the left side of the horizon well out of the way for the detector when we used the modes $g_{k_y k_z \Omega}$, no longer vanish on the right side of the horizon. Instead, they are equal to $e^{-\pi|\Omega|} g^*_{(-k_y)(-k_z)(-\Omega)}$ there. Therefore there are now positive norm modes with negative Rindler energy on the right side of the horizon. Some or most of these modes (depending on ω and their k_y-values) will give rise to a set of modes with negative energy $\epsilon_{k_y k_z \Omega}$ which will cause the same excitations as those that we originally interpreted to be caused by thermal Rindler particles.

7.5 Clearing up a few paradoxes

A little before the end of the last section, we noted that it does not seem to make physical sense that a detector should be able to throw particles with negative energy around and gain energy for itself just because it happens to be following some peculiar trajectory. The reason why this should seem strange, apart from everyday experience and common sense, can be seen if we think of the field modes as classical particles for a moment.

First take $a < \omega$. The particle is characterized by the energy E and the angular momentum m around the z-axis and the momentum k_z in the z-direction. If we now assume that the particle is following a circular orbit at a distance r from the z-axis, then the magnitude of its angular momentum will be $|m| = r k_{x,y}$, where $k_{x,y}$ is the momentum parallel to its orbit (i.e. a combination of the x- and y-direction). If the orbital speed of the particle is u, then we have $k_{x,y} \leq Eu \leq E$, since $u \leq 1$ (the speed of light). This means that $|m| \leq rE$. If the energy $\frac{\omega}{\rho}\left(E - \frac{\rho^2}{\omega} m\right)$ is supposed to be negative, we must have $m > \frac{\omega}{\rho^2} E$, which implies that $r > \frac{\omega}{\rho^2}$. This is beyond the static limit, and thus beyond the reach of the detector (even if the detector were extended and not pointlike, it could not extend rigidly beyond the static limit, since then part of it would be moving faster than the speed of light). A particle with negative energy is therefore confined to a region of space where it can never collide with or be thrown off the detector.

The same kind of analysis applies when $a > \omega$, although it is not quite so transparent. In this case, Ω is the eigenvalue of the field mode with respect to $i\frac{\partial}{\partial \chi} = i\xi\left(\cosh\chi \frac{\partial}{\partial t} + \sinh\chi \frac{\partial}{\partial x}\right)$. This means that the Ω-value for the particle will be $\Omega = \xi\left(\cosh\chi E - \sinh\chi k_x\right)$, where ξ is the ξ-value at which the particle is located. E and k_x of course change with time as the particle accelerates, but in such a way that Ω is constant. For simplicity, choose $\chi = 0$

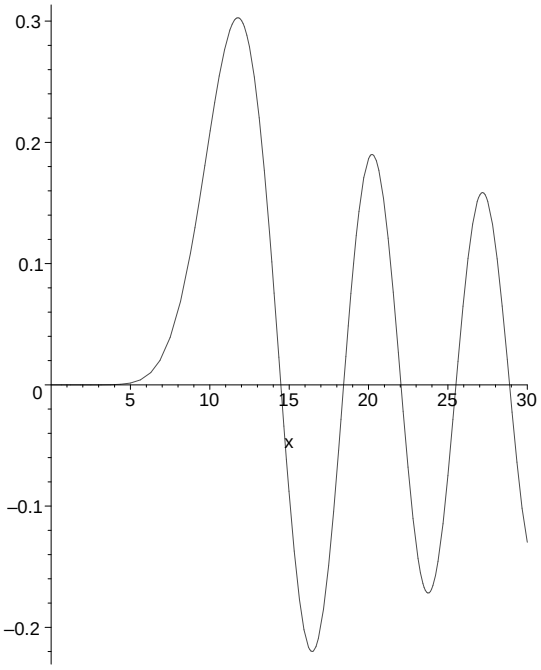

Figure 7.3: Plot of $J_{10}(x)$

so that $\Omega = \xi E$. Since $E = \sqrt{k_x^2 + k_x^2 + k_z^2 + M^2} \geq |k_y|$, we have that $\Omega \geq \xi |k_y|$. But if the energy $\sigma\left(\Omega + \frac{\omega}{\rho^2} k_y\right)$ is supposed to be negative, then we must have $k_y < -\frac{\sigma^2}{\omega}\Omega$ (supposing Ω is positive so that we are on the same side of the horizon as the detector), i.e. $|k_y| > \frac{\sigma^2}{\omega}\Omega$. For these inequalities to be reconciled, we must have $\xi < \frac{\omega}{\sigma^2}$, which means that the particle must be located below the static limit and thus be out of reach of the detector.

The observation that a particle with negative energy should be located beyond the static limit of course makes sense; it is here that the integral lines of $\frac{\partial}{\partial \tau}$ become spacelike, so hence it is only here that the inner product between $\frac{\partial}{\partial \tau}$ and the timelike, future directed four-momentum vector of the particle can be negative.

