The Ekpyrotic Universe

Torquil MacDonald Sørensen

Thesis submitted for the degree of
Candidatus Scientiarum

Department of Physics
University of Oslo

May 2005

Version 1.0, 2005-05-25

Acknowledgments

I would like to thank my supervisor Prof. Øyvind Grøn for the opportunity to work in this exciting area of physics, and for valuable discussions and encouragement during my work. Also, Prof. Finn Ravndal has encouraged me to give lectures and seminars, for which I am very grateful.

Thanks to all the students at the theory group. I especially mention my brother Atle, Eirik, Gerald, Ingunn, Mats, Morad, Nicolay, Olav and Tore. Most of all I would like to thank my family for their support.

Contents

1 Introduction **9**
- 1.1 Space and Time . . . 9
- 1.2 Ekpyrotic Universe . . . 10
- 1.3 Summary of the thesis . . . 11

2 Strings and Branes **13**
- 2.1 The bosonic string action . . . 13
- 2.2 Quantization . . . 24
- 2.3 Spectrum of physical states . . . 27
- 2.4 Unoriented strings . . . 30
- 2.5 Compactification . . . 32
- 2.6 Closed strings and one compact dimension . . . 34
- 2.7 T-Duality . . . 36
- 2.8 Several branes . . . 38
- 2.9 Unoriented strings and orientifolds . . . 39
- 2.10 D-brane action . . . 40
- 2.11 Super-symmetric strings . . . 41
- 2.12 M-theory . . . 42

3 Big Bang and Inflation **43**
- 3.1 Introduction . . . 43
- 3.2 The Robertson-Walker metric . . . 43
- 3.3 Friedmann equations and the past singularity . . . 45
- 3.4 Cosmological problems . . . 48
- 3.5 Inflation . . . 50
- 3.6 Drawbacks of inflation . . . 52

4 Brane Universe **53**
- 4.1 Introduction . . . 53
- 4.2 Motivation . . . 55
- 4.3 The induced Einstein equation . . . 56
- 4.4 Brane coupling to bulk gauge field . . . 60

5 Ekpyrotic Universe **63**
5.1 The cosmological problems 65
5.2 The ekpyrotic universe action 68
5.3 Einstein equations 72
5.4 Equations of motion for ϕ and $\mathcal{A}$ 77
5.5 Boundary conditions 79
5.6 Time-independent solution 80
5.7 The metric signature 83
5.8 Geometric properties of the static solution 84
5.9 Brane tensions 86

6 Moduli Space Approximation and Numerical Results **87**
6.1 Field theory of real, scalar fields 87
6.2 The ekpyrotic universe 89
6.3 Bulk brane potential 96
6.4 Field equations for the ekpyrotic universe 99
6.5 Numerical results 102
6.6 Stability 114
6.7 Dimensional bounce scenario 116
6.8 Action with no bulk brane 116
6.9 Brane matter 118

7 Conclusion **121**

A Moduli space approximation **123**

B Einstein tensor **127**

Bibliography **131**

Units and conventions

- We use units with $c = 1$ and $\hbar = 1$, and a metric signature of $(-,+,+,+)$, which gives positive square lengths for space-like vectors.
- The Ricci tensor is obtained from the Riemann tensor as follows:
$$R_{\beta\delta} := R^{\alpha}_{\ \beta\alpha\delta}.$$
- We define the "rationalized" Planck mass M_D in D space-time dimensions by demanding that Einstein-Hilbert action is
$$S_{EH} := \frac{M_D^{D-2}}{2} \int d^D x \sqrt{-g}\mathcal{R},$$
where g is the determinant of the metric and $\mathcal{R}$ is the Ricci curvature scalar.
- The energy-momentum tensor is defined as a functional derivative of the action. It is
$$T_{\alpha\beta} := -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\alpha\beta}},$$
where S_m is the action for the gravitational source, e.g. matter, radiation and vacuum energy.
- The Einstein equation in D-dimensional space-time then become
$$E_{\alpha\beta} = \frac{1}{M_D^{D-2}} T_{\alpha\beta},$$
where $E_{\alpha\beta}$ is the Einstein tensor.
- The symmetric and antisymmetric part of a matrix $M_{\alpha\beta}$ is denoted by parentheses and square brackets, respectively, around the indices in question:
$$M_{(\alpha\beta)} := \frac{1}{2}\left(M_{\alpha\beta} + M_{\beta\alpha}\right) \quad \text{and} \quad M_{[\alpha\beta]} := \frac{1}{2}\left(M_{\alpha\beta} - M_{\beta\alpha}\right).$$
For any matrix we then have
$$M = \frac{1}{2}\left(M_{(\alpha\beta)} - M_{[\alpha\beta]}\right).$$
- We define the wedge product between two one-forms to be
$$\omega \wedge \eta := \left(\omega \otimes \eta - \eta \otimes \omega\right),$$
and similar for combinations of several one-forms.
- When an operator acts on a product of quantities, we use a parenthesis to indicate the fact: $\partial\phi\partial\phi := (\partial\phi)(\partial\phi) \neq \partial(\phi\partial\phi)$.

Chapter 1

Introduction

1.1 Space and Time

It is interesting to think about the evolution of our understanding of space-time, both in the accepted scientific theories and in the leading candidate for a universal descriptions of nature, string theory.

Newtonian viewpoint The Newtonian viewpoint is that we have a universal time all over the universe, which is not affected by physical events. Time is simply a quantity used to parametrize the motion of objects. The difference in the coordinates of an event as seen from two inertial systems moving with constant speed with respect to each other is described by the Galilean transformations, under which the time coordinate is invariant. In the Newtonian world, gravitational influences between different objects propagate infinitely fast, since the gravitational field of an object depends on spatial distance, but not time.

Einsteinian viewpoint Here time is an intrinsic, dynamical part of the geometry of the universe, and is not invariant under transformations between two inertial systems in relative motion. A clock that is moving with respect to the observer's reference system will run slower as seen by that observer. Another difference from the Newtonian viewpoint is that to preserve causality, no information can propagate faster that the speed of light, which is a constant velocity independent of the movements of the observer. Although possible also in the Newtonian world, here gravity is most often represented as curvature of space-time. The difference is that in the Einsteinian world the time coordinate is also affected by gravitational sources and the laws governing the four dimensional geometry (Einstein's field equations) are different from the corresponding three dimensional laws in a possible geometrical formulation of Newton's gravitation. Einstein gravity reduces to Newtonian gravity when only small masses and speeds are involved.

Stringy viewpoint Although string theory can not be an accepted theory of nature since it has not been experimentally tested at the present time, it is certainly the best candidate we have for a theory of everything, by which we mean a theory which offers a quantum formulation of all the observed forces: electromagnetic force, weak nuclear force, strong nuclear force and the gravitational force. Nevertheless, since the ekpyrotic universe is based on string theory, we will assume string theory to be true throughout this thesis. That being said, string theory has an interesting new perspective on the geometry of space-time.

Since string theory contains a quantum theory of gravitation, it predicts that space-time should be fuzzy at short distances. The fuzziness comes from the quantum corrections to the classical states, which are smooth.

Another geometrical difference from theories with particles and curved space-time is how the laws of nature change when a dimension is compactified and shrunk to zero size. If we do this in standard quantum field theory, we get a theory with one less dimension. In string theory, due to the string's ability to wrap around the compactified dimension, we get back a theory with the same number of dimensions with which we started. A world in which one direction is compactified to the radius R can be interpreted as another world with a different string theory compactified with a radius of α'/R, where $\sqrt{\alpha'}$ is the typical length of a string. This suggests that there is a minimal measurable length $\sqrt{\alpha'}$.

In string theory one has extended objects of various dimensions called D-branes (a generalization of the term membrane). In the realm of quantum mechanics, we know that observables corresponding to length in a certain direction, and momentum in the same direction don't commute. But we are accustomed to thinking that measurements of lengths in orthogonal directions commute. In string theory we can have a brane on which this isn't true, e.g. measuring the x-coordinate of an object living on the brane leads to an uncertainty when we try to measure its y-coordinate. This could be an indication that the underlying space-time is foamy instead of smooth.

In general relativity the curvature of space-time is dynamic, but its topological properties must be constant throughout the evolution of the geometry. String theory allows for changes in the space-time topology. Certain topological properties must stay fixed, however [1, 2, 3]. These topological transitions are by definition discontinuous, and can e.g. change the number of holes in space-time.

1.2 Ekpyrotic Universe

Ever since it was discovered that super-symmetric string theory demands 10 space-time dimensions, it has been assumed by string theorists that at the low energies accessible now, six of those dimensions are curled up into a tiny compact manifold. In this picture our universe fills out all space, also the curled up dimensions. In the nineties it was discovered that the five existing string theories were different limits of a single theory, called M-theory. It is 11 dimensional and has 11 dimensional super-gravity as its low energy limit. The fundamental objects it contains are not zero dimensional as the particles in quantum field theory, but M2- and M5-branes (having 2 and 5 spatial dimensions), and the objects in the five familiar 10 dimensional string theories (strings and branes) are in fact obtained by wrapping these M2- and M5-branes around a small compact dimension. The non-perturbative part of M-theory is mostly unknown at the present time, but its low energy limit is known as 11 dimensional super-gravity, and that theory has been studied since the 1970s.

From string theory we know that we get super-symmetric Yang-Mills gauge theories on these M-branes, and from this we might get the standard model at low energy with the aid of super-symmetry breaking. At first sight this looks rather promising. It is possible that the entire visible universe is a three dimensional brane floating around in the 11-dimensional background geometry of M-theory. This idea first appeared in [4] and [5] outside the context of string theory. There was little research activity until two articles by Horava and Witten [6, 7] in which they conjectured the 11 dimensional M-theory.

The study of the cosmological properties of these branes is called brane cosmology and is an active area of research nowadays. From current experimental results there can be no doubt that the universe is expanding. Recent observations even suggest that the expansion rate is increasing with time. In standard cosmology this is explained by the introduction of exotic forms of energy, called dark energy. In brane cosmology, the dark radiation component is automatically present because it is the so-called brane tension.

The current cosmological paradigm is that the universe began with a big bang and has expanded ever since. In brane cosmology we now have another opportunity, namely that the universe existed forever, but the hot, expanding phase of its evolution began from an inelastic collision between two branes. The energy content of the current universe is proposed to be the remnants of the energy deposited in that collision. This is called the ekpyrotic universe, and its viability will be investigated in this thesis. Specifically, it says that the visible universe is a 3-brane, floating in a five dimensional space-time. Thus there is one extra spatial dimension. The ekpyrotic universe is a new proposal put forth in 2001 [8].

The 3-branes used in the ekpyrotic scenario are actually M5-branes with two of their dimensions wrapped around a compactified six dimensional complex manifold. There is a regime in M-theory where we can neglect those 6 curled up dimensions, and therefore treat space-time as having five dimensions, and containing 3-branes. Also, instead of our universe filling out all of this space, it can instead be a 3-brane. So this strange fifth extra dimension corresponds to the 11th dimension of the full M-theory.

In addition to replacing the big bang with a brane collision, thus eliminating the big bang singularity from the mathematics describing the evolution of the universe, it addresses and aims to solve many of the cosmological problems.

1.3 Summary of the thesis

We begin by studying the properties of bosonic string theory in chapter 2, deducing the fundamental properties of strings. The emergence of a duality symmetry in the theory then leads us to the existence of D-branes. After this, we look at the world-volume action that describes the dynamics of the D-brane.

Since we will apply the D-branes in a string based cosmological theory, we look at standard cosmology in chapter 3. The well-known big bang singularity is derived in the case of a homogeneous, isotropic radiation-dominated Robertson-Walker cosmology. Inflation is then introduced to explain the flatness and horizon problems.

In chapter 4, we introduce the concept of brane-worlds, a universe constrained to a D-brane from string theory, and find the effective Einstein equations induced on such a brane from the Einstein equations in the surrounding bulk space.

Subsequently, we introduce the Ekpyrotic Universe, and its mathematical foundation in chapter 5. The equations of motion are deduced from varying the action functional, and we show the validity of a static solution containing parallel, static branes.

These vacuum solutions are the basis of the moduli space approximation introduced in chapter 6. After applying the approximation to a simple example, we deduce the moduli approximation action for the Ekpyrotic Universe. After deducing the new equations of motion, and realizing that an analytic approach is futile, we look at numerical solutions for different parameters and potentials. A common property of all the solutions is that no brane collision occurs. This shows it to be likely that the ekpyrotic universe in the moduli space approximation fails to provide an alternative to the four-dimensional big bang theory. We have published some of our numerical results, see [9].

In chapter 7, we give a short summary of the thesis, and supply our conclusions.

Appendices A and B contain Mathematica programs to calculate the Einstein tensor in arbitrary dimensions, and a program to solve the moduli space approximation equations of motion numerically, and plot the solutions.

Chapter 2

Strings and Branes

In this chapter we will learn what branes are in the context of string theory, since this is the underlying theoretical basis for the ekpyrotic scenario. First, we will see how Dirichlet branes (D-branes) arise in bosonic string theory, and then we will mention that similar objects must exist in the super-symmetric string theories and M-theory. For good introductions to string theory, see [10, 11, 12, 13].

2.1 The bosonic string action

Let us try to make a quantum theory of strings by quantizing their movements and vibrations. We will use the elegant principle of least action to accomplish this feat.

2.1.1 Point particle action

When we formulate the action principle for the usual zero dimensional point particle, we use as the relativistic action the length of the particle's world-line. We use a parameter $\tau \in \mathbf{R}$ which is monotonically increasing along the particle's world-line, so that its path can be described by a vector-valued function $X(\tau)$. The action of a point particle is

$$S[X] := -m \int ds = -m \int d\tau \sqrt{-\dot{X}(\tau) \cdot \dot{X}(\tau)}, \tag{2.1}$$

where m is the rest mass of the particle, and $\dot{X}(\tau) := dX(\tau)/d\tau$.

We now analyze how this action behaves under a diffeomorphism transformation, a reparametrisation of the world-line. Since τ now is a function of a new parameter, say τ', we can easily obtain the transformation of the integration measure in the action.

$$d\tau \mapsto \left(\frac{d\tau}{d\tau'}\right) d\tau' \tag{2.2}$$

In the new action, fields must be written with derivatives with respect to τ', so we must make use of the chain rule for differentiation on $\dot{X}(\tau)$ to find its transformation rule. We get

$$\frac{dX}{d\tau}(\tau) \mapsto \left(\frac{d\tau'}{d\tau}\right)\frac{dX}{d\tau'}(\tau'). \tag{2.3}$$

We see that the action is invariant under this reparametrisation, since the factors of $d\tau/d\tau'$ cancel against each other. This invariance property is called diffeomorphism symmetry.

The action (2.1) contains a square root, which makes the theory difficult to quantize both in the Hamiltonian and in the path-integral formalism. Also, it is meaningless when $m = 0$, so it only works for massive particles. It is, however, equivalent to a different action that does not contain a square root, and which works for massless states. To obtain this action, we need to introduce an additional auxiliary field $\eta(\tau)$. The action we propose is

$$S[X,\eta] := \frac{1}{2}\int d\tau \left(\frac{1}{\eta}\dot{X}\cdot\dot{X} - \eta m^2\right). \tag{2.4}$$

The Euler-Lagrange equation for the field $\eta(\tau)$ has the solution

$$\eta = \sqrt{-\dot{X}\cdot\dot{X}}/m. \tag{2.5}$$

Plugging this back into the action, we get

$$\begin{aligned} S[X,\eta = \sqrt{-\dot{X}\cdot\dot{X}}/m] &= \frac{1}{2}\int d\tau \left(\frac{m}{\sqrt{-\dot{X}\cdot\dot{X}}}\dot{X}\cdot\dot{X} - \frac{\sqrt{-\dot{X}\cdot\dot{X}}}{m}m^2\right) \\ &= \frac{1}{2}\int d\tau \left(-m\sqrt{-\dot{X}\cdot\dot{X}} - m\sqrt{-\dot{X}\cdot\dot{X}}\right) \\ &= -m\int d\tau\sqrt{-\dot{X}\cdot\dot{X}} = S[X], \end{aligned} \tag{2.6}$$

and recover the original action. Thus we have shown that the two actions are equivalent at the classical level. Since this action must also be invariant under a reparametrisation, we easily see from the fact that the term $\eta d\tau$ must be invariant, and thus that the η field transforms like

$$\eta(\tau) \to \frac{d\tau'}{d\tau}\eta(\tau'). \tag{2.7}$$

This new action also works when $m = 0$, and as a bonus, it does not contain a square root. It is in fact a polynomial of degree two, which makes it suitable for path-integral quantization. Since η transforms non-trivially under the reparametrisation transformation, we can choose a parametrization where $\eta = 1$. Dropping the insignificant constant term -$m^2/2$, the action then reduces to the familiar form

$$S[X,\eta = 1] = \frac{1}{2}\int d\tau\dot{X}\cdot\dot{X}, \tag{2.8}$$

which looks like a nice relativistically invariant generalization of the standard action $v^2/2$ used for a point particle in classical mechanics.

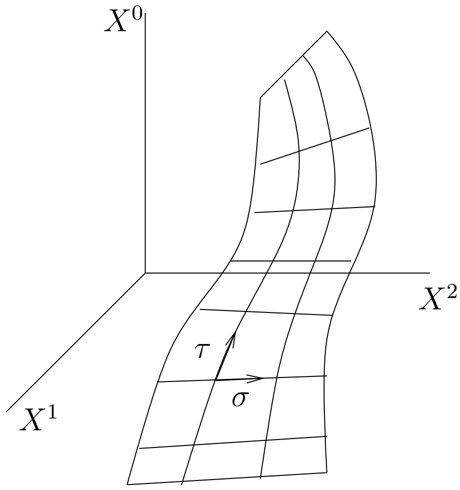


Figure 2.1: The world-sheet of an open string

2.1.2 String action

Now on to the string action. A string will not sweep out a world-line during the course of its time evolution, but a two-dimensional world-sheet $\mathcal{M}$. We parametrize the world-sheet $\mathcal{M}$ with two real numbers $\sigma^a = (\tau, \sigma)$, and use the area of this world-sheet as our action. The action so constructed is called the Nambu-Goto action. The world-sheet of the string is embedded into our space-time of unspecified dimension D by means of a vector-valued function $X(\tau, \sigma)$, giving a point in D-dimensional space-time for each point (τ, σ) on the world-sheet.

We choose the D dimensional space-time to be flat, with the Minkowski metric $\eta = \mathrm{diag}(-1, 1, 1, .., 1)$. The induced metric on the world-sheet, the pullback of the space-time metric, is

$$h_{ab} = \partial_a X \cdot \partial_b X = \partial_a X^\alpha \partial_b X^\beta \eta_{\alpha\beta}, \tag{2.9}$$

where $\partial_a := \partial/\partial\sigma^a$. Defining $h := \det(h_{ab})$ and $d\sigma^2 := d\sigma d\tau$, the action for the string is

$$S[X] = -\frac{1}{2\pi\alpha'} \int_{\mathcal{M}} d\sigma^2 \sqrt{-h}. \tag{2.10}$$

This is the correct expression for the area of a two dimensional surface, multiplied by the string tension $1/2\pi\alpha'$. As with the first action we spoke of for point particles, equation (2.1), this one also contains a square root, which makes quantization difficult. Therefore, as with the point particle, we will introduce an auxiliary field, thereby allowing us to use a different action that is a polynomial in X and its first derivatives. The essential point behind this trick is that since the auxiliary fields are non-dynamical, their equation of motion is purely algebraic. This makes it easy to manufacture an action that is a polynomial in X. One way of doing this is to introduce an independent world-sheet metric γ_{ab}, and use the following action called the Polyakov action.

$$S[X,\gamma] = -\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d\tau d\sigma \sqrt{-\gamma}\gamma^{ab}h_{ab} \tag{2.11}$$

We now solve the equation of motion for γ_{ab}, and plug the solution back into the action, to show that this action is equivalent to the Nambu-Goto action. We need the formulas for the variation of the determinant of a matrix,

$$\delta\gamma = \gamma\gamma^{ab}\delta\gamma_{ab} = -\gamma\gamma_{ab}\delta\gamma^{ab}, \tag{2.12}$$

which enables us to find the corresponding variation of the action,

$$\delta S = -\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d\sigma^2 \left(\frac{\gamma\gamma_{cd}}{2\sqrt{-\gamma}}\gamma^{ab}h_{ab} + \sqrt{-\gamma}h_{cd}\right)\delta\gamma^{cd}. \tag{2.13}$$

Demanding that this is zero gives us the equation of motion for γ_{ab},

$$\frac{\gamma_{cd}}{2}\gamma^{ab}h_{ab} = h_{cd}. \tag{2.14}$$

Notice that the quantity on the left is just a number multiplying the matrix γ_{cd}. We then use the fact that $\det(aM) = a^m\det(M)$ for a number a and a quadratic matrix M, where m is the number of rows or columns in M. Taking the determinant of both sides with respect to the indices c and d then gives

$$h = (\frac{1}{2}\gamma^{ab}h_{ab})^2\gamma. \tag{2.15}$$

Thus we get

$$\gamma^{ab}h_{ab} = 2(-h)^{1/2}(-\gamma)^{-1/2}. \tag{2.16}$$

Putting this into (2.11) gives us the Nambu-Goto action (2.10), so the two actions are classically equivalent.

2.1.3 Symmetries of the string action

The Polyakov action (2.11) has an enormous amount of symmetry. We have a global D-dimensional Poincare symmetry under which X is a vector, and γ_{ab} are invariant. X transforms as

$$X \to \Lambda X + A, \tag{2.17}$$

where Λ is a Lorentz transformation matrix , and A is a translation vector. This symmetry is global from the two dimensional perspective of the string world-sheet because it does not depend on τ or σ. We also have the diffeomorphism symmetry (reparametrisation symmetry), under which X are invariant, and γ_{ab} is a rank two covariant tensor

$$\gamma_{ab}(\tau,\sigma) \to \frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}\gamma_{cd}(\tau,\sigma), \tag{2.18}$$

for new coordinates $\sigma'^a(\tau,\sigma)$. Lastly, we have the Weyl symmetry under which X are invariant, and

$$\gamma_{ab}(\tau,\sigma) \rightarrow \exp(2\omega(\tau,\sigma))\gamma_{ab}(\tau,\sigma), \tag{2.19}$$

for an arbitrary function $\omega(\tau,\sigma)$. Since the Weyl-symmetry rescales lengths, invariance implies that there is no inherent length-scale in the theory, so in particular it must contain only massless fields. In total, we have three gauge symmetries on the world-sheet. The fact that the world-sheet metric is a 2 by 2 matrix, and must be symmetric, means that it has three independent degrees of freedom. Thus we can use the three gauge symmetries to fix a gauge where the world-sheet metric is diagonal, $\gamma_{ab} = \mathrm{diag}(-1,1)$. This gauge is called conformal gauge. In this case, we have $\sqrt{-\gamma} = 1$, so the Polyakov action simplifies to

$$S = -\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d\tau d\sigma \gamma^{ab} h_{ab}. \tag{2.20}$$

We have not yet completely eliminated all the gauge degrees of freedom. There is still an infinitely large "conformal" symmetry left in the action. We will later use the Gupta-Bleuler quantization method which ensures that the spectrum of states is physical by demanding that the conformal symmetry generators annihilate the states.

2.1.4 Einstein-Hilbert action and cosmological constant

It is possible to add more terms to the string action (2.11). Such possibilities include a two dimensional Einstein-Hilbert term, with a boundary term associated with the boundary of the string world sheet in the case of the open string. Another possibility is to include a two dimensional "cosmological constant", which must not be confused with the space-time cosmological constant used in cosmology. The first term looks like

$$S' = \frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\sqrt{-\gamma}\mathcal{R} + \frac{1}{2\pi\alpha'}\int_{\partial\mathcal{M}} ds K, \tag{2.21}$$

where $\mathcal{R}$ is the two dimensional Ricci scalar curvature, and K is the extrinsic curvature of the edge of the world-sheet, as measured from the world-sheet.

The cosmological constant term looks like

$$S'' = \frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\sqrt{-\gamma}. \tag{2.22}$$

We see immediately that the cosmological term is not Weyl transformation invariant. It is customary to demand Weyl invariance from the string world-sheet theory, so we drop this term.

The two dimensional gravity term does not contribute any dynamics to the total world-sheet field theory. This comes about because its action is a topological quantity, and indeed the value of the Einstein-Hilbert term in two dimensions is not dependent on the local curvature of the world-sheet, but only on the global topological property of how many holes there are in the world-sheet surface. It counts the genus of the surface. Depending on the genus of the world-sheet describing a particular interaction, this term will suppress or enhance the amplitude by a constant λ, which itself turns out to be the vacuum expectation value of a certain field ϕ that occurs in string theory called the dilaton field. So there is still only one parameter (α') in the string theory. In fact, since this is a dimensionful parameter, it can be set to any value by a choice of measuring scale, so there is in fact no free parameters in string theory, only constants of nature $\hbar$, c, M, α'. The full two dimensional string world sheet action is then

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\sqrt{-\gamma}\gamma^{ab}\partial_a X^\delta \partial_b X_\delta \\ &\quad +\lambda\left(\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\sqrt{-\gamma}\mathcal{R} + \frac{1}{2\pi\alpha'}\int_{\partial\mathcal{M}} dsK\right). \end{aligned} \tag{2.23}$$

When we calculate the equation of motion for γ_{ab} we will ignore the two dimensional gravity term because it does not vary under small perturbations of γ_{ab}.

2.1.5 Equations of motion for X

To find the equations of motion, we vary the Polyakov action in conformal gauge (2.20) with respect to the fields X^α

$$\begin{aligned} \delta S &= -\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d\sigma^2\gamma^{ab}\left(\partial_a\left(\delta X\right)\cdot\partial_b X + \partial_a X\cdot\partial\left(\delta X\right)\right) \\ &= -\frac{1}{2\pi\alpha'}\int_{\mathcal{M}} d\sigma^2\gamma^{ab}\partial_a X\cdot\partial_b\left(\delta X\right). \end{aligned} \tag{2.24}$$

Since we assume the string world-sheet to be infinitely long in the τ direction, we can assume that δX goes to zero in the limits $\tau \to \pm\infty$, allowing the variation to be nonzero on the boundaries in the σ direction. This means that when we now do an integration by parts, we get a surface contribution from the boundaries at $\sigma \in \{0, \pi\}$, since we defined the σ coordinate to be in $[0, \pi]$.

$$\begin{aligned} \delta S &= \frac{1}{2\pi\alpha'}\int_{\mathcal{M}} d\sigma^2\delta X\cdot\partial_b\left(\gamma^{ab}\partial_a X\right) - \frac{1}{2\pi\alpha'}\int_{\partial\mathcal{M}} d\tau\delta X\cdot\left(\partial_\sigma X\right)|_{\sigma=0}^{\pi} \\ &= \frac{1}{2\pi\alpha'}\int_{\mathcal{M}} d\sigma^2\delta X\cdot\partial^2 X - \frac{1}{2\pi\alpha'}\int_{\partial\mathcal{M}} d\tau\delta X\cdot\left(\partial_\sigma X\right)|_{\sigma=0}^{\pi} \end{aligned} \tag{2.25}$$

For this to be zero for an arbitrary variation δX, we must have the following equation of motion:

$$\partial^2 X := (-\partial_\tau^2 + \partial_\sigma^2)X = 0. \tag{2.26}$$

together with the following condition to eliminate the boundary term

$$\delta X\cdot\left(\partial_\sigma X\right)|_{\sigma=0}^{\pi} = 0. \tag{2.27}$$

Notice that the boundary term and the internal term can't cancel each other out without making the theory non-local, which is something we will avoid doing. For the boundary conditions in the σ direction we have a few choices corresponding to the open string, and one possibility for a closed string solution. Notice that boundary conditions for different components of the space-time coordinates X of the string are specified independently, so a string can have different boundary conditions in different space-time directions.

2.1.6 Open strings

In a local theory, the boundary conditions at each edge of the open string must be satisfied independently. Therefore we discuss the boundary conditions at each edge separately. For specificness, we discuss the condition at $\sigma = 0$ in the space-time direction with index α. Leaving the variation δX^α at $\sigma = 0$ completely general, we must have

$$\partial_\sigma X_\alpha(\tau, 0) = 0. \tag{2.28}$$

This is called a Neumann boundary condition of the string in the α direction. It describes an open string endpoint that is freely moving in the α direction of space-time. The σ derivative of the X_α field represents a momentum flux density on the string world-sheet, and the boundary condition demands that the component of the momentum flux density in the α direction on the edge of the string be zero, indicating momentum conservation in the α direction at the edge of the string world-sheet. This momentum conservation fits with the fact that the this boundary condition preserves the translational symmetry in the same direction in space-time.

Another possibility is to restrict the possible variations δX^α to be zero. This is a string state that breaks the translational symmetry of the theory in this particular direction. It corresponds to a string endpoint that is stuck on a hyper-surface normal to the α direction. This is called a Dirichlet boundary condition, and is written as

$$X_\alpha(\tau, 0) = b_\alpha, \tag{2.29}$$

where b_α is some constant. A more general choice is that the open string endpoint moves on a curved hyper-surface of the total space-time.

Since we can specify boundary conditions separately for each edge of the open string, we indicate an open string's boundary conditions by calling it a NN, ND, DN or DD string, the letters N and D corresponding to Neumann or Dirichlet boundary conditions, and the first letter indicating the condition at $\sigma = 0$, and the second at $\sigma = \pi$. Of course, in the case of Dirichlet conditions, it must also be specified which surface the string endpoint is attached to. We will only quantize the NN open string, but we will later see the appearance of DD open strings, indicating the existence of physical objects that correspond to the surfaces on which the endpoints of those string are attached.

The following expressions for $X^\beta(\tau, \sigma)$ solve the equations of motion deduced from the action (2.11).

$$X^\beta(\tau,\sigma) = x^\beta + 2\alpha' p^\beta \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\beta e^{-in\tau}\cos(n\sigma) \tag{2.30}$$

for the open string. Since the quantity X^β is to represent a position in space-time, it must be real. To accomplish that, we must impose $(\alpha_{-n}^\beta)^* = \alpha_n^\beta$. We now show that it satisfies the equation of motion,

$$\begin{aligned}
(-\partial_\tau^2 + \partial_\sigma^2)X_\alpha &= -\partial_\tau\left(2\alpha' p^\beta + \sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\beta e^{-in\tau}\cos(n\sigma)\right) \\
&\quad +\partial_\sigma\left(-i\sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\beta e^{-in\tau}\sin(n\sigma)\right) \\
&= -\sqrt{2\alpha'}\sum_{n\neq 0}(-in)\alpha_n^\beta e^{-in\tau}\cos(n\sigma) \\
&\quad -i\sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\beta e^{-in\tau}n\cos(n\sigma) = 0.
\end{aligned} \tag{2.31}$$

The mode expansion (2.30) satisfies Neumann boundary condition (2.28) on both ends of the string since $\sin(n\pi) = 0$ for $n \in \mathbf{Z}$:

$$\partial_\sigma X_\alpha(\tau,0) = i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\beta e^{-in\tau}(-\sin(0)) = 0 \tag{2.32}$$

$$\partial_\sigma X_\alpha(\tau,\pi) = i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\beta e^{-in\tau}(-\sin(n\pi)) = 0. \tag{2.33}$$

2.1.7 Closed strings

For the second choice of boundary conditions, corresponding to closed strings, we also choose that the σ goes from 0 to π,

$$\delta X_\alpha(\tau,0) = \delta X_\alpha(\tau,\pi) \tag{2.34}$$
$$\partial_\sigma X_\alpha(\tau,0) = \partial_\sigma X_\alpha(\tau,\pi). \tag{2.35}$$

Since the variation is arbitrary, we must impose $X_\alpha(\tau,0) = X_\alpha(\tau,\pi)$ to achieve the first of these. This implies

$$X_\alpha(\tau,0) = X_\alpha(\tau,\pi) \tag{2.36}$$
$$\partial_\sigma X_\alpha(\tau,0) = \partial_\sigma X_\alpha(\tau,\pi). \tag{2.37}$$

Thus the two ends of the string are always located at the same point in space-time, and it describes a closed loop. The string called a closed string.

As a closed string solution, we have

$$X^\beta(\tau,\sigma) = x^\beta + 2\alpha' p^\beta \tau + i\sqrt{\alpha'/2}\sum_{n\neq 0}\left(\frac{1}{n}\alpha_n^\beta e^{-2in\sigma^-} + \frac{1}{n}\tilde{\alpha}_n^\beta e^{-2in\sigma^+}\right). \tag{2.38}$$

Here $-\infty < \tau < \infty$ and $0 \leq \sigma \leq \pi$ are the coordinates on the world-sheet of the string. To make the position vector X^α real, we have the conditions $(\alpha_{-n}^\beta)^* = \alpha_n^\beta$ and $(\tilde{\alpha}_{-n}^\beta)^* = \tilde{\alpha}_n^\beta$. As is obvious from the above formula, the closed string modes have been decomposed into right and left moving parts which only depends on τ and σ through the combinations $\sigma^\pm = \tau \pm \sigma$. Let us check if the boundary condition (2.36) is satisfied

$$\begin{aligned}
X^\beta(\tau,\pi) &= x^\beta + 2\alpha' p^\beta \tau + i\sqrt{\alpha'/2}\sum_{n\neq 0}\left(\frac{1}{n}\alpha_n^\beta e^{-2in(\tau-\pi)} + \frac{1}{n}\tilde{\alpha}_n^\beta e^{-2in(\tau+\pi)}\right)\\
&= x^\beta + 2\alpha' p^\beta \tau + i\sqrt{\alpha'/2}\sum_{n\neq 0}\left(\frac{1}{n}\alpha_n^\beta e^{-2in\tau} + \frac{1}{n}\tilde{\alpha}_n^\beta e^{-2in\tau}\right)\\
&= X^\beta(\tau,0).
\end{aligned} \tag{2.39}$$

We then check the second boundary condition (2.37) for the closed string,

$$\begin{aligned}
\partial_\sigma X^\beta(\tau,\pi) &= i\sqrt{\alpha'/2}\sum_{n\neq 0} 2i\left(\alpha_n^\beta e^{-2in(\tau-\pi)} - \tilde{\alpha}_n^\beta e^{-2in(\tau+\pi)}\right)\\
&= i\sqrt{\alpha'/2}\sum_{n\neq 0} 2i\left(\alpha_n^\beta e^{-2in\tau} - \tilde{\alpha}_n^\beta e^{-2in\tau}\right)\\
&= \partial_\sigma X^\beta(\tau,0).
\end{aligned} \tag{2.40}$$

It is simplifying to make the following definitions

$$\alpha_0^\beta := \sqrt{2\alpha'}p^\beta \quad \text{, open string} \tag{2.41}$$
$$\alpha_0^\beta := \sqrt{\alpha'/2}p^\beta \quad \text{, closed string} \tag{2.42}$$
$$\tilde{\alpha}_0^\beta := \sqrt{\alpha'/2}p^\beta \quad \text{, closed string.} \tag{2.43}$$

As shown by the following calculation, the zero mode part ($n = 0$) of equation (2.30), $x^\beta + 2\alpha' p^\beta \tau$ is the string's average position in space-time at world-sheet time τ,

$$\begin{aligned}
< X^\beta > &= \frac{1}{\pi}\int_0^\pi d\sigma X^\beta\\
&= x^\beta + 2\alpha' p^\beta \tau + \frac{1}{\pi} i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\beta e^{-in\tau}\int_0^\pi d\sigma \cos(n\sigma)\\
&= x^\beta + 2\alpha' p^\beta \tau + \frac{1}{\pi} i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\beta e^{-in\tau} n \sin(n\sigma)|_{\sigma=0}^\pi\\
&= x^\beta + 2\alpha' p^\beta \tau.
\end{aligned} \tag{2.44}$$

Doing the same for the closed string, we get the same result,

$$< X^\beta >= \frac{1}{\pi}\int_0^\pi d\sigma X^\beta = x^\beta + 2\alpha' p^\beta \tau. \tag{2.45}$$

The average positions of these strings thus move in straight lines through space-time. This is natural, since we have not included any nontrivial background fields that the strings would interact with. Had we e.g. included a nontrivial gravitational background configuration, the center of mass of the string would move along a geodesic.

2.1.8 Equation of motion for γ

Varying the action with respect to γ_{ab} gives the equation of motion for the world-sheet metric

$$\frac{\delta S[X,\gamma]}{\delta \gamma^{ab}} = 0 \Rightarrow -\frac{\sqrt{-\gamma}}{4\pi\alpha'}\left(\partial_a X \cdot \partial_b X - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X \cdot \partial_d X\right) = 0. \tag{2.46}$$

The Einstein-Hilbert term in the action does not contribute to this equation of motion, since that is a topological term invariant under small variations of the metric. Notice the trace of this equation is satisfied irrespective of the values of X and γ, which follows from the Weyl symmetry (2.19) of the two dimensional field theory.

2.1.9 Energy-momentum tensor and Virasoro operators

The energy-momentum tensor is proportional to the variation of the action with respect to the metric γ_{ab}.

$$T_{ab} := -\frac{2}{\sqrt{-\gamma}}\frac{\delta S[X,\gamma]}{\delta \gamma^{ab}} = \frac{1}{2\pi\alpha'}\left(\partial_a X \cdot \partial_b X - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X \cdot \partial_d X\right) \tag{2.47}$$

As with the equation of motion for γ_{ab}, the trace of this tensor is identically zero independently of the fields X and γ, which follows from the Weyl symmetry. From the equation of motion for γ_{ab} (2.46), this quantity is equal to zero. This property should be demanded from the quantized version of this theory, as operator equations. Thus we will try to demand that the energy-momentum operator should annihilate all physical states in the Fock space.