So the detector should not be able to excite itself by throwing out negative energy particles, since it can never be directly in contact with the part of spacetime where negative energy particle can exist. Classically, that is. However, a quantum mechanical particle that corresponds to one of the particle modes that we are using, is not located at any sharp position, but is rather described by a wave function equal to the field mode function. The radial or ξ-part of this wave function is $J_m(Qr)$ for $a < \omega$ and $K_{i\Omega}(\Xi\xi)$ for $a > \omega$, where $Q = \sqrt{E^2 - k_z^2 - M^2}$, and $\Xi = \sqrt{k_y^2 + k_z^2 + M^2}$.

Figs. 7.5 and 7.4 show a plot of $J_{10}(x)$ and $K_{10i}(x)$ respectively. They both show a feature that is characteristic for all Bessel function and modified Bessel functions. The Bessel functions $J_m(x)$ oscillate for $x > m$, but drop off in an exponential-like fashion inside of this limit. Correspondingly, the modified Bessel functions $K_{i\Omega}(x)$ oscillate for $x < \Omega$ (wildly so when $x \to 0$, which corresponds to approaching the horizon in the accelerated reference frame), and then taper of exponential-like to zero for $x > \Omega$. For a wave function, oscillating behaviour is usually associated with regions where the particle is allowed

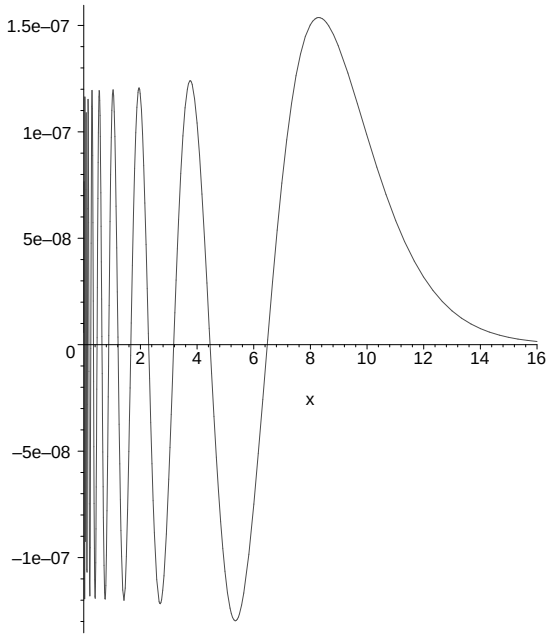

Figure 7.4: Plot of $K_{10}(x)$

to exist classically, whereas exponential-like decaying behaviour is usually observed in regions where the particle cannot exist classically, but where there is still a finite chance of finding it due to quantum uncertainty about its position. This fits well with what we find upon closer inspection: For $a < \omega$, the argument of the wave function at the static limit r_{stat} is $\sqrt{E^2 - k_z^2 - M^2} r_{stat} < E r_{stat}$. If the mode is supposed to have negative energy, then $m > \frac{\omega}{\rho^2} E$, so $E r_{stat} < \frac{\rho^2}{\omega} m r_{stat} = m$, since $r_{stat} = \frac{\omega}{\rho^2}$. $x = m$ is at the boundary where the behaviour of $J_m(x)$ becomes exponentially decaying, so this means that all $r < r_{stat}$ are in the decaying region of the wave function, which is what we would expect. Correspondingly, for $a > \omega$ the argument of the wave function at the static limit ξ_{stat} is $\sqrt{k_y^2 + k_z^2 + M^2} \xi_{stat} > k_y \xi_{stat}$. Since we must have $k_y > \frac{\sigma^2}{\omega} \Omega$ for a negative energy mode, this is greater than $\frac{\sigma^2}{\omega} \Omega \xi_{stat} = \Omega$, since $\xi_{stat} = \frac{\omega}{\sigma^2}$. This once again is the limit $x = \Omega$ where $K_{i\Omega}(x)$ becomes exponentially decaying, so that the wave function is decaying for $\xi < \xi_{stat}$. Thus we can conclude that the excitation of the detector by emitting particles with negative energy is a quantum mechanical effect arising from the fact that, quantum mechanically, these particles can exist with finite probability in the vicinity of the detector, even though they cannot do so classically.

To round off, I would also like to point out one last apparent paradox about the antics of our detector. From its own perspective and its own definition of energy, the detector seems to be playing by the rules and only becoming excited by taking its excitation energy from the field, either by absorbing particles that are actually there (thermal Rindler particles) or by creating particles with negative energy in the field. But if we view this from the outside, as inertial observers watching the detector follow some cycloid-like path for $a < \omega$ or some twisted hyperbolic path for $a > \omega$, the detector seems to be giving energy conservation a long nose. First, it becomes excited itself, which must obviously represent

an increase in energy with respect to the inertial definition of energy as well as that of the detector's stationary reference frame (otherwise, the inertial time unit vector would have to be antiparallel to the four-momentum of the detector, in which case the detector would be travelling backwards in time!). And second, it changes the state of the field from the Minkowski vacuum state into some other state. With respect to the inertial definition of energy, this change must entail that the energy of the field *increases* regardless of what the detector claims, since the Minkowski vacuum state is the state of minimum energy for the field as far as any inertial observer is concerned. So when the detector "detects" something, regardless of whether it is interpreted as being a thermal Rindler particle or a negative energy state, both the field and the detector gain energy as defined by an inertial observer, and energy conservation must obviously be thrown overboard.

This should not really come as a surprise, though. When we normally expect energy to be conserved, it is because there is usually invariance under time translation in the system; energy is conserved because, and only because the Hamiltonian (or if you prefer, the Lagrangian) is not explicitly time dependent. The presence of the detector completely breaks this symmetry. The interaction Hamiltonian density $\mathcal{V}(x') \sim \phi(x')\delta(x' - x(\tau))$ is not at all invariant under translations of the inertial time t for any stationary trajectory $x(\tau)$ except the inertial trajectory $a = 0$. So with the detector present there is no longer any reason why the energy of the system should be conserved in the eyes of an inertial observer. If we want inertial energy to be conserved after all, we will have to add some other symmetry breaking term to the Hamiltonian which exactly cancels the time translation symmetry breaking of $V(\tau)$. This term would presumably represent an external force that keeps the detector on its stationary trajectory in spite of its inertia, something we chose to ignore when we designed our detector interaction. Thus, if we want to have energy conservation in the system, the energy must necessarily be supplied by whatever mechanism that propels the detector.

One final apparent paradox before we quit: it has been noted that the detection of Rindler particles by the detector when $a > \omega$ could get us into trouble with causality, since it apparently affects the state of the field on the left side of the horizon as well, which is not causally connected to the detector. On the face of it, this is indeed a problem. If the detector detects a Rindler particle with momenta and Rindler energy (k_y, k_z, Ω) on the right side of the horizon, then we know from eq. 6.14 that there must be a particle with momenta and energy $(-k_y, -k_z, -\Omega)$ on the other side of the horizon. Or, if we choose to view things in terms of the field modes $\tilde{g}_{k_y k_z \Omega}$ which span across the horizon, the excitation of the detector creates a negative energy particle whose wave function stretches across the horizon and therefore seems to affect the state of the field there. In either case it appears that when the detector is excited we either gain information about or alter the state of the field in a region of spacetime that is causally disconnected from the detector. A more thorough analysis however, would reveal that this apparent paradox is of the same kind as the famous Einstein-Podolsky-Rosen (EPR) paradox concerning measurement on a member of an entangled pair of particles. The state of the field on one side of the horizon is entangled with that on the other side, as we can see from eq. 6.14, but this still cannot be used to transmit information across the horizon. Thus we do have correlations across the horizon, but these still do not violate causality. Going into the details of this goes slightly beyond what the constraints of time allowed for this thesis, but a fuller analysis, as well as a more thorough discussion on how the presence of the detector affects the quantum field

as a whole, can be found in the article [15] by Unruh and Wald.

Chapter 8

Conclusions — To be, not to be, maybe. . .

Although I won't vouch for what the reader may or may not have concluded from the material in this thesis, we seem to have provided a thorough step-by-step discussion of what particles and vacua in quantum fields are and how they should be defined, how the Unruh effect can be derived from this and how many inequivalent ways there really are of perceiving the Minkowski vacuum. We have also discussed why a model particle detector detects something other than the particles that we have defined and found that this actually does not contradict any of our particle definitions after all.

First of all, we have learnt that particles are little packets of some physical observable or a set of mutually commuting observables in a quantum field, corresponding to eigenvalues of the transformations that these observables generate on the phase space of the field. We have learnt that we can define a total number operator $\hat{\mathcal{N}}$ whose corresponding eigenstates $|\mathcal{N}\rangle$ contain a definite total number of $\mathcal{N}$ particles in the field.

Furthermore, we have learnt that the total number operators defined by two different sets of observables will be the same if and only if the positive-norm eigenfunctions of the first set can be decomposed into a combination of purely positive-norm eigenfunctions of the other set, without mixing any negative-norm eigenfunctions into them. We have then argued that it is desirable to define particles in terms of the Hamiltonian H_{RF} of a stationary reference frame. With this in hand, we went on to ostensibly rederive the result from Letaw and Pfautsch's article [4], that there are only two inequivalent total number operators and hence only two distinct vacuum states in Minkowski spacetime, namely the Minkowski vacuum and the Rindler-Fulling vacuum.

However, we have also learnt that the definition of the total-number operator for any observable can be ambiguous if it has degenerate eigenvalues. If there are degeneracies between eigenfunctions with positive and negative norm, then we can define a new set of eigenfunctions for the same observable by mixing eigenfunctions with positive and negative norm, which means that the observable will have several inequivalent number operators associated with it. It will then not be possible to uniquely distinguish the act of creating a

particle from the act of annihilating one. The observable may therefore not be able to agree even with itself on what the particle and vacuum states are!