We will now count the degrees of freedom of the energy-momentum tensor before imposing the equations of motion. Since the world-sheet metric has to be symmetric, it has three independent degrees of freedom (three functions). As the energy-momentum tensor is defined as the variation of a single function (the action) with respect to the metric, it also has three independent degrees of freedom. But we must remember that the Weyl symmetry puts limitations on the energy-momentum tensor by demanding that its trace be zero, so we are down to two degrees of freedom. We will Fourier expand these components, and define the Virasoro generators L_m and $\tilde{L}_m$ as being proportional to the Fourier coefficients in this expansion. These coefficients are generators of the conformal symmetry of the world-sheet field theory.

We choose to define the Virasoro generators by expanding the following combinations of the energy-momentum components

$$\tilde{L}_m := \frac{1}{4}\int_0^\pi d\sigma (T_{\tau\tau} + T_{\tau\sigma}) e^{2im\sigma^+} \tag{2.48}$$

$$L_m := \frac{1}{4}\int_0^\pi d\sigma (-T_{\sigma\tau} + T_{\sigma\sigma}) e^{2im\sigma^-}. \tag{2.49}$$

For the closed string, both these expressions will give nonzero Virasoro generators, and for the open string, only L_m will be nonzero. We now do the calculation to obtain the Virasoro generators as functions of the α's in the mode expansion of the closed string. We need the expression for the energy-momentum tensor given in equation (2.47).

$$\begin{aligned} T_{\tau\tau} + T_{\tau\sigma} &= \frac{1}{2\pi\alpha'}\left(\partial_\tau X \cdot \partial_\tau X - \frac{1}{2}\gamma_{\tau\tau}\gamma^{cd}\partial_c X \cdot \partial_d X + \partial_\tau X \cdot \partial_\sigma X\right) \\ &= \frac{1}{2\pi\alpha'}\left(\partial_\tau X \cdot \partial_\tau X + \frac{1}{2}(-\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X) + \partial_\tau X \cdot \partial_\sigma X\right) \\ &= \frac{1}{4\pi\alpha'}(\partial_\tau X + \partial_\sigma X)^2 \end{aligned} \tag{2.50}$$

We now use the mode expansion for the closed string (2.38) to calculate this quantity (suppressing the space-time Lorentz indices). We get

$$\begin{aligned} \partial_\tau X + \partial_\sigma X &= 2\alpha' p + i\sqrt{\alpha'/2}\sum_{n\neq 0}(-2i)\left(\alpha_n e^{-2in\sigma^-} + \tilde{\alpha}_n e^{-2in\sigma^+}\right) \\ &\quad + i\sqrt{\alpha'/2}\sum_{n\neq 0}(-2i)\left(-\alpha_n e^{-2in\sigma^-} + \tilde{\alpha}_n e^{-2in\sigma^+}\right) \\ &= 2\alpha' p + 2\sqrt{2\alpha'}\sum_{n\neq 0}\tilde{\alpha}_n e^{-2in\sigma^+} = 2\sqrt{2\alpha'}\sum_{n\in\mathbf{Z}}\tilde{\alpha}_n e^{-2in\sigma^+}, \end{aligned} \tag{2.51}$$

where we have used equation (2.43) to write everything as α operators. Continuing now, we get

$$\begin{aligned} \tilde{L}_k &:= \frac{1}{4}\int_0^\pi d\sigma (T_{\tau\tau} + T_{\tau\sigma}) e^{2ik\sigma^+} = \frac{1}{16\pi\alpha'}\int_0^\pi d\sigma (\partial_\tau X + \partial_\sigma X)^2 e^{2ik\sigma^+} \\ &= \frac{1}{2\pi}\sum_{m,n\in\mathbf{Z}}\tilde{\alpha}_n \cdot \tilde{\alpha}_m \int_0^\pi d\sigma e^{2i(k-m-n)\sigma^+} = \frac{1}{2}\sum_{m,n\in\mathbf{Z}}\tilde{\alpha}_n \cdot \tilde{\alpha}_m \times \pi\delta_{k,m+n} \\ &= \frac{1}{2}\sum_{m\in\mathbf{Z}}\tilde{\alpha}_{k-m}\cdot\tilde{\alpha}_m. \end{aligned} \tag{2.52}$$

Doing the same calculation for the L_k, we find that it is given in the same way using the α's instead.

$$L_k = \frac{1}{2}\sum_{m\in\mathbf{Z}}\alpha_{k-m}\cdot\alpha_m \tag{2.53}$$

$$\tilde{L}_k = \frac{1}{2}\sum_{m\in\mathbf{Z}} \tilde{\alpha}_{k-m}\cdot\tilde{\alpha}_m \tag{2.54}$$

For the open string, the $\tilde{L}_k$ are trivially zero, so we only get one set of Virasoro generators.

There are a countably infinite number of independent Virasoro generators, which act as generators for the therefore very large conformal symmetry of the world-sheet field theory. This field theory thus has infinitely many conserved charges. In the classical theory, the energy-momentum tensor is equal to zero, so all the Virasoro generators must also be equal to zero.

2.2 Quantization

To quantize the motion and oscillations of the string, we must find the conjugate momentum density, which are given by the standard formula

$$\begin{aligned} \Pi_\beta &:= \frac{\delta}{\delta(\partial_\tau X^\beta)}S = \frac{\partial}{\partial(\partial_\tau X^\beta)}\left(\frac{-1}{4\pi\alpha'}\gamma^{ab}\partial_a X^\gamma \partial_b X_\gamma\right) \\ &= \frac{-1}{2\pi\alpha'}\gamma^{a\tau}\partial_a X_\beta = \frac{1}{2\pi\alpha'}\partial_\tau X_\beta. \end{aligned} \tag{2.55}$$

We can calculate this explicitly for the open and the closed strings. The open string has the mode expansion (2.30), which gives

$$\begin{aligned} \Pi^\beta &= \frac{1}{2\pi\alpha'}\left(2\alpha' p^\beta + \sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\beta e^{-in\tau}\cos(n\sigma)\right) \\ &= \frac{1}{\pi\sqrt{2\alpha'}}\sum_{n\in\mathbf{Z}}\alpha_n^\beta e^{-in\tau}\cos(n\sigma). \end{aligned} \tag{2.56}$$

For the closed string, we get contributions both from the right-moving and the left-moving waves on the string

$$\begin{aligned} \Pi^\beta &:= \frac{1}{2\pi\alpha'}\left(2\alpha' p^\beta + i\sqrt{\alpha'/2}\sum_{n\neq 0}(-2i)\left(\alpha_n^\beta e^{-2in\sigma^-} + \tilde{\alpha}_n^\beta e^{-2in\sigma^+}\right)\right) \\ &= \frac{1}{\pi\sqrt{2\alpha'}}\sum_{n\in\mathbf{Z}}\left(\alpha_n^\beta e^{-2in\sigma^-} + \tilde{\alpha}_n^\beta e^{-2in\sigma^+}\right). \end{aligned} \tag{2.57}$$

We have used the definitions in equations (2.41) - (2.43) to simplify these expressions.

The Hamiltonian density for the string is

$$\mathcal{H} := \partial_\tau X^\beta \Pi_\beta - \mathcal{L} = \frac{1}{4\pi\alpha'}\left(\partial_\tau X^\beta \partial_\tau X_\beta + \partial_\sigma X^\beta \partial_\sigma X_\beta\right). \tag{2.58}$$

By integrating this quantity along the σ direction, we get the Hamiltonian of the string. We do it explicitly for the open string only,

$$\begin{aligned} H &= \int_0^\pi d\sigma\mathcal{H} = \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma\left(\partial_\tau X^\beta\partial_\tau X_\beta + \partial_\sigma X^\beta\partial_\sigma X_\beta\right) \\ &= \frac{2\alpha'}{4\pi\alpha'}\int_0^\pi d\sigma\left[\left(i\sum_{n\in\mathbf{Z}}\alpha_n(-i)e^{-in\tau}\cos(n\sigma)\right)^2 + \left(\sum_{n\in\mathbf{Z}}\alpha_n e^{-in\tau}(-1)\sin(n\sigma)\right)^2\right] \\ &= \frac{1}{2\pi}\sum_{n,m\in\mathbf{Z}}\alpha_n\cdot\alpha_m e^{-2i(n+m)\tau}\int_0^\pi d\sigma\left(\cos(n\sigma)\cos(m\sigma)+\sin(n\sigma)\sin(m\sigma)\right) \\ &= \frac{1}{2\pi}\sum_{n,m\in\mathbf{Z}}\alpha_n\cdot\alpha_m e^{-2i(n+m)\tau}\int_0^\pi d\sigma\cos((n+m)\sigma). \end{aligned} \tag{2.59}$$

The integral in the last line is nonzero only when $n+m=0$, so we get

$$H = \frac{1}{2}\sum_{m\in\mathbf{Z}}\alpha_{-m}\cdot\alpha_m = L_0. \tag{2.60}$$

Thus we see that the zero-mode Virasoro generator acts as the Hamiltonian of the string.

The results for both open and closed strings are

$$H = \begin{cases} L_0 & \text{, open string} \\ L_0+\tilde{L}_0 & \text{, closed string.} \end{cases} \tag{2.61}$$

We can now quantize by introducing the equal time Poisson bracket between X^β and Π^γ, which by definition is

$$[X^\beta(\tau,\sigma),\Pi^\gamma(\tau,\sigma')]_P = \eta^{\beta\gamma}\delta(\sigma-\sigma'). \tag{2.62}$$

In the quantum theory, we promote X^β and Π^γ to operators with commutators given by i times the Poisson bracket above. Thus we get

$$[X^\beta(\tau,\sigma),\Pi^\gamma(\tau,\sigma')] = i\eta^{\beta\gamma}\delta(\sigma-\sigma'). \tag{2.63}$$

From this we can derive the nonzero commutators for x^β and the operators α_m^β in the mode expansion of X^β. A detailed calculation is provided in [13].

$$[x^\beta,p^\gamma] = i\eta^{\beta\gamma} \tag{2.64}$$

$$[\alpha_m^\beta,\alpha_n^\gamma] = m\delta_{m,-n}\eta^{\beta\gamma} \tag{2.65}$$

There is a strong resemblance here to a system of harmonic oscillators, one for each value of m. Thus we conclude that the α's act as creation and annihilation operators on the quantum state of the string. The α_n^β are creation operators for negative n and annihilation operators for positive n for vibration modes on the string.

The order of non-commuting operators is important in the quantized theory. Operators must be normal ordered, with all annihilation operators on the right and the creation operators on the left. This is nontrivial to do only on operators that don't commute. The only operators that don't commute are the pairs with opposite index values. We must therefore be careful when defining the quantum operator corresponding to L_0 and $\tilde{L}_0$ since they contain a sum of pairs of non-commuting operators; see equations (2.53) and (2.54). Using the commutators for the α's in equation (2.65), we can move all annihilation operators (α's with positive index) to the right, and obtain the correct quantum operators (up to an additive infinite constant that we have ignored)

$$\tilde{L}_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \tag{2.66}$$

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n. \tag{2.67}$$

The quantized Virasoro generators satisfy the following Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m-n} + \frac{D}{12}m(m^2-1)\delta_{m+n,0}, \tag{2.68}$$

with the same algebra for the $\tilde{L}_n$ operators. Also, all commutators with one right-going and one left-going Virasoro operator vanish. D is the number of space-time dimensions. The non-operator, or central, term in the Virasoro algebra originates from the normal ordering of the zero-mode operators, and represents an anomaly in the conformal symmetry, i.e. a breakage of the original symmetry by the quantization procedure. In our quantization method, we should really also include ghost states to cancel the negative norm states that is possible in the spectrum. The creation and annihilation operators for these ghost states would then also contribute to the full Virasoro operators, in such a way that the central term disappears. This indicates the the conformal symmetry is still present in the quantized bosonic string.

2.2.1 The values of D and a

Since the $X^\alpha(\tau,\sigma)$ fields represent the coordinates in space-time of a point (τ,σ) on the world-sheet, a restriction on the number of values the α index can take is a restriction on the number of space-time dimensions. Demanding that the quantized theory should exhibit the symmetries of the classical action (Lorentz, diffeomorphism and Weyl), leads to the conclusion that $D = 26$ and $a = 1$, but we will not derive this result in this thesis.

The constant a is a regulated constant coming from the normal ordering of creation and annihilation operators. It appears as a negative contribution to the zero-mode Virasoro operator, which we saw was equal to the Hamiltonian of the string. This means that it represents a negative contribution to the energy of the string. As one of the dimensions of the two dimensional field theory on the world-sheet is compact (closed line element for the open string, and $\mathbf{S}_1$ for the closed string), this negative contribution is in fact the Casimir energy from the compact dimension. In a super-symmetric string theory, the fermions also contribute to this constant.

More surprising is the restriction on the space-time dimensionality. This restriction seems to rule out bosonic string theory as a realistic model, but this we already know from that lack of fermions in the theory. Instead, we must look to super-string theory for more realism. In that case, the space-time dimension is 10, and to obtain a three dimensional world, we must make six dimensions small and compact, or assume that the visible universe is constrained on a three dimensional hyper-surface, a 3-brane.

2.3 Spectrum of physical states

The vanishing of the energy momentum tensor in the classical theory must be implemented in the quantum theory as an operator equation. This means that all matrix elements of the energy-momentum tensor operator must vanish. To do that we demand that only the positive index Virasoro generators annihilate all physical states, and that the zero-mode Virasoro operator minus the intercept annihilates the physical states.

$$L_m|\phi >= \tilde{L}_m|\phi >= 0 \quad m > 0 \tag{2.69}$$

$$L_0 - 1|\phi >= \tilde{L}_0 - 1|\phi >= 0 \tag{2.70}$$

In analogy with standard harmonic oscillator quantization, we define the weighted number of exited states (weighted by the energy of the occupied state compared to the first exited state) as

$$N := \sum_{n>0} \alpha_{-n} \cdot \alpha_n \tag{2.71}$$

$$\tilde{N} := \sum_{n>0} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n, \tag{2.72}$$

and we refer to these operators as number operators. From equation (2.70), using the quantum L_0 operators in (2.66) and (2.67), it is easy to derive the following mass formulas for the quantized string, using $M^2 = -p^2$ and the definitions in equations (2.41) - (2.43),

$$M^2 = \begin{cases} (N-1)/\alpha' & \text{, open string} \\ 2(N + \tilde{N} - 2)/\alpha' & \text{, closed string.} \end{cases} \tag{2.73}$$

We see that the negative normal ordering constant implies the existence of states with negative square mass. Such states are called tachyons, and indicate an instability in the theory. We will ignore this complication, since the tachyon doesn't appear in the super-symmetric string theory that is the basis for the cosmological model we will discuss in later chapters.

We see from equation (2.70) that acting on a physical closed string state with $L_0 - \tilde{L}_0$ must give zero. The symmetry origin of this restriction is that this combination of the zero-mode Virasoro operators generate a translation around the closed string, and since this is a part of the diffeomorphism symmetry of the theory, it must give zero when acting on a physical state. Writing this in terms of the α operators for a closed string shows that the number operator for the right- and left-going excitations, must have the same value.

$$N = \tilde{N} \tag{2.74}$$

This is called the level-matching condition. This means that we can interpret the closed string oscillations as a sum of two open string sectors, with the level-matching constraint enforced.

2.3.1 The tachyon states

With no oscillations on the string excited, the number operator(s) have value zero. From the mass formulas in equation (2.73), we see that the state has negative square mass.

$$M^2 = \begin{cases} -1/\alpha' & \text{, open string tachyon} \\ -4/\alpha' & \text{, closed string tachyon} \end{cases} \tag{2.75}$$

The negative squared masses mean that these particles move faster than the speed of light. In the past, it was customary to discount such states as unphysical, and consider them as an inconsistency of the theory. The modern interpretation is that if a certain state includes a tachyon, that state is unstable, and will decay into some other state of lower energy. In fact, even in the super-symmetric string theories, tachyons can appear in the presence of unstable systems of D-branes.

This can be illustrated by an example with a scalar field in a potential. Since the double derivative of the potential give the square mass of the state, this means that a state where the scalar field sits on top of a bump in the potential, has negative squared mass. This is of course accompanied by the instability of sitting on top of the bump in the potential. The negative mass squared indicates that the field will fall down one of the sides of the bump in the potential.

Since the tachyon states do not contain any creation operators, they have no Lorentz indices to affect their transformation under a Lorentz transformation. This means that they are scalar particles.

2.3.2 Massless states

From (2.73) we see that we get $M^2 = 0$ when $N = 1$ for the open string, and when $N + \tilde{N} = 2$ for the closed string. Thus we can construct the following massless states:

- Open string vector state: $\zeta \cdot \alpha_{-1}|0, k>$
  - $|0, k>$ represents the open string with no oscillators excited and momentum k
  - We act on this state once with the creation operator α_{-1} , thus getting $N = 1$ for this state. We then project this state along a polarization vector ζ. Since the state is massless, we have $k^2 = 0$. The Virasoro constraint $L_1|\phi>= 0$ gives the constraint $\zeta \cdot k = 0$.

  - This state is called the photon.

- Closed string state: $\xi_{\beta\gamma}\alpha^{\beta}_{-1}\tilde{\alpha}^{\gamma}_{-1}|0,k>$

  - We act on the state with on left- and one right-going creation operator, to satisfy the level matching condition (2.74), getting $N = \tilde{N} = 1$. As with the photon state, we have $k^2 = 0$.
  - If the ξ is symmetric and traceless; it is called a graviton state.
  - If ξ is antisymmetric, it is called a Kalb-Ramond state.
  - If ξ is proportional to the identity matrix, it is called a dilaton state.

The thing to notice here, is that we have many massless fields that weren't present in the classical string theory, which only had one scalar massless state (the one with no oscillators excited). Since string theory hopes to be a correct quantum theory of gravity, it is good to see that the graviton is included in the spectrum.

2.3.3 Massive states

From the mass formulas in equation (2.73), we see that the mass squared of the next level of states are of $1/\alpha'$. Since the fundamental string length is very small, this means that these states are very massive indeed. In fact, it is the massless string states that are candidates to be the experimentally observed particle states, even though these are massive. The states will have to gain mass through some sort of symmetry breaking mechanism, as is done in the standard model of particle physics. We will almost entirely ignore these massive string states for that reason throughout the rest of this thesis.

2.3.4 Space-time gauge symmetry

A state that is orthogonal to any physical state is called spurious. A state with zero norm is a null state. We define the spurious open string state

$$L_{-1}|0,k> . \tag{2.76}$$

To check that it is spurious, we take the inner product of it with an arbitrary physical state $|\psi>$

$$<\psi|L_{-1}|0,k>=<0,k|L_1|\psi>^*=0, \tag{2.77}$$

since all Virasoro generators with positive index annihilate physical states. The following shows that this spurious state is null since $k^2 = 0$.

$$<0,k|L_1L_{-1}|0,k>=<0,k|2L_0|0,k>=\alpha'k^2=0 \tag{2.78}$$

where we have used the Virasoro algebra (2.68). It can also be verified that it is physical, and thus it must have zero norm to be orthogonal to all physical states.

Let us see what happens if we add this spurious state to the open string photon state. Since the spurious state has a vanishing inner product with any physical state, all matrix elements are preserved under this transformation. This is thus a $U(1)$ space-time gauge symmetry of the theory, with the photon state as its gauge field.

In open string theory we are allowed to endow the ends of the strings with extra non-dynamical degrees of freedom called Chan-Paton factors [14]. Let us demand that for each open string, the two ends are in states labeled by an integer from 1 to N. So in addition to being characterized by momentum and the configuration of excitations, each string carries a label ij, where $i, j \in 1, 2, .., N$. This means that an open string state will decompose into states labeled by i and j

$$|k, a >= \sum_{i,j=1}^{N} |k, ij > \lambda^a_{ij}, \tag{2.79}$$

where the λ^a are $N \times N$ matrices. Now, every open string interaction amplitude will contain the trace of products of such matrices, and therefore each interaction vertex are invariant under a $\mathbf{U}(N)$ transformation, $\lambda^a \to U\lambda^a U^{-1}$, of these Chan-Paton factors. This is at first sight just a global symmetry of the open string theory. But if we check how the massless vector state of the open string transforms under this $\mathbf{U}(N)$ transformation, we find that it transforms in the adjoint representation of $\mathbf{U}(N)$. So we have obtained a $\mathbf{U}(N)$ space-time gauge symmetry for the open strings.

The Hamiltonian of the string is independent of the Chan-Paton factors, since they are non-dynamical. If we could formulate string theory as a field theory in 26-dimensional space-time, with a Lagrangian built up from creation and annihilation operators for strings of specific vibrational configuration, it would be possible to formulate this gauge symmetry with covariant derivatives in the same manner as with field theory for point particles. That kind of a construction is called a string field theory, but has not yet been successfully formulated.

2.4 Unoriented strings

Now we will investigate the possibility of having only unoriented strings in our spectrum. This puts extra restrictions on the allowed modes of vibration of the string. We look at open and closed strig separately.

2.4.1 Unoriented open strings

To make a theory of unoriented strings, we invent a parity operator Ω on the world-sheet and demand that all states in the spectrum be invariant with respect to its action. Let Ω be the operator that takes a point (τ, σ) on the world-sheet to the point $(\tau, \pi - \sigma)$. In the case of open strings, its action of the string expansion $X^\beta(\tau, \sigma)$ is

$$\Omega X^\beta(\tau, \sigma)\Omega^{-1} = X^\beta(\tau, \pi - \sigma). \tag{2.80}$$

In terms of the open string mode expansion (2.30), and using

$$\cos(n(\pi - \sigma)) = (-1)^n \cos(n\sigma),$$

we find the following action on x and α_m,

$$x^\beta \mapsto \Omega x^\beta \Omega^{-1} = x^\beta \tag{2.81}$$
$$\alpha_m^\beta \mapsto \Omega \alpha_m^\beta \Omega^{-1} = (-1)^m \alpha_m^\beta, \tag{2.82}$$

for all m. Before we can use Ω to project out non-invariant states, we need to know what kind of action it can have on the Chan-Paton part of the states. The most general action that can be a symmetry of amplitudes that contain a trace of products of Chan-Paton matrices is

$$\Omega : \lambda_{ij}|k, ij> \;\; \mapsto \;\; (M\lambda^T M^{-1})_{ij}|k, ji> . \tag{2.83}$$

We have transposed the λ since Ω exchanges the ends of the string. We know that $\Omega^2 = 1$, so we must have $\lambda = M(M^{-1})^T\lambda M^T M^{-1}$. Writing this as $\lambda M(M^T)^{-1} = M(M^{-1})^T\lambda$, and noticing that $(M^T)^{-1} = (M^{-1})^T$, we see that $M(M^{-1})^T$ commutes with the arbitrary matrix λ. Thus $M(M^{-1})^T$ must be proportional to the identity matrix, i.e. $M(M^{-1})^T \propto I_N$. This is in fact Schur's lemma from linear algebra. Hence there are two possibilities:

- M is symmetric, and equal to the identity after a change of basis, $M = I_N$. If we then want the photon state $\lambda_{ij}\alpha_{-1}^\alpha|k, ij>$ in the spectrum, we need $\lambda = -\lambda^T$. Since these matrices act as generator of the space-time gauge symmetry group, this means that our gauge group is $SO(N)$. It turns out that these states are the gauge bosons for this gauge group.

- M is antisymmetric, and we have $M = -M^T = i\begin{pmatrix} 0 & I_{N/2} \\ -I_{N/2} & 0 \end{pmatrix}$ after a change of basis, with $\lambda = -M\lambda^T M$. From these formulas, we recognize that λ is a generator for the gauge group $Sp(N/2)$, which is the symplectic group, meaning quaternionic $(N/2) \times (N/2)$ matrices with $AA^* = 1$, where the star is quaternionic conjugation: $1^* = 1$, $i^* = -i$, $j^* = -j$, and $k^* = -k$.

2.4.2 Unoriented closed strings

We saw in (2.38) that we could write the mode expansion for the closed string as a sum over left- and right-going modes. Since we have translation symmetry around the closed string, we can represent the parity operator as $\sigma \mapsto -\sigma$ instead of $\sigma \mapsto \pi - \sigma$. So, in the closed string case, we have

$$\Omega X^\beta(\tau, \sigma)\Omega^{-1} = X^\beta(\tau, -\sigma). \tag{2.84}$$

Looking at the closed string mode expansion (2.38), we see that it must have the following action on the operators,

$$x^\beta \mapsto \Omega x^\beta \Omega^{-1} = x^\beta \tag{2.85}$$
$$\alpha_m^\beta \mapsto \Omega \alpha_m^\beta \Omega^{-1} = \tilde{\alpha}_m^\beta \tag{2.86}$$
$$\tilde{\alpha}_m^\beta \mapsto \Omega \tilde{\alpha}_m^\beta \Omega^{-1} = \alpha_m^\beta. \tag{2.87}$$

From this it follows that states in the unoriented closed string theory must be constructed from combinations of creation operators that are symmetric under the interchange $\alpha_m^\beta \leftrightarrow \tilde{\alpha}_m^\beta$. Hence we see that the tachyon $|k>$ is unaffected since it contains no oscillations. Also, we see that the graviton $(\alpha_{-1}^\beta \tilde{\alpha}_{-1}^\gamma + \alpha_{-1}^\gamma \tilde{\alpha}_{-1}^\beta)|k>$ and the dilaton survive the projection, since they are symmetric. The antisymmetric Kalb-Ramond field, does not, since it is antisymmetric. So even in the unoriented string theories, we have a graviton state.

2.5 Compactification

Ordinarily, we think of space and time as being infinitely large in each possible direction, as we have done up to now in this chapter. We haven't put any restrictions on which values the X^α can take. Let us now consider the possibility of letting one dimension become small, or "compact".

Compactness is a topological property, which means that is is independent of any continuous stretching without tearing of the space in question. Mathematically, it means that if you have any family of open sets whose union cover a space M, you can always throw away some of them obtaining a finite amount of open sets still covering M. Then M is compact. This is a topological property because the definition only refers to the open sets on the space M, and it doesn't care if M has a differentiable structure or not. Now, cast in such an abstract form it is not very obvious how the property of compactness relates to physics. For subsets of $\mathbf{R}^n$, there is the Heine-Borel theorem, which says that any closed and bounded subset of $\mathbf{R}^n$ is compact. $\mathbf{S}_1$ (circle), $\mathbf{S}_2$ (sphere), $\mathbf{T}_2 = \mathbf{S}_1 \times \mathbf{S}_1$ (torus), and the closed interval $[0, 1]$ are examples of compact spaces. An important example of how to make the topological space $\mathbf{R}$ compact is the following. You simply add the "point at infinity", and add new open neighborhoods around this point. This new space $\mathbf{R}^*$ is compact because it is topologically equivalent (homeomorphic) to the circle $\mathbf{S}_1$, which is compact. Mathematicians call it the one-point compactification of $\mathbf{R}$.

Compact dimensions are used in string theory mainly because consistency of super-symmetric string theory demands 10 dimensions, and with compactification we make six of those dimensions compact, and smaller than the current experimental resolution, so that we get an effective four dimensional theory. In its simplest form, which is not what is used in super-symmetric string theory, we restrict one of the dimensions to be $\mathbf{S}_1$ instead of the usual $\mathbf{R}$. Compactification is one aspect of string theory which is very different from theories with ordinary point particles, because strings can wind around these compact dimensions.

2.5.1 Orbifolds

An orbifold is obtained from a compact manifold by making identifications between different points in the manifold. We can e.g. make the orbifold $\mathbf{S}_1/\mathbf{Z}_2$ from the manifold $\mathbf{S}_1$ as follows: Let $y \in [-R, R]$ with identification $R \sim -R$ be a coordinate on $\mathbf{S}_1$. Let $\mathbf{Z}_2$ be the group of reflections of $\mathbf{S}_1$ about $y = 0$. We use the actions $y \mapsto y$ and $y \mapsto -y$ of $\mathbf{Z}_2$ on $\mathbf{S}_1$ to define the new manifold $\mathbf{S}_1/\mathbf{Z}_2$ by identifying points on $\mathbf{S}_1$ that are related by an action of $\mathbf{Z}_2$. Since we therefore have the identification $y \sim -y$, the coordinate on $\mathbf{S}_1/\mathbf{Z}_2$ is $y \in [0, R]$, so we can think of it as a closed line element $[0, R]$. $\mathbf{S}_1/\mathbf{Z}_2$ is called an orbifold, since it consists of the orbits of the group action of $\mathbf{Z}_2$ on $\mathbf{S}_1$. This reduces the circle to the line interval $[0, \pi]$ with fixed points 0 and π. These points are called fixed because they are identified with themselves in the orbifold construction. Notice that we have gone from a differentiable manifold $\mathbf{S}_1$ without a boundary, to an orbifold with two boundaries at 0 and π, one at each fixed point of the $\mathbf{Z}_2$ group action on $\mathbf{S}_1$.

An orbifold string theory is equivalent to a theory on the original manifold, where we only keep the states that are invariant under the orbifold identification. This means that when we formulate a theory of gravitation on an orbifold, we can think of it as a theory on the original manifold without a boundary, provided that we demand that e.g. the metric (and thus the curvature) is symmetric about the fixed points. Notice, however, that not all quantities should be symmetric under such reflections. The derivative of, say a metric component, in the orbifold direction, should be antisymmetric, since the definition of the direction of increasing coordinate values is reversed in the identification process. This is easy to see from the fact that a graph that is symmetric about the origin automatically has an antisymmetric derivative.

2.5.2 The Calabi-Yau manifold

Here we describe some properties of the six-dimensional compact Calabi-Yau manifolds that are commonly used to compactify six of the nine spatial dimensions in string theory. It is advantageous to describe this in the framework of algebraic geometry.

To construct the manifold, we must use the algebraically complete number system $\mathbf{C}$, the set of all complex numbers. It is algebraically complete because every polynomial equation with coefficients in $\mathbf{C}$ has a solution i $\mathbf{C}$. When speaking about dimensions of objects, it is common to count the dimension over the complex numbers. E.g. complex dimension 3 is the same as real dimension 6, and the vector space of complex numbers has complex dimension 1. We define the complex affine n-space $\mathbf{C}^n$ to be the set of n-tuples of numbers in $\mathbf{C}$, and n indicates its complex dimension, twice its real dimension. An element in $\mathbf{C}^n$ can be described by coordinates $(z_1, z_2, \ldots, z_n)$. Next, we define the complex projective n-space $\mathbf{CP}^n$, which is the space of lines in $\mathbf{C}^{\mathbf{n+1}}$ through the origin. A point in $\mathbf{CP}^n$ can be represented with homogeneous coordinates $(z_0, z_1, \ldots, z_n)$, where coordinates which are related by a multiplication of all the components of the homogeneous coordinates by a non-zero complex number represent the same element in $\mathbf{CP}^n$. This corresponds to choosing a point on the line in $\mathbf{C}^{n+1}$ representing the chosen element of $\mathbf{CP}^n$, outside the origin of $\mathbf{C}^{n+1}$, of course, since that would not correspond to a specific line. A generalization of this projective space is the complex weighted projective n-space $\mathbf{CP}^n(k_0, k_1, \ldots, k_n)$ where points are given with coordinates $(z_0, z_1, \ldots, z_n)$, but where coordinates $\vec{z}$ and $\vec{v}$ with components related by related by $v_i = \lambda^{k_i} z_i$ represent the same point.

All Calabi-Yau spaces can be constructed as intersections of hyper-surfaces in complex weighted projective spaces. To construct an example, we start with $\mathbf{CP}^4(1, 1, 2, 2, 6)$, which has four complex dimensions, or eight real ones. To define a hyper-surface by the locus of zeros of a polynomial $P(z_0, z_1, z_2, z_3, z_4)$, we must choose a P with terms of degrees that make the locus of zeros independent of the representatives we use for coordinates: If $P(\vec{z}) = 0$, then $P(\vec{v})$ must be zero if $v_i = \lambda^{k_i} z_i$, since then $\vec{z}$ and $\vec{v}$ is really the same point in $\mathbf{CP}^4(1, 1, 2, 2, 6)$. A simple example is $P(\vec{z}) = z_0^{12} + z_1^{12} + z_2^6 + z_3^6 + z_4^2$ for which $P(\vec{v}) = \lambda^{12} P(\vec{z})$, so the zeros locus is well-defined since $\lambda \neq 0$. Defining a hyper-surface in complex projective space with a complex polynomial lowers the complex dimension by one, and the real dimension by 2, so we end up with six real dimensions, and this is our sought after six-dimensional Calabi-Yau manifold.

Millions of six-dimensional Calabi-Yau manifolds exists, and string theory doesn't choose one over the other when going to low energy. This is the moduli space problem in string theory compactification. It is similar to spontaneous breaking of rotational symmetry of a ferromagnetic block when the temperature is lowered. Nobody can predict which direction the magnetic field will point. Since different Calabi-Yau manifolds gives different spectra of elementary particles and masses, this is a very important problem in string theory.

2.6 Closed strings and one compact dimension

Let us make the X^{25} space dimension into a circle with radius R. This we can achieve by making the identification $X^{25} \sim X^{25} + 2\pi R$. It means that when we let σ run from 0 to π, X^{25} is allowed to change by an amount $2\pi R w$, since that corresponds to the same point in space-time, where $w \in \mathbf{Z}$ is called the winding number for obvious reasons. Since the space-time topology has changed, there are more ways of fulfilling the boundary conditions than before, so we need to start with a more general mode expansion for the closed string. This mode expansion contains a more general zero-mode part. Before, α_0 and $\tilde{\alpha}_0$ were identical, but now we can allow them to differ.

$$X^{25}(\tau, \sigma) = x^{25} + \sqrt{2\alpha'} \left(\alpha_0^{25} + \tilde{\alpha}_0^{25} \right) \tau - \sqrt{2\alpha} \left(\alpha_0^{25} - \tilde{\alpha}_0^{25} \right) \sigma + \text{oscillator terms}$$

All the oscillator terms in the sum are periodic with respect to the transformation of σ, so we don't write them explicitly. In addition to this we have that the momentum in the compact direction p^{25} only takes values that are integer multiples of $1/R$. In terms of the mode expansions, this is

$$X^{25}(\tau, \sigma + \pi) - X^{25}(\tau, \sigma) \quad = \quad \pi\sqrt{2\alpha'}(\alpha_0^{25} - \tilde{\alpha}_0^{25}) = 2\pi R w \tag{2.88}$$

$$p^{25} = n/R \;\;\Rightarrow\;\; \sqrt{1/2\alpha'}(\alpha_0^{25} + \tilde{\alpha}_0^{25}) = n/R, \tag{2.89}$$

for $n, w \in \mathbf{N}$. Solving these equations for the α's gives

$$\alpha_0^{25} = \sqrt{\alpha'/2}(n/R + wR/\alpha') \tag{2.90}$$
$$\tilde{\alpha}_0^{25} = \sqrt{\alpha'/2}(n/R - wR/\alpha'). \tag{2.91}$$

Now, we find the new mass spectrum. We use the zero-mode Virasoro constraints $L_0 + L_0 - 2 = 0$ and $L_0 - \tilde{L}_0 = 0$ when acting on physical states. These are different now because the zero-mode α operators now are different from before because of the compactification. We need equations (2.66), (2.67), (2.90), (2.91). When the compact dimension is small, the square of the momentum vector in the dispersion relation for the string, $p^2 = -M^2$, must not sum over the components in the compact dimension, since no movement is observed in that direction. This is the reason for the change in the mass spectrum. The condition $L_0 + L_0 - 2 = 0$ now becomes

$$M^2 = (2/\alpha')(\alpha_0^{25})^2 + (2/\alpha')(\tilde{\alpha}_0^{25})^2 + (4/\alpha')(N-1) + (4/\alpha')(\bar{N}-1). \tag{2.92}$$

The level matching condition becomes

$$(2/\alpha')(\alpha_0^{25})^2 - (2/\alpha')(\tilde{\alpha}_0^{25})^2 + (4/\alpha')(N - \bar{N}) = 0 \tag{2.93}$$

When we write this in terms of n and w, we get

$$M^2 = (n/R)^2 + (wR/\alpha')^2 + (2/\alpha')(N + \bar{N} - 2) \tag{2.94}$$
$$0 = nw + N - \bar{N}, \tag{2.95}$$

so we see that we get physical massless states when $(n, w, N, \bar{N}) = (0, 0, 1, 1)$.