In particular, this applies to H_{RF} in the cases when $\mathbf{a} \not\perp \boldsymbol{\omega}$ or $a > \omega$, i.e. the cases in which the particle states defined by H_{RF} were supposed to be the Rindler particle states. In that case there are negative-norm eigenfunctions on the left side of the horizon that have exactly the same eigenvalues for all the characteristic observables as a corresponding positive-norm eigenfunction on the right side of the horizon. We found that by mixing these together, we could form just about *any* set of particle states. The Minkowski vacuum could therefore be made to look like any superposition of particle states with respect to H_{RF}. The particle states hence do not depend only on what observable we use to define them or what reference frame we are in, but may also depend on which of several possible sets of eigenfunctions that we fancy using. As a result, there would seem to be infinitely many inequivalent sets of particles states and infinitely many inequivalent vacua in Minkowski spacetime. Specifying the motion of the observer will not necessarily tell us which one to use. The vacuum is therefore not just subjective in the sense that observers in different reference frames may define it differently, it is even subjective to the extent that a single observer in some cases can pick and choose between an infinite number of inequivalent vacua.

We also found out that the root of this vast ambiguity is the presence of an event horizon in the reference frames where $\mathbf{a} \not\perp \boldsymbol{\omega}$ or $a > \omega$. Because the field equation expressed in these reference frames is singular at the horizon we are free to join the eigenfunctions on either side of the horizon in any way we want to form new eigenfunctions. The horizon was also the reason why an observer could see the Minkowski vacuum as a statistical ensemble (thermal or otherwise) of Rindler or other kinds of particles, even though the Minkowski vacuum is of course a pure and not a mixed state no matter which definition we use for the total-number operator. Because the horizon divides the spacetime into two causally disconnected parts, the observer can never know anything about what the state of the field is on the other side of the horizon. As far as the observer is concerned we may as well therefore express the Minkowski vacuum as a density matrix and perform a partial trace over all states localized to the other side of the horizon, thereby "forgetting" about what the state of the field is there and producing a mixed state.

We noted briefly that this phenomenon of the pure vacuum state of one reference frame being viewed as a statistical ensemble or a mixed state in another reference frame is in general tied to the presence of a horizon. Turning a pure state into a mixed one requires ignoring and tracing out over states in a part of spacetime that the observer cannot know anything about, so without a horizon a pure state must always remain pure. This is at the heart of the Hawking effect for black holes as well, and holds generally in any spacetime, and the Hawking effect and the Unruh effect are close relatives. Both are produced by event horizons, and both take place in a reference frame in which you have either a true gravitational field, or a fictitious gravitational field caused by acceleration. It is in fact even possible to find an explicit isometry mapping from the Rindler reference frame where $\mathbf{a} \neq 0$ and $\boldsymbol{\omega} = 0$ to the region of Schwarzschild spacetime close to the horizon of a static black hole in the limit where the mass of the hole becomes infinitely large (so that spacetime becomes flat). The two situations are therefore actually identical. During my work I also verified that it is possible to find a corresponding isometry between the reference frame $\mathbf{a} \perp$

$\boldsymbol{\omega}$, $a > \omega$ and the region of Kerr spacetime close to the equator of the horizon of a rotating black hole. Unfortunately time and space (pun not intended, at least not entirely...) did not suffice to include this in the thesis. Another possible link between stationary reference frames in Minkowski spacetime and black holes is that there could be an isometry from a stationary reference frame where $\mathbf{a} \not\parallel \boldsymbol{\omega}$ to the region close to the *pole* of a rotating black hole in Kerr spacetime. The accelerating helical motion of the observer in the frame $\mathbf{a} \not\parallel \boldsymbol{\omega}$ seems to mimic how the motion of an observer close to the pole at rest with respect to the standard coordinates in Kerr spacetime would look to a freely falling observer. This is a possibility that might be worth a little further investigation, though I unfortunately could not find the time for it.

Finally we also noted that in Minkowski spacetime, for a stationary observer with $\mathbf{a} \not\parallel \boldsymbol{\omega}$ or $a > \omega$ it may seem that we are free to choose how we join eigenfunctions across the horizon. But if we take a step back and view the eigenfunctions from the point of view of an external inertial observer, there are no singularities in the field equation. This observer would therefore insist that we can only join the eigenfunctions in such a way as to form the set $\tilde{g}_{k_y k_z \Omega}$ of eigenfunctions, which were found to be maximally analytic across the horizon, and which gives us a vacuum and a set of particle states that are identical to the inertial definition. Since the field equation is not singular anywhere, anything else would be unphysical and just a quirk caused by the ignorance of the accelerating observer (a notable exception is if we have certain boundary conditions on the spacetime, e.g. the famous example of an accelerating mirror, a version of which can be found in Birrell and Davies' book [12]. How to construct an infinitely large perfectly reflective mirror and accelerate it at 10^{20}-something m/s^2 is left as an exercise to the reader). From this point of view it therefore seems that we really have just *one* vacuum in Minkowski spacetime, the Minkowski vacuum itself.