When we compactify a dimension, we break the original Lorentz group $\mathbf{SO}(1,25)$ into the subgroup $\mathbf{SO}(1,24)$. The states we observe transform in irreducible representations of this subgroup. These massless states are (where free indices run from 0 to 24):

- Graviton field $G^{\gamma\delta}$:$(\alpha_{-1}^{\gamma}\tilde{\alpha}_{-1}^{\delta} + \alpha_{-1}^{\delta}\tilde{\alpha}_{-1}^{\gamma})|k>$
- Kalb-Ramond field $B^{\gamma\delta}$:$(\alpha_{-1}^{\gamma}\tilde{\alpha}_{-1}^{\delta} - \alpha_{-1}^{\delta}\tilde{\alpha}_{-1}^{\gamma})|k>$
- Dilaton field Φ:$\alpha_{-1\delta}\tilde{\alpha}_{-1}^{\delta}|k>$
- Left $U(1)$ gauge field A_L^{γ}: $\alpha_{-1}^{\gamma}\tilde{\alpha}_{-1}^{25}|k>$
- Right $U(1)$ gauge field A_R^{γ}: $\tilde{\alpha}_{-1}^{\gamma}\alpha_{-1}^{25}|k>$
- Scalar field $G^{25,25}$: $\alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}|k>$

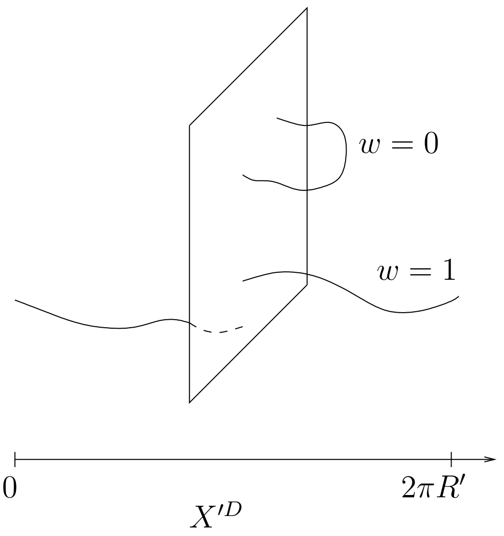


Figure 2.2: Open strings with winding numbers 0 and 1

The graviton, Kalb-Ramond and dilaton states are lower-dimensional versions of the original fields where the oscillations point in the non-compact directions. The components of the original graviton and Kalb-Ramond with one of the oscillators in the compact direction can be combined into two $\mathbf{SO}(1,24)$ vector gauge fields. This gives a $\mathbf{U}(1) \times \mathbf{U}(1)$ gauge symmetry in the lower dimensional theory. The component of the graviton with both oscillators in the compact dimension appears as a scalar, related to the size of the compact direction. As it is originally a scalar, the dilaton remains a scalar in the compactified theory.

If we had $R = \sqrt{\alpha'}$ we see from (2.94) that we would have additional massless states with $(n, w, N, \bar{N}) = (2,0,0,0)$,$(0,2,0,0)$,$(1,1,1,0)$ and $(1,1,0,1)$, in fact providing enough states with the right transformation properties to combine with the former $\mathbf{U}(1)$ fields to enlarge the symmetry to $\mathbf{SU}(2) \times \mathbf{SU}(2)$.

2.7 T-Duality

2.7.1 Closed strings

Lets find out what happens to the closed string when R approaches infinity and zero.

- $R \to \infty$: In this case we see from (2.94) that the mass of the modes with nonzero winding number w go to infinity, and we are left with only $w = 0$ states in the theory. In addition, the discrete quantum number n which enumerates the momentum in the compact dimension, goes over into a continuum, and therefore we are back to our original 26 dimensional theory

- $R \to 0$: Here, by the same argument as for $R \to \infty$, we are left with only $n = 0$ states, and the different w states become a continuum, and act as a new momentum, and we are also in this case back to a 26 dimensional theory. This fact is a purely stringy phenomenon, with no analog in theories of point particles.

From (2.94) we also see that if we do the transformation $n \leftrightarrow w$, $R \leftrightarrow \alpha'/R$, we get the same spectrum of states as before. Stated in terms on the $\alpha's$ in the mode expansion (2.38), this is $\alpha_0^{25} \leftrightarrow \alpha_0^{25}$, $\tilde{\alpha}_0^{25} \leftrightarrow -\tilde{\alpha}_0^{25}$. We now extend this transformation rule to the non zero-mode sector. By defining $X'^{25}(\tau,\sigma) = X_L^{25}(\tau+\sigma) - X_R^{25}(\tau-\sigma)$, we can see that $X^{25}(\tau,\sigma) \to X'^{25}(\tau,\sigma)$ is the same transformation. It can be shown that the energy momentum tensor and all operator product expansions in the theory are the same, so it is a symmetry of the perturbative string theory. This means that in terms of this new variable $X'^{25}(\tau,\sigma)$, we have a theory with the same spectrum as before, but compactified on a circle of radius α'/R instead. This is the so-called T-dual of the theory we started with, and is just a different description of the old theory, but on a compactified circle of a different radius and with the X' variable instead of the X. From $X'^{25}(\tau,\sigma) = X_L^{25}(\tau+\sigma) - X_R^{25}(\tau-\sigma)$ we see that this transformation is a space-time parity transformation of the right-moving degrees of freedom, since that part of the mode expansion changes sign.

2.7.2 Open strings

The open strings don't have a conserved quantum number like the winding number for the closed strings. Therefore we don't get the 26th dimension back like we did with the closed string. It therefore looks as if we in this case get a 25 dimensional theory as $R \to 0$. Since open string theories also necessarily contain closed strings, this seems like a paradox. But the surprising conclusion from a closer examination is that it is the endpoints of the open strings that are confined to 24 dimensional hyperplanes, and that the interior of the strings can move in all the 25 spatial dimensions. This can be seen be finding the mode expansions in the T-dual theory.

$$\begin{aligned} X'^{25}(\tau,\sigma) &= X_L^{25}(\tau+\sigma) - X_R^{25}(\tau-\sigma) \\ &= x'^{25} + 2\alpha' p^{25}\sigma - i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-in\tau}\sin(n\sigma) \\ &= x'^{25} + 2\alpha'\frac{n}{R}\sigma - i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-in\tau}\sin(n\sigma), \end{aligned} \tag{2.96}$$

which means that

$$\partial_\tau X'^{25}(\sigma = 0,\pi) = -i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}(-in)e^{-in\tau}\sin(n\sigma)|_{\sigma=0,\pi} = 0. \tag{2.97}$$

This means that in the T-dual description on the theory, the endpoints of open strings have a constant coordinate in the compactified direction. Notice that instead of Neumann boundary conditions, we now have Dirichlet conditions on X' in the T-dualized direction.

$$X'^{25}(\sigma = 0) = x'^{25} \quad \text{and} \quad X'^{25}(\sigma = \pi) = x'^{25} + 2\pi\alpha'\frac{n}{R} \tag{2.98}$$

We can give the difference in the X'^{25} for the two ends of the string in terms of the compactification radius of the T-dual coordinate.

$$X'^{25}(\sigma = \pi) - X'^{25}(\sigma = 0) = 2\pi n R' \tag{2.99}$$

Since the radius of the circle in the T-dual theory is R', this says that both ends of the open string end on the same hyperplane, a so-called D24-brane, since it extends itself in 24 spatial dimensions; the directions we have not T-dualized.

The possibility of interaction between different strings implies that all string endpoints end on the same hyperplane. Imagine a graviton exchange between two arbitrary open strings. The world-sheet of this process is two different sheets (one for each string), with a tube connecting the two. Choosing an appropriate coordinate system on the world-sheet so that we can take a constant time cross-section that contains both strings and the tube in between, that above calculation must still be valid. This means that the above argument is valid for any two open string endpoints, hence all open strings end on this D24-brane.

By compactifying several orthogonal dimensions, we can make D-branes of arbitrary dimensions. A dualization parallel to the tangent space of a brane reduces its dimension by one, and a dualization orthogonal to the tangent space increases the dimension by the same amount.

The D-brane in bosonic string theory is this hyperplane to which the string endpoints are attached. In this case it is a flat, static plane, but a general brane is a dynamic surface moving around in the higher dimensional space. With branes as a part of our string theory vocabulary, we can explain the significance of the tachyonic state we found in the spectrum earlier; see section (2.3.1). The state signifies the instability of the D25 space-filling brane in the bosonic string theory. All the bosonic branes are unstable, and therefore cannot be used as brane universes. In super-symmetric string theory there is no tachyon, and many brane states are stable. The crucial difference is that the super-symmetric branes carry a conserved charge. The conservation of this charge prohibits the branes from decaying. They are therefore possible candidates for brane universes.

2.8 Several branes

Let us look at an oriented open string. We found that we had $\mathbf{U}(N)$ gauge symmetry in this case. Also, assume that the X^{25} direction has been compactified with radius R. We now choose a configuration for the $\mathbf{U}(N)$ gauge field A^{α} with

$$A_{25} = \frac{1}{2\pi R}\text{diag}\{\theta_1, \theta_2, ..., \theta_N\} = -i\Lambda^{-1}\partial_{25}\Lambda, \tag{2.100}$$

where

$$\Lambda = \text{diag}\{e^{\frac{iX^{25}\theta_1}{2\pi R}}, e^{\frac{iX^{25}\theta_2}{2\pi R}}, ..., e^{\frac{iX^{25}\theta_N}{2\pi R}}\}. \tag{2.101}$$

Any field configuration that can be put in the preceding form, will have a vanishing field strength tensor, and is therefore called a "pure gauge" configuration. By demanding this field configuration we are breaking the gauge symmetry from $\mathbf{U}(N)$ to $\mathbf{U}(1)^N$, except in that cases where some of the angles θ_i coincide. This is analogous to the breaking of the electroweak gauge symmetry in the standard model when the Higgs field acquires a non-zero vacuum expectation value. If we now make a gauge transformation so that A disappears everywhere, we will on account of the nontrivial topology of a compactified dimension modify the periodicity of all fields that transform non-trivially under this gauge symmetry. This effect is analogous to the Aharanov-Bohm effect in quantum mechanics. We get $|k, ij> \mapsto e^{i(\theta_i - \theta_j + 2\pi n)/2\pi R}|k, ij>$ when $X^{25} \mapsto X^{25} + 2\pi R$, and this means that the state $|k, ij>$ has fractional momentum $(\theta_i - \theta_j + 2\pi n)/2\pi R$. In the dual picture this manifests itself as

$$X'^{25}(\sigma = \pi) - X'^{25}(\sigma = 0) = (2\pi R + \theta_i - \theta_j)R', \tag{2.102}$$

for a state with Chan-Paton states i and j. This means that the string endpoints no longer are restricted to the same D24-brane, but that we have one D24-brane for each unique θ_i. If we had chosen a gauge field configuration with not all angles unique, e.g. $\theta_1 = \theta_2$, the gauge group is not broken all the way down to $\mathbf{U}(1)^N$, but to e.g. $\mathbf{U}(2) \times \mathbf{U}(1)^{N-2}$ in our example. This corresponds to having two branes on top of each other, and the extra massless open string necessary to be the gauge bosons for the $\mathbf{U}(2)$ part of the gauge group states appears, since we in addition to strings going to and from the same brane also have massless strings going from one brane to the other. An endpoint in the Chan-Paton state i ends on $X'^{25} = \theta_i R = 2\pi\alpha' A_{25.ii}$.

2.9 Unoriented strings and orientifolds

Now we will use the projection operator Ω in equation (2.84) to make an unoriented string theory of closed strings, and see what happens in the dual theory. From equations (2.86) and (2.87), we see that the action on the left- and right-moving parts are

$$\begin{aligned} X_L(\tau, \sigma) &\mapsto \Omega X_L(\tau, \sigma)\Omega^{-1} = X_R(\tau, -\sigma) & (2.103)\\ X_R(\tau, \sigma) &\mapsto \Omega X_R(\tau, \sigma)\Omega^{-1} = X_L(\tau, -\sigma). & (2.104) \end{aligned}$$

For the dual coordinate $X' = X_R - X_L$, this implies

$$X'^{25}(\tau, \sigma) \mapsto \Omega X'^{25}(\tau, \sigma)\Omega^{-1} = -X'^{25}(\tau, -\sigma), \tag{2.105}$$

which says that the dual theory of the theory of unoriented closed strings is a theory which is invariant under the product of a space-time and a world-sheet parity operation. Upon compactifying the original theory on a circle, this means that the dual theory lives in a space-time $\mathbf{R}^{25} \times \mathbf{S}^1/\mathbf{Z}_2$. At the fixed points of the space-time parity operation, we have symmetry planes called orientifold planes, and the states of the theory have to be symmetric under a reflection through these planes. In this case the orientifold plane extends itself into 24 spatial dimensions, and is called a O24-plane. Since the product of the space-time and world-sheet parity operations only gives a nonlocal correspondence between states, this means that locally, away from the orientifold planes, the string theory looks like an ordinary oriented string theory. Since we demand that the states are symmetric about these O-planes, and that locally it looks like an oriented theory, we can have arbitrary configurations of D-branes subject to the condition that there is an identical D-brane configuration at the opposite side of each O-plane.

2.10 D-brane action

The action for a Dp-brane is

$$S = -T_p \int d^{p+1}\xi e^{-\phi} \sqrt{\det(G + B + 2\pi\alpha' F)}. \tag{2.106}$$

T_p is the brane tension, which can be said to be the energy density per volume inherent in the brane. Since a Dp-brane has p spatial dimensions, the integral over its world-volume is a $p+1$-dimensional integral. The factor $\exp(-\phi)$ describes the brane's coupling to the bulk dilaton field ϕ. $G_{ab}(\xi) := \partial_a X^\alpha \partial_b X^\beta G_{\alpha\beta}(X(\xi))$ is the pullback of the space-time metric from space-time to the brane volume, and similarly for the antisymmetric B_{ab} Kalb-Ramond bulk two-form field. The F_{ab} is the antisymmetric field strength tensor of the $\mathbf{U}(1)$ gauge field on the brane. Usually, the action is referred to as the Dirac-Born-Infeld action.

The action can be derived by finding a solution of string theory on a background containing a Dp-brane, and then demanding that it satisfies conformal invariance. This derivation is out of the scope of this thesis. However, we can illustrate the type of argument used when deriving the action. The fact that G and F must appear in the combination $G + 2\pi\alpha' F$ can be motivated by showing that having a certain value of the gauge field A on the brane corresponds in the T-dual theory to a tilting of the brane which changes the induced metric to the form $G + 2\pi\alpha' F$. The combination $B + 2\pi\alpha' F$ is necessary to maintain gauge invariance of the action, since A and B transform under this symmetry as

$$\delta A_\alpha = -\zeta_\alpha / 2\pi\alpha' \quad \text{and} \quad \delta B_{\alpha\beta} = \partial_\alpha \zeta_\beta - \partial_\beta \zeta_\alpha. \tag{2.107}$$

The exponential dilaton factor is necessary since the action is the result of tree-level open string diagrams, and must therefore have a certain dependence on the string coupling constant $\exp(-\phi)$ [11]. This factor results from the topology of the interaction world-sheets, as discussed in section 2.1.4.

A few things stand out when looking at the Dirac-Born-Infeld action (2.106). Firstly, if we Taylor expand the square root, we get to lowest order in α' a F^2 Yang-Mills gauge field strength term. Thus at low energy, we have standard gauge theory on the brane. In addition, we see that the action is not well-defined when the contents of the square root is zero or negative, since in those cases it either doesn't have a well-defined derivative and thus we cannot vary it to obtain an equation of motion, or it is imaginary. This occurrence signifies that the action is no longer valid for the system in question (the D-brane), and the fact that it is not universally valid is only natural, considering the fact that it only couples to massless fields. It is only valid at low energy, and in that regime the contents of the square root will always be positive and small.

The Dirac-Born-Infeld action has been used cosmologically, e.g. to analyze how the brane gauge theory influences the brane geometry in the case of a randomly oriented electromagnetic field of fixed norm as each particular time, $\vec{E}^2 = \vec{B}^2 = \epsilon(t)$ [15, 16]. The authors run into this singularity when following the solution of the problem backward in time, and propose the occurrence of an S-brane, i.e. a brane whose world-volume is only extended in spatial dimensions and not the time dimension, at the moment where the action breaks down. We attribute the breakdown of the action not to the occurrence of an S-brane, but to the non-validity of this action at high energies.

2.11 Super-symmetric strings

From equation (2.11) we see that we can think of bosonic string theory as being a two dimensional quantum field theory with 26 bosonic fields. In fact, working with the two dimensional field theory on the world-sheet is by far the most common viewpoint employed by string theorists. Now, in a quantum field theory, we demand that each field representing a fundamental degree of freedom transform identically to an irreducible representation of the Poincare group of translations and rotations of space-time. But since spin really is one of the numbers we use to enumerate the different representations, we see that under a Poincare transformation, these fields retain their spin value. This means that with only Poincare symmetry, there is no connection between e.g. the masses of bosons and fermions. Nevertheless, there exists an extension of this symmetry group, called the Super-Poincare group, which does transform bosons into fermions and visa versa. Also, if introduced into the standard model of elementary particles, it partially solves some of the problems with that model, namely the hierarchy problem and the problem of vacuum energy. In string theory, super-symmetry is introduced in order to get fermions into the spectrum of the theory. The method is to demand that the two dimensional world-sheet quantum field theory has super-symmetry, and there are different ways of doing that, leading to the five known consistent string theories (Type I of open and closed unoriented strings, Type IIA or IIB of oriented closed strings, $\mathbf{E}_8 \times \mathbf{E}_8$ or $\mathbf{SO}(32)$ Heterotic of oriented closed strings).

All of these five theories have 10 space-time dimensions. However, having a fermionic field on the two dimensional world-sheet only means that the fields has fermionic transformation properties with respect to the two dimensional Lorentz transformations on the world-sheet, which is not the same as the 10 dimensional space-time Lorentz transformations. In order to also obtain space-time super-symmetry, the spectrum of states of all these theories has to be truncated by projecting the space of states down to a subspace with the so-called GSO projection, which also eliminates the tachyon from the spectrum. This truncation of the spectrum is also necessary for consistency of the theory, i.e. to remove anomalies that emerge during quantization.

Name	Gauge Symmetry	String Types
Type I	$\mathbf{SO}(32)$	Open and closed, unoriented
Type IIA	-	Closed, oriented
Type IIB	-	Closed, oriented
Heterotic	$\mathbf{SO}(32)$	Closed, oriented
Heterotic	$\mathbf{E}_8 \times \mathbf{E}_8$	Closed, oriented

Table 2.1: Super-symmetric string theories

2.12 M-theory

It was discovered in the mid-nineties that a strong coupling limit of the Type IIA string theory was 11 dimensional. This along with duality relations among the different string theories and the existence of a 11 dimensional theory of super-gravity is strong circumstantial evidence for the existence of an 11 dimensional theory that reduces to these five different string theories at different weak coupling limits. Also, it reduces to 11 dimensional super-gravity at low energy. In some sense, the M-theory can be thought of as a theory of branes, since it appears that the Type IIA string is really an M2-brane wrapped around a compact dimension. This comes out of the fact that the strong coupling regime of the Type IIA string theory is M-theory compactified on $\mathbf{S}_1$. Also, the strong coupling regime of the Type IIB string theory is M-theory compactified on $\mathbf{S}_1/\mathbf{Z}_2$.

In the standard model of elementary particles, the leptons couple chirally to the weak force. It is possible to get chiral coupling from the Type IIB string theory, and therefore this is regarded as the best candidate of the five theories to unify the standard model with gravity. Since this string theory is obtained from M-theory by compactifying on $\mathbf{S}_1/\mathbf{Z}_2$, it should be possible to use this manifold as a background geometry for a string-based cosmological model. We will do just that later, in chapter 5.

Chapter 3

Big Bang and Inflation

3.1 Introduction

Since we will work on an alternative cosmological model, mainly to provide an alternative to the big bang, we give a short review to the basics of standard cosmology at very early times.

In the standard model of cosmology, the universe starts with an explosion called the "Big Bang", at which according to Einstein's equations the density and temperature diverges. After that, there is a period of exponential expansion that increases the size of the universe by a factor of around e^{60}, or 60 e-foldings. This is called inflation, and is necessary to account for the flatness and isotropy of the universe on distance scales that would otherwise never have been in contact with each other, and therefore would be expected to have very different properties. This way, all observed regions of the universe will have been in thermodynamic equilibrium at some early time, thus accounting for their similar value of the temperature of the cosmic background radiation.

3.2 The Robertson-Walker metric

Current observations show that the universe looks the same in every direction, on scales large in comparison with the size of galactic clusters. This is the same as saying that there is no preferred direction in the universe, on these scales. We say that the universe is cosmologically isotropic.

Observations also suggest that we do not live in a special part of the universe. When we analyze the spectral properties of light from other galaxies, we see that it it looks the same as light emitted from earthly substances, only red-shifted to lower frequencies. This redshift is a natural consequence of the expansion of the universe, and the individual movements of the observed objects. In addition, it looks as if the distribution of galaxies across the sky is even on large scales. Thus we can assume that the universe is cosmologically isotropic about every other point as as well. It can be shown geometrically that if a space if isotropic about every point, it is similar at every point. It is cosmologically homogeneous.

We must stress that there may be unobserved parts of the universe that are wildly different from our own neighborhood, but in the cosmological standard model, this is assumed not to be the case.

We use the homogeneity and isotropy of the universe to construct a metric Ansatz of the form

$$ds^2 = -dt^2 + a(t)^2 \left(d\chi^2 + r(\chi)^2 d\Omega_2^2\right). \tag{3.1}$$

Isotropy will be used to determine the function $r(\chi)$. We calculate the Ricci tensor R^{α}_{β} (see appendix B), and get the non-zero components

$$\begin{aligned} R^t_{\ t} &= 3\frac{\ddot{a}}{a} && (3.2)\\ R^{\chi}_{\ \chi} &= \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - \frac{2r''}{a^2 r} && (3.3)\\ R^{\theta}_{\ \theta} &= \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - \frac{r''}{a^2 r} - \frac{(r')^2}{a^2r^2} + \frac{1}{a^2r^2} && (3.4)\\ R^{\phi}_{\ \phi} &= \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - \frac{r''}{a^2 r} - \frac{(r')^2}{a^2r^2} + \frac{1}{a^2r^2}. && (3.5) \end{aligned}$$

Since we want isotropy, the three spatial components of the Ricci tensor in orthonormal coordinates should be equal. The above components are not in orthonormal coordinates, but it is easy to understand that when the metric is diagonal, these contravariant-covariant diagonal components are the same as the corresponding orthonormal components. Thus we demand equality of (3.3), (3.4) and (3.5), which gives

$$-2\frac{r''}{r} = -\frac{r''}{r} - \frac{(r')^2}{r^2} + \frac{1}{r^2} \Rightarrow r''r - (r')^2 = -1.$$

This equation can be integrated to give

$$r'(\chi) = \sqrt{1 - kr^2}, \tag{3.6}$$

where k is an integration constant that has mass-dimension 2. It is customary to re-parametrize the mass-scale so that that numeric value of k is 1,0 or -1. We can integrate (3.6) to get

$$r(\chi) = \begin{cases} \frac{1}{\sqrt{k}}\sin(\sqrt{k}\chi) & ,k > 0 \\ \chi & ,k = 0 \\ \frac{1}{\sqrt{-k}}\sinh(\sqrt{-k}\chi) & ,k < 0. \end{cases}$$

In the three cases spatial hyper-surfaces at constant t have constant positive, vanishing, and negative curvature. Now we can easily transform the metric to the coordinate system (t, r, θ, ϕ), getting

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_2^2\right) \tag{3.7}$$

In the case with positive spatial curvature, $k > 0$, a spatial hyper-surface is the three dimensional sphere of constant radius S_3, on which our coordinate system cover only one of the hemispheres, and $r = 1/\sqrt{k}$ corresponds to the equator. When $k = 0$, the spatial hyper-surfaces at constant time are flat Euclidean three dimensional space E_3, and the coordinates cover the entire space. When $k < 0$, we get a hyperbolic space H_3, which is a three dimensional generalization of the saddle surface with constant negative curvature, covered entirely by our coordinate system. The Ricci curvature scalar for spatial hyper-surfaces of constant time is independent of spatial coordinates, and is given by

$$\mathcal{R}_3 = \frac{6k}{a(t)^2}. \tag{3.8}$$

From this we see that the curvature of spatial sections of space-time diverge in the limit where the scale factor $a(t)$ appraches zero.

3.3 Friedmann equations and the past singularity

The Einstein tensor calculated from the metric in equation (3.7) is (see appendix B)

$$E^t{}_t = -3\frac{a'^2 + k}{a^2} \tag{3.9}$$

$$E^r{}_r = E^\theta{}_\theta = E^\phi{}_\phi = -2\frac{a''}{a} - \frac{a'^2 + k}{a^2}. \tag{3.10}$$

It is assumed that the matter contents is given by a perfect fluid, a fluid with no viscosity or heat conduction. The energy-momentum tensor for a homogeneous distribution of such a fluid with movements given by a 4-velocity field u^α, is

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + pg^{\alpha\beta}, \tag{3.11}$$

where ρ and p are energy density and pressure, respectively, of the fluid. To keep the models homogeneous, we assume them to be independent of spatial coordinates. In terms of components of the energy-momentum tensor, this gives

$$T^t{}_t = -\rho \tag{3.12}$$

$$T^r{}_r = T^\theta{}_\theta = T^\phi{}_\phi = p. \tag{3.13}$$

The Einstein equations $E^{\alpha\beta} = 1/M_4^2 \times T^{\alpha\beta}$ gives two independent equations, called the Friedmann equations. Here, M_4 is the four-dimensional Planck mass.

$$3\frac{\dot{a}^2 + k}{a^2} = \frac{1}{M_4^2}\rho \tag{3.14}$$

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} = \frac{1}{M_4^2}p \tag{3.15}$$

Combining the two first Friedmann equations with the time derivative of the first one with respect to time, we can derive

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{3.16}$$

This is the energy conservation equation for the perfect fluid. It can also be obtained by equating the covariant divergence of the energy momentum tensor to zero, which also must be satisfied, since the covariant divergence of the Einstein tensor is zero by its geoemtric construction. This follows from contraction of the purely geometric Bianchi second identity $R^{\alpha}{}_{\beta[\gamma\delta,\epsilon]} = 0$, where the square brackets means total antisymmetrisation.

We can also easily derive the acceleration equation from the two Friedmann equations,

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_4^2}(\rho + 3p). \tag{3.17}$$

The right side of this equation is negative for all ordinary matter. Since we observe a slightly positive second derivative on the scale factor today, this means that if we trust Einstein's theory of gravitation, the expansion of the universe today is mainly driven by an exotic form of matter, for which $\rho + 3p < 0$.

From here on we assume a simple equation of state for the perfect fluid

$$p = w\rho, \tag{3.18}$$

where w is an arbitrary constant. If some form of matter is to cause a cosmic acceleration, and it has an equation of state of this form, it must have $w < -1/3$ to satisfy $\rho + 3p < 0$.

A general argument showing the existence of a past singularity in any model with an equation of state parameter satisfying $w > -1/3$ goes as follows. When $w > -1/3$, we see from equation (3.17) that this gives $\ddot{a} < 0$. Given that $\dot{a}$ is positive and finite today, this means that $\dot{a}$ must always have been positive, and even larger, in the past. Thus a must have been zero not more that a time $a/\dot{a} := H^{-1}$ ago. The time at which $a = 0$ is a geometric singularity, since the curvature of spatial sections diverge at this time.

With our expression for the equation of state, equation (3.16) now becomes

$$\dot{\rho} + 3(1+w)\frac{\dot{a}}{a}\rho = 0 \Rightarrow \frac{d\rho}{\rho} = -3(1+w)\frac{da}{a},$$

which has the solution

$$\rho = \rho_0 a^{-3(1+w)}, \tag{3.19}$$

where ρ_0 is the energy density when $a(t) = 1$. Thus the value of w dictates how the energy density and pressure varies as a function of the scale factor. For radiation it can be shown that $w = 1/3$, so we have

$$\rho_r = \rho_{0,r} a^{-4}. \tag{3.20}$$

For pressure-less matter, with $w = 0$, we get

$$\rho_m = \rho_{0,m} a^{-3}. \tag{3.21}$$

We see that the radiation energy density falls faster than the matter energy density. Given the values for both quantities today, we can extrapolate backwards in time to see that there must have been an epoch when radiation dominated the evolution of the universe.

To see how the scale factor varied during that early epoch, we insert (3.20) into the first Friedmann equation (3.14).

$$\dot{a}^2 a^2 + ka^2 = \frac{\rho_0}{3M_4^2} \tag{3.22}$$

This can be written as

$$\frac{ada}{\sqrt{\rho_0/3M_4^2 - ka^2}} = \pm dt \tag{3.23}$$

Since we know that $a = 0$ sometime in the past, we choose the time coordinate so that this happens at $t = 0$. Integrating equation (3.23) from 0 to t gives

$$-\frac{1}{2k}\sqrt{\rho_0/3M_4^2 - ka^2} = \pm t - \frac{1}{2k}\sqrt{\rho_0/3M_4^2}, \tag{3.24}$$

which when solving for a^2 gives

$$a^2 = -kt^2 \pm 2\sqrt{\frac{\rho_o}{3M_4^2}}t. \tag{3.25}$$

Since the left side is positive at all times, particularly at small t, we must choose the plus sign on the right side to get a non-imaginary scale factor. The expression is valid for all values of k. The expansion rate $\dot{a}$ is given by the expression

$$\dot{a} = \frac{\sqrt{\rho_0/3M_4^2} - kt}{\sqrt{2\sqrt{\rho_0/3M_4^2}t - kt^2}}, \tag{3.26}$$

which we see diverges towards infinity when $t \to 0^+$, since the denominator approaches zero, while the numerator does not. This means that at the big bang, space expanded infinitely fast.

In the case of $w = -1$, the behaviour of a cosmological constant or a slow-rolling scalar field used in inflation, the continuity equation (3.16) informs us that the energy density is independent of the scale factor, so we can write

$$\rho = \rho_0 \quad \text{and} \quad p = -\rho_0. \tag{3.27}$$

Inserting this into the acceleration equation (3.17), we get

$$\ddot{a} - \frac{\rho_0}{3M_4^2}a = 0, \tag{3.28}$$

which has the solution

$$a(t) = a_0 \exp(Ht) \quad \text{where} \quad H := \sqrt{\frac{\rho_0}{3M_4^2}}. \tag{3.29}$$

Here we have dropped the decaying part of the general solution, since it has no cosmological significance. In this case the universe still has a geometric singularity, but in the infinite past.

From the Ricci tensor (3.2) - (3.5), we find the Ricci curvature scalar

$$\mathcal{R} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \tag{3.30}$$

This quantity diverges towards infinity when a approaches zero. Since the Ricci curvature scalar is a coordinate independent quantity, this is a geometric singularity of the four-dimensional space-time manifold, at which also the Riemann and Ricci tensor diverges. Thus the solutions to Einstein equations are singular, meaning that we can't trust them at near the singularity. We must assume that general relativity is not valid at all energy scales, let alone when the energy density diverges towards infinity, as is the case with the Big Bang. It is natural to assume that general relativity is an effective theory that is the low energy limit of a quantum gravity. This quantum gravity could be string theory. But it must be mentioned that the big bang singularity problem has not been resolved in the framework of string theory. This could of course change in time after string theory becomes better understood.

3.4 Cosmological problems

3.4.1 Flatness

The observed universe is very close to being flat, i.e. k is very close to zero. Neglecting k in equation (3.14), this gives a relation between the Hubble parameter $H(t) := \dot{a}(t)/a(t)$ and the energy density ρ_c in a flat universe.

$$\rho_c(t) = 3M_4^2 H(t)^2 \tag{3.31}$$

This is called the critical density; it is the energy density needed to maintain a flat universe. We define the density parameter as the ratio between the density in the universe and the critical density,

$$\Omega(t) := \frac{\rho(t)}{\rho_c(t)}. \tag{3.32}$$

The first Friedmann equation can then be rewritten. We take absolute values on both sides.

$$|\Omega(t) - 1| = \frac{|k|}{3H(t)^2 a(t)^2} \tag{3.33}$$

With a matter or radiation dominated universe, the denominator in this expression diminishes with time, since the gravitational attraction on ordinary matter acts as a break on the scale factor. This means that the density parameter moves away from unity as time goes by. Since it is close to unity now, this means that it must have been extremely close to unity at early times. Assuming e.g. a radiation dominated universe, giving $a(t) \propto \sqrt{t}$, we get

$$|\Omega(t) - 1| \propto t. \tag{3.34}$$

This implies that at nucleosynthesis ($t \approx 1$s), we must have had $|\Omega - 1| \approx 10^{-16}$. This is an fine-tuning problem for non-inflationary cosmology.

3.4.2 Horizon/Homogeneity/Isotropy

Since the cosmic microwave background (CMB) radiation temperature is the same across the entire sky to an accuracy of a few parts in one million, it seems that the visible universe must have been in thermodynamic equilibrium at some point in the past before the radiation was emitted, since the radiation has been effectively non-interacting after that point in time.

The maximum comoving distance light could have traveled since the Big Bang at $t = 0$ until the time t_{dec} of photon decoupling at the last scattering surface is

$$\int_0^{t_{dec}} \frac{dt}{a(t)}. \tag{3.35}$$

However, the comoving distance over which we have observed the universe to be homogeneous and isotropic is given by the distance light have traveled since the last scattering surface, and is

$$\int_{t_{dec}}^{t_0} \frac{dt}{a(t)}. \tag{3.36}$$

The horizon problem is the fact that the first of these is much smaller than the second for a matter dominated, and in fact also a radiation dominated universe. These comoving scales correspond to angular separations on the map of the CMB radiation across the sky, and according to 3.35, regions separated by more than 2 degrees have not been in causal contact, but the CMB is approximately the same across the whole sky within a few parts in a million.

Thus, assuming a matter or radiation dominated universe model, the regions of space that emitted the CMB radiation coming from opposite sides of the sky have never been in thermal equilibrium. In other words, the visible universe was must larger than the causal horizon at the time when the radiation was emitted. The fact that the universe is also homogeneous and isotropic is also a mystery for the same reason.

3.4.3 Mass

The total mass of the universe is of the order of $10^{60} M_4$, where M_4 is the four-dimensional Planck mass. We have $M_4 \approx 5.5 \times 10^{-8}$kg, so we get a universe mass of the order of 10^{52}kg. Where did all this mass come from? No cosmological theory can give an explanation of this fact. One simply assumes that this energy have always existed since the beginning of time. But if time has a beginning, this is a problem from the point of view of the principle of energy conservation, if this principle can be applied to something as exotic as the big bang.

3.4.4 Structure formation

The universe is obviously not completely homogeneous. How were the inhomogeneities created? It is usually explained as originating from quantum fluctuations during the early evolution of the universe. There are well-understood processes by which these microscopic fluctuations can become macroscopic at later times in cosmological evolution. This can happen when the wavelength of the fluctuations is larger than the horizon, and are streched out to macroscopic sizes by the expansion of space-time.

3.4.5 Monopoles

Monopoles are extended gauge field configurations, meaning the gauge field is not localized to a point in space, but is non-zero throughout space. This means that the field value approaches a non-zero value, a pure gauge, at infinity. The corresponding field strength tensor approaches zero at infinity, however, so the energy of the field is bounded.

Production of these states is expected to occur frequently at high energy. At very high energy in the early universe we would presumably have production of monopoles, which are stable against decay. These topological defects could e.g. appear as a long string-like discontinuity in the space-time geometry. The lack of observed monopoles is called the cosmological monopole problem.

3.5 Inflation

In 1981 Guth presented an article entitled "The Inflationary Universe: A Possible Solution To The Horizon And Flatness Problems" [17]. It was based on the possibility of a phase transition around the GUT (=Grand Unified Theory) scale of about 10^{15}GeV, that caused exponential expansion. It offers a good explanation for the questions of homogeneity, isotropy, flatness, horizon and structure formation in the universe. This was a big breakthrough, and ever since most cosmologists have taken the theory of inflation to their hearts.

The flatness problem can be solved by adding an epoch in the history of the universe where the denominator in (3.33) grows rapidly, pushing the density parameter very close to $\Omega = 1$; see equation (3.33). It also solves the horizon problem if the scale factor behavior is modified at early times so that the value of (3.35) increases far beyond the value of (3.36).

We define inflation as the geometric property $\ddot{a} > 0$, thus we also call the increasing expansion velocity observed today for the current universe inflation. In standard cosmological evolution with non-exotic matter, we have $\ddot{a} < 0$. We also know that in standard evolution (3.35) is much smaller than (3.36), and we have a horizon problem. However, we can see qualitatively how demanding $\ddot{a} < 0$ during some early epoch before last scattering will modify this result. At last scattering, the universe has some known size, independent of previous inflation or not. If we change the behavior of a from $\ddot{a} < 0$ to $\ddot{a} > 0$ during some earlier epoch, this will suppress the scale factor more as we go back in time during that period, thus making the integrand larger. By having such an epoch we can thus make the integral (3.35) blow up to any value we want, and thus solve the horizon problem. It turns out that it is necessary to have an inflationary period where the scale factor increases by a factor around e^{60} to solve the horizon problem.

The mechanism of structure formation is well understood in inflationary models, and that monopoles are not observed can be explained by the fact that they are only produced at energies accessible before inflation, and are therefore significantly diluted by the enormous expansion to negligible densities, since it is known that their energy density scales as $\rho \propto a^{-3}$.