Nevertheless, a detector travelling through the Minkowski vacuum on a stationary trajectory was found to respond as if it was detecting a perfectly thermal spectrum of particles when $\omega = 0$ and a more or less thermal spectrum when $\omega \neq 0$, even when $\omega > a$ as long as $\frac{\omega}{a}$ is not too large. We found that this in fact did not contradict any of our particle definitions. Any discrepancy can always be attributed to the presence of modes with negative energy in the field that will excite the detector regardless of what particles are present. But this cautions us that particle states are actually rather abstract quantities that do not necessarily give us a very physical picture of what the result of any detection experiment will be. A thermometer travelling through the Minkowski vacuum may very well detect an approximate temperature, even when there are not supposed to be any particles at any temperature present in its reference frame. Why the detected spectrum is so nearly thermal, even when ω is somewhat larger than a, and why it is nearly thermal but with the wrong temperature when $a > \omega$, would be an interesting subject for further study.

So what are we to make of all this? Not only may the vacuum and particle states of Minkowski spacetime not be uniquely defined, even the way in which they may not be uniquely defined is not uniquely defined! And hardly any of the particle definitions give us a very good picture of how a particle detector will respond. Perhaps we can learn a lesson from how our model particle detector solves the problem. It does not really detect particles at all. It just becomes excited whenever it has an opportunity to scoop up energy from the field, regardless of whether that energy comes from absorbing particles with positive

energy or from emitting particles with negative energy. In essence, the detector really just responds to the field, not to any particles that we define as being or not being there. This obviously resolves all of these paradoxes. No matter what reference frame we view it from or what definitions we use, the state of the field is always the same. Only our definition of how many particles there are in that state changes.

If I wished to end this thesis on a pompous note I could paraphrase Leopold Kronecker[1] and say that "*God made quantum fields, man made all the rest*". However, to avoid being quite so dogmatic, I will just note that in the bewildering context of swapping between different reference frames and observers with irreconcilable points of view, the quantum field *is*. Particles are just maybe...

[1] "*Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk*", citation from lecture at Berliner Naturforscher-Versammlung, 1893

Appendix A

Tricky calculations

A.1 Integration of a product of Bessel functions

We will here calculate the integral

$$\int_0^\infty dr\, r J_m(Qr) J_{m'}(Q'r) \tag{A.1}$$

from eq. 5.32. First we note that unless $m' = m'$, the factor $\delta_{m,m'}$ in eq. 5.32 will cause $\langle f_i, f_j \rangle$ to vanish, so we need only consider the case where $m = m' \equiv m$. The corresponding indefinite integral can then be found e.g. in [17]. It is:

$$\int dr\, r J_m(Qr) J_m(Q'r) = r\, \frac{Q' J_m(Qr) J_{m-1}(Q'r) - Q J_{m-1}(Qr) J_m(Q'r)}{Q^2 - Q'^2} \tag{A.2}$$

The lower limit of the definite integral then vanishes, so that eq. A.1 is equal to the limit of A.2 when $r \to \infty$. This limit can be found from the assymptotic behaviour of the Bessel functions when $r \to \infty$:

$$J_m(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) = \begin{cases} (-1)^{\frac{m}{2}} \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right) & m \text{ even} \\ (-1)^{\frac{m-1}{2}} \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4}\right) & m \text{ odd} \end{cases} \tag{A.3}$$

(see e.g. [18]) Regardless of whether m is even or odd, each term in the numerator of eq. A.2 will then have one sine and one cosine factor. The only difference will be the sign and which factor becomes sine or cosine in which term. If e.g. m is even and a multiple of 4, the result will be (where the limit $r \to \infty$ is understood):

$$\begin{aligned}
&\int dr\, r J_m(Qr) J_m(Q'r) \\
&\quad \to \frac{2}{\pi\sqrt{QQ'}} \frac{-Q'\cos\left(Qr-\frac{\pi}{4}\right)\sin\left(Q'r-\frac{\pi}{4}\right)+Q\sin\left(Qr-\frac{\pi}{4}\right)\cos\left(Q'r-\frac{\pi}{4}\right)}{Q^2-Q'^2} \\
&\quad = \frac{1}{\pi\sqrt{QQ'}} \frac{(Q-Q')\sin\left((Q+Q')r-\frac{\pi}{2}\right)+(Q+Q')\sin\left((Q-Q')r\right)}{Q^2-Q'^2} \\
&\quad = \frac{1}{\pi\sqrt{QQ'}} \left(-\frac{\cos\left((Q+Q')r\right)}{Q+Q'}+\frac{\sin\left((Q-Q')r\right)}{Q-Q'}\right)
\end{aligned} \tag{A.4}$$

This expression has no well defined limit for $r \to \infty$. However, the inner product between two free-particle eigenfunctions should not be understood as a proper function of Q and Q', but rather as a distribution, which is only well defined in terms of how it behaves when smeared out with a "sufficiently well-behaved" trial function, or combined to form wave packets, and integrated over Q. When viewed as distributions, one can show that the following limits hold:

$$\lim_{r\to\infty} \frac{\sin\left((Q-Q')r\right)}{Q-Q'} = \pi\delta\left(Q-Q'\right) \qquad \lim_{r\to\infty} \frac{\cos\left((Q+Q')r\right)}{Q+Q'} = 0 \tag{A.5}$$