In order to create an inflation mechanism we introduce a scalar field with a potential making the scale factor increase exponentially. This scalar field is often called the inflaton. The action and Lagrangian for a scalar field $\phi(t)$ is

$$S_\phi = \int d^4x\sqrt{-g}\mathcal{L}_\phi, \tag{3.37}$$

where

$$\mathcal{L}_\phi = -\frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi - V(\phi). \tag{3.38}$$

The covariant derivative of a scalar field reduces to the ordinary coordinate derivative $\nabla_\alpha\phi = \partial_\alpha\phi$. Varying the action with respect to ϕ, and dividing the result by $\sqrt{-g}$, we get the Euler-Lagrange equation

$$\frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}g^{\alpha\beta}\partial_\beta\phi\right) + V'(\phi) = 0 \tag{3.39}$$

Substituting the metric (3.7) into this expression, and assuming that ϕ doesn't depend on the spatial coordinates to satisfy the condition of homogeneity, we get

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0. \tag{3.40}$$

We see that a positive expansion rate ($\dot{a} > 0$) acts as a brake for the movement of the scalar field in its potential. The energy momentum contribution of the scalar field is obtained by the usual formula,

$$\begin{aligned} T_\phi^{\alpha\beta} &= -2\frac{\partial\mathcal{L}_\phi}{\partial g_{\alpha\beta}} + \mathcal{L}_\phi g^{\alpha\beta} \\ &= \partial^\alpha\phi\partial^\beta\phi + \left[-\frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi - V(\phi)\right]g^{\alpha\beta} \\ &= \partial^\alpha\phi\partial^\beta\phi + \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)g^{\alpha\beta}. \end{aligned} \tag{3.41}$$

From this we get the energy density and pressure,

$$\rho_\phi = -T^t_{\ t} = -\partial^t\phi\partial_t\phi - \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.42}$$

$$\begin{aligned} p_\phi &= \frac{1}{3}T^i_{\ i} = \frac{1}{3}\left[\partial^i\phi\partial_i\phi + 3\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)\right] \\ &= \frac{1}{2}\dot{\phi}^2 - V(\phi). \end{aligned} \tag{3.43}$$

Both the kinetic and potential energy of the scalar field contribute to these quantities. For the scalar field, the equation of state is not as simple as equation (3.18), since p is not proportional to ρ. Instead, we get a varying w parameter

$$w := \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{3.44}$$

We can see that when the potential is small compared to the kinetic energy we get $w \approx 1$, which is called stiff fluid. When the kinetic energy is small compared to the energy density we have $w \approx -1$, as for a cosmological constant. In the latter case, we get an exponential expansion if the scalar field dominates the energy density of the universe, as seen from equation (3.29). By choosing the right potential, a realistic inflationary cosmological model can be constructed.

3.6 Drawbacks of inflation

Even though inflation solves many of the cosmological problems, it has some drawbacks. Inflation postulates the inflaton scalar field, which has not been observed in the laboratory. The fact that this field is of unexplained origin and that it must contain all the energy of the universe as kinetic and potential energy, means that inflation doesn't solve the mass problem. One must also explain theoretically the origin and detailed structure of the inflaton potential. In addition, fine-tuning of this potential is needed to obtain the correct number of e-foldings in the expansion.

The fact that the space-time expansion can move objects apart with super-luminal speeds, and that this is used in inflation, is sometimes mentioned as a drawback, and is e.g. used as a motivation for introducing the ekpyrotic scenario. There is, however, no restriction on this type of behaviour in general relativity, as can e.g. be seen from the fact that $\dot{a}$ is not bounded from above in e.g. a radiation dominated cosmological model. In that case the expansion speed diverges towards infinity as we follow the solution the backwards in time towards the big bang.

Chapter 4

Brane Universe

4.1 Introduction

The current universe seems to have four space-time dimensions. There are, however, some problems that can be solved by introducing extra dimensions. In addition, there is theoretical motivation to investigate extra dimensions, coming from string theory, where there are ten. In M-theory, there are eleven space-time dimensions. Under the assumption that extra dimensions exist, we must be able to explain why only four dimensions are observed. There are two ways of doing this. The first method is to say that the extra dimensions are compactified, thus too small to be observed in current experiments. The other way is to say that our visible universe is stuck on a 3-brane that floats around in the higher dimensional space.

4.1.1 Compactification

In the case of compactifying the ten dimensions of string theory, the problem is that there are many different ways to compactify the six dimensions, and string theory doesn't say which realization must be found in nature. It only puts some restraints on the allowed types of compact dimensions. They must form a special kind of manifold, called a Calabi-Yau space. Thus one way of fixing the number of dimensions to four is to compactify six dimensions to a tiny Calabi-Yau space, which is smaller than we can detect with current technology. But there are still millions of different Calabi-Yau spaces to choose among.

4.1.2 3-branes

Another possibility is that the universe is a brane floating around in the nine spatial dimensions of string theory. By the laws of string theory, light from an object on the brane can not leave the brane, so from the electromagnetic point of view, this is consistent with current observations. In general, there is a Yang-Mills gauge theory confined to the brane. But gravity is allowed to leave the brane, so in order for the brane-universe to be consistent with observations, the extra dimensions have to be tiny, or some mechanism is needed to stop gravity from escaping the brane. One such mechanism is to use a negative cosmological constant in the bulk space outside the brane, as in the Randall-Sundrum scenario [18]. In this model, the bulk geometry is warped in such a way that the physical length of the extra dimension is finite and small.

4.1.3 Ekpyrotic universe

The ekpyrotic universe uses a combination of the two ideas above. We start with the eleven dimensions of M-theory. The strong string coupling limit of the heterotic $\mathbf{E}_8 \times \mathbf{E}_8$ string theory is M-theory on a $\mathbf{R}^{10} \times \mathbf{S}^1/\mathbf{Z}_2$ space-time [6]. The $\mathbf{E}_8 \times \mathbf{E}_8$ string theory is considered to be the most realistic candidate for a successful high-energy generalization of the standard model and quantum gravity. It includes the possibility of chiral couplings (interactions that only affect left-handed particles), which is an essential ingredient in the standard model of elementary particles. On each boundary of this space-time we get a Yang-Mills gauge theory with an $\mathbf{E}_8$ gauge group. We have one copy of such a gauge theory on each boundary, contained in separate M5-branes. Then we compactify six spatial dimensions into a small, compact Calabi-Yau space. This leaves us with a five dimensional space. Two of the six compactified dimensions are parallel to the M5-branes, so in the five-dimensional theory, the 5-branes look like 3-branes.

Next, we must make the fifth dimension small, but larger than the Calabi-Yau manifold. We compactify it into a circle $\mathbf{S}_1$. After that, we define a $y \in [-R, R]$ angular coordinate on the $\mathbf{S}_1$ and identify points $y \sim -y$. What we then get is called $\mathbf{S}^1/\mathbf{Z}_2$. At the same time, we put the M5-branes at the two fix-points of the identification, thus at $y_i \in \{0, R\}$. The length of the $\mathbf{S}^1/\mathbf{Z}_2$ part of $\mathbf{R}^{10} \times \mathbf{S}^1/\mathbf{Z}_2$ is an order of magnitude bigger than the six Calabi-Yau manifold dimensions [7]. This estimation was made by demanding a reasonable value of the Newtonian gravity constant in the resulting GUT at low energy (low compared to the string energy scale) on the boundary branes. Thus there is a regime with low energy where we can neglect the Calabi-Yau dimensions, and get an effective five-dimensional theory on a $\mathbf{R}^4 \times \mathbf{S}^1/\mathbf{Z}_2$ orbifold. This effective theory is called the heterotic M-theory. Since two of the 5 spatial dimensions of these M5-branes are in the compact Calabi-Yau dimensions, we now call them 3-branes. We have now constructed the so-called Horava-Witten theory. The two 3-branes are on each boundary of the fifth dimension, which is topologically the same as a line element with length R. The boundary 3-branes each have the possibility of having a $\mathbf{E}_8$ Yang-Mills gauge theory on them.

We consider the possibility that our universe is stuck on one of them, and that there is a "hidden universe" on the other boundary brane. The construction opens up a possibility where a third brane, the bulk brane, can move inside the bulk space between the visible and hidden universe, and collide with the visible universe. This hypothetical event could be an alternative to the standard big bang model, thus avoiding the geometric singularity associated with that phenomenon.

4.2 Motivation

4.2.1 String theory

One of the motivations for working on this cosmological model is that it is based on string theory. Ordinary cosmology is based on Einsteins theory of general relativity, but as we have seen, it leads to singularities in the space-time geometry like the big bang and the centers of black holes. These singularities are expected to be smoothed out in a quantum theory of gravitation. Since string theory is the leading candidate for a quantum gravity, a string based cosmology could present us with non-singular alternatives to the big bang and black holes. Specifically, the ekpyrotic universe eliminates the initial big bang singularity.

4.2.2 The hierarchy problem

The hierarchy problem can be stated as follows: why is the gauge symmetry group of the field theory that is supposed to describe all the phenomena of nature broken not at one, but at two completely different energy levels? For example, in string theory, one would expect it to be broken at the Planck energy, but in fact, part of the symmetry group (the standard model electroweak symmetry group $\mathbf{SU}(2) \times \mathbf{U}(1)$) is broken at a much lower energy scale. The energy level at which the electroweak interaction is broken by spontaneous symmetry breaking is approximately equal to the mass of the gauge bosons for the broken part of the gauge group, i.e. the $W^{\pm}$ and Z bosons. This mass is roughly 100GeV. On the other hand, the energy level at which the strength of the electroweak, strong and gravitational interactions are believed to be the same, is about 10^{18}GeV, so the full symmetry group of a GUT (Grand Unified Theory) is broken at that scale. This difference in energy scales is the root of the so called hierarchy problem. Another way of looking at it is in terms of renormalization of particle masses. The renormalization of the mass of the Higgs boson is proportional to the square of the cutoff, the bare mass of the Higgs boson needs to be adjusted with a fine-tuning precision of one part in 10^{14} to obtain a mass of the order of 100GeV. Now we have formulated the hierarchy problem as a problem of fine-tuning the value of a constant. The hierarchy problem is an aesthetic problem with the standard model, since no formal mathematical complaint can be made against this fine-tuning. Nevertheless, this problem is among many particle physicists regarded as one of the most important problems of the standard model. In a super-symmetric theory, additional contributions to the Higgs boson mass renormalization, by super-partner particles, would cancel this quadratic dependence on the cutoff.

Extra dimensions can solve the hierarchy problem because they can lower the value of the Planck mass. The mathematics of extra dimensions shows that the fundamental higher dimensional Planck mass is connected with the effective four dimensional Planck mass through the length of the extra dimension. In the case of a brane universe, the connection is given by the brane tension together with the geometry of the extra dimension. With a small extra dimension, the fundamental Planck mass differs from the observed one by many orders of magnitude, thereby lowering the gap between it and the weak interaction energy scale, maybe even making the two equal.

4.2.3 Dark energy

Recent observations show that the size of the universe is accelerating, i.e. the scale factor of the universe has a positive second time derivative. This phenomenon can be accounted for by including types of energy which have an equation of state with $w < -1/3$. This is called dark energy. Examples can be a cosmological constant ($w = -1$), or a dynamic scalar field with an appropriate potential. This is called quintessence. Types of energy for which $w < -1$ is called phantom energy, and will cause the scale factor of the universe to diverge at a finite time in the future. Notice that this singularity is can be avoided in models where the w parameter approaches -1 as the system evolves.

The brane universe theories give an alternative origin of the cosmological constant. In four-dimensional quantum field theory, the quantum contributions to the cosmological constant give an over-estimate of a factor of 10^{120}. Introducing super-symmetry at higher energies, this can be reduced to 10^{60}. Obviously, something is wrong here. The brane universe solution for this is that the tension of the universe brane as well as the cosmological constant in the bulk will give a contribution to the cosmological constant. If the quantum corrections instead don't contribute to the cosmological constant, we can choose the brane tension to correspond with the observed value. In a theory of extra dimensions without branes, we would choose the radius of the compact dimension such that the effective cosmological constant has the appropriate value.

The quintessence field can also be explained as being the result of a higher dimensional degree of freedom which behaves like a scalar field with a potential in the effective four dimensional theory. Its origin can be e.g. the length of the extra dimension, or some other bulk field coming from string theory.

4.3 The induced Einstein equation

We will in this section deduce the Einstein tensor of a time-like four dimensional hyper-surface embedded in a five dimensional bulk space [19, 20]. Since heterotic M-theory is the underlying theoretical motivation for our universe model, we will assume that we have a $\mathbf{Z}_2$ reflection symmetry with our hyper-surface as the fixed point of this symmetry operation. The bulk space is denoted by $\mathcal{M}_5$, and the hyper-surface by $\mathcal{M}_4$. The unit normal vector on the hyper-surface is denoted $\mathbf{n}$, and since the hyper-surface is time-like, this vector is space-like: $\mathbf{n}^2 = 1$. The basis vectors in the bulk space are $\mathbf{e}_\beta$, and ∇_α is the corresponding covariant derivative in the direction of $\mathbf{e}_\alpha$. The projection tensor from five to four dimensional space-time is $h_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta$, and this is also the induced metric on the surface, if we restrict the indices to directions tangent to the surface. We denote the four-dimensional tensors with a tilde.

The extrinsic curvature tensor measures the curvature of the hyper-surface as seen from the bulk space, and is defined as

$$K_{\alpha\beta} := -\mathbf{e}_\beta \cdot \nabla_\alpha \mathbf{n}, \tag{4.1}$$

evaluated at the positions on the hyper-surface. The Gauss Theorema Egregium [21] relates the Riemann tensor to the induced Riemann tensor on the hyper-surface and its extrinsic curvature, and is

$$\tilde{R}^\alpha_{\beta\gamma\delta} = R^\mu_{\nu\rho\sigma} h^\alpha_{\ \mu} h^\nu_{\ \beta} h^\rho_{\ \gamma} h^\sigma_{\ \delta} + K^\alpha_{\ \gamma} K_{\beta\delta} - K^\alpha_{\ \delta} K_{\beta\gamma}. \tag{4.2}$$

Contracting (4.2) on α and γ gives the Ricci tensor on the hyper-surface,

$$\begin{aligned}
\tilde{R}_{\beta\delta} &= R^\mu_{\nu\rho\sigma} h^\alpha_{\ \mu} h^\nu_{\ \beta} h^\rho_{\ \alpha} h^\sigma_{\ \delta} + K^\alpha_{\ \alpha} K_{\beta\delta} - K^\alpha_{\ \delta} K_{\beta\alpha} \\
&= R^\mu_{\nu\rho\sigma} h^\rho_{\ \mu} h^\nu_{\ \beta} h^\sigma_{\ \delta} + K K_{\beta\delta} - K^\alpha_{\ \delta} K_{\beta\alpha} \\
&= R^\mu_{\nu\rho\sigma} (g^\rho_{\ \mu} - n^\rho n_\mu) h^\nu_{\ \beta} h^\sigma_{\ \delta} + K K_{\beta\delta} - K^\alpha_{\ \delta} K_{\beta\alpha}.
\end{aligned} \tag{4.3}$$

Thus we get

$$\tilde{R}_{\beta\delta} = R_{\nu\sigma} h^\nu_{\ \beta} h^\sigma_{\ \delta} - R^\mu_{\nu\rho\sigma} n^\rho n_\mu h^\nu_{\ \beta} h^\sigma_{\ \delta} + K K_{\beta\delta} - K^\alpha_{\ \delta} K_{\beta\alpha}, \tag{4.4}$$

where $K := K^\alpha_{\ \alpha}$ is the trace of the extrinsic curvature. We see that the intrinsic Ricci tensor of the hyper-surface depends quadratically on the extrinsic curvature of the surface. We can now calculate the intrinsic Ricci scalar of the surface by contracting again,

$$\begin{aligned}
\tilde{R} &= R_{\nu\sigma} h^{\nu\delta} h^\sigma_{\ \delta} - R^\mu_{\nu\rho\sigma} n^\rho n_\mu h^{\nu\delta} h^\sigma_{\ \delta} + K K^\delta_{\ \delta} - K^\alpha_{\ \delta} K^\delta_{\ \alpha} \\
&= R_{\nu\sigma} h^{\nu\sigma} - R^\mu_{\nu\rho\sigma} n^\rho n_\mu h^{\nu\sigma} + K^2 - K^\alpha_{\ \delta} K^\delta_{\ \alpha} \\
&= R_{\nu\sigma} (g^{\nu\sigma} - n^\nu n^\sigma) - R^\mu_{\nu\rho\sigma} n^\rho n_\mu (g^{\nu\sigma} - n^\nu n^\sigma) + K^2 - K^\alpha_{\ \delta} K^\delta_{\ \alpha} \\
&= R - 2R_{\nu\sigma} n^\nu n^\sigma + R^\mu_{\nu\rho\sigma} n^\rho n_\mu n^\nu n^\sigma + K^2 - K^\alpha_{\ \delta} K^\delta_{\ \alpha}.
\end{aligned} \tag{4.5}$$

Using $R^\mu_{\nu\rho\sigma} n^\rho n_\mu n^\nu n^\sigma = 0$ because of the antisymmetry of the Riemann tensor, we get

$$\tilde{R} = R - 2R_{\nu\sigma} n^\nu n^\sigma + K^2 - K^\alpha_{\ \delta} K^\delta_{\ \alpha}. \tag{4.6}$$

Now we have the ingredients we need to find the intrinsic Einstein tensor on the hyper-surface. It is defined in the standard way in terms of the intrinsic Ricci tensor and Ricci scalar,

$$\begin{aligned}
\tilde{E}_{\beta\delta} &:= \tilde{R}_{\beta\delta} - \frac{1}{2} h_{\beta\delta} \tilde{R} \\
&= R_{\nu\sigma} h^\nu_{\ \beta} h^\sigma_{\ \delta} - R^\mu_{\nu\rho\sigma} n^\rho n_\mu h^\nu_{\ \beta} h^\sigma_{\ \delta} + K K_{\beta\delta} - K^\alpha_{\ \delta} K_{\beta\alpha} \\
&\quad - \frac{1}{2} h_{\beta\delta} \left(R - 2R_{\nu\sigma} n^\nu n^\sigma + K^2 - K^\alpha_{\ \delta} K^\delta_{\ \alpha} \right) \\
&= R_{\nu\sigma} h^\nu_{\ \beta} h^\sigma_{\ \delta} - \frac{1}{2} h_{\beta\delta} R + h_{\beta\delta} R_{\nu\sigma} n^\nu n^\sigma + K K_{\beta\delta} \\
&\quad - K^\alpha_{\ \delta} K_{\beta\alpha} - \frac{1}{2} h_{\beta\delta} \left(K^2 - K^\alpha_{\ \delta} K^\delta_{\ \alpha} \right) - R^\mu_{\nu\rho\sigma} n^\rho n_\mu h^\nu_{\ \beta} h^\sigma_{\ \delta}.
\end{aligned} \tag{4.7}$$

Using the definition of the five dimensional Einstein tensor, we then get

$$\begin{aligned}\tilde{E}_{\beta\delta} &= E_{\nu\sigma}h^{\nu}{}_{\beta}h^{\sigma}{}_{\delta} + h_{\beta\delta}R_{\nu\sigma}n^{\nu}n^{\sigma} + KK_{\beta\delta} - K^{\alpha}{}_{\delta}K_{\beta\alpha} \\ &\quad -\frac{1}{2}h_{\beta\delta}\left(K^2 - K^{\alpha}{}_{\delta}K^{\delta}{}_{\alpha}\right) - R_{\mu\nu\rho\sigma}n^{\mu}h^{\nu}{}_{\beta}n^{\rho}h^{\sigma}{}_{\delta}.\end{aligned} \tag{4.8}$$

The Riemann tensor can always be decomposed into the Ricci tensor $R_{\beta\alpha}$, the Ricci scalar R, and the Weyl tensor $C_{\mu\alpha\nu\beta}$ [21]. As an aside, we note that in empty space, the Weyl term is the only non-zero contribution in this decomposition, since it is easy to see from the Einstein equation that $R_{\alpha\beta}$ and R are zero in empty space.

The Riemann tensor decomposition is

$$R_{\mu\nu\rho\sigma} = \frac{2}{3}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) - \frac{1}{6}g_{\mu[\rho}g_{\sigma]\nu}R + C_{\mu\nu\rho\sigma}. \tag{4.9}$$

We use equation (4.9) to rewrite the last term in equation (4.8), which gives (setting terms to zero using $h^{\alpha}_{\beta}n^{\beta} = 0$)

$$\begin{aligned}-R_{\mu\nu\rho\sigma}n^{\mu}h^{\nu}{}_{\beta}n^{\rho}h^{\sigma}{}_{\delta} &= -\left(\frac{2}{3}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) - \frac{1}{6}g_{\mu[\rho}g_{\sigma]\nu}R + C_{\mu\nu\rho\sigma}\right)n^{\mu}h^{\nu}{}_{\beta}n^{\rho}h^{\sigma}{}_{\delta} \\ &= -\frac{1}{3}g_{\mu\rho}R_{\sigma\nu}n^{\mu}h^{\nu}{}_{\beta}n^{\rho}h^{\sigma}{}_{\delta} + \overbrace{\frac{1}{3}g_{\mu\sigma}R_{\rho\nu}n^{\mu}h^{\nu}{}_{\beta}n^{\rho}h^{\sigma}{}_{\delta}}^{=0} \\ &\quad + \overbrace{\frac{1}{3}g_{\nu\rho}R_{\sigma\mu}n^{\mu}h^{\nu}{}_{\beta}n^{\rho}h^{\sigma}{}_{\delta}}^{=0} - \frac{1}{3}g_{\nu\sigma}R_{\rho\mu}n^{\mu}h^{\nu}{}_{\beta}n^{\rho}h^{\sigma}{}_{\delta} \\ &\quad + \frac{1}{12}g_{\mu\rho}g_{\sigma\nu}Rn^{\mu}h^{\nu}{}_{\beta}n^{\rho}h^{\sigma}{}_{\delta} - \overbrace{\frac{1}{12}g_{\mu\sigma}g_{\rho\nu}Rn^{\mu}h^{\nu}{}_{\beta}n^{\rho}h^{\sigma}{}_{\delta}}^{=0} - \mathcal{E}_{\beta\delta},\end{aligned}$$

where we have defined

$$\mathcal{E}_{\beta\delta} \equiv C_{\mu\nu\rho\sigma}n^{\mu}h^{\nu}{}_{\beta}n^{\rho}h^{\sigma}{}_{\delta}, \tag{4.10}$$

which is often called the electric part of the Weyl tensor. This gives us

$$-R_{\mu\nu\rho\sigma}n^{\mu}h^{\nu}{}_{\beta}n^{\rho}h^{\sigma}{}_{\delta} = -\frac{1}{3}R_{\sigma\nu}h^{\nu}{}_{\beta}h^{\sigma}{}_{\delta} - \frac{1}{3}R_{\rho\mu}n^{\mu}n^{\rho}h_{\beta\delta} + \frac{1}{12}Rh_{\beta\delta} - \mathcal{E}_{\beta\delta}. \tag{4.11}$$

Inserting (4.11) into (4.8) gives

$$\begin{aligned}\tilde{E}_{\beta\delta} &= E_{\nu\sigma}h^{\nu}{}_{\beta}h^{\sigma}{}_{\delta} + h_{\beta\delta}R_{\nu\sigma}n^{\nu}n^{\sigma} + KK_{\beta\delta} - K^{\alpha}{}_{\delta}K_{\beta\alpha} \\ &\quad -\frac{1}{2}h_{\beta\delta}\left(K^2 - K^{\alpha}{}_{\delta}K^{\delta}{}_{\alpha}\right) - \frac{1}{3}R_{\sigma\nu}h^{\nu}{}_{\beta}h^{\sigma}{}_{\delta} \\ &\quad +0+0-\frac{1}{3}R_{\rho\mu}n^{\mu}n^{\rho}h_{\beta\delta} + \frac{1}{12}Rh_{\beta\delta} - \mathcal{E}_{\beta\delta} \\ &= \left(E_{\nu\sigma} - \frac{1}{3}R_{\sigma\nu}\right)h^{\nu}{}_{\beta}h^{\sigma}{}_{\delta} + \frac{2}{3}R_{\nu\sigma}n^{\nu}n^{\sigma}h_{\beta\delta} + \frac{1}{12}Rh_{\beta\delta} \\ &\quad +KK_{\beta\delta} - K^{\alpha}{}_{\delta}K_{\beta\alpha} - \frac{1}{2}h_{\beta\delta}\left(K^2 - K^{\alpha}{}_{\delta}K^{\delta}{}_{\alpha}\right) - \mathcal{E}_{\beta\delta}.\end{aligned} \tag{4.12}$$

We can replace the five dimensional Ricci scalar with the trace of the five dimensional Einstein tensor

$$E := E^{\alpha}_{\ \alpha} = R(1 - D/2). \tag{4.13}$$

Using this for $D = 5$, which is appropriate for our cosmological model, we get

$$\begin{aligned} \tilde{E}_{\beta\delta} &= \frac{2}{3}E_{\nu\sigma}h^{\nu}_{\ \beta}h^{\sigma}_{\ \delta} + \frac{2}{3}E_{\nu\sigma}n^{\nu}n^{\sigma}h_{\beta\delta} - \frac{1}{6}Eh_{\beta\delta} \\ &\quad + KK_{\beta\delta} - K^{\alpha}_{\ \delta}K_{\beta\alpha} - \frac{1}{2}h_{\beta\delta}\left(K^2 - K^{\alpha}_{\ \delta}K^{\delta}_{\ \alpha}\right) - \mathcal{E}_{\beta\delta}. \end{aligned} \tag{4.14}$$

Now that we know the Einstein tensor on the hyper-surface (or brane), we need to know which field equation it satisfies. To accomplish this, we only need to apply the five dimensional field equations in the expression for the four dimensional Einstein tensor, to remove the references to the five dimensional Einstein tensor by replacing it with the energy-momentum tensor. The five dimensional Einstein equation is $E_{\rho\sigma} = M_5^{-3}T_{\rho\sigma}$, where M_5 is the five dimensional Planck mass. Substituting it into equation (4.14), we get the following field equation on the brane,

$$\begin{aligned} \tilde{E}_{\beta\delta} &= \frac{2}{3}M_5^{-3}\left(T_{\nu\sigma}h^{\nu}_{\ \beta}h^{\sigma}_{\ \delta} + \left(T_{\nu\sigma}n^{\nu}n^{\sigma} - \frac{1}{4}T\ \right)h_{\beta\delta}\right) \\ &\quad + KK_{\beta\delta} - K^{\alpha}_{\ \delta}K_{\beta\alpha} - \frac{1}{2}h_{\beta\delta}\left(K^2 - K^{\alpha}_{\ \delta}K^{\delta}_{\ \alpha}\right) - \mathcal{E}_{\mu\nu}, \end{aligned} \tag{4.15}$$

where we have defined $T := T^{\alpha}_{\ \alpha}$. We see that the equation contains terms quadratic in the extrinsic curvature, which leads to an important modification of the Friedmann equations on the brane. In general, it is not possible to solve for all the unknown functions using this equation alone. The bulk energy-momentum tensor and the electric Weyl tensor depends on the details of the bulk dynamics, described by the five dimensional Einstein equation. We describe this situation by saying that the system of equations is not closed.

In the case of a flat brane at $y = 0$, with $\mathbf{Z}_2$ reflection symmetry across the brane, and with only a cosmological constant Λ in the bulk, we can simplify this equation somewhat. We assume that the total energy momentum tensor can be decomposed as follows,

$$T_{\alpha\beta} = -\Lambda g_{\alpha\beta} + S_{\alpha\beta}\delta(y). \tag{4.16}$$

We have assumed that the brane is infinitely thin, hence the delta function. Furthermore, we assume that we can divide the brane tensor into a brane cosmological constant λ plus something else denoted by $\tau_{\alpha\beta}$,

$$S_{\alpha\beta} = -\lambda h_{\alpha\beta} + \tau_{\alpha\beta}. \tag{4.17}$$

The Israel junction conditions [22] then give us the jump discontinuities in the metric and the extrinsic curvature of the brane,

$$[h_{\alpha\beta}] = 0 \quad , \quad [K_{\alpha\beta}] = -M_5^{-3}\left(S_{\alpha\beta} - \frac{1}{3}h_{\alpha\beta}S\right), \tag{4.18}$$

where $[A] := \lim_{y\to o^+} A - \lim_{y\to 0^-} A$ and $S := S^\alpha_{\ \alpha}$. The metric discontinuity is of course always zero, but the singular contribution of the brane energy momentum gives a non-zero discontinuity in the extrinsic curvature, since the bulk curvature has a jump discontinuity at the position of the brane. In the case of reflection symmetry $y \mapsto -y$, we can simplify this by writing it in terms of limits on the positive side of the brane only,

$$K^+_{\alpha\beta} = -\frac{1}{2}M_5^{-3}\left(S_{\alpha\beta} - \frac{1}{3}h_{\alpha\beta}S\right), \tag{4.19}$$

and we will drop the + superscript in the sequel. By substituting this value for the extrinsic curvature together with the expressions for the energy momentum tensors above into the Einstein equation (4.15), and doing quite a bit of algebra, we obtain

$$\tilde{E}_{\alpha\beta} = -\Lambda_4 h_{\alpha\beta} + 8\pi G_N \tau_{\alpha\beta} + M_5^{-6}\pi_{\alpha\beta} - \mathcal{E}_{\alpha\beta}, \tag{4.20}$$

where we have defined the quantities

$$\Lambda_4 = \frac{1}{2}M_5^{-3}\left(\Lambda + \frac{1}{6}M_5^{-3}\lambda^2\right) \tag{4.21}$$

$$G_N = \frac{\lambda}{48\pi M^6} \tag{4.22}$$

$$\pi_{\alpha\beta} = -\frac{1}{4}\tau_{\alpha\gamma}\tau_\beta^{\ \gamma} + \frac{1}{12}\tau\tau_{\alpha\beta} + \frac{1}{8}h_{\alpha\beta}\tau_{\gamma\delta}\tau^{\gamma\delta} - \frac{1}{24}h_{\alpha\beta}\tau^2. \tag{4.23}$$

It is important to notice here some general features of brane-world cosmological theories. First, both the bulk and the brane cosmological constants contributes to the cosmological constant in the effective four-dimensional theory on the brane. The brane cosmological constant can originate from the brane tension as well as additional contributions from brane matter. Second, the Newtonian gravitational constant in the effective theory depends on the brane tension (or cosmological constant). If the brane tension is negative, we get the wrong sign on our gravitational constant, and gravitational forces are reversed for ordinary matter on the brane.

The $\pi_{\alpha\beta}$ tensor contains the quadratic contributions from the brane matter, which will strongly modify cosmology on the brane when the energy density is large. The electric Weyl tensor corresponds to bulk effects influencing the gravitational equations on the brane, and is often called dark radiation because in the Friedmann equations, it will scale in the same way as radiation with respect to changes in the scale factor. Its influence depends on the bulk geometry, and can in some cases vanish, and in other cases contribute non-local effects on the brane.

4.4 Brane coupling to bulk gauge field

The world sheet of a point-particle is one-dimensional, as can be seen from equation (2.1). If the particle has electric charge, it interacts with the electromagnetic field potential, which is a one-form field. This interaction terms in the action is given in a Lorentz-invariant way as the one-dimensional integral of this one-form along the world-sheet,

$$S_{int} = q \int A_\alpha dx^\alpha \tag{4.24}$$

From differential geometry, we know that a p-form can be integrated over a p-dimensional manifold. This enables the possibility for (p-1)-dimensional objects (with p-dimensional world-volumes) to couple to p-form fields, which is just what happens to Dp-branes in string theory. In our situation, we have a 3-brane, which can couple to a four-form bulk field. The interaction term in the action is simply of the form

$$S_{int} \propto \int_{\mathcal{M}_4} \mathcal{A}, \tag{4.25}$$

where $\mathcal{M}_4$ is the world-volume of the brane-universe, and $\mathcal{A}$ is the four-form bulk field. This term is gauge invariant, because under a gauge transformation, the gauge field transforms as $\mathcal{A} \mapsto \mathcal{A} + d\lambda$ for an arbitrary three-form λ. The variation of the integral term under this transformation is proportional to

$$\int_{\mathcal{M}_4} d\lambda, \tag{4.26}$$

which is zero since we can use Gauss' theorem to evaluate this at infinity, where we can take λ in the gauge transformation to be zero.

Chapter 5

Ekpyrotic Universe

In the ekpyrotic universe [8, 23] we have a five dimensional space-time, the fifth dimension being a finite orbifold dimension, and the big bang is realized as a collision between a visible brane on which we live now, and another brane, a bulk brane, which interacts with the visible brane through stringy effects, including gravity. There are different versions of this scenario, with different brane configurations.

In the first version, [8], there are three branes: the visible brane, the hidden brane, and the bulk brane, and they are located at each end of the orbifold dimension, and in the bulk, respectively. The bulk brane is spontaneously produced close to the hidden brane by a mechanism similar to bubble nucleation and is thereafter moving towards the visible brane. The bulk brane is proposed to be light compared to the boundary branes. During the bulk brane movement it is under the influence of a potential created by the exchange of appropriate M-theory fields between the three branes. The subsequent collision of the bulk brane with the visible brane is responsible for depositing enough energy on the visible brane for it to evolve into what we are living in today. We will call this version the bulk brane scenario.

The newer ekpyrotic scenario [23] has only two branes, one visible and one hidden. Here, the collision is brought about by a different mechanism: the fifth dimension contracts, vanishes, and grows again. In the new scenario, since the fifth dimension vanishes at the instant of collision, a full description demands a rigorous treatment of this singularity in string theory/M-theory. We will call this version the dimensional bounce scenario. It is immediately obvious that, in this version, we could imagine the possibility of a cyclic universe, in which the fifth dimension undergoes a cycle of contraction and expansion a number of times, or indefinitely. This is called the cyclic model, and was proposed by Steinhardt and Turok in [24]. The ekpyrotic universe is fundamentally different from standard cosmology, and offers radically different explanations for the cosmological problems.

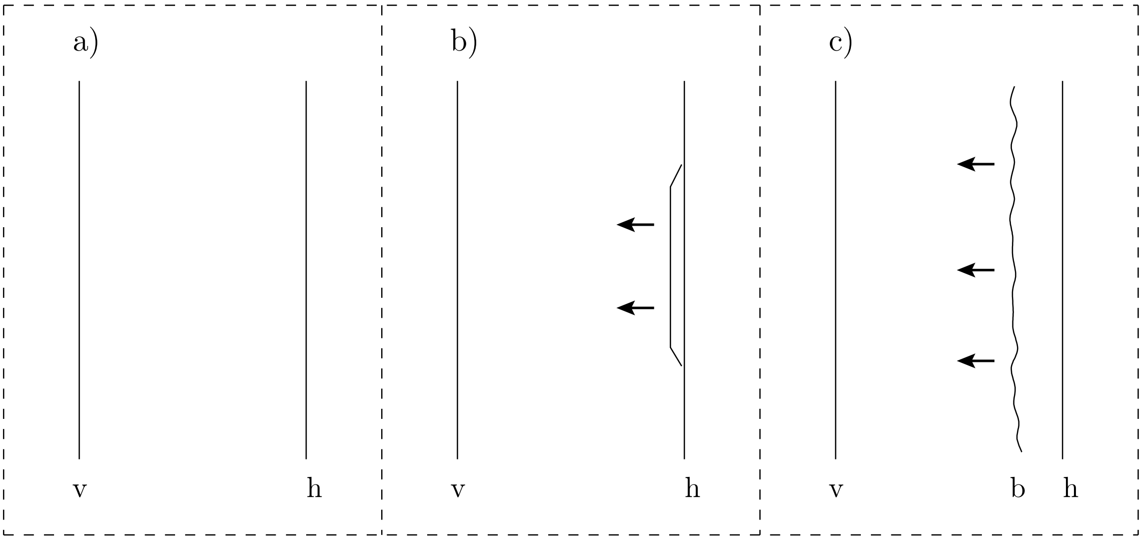


Figure 5.1: The chain of events in the bulk brane universe. The visible, hidden and bulk branes are indicated with letters v, h, b, respectively. a) The initial state of two parallel static branes, b) the hidden brane spontaneously produces a bulk brane, c) the bulk brane travels towards the visible brane, acquiring quantum fluctuations, and eventually impacting inelastically on it and being absorbed.

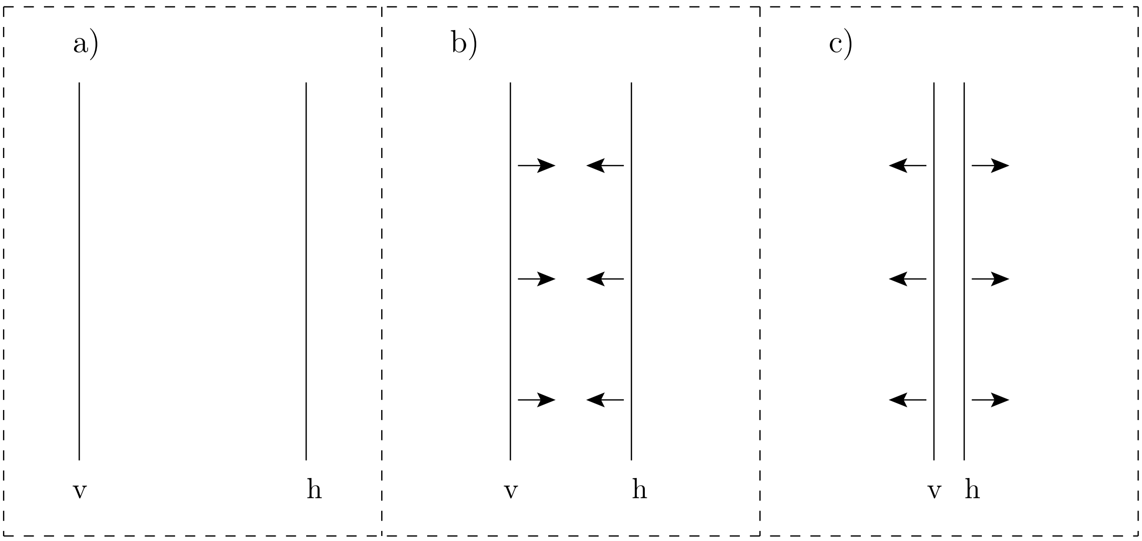


Figure 5.2: Dimensional bounce scenario. a) The initial state, b) the fifth dimension starts to contract, c) the fifth dimension has bounced at $y = 0$, and is expanding again.