Hence the integral becomes:

$$\int_0^\infty dr\, r J_m(Qr) J_{m'}(Q'r) = \frac{\delta(Q-Q')}{\pi^2 Q} \tag{A.6}$$

A.2 Integration of a product of modified Bessel functions

In this section we will calculate the integral

$$\int_0^\infty d\xi\, \frac{1}{\xi} K_{i\Omega_i}(\Xi\xi) K_{\Omega'}(\Xi'\xi) \tag{A.7}$$

from eq. 5.53. First, note that the value of the intergral is only relevant if $\Xi = \Xi'$, i.e. $k_{y'} = k'_{y'}$ and $k_{z'} = k'_{z'}$, otherwise the delta-functions in eq. 5.53 will cause the whole inner product to vanish, so we will assume that Ξ and Ξ' are in fact equal. We then use the following indefinite integral, which can be found in [19]:

$$\begin{aligned}
\int d\xi\, \frac{1}{\xi} K_{i\Omega}(\Xi\xi) K_{i\Omega'}(\Xi\xi) \;= &\Xi\xi\, \frac{K_{i\Omega-1}(\Xi\xi)K_{i\Omega'}(\Xi\xi) - K_{i\Omega}(\Xi\xi)K_{i\Omega'-1}(\Xi\xi)}{\Omega'^2-\Omega^2} \\
&- \frac{K_{i\Omega}(\Xi\xi)K_{i\Omega'}(\Xi\xi)}{i\Omega+i\Omega'}
\end{aligned} \tag{A.8}$$

To find the definite integral A.7, we use the fact that when $x \to \infty$ the modified Bessel function $K_{i\nu}(x)$ behaves assymptotically as follows:

$$K_{i\nu}(x) \sim \sqrt{\frac{\pi}{2x}}\, e^{-x} \tag{A.9}$$

(see [20]) Inserting this into eq. A.8 gives us zero contribution to the integral when $\xi \to \infty$. In the limit $x \to 0$, we have:

$$K_{i\Omega}(x) \sim \cos(\Omega \ln x) \tag{A.10}$$

which can be inferred directly from eq. 5.48 by ignoring the term $\Xi^2_{k_{y'}k_{z'}} G_{k_{y'}k_{z'}\Omega}(\xi)$, which we may do in the limit $\xi \to 0$ (assuming that the solution itself is bounded in this limit, whereas its first and second derivatives are not, which turns out to be true). The equation then has the simple solution $G_{k_{y'}k_{z'}\Omega}(\xi) = C_1\xi^{i\Omega} + C_2\xi^{-i\Omega} = C_1 e^{i\Omega\ln\xi} + C_2 e^{-i\Omega\ln\xi}$, which yields eq. A.10 when we demand that $\tilde{F}_i(\xi)$ be real (since $K_{i\nu}(x)$ is defined to be real for real x). We could of course have chosen a sine solution instead, but this does not really matter for our conclusion. The cosine solution is the one which coincides numerically with $K_{i\nu}$. This further gives us:

$$K_{i\Omega-1}(x) \overset{x\to 0}{\sim} \cos((\Omega+i)\ln x) = \cos(\Omega\ln x)\cosh(\ln x) - i\sin(\Omega\ln x)\sinh(\ln x) \tag{A.11}$$

Inserting these results into eq. A.7 we get (where it is implicitly understood that $\xi \to 0$):

$$\begin{aligned}
&\int d\xi\, \frac{1}{\xi} K_{i\Omega}(\Xi\xi) K_{i\Omega'}(\Xi\xi) \\
&\quad \sim -i\Xi\xi \sinh(\ln(\Xi\xi)) \frac{\cos(\Omega\ln(\Xi\xi))\sin(\Omega'\ln(\Xi\xi)) - \sin(\Omega\ln(\Xi\xi))\cos(\Omega'\ln(\Xi\xi))}{\Omega'^2 - \Omega^2} \\
&\qquad + i\,\frac{\cos(\Omega\ln(\Xi\xi))\cos(\Omega'\ln(\Xi\xi))}{\Omega+\Omega'} \\
&\quad \simeq \frac{i}{2}\frac{\sin((\Omega'-\Omega)\ln(\Xi\xi))}{(\Omega'-\Omega)(\Omega'+\Omega)} + \frac{i}{2}\frac{\cos[(\Omega'+\Omega)\ln(\Xi\xi)] - \cos[(\Omega'-\Omega)\ln(\Xi\xi)]}{\Omega+\Omega'}
\end{aligned} \tag{A.12}$$

When $\xi \to 0$, we get $\ln(\Xi\xi) \to -\infty$. Once again, the limits are not well-defined, but jus as in A.1, the results should be viewed as distributions rather than functions of Ω and Ω'. In that case the first term gives us a term proportional to $\delta(\Omega'-\Omega)$, whereas the last two terms vanish. We thus get:

$$\int_0^\infty d\xi\, \frac{1}{\xi} K_{i\Omega_i}(\Xi\xi) K_{\Omega'}(\Xi'\xi) \sim \delta(\Omega'-\Omega) \tag{A.13}$$

Since we are really just inerested in confirming that the eigenfunctions we are using are indeed orthogonal, and the sign of the norm can be inferred by other means, I do not bother to explicitly calculate the proportionality constants.

A.3 Integration of a product of a product of a rational function, an exponential and a modified Bessel function

Finally we need to calculate the following integral

$$\int_0^\infty \left(\pm\frac{\Omega}{\xi} - E_{\mathbf{k}'}\right) e^{\pm i k'_{x'}\xi} K_{i\Omega_i}(\Xi_{k_{y'}k_{z'}}\xi)\, d\xi \tag{A.14}$$

from eq. 5.58. This integral does not appear to be listed in most standard reference works, but can be solved using analytical software. I used Maple to find the following expression (using $\Xi \equiv \Xi_{k_{y'}k_{z'}}$, $E \equiv E_{\mathbf{k}'}$, $k \equiv k'_{x'}$ to simplify the notation):

$$\begin{aligned}\frac{\frac{i}{2}\Omega - 1}{2i + \Omega}\left(\frac{E}{\Xi\sqrt{\frac{k^2}{\Xi^2}+1}\,\sinh\frac{\Omega\pi}{2}} \mp \frac{1}{\cosh\frac{\Omega\pi}{2}}\right)\pi\sin\left(\Omega\,\mathrm{arsinh}\,\frac{\pm k}{\Xi}\right)\\ + \frac{i + \frac{1}{2}\Omega}{2i+\Omega}\left(\frac{E}{\Xi\sqrt{\frac{k^2}{\Xi^2}+1}\cosh\frac{\Omega\pi}{2}} \mp \frac{1}{\sinh\frac{\Omega\pi}{2}}\right)\pi\cos\left(\Omega\,\mathrm{arsinh}\,\frac{\pm k}{\Xi}\right)\end{aligned} \tag{A.15}$$

We can simplify A.15 by putting everything on a common denominator and using the fact that $\cosh\frac{\Omega\pi}{2}\sinh\frac{\Omega\pi}{2} = \frac{1}{2}\sinh(\Omega\pi)$. We also have that

$$\mathrm{arsinh}\,\frac{\pm k}{\Xi} = \ln\left(\pm\frac{k}{\Xi} + \sqrt{\frac{k^2}{\Xi^2}+1}\right) \tag{A.16}$$

and

$$\cos\left(\Omega\ln\left(\pm\frac{k}{\Xi} + \sqrt{\frac{k^2}{\Xi^2}+1}\right)\right) = \frac{1}{2}\left[\left(\pm\frac{k}{\Xi} + \sqrt{\frac{k^2}{\Xi^2}+1}\right)^{i\Omega} + \left(\pm\frac{k}{\Xi} + \sqrt{\frac{k^2}{\Xi^2}+1}\right)^{-i\Omega}\right] \tag{A.17}$$

as well as

$$\sin\left(\Omega\ln\left(\pm\frac{k}{\Xi} + \sqrt{\frac{k^2}{\Xi^2}+1}\right)\right) = \frac{1}{2i}\left[\left(\pm\frac{k}{\Xi} + \sqrt{\frac{k^2}{\Xi^2}+1}\right)^{i\Omega} - \left(\pm\frac{k}{\Xi} + \sqrt{\frac{k^2}{\Xi^2}+1}\right)^{-i\Omega}\right] \tag{A.18}$$

If we apply this to eq. A.15 we get

$$\frac{\pi}{2}\,\frac{E\cosh\frac{\Omega\pi}{2} \mp \sqrt{k^2+\Xi^2}\,\sinh\frac{\pi\Omega}{2}}{\sqrt{k^2+\Xi^2}\,\sinh(\Omega\pi)}\left[\left(\pm\frac{k}{\Xi} + \sqrt{\frac{k^2}{\Xi^2}+1}\right)^{i\Omega} - \left(\pm\frac{k}{\Xi} + \sqrt{\frac{k^2}{\Xi^2}+1}\right)^{-i\Omega}\right]$$

$$+\frac{\pi}{2}\,\frac{E\sinh\frac{\Omega\pi}{2}\mp\sqrt{k^2+\Xi^2}\,\cosh\frac{\Omega\pi}{2}}{\sqrt{k^2+\Xi^2}\,\sinh(\Omega\pi)}\,\left[\left(\pm\frac{k}{\Xi}+\sqrt{\frac{k^2}{\Xi^2}+1}\right)^{i\Omega}+\left(\pm\frac{k}{\Xi}+\sqrt{\frac{k^2}{\Xi^2}+1}\right)^{-i\Omega}\right] \tag{A.19}$$