5.1 The cosmological problems

In section 3.4, we mentioned some fundamental cosmological problems concerning the whole of the universe. These are explained in the context of the ekpyrotic universe in the next few sections. It seems that the ekpyrotic universe resorts more to the anthropic type of argument (see section 5.1.6) than the inflationary universe, because the flatness and isotropy of the universe are explained by choosing an appropriate initial configuration. This is not so in inflationary theory, where the observed universe can be the outcome even if the initial conditions are very far from flat and isotropic. If try to weigh these theories agaist each other, this fact should count to the advantage of the inflationary theory.

5.1.1 Homogeneity, isotropy and flatness

We have one orbifold plane at each boundary of the fifth dimension. As we have already seen, these orbifold planes are static. On top of each of these we have the visible and hidden branes. Each of these are assumed to be in a nearly BPS state . The defining property of a BPS state is that it does not violate the super-symmetry totally, i.e. there exists some super-symmetry transformation that leaves the state invariant. An exact BPS state would be time translation invariant. This is an assumption about the initial state, and has been criticized by the opponents of these scenarios [25]. One of the virtues of inflationary cosmology is that the homogeneous and isotropic result of inflation is achieved for a lot of different inflation potentials.

This BPS property provides us with a $N = 1$ super-symmetric Yang-Mills field theory with $\mathbf{E}_8$ gauge group at low energy on the visible brane. In a brane collision e.g. between the bulk brane and the visible brane, a so-called small instanton transition can occur, breaking the gauge symmetry down to e.g. $\mathbf{SU}(3) \times \mathbf{SU}(2) \times \mathbf{U}(1)$ [26]. It can also change the number of particle generations on the visible branes, e.g. to three generations. The nature of the small instanton transition depends on the type of the vector bundles on the Calabi-Yau manifold dimensions. Since the initial state of the visible brane is approximately homogeneous and isotropic, and the approaching brane is also homogeneous and isotropic, we have solved these problems by construction of the initial state, provided that the attracting potential between the branes is independent of the brane coordinate directions, flat near the hidden brane, and that the quantum fluctuations arising while the branes move towards each other are not to too big.

In the dimensional bounce version of the scenario, the colliding branes will by the properties of the bulk space-time be approximately parallel, so here there is no problem. This also goes for the cyclic version.

In the bulk brane version, we have to investigate the creation of the bulk brane to see if it will be homogeneous and isotropic. If the bulk brane is spontaneously produced parallel to the hidden brane it is OK. If it is born from the hidden brane, the process will presumably start at a point on the brane, and expand outward at light speed, and for this to make the bulk brane flat enough, the motion in the fifth dimension has to be very slow compared to the motion outwards of the separation point between the bulk and the hidden brane. So the potential slope has to be small in the vicinity of the hidden brane. This is an assumption about the potential again, and can presumably be checked with M-theory calculations of the non-perturbative potential between two boundary branes.

Flatness of the resulting visible brane also follows from the same considerations, since the BPS state is flat.

5.1.2 Horizon

In these scenarios, widely separated regions look alike because their properties arise from a common cause, the colliding brane. Put in another way, if A and B are to widely separated points on the visible brane, they will look alike even though there has been no propagating effect from A to B that would require super-luminal speed because their properties are both the result of an effect propagating from a common third place, namely some place in the bulk at an earlier time.

5.1.3 Monopoles

By the choice of the potential between the branes, which will be motivated from M-theory but not calculated exactly, one should be able to calculate the energy released in the collision, and find that this energy is below the threshold of monopole production, according to current opinions about GUT theories. In reality, since we have to guess the detailed form of the potential, this problem is solved if we can adjust the potential so that the energy deposit in the collision is not too large.

5.1.4 Structure

The quantum fluctuations in the bulk brane geometry will make the collision happen at slightly different times at different positions on the visible brane, and thus we obtain an inhomogeneity in the temperature and energy density levels on the brane. To conform to current experimental observations, this spectrum has to be quite close to scale invariant. Since the fluctuations originate from quantum fluctuations of the traveling brane, this spectrum will contain fluctuations with wavelengths larger than the causal horizon on the visible brane. The spectrum of these fluctuations will be dependent on the form of the brane-brane potential.

The spectrum of gravitational radiation, of which we currently have no experimental observations, will be strongly blue (higher amplitude at lower wavelength) in brane-world models. The spectrum of gravitational radiation in the universe will measured by the future probe LIGO, and these measurements could be decisive in the question of standard cosmology versus brane cosmology, since the two theories predict very different spectra of gravitational waves.

5.1.5 Mass problem

All energy in the visible universe is given as either brane tension or radiation/matter deposited in the bulk brane collision. This energy depends on the brane tensions and the kinetic energy of the bulk brane collision, thus also on the form of the inter-brane potential which governs the bulk brane movement. Since the bulk brane originates from the hidden brane, we can say that, in the ekpyrotic universe, the mass of the visible universe has existed as brane tension since the infinite past.

5.1.6 Anthropic Principle and String Landscape

The anthropic principle says roughly that the universe looks the way it looks because otherwise we would not be here to observe it. This argument is sometimes used when one is unable to come up with a scientific explanation for some of the fundamental cosmological properties, e.g. isotropy of the universe. A scientific statement is one that can be tested with experiments, so invoking the anthropic principle is not a scientific explanation. It is sometimes said that the universe is e.g. isotropic because if it was not, we could not exist to observe it. But it is probably not possible to show the improbability or impossibility of our existence in an anisotropic universe. The anthropic principle is not a scientific argument, except maybe in a few cases involving quantum mechanics, where nature is fundamentally indeterministic. It can be applied to many questions, but one does not gain any knowledge by using it. We can e.g. ask: Why can the fundamental particles form bound states? If we answer by saying that if they did not, we could not exist, we would not have gained any knowledge, so we have not really explained anything. The real answer is that they can form bound states because of the detailed properties of their interactions. They would form bound states independently of intelligent lifeforms observing them.

Quantum mechanics contains randomness. It is impossible to predict in which direction an electron emitted during the decay of a neutron will go. Thus, if an observed property depends manifestly on such a random event, then we can't give a reason for why it is the way it is, since it is based on a random quantum event. If, in addition to this, our existence depended on this specific outcome, the anthropic principle is the best we can do as regards an explanation. So, before using the anthropic principle as an explanation to why the universe is e.g. flat, we must show that the flatness is the outcome of a truly random quantum event. This could be the case in a many-world universe interpretation of quantum mechanics.

The ekpyrotic universe model does not explain why the universe is flat and isotropic, it merely encodes these properties into the initial conditions and the inter-brane potential. It uses a specific choice of string theory vacuum solution as a basis for a cosmological model. Lately, a new concept in string theory has emerged called "string landscape", which is a study of all the different vacuum solutions to string theory, and why our particular vacuum is preferred. The space of such vacua is estimated to be finite and countable, and is therefore called a "discretuum", as opposed to a "continuum", with something of the order of 10^{100} different points. So it is indeed a monumental (and maybe impossible) task to try fully explore this space. The problem of finding the mechanism in string theory that selects a particular vacuum solution amount to finding an effective potential with a unique minimum on this "discretuum". Failure to find this potential will mean that string theory has lost much predictive power, since one can then just pick and choose the vacuum solution that best describes the world we observe.

5.2 The ekpyrotic universe action

5.2.1 Simplified Heterotic M-theory

The ekpyrotic scenario is mathematically based on simplified heterotic M-theory, which means heterotic M-theory with many of the fields set equal to zero [27, 28] to simplify the problem. Only fields whose equation of motion allow it, are set to zero, of course. The action of simplified heterotic M-theory is [8, 29, 28, 27]

$$\begin{aligned} S &= \frac{M_5^3}{2}\int_{\mathcal{M}_5} d^5x\sqrt{-g}\left(\mathcal{R} - \frac{1}{2}\partial_A\phi\partial^A\phi - \frac{3}{2}\frac{1}{5!}e^{2\phi}\mathcal{F}^2\right) \\ &\quad - \sum_{i=1}^{N}\frac{3}{2}\alpha_i M_5^3\int_{\mathcal{M}_4^{(i)}} d^4\xi_{(i)}\left(\sqrt{-h_{(i)}}e^{-\phi}\right. \\ &\quad \left. - \frac{1}{4!}\epsilon^{\alpha\beta\gamma\delta}\mathcal{A}_{ABCD}\partial_\alpha X^A_{(i)}\partial_\beta X^B_{(i)}\partial_\gamma X^C_{(i)}\partial_\delta X^D_{(i)}\right). \end{aligned} \tag{5.1}$$

The theory is defined on the orbifold $\mathbf{R}^4 \times \mathbf{S}_1/\mathbf{Z}_2$, a manifold with a boundary which we call the "bulk space". Since differentiation is not well-defined in the direction normal to a boundary of a manifold, we define the action of the theory on the manifold $\mathbf{R}^4 \times \mathbf{S}_1$, which has no boundary, and differentiation is well-defined everywhere. We then have to demand $\mathbf{Z}_2$ reflection symmetry $x^4 \sim -x^4$ from all the fields and sources in the Lagrangian. The fifth dimension coordinate is $x^4 \equiv y \in [-R, R]$ with identification $R \sim -R$ from the definition of $\mathbf{S}_1$.

Uppercase Latin indices refer to a coordinate system in the bulk space, and small Greek indices refer to a coordinate system on a brane. M_5 is the five dimensional Planck mass. g_{AB} is the metric on the bulk space, $\mathcal{R}$ is the five dimensional scalar curvature, e^ϕ is the size of the compactified Calabi-Yau six dimensional manifold,$\mathcal{A}_{ABCD}$ is a 4-form gauge field, and $\mathcal{F} = d\mathcal{A}$ is its field strength ($\mathcal{F}^2$ is short for $\mathcal{F}_{ABCDE}\mathcal{F}^{ABCDE}$). Furthermore, we have the contribution from an unspecified number of branes, which couple to the bulk fields ϕ and $\mathcal{A}$. These branes have tensions $\frac{3}{2}\alpha_i M_5^3$, and their action is given by the four dimensional integrals over their world-volumes $\mathcal{M}_4^{(i)}$, and the $\xi_{(i)}$ are coordinates on the branes. The corresponding integrands also contain the induced metric on their world-volume $h^{(i)}_{\alpha\beta}$, which is the pull-back of the bulk metric onto $\mathcal{M}_4^{(i)}$. $X^A_{(i)}(\xi^\alpha_{(i)})$ are the coordinates in $\mathcal{M}_5$ of a point with coordinates $\xi^\alpha_{(i)}$ in $\mathcal{M}_4^{(i)}$. It looks complicated, but $\mathcal{A}_{ABCD}\partial_\alpha X^A_{(i)}\partial_\beta X^B_{(i)}\partial_\gamma X^C_{(i)}\partial_\delta X^D_{(i)}$ is simply the pull-back to the brane (i) world-volume of the bulk field $\mathcal{A}$. Anomaly cancellation, which must be satisfied to enable a consistent quantization of the theory, demands that the brane tensions α_i sum up to zero [30], and therefore we parametrize them as $\alpha_1 = -\alpha$, $\alpha_2 = \alpha - \beta$ and $\alpha_3 = \beta$.

Normally, the ekpyrotic scenario is presented with two or three branes living in $\mathbf{R}^4 \times \mathbf{S}_1/\mathbf{Z}_2$, but to make sense of differentiation on the boundary, and avoid problems due to using the variational principle on a manifold with a boundary, we will treat the model as containing four of six branes living in $\mathbf{R}^4 \times \mathbf{S}_1$, and impose reflection symmetry about $y = 0$. We restrict the model to contain only branes parallel to the $\mathbf{R}^4 \times \mathbf{S}_1/\mathbf{Z}_2$ boundaries, thus normal to the y-direction. There is one brane on each boundary, at $y = 0$ and R, not moving. There is also a bulk brane at $y = Y(t)$, which is allowed to move in the y-direction. Since we define the model on $\mathbf{R}^4 \times \mathbf{S}_1$, we put in six branes located at $y_i \in \{0, R, Y(t), 0, -R, -Y(t)\}$, but since $-R$ and R is the same point on the space we integrate over the extra dimension in the action, this is the same as $y_i \in \{0, R, Y(t), 0, R, -Y(t)\}$. The branes at $y = 0$, R, and $Y(t)$ are called the visible, hidden, and bulk branes, respectively. We call the other branes "mirror branes".

Since the action is measured in quantity of $\hbar$, and we have $\hbar = 1$ in our choice of units, the action must be dimensionless. In particular, it must have mass dimension zero. This implies that the quantities in the action have the following mass-dimensions. $\mathcal{R}$ has mass-dimension 2, which is correct. ϕ has dimension 0, $\mathcal{F}$ has 1, $\mathcal{A}$ has 0, and α_i has 1.

By introducing delta functions in the y-direction, we can rewrite every part the action as an integral over the bulk space. We must first choose the coordinates on the branes to be the same as the bulk coordinates t, x^1, x^2 and x^3.

$$S = \int_{\mathcal{M}_5} d^5x\sqrt{-g}\left(\mathcal{L}_{\mathcal{R}} + \mathcal{L}_\phi + \mathcal{L}_{\mathcal{A}} + \sum_{\text{branes}} \mathcal{L}_{(i)}\right), \tag{5.2}$$

where

$$\mathcal{L}_{\mathcal{R}} = \frac{M_5^3}{2}\mathcal{R} \tag{5.3}$$

$$\mathcal{L}_\phi = -\frac{M_5^3}{4}\partial_A\phi\partial^A\phi \tag{5.4}$$

$$\mathcal{L}_{\mathcal{A}} = -\frac{3M_5^3}{4}\frac{1}{5!}e^{2\phi}\mathcal{F}^2 \tag{5.5}$$

$$\mathcal{L}_{(i)} = -\frac{3M_5^3}{2\sqrt{-g}}\alpha_i\delta(y-y_i)\left(\sqrt{-h_{(i)}}e^{-\phi} - \frac{1}{4!}\epsilon^{\alpha\beta\gamma\delta}\mathcal{A}_{ABCD}\partial_\alpha X^A_{(i)}\partial_\beta X^B_{(i)}\partial_\gamma X^C_{(i)}\partial_\delta X^D_{(i)}\right). \tag{5.6}$$

The individual parts of the action can be described as a kinetic term for the gravitational field 5.3 of Einstein-Hilbert form leading the standard gravitational equations, kinetic term for the bulk fields ϕ and $\mathcal{A}$, and the Lagrangian $\mathcal{L}_{(i)}$ for the each brane.

The brane Lagrangian is of Dirac-Born-Infeld type (2.106) with the electromagnetic field $\mathbf{F}$ and Kalb-Ramond field $\mathbf{B}$ set to zero. The bulk scalar field interacts in the same way as the dilaton field in the Dirac-Born-Infeld action. In this action, the branes also couple to the bulk four-form field through the standard mechanism explained in section (4.4), but written explicitly in terms of components.

We observe that the action is written in the Einstein frame. Redefining the metric by means of a Weyl-transformation leads to different choices of frames. In the Einstein frame we obtain standard gravitational equations. Defining a new metric $\tilde{g} = \exp(-2\phi)g$, and formulating the action with a Ricci scalar computed from the new metric, would lead to an action written in the string frame, but then the gravitational equations would be modified since the Einstein-Hilbert term would include an $\exp(-2\phi)$ factor.

5.2.2 A note on the 4-form field

The action contains a 4-form (antisymmetric) field. We will now show that this field represents no dynamical degrees of freedom in the theory. Indeed, the rank of the corresponding field strength tensor is the same as the number of dimension of space-time, namely five. If we write the field equation of a free 4-form field $\mathcal{A}_4$ in five dimensions, we get

$$d \star \mathcal{F}_5 = 0, \tag{5.7}$$

where $\star$ is the Hodge duality operator, called the Hodge star. Since the Hodge star maps p-forms to (D-p)-forms, $\star\mathcal{F}_5$ is a 0-form, a scalar function. Acting on a scalar function, the exterior derivative is simply the gradient. Thus the above equation of motion says that $\star\mathcal{F}_5$ is constant throughout space-time. This means that there is no dynamical (propagating) degrees of freedom represented by the 4-form field $\mathcal{A}$.

5.2.3 Non-covariant brane Lagrangian

The 4-form interaction term in the brane Lagrangian (5.6) contains many contractions, between the Levi-Cevita tensor, the 4-form field components, and derivatives of the brane embedding function X. We can write these contractions out to obtain a non-covariant form of the Lagrangian.

For a flat brane normal to the orbifold dimension, with y-coordinate $Y(t)$, the embedding function is $X = (t, \vec{x}, Y(t))$, so the differentials are

$$\partial_\alpha X^A = \begin{cases} \delta^A_\alpha & , A \in \{0,1,2,3\} \\ 0 & , A = 5, \alpha \in \{1,2,3\} \\ \dot{Y} & , A = 5, \alpha = 0. \end{cases} \tag{5.8}$$

From this we can simplify the four-form field term in the brane actions, as follows. We utilize heavily that $\mathcal{A}_{ABCD}$ is totally antisymmetric. We can split the contractions up into one part where $A, B, C, D \neq 4$, and one part where only one of them are equal to 4. If two were equal to 4, the corresponding two $\alpha, \beta, \gamma, \delta$ would have to be equal to 0, and then the Levi-Cevita tensor vanishes. So we have

$$\frac{1}{4!}\epsilon^{\alpha\beta\gamma\delta}\mathcal{A}_{ABCD}\partial_\alpha X^A \partial_\beta X^B \partial_\gamma X^C \partial_\delta X^D = \mathcal{A}_{0123} + \frac{1}{3!}\epsilon^{\alpha\beta\gamma\delta}\mathcal{A}_{ABC4}\partial_\alpha X^A \partial_\beta X^B \partial_\gamma X^C \partial_\delta X^4$$

Now, we only get contributions when $\delta = 0$, since X. Using (5.8), and the fact that we then get contributions from all combinations of $\alpha, \beta, \gamma \in \{1, 2, 3\}$, non of them equal.

$$\mathcal{A}_{0123} + \frac{1}{3!}\epsilon^{\alpha\beta\gamma 0}\mathcal{A}_{ABC4}\delta^A_\alpha\delta^B_\beta\delta^C_\gamma\partial_0 X^4 = \mathcal{A}_{0123} + \frac{1}{3!}\epsilon^{\alpha\beta\gamma 0}\mathcal{A}_{\alpha\beta\gamma 4}\dot{Y}$$

$$= \mathcal{A}_{0123} + \epsilon^{1230}\mathcal{A}_{1234}\dot{Y} = \left(\mathcal{A}_{0123} + \mathcal{A}_{1234}\dot{Y}\right) \tag{5.9}$$

We then get the following non-covariant expression for the brane Lagrangian,

$$\sqrt{-g}\mathcal{L} = -\frac{3}{2}\alpha_i M_5^3 \delta(y - Y(t))\left(\sqrt{-h}e^{-\phi} - \mathcal{A}_{0123} - \mathcal{A}_{1234}\dot{Y}\right). \tag{5.10}$$

In the case of static boundary branes, we must of course use the proper delta function and use $\dot{Y} = 0$.

5.2.4 Brane matter and brane interaction

To allow for matter on the branes, we could introduce the following extra contribution to the action

$$S_m = \sum_{\text{branes}} \int d^4x \sqrt{-h_{(i)}}\mathcal{L}_{m(i)}. \tag{5.11}$$

These contributions would represent a perfect cosmological fluid on the branes, which is implied by the assumption of homogeneity and isotropy of the branes. In the ekpyrotic universe, it is assumed that the branes are devoid of internal matter before the collision occurs (ekpyrosis). It is not obvious that this is possible, since there is an interaction among the branes as the system evolves. We will at least assume that there is no brane matter in the distant past, thus matter will have to be generated by the brane interactions/collision in the course of the system evolution. To simplify we also assume that the bulk brane has no matter. In this thesis, we analyze numerical solutions of the original proposal, and we will be concerned with what happens before ekpyrosis, so we will not speculate on the existence of extra matter on the branes.

Without an attractive force between the branes that are supposed to collide, there is no ekpyrotic universe, at least in the usual sense. Therefore we will introduce a term in the action that represents the interaction between the branes. Perturbatively, there is no interaction since the branes are BPS states. The interaction is therefore postulated to arise from non-perturbative effects in the heterotic M-theory. Since this is not a well-known subject, the details of this interaction are unknown, but it is argued in [8] that the interaction will generate an effective potential between the branes of exponential character. To represent this interaction, we could include in the full action an unknown term. This potential will be introduced into the action after we have used an approximation scheme called moduli space approximation in the next chapter.

5.3 Einstein equations

We will now construct the Einstein equations from the action (5.1). To so this, we need the Einstein tensor and the energy-momentum tensor.

5.3.1 Einstein tensor

Assuming homogeneity and isotropy in the spatial directions parallel to the visible universe hyper-surface restricts the 5D bulk metric to the form

$$ds^2 = -\tilde{n}(\tilde{t}, y)^2 d\tilde{t}^2 + \tilde{a}(\tilde{t}, y)^2 \sum (dx^i)^2 + \tilde{b}(\tilde{t}, y)^2 dy^2 + 2\tilde{f}(\tilde{t}, y) d\tilde{t} dy. \tag{5.12}$$

By changing the $\tilde{t}$ coordinate and writing it as a function of a new time coordinate t and y, as $\tilde{t}(t, y)$, we can transform the cross-term $\tilde{f}$ out of the metric. We have

$$d\tilde{t} = \frac{\partial \tilde{t}}{\partial t} dt + \frac{\partial \tilde{t}}{\partial y} dy. \tag{5.13}$$

Inserting this into the metric, we get

$$\begin{aligned}
ds^2 &= -\tilde{n}(\tilde{t}(t, y), y)^2 \left(\frac{\partial \tilde{t}}{\partial t} dt + \frac{\partial \tilde{t}}{\partial y} dy \right)^2 + \tilde{a}(\tilde{t}(t, y), y)^2 \sum (dx^i)^2 \\
&\quad + \tilde{b}(\tilde{t}(t, y), y)^2 dy^2 + 2\tilde{f}(\tilde{t}(t, y), y) \left(\frac{\partial \tilde{t}}{\partial t} dt + \frac{\partial \tilde{t}}{\partial y} dy \right) dy \\
&= -\left(\frac{\partial \tilde{t}}{\partial t} \right)^2 \tilde{n}(\tilde{t}(t, y), y)^2 dt^2 + \tilde{a}(\tilde{t}(t, y), y)^2 \sum (dx^i)^2 \\
&\quad + \left\{ \tilde{b}(\tilde{t}(t, y), y)^2 - \left(\frac{\partial \tilde{t}}{\partial y} \right)^2 \tilde{n}(\tilde{t}(t, y), y)^2 + 2 \left(\frac{\partial \tilde{t}}{\partial y} \right) \tilde{f}(\tilde{t}(t, y), y) \right\} dy^2 \\
&\quad + 2 \left(\frac{\partial \tilde{t}}{\partial t} \tilde{f}(\tilde{t}(t, y), y) - \frac{\partial \tilde{t}}{\partial t} \frac{\partial \tilde{t}}{\partial y} \tilde{n}(\tilde{t}(t, y), y)^2 \right) dt dy.
\end{aligned} \tag{5.14}$$

Equating the quantity in front of the $dtdy$ term in this line element to zero gives us a first order partial differential equation in two independent variables with one unknown function $\tilde{t}(t, y)$.

$$\frac{\partial \tilde{t}}{\partial t} \tilde{f}(\tilde{t}(t, y), y) - \frac{\partial \tilde{t}}{\partial t} \frac{\partial \tilde{t}}{\partial y} \tilde{n}(\tilde{t}(t, y), y)^2 = 0 \tag{5.15}$$

Notice that a trivial transformation independent of t will do the trick since then $\tilde{t}$ is independent of t and $\partial \tilde{t}/\partial t = 0$, but this is a singular transformation with zero Jacobian, which means that $\tilde{t}$ and y is not independent, and we only get a coordinate system that is valid on a hyper-surface of the total space we are interested in. Thus $\partial \tilde{t}/\partial t \neq 0$, and we can divide out $\partial\tilde{t}/\partial t$ from the equation. The problem of how to transform the line element to diagonal form is equivalent to solving the following partial differential equation.

$$\tilde{f}(\tilde{t}(t,y),y) - \frac{\partial\tilde{t}}{\partial y}\tilde{n}(\tilde{t}(t,y),y)^2 = 0. \tag{5.16}$$

We assume this to be possible, and give the names $-n(t,y)^2$, $a(t,y)^2$ and $b(t,y)^2$ to the coefficients in front of the expressions dt^2, $\sum(dx^i)^2$ and dy^2 in equation (5.14), respectively. Thus we have the following diagonal line element

$$ds^2 = -n(t,y)^2dt^2 + a(t,y)^2\sum(dx^i)^2 + b(t,y)^2dy^2. \tag{5.17}$$

From equation (5.17), the nonzero components of the Einstein tensor are (see appendix (B))

$$\begin{aligned}
E^t_{\ t} &= \frac{3}{b^2}\left(\frac{a''}{a} + \frac{a'}{a}\left(\frac{a'}{a} - \frac{b'}{b}\right)\right) - \frac{3}{n^2}\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b}\right) && (5.18)\\
E^i_{\ i} &= \frac{1}{b^2}\left(2\frac{a''}{a} + \frac{n''}{n} + \frac{a'}{a}\left(\frac{a'}{a} + 2\frac{n'}{n}\right) - \frac{b'}{b}\left(\frac{n'}{n} + 2\frac{a'}{a}\right)\right) && (5.19)\\
&\quad -\frac{1}{n^2}\left(2\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} - 2\frac{\dot{n}}{n}\right) + \frac{\dot{b}}{b}\left(2\frac{\dot{a}}{a} - \frac{\dot{n}}{n}\right)\right) &&\\
E^y_{\ y} &= \frac{3}{b^2}\frac{a'}{a}\left(\frac{a'}{a} + \frac{n'}{n}\right) - \frac{3}{n^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} - \frac{\dot{n}}{n}\right)\right) && (5.20)\\
E^t_{\ y} &= \frac{3}{n^2}\left(\frac{n'}{n}\frac{\dot{a}}{a} - \frac{\dot{a}'}{a} + \frac{a'}{a}\frac{\dot{b}}{b}\right). && (5.21)
\end{aligned}$$

5.3.2 Energy momentum tensor

To formulate the Einstein equations, we need to calculate the energy momentum tensor from the action (5.2). When the gravitational source terms in the Lagrangian don't contain derivatives of the metric, it is given by the expression

$$T^L_N := \frac{-2g^{LM}}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{MN}} = -2g^{LM}\frac{\partial\mathcal{L}}{\partial g^{MN}} + \mathcal{L}\delta^L_N. \tag{5.22}$$

Contribution from the scalar field

The energy momentum contribution from the bulk scalar field ϕ is obtained by inserting $\mathcal{L}_\phi$, given in equation (5.4), into this expression.

$$T_{\phi\ N}^{\ L} := -2g^{LM}\frac{\partial\mathcal{L}_\phi}{\partial g^{MN}} + \mathcal{L}_\phi\delta^L_N$$

$$
\begin{aligned}
&= -2g^{LM}\left(-\frac{M_5^3}{4}\partial_M\phi\partial_N\phi\right)+\left(-\frac{M_5^3}{4}g^{AB}\partial_A\phi\partial_B\phi\right)\delta^L_{\ N} \\
&= M_5^3\left(\frac{1}{2}g^{LM}\partial_M\phi\partial_N\phi-\frac{1}{4}\delta^L_{\ N}g^{AB}\partial_A\phi\partial_B\phi\right) \qquad (5.23)
\end{aligned}
$$

We have used the relation

$$
\frac{\partial}{\partial g^{MN}}g^{AB}\partial_A\phi\partial_B\phi=\partial_M\phi\partial_N\phi. \qquad (5.24)
$$

Contribution from the four-form field

Looking at (5.5), we need the following relation to calculate the contribution to the energy momentum tensor from the bulk four-form field.

$$
\begin{aligned}
g^{LM}\frac{\partial}{\partial g^{MN}}\mathcal{F}^2 &= g^{LM}\frac{\partial}{\partial g^{MN}}\left(g^{AF}g^{BG}g^{CH}g^{DI}g^{EJ}\right)\mathcal{F}_{ABCDE}\mathcal{F}_{FGHIJ}= \\
&= g^{LM}\left(\delta^A_{\ M}\delta^F_{\ N}\mathcal{F}_{ABCDE}\mathcal{F}_F^{\ BCDE}+\delta^B_{\ M}\delta^G_{\ N}\mathcal{F}_{ABCDE}\mathcal{F}_G^{\ A\ CDE}+\ldots\right) \\
&= g^{LM}\left(\mathcal{F}_{MBCDE}\mathcal{F}_N^{\ BCDE}+\mathcal{F}_{AMCDE}\mathcal{F}_N^{\ A\ CDE}+\ldots\right) \\
&= 5g^{LM}\mathcal{F}_{MBCDE}\mathcal{F}_N^{\ BCDE}, \qquad (5.25)
\end{aligned}
$$

where we have used the total antisymmetry of $\mathcal{F}$ in the last line. Furthermore, we get

$$
\begin{aligned}
g^{LM}\frac{\partial}{\partial g^{MN}}\mathcal{F}^2 &= 5g^{LM}\mathcal{F}_{MBCDE}\mathcal{F}_N^{\ BCDE}=5\mathcal{F}^{LBCDE}\mathcal{F}_{NBCDE} \\
&= 5\delta^L_{\ N}\mathcal{F}^{NBCDE}\mathcal{F}_{NBCDE}=\delta^L_{\ N}\mathcal{F}^2, \qquad (5.26)
\end{aligned}
$$

where there is no sum over N in the next to last expression, and we have used that all the indices $BCDE$ must be unequal to N (since $\mathcal{F}$ is antisymmetric), and therefore only the term with $L=N$ contributes. The contribution from the kinetic term for the 4-form gauge field $\mathcal{A}$ is thus

$$
\begin{aligned}
T^{\ L}_{\mathcal{A}\ N} &:= -2g^{LM}\frac{\partial\mathcal{L}_\mathcal{A}}{\partial g^{MN}}+\mathcal{L}_\mathcal{A}\delta^L_{\ N} \\
&= -2g^{LM}\left(-\frac{3M_5^3}{4}\frac{1}{5!}e^{2\phi}\frac{\partial}{\partial g^{MN}}\mathcal{F}^2\right)+\left(-\frac{3M_5^3}{4}\frac{1}{5!}e^{2\phi}\mathcal{F}^2\right)\delta^L_{\ N} \\
&= \frac{3M_5^3}{2}\frac{1}{5!}e^{2\phi}\delta^L_{\ N}\mathcal{F}^2-\frac{3M_5^3}{4}\frac{1}{5!}e^{2\phi}\mathcal{F}^2=\frac{3M_5^3}{4}\frac{1}{5!}e^{2\phi}\delta^L_{\ N}\mathcal{F}^2. \qquad (5.27)
\end{aligned}
$$

Contribution from the branes

We first calculate the contribution to the energy momentum tensor from a moving brane at $y=Y(t)$, which could be either a bulk or a boundary brane. After that we look at the special case of boundary branes, for which $\dot{Y}=0$.

To do this, we need to find the induced metrics on the brane, thus finding the pullback of the bulk metric tensor to the brane through the brane embedding function X. Choosing $(t, \vec{x})$ as coordinates on the branes, this embedding function is

$$X : (t, \vec{x}) \mapsto (t, \vec{x}, Y(t)). \tag{5.28}$$

The expression for the pullback of the metric onto the brane is, as for any 2-tensor,

$$h_{\alpha\beta} = \frac{\partial X^M}{\partial x^\alpha}\frac{\partial X^N}{\partial x^\beta} g_{MN}, \tag{5.29}$$

where one must also of course evaluate the quantities at a position on the brane. The induced metric on a moving brane at $y = Y(t)$ has an non-trivial contribution to the 00 component, since in this case $X^4 := Y(t)$ depends on time and contributes non-trivially to the above sum:

$$h_{00} = \frac{\partial X^M}{\partial t}\frac{\partial X^N}{\partial t} g_{MN} = \frac{\partial t}{\partial t}\frac{\partial t}{\partial t} g_{00} + \frac{\partial Y}{\partial t}\frac{\partial Y}{\partial t} g_{44} = g_{00} + \dot{Y}^2 g_{44}. \tag{5.30}$$

For the other components, it is simpler:

$$h_{ij} = \frac{\partial X^M}{\partial x^i}\frac{\partial X^N}{\partial x^j} g_{MN} = \frac{\partial X^i}{\partial x^i}\frac{\partial X^j}{\partial x^j} g_{ij} = g_{ij}. \tag{5.31}$$

The induced metric on the moving brane is thus

$$h_{\alpha\beta} = \left\{ \begin{array}{ll} g_{00} + \dot{Y}^2 g_{44} & , \alpha = \beta = 0 \\ g_{ij} & , \alpha = i \neq 0, \beta = j \neq 0. \end{array} \right. \tag{5.32}$$

Note that the 00 component of the induced brane metric (5.32) approaches zero as the brane velocity approaches the speed of light, since in the bulk system this movement corresponds to $ds^2 = 0 \Rightarrow g_{00}dt^2 + g_{44}dY^2 \Rightarrow g_{00} + \dot{Y}^2 g_{44} = 0$. The determinant of (5.32) in terms of the bulk metric is

$$h = g \times \frac{g_{00} + \dot{Y}^2 g_{44}}{g_{00} g_{44}} = g(g^{44} + \dot{Y}^2 g^{00}) \tag{5.33}$$

Since the brane part of the action contains $\sqrt{-h}$, we need the following formula. To calculate, we use equation (2.12), and get

$$\begin{aligned} \frac{\partial h}{\partial g^{MN}} &= \frac{\partial}{\partial g^{MN}} \left(g g^{44} + \dot{Y}^2 g g^{00} \right) \\ &= -g g_{MN} g^{44} + g \delta^4_{\ M} \delta^4_{\ N} - \dot{Y}^2 g g_{MN} g^{00} + \dot{Y}^2 g \delta^0_{\ M} \delta^0_{\ N} \\ &= -h g_{MN} + g \left(\delta^4_{\ M} \delta^4_{\ N} + \dot{Y}^2 \delta^0_{\ M} \delta^0_{\ N} \right) \end{aligned} \tag{5.34}$$

For the square root, we now get

$$\frac{\partial\sqrt{-h}}{\partial g^{MN}} = -\frac{1}{2\sqrt{-h}}\frac{\partial h}{\partial g^{MN}} = -\frac{1}{2}\left(\sqrt{-h} g_{MN} + \frac{g}{\sqrt{-h}} \left(\delta^4_{\ M} \delta^4_{\ N} + \dot{Y}^2 \delta^0_{\ M} \delta^0_{\ N} \right) \right). \tag{5.35}$$

The 4-form field part of the brane action is independent of the bulk metric and drops out in the variation with respect to the bulk metric. This allows us to give the contribution to the energy momentum tensor from the moving brane with tension α_i.

$$\begin{aligned}
T^L_{\ N} &:= \frac{-2g^{LM}}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{MN}} = \frac{-2g^{LM}}{\sqrt{-g}}\frac{\partial}{\partial g^{MN}}\left(-\frac{3M_5^3}{2}\alpha_i\delta(y-Y(t))\sqrt{-h}e^{-\phi}\right) \\
&= \frac{3M_5^3\alpha_i g^{LM}}{\sqrt{-g}}\delta(y-Y(t))e^{-\phi}\frac{\partial\sqrt{-h}}{\partial g^{MN}} \\
&= -\frac{3M_5^3\alpha_i\sqrt{-h}}{2\sqrt{-g}}\delta(y-Y(t))e^{-\phi}\left(\delta^L_{\ N}-\frac{g}{h}\left(g^{L4}\delta^4_{\ N}+g^{L0}\delta^0_{\ N}\dot{Y}^2\right)\right)
\end{aligned} \tag{5.36}$$

In the case of static boundary branes, e.g. the visible brane at $y=0$ with tension α_1, this simplifies somewhat. We then get

$$T^L_{\ N} = -\frac{3M_5^3\alpha_1\sqrt{-h}}{2\sqrt{-g}}\delta(y)e^{-\phi}\left(\delta^L_{\ N}-\frac{g}{h}g^{L4}\delta^4_{\ N}\right), \Rightarrow T^0_{\ 0} = -\frac{3M_5^3\alpha_1}{2\sqrt{g_{44}}}\delta(y)e^{-\phi}.$$

5.3.3 Einstein equations

The Einstein tensor is given in equations (5.18) - (5.21). We found the contributions to the energy momentum tensor in the subsection above, given in equations (5.23), (5.27) and (5.36). Written with free indices, we have the Einstein equations

$$\begin{aligned}
E^L_{\ N} &= \frac{1}{2}g^{LM}\partial_M\phi\partial_N\phi - \frac{1}{4}\delta^L_{\ N}g^{AB}\partial_A\phi\partial_B\phi + \frac{3M_5^3}{4}\frac{1}{5!}e^{2\phi}\delta^L_{\ N}\mathcal{F}^2 \\
&\quad -\sum_i \frac{3M_5^3\alpha_i\sqrt{-h_i}}{2\sqrt{-g}}\delta(y-y_i)e^{-\phi}\left(\delta^L_{\ N}-\frac{g}{h_i}\left(g^{L4}\delta^4_{\ N}+g^{L0}\delta^0_{\ N}\dot{y}_i^2\right)\right).
\end{aligned} \tag{5.37}$$

Writing out the 00 component of the Einstein equations, including the detailed expressions for the Einstein tensor, we have

$$\begin{aligned}
&E^0_{\ 0} := \frac{3}{b^2}\left(\frac{a''}{a}+\frac{a'}{a}\left(\frac{a'}{a}-\frac{b'}{b}\right)\right)-\frac{3}{n^2}\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}\right) \\
&= -\frac{\dot{\phi}^2}{4n^2}-\frac{\phi'^2}{4b^2}+\frac{3M_5^3}{4}\frac{1}{5!}e^{2\phi}\mathcal{F}^2 \\
&-\sum_i \frac{3M_5^3\alpha_i\sqrt{(g^{44}+\dot{y}_i^2g^{00})}}{2}\delta(y-y_i)e^{-\phi}\left(1+\frac{\dot{y}_i^2}{n^2(g^{44}+\dot{y}_i^2g^{00})}\right)
\end{aligned}$$

where we have assumed that quantities are at most functions of t and y, and prime means y-differentiation.

5.4 Equations of motion for ϕ and $\mathcal{A}$

The ekpyrotic universe contains two bulk fields, one scalar and one four-form. The scalar field is connected to the size of the 6 compact dimensions (the Calabi-Yau manifold), $\mathrm{Vol}(\mathcal{CY}) \propto e^{\phi}$. The four-form has no physical gauge degrees of freedom, as seen in subsection 5.2.2, and it is possible to solve for it and then eliminate it from the action.

5.4.1 Equation of motion for ϕ

The field equation for ϕ is obtained from the action (5.2) by varying it with respect to ϕ. We obtain the following Euler-Lagrange equation

$$\begin{aligned} 0 &\overset{!}{=} \partial_A \left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_A \phi)} \right) - \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial \phi} \\ &= \partial_A \left(-\frac{M_5^3}{2}\sqrt{-g}\partial^A \phi \right) - \left(-\frac{M_5^3}{2}\frac{3}{5!}\sqrt{-g}e^{2\phi}\mathcal{F}^2 + \sum_{\text{branes}} \frac{3}{2}\alpha_i M_5^3 \delta(y-y_i)\sqrt{-h_{(i)}}e^{-\phi} \right) \\ &= -\frac{M_5^3}{2}\left(\partial_A \left(\sqrt{-g}\partial^A\phi\right) - \frac{3}{5!}\sqrt{-g}e^{2\phi}\mathcal{F}^2 + \sum_{\text{branes}} 3\alpha_i \delta(y-y_i)\sqrt{-h_{(i)}}e^{-\phi} \right). \end{aligned}$$

Thus the equation of motion for ϕ is

$$\Box\phi - \frac{3}{5!}e^{2\phi}\mathcal{F}^2 + \sum_{\text{branes}} 3\alpha_i \delta(y-y_i)\frac{\sqrt{-h_{(i)}}}{\sqrt{-g}}e^{-\phi} = 0. \tag{5.38}$$

where $\Box$ is the covariant d'Alembert operator

$$\Box\phi := \frac{1}{\sqrt{-g}}\partial_A\left(\sqrt{-g}\partial^A\phi\right). \tag{5.39}$$

We see that the four-form field and the brane-scalar interaction act as sources for the scalar field.

5.4.2 Equation of motion for $\mathcal{A}$

Before we obtain the equation of motion for $\mathcal{A}$, we investigate the connection between $\mathcal{A}$ and its field strength $\mathcal{F}$. We define the components of $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}_{ABCD}dx^A \otimes dx^B \otimes dx^C \otimes dx^D = \frac{1}{4!}\mathcal{A}_{ABCD}dx^A \wedge dx^B \wedge dx^C \wedge dx^D, \tag{5.40}$$

where the components $\mathcal{A}_{ABCD}$ are totally antisymmetric in the indices. The definition of the field strength is $\mathcal{F} = d\mathcal{A}$. Using the definition of exterior differentiation, we get

$$\begin{aligned}\mathcal{F} &= d\mathcal{A} = \frac{1}{4!}\partial_E \mathcal{A}_{ABCD} dx^E \wedge dx^A \wedge dx^B \wedge dx^C \wedge dx^D \\ &= \frac{1}{4!}\partial_A \mathcal{A}_{BCDE} dx^A \wedge dx^B \wedge dx^C \wedge dx^D \wedge dx^E \\ &= \frac{1}{4!}\partial_{[A} \mathcal{A}_{BCDE]} dx^A \wedge dx^B \wedge dx^C \wedge dx^D \wedge dx^E, \end{aligned} \tag{5.41}$$

where $\partial_{[A}\mathcal{A}_{BCDE]}$ is the totally antisymmetric part of $\partial_A \mathcal{A}_{BCDE}$, which we could substitute into expression because $\partial_A \mathcal{A}_{BCDE}$ was contracted with a totally antisymmetric quantity. From the definition of the components of a form when written in a wedge-product basis, we have

$$\mathcal{F} := \frac{1}{5!}\mathcal{F}_{ABCDE} dx^A \wedge dx^B \wedge dx^c \wedge dx^d \wedge dx^E. \tag{5.42}$$

So the components of $\mathcal{F}$ are related to the components of $\mathcal{A}$ by the formula

$$\mathcal{F}_{ABCDE} = 5\partial_{[A}\mathcal{A}_{BCDE]}. \tag{5.43}$$

Since the field strength is totally antisymmetric, it has only one independent component. Therefore we can write the square field strength in terms of one component, which we choose to be $\mathcal{F}_{01234}$.

$$\mathcal{F}^2 = 5!\mathcal{F}_{01234}\mathcal{F}^{01234} \tag{5.44}$$

To find the equation of motion, we must differentiate with respect to $\mathcal{A}$, not $\mathcal{F}$. Using the chain rule for differentiation, we get the following relations.

$$\frac{\partial}{\partial(\partial_F \mathcal{A}_{GHIJ})}\left(\mathcal{F}^2\right) = 2\mathcal{F}^{ABCDE}\frac{\partial}{\partial(\partial_F \mathcal{A}_{GHIJ})}\mathcal{F}_{ABCDE} \tag{5.45}$$

$$\frac{\partial}{\partial(\partial_F \mathcal{A}_{GHIJ})}\mathcal{F}_{ABCDE} = 5\frac{\partial}{\partial(\partial_F \mathcal{A}_{GHIJ})}\partial_{[A}\mathcal{A}_{BCDE]} = 5\delta^F_{[A}\delta^G_B\delta^H_C\delta^I_D\delta^J_{E]} \tag{5.46}$$

$$\frac{\partial}{\partial(\partial_F \mathcal{A}_{GHIJ})}\left(\mathcal{F}^2\right) = 10\mathcal{F}^{ABCDE}\delta^F_{[A}\delta^G_B\delta^H_C\delta^I_D\delta^J_{E]} = 10\mathcal{F}^{[FGHIJ]} = 10\mathcal{F}^{FGHIJ} \tag{5.47}$$

The equation of motion is

$$\begin{aligned} 0 &\overset{!}{=} \partial_F\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial(\partial_F \mathcal{A}_{GHIJ})}\right) - \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial \mathcal{A}_{GHIJ}} = \partial_F\left(-\sqrt{-g}\frac{3M_5^3}{4}\frac{1}{5!}e^{2\phi}\times 10\mathcal{F}^{FGHIJ}\right) \\ &- \sum_{\text{branes}} \frac{3M_5^3}{2}\alpha_i\delta(y-y_i)\frac{1}{4!}\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha X^G_{(i)}\partial_\beta X^H_{(i)}\partial_\gamma X^I_{(i)}\partial_\delta X^J_{(i)}. \end{aligned} \tag{5.48}$$

Thus the equation of motion for $\mathcal{A}$becomes

$$\nabla_A\left(e^{2\phi}\mathcal{F}^{ABCDE}\right) + \sum_{\text{branes}}\alpha_i\delta(y-y_i)\frac{1}{\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha X^B_{(i)}\partial_\beta X^C_{(i)}\partial_\gamma X^D_{(i)}\partial_\delta X^E_{(i)} = 0 \tag{5.49}$$

where ∇_A is the covariant derivative

$$\nabla_A \left(e^{2\phi} \mathcal{F}^{ABCDE} \right) := \frac{1}{\sqrt{-g}} \partial_A \left(\sqrt{-g} e^{2\phi} \mathcal{F}^{ABCDE} \right) . \tag{5.50}$$

5.5 Boundary conditions

In [31], the author considers the fact that if $T^{ty} \neq 0$ at the edges of the orbifold dimension, this corresponds to a flow of energy off the five dimensional space-time, disappearing into nowhere. The author considers this a physical impossibility, and demands that this quantity is zero as a boundary condition. However, in the case where we allow for matter on the edge branes, we have the possibility of these branes absorbing the energy in question. It is at least probable that the edge branes will interact with this energy flow, and absorb some or all of it. The relations obtained by demanding that it be zero will then be modified to include the energy density and pressure of the edge branes, and one would need to add extra matter terms to the total action.

When we view the model as not defined on the line element $y \in [0, R]$, but as defined on the S_1 extra dimension before making the orbifold identification, and with only symmetric states, we can try to make sense of what happens at the edge of the orbifold direction when $T^{ty} \neq 0$. With this picture, T^{ty} is an antisymmetric function that has a jump discontinuity at $y = 0$.

In the case where $T^{ty} > 0$ for positive y, this corresponds to having an energy flow in the direction of increasing y, at positive y, and a flow towards the left for negative y. In this picture, there is obviously energy conservation everywhere since the action has no explicit time-dependence, so this energy flow must correspond to energy being emitted by the brane at $y = 0$, flowing into the bulk. The same argument is valid for the brane at $y = R$.

Thus, by construction, there can be no energy flow out of the universe at $y = 0$. Hence this energy flow must be accompanied by changes in the matter content of the edge branes. In the original ekpyrotic universe model, there is no matter on any of the branes before ekpyrosis, since the action of the branes (5.6) are given simply as world-volume integrals without any non-zero gauge field strength living on the brane, like in the Dirac-Born-Infeld action (2.106). Allowing for a non-zero gauge field strength tensor $\mathbf{F}$ in the brane action would complicate the mathematics significantly, but produce a more realistic model. In our numerical analysis of the equations of motion, we will not include this term, but we will argue that it would not change our results qualitatively.

It is interesting to speculate on how this antisymmetric function T^{ty} would be modified in a quantized version of the heterotic M-theory. It is reasonable then to expect the branes to have a non-zero thickness of the order of the Planck length, and loop graviton contributions should smooth out any discontinuities like the one in T^{ty} at $y = 0$. The range of the y variable in the vicinity of $y = 0$, at which the value of T^{ty} moves from the classical, finite value, to the quantum value of zero at $y = 0$, corresponds to the part of space where the energy flow is emitted or deposited, according to the sign of T^{ty}. If the

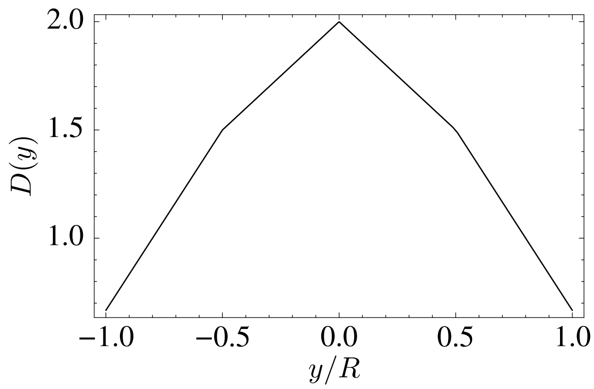


Figure 5.3: $D(y)$ in the case of $R = 1/M_5$, $\alpha = -M_5$, $\beta = M_5/2$, $Y = R/2$, $C = 2$. It has a discontinuous derivative at each brane's position. The function must be everywhere positive to ensure that the signature of the metric doesn't change anywhere in the bulk space

energy indeed originates from the edge brane, this distance must be equal or smaller than the thickness of the brane. It seems natural that both these lengths would be of the order of the Planck length.

5.6 Time-independent solution

One possible vacuum solution of this theory contains flat, static, parallel and isotropic branes that are Poincare invariant in the directions tangent to the branes. This means that we can express their embeddings as ($t := x^0$, $y := x^4$)

$$X^A_{(i)} = (t, x^1, x^2, x^3, y_i). \tag{5.51}$$

A static solution of the equations of motion is [8]

$$\begin{aligned} ds^2 &= D(y)\left(-N^2dt^2 + A^2\sum_{i=1}^{3}(dx^i)^2\right) + B^2D(y)^4dy^2 & (5.52)\\ e^{\phi(y)} &= BD(y)^3 & (5.53)\\ \mathcal{F}_{01234}(y) &= -NA^3B^{-1}D(y)^{-2}D'(y), & (5.54)\end{aligned}$$

where

$$D(y) = \begin{cases} \alpha y + C & ,0 < y < Y \\ \alpha y - \beta(y-Y)\theta(y-Y) + C & ,Y < y < R \\ D(-y) & ,y < 0, \end{cases} \tag{5.55}$$

and N, A, B, C and Y are constants. $D(y)$ is plotted in figure 5.3.

We will now show that this solution minimizes the action (5.2), using the relations

$$\phi' = 3\frac{D'(y)}{D(y)} \tag{5.56}$$
$$g = -N^2A^6B^2D(y)^8. \tag{5.57}$$

Einstein equations and energy momentum tensor We find the nonzero components of the Einstein tensor from equation (5.52) by substituting $n(t,y) = N\sqrt{D(y)}$, $a(t,y) = A\sqrt{D(y)}$ and $b(t,y) = BD(y)^2$ into equations (5.18), (5.19), (5.20) and (5.21) to obtain

$$E^t_{\ t} = 3B^{-2}D(y)^{-5}\left(\frac{1}{2}D''(y) - D(y)^{-1}D'(y)^2\right) \tag{5.58}$$
$$E^i_{\ i} = 3B^{-2}D(y)^{-5}\left(\frac{1}{2}D''(y) - D(y)^{-1}D'(y)^2\right) \tag{5.59}$$
$$E^y_{\ y} = \frac{3}{2}B^{-2}D(y)^{-6}D'(y)^2 \tag{5.60}$$
$$E^t_{\ y} = 0. \tag{5.61}$$

The components of the energy momentum tensor are obtained from the contributions calculated in section 5.3.2 in the static case. The contributions from all branes are now equal, except that the functions are evaluated at different y-values, since this is a time-independent solution. We get

$$\begin{aligned}
\frac{1}{M_5^3}T^t_{\ t} &= -\frac{1}{4}B^{-2}D(y)^{-4}\phi'^2 + \frac{3}{4}\frac{1}{5!}e^{2\phi}\mathcal{F}^2 - \sum_{\text{branes}}\frac{3}{2}\alpha_i\delta(y-y_i)B^{-1}D(y)^{-2}e^{-\phi} \\
&= -3B^{-2}D(y)^{-6}D'(y)^2 - \sum_{\text{branes}}\frac{3}{2}\alpha_i\delta(y-y_i)B^{-2}D(y)^{-5} \qquad (5.62)\\
\frac{1}{M_5^3}T^i_{\ i} &= -\frac{1}{4}B^{-2}D(y)^{-4}\phi'^2 + \frac{3}{4}\frac{1}{5!}e^{2\phi}\mathcal{F}^2 - \sum_{\text{branes}}\frac{3}{2}\alpha_i\delta(y-y_i)B^{-1}D(y)^{-2}e^{-\phi} \\
&= -3B^{-2}D(y)^{-6}D'(y)^2 - \sum_{\text{branes}}\frac{3}{2}\alpha_i\delta(y-y_i)B^{-2}D(y)^{-5} \qquad (5.63)\\
\frac{1}{M_5^3}T^y_{\ y} &= \frac{1}{4}B^{-2}D(y)^{-4}\phi'^2 + \frac{3}{4}\frac{1}{5!}e^{2\phi}\mathcal{F}^2 = \frac{3}{2}B^{-2}D(y)^{-6}D'(y)^2 \qquad (5.64)\\
T^t_{\ y} &= 0. \qquad (5.65)
\end{aligned}$$

After constructing the Einstein equations $E^\alpha_{\ \beta} = 1/M_5^3 T^\alpha_{\ \beta}$ using equations (5.58) - (5.61) and (5.62) - (5.65), we see that the ty and yy component are satisfied independently of the details of the $D(y)$ function. The other Einstein equations both boil down to the same equation. We demonstrate this only for the 00 component equation. We get

$$0 = E^t_{\ t} - \frac{1}{M_5^3}T^t_{\ t} = 3B^{-2}D(y)^{-5}\left(\frac{1}{2}D''(y) - D(y)^{-1}D'(y)^2\right)$$

$$
\begin{aligned}
&+3B^{-2}D(y)^{-6}D'(y)^2 + \sum_{\text{branes}} \frac{3}{2}\alpha_i\delta(y-y_i)B^{-2}D(y)^{-5} \\
= \quad & \frac{3}{2}B^{-2}D^{-5}\left(D'' + \sum_{\text{branes}} \alpha_i\delta(y-y_i)\right), \qquad (5.66)
\end{aligned}
$$

which implies

$$
D''(y) + \sum_{\text{branes}} \alpha_i\delta(y-y_i) = 0. \qquad (5.67)
$$

We will later show that this equation is satisfied for the $D(y)$ given in the static solution.

5.6.1 Equation of motion for ϕ

Inserting the static solution into the field equation (5.38) for ϕ, we get

$$
\Box\phi - \frac{3}{5!}e^{2\phi}\mathcal{F}^2 + \sum_{\text{branes}} 3\alpha_i\delta(y-y_i)B^{-1}D(y)^{-2}e^{-\phi} = 0. \qquad (5.68)
$$

Since $\mathcal{F}$ is a five-form in a five dimensional space, it has only one independent component, which we can take as $\mathcal{F}_{01234}$. Substituting equations (5.52), (5.53), (5.56) and (5.44) into equation (5.38), we obtain

$$
\begin{aligned}
&3B^{-2}D(y)^{-4}\partial_y\left(\frac{D'(y)}{D(y)}\right) + 3B^{-2}D(y)^{-6}D'(y)^2 + \sum_{\text{branes}} 3\alpha_i\delta(y-y_i)B^{-2}D(y)^{-5} \\
&= 3B^{-2}D(y)^{-5}\left(D''(y) - \frac{D'(y)^2}{D(y)} + \frac{D'(y)^2}{D(y)} + \sum_{\text{branes}} \alpha_i\delta(y-y_i)\right) \\
&= 3B^{-2}D(y)^{-5}\left(D''(y) + \sum_{\text{branes}} \alpha_i\delta(y-y_i)\right).
\end{aligned}
$$

Thus equation (5.38) also reduces to equation (5.67).

5.6.2 Equation of motion for $\mathcal{A}$

We will now show that the equation of motion for $\mathcal{A}$ reduces to equation (5.67). Equation (5.49) is totally antisymmetric in the indices B, C, D, E, and in the sum over branes only components with $A, B, C, D \neq 4$ contributes. Only the derivative with respect to y survives, son we can choose $(A, B, C, D) = (0, 1, 2, 3)$, and get

$$
\nabla_4(e^{2\phi}\mathcal{F}^{40123}) + \sum_{\text{branes}} \alpha_i\delta(y-y_i)\frac{1}{\sqrt{-g}}\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha X^0\partial_\beta X^1\partial_\gamma X^2\partial_\delta X^3 = 0. \qquad (5.69)
$$

In the trace inside the sum, only one term contributes, $(\alpha, \beta, \gamma, \delta) = (0, 1, 2, 3)$. Using equations (5.52), (5.54), (5.53) and (5.57) we get

$$\partial_4(\sqrt{-g}B^2D^6\mathcal{F}^{01234}) + \sum_{\text{branes}} \alpha_i\delta(y - y_i) = 0$$

$$\Rightarrow \partial_4(NA^3BD^4B^2D^6(-N^2A^6B^2D^8)^{-1}(-NA^3B^{-1}D^{-2}D')) + \sum_{\text{branes}} \alpha_i\delta(y - y_i) = 0$$

$$\Rightarrow D''(y) + \sum_{\text{branes}} \alpha_i\delta(y - y_i) = 0 \tag{5.70}$$

The static solution gives an expression for the field strength, not the gauge field itself. Using the relation $\mathcal{F} := d\mathcal{A}$, we can find the gauge field. We use the fact that only y-derivatives survive in the static solution. Using equation (5.43), we get

$$\mathcal{F} := d\mathcal{A} \Rightarrow \mathcal{F}_{01234} = 5\partial_{[0}\mathcal{A}_{1234]} = 5\frac{1}{5!}4!\partial_4\mathcal{A}_{0123},$$

where the 4! comes from the 4! permutations of the indices 0,1,2,3, which all contributes equally. Given the value of $\mathcal{F}_{01234}$ in equation (5.54), this relation is satisfied if we choose

$$\mathcal{A}_{0123} = NA^3B^{-1}D(y)^{-1}, \tag{5.71}$$

and components with an index equal to four, we choose to be zero, since these don't affect the value of the field strength tensor.

Note that the expression for the field strength in equation (6.11) is antisymmetric with respect to mirroring about $y = 0$. Since it is the $y-$derivative of $\mathcal{A}_{0123}$, $\mathcal{A}_{0123}$ must be symmetric about $y = 0$.

5.6.3 The equation $D''(y) + \sum \alpha_i\delta(y - y_i) = 0$

Analyzing the $D(y)$ in (5.55), we see that $D''(y)$ is clearly zero away from the branes, since $D(y)$ is linear in those intervals. Near $y = 0$, the slope is $-\alpha$ for $y < 0$ and α for $y > 0$, so $D'(y) = 2\alpha\theta(y) - \alpha$ in the vicinity of $y = 0$. Thus $D''(0) = 2\alpha\delta(y) = -2\alpha_1\delta(y)$ around $y = 0$. Applying the same argument at $y = R$ and $y = \pm Y$ shows that $D''(y) = 2\left(\alpha\delta(y) - (\alpha - \beta)\delta(y - R)\right) - \beta\left(\delta(y - Y) + \delta(y + Y)\right)$. Since $\alpha_1 = -\alpha$, $\alpha_2 = \alpha - \beta$, and $\alpha_3 = \beta$, this is the same as $-\sum \alpha_i\delta(y - y_i)$, and thus equation (5.67) is satisfied.

5.7 The metric signature

From the metric in equation (5.52), we see that $D(y)$ must be everywhere positive so as to not change the signature of the metric. Since it is symmetric about the point $y = 0$, we only have to worry about positive y-values. We have

$$D(y) = \begin{cases} \alpha y + C & , y \leq Y \\ (\alpha - \beta)y + C + \beta Y & , y \geq Y. \end{cases}$$

$D(y)$ is a piecewise linear function of y, the two pieces being separated at the point $y = Y$. So it is enough to ascertain that it is positive in the points $y \in \{0, Y, R\}$. At $y = 0$, we have $D(y) = C$, giving us the demand $C > 0$. At $y = Y$, we have $D(Y) = \alpha Y + C$, giving us $C > -\alpha Y$. At $y = R$, we get the condition $C > -\alpha R$. In sum, we get the following condition on the constant C

$$C > \begin{cases} 0 & , \alpha \geq 0 \\ -\alpha R & , \alpha \leq 0. \end{cases}$$

A naked singularity is a geometric singularity not shielded by a horizon, thus light originating from a point arbitrarily near the singularity can escape to infinity. Penrose put forth in 1969 a conjecture called cosmic censorship, which states that no naked geometric singularities can exist. The conjecture has been shown to be false for some cosmological models, but those models are considered to be physically unreasonable. In ordinary gravitational theory, we have black holes with central geometric singularities which are shielded by a horizon, located at $r = 2GM$ for a Schwarzchild black hole measured in Schwarzchild coordinates. M is the mass of the black hole and G is the four dimensional Newtonian gravitational constant. So black holes are of course in line with the cosmic censorship conjecture.

In the case of the static solution of the simplified heterotic M-theory action, notice that if the orbifold dimension was not truncated at the positions of the visible and hidden branes, we would have a naked geometric singularity in the space-time, where $D(y) = 0$. We can say that the visible or hidden brane, depending on which side the singularity is located, shields the singularity.

5.8 Geometric properties of the static solution

A conformal transformation is a coordinate dependent non-vanishing rescaling of all vectors, preserving of course the metric signature. This means that it can be represented by the following operation on the metric.

$$\mathbf{g} \mapsto \mathbf{g}_c := \exp(\omega)\mathbf{g} \tag{5.72}$$

where ω is an arbitrary function of position in space-time. Since light-like geodesics follow paths which have tangent vectors of length zero, this transformation don't affect them. We say that a metric is conformally flat if it can be made into a flat metric by a conformal transformation. Transforming the metric in the static solution (5.52), using (5.72) with $\omega = -\ln(D(y))$,we get the flat metric

$$ds_c^2 = N^2 dt^2 + A^2 \sum_{i=1}^{3} (dx^i)^2 + B^2 D(y)^3 dy^2. \tag{5.73}$$

It is easy to see that it is flat because the only coordinate dependence in its coefficients is the y-dependence in g_{44}. This dependence is removable by a change of y-coordinate to a coordinate $\bar{y}$ that satisfies the relation $d\bar{y}^2 = D(y)^3 dy^2$, thus one only has to integrate the

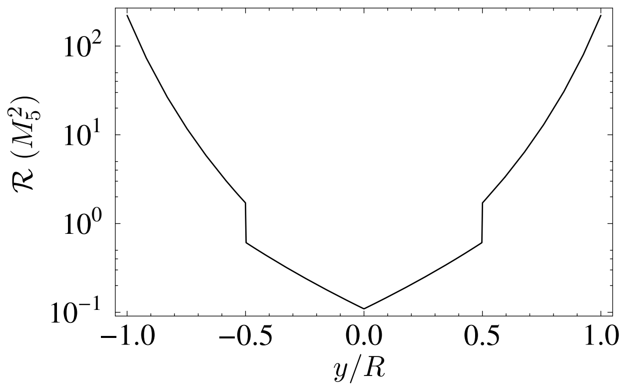


Figure 5.4: Logarithmic plot of the Ricci scalar in the static solution for parameter values $\alpha = -1$, $\beta = 1/2$, $Y = R/2$, $B = 1$, $C = 2$. The delta function character of the bulk brane in the y-direction, gives a discontinuity in the space-time curvature at $y = \pm R/2$.

differential equation $d\bar{y}/dy = D(y)^{3/2}$. This shows that the static solution is conformally flat. Since the light-cone is independent of any transformation that just multiplies the metric by an arbitrary function, this means that the light-cone is the same as in flat space-time.

The 5-dimensional Ricci scalar curvature of the static solution (see appendix (B)) is

$$\mathcal{R} = \frac{7D'(y)^2 - 4D(y)D''(y)}{B^2 D(y)^6}. \tag{5.74}$$

Since the branes give a discontinuity in the derivatives of the metric, the Ricci scalar is not well-defined at the location of the branes. Evaluated away from the branes, the double y-derivative of D is zero, and we get

$$\mathcal{R} = \begin{cases} 7\alpha^2 B^{-2}(\alpha y + C)^{-6} & ,0 < y < Y \\ 7(\alpha - \beta)^2 B^{-2}((\alpha - \beta)y + \beta Y + C)^{-6} & ,Y < y < R. \end{cases}$$

We see that the Ricci scalar curvature increases at small y when α is positive, which corresponds to negative visible brane tension, and from figure 5.4, we see that the scalar curvature increases towards larger y when the visible brane tension is positive ($\alpha < 0$).

Let us also look at constant-time slices of the space-time. On hyper-surfaces of constant time, the metric is obtained by dropping the components in the time directions. These metrics are of course also conformally flat. We get

$$ds_t^2 = D(y)A^2 \sum_{i=1}^{3} (dx^i)^2 + B^2 D(y)^4 dy^2. \tag{5.75}$$

The physical 4-volume of an infinitesimal spatial comoving volume element $d^3x dy$ is therefore $V_{\rm cm} = A^3 B D(y)^{7/2} d^3x dy$. If α in $D(y)$ is negative, corresponding to positive tension on the visible universe, this quantity decreases towards larger y, so in that case the visible brane is in the large spatial volume region of the extra dimension. The 4-dimensional Ricci scalar curvature of hyper-surfaces of constant time is (see appendix (B))

$$\mathcal{R}_t = \frac{6D'(y)^2 - 3D(y)D''(y)}{B^2 D(y)^6} \tag{5.76}$$

Since the double y-derivative of D is zero away from the branes, this is proportional to the 5-dimensional Ricci scalar, which increases with increasing y for negative α. It is no accident that the region of large spatial comoving volume coincides with the region of small spatial scalar curvature.

5.9 Brane tensions

If we were to add matter to the visible brane in the action (5.1), it would appear also multiplied by the visible brane tension, just as the brane gauge field in the Dirac-Born-Infeld action (2.106) appears in an expression multiplied by the tension. Looking at the effective four-dimensional Einstein equation (4.15) on the brane, we see that if this tension were negative, this matter would contribute with the opposite sign in the Einstein equation, compared to standard matter in standard four-dimensional Einstein gravity. This means that gravity would be repulsive on the visible brane. Since we know that gravity is attractive for normal matter, we must choose a positive visible brane tension in a realistic model.

It is claimed in [8] that it is straightforward to show that the bulk brane must have positive tension. Since the three brane tensions must sum to zero, this means that the hidden brane must have negative tension. Several examples for the ekpyrotic universe of brane embeddings and calculations of brane tensions are given in [30], and in all these examples the boundary branes have tensions of opposite signs, and the bulk brane has positive tension.

Chapter 6

Moduli Space Approximation and Numerical Results

In this chapter, we will introduce an approximation scheme called the moduli space approximation. In the original proposition of the ekpyrotic universe, [8], it was argued that if the system evolves very slowly, we can approximate the evolution by assuming that it moves between different vacuum solutions. The moduli space of vacua is in our case a five-dimensional space, or after we choose the gauge $N(t) = 1$, a four-dimensional space. At each point in the moduli space, we have a heterotic M-theory static vacuum solution of the type described in section 5.6, with particular values for A, B, C and Y.

In that case the constants A, B, C and the constant position Y of the bulk brane in the time independent solution will vary as the system of branes changes with time. The moduli space approximation consists of promoting these constants to functions depending slowly on coordinates parallel to the branes, and since we assume isotropy and homogeneity, this means that they only depend on time.

After promoting the constants to functions of time, we insert the solution into the original action, and then integrate over the orbifold dimension to obtain a new action. From this action we can obtain new equations of motion for the now time-dependent functions A, B, C and Y. To warm up, we first present a simple example of the moduli space approximation, before applying the approximation scheme to the action for the ekpyrotic universe.

When we have obtained the new equations of motion, we solve them numerically for different parameter values. From the plots, we can analyze the behaviour of the model.

6.1 Field theory of real, scalar fields

The moduli space of a field theory is the space of vacuum solutions. As an example, we start with a field theory of real, scalar fields. The action for the theory is

$$S = \int d^4x \left(-\frac{1}{2}\partial_\alpha \vec{\phi} \cdot \partial^\alpha \vec{\phi} - V(\vec{\phi}) \right). \tag{6.1}$$

The moduli space is the set of minima of the potential $V(\phi)$. If we have a potential$V(\vec{\phi}) = \frac{\lambda}{4}\left(\vec{\phi}\cdot\vec{\phi} - v^2\right)^2$ with positive λ, the set of minima is given by the solutions of $\vec{\phi}\cdot\vec{\phi} - v^2 = 0$. If we have N scalar fields in the vector $\vec{\phi}$, the moduli space is the $(N-1)$-sphere $\mathbf{S}_{N-1}$.

An element in the moduli space is a time independent solution, and the set of time independent solutions will be parametrized by as many parameters as the dimension of the moduli space. In the case of a theory with only two real scalar fields, it will be one angular parameter since the moduli space in this case is $\mathbf{S}_1$. The point of the moduli space approximation is to generate time dependent solutions by letting those parameters be functions of time, and then inserting the solution back into the action. Such a time dependent ansatz will only be a solution to a certain accuracy, only valid at low speeds.

The Euler-Lagrange equation generated by the action (6.1) with the potential mentioned above is

$$-\partial_\alpha\partial^\alpha \vec{\phi} + \lambda\left(\vec{\phi}\cdot\vec{\phi} - v^2\right)\vec{\phi} = 0. \tag{6.2}$$

The vacuum solutions will be constant in both space and time, so $\partial_\alpha\partial^\alpha \vec{\phi}$ is zero, and we get the condition

$$\vec{\phi}\cdot\vec{\phi} = v^2. \tag{6.3}$$

It $N = 2$, this tells us that the coordinate independent solutions can be written

$$\vec{\phi} = v\left(\cos(\theta), \sin(\theta)\right), \tag{6.4}$$

parametrized by an angle θ. Thus the moduli space is $\mathbf{S}_1$.

To find a time dependent solution with the moduli space approximation, we let the parameter θ depend on time: $\theta \to \theta(t)$, which gives us a new expression for the solution:

$$\vec{\phi}_M = \lambda\left(\cos(\theta(t)), \sin(\theta(t))\right). \tag{6.5}$$

We substitute this into the action, and treat $\theta(t)$ as an independent field. The space-derivatives of this expression give zero, and we get the action

$$S = \int d^4x \frac{v^2}{2}\dot{\theta}(t)^2, \tag{6.6}$$

which gives the following Euler-Lagrange equation and solution for $\theta(t)$:

$$\partial_t^2\theta(t) = 0 \Rightarrow \theta(t) = \omega t + \theta_0 \Rightarrow \vec{\phi}_M(t) = v\left(\cos(\omega t + \theta_0), \sin(\omega t + \theta_0)\right). \tag{6.7}$$

We can test whether this is a solution of the original equation of motion, by inserting this $\vec{\phi}_M(t)$ into the original Euler-Lagrange equation (6.2). This gives

$$-\partial_\alpha\partial^\alpha \vec{\phi}_M + \lambda\left(\vec{\phi}_M\cdot\vec{\phi}_M - v^2\right)\vec{\phi}_M = \partial_t^2\vec{\phi}_M = -\omega^2\vec{\phi}_M \neq 0,$$

which is not zero. But is zero to first order in the ω parameter. Since ω is a measure of the speed of the system, this is as expected only a solution to a certain accuracy depending on the speed of the evolution. It is of course an exact solution when $\omega = 0$.

At first sight, the fact that the Euler-Lagrange equations are not satisfied may seem strange since it should always be possible the change variables by writing the original fields in the Lagrangian as functions of new fields, as it seems that we have done in equation (6.5). The reason it doesn't work quite that way in this approximation is that we are reducing the number of degrees of freedom. In the previous model we start with N degrees of freedom, and construct an ansatz that is constrained to move on the moduli space, which is of lower dimension, in this case $N - 1$. In this case our moduli space solution (6.7) is not a solution of (5.38), but is a solution of the same action with the additional requirement $\vec{\phi}_M \cdot \vec{\phi}_M - v^2 = 0$. The original action together with this requirement is called a sigma-model field theory, and is used i.e. when describing systems of spins in a magnetic field where the requirement expresses the constant length of the spin vectors, and in string theory where the requirement on $\vec{\phi}_M$ represents a compact space-time manifold since in this case the values of $\vec{\phi}_M$ are interpreted as values of space-time coordinates.

6.2 The ekpyrotic universe

Our static solution in equations (5.55), (5.52), (5.53) and (5.54) of the equations of motion obtained from the action (5.2) contains five arbitrary constants, or moduli: N, A, B, C, and Y. We want to construct the moduli space action, analogous to (6.6) for the scalar field theory above. To do this we need to promote our moduli to time-dependent functions, and calculate the new action. We must thus insert

$$\begin{aligned} D(y) &= \alpha|y| - \beta(y - Y(t))\theta(y - Y(t)) \\ &\quad -\beta(y + Y(t))\theta(y + Y(t)) + C(t) \end{aligned} \tag{6.8}$$

$$ds^2 = D(y)\left(-N(t)^2dt^2 + A(t)^2\sum_{i=1}^{3}(dx^i)^2\right) + B(t)^2D(y)^4dy^2 \tag{6.9}$$

$$e^{\phi(y)} = B(t)D(y)^3 \tag{6.10}$$

$$\mathcal{F}_{01234}(y) = -N(t)A(t)^3B(t)^{-1}D(y)^{-2}D'(y) \tag{6.11}$$

into the action (5.2).

To keep things simple, we will keep the $D(y)$ in our expressions for now, not substituting in its specific dependence on y, $Y(t)$ and $C(t)$ quite yet, and we don't indicate its dependence on $C(t)$ and $Y(t)$ explicitly. We also do the calculation separately for the bulk and brane parts of the action.