We have that $\Xi=\sqrt{k_{y'}^2+k_{z'}^2}$. But in eq. 5.58 we have a factor of $\delta\big(k'_{y'}+k_{y'}\big)\delta(k'_{z'}+k_{z'})$, so that $\beta(k_{y'},k_{z'},\Omega;\mathbf{k}')$ will vanish unless $k'_{y'}=-k_{y'}$ and $k'_{z'}=-k_{z'}$. We can therefore assume hat $\Xi^2=k_{y'}^{\prime 2}+k_{z'}^{\prime 2}$, so that $\sqrt{\Xi^2+k^2}\equiv\sqrt{\Xi^2k_{x'}^{\prime 2}}=\sqrt{k_{x'}^{\prime 2}+k_{y'}^{\prime 2}+k_{z'}^{\prime 2}}=E'\equiv E$. This also means that $\Xi=\sqrt{k_{y'}^2+k_{z'}^2}=\sqrt{E_{\mathbf{k}'}^2-k_{x'}^{\prime 2}}\equiv\sqrt{E^2-k^2}$. Inserting into eq. A.19 we get:

$$\begin{aligned}\pi E\,\frac{\sinh\frac{\Omega\pi}{2}\mp\cosh\frac{\Omega\pi}{2}}{E\sinh(\Omega\pi)}\left(\frac{E\pm k}{\Xi}\right)^{\mp i\Omega}&=\pi E\,\frac{\sinh\frac{\Omega\pi}{2}\mp\cosh\frac{\Omega\pi}{2}}{E\sinh(\Omega\pi)}\left(\frac{E\pm k}{E\mp k}\right)^{\mp\frac{i\Omega}{2}}\\&=\pi\,\frac{\sinh\frac{\Omega\pi}{2}\mp\cosh\frac{\Omega\pi}{2}}{\sinh(\Omega\pi)}\left(\frac{E-k}{E+k}\right)^{\frac{i\Omega}{2}}\end{aligned} \tag{A.20}$$

Expanding the hyperbolic sines and cosines in the numerator in terms of exponentials, we finally obtain:

$$\mp\frac{\pi e^{\mp\frac{\pi\Omega}{2}}}{\sinh(\Omega\pi)}\left(\frac{E-k}{E+k}\right)^{\frac{i\Omega}{2}} \tag{A.21}$$

Written out in full, this is:

$$\int_0^\infty\left(\frac{\Omega}{\xi}-E_{\mathbf{k}'}\right)e^{ik'_{x'}\xi}K_{i\Omega}(\Xi_{k_{y'}k_{z'}\Omega}\,\xi)\,d\xi=\mp\frac{\pi e^{\mp\frac{\pi\Omega}{2}}}{\sinh(\pi\Omega)}\left(\frac{E_{\mathbf{k}'}-k'_{x'}}{E_{\mathbf{k}'}+k'_{x'}}\right)^{\frac{i\Omega}{2}} \tag{A.22}$$

Appendix B

Matlab program for numerical integration of transition rates

In this appendix I have included a listing of some very simple Matlab functions that I used to numerically integrate the integrals 7.30 and 7.41 in order to calculate the excitation spectra for the monopole detector in chapter 7. It is nothing fancy and prone to numerical instabilities for some combinations of the parameters, but it agrees with analytical results for $\omega = 0$ and did the job of producing the graphs in figs 7.1 and 7.2. Note that although the correlation function here only seems to be valid for $a > \omega$, it can in fact be used for $a > \omega$ as well because Matlab can deal with complex numbers and can hence continue the correlation function for $a > \omega$ continuously into that of $a < \omega$.

B.1 Main function: `transrate`

This is the main function, which passes parameters on to the functions that do the actual integration.

```
function gamma = transrate(a, omega, E)
% TRANSRATE Calculate transition rate for de Witt detector
% TRANSRATE(a, omega, E) calculates the transition from one
% energy level to another with energy difference E for a
% de Witt detector following a stationary trajectory with
% parameters a and omega.


gamma = real(quad('scalar_integrand_trans', 0,
                  pi/2, 1e-10, [], a, omega, E));
```

B.2 The function `scalar_integrand_trans`

This function transforms the integration interval from $[0, \frac{\pi}{2}]$ to $[0, \infty]$ by substituting $x = \tan t$.

```
function result = scalar_integrand_trans(t, a, omega, E)


x = tan(t);
result = (1 + x.^2).*scalar_integrand(x, a, omega, E);
```

B.3 The function `scalar_integrand`

```
function I = scalar_integrand(t, a, omega, E)
% SCALAR_INTEGRAND Scalar transition rate integrand


sigma = sqrt(a.^2 - omega.^2);
upperlims = [ acos(omega.^2./a.^2)./sigma ,
              2*acos(omega./a)./sigma ];
y = abs(min(upperlims));
x = t - i*y;
I = -(1/(4*pi*pi))*exp(-i*E.*x).*scalar_correl(a, omega, x);
```

B.4 The correlation function: `scalar_correl`

```
% SCALAR_CORREL Scalar field correlation function
% SCALAR_CORREL(a, omega, t) provides the value of the
% scalar field correlation function between to points
% separated by the proper time interval t on the the
% path of a stationary trajectory with parameters
% a and omega.


s = sqrt(a.^2 - omega.^2);

y = 1./((4*a.^2.*(sinh(s*t/2)).^2)/s.^4 - omega^2*t.^2/s.^2);
nanvec = isnan(y);
for k = 1:length(y)
 if nanvec(k) == 1
  y(k) = 0;
 end
end
```

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