6.2.1 Bulk moduli space approximation Lagrangian

The bulk part of the Lagrangian in equation (5.2) is

$$\begin{aligned}
\frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_b &:= \frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_{\mathcal{R}} + \frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_\phi + \frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_{\mathcal{A}} \\
&= \frac{1}{2}\sqrt{-g}\left(\mathcal{R} - \frac{1}{2}\partial_A\phi\partial^A\phi - \frac{3}{2}\frac{1}{5!}e^{2\phi}\mathcal{F}^2\right). \qquad (6.12)
\end{aligned}$$

First, we insert the time-dependent solution into the kinetic scalar part, and obtain

$$\begin{aligned}
\frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_\phi &:= -\frac{1}{4}\sqrt{-g}\partial_A\phi\partial^A\phi = -\frac{1}{4}NA^3BD^4\left(-N^{-2}D^{-1}\dot{\phi}^2 + B^{-2}D^{-4}\phi'^2\right) \\
&= \frac{1}{4}N^{-1}A^3BD^3\left(B^{-1}\dot{B} + 3D^{-1}\dot{D}\right)^2 - \frac{9}{4}NA^3B^{-1}D^{-2}D'^2 \\
&= \frac{1}{4}N^{-1}A^3B^{-1}\dot{B}^2D^3 + \frac{3}{2}N^{-1}A^3\dot{B}D^2\dot{D} \\
&\quad +\frac{9}{4}N^{-1}A^3BD\dot{D}^2 - \frac{9}{4}NA^3B^{-1}D^{-2}D'^2. \qquad (6.13)
\end{aligned}$$

Similarly for the kinetic four-form part of the Lagrangian, we get

$$\begin{aligned}
\frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_{\mathcal{A}} &:= -\frac{1}{2}\sqrt{-g}\frac{3}{2}\frac{1}{5!}e^{2\phi}\mathcal{F}^2 = -\frac{3}{4}\frac{1}{5!}NA^3BD^4B^2D^6 5!\mathcal{F}_{01234}\mathcal{F}^{01234} \\
&= -\frac{3}{4}NA^3B^3D^{10}\left(-NA^3B^{-1}D^{-2}D'\right)^2\left(-N^2A^6B^2D^8\right)^{-1} \\
&= \frac{3}{4}NA^3B^{-1}D^{-2}D'^2. \qquad (6.14)
\end{aligned}$$

To find the Einstein-Hilbert part, we first calculate the Ricci scalar in terms of the time-dependent metric Ansatz. We can use the program in appendix B to calculate it as a function of the metric (6.9).

$$\begin{aligned}
\mathcal{R} &= 6N^{-2}A^{-2}\dot{A}^2D^{-1} + 6N^{-2}A^{-1}\dot{A}B^{-1}\dot{B}D^{-1} - 6N^{-3}\dot{N}A^{-1}\dot{A}D^{-1} \\
&\quad -2N^{-3}\dot{N}B^{-1}\dot{B}D^{-1} + 6N^{-2}A^{-1}\ddot{A}D^{-1} + 2N^{-2}B^{-1}\ddot{B}D^{-1} \\
&\quad 7B^{-2}D^{-6}D'^2 - 4B^{-2}D^{-5}D'' + 21N^{-2}A^{-1}\dot{A}D^{-2}\dot{D} + 10N^{-2}B^{-1}\dot{B}D^{-2}\dot{D} \\
&\quad -7N^{-3}\dot{N}D^{-2}\dot{D} + \frac{13}{2}N^{-2}D^{-3}\dot{D}^2 + 7N^{-2}D^{-2}\ddot{D} \qquad (6.15)
\end{aligned}$$

Thus the Einstein-Hilbert part of the Lagrangian is

$$\begin{aligned}
\frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_{\mathcal{R}} &:= \frac{1}{2}\sqrt{-g}\mathcal{R} = \frac{1}{2}NA^3BD^4\mathcal{R} \\
&= 3N^{-1}A\dot{A}^2BD^3 + 3N^{-1}A^2\dot{A}\dot{B}D^3 - 3N^{-2}\dot{N}A^2\dot{A}BD^3 - N^{-2}\dot{N}A^3\dot{B}D^3 \\
&\quad +3N^{-1}A^2\ddot{A}BD^3 + N^{-1}A^3\ddot{B}D^3 + \frac{7}{2}NA^3B^{-1}D^{-2}D'^2
\end{aligned}$$

$$-2NA^3B^{-1}D^{-1}D'' + \frac{21}{2}N^{-1}A^2\dot{A}BD^2\dot{D} + 5N^{-1}A^3\dot{B}D^2\dot{D}$$
$$-\frac{7}{2}N^{-2}\dot{N}A^3BD^2\dot{D} + \frac{13}{4}N^{-1}A^3BD\dot{D}^2 + \frac{7}{2}N^{-1}A^3BD^2\ddot{D} \qquad (6.16)$$

It contains second derivatives with respect to both t and y. It is convenient to use partial integration to convert these to only first derivatives, since we can then use the standard formulas for the Euler-Lagrange equations. As an example, we display partial integration on the fifth term in the expression above.

$$\begin{aligned} 3N^{-1}A^2\ddot{A}BD^3 &= 3\partial_t(N^{-1}A^2\dot{A}BD^3) + 3N^{-2}\dot{N}A^2\dot{A}BD^3 \\ &\quad -6N^{-2}A\dot{A}^2BD^3 - 3N^{-1}A^2\dot{A}\dot{B} - 9N^{-1}A^2\dot{A}BD^2\dot{D} \end{aligned}$$

Using partial integration with respect to time on the double time derivatives, and dropping total derivatives, we get

$$\begin{aligned} \frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_{\mathcal{R}} &= N^{-1}A^3BD^3\left(-3\left(\frac{\dot{A}}{A}\right)^2 - 3\frac{\dot{A}}{A}\frac{\dot{B}}{B} - 9\frac{\dot{A}}{A}\frac{\dot{D}}{D} - \frac{3}{2}\frac{\dot{B}}{B}\frac{\dot{D}}{D} - \frac{15}{4}\left(\frac{\dot{D}}{D}\right)^2\right) \\ &\quad +\frac{7}{2}NA^3B^{-1}D^{-2}(D')^2 - 2NA^3B^{-1}D^{-1}D''. \end{aligned} \qquad (6.17)$$

In the end, the total bulk Lagrangian becomes

$$\begin{aligned} \frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_b &:= \frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_{\mathcal{R}} + \frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_\phi + \frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_{\mathcal{A}} \\ &= N^{-1}A^3BD^3\left(-3\left(\frac{\dot{A}}{A}\right)^2 - 3\frac{\dot{A}}{A}\frac{\dot{B}}{B} - 9\frac{\dot{A}}{A}\frac{\dot{D}}{D} - \frac{3}{2}\left(\frac{\dot{D}}{D}\right)^2 + \frac{1}{4}\left(\frac{\dot{B}}{B}\right)^2\right) \\ &\quad +2NA^3B^{-1}\left(D^{-2}(D')^2 - D^{-1}D''\right). \end{aligned} \qquad (6.18)$$

Since we have $D^{-2}(D')^2 - D^{-1}D'' = -(D^{-1}D')'$, the two last terms are just a total derivative. When we do the integral over $y \in [-R, R]$ in the action, there are no boundary contributions, since $y = -R$ and $y = R$ are identified as the same point. We can therefore drop this total derivative from the Lagrangian without hesitation.

We want to do the integral over the orbifold dimension, and must therefore use the explicit form of the $D(y)$ function. We have

$$\dot{D} = \beta\dot{Y}\theta(y - Y) + \dot{C}. \qquad (6.19)$$

Inserting this into (6.18) gives

$$\begin{aligned} \frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_b &= N^{-1}A^3BD^3\Bigg(-3\left(\frac{\dot{A}}{A}\right)^2 - 3\frac{\dot{A}}{A}\frac{\dot{B}}{B} - 9\frac{\dot{A}}{A}\left(\frac{\beta\dot{Y}\theta(y-Y)+\dot{C}}{D}\right) \\ &\quad +\frac{1}{4}\left(\frac{\dot{B}}{B}\right)^2 - \frac{3}{2}\left(\frac{\beta\dot{Y}\theta(y-Y)+\dot{C}}{D}\right)^2 + \frac{1}{4}\left(\frac{\dot{B}}{B}\right)^2\Bigg). \end{aligned} \qquad (6.20)$$

We now integrate this over $y \in [-R, R]$ to obtain a four-dimensional Lagrangian. Define the integrals over the orbifold dimension

$$\begin{aligned} I_{m,a} &:= 2\int_0^Y D^m dy = \frac{2}{\alpha(m+1)}\left((\alpha Y + C)^{m+1} - C^{m+1}\right) && (6.21)\\ I_{m,b} &:= 2\int_Y^R D^m dy = \frac{2}{(\alpha-\beta)(m+1)} \times && (6.22)\\ &\quad \left(((\alpha-\beta)R + C + \beta Y)^{m+1} - (\alpha Y + C)^{m+1}\right) \\ I_m &:= I_{m,a} + I_{m,b}. && (6.23)\end{aligned}$$

Notice that the value of $I_{m,a}$ when $Y \to R$ is not the same as $I_{m,b}$ when $Y \to 0$ since the D function is not equal on both sides of the bulk brane at $y = Y$. These integrals are positive since $D(y)$ must be positive to preserve the metric signature. $D(y)$ is dimensionless, so the integrals have the dimensions of length. From these definitions and the definition of the D function we calculate

$$I_{0,a} = 2Y \qquad I_{0,b} = 2(R - Y) \qquad I_0 = 2R \tag{6.24}$$

$$\tfrac{d}{dC} D = 1 \qquad \tfrac{d}{dY} D = \alpha. \tag{6.25}$$

Derivatives of the I_m integrals with respect to the independent functions are needed when we find the equations of motion from our new action.

$$\begin{aligned} \frac{dI_{m,a}}{dC} &= \frac{2}{\alpha}\left((\alpha Y + C)^m - C^{m+1}\right) = mI_{m-1,a} && (6.26)\\ \frac{dI_{m,a}}{dY} &= \frac{2}{\alpha}\alpha(\alpha Y + C)^m = 2(\alpha Y + C)^m = 2D^m && (6.27)\\ \frac{dI_{m,b}}{dC} &= \frac{2}{(\alpha-\beta)}\left(((\alpha-\beta)R + C + \beta Y)^m - (\alpha Y + C)^m\right) = mI_{m-1,b} && (6.28)\\ \frac{dI_{m.b}}{dY} &= \frac{2}{(\alpha-\beta)}\left(\beta\left((\alpha-\beta)R + C + \beta Y\right)^m - \alpha(\alpha Y + C)^m\right) \\ &= m\beta I_{m-1,b} - 2D^m && (6.29)\\ \frac{dI_m}{dC} &= \frac{dI_{m,a}}{dC} + \frac{dI_{m,b}}{dC} = mI_{m-1,a} + mI_{m-1,b} = mI_{m-1} && (6.30)\end{aligned}$$

$$\begin{aligned} \frac{dI_m}{dY} &= \frac{dI_{m,a}}{dY} + \frac{dI_{m,b}}{dY} = 2(\alpha Y + C)^m + \\ &\quad + \frac{2}{(\alpha-\beta)}\left(\beta\left((\alpha-\beta)R + C + \beta Y\right)^m - \alpha(\alpha Y + C)^m\right) \\ &= \frac{2}{(\alpha-\beta)}\left((\alpha-\beta)(\alpha Y + C)^m + \beta\left((\alpha-\beta)R + C + \beta Y\right)^m - \alpha(\alpha Y + C)^m\right) \\ &= \frac{2\beta}{(\alpha-\beta)}\left(((\alpha-\beta)R + C + \beta Y)^m - (\alpha Y + C)^m\right) = m\beta I_{m-1,b}. && (6.31)\end{aligned}$$

Thus we have

$$\dot{I}_m = \frac{dI_m}{dC}\dot{C} + \frac{dI_m}{dY}\dot{Y} = mI_{m-1}\dot{C} + m\beta I_{m-1,b}\dot{Y}. \tag{6.32}$$

We now have the contribution of the bulk part of the five-dimensional simplified heterotic M-theory Lagrangian to the four-dimensional moduli space approximation Lagrangian.

$$\mathcal{L}_b^{mod} := \int_{-R}^{R} dy \frac{1}{M_5^3}\sqrt{-g}\mathcal{L}_b \tag{6.33}$$

Thus we have

$$\begin{aligned}\mathcal{L}_b^{mod} &= -\frac{3M_5^3 A^3 B}{N}\Bigg[I_3\left(\frac{\dot{A}}{A}\right)^2 + \frac{\dot{A}}{A}\left(I_3\left(\frac{\dot{B}}{B}\right) + 3I_2\dot{C} + 3\beta I_{2,b}\dot{Y}\right) \\ &\quad -\frac{1}{12}I_3\left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{2}I_1\dot{C}^2 + \beta I_{1,b}\dot{Y}\left(\dot{C} + \frac{\beta}{2}\dot{Y}\right)\Bigg].\end{aligned} \tag{6.34}$$

Since we have absorbed the square root of the determinant of the metric into the Lagrangian in these calculations, the action obtained from this Lagrangian is given by

$$S_b^{mod} = \int d^4x \mathcal{L}_b^{mod}. \tag{6.35}$$

This action now needs to be supplemented with the contributions from the boundary and bulk branes, and after that we will add a potential-term to drive the evolution of the system.

6.2.2 Boundary brane moduli space Lagrangian

Now we must convert the boundary brane part of the action (5.2). To do this we need an expression for the 4-form field $\mathcal{A}$. This expression was found in the static case in equation (5.71), and this will be unchanged in the moduli space approximation, apart from making the constants time-dependent.

We must insert this into the Lagrangian for the boundary branes. We use the simplified, non-covariant version given in equation (5.10). For the visible brane, we get

$$\begin{aligned}\sqrt{-g}\mathcal{L} &= -\frac{3}{2}\alpha_1 M_5^3\delta(y)\left(\sqrt{-h}e^{-\phi} - \mathcal{A}_{0123}\right) \\ &= -\frac{3}{2}\alpha_1 M_5^3\delta(y)\left(NA^3D^2B^{-1}D^{-3} - NA^3B^{-1}D^{-1}\right) = 0,\end{aligned} \tag{6.36}$$

where we have used the determinant of the induced metric (5.32). The same goes for the hidden brane, of course.

So the boundary branes don't contribute to the moduli space action. This fits well with the fact that they are static, so they don't contribute to the kinetic energy part of the Lagrangian. By design, they have no matter content, so their contribution must then be zero.

6.2.3 Bulk brane moduli space Lagrangian

Since the bulk brane is moving, the induced metric is modified, and this gives a kinetic energy term for Y in the bulk brane in the moduli space action. The brane movement causes the time component of the induced metric on the brane to differ from the bulk time component of the bulk metric.

$$h_{tt} = g_{MN}\frac{\partial X^M}{\partial \xi^0}\frac{\partial X^N}{\partial \xi^0} = g_{00} + g_{44}\left(\frac{\partial Y}{\partial t}\right)^2 = g_{00} + g_{44}\dot{Y}^2 \tag{6.37}$$

This leads to a new expression for $\sqrt{-h}$ which we expand in the small velocity $\dot{Y}$ to get

$$\begin{aligned}\sqrt{-h} &= \sqrt{N^2 D - B^2 D^4 \dot{Y}^2} A^3 D^{3/2} \\ &\approx NA^3D^2 - \frac{1}{2}N^{-1}A^3B^2D^5\dot{Y}^2.\end{aligned} \tag{6.38}$$

We must use the gauge choice (5.71), where we chose $\mathcal{A}_{1234} = 0$, which eliminates the coupling between the bulk brane speed $\dot{Y}$ and the gauge field. Inserting this into the Lagrangian (5.10), we obtain the five-dimensional Lagrangian

$$\begin{aligned}\mathcal{L}^{mod}_{\beta,5d} &\approx -\frac{3}{2}\sum \beta M_5^3 \delta(y-Y)\Big(NA^3B^{-1}D^{-1} - \frac{1}{2}N^{-1}A^3BD^2\dot{Y}^2 \\ &\quad -NA^3B^{-1}D^{-1} + \mathcal{A}_{1234}\dot{Y}\Big) \\ &= -\frac{3}{2}\beta M_5^3\delta(y-Y)\left(-\frac{1}{2}N^{-1}A^3BD^2\dot{Y}^2 + \mathcal{A}_{1234}\dot{Y}\right) \\ &= \frac{3}{4}\beta M_5^3 \delta(y-Y)N^{-1}A^3BD^2\dot{Y}^2.\end{aligned} \tag{6.39}$$

When doing the integral of this over the orbifold dimension to obtain the four-dimensional Lagrangian, we get equal contributions from the bulk brane and the mirror bulk brane at $y = -Y$. The effect is that the delta function vanishes, and we get an extra factor of 2.

$$\mathcal{L}^{mod}_{\beta} = \frac{3}{2}\beta M_5^3 N^{-1}A^3BD^2\dot{Y}^2 \tag{6.40}$$

This acts as a kinetic term for the bulk brane movement. The bulk brane tension acts as a mass for the bulk brane in this expression.

6.2.4 Moduli space approximation Lagrangian for the ekpyrotic universe

The total Lagrangian is the sum of the bulk and the brane parts

$$\mathcal{L}_{mod} = \mathcal{L}^{mod}_b + \mathcal{L}^{mod}_\beta$$

$$= \quad -\frac{3M_5^3 A^3 B}{N}\Bigg(I_3 \left(\frac{\dot{A}}{A}\right)^2 + \frac{\dot{A}}{A}\left(I_3 \left(\frac{\dot{B}}{B}\right) + 3I_2\dot{C} + 3\beta I_{2,b}\dot{Y}\right)$$

$$-\frac{1}{12} I_3 \left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{2} I_1 \dot{C}^2 + \beta I_{1,b}\dot{Y}\left(\dot{C} + \frac{\beta}{2}\dot{Y}\right) - \frac{\beta}{2} D^2 \dot{Y}^2 \Bigg). \tag{6.41}$$

As should be, no potential terms have arisen for the moduli N, A, B, C, Y. This expression should be interpreted as a non-trivial kinetic term of the same form as in sigma-models in field theory, where one could have a kinetic term of the form $G(\vec{\phi})_{MN}\dot{\phi}_M\dot{\phi}_N$ where G is interpreted as a metric in field-space.

Since we now have a four-dimensional effective theory for the ekpyrotic universe, we can deduce the four-dimensional Planck mass M_4 by comparing this action to the one obtained from the four-dimensional metric

$$ds^2 = D(0)\left(-N(t)^2 dt^2 + A(t)^2 \sum_{i=1}^{3} (dx^i)^2\right). \tag{6.42}$$

Calculating the four-dimensional Ricci scalar from this leads us to a four-dimensional Einstein-Hilbert Lagrangian (dropping irrelevant terms),

$$\mathcal{L}^{4D} = -\frac{3M_4^2 A^3 D}{N}\left(\frac{\dot{A}}{A}\right)^2 + \dots. \tag{6.43}$$

The factor multiplying the scale factor A in our moduli space approximation must be equal to the corresponding factor in standard four-dimensional gravity. Thus we have the four-dimensional Planck mass as a function of the five-dimensional one and the moduli fields

$$M_4^2 = M_5^3 \left(\frac{BI_3}{D}\right). \tag{6.44}$$

If the four-dimensional Planck mass is frozen to some value at ekpyrosis, as suggested in ([8]), we must evaluate this at ekpyrosis, where $Y = 0$. We then obtain

$$M_4^2 = M_5^3 \frac{B(t)I_3(t)}{C(t)}, \tag{6.45}$$

where we have chosen ekpyrosis to occur at a time t_c. In the case of ordinary Kaluza-Klein compactification without brane-worlds, the relation between the five and the four-dimensional Planck masses is different [32]. It then only depends on the radius of the compact dimension as

$$M_4^2 \sim M_5^3 R. \tag{6.46}$$

In the ekpyrotic universe the relation depends on the brane tensions and length of the extra dimension through the integral I_3, but also explicitly on the functions $B(t)$ and $C(t)$.

The question of whether the four dimensional gravitational coupling strength is a constant or not, depends on the behavior of the five dimensional fields. If the Planck masses are related by (6.44) also at times after the collision, and the $B(t)$ and $C(t)$ vary, then the Newtonian gravitational constant will vary with time. In that case, since we observe a constant gravitational constant, the B and C must stabilize on some values by some mechanism.

6.3 Bulk brane potential

The moduli space action contains no potentials; the fields are almost free. But they are required to move on the moduli space manifold, hence the non-trivial kinetic term. To make the ekpyrotic universe model work, we need an attractive potential that pulls the bulk brane towards the visible brane. If the branes were exact BPS objects, there would not be any forces between them. In string theory, the attractive forces between two BPS branes coming from the graviton and dilaton exchange are canceled by the repulsive forces of Ramond string interactions. Thus we must postulate that the branes are not exact BPS objects.

The potential must satisfy the following requirements:

- It must depend only on the coordinate in the orbifold dimension direction, since the bulk brane movement in the orbifold direction must be independent of all the coordinates except time. This is so because it must remain flat during its journey across the bulk space. Otherwise it would hit the visible brane at significantly different times on different points in the visible universe, causing big inhomogeneities in our universe, which is experimentally incorrect.

- The derivative of the potential must be small near $y = R$ because otherwise the bulk brane will not be flat. As in Newton's second law, the derivative of the potential gives the acceleration of the position of the object in question. The acceleration must be small because the bulk brane nucleation process presumably starts only at one point on the hidden brane, and grows outwards in all three space-like directions tangent to hidden brane. For the bulk brane to become flat, the nucleation growth rate tangent to the hidden brane must be much faster than the movement of the bulk brane in the orbifold direction.

- We must have $V(R) = 0$ since otherwise the hidden brane state would not be static, but would undergo some kind of inflation. In that case it could not have been in a static configuration in the infinitely distant past.

- We must have $V(0) \approx 0$ since otherwise it would contribute to a cosmological constant in the visible universe after the bulk brane collides.

- In [8] it is argued that the potential must come from non-perturbative interactions. Little is known about the non-perturbative part of string theory, so it is not currently possible to calculate the potential from first principles. But we will learn about the theory by trying different potentials.

6.3.1 Velocity dependence

In 10 dimensional perturbative string theory, there is no interaction between static, parallel Dp-branes of equal tension. There is a cancellation between attractive and repulsive forces between the branes which has its origin in the BPS property of the static state. When they are moving there is, however, a velocity dependent force between parallel Dp-branes moving in a direction transverse to the branes. The potential between them is at large distances and to leading order in the velocity [11]

$$V(r,v) = -\frac{v^4}{r^{7-p}}\frac{V_p}{\alpha'^{p-3}} \times 2^{2-2p}\pi^{(5-3p)/2} \times \Gamma(\frac{7-p}{2}) + O(v^6), \tag{6.47}$$

where v is the transverse relative brane velocity, r the brane separation, V_p is a space-time volume factor which must be divided away to obtain the potential density, which is finite in the limit of infinitely large dimensions. α' is the string constant, and Γ is the gamma function. This is an attractive potential, and the cancellation of forces is no longer there since the movement in the brane configuration breaks the super-symmetry, and thus also the BPS property which protects the branes from interactions. But this is a perturbative force, and is highly suppressed at low velocities, which is the velocity-region we are interested in, but it does suggest the possibility of a velocity dependent potential between the branes.

6.3.2 Adding a potential to the action

We now introduce a potential into the theory by adding a new term to the Lagrangian. The term must depend on N, A and B in a manner consistent with general coordinate invariance. Since we have absorbed $\sqrt{-g}$ into the Lagrangian, it must be proportional to that quantity, which in terms of the metric components N, A and B, is NA^3B.

$$\mathcal{L}_V = -3\beta M_5^3 NA^3BV(Y). \tag{6.48}$$

The potential $V(Y)$, so defined, is dimensionless. The full Lagrangian with potential is

$$\begin{aligned}\mathcal{L}_{mod} \;&=\; -\frac{3M_5^3A^3B}{N}\Bigg[I_3\left(\frac{\dot{A}}{A}\right)^2 + \frac{\dot{A}}{A}\left(I_3\frac{\dot{B}}{B} + 3I_2\dot{C} + 3\beta I_{2,b}\dot{Y}\right) - \frac{1}{12}I_3\left(\frac{\dot{B}}{B}\right)^2 \\ &\quad + \frac{1}{2}I_1\dot{C}^2 + \beta I_{1,b}\dot{Y}\left(\dot{C} + \frac{\beta}{2}\dot{Y}\right) - \beta\left(\frac{1}{2}D^2\dot{Y}^2 - N^2V(Y)\right)\Bigg]. \end{aligned} \tag{6.49}$$

We see that multiplying A and B by arbitrary non-zero constants, is the same as multiplying the Lagrangian by an overall constant. This has no effect on the classical solutions. This is a consequence of the invariance with respect to re-scalings of the coordinates. We

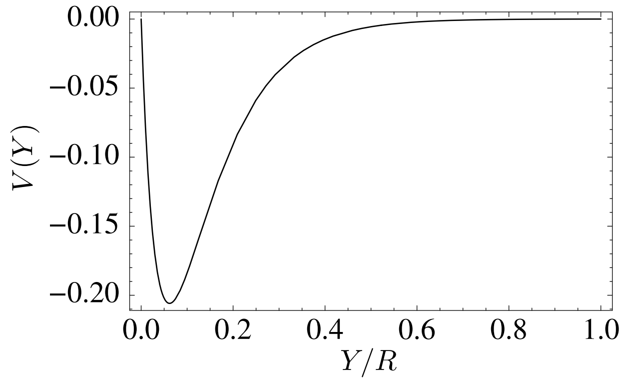


Figure 6.1: Example potential for the bulk brane. It satisfies the conditions $\lim_{Y\to R} V(Y) = \lim_{Y\to 0} V(Y) = 0$, and $\lim_{Y\to R} V'(Y) = 0$.

can therefore choose $A(0) = B(0) = 1$ as initial conditions later in our numerical treatment. We see that the kinetic terms, e.g. the $\dot{A}^2$ and $\dot{B}^2$ terms, have different signs, which could lead to unstable behavior in the solutions.

6.3.3 Examples

A potential that does not satisfy the condition of having a small derivative near $Y = R$, is a parabolic potential

$$V(Y) = M_5^2 Y(Y - R). \tag{6.50}$$

Although this potential does not not satisfy the condition mentioned above, we will use it in numerical simulations to show that certain qualitative properties of the solutions are independent of the detailed for of the potential.

A more realistic potential that satisfies all the conditions is

$$V(Y) = -k_1 \left(e^{-k_2 Y} - e^{-k_2 R}\right)\left(1 - \frac{1}{k_3 y/R + 1}\right), \tag{6.51}$$

which for values $R = 1/M_5$, $k_1 = 1$, $k_2 = 10/R$ and $k_3 = 10$ is plotted in figure 6.1. We call this the "exponential" potential because it is exponential at large y, but not for small y.

6.3.4 Scalar field in potential well in contracting space-time

In this section we illustrate the fact that in a contracting space-time, fields gain extra kinetic energy from the gravitational field. This is just the gravitational Doppler effect that in the case of photons would blue-shift them by shrinking their wavelength as the space contracts. Let the metric be of Robertson-Walker type,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2. \tag{6.52}$$

The equation of motion for a space independent scalar field ϕ in this curved space with scale factor $a(t)$ is

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0. \tag{6.53}$$

Let us assume that the scalar field's energy density is small, so we don't have to worry about it's back-reaction on the geometry. This would correspond to small potential and low velocities. Also assume that the scale factor is given by $a(t) = e^{-t}$ (giving $\dot{a}/a = -1$), and $V(\phi) = \phi(\phi - 1)$. Note that an infinitesimal comoving spatial volume element d^3x has physical volume $e^{-3t}d^3x$, which is decreasing with time. The equation of motion then becomes

$$\ddot{\phi} - 3\dot{\phi} + 2\phi = 1. \tag{6.54}$$

The homogeneous equation has solutions

$$\phi_H(t) = Ae^t + Be^{2t}. \tag{6.55}$$

We need to also find a particular solution; it is easy to see that $\phi_P(t) = 1/2$ fits the bill. Choosing initial conditions $\phi(0) = 1$ and $\phi'(0) = 0$, the full solution is

$$\phi(t) = e^t - \frac{1}{2}e^{2t} + \frac{1}{2}. \tag{6.56}$$

We see that this solution will decrease exponentially towards minus infinity. We can see in figure (6.2) that the field falls down the potential and rises indefinitely up the potential on the negative side. This is in contrast to flat space mechanics, where energy conservation would constrain $\phi(t)$ to the region $\phi(t) \in [-1, 1]$. The energy conservation law is different in curved space, an extra contribution coming from the field's interaction with the gravitational field. This interaction is unavoidable, since gravitation interacts with all fields having non-zero energy. Integrating equation (6.53), we get the modified conservation of energy equation

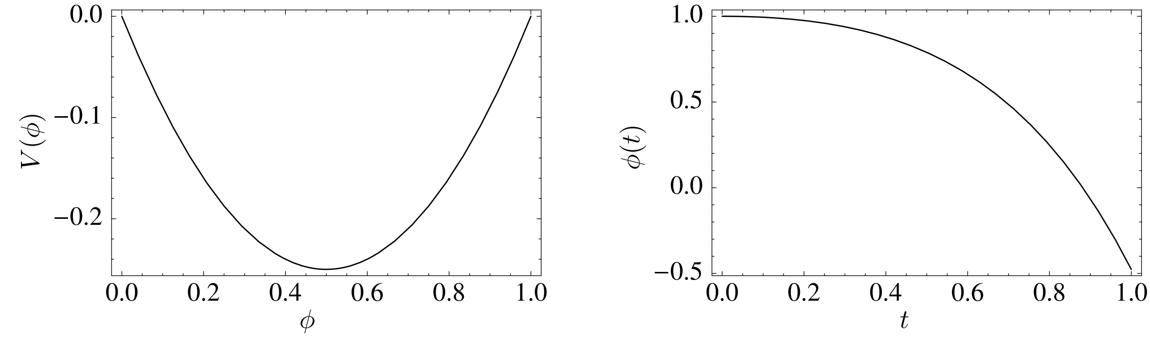
$$\frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) = -3\frac{\dot{a}}{a}\dot{\phi}^2, \tag{6.57}$$

from which we see that the coupling to gravitation contributes to the total energy of the field. It gives a positive contribution when $\dot{a} < 0$, so we see that the field gains energy when moving in the direction of decreasing spatial volume.

6.4 Field equations for the ekpyrotic universe

In this section we calculate the field equations for N, A, B, C and Y from the moduli space Lagrangian in equation (6.49). We keep in mind that the moduli space approximation is

Figure 6.2: Scalar field in contracting space-time



(a) Parabolic potential for $\phi(t)$. In Minkowski space it always leads to oscillatory behaviour, but not neccesarily in a contracting space-time.

(b) The scalar field can move past the point $\phi = 0$, even though the potential is higher there than where the scalar field started it's movement.

only valid for almost static systems, although we don't implement this mathematically in the equations by Taylor expanding them in some velocity-related small parameter. The functions' dependency on time has been suppressed to simplify the notation.

6.4.1 Gauge fixing

Since the functions N, A and B are scale-factors in the metric, they are modified during a change of coordinates. This is a gauge symmetry of the theory. Since N only depends on time, we can assume that we have chosen a time coordinate so that $N(t) = 1$.

6.4.2 Equation of motion for $N(t)$

The equation of motion for $N(t)$ is the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}_{mod}}{\partial \dot{N}}\right) - \frac{\partial \mathcal{L}_{mod}}{\partial N} = 0. \tag{6.58}$$

The action contains no time derivatives of $N(t)$, so the first term vanishes. From (6.49), we get

$$\begin{aligned} &\frac{3M_5^3 A^3 B}{N^2}\Bigg[I_3\left(\frac{\dot{A}}{A}\right)^2 + \frac{\dot{A}}{A}\left(I_3\frac{\dot{B}}{B} + 3I_2\dot{C} + 3\beta I_{2b}\dot{Y}\right) - \frac{1}{12}I_3\left(\frac{\dot{B}}{B}\right)^2 \\ &+\frac{1}{2}I_1\dot{C}^2 + \beta I_{1b}\dot{Y}\left(\dot{C} + \frac{\beta}{2}\dot{Y}\right) - \beta\left(\frac{1}{2}D^2\dot{Y}^2 + N^2V(Y)\right)\Bigg] = 0 \end{aligned} \tag{6.59}$$

Choosing the gauge $N(t) = 1$ and eliminating by the prefactor common to all terms, this becomes

$$\begin{aligned} & I_3 \left(\frac{\dot{A}}{A}\right)^2 + \frac{\dot{A}}{A}\left(I_3\frac{\dot{B}}{B} + 3I_2\dot{C} + 3\beta I_{2b}\dot{Y}\right) - \frac{1}{12}I_3\left(\frac{\dot{B}}{B}\right)^2 \\ & + \frac{1}{2}I_1\dot{C}^2 + \beta I_{1b}\dot{Y}\left(\dot{C} + \frac{\beta}{2}\dot{Y}\right) - \beta\left(\frac{1}{2}D^2\dot{Y}^2 + N^2V(Y)\right) = 0 \end{aligned} \tag{6.60}$$

Since this equation comes from varying the time component of the metric in the action, this corresponds to the usual first Friedmann equation in cosmology. As the first Friedmann equation, this one is also of first order in time-derivatives.

6.4.3 Equation of motion for $A(t)$

In the gauge $N(t) = 1$, the equation of motion for $A(t)$ is

$$\begin{aligned} & \frac{d}{dt}\left[-A^2B\left(2I_3\frac{\dot{A}}{A} + I_3\frac{\dot{B}}{B} + 3I_2\dot{C} + 3\beta I_{2b}\dot{Y}\right)\right] + A^2B\Bigg[I_3\left(\frac{\dot{A}}{A}\right)^2 \\ & + 2\frac{\dot{A}}{A}\left(I_3\frac{\dot{B}}{B} + 3I_2\dot{C} + 3\beta I_{2b}\dot{Y}\right) + 3\Bigg[-\frac{1}{12}I_3\left(\frac{\dot{B}}{B}\right)^2 \\ & + \frac{1}{2}I_1\dot{C}^2 + \beta I_{1b}\dot{Y}\left(\dot{C} + \frac{\beta}{2}\dot{Y}\right) - \beta\left(\frac{1}{2}D^2\dot{Y}^2 - V(Y)\right)\Bigg]\Bigg] = 0 \end{aligned} \tag{6.61}$$

This equation corresponds to the Friedmann equation for the scale factor in the spatial directions tangent to the branes. It is a second order differential equation, just as the ordinary second Friedmann equation.

6.4.4 Equation of motion for $B(t)$

In the same gauge, the equation of motion for $B(t)$ is

$$\begin{aligned} & \frac{d}{dt}\left[-A^3I_3\left(\frac{\dot{A}}{A} + \frac{1}{6}\frac{\dot{B}}{B}\right)\right] + A^3\Bigg[I_3\left(\frac{\dot{A}}{A}\right)^2 + \frac{\dot{A}}{A}\left(3I_2\dot{C} + 3\beta I_{2b}\dot{Y}\right) \\ & + \frac{1}{12}I_3\left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{2}I_1\dot{C}^2 + \beta I_{1b}\dot{Y}\left(\dot{C} + \frac{\beta}{2}\dot{Y}\right) - \beta\left(\frac{1}{2}D^2\dot{Y}^2 - V(Y)\right)\Bigg] = 0 \end{aligned} \tag{6.62}$$

6.4.5 Equation of motion for $C(t)$

This case is complicated because the D and the I_m functions (6.21) and (6.22) depend on $C(t)$ and $Y(t)$. But we have calculated the relevant quantities in equations (6.25), (6.28) and (6.31). Using these results, we get

$$\frac{d}{dt}\left[-A^3B\left(3I_2\frac{\dot{A}}{A}+I_1\dot{C}+\beta I_{1b}\dot{Y}\right)\right]+A^3B\left[3I_2\left(\left(\frac{\dot{A}}{A}\right)^2+\frac{\dot{A}}{A}\frac{\dot{B}}{B}-\frac{1}{12}\left(\frac{\dot{B}}{B}\right)^2\right)\right.$$
$$\left.+6\frac{\dot{A}}{A}\left(I_1\dot{C}+\beta I_{1b}\dot{Y}\right)+\frac{1}{2}I_0\dot{C}^2+\beta I_{0b}\dot{Y}\left(\dot{C}+\frac{\beta}{2}\dot{Y}\right)-\beta D\dot{Y}^2\right]=0. \tag{6.63}$$

This equation can be interpreted as coming from the equation of motion for ϕ in our original variables. It is of second order.

6.4.6 Equation of motion for $Y(t)$

Using equations (6.25), (6.29) and (6.31), we get

$$\frac{d}{dt}\left[-A^3B\left(3I_{2,b}\frac{\dot{A}}{A}+I_{1,b}\left(\dot{C}+\beta\dot{Y}\right)-D^2\dot{Y}\right)\right]+A^3B\left[3I_{2,b}\left\{\left(\frac{\dot{A}}{A}\right)^2\right.\right.$$
$$\left.+\frac{\dot{A}}{A}\frac{\dot{B}}{B}-\frac{1}{12}\left(\frac{\dot{B}}{B}\right)^2\right\}+6\frac{\dot{A}}{A}\left(I_{1,b}\dot{C}+\left(\beta I_{1,b}-D^2\right)\dot{Y}\right)+\left(R-Y\right)\dot{C}^2$$
$$\left.+2\left(\beta\left(R-Y\right)-D\right)\dot{Y}\left(\dot{C}+\frac{\beta}{2}\dot{Y}\right)-\left(\alpha D\dot{Y}^2-V'(Y)\right)\right]=0 \tag{6.64}$$

This is the equation of motion for the bulk brane position Y. For just that reason it contains the Y-derivative of the potential $V(Y)$, which is related to the force on the bulk brane from the bulk brane potential.

6.5 Numerical results

6.5.1 Free parameters and initial conditions

All coordinates can be rescaled as we wish, so we can adjust the initial value for the functions $A(t)$ and $B(t)$. We will choose $A(0) = B(0) = 1$. Since the bulk brane is supposed to start at the hidden brane, we must by construction choose $Y(0) = R$. The system must begin with a static configuration, corresponding to the limit $t \to -\infty$. Thus we use $A'(0) = B'(0) = C'(0) = Y'(0) = 0$. In addition, we can choose the mass scale so that $R = 1/M_5$, but we will keep R explicitly in our expressions. Thus the remaining degrees of freedom α, β, $C(0)$ and the potential correspond to different physical systems. To have a well-defined metric in the entire bulk space, the function $D(y)$ must be positive everywhere in $y \in [0, R]$ for all values of the functions $C(t)$ and $Y(t)$ in the solution of the equations of motion, otherwise the metric changes its signature, as can be seen from the definition (6.8). So we must keep this in mind when choosing $C(0)$.

6.5.2 Negative tension on the visible brane and a parabolic potential

We choose $R = 1/M_5$, $\alpha = 100M_5$, $\beta = 1/1000 \times M_5$ and $C(0) = 200$. The potential is chosen to be parabolic,

$$V(Y) = \frac{M_5^2}{10} Y(Y - R). \tag{6.65}$$

Using the program in appendix A, we calculate the equations of motion, and solve those equations numerically from $t = 0$ to $t = 8000/M_5$. The solutions are given in figure 6.3.

We notice that while the bulk brane position varies wildly between 0 and R, the variations in the three other unknown functions are small. This follows from the smallness of the bulk brane tension compared to the boundary brane tensions. Since the bulk brane has little energy, it does not influence the functions in the metric much.

In the course of the evolution, the scale factor $A(t)$ increases monotonically, as do $B(t)$. $C(t)$, however, decreases monotonically. These three functions show no oscillatory tendencies, as opposed to $Y(t)$, which moves back and forth in the bulk space during the same time interval, without hitting the visible brane, at $y = 0$. It also does not bump into the hidden brane at $y = R$ at the local maxima on the graph of its movement. This means that in this configuration, with these initial values for the unknown functions, there is no ekpyrosis (bulk brane - visible brane crash).

From the figure for $Y(t)$, it seems that the bulk brane will continue to oscillate back and forth in the bulk space, but the numeric solution actually breaks down around $t = 235000/M_5$. The oscillations are actually damped, but this is only apparent when β is larger so the local minima of the bulk brane position $Y(t)$ increases with increasing time, and the local maxima decreases with increasing time.

Analysis of the numeric solution (see appendix A) shows that the bulk brane almost but not quite hits the visible brane. The minimum value of the bulk brane y coordinate in this example is $y \approx 1.32 \times 10^{-5}/M_5$ at $t \approx 1.75 \times 10^3/M_5$ on its first visit to the neighborhood of the visible brane.

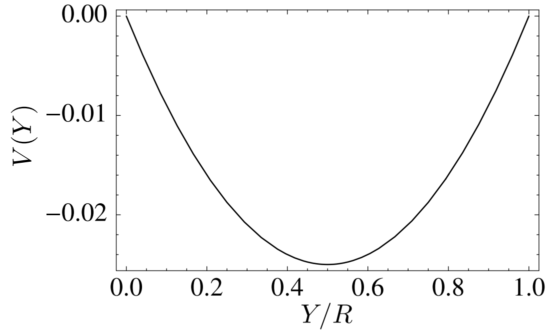
6.5.3 Comparing with the results in [8]

In [8], the authors choose the functions $B(t)$ and $C(t)$ to be constant to simplify the problem, and make it susceptible to an analytic approach, and obtain closed expressions for the unknown functions $A(t)$ and $Y(t)$. This choice is in conflict with the equations of motion, and an approximation to justify this choice was not introduced.

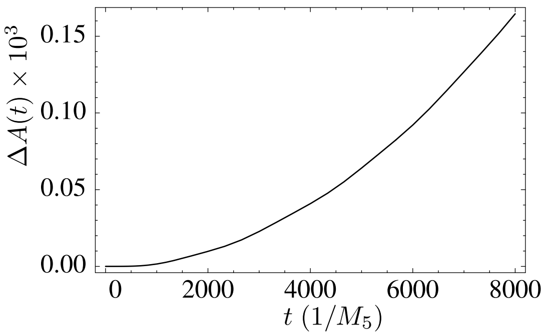
In this section, we will show that the solution of the model with $B(t)$ and $C(t)$ constant differs from the full moduli space approximation in that there exists choices of parameters and potentials that generates ekpyrosis in this model, but not in the full model.

Using the same parameters as in the previous section, we now demonstrate that this model does indeed lead to a bulk brane - visible brane crash. The solution is plotted in figure 6.4.

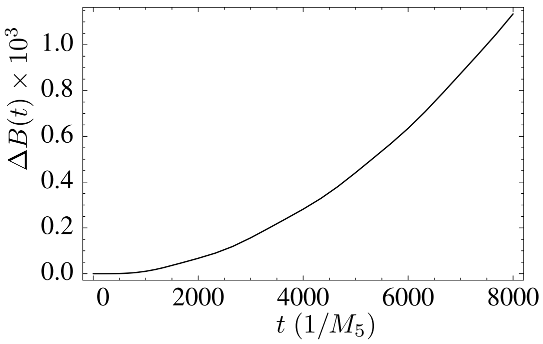
Figure 6.3: Negative tension, parabolic potential. Parameters $\alpha = 100M_5$, $\beta = 1/1000 \times M_5$, and $C(0) = 200$.



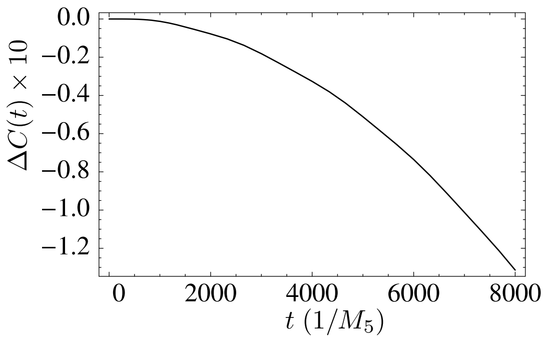
(a) A parabolic bulk brane potential.



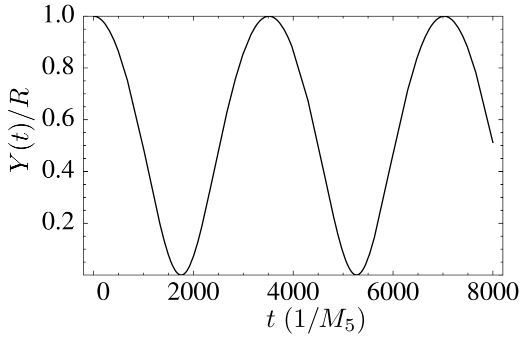
(b) $\Delta A := A(t) - 1$. $A(t)$ increases slowly during the evolution of the system.



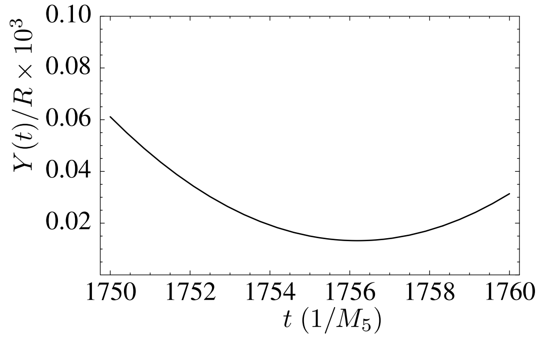
(c) $\Delta B := B(t) - 1$. $B(t)$ also increases as time passes.



(d) $\Delta C := C - 200$. $C(t)$ decreases during the evolution.

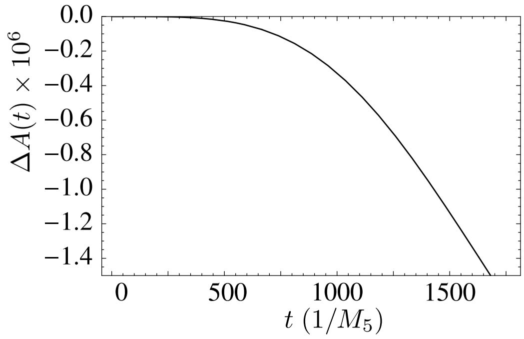


(e) $Y(t)$. The bulk brane oscillates back and forth in the bulk space without hitting the visible brane.

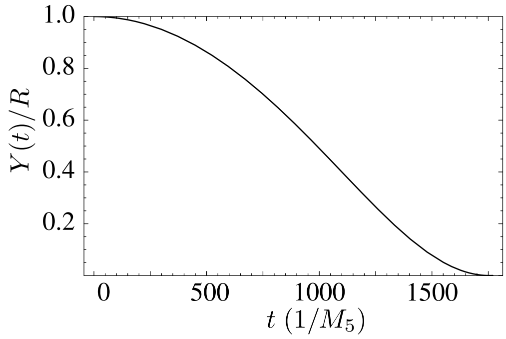


(f) Bulk brane movement near it's first encounter with the visible brane.

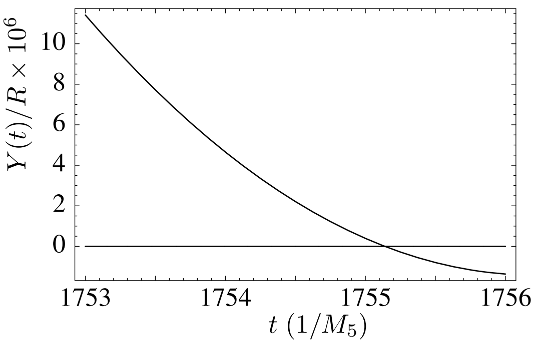
Figure 6.4: Numerical results with parabolic potential and constant $B(t)$ and $C(t)$. Parameter values $\alpha = 100M_5$, $\beta = 1/1000 \times M_5$.



(a) $\Delta A := (A(t) - 1)$. Since β is very small, $A(t)$ varies very slowly.



(b) $Y(t)$ falls steadily towards the visible brane.



(c) Bulk brane movement near impact. At impact the bulk brane is absorbed into the visible brane.

The figure shows that the bulk brane in this model actually hits the visible brane. Numerical analysis of the solution shows that ekpyrosis occurs at $t \approx 1.75 \times 10^3/M_5$ with an impact speed of $Y'(t) \approx -2.66 \times 10^{-6}$. But it also shows a decreasing scale factor $A(t)$, just the opposite behavior from the full model with non-constant $B(t)$ and $C(t)$. So this model gives an very erroneous description of the ekpyrotic universe.

To show that our numerical solution of this model agrees with the analytic expression for small β given in [8], we compare the values obtained for the conformal Hubble factor H_c at impact in each case. In [8], equation (27) gives the following expression for this quantity

$$H_c := \left|\frac{\dot{a}}{a^2}\right| = \frac{3\beta M_5}{\sqrt{2B}(I_3 M_5)^{3/2}} \int_{Y=0}^{R} D(Y)\sqrt{-V(Y)}dY \tag{6.66}$$

valid in the small β limit. $a(t)$ in this expression is a modified scale factor defined as

$$a(t) = A(t)\sqrt{BI_3M_5} \tag{6.67}$$

which in the constant B and C model turns out to be a scale factor that simplifies the equations of motion for $Y(t)$. Inserting our chosen values $R = 1/M_5$, $\alpha = 100M_5$, $\beta = 1/1000 \times M_5$ and $C(0) = 200$, and the potential (6.65), we calculate this to be

$$\begin{aligned} H_c &= \frac{3/1000 \times M_5}{\sqrt{2}\sqrt{10}(32500000)^{3/2}} \int_{Y=0}^{1} (100Y + 200)\sqrt{-Y(Y-1)}dY \\ &\approx 3.554 \times 10^{-12} M_5 \approx 3.6 \times 10^{-12} M_5 \end{aligned} \tag{6.68}$$

Our numerical result is

$$H_c^{num} \approx 3.585 \times 10^{-12} M_5 \approx 3.6 \times 10^{-12} M_5 \tag{6.69}$$

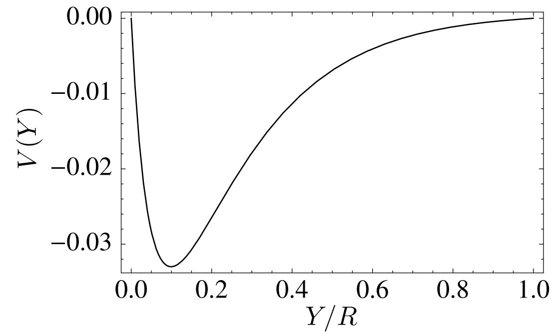
According to our numerical simulations, H_c^{num} converges towards H_c in the limit where $\beta \to 0$. This is to be expected since (6.66) is only valid in this limit.

6.5.4 Negative tension on the visible brane and an "exponential" potential

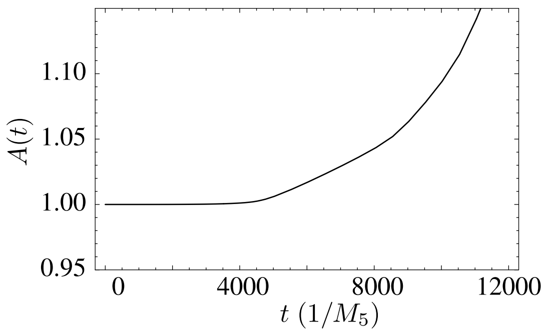
To make a more realistic model, we try a different potential that allows for a bulk brane motion that starts out more softly. We make sure the derivative of the potential at $y = R$, i.e. the acceleration of the bulk brane at $t = 0$, is very small. Since our model works for any value of β, we use a larger value than in the previous example, to get bigger variations in the unknown functions. We choose the parameters $R = 1/M_5$, $\alpha = 100M_5$, $\beta = M_5$, $B(0) = 1$, $C(0) = 200$. As a potential, we use the following, which is of the same form as in equation (6.51).

$$V(Y) = -\frac{1}{10}(e^{-5Y/R} - e^{-5})(1 - \frac{1}{10Y/R + 1})(\frac{10+1}{10}) \tag{6.70}$$

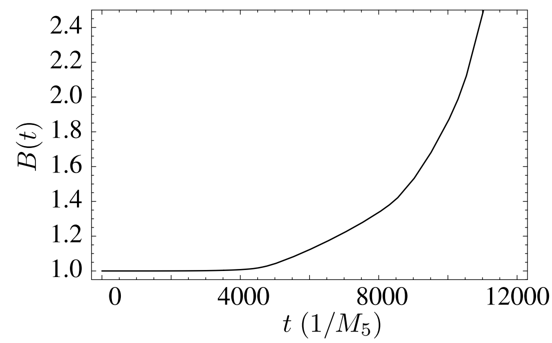
Figure 6.5: Negative tension and "exponential" potential



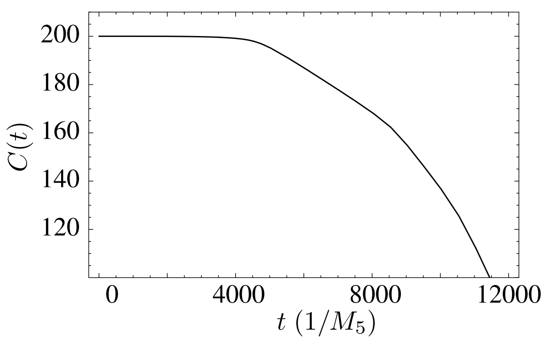
(a) Potential. It is exponential at large brane separations, and disappears at small separations.



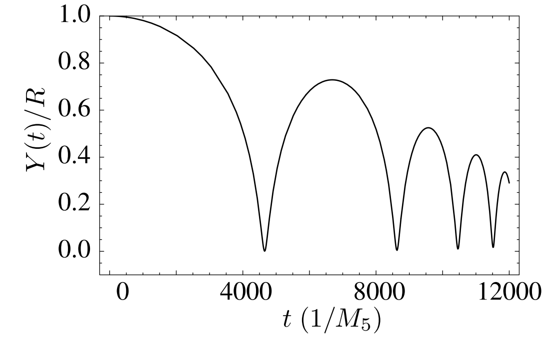
(b) $A(t)$ climbs as the bulk brane gets close to the visible brane.



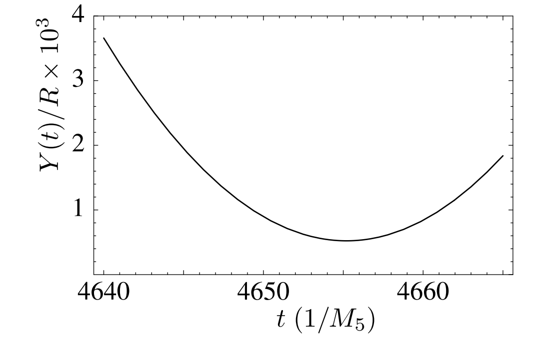
(c) $B(t)$ climbs as the bulk brane gets close to the visible brane.



(d) $C(t)$ falls significantly after the bulk brane's first visit near the visible brane.



(e) Bulk brane position. It bounces back and forth in the bulk space.



(f) Bulk brane position near it's first encounter with the visible brane. It almost crashes with the visible brane.

The results are given in figure 6.5.

Since β is much larger now, the functions vary much more. We get qualitatively the same kind of motion as with the parabolic potential. The $A(t)$, $B(t)$ and $C(t)$ still vary monotonically, as before. But in this case, we notice that the functions $A(t)$, $B(t)$ and $C(t)$ seem to demonstrate a similarity to somewhat smoothed out stepwise linear functions. The second derivatives of these functions are largest when the bulk brane are near the visible brane, and the bulk brane movement has a large second derivative. The bulk brane still oscillates, but no longer moves in a sinusoidal way, but like a damped cycloidal motion with decreasing wavelength. This wavelength decreases until the solution breaks down at around zero wavelength, which is not shown in the plot. Analyzing the local minima of the solution, we find that there is no ekpyrosis in this case either. An example is seen in the last plot, where the bulk brane comes very close, but turns back into the bulk space.

6.5.5 Positive tension on the visible brane and parabolic potential

We use the parabolic potential in equation (6.65), and the same initial conditions as before. The potential and the numerical results can be seen in figure 6.6.

The plots show the same type of behavior as the other examples, the typical damped oscillatory movement of the bulk brane without hitting the visible brane. A striking difference is how the y values of the local minima in the graph of $Y(t)$ increases significantly, so the bulk brane converges towards the middle of the bulk space. This follows of course from the parabolic potential being symmetric about its minimum at $y = R/2$. It would also have been seen in section 6.5.2, had not β been so small in that example. A continuous kink is seen in the solution at around $t \approx 3000/M_5$ which is connected to the fact that the bulk brane movement at this point displays a large negative second derivative. There is no ekpyrosis in this case either.

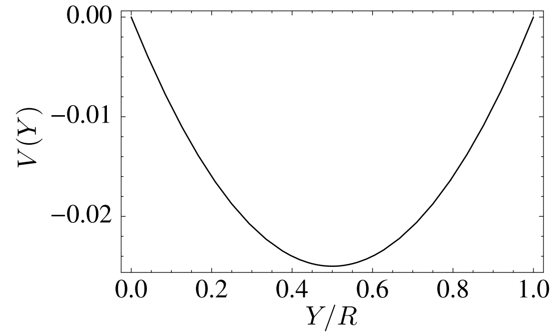
6.5.6 Positive tension on the visible brane and "exponential" potential

The last example is with positive brane tension and our "exponential" potential (6.51). The initial conditions are the same as in the previous examples. Figure 6.7 shows the results graphically, and they do not seem to provide us with any interesting new behavior. This is the most realistic case we have considered up to now, since the visible brane tension is positive, and the potential satisfies the conditions given in section 6.3.

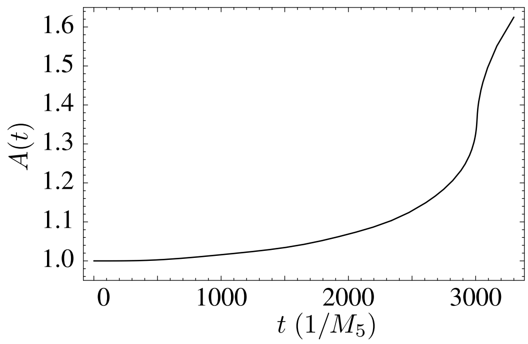
6.5.7 Many different parameter values

To show that these types of behavior are typical, we find the solutions for the bulk brane position for many different values of the parameters α, β and $C(0)$. The solutions for $\alpha \in \{-100, -80, -60, -40, -20\} \times M_5$, $\beta \in \{1, 5, 9, 13, 17\} \times M_5$ and $C(0) \in \{120, 140, 160, 180, 200\}$ are plotted in figure 6.8.

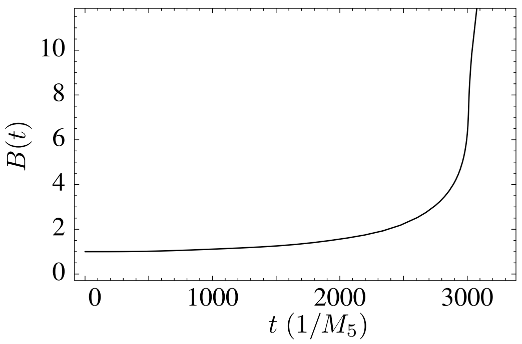
Figure 6.6: Positive tension and parabolic potential



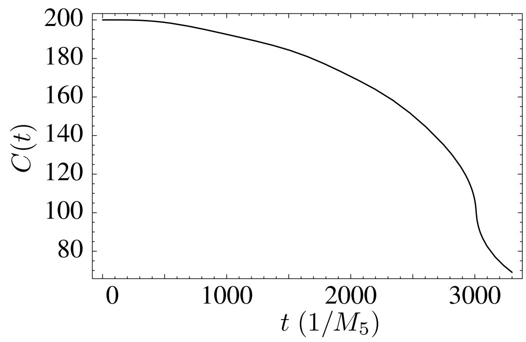
(a) A simple parabolic potential.



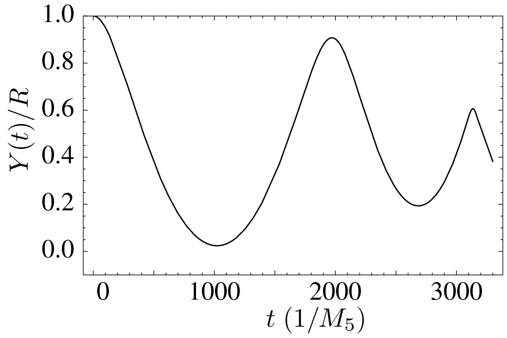
(b) $A(t)$ increases monotically.



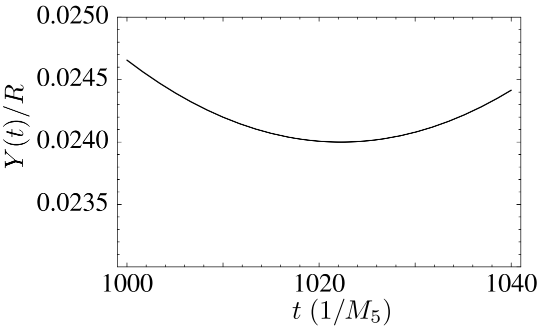
(c) $B(t)$ increases monotically.



(d) $C(t)$ decreases monotically.

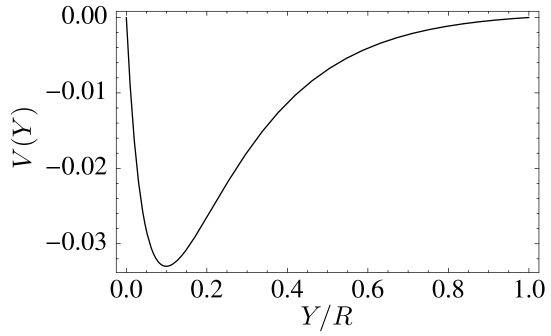


(e) $Y(t)$ oscillates without hitting the visible brane.

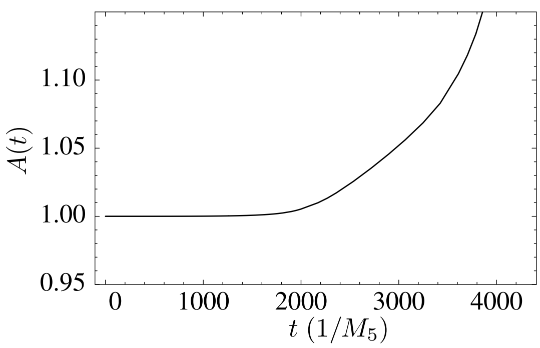


(f) $Y(t)$, near first turnaround point.

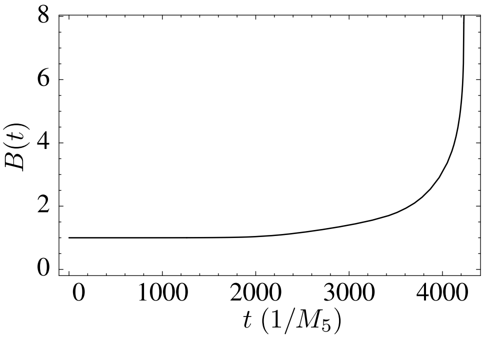
Figure 6.7: Positive tension and "exponential" potential



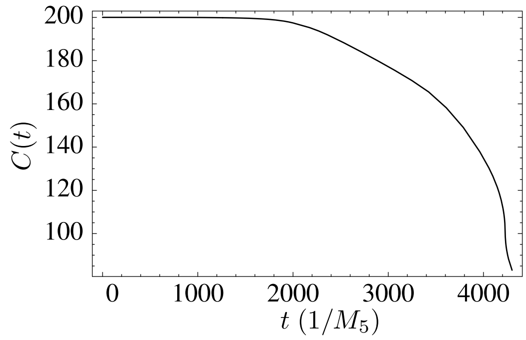
(a) "Exponential" potential.



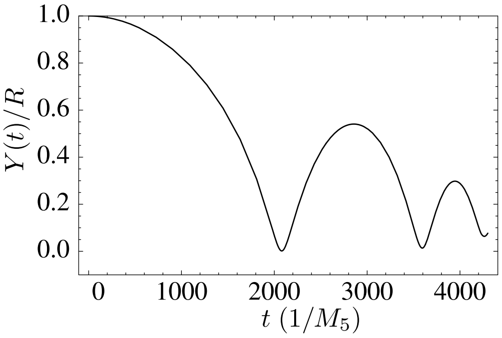
(b) $A(t)$ increases.



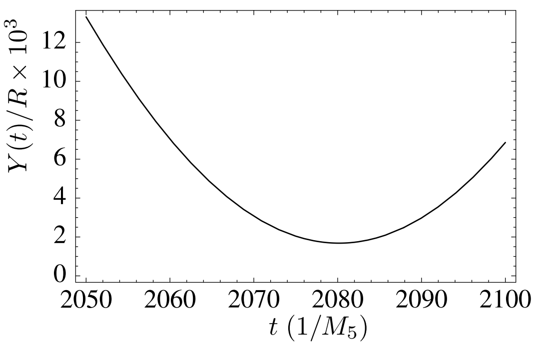
(c) $B(t)$ increases.



(d) $C(t)$ decreases.

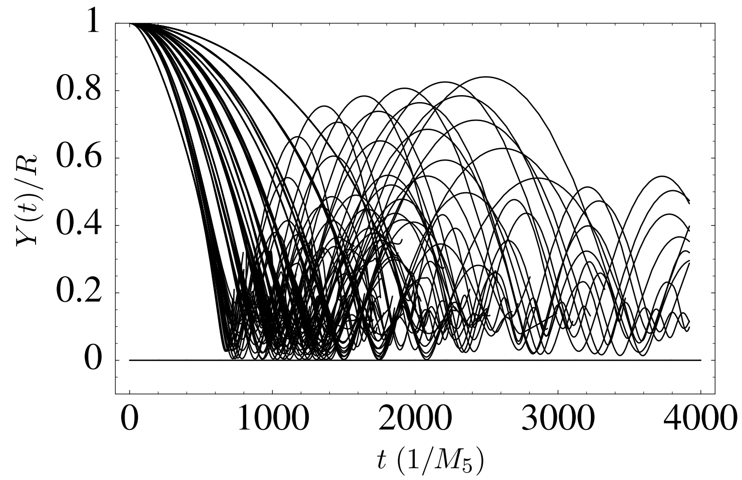


(e) $Y(t)$ with the usual damped oscillations.

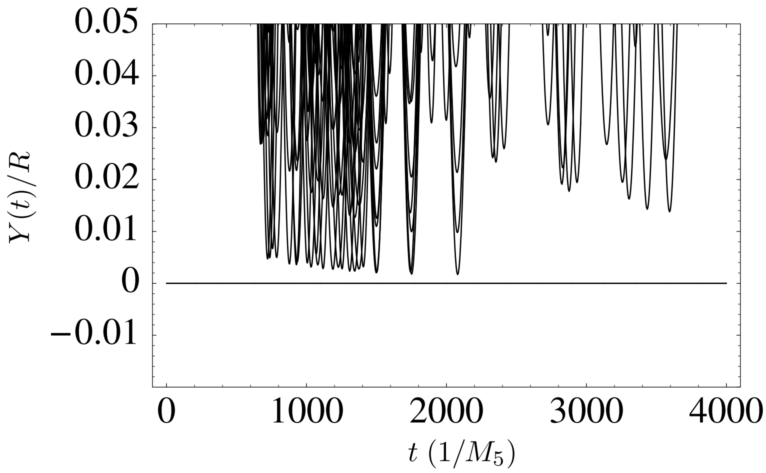


(f) $Y(t)$, near first turnaround point. It does not bump into the visible brane.

Figure 6.8: $Y(t)$ for 125 different parameter values, all combinations of $\alpha \in \{-100, -80, -60, -40, -20\} \times M_5$, $\beta \in \{1, 5, 9, 13, 17\} \times M_5$ and $C(0) \in \{120, 140, 160, 180, 200\}$.



(a) All solutions bounce away from the visible universe. Note that most of the solutions stop before $t \approx 4000$, since they become singular before that point.



(b) The same plot, but for $-0.02 > y/R > 0.05$. None of the solutions hit the visible universe at $y = 0$.

6.5.8 Physical length of the orbifold dimension

From the numeric results for $B(t)$, $C(t)$ and $Y(t)$, we now also know $b(t,y) = B(t)D(y)^2$, since $D(y)$ is a function of y, $C(t)$ and $Y(t)$. Since $b(t,y)$ is the scale factor in the orbifold direction, we can calculate the physical length of this dimension. We get

$$\begin{aligned}\text{Length} &= \int_0^R dy b(t,y) = B(t)\int_0^R dy D(y)^2 = \frac{1}{2}B(t)I_2 \\ &= \frac{1}{3}B(t)\left[\frac{1}{\alpha}\left\{(\alpha Y(t)+C(t))^3 - C(t)^3\right\}\right. \\ &\quad \left.+\frac{1}{(\alpha-\beta)}\left\{((\alpha-\beta)R + C(t) + \beta Y(t))^3 - (\alpha Y(t)+C(t))^3\right\}\right], \quad (6.71)\end{aligned}$$

where we have used equations (6.21), (6.22) and (6.23). Plots of the numerical results for the physical length of the orbifold dimension are given in figures 6.9 and 6.10.

They show that the physical length increases with time in all the four cases we have treated. Qualitatively, the motion is alike in all four cases, being rather like a smoothed out stepwise linear curve. The curves in the cases with exponential potentials have small dips in the values at the points in time corresponding the the bulk brane changing direction rapidly. These are of course also the times at which there are local minima on the graph of the bulk brane movement.

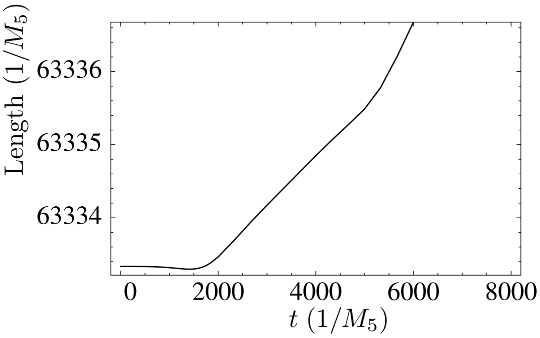
Calculations based on the model with a constant $B(t)$ and $C(t)$ gave opposite results (decreasing length) when the visible brane tension was negative, and this was partly the cause for introducing the cyclic model in [23]. Our numerical solution for the physical length in this model is given in figure 6.9, and is consistent with this result.

There have been a lot of discussion in the literature about the nature of the singularity that appears if the orbifold dimension had contracted away to nothing. In [33] it is shown even the presence of only one particle in such a universe would trigger a collapse of the entire universe. The singular space obtained when the orbifold dimension length is zero, is equivalent to an orbifold of space-time with an arbitrary simple Lorentz boost, and the gravitational energy of one particle in this space-time is then the same as the total gravitational energy of this particle together with all its images under the orbifold identification. This means that a calculation of the energy density would need to include the energies of all these mirror particles, and this is large enough to create an all enveloping black hole! Many had thought that these orbifold singularities were less severe than the black hole singularities in general relativity, but alas, this is apparently not the case. Our numerical simulations show that in the cases we have treated, the physical length of the orbifold dimension increases, so this will not be a concern here.

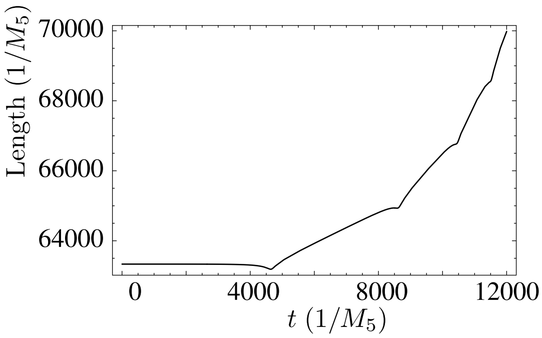
6.5.9 The visible brane scale factor

From the moduli space metric (6.9), we see that since we obtain the metric in the visible brane universe by just dropping the dy^2 term in this expression, we get an expression of

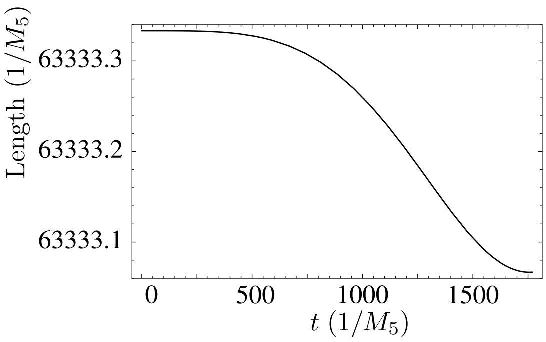
Figure 6.9: Physical length of the orbifold dimension, with negative tension. They correspond to the examples in sections 6.5.2, 6.5.4 and 6.5.3.



(a) Negative tension, parabolic potential. The physical orbifold length increases with time.

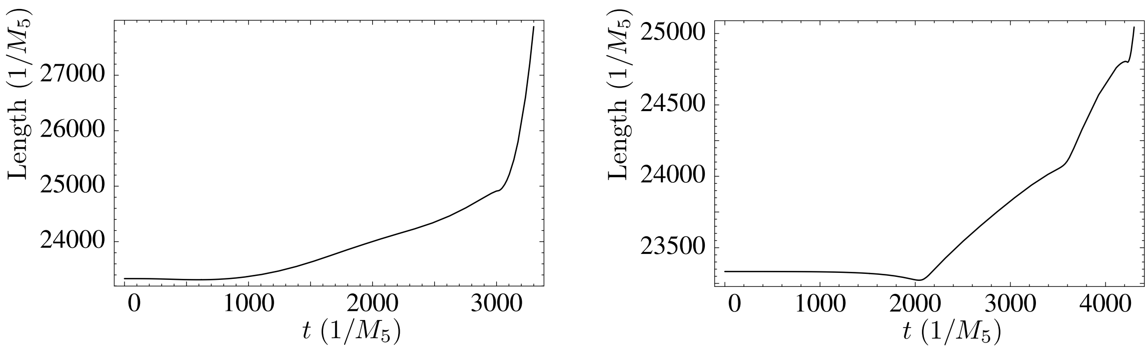


(b) Negative tension, "exponential" potential. The physical length grows.



(c) Model with constant $B(t)$ and $C(t)$, the orbifold dimension shrinks. Negative tension on the visible brane, and parabolic potential.

Figure 6.10: Physical length of the orbifold dimension, with positive tension, corresponding to the examples in sections 6.5.5 and 6.5.6



(a) Positive tension, parabolic potential.

(b) Positive tension, "exponential" potential.

the four-dimensional scale factor $a_{eff}(t)$ as a function of the five-dimensional quantities. The result is

$$a_{eff}(t) = \sqrt{D(0)}A(t) = \sqrt{C(t)}A(t) \tag{6.72}$$

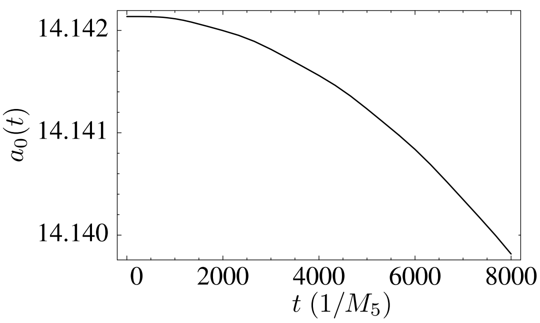
From our numerical solutions we know that $A(t)$ increases and $C(t)$ decreases. $C(t)$ decreases more strongly than $A(t)$ increases, but there is a square root around $C(t)$ in (6.72), so it is not obvious whether the visible brane scale factor increases or decreases during the evolution before a possible ekpyrosis. The visible scale factor is plotted in figures 6.11 and 6.12.

In all cases, the effective four-dimensional scale factor decreases with time. To obtain realistic cosmology after an eventual ekpyrosis, this behavior would have to be reversed. As an aside, the model with constant $B(t)$ and $C(t)$ also gives a contracting effective scale factor.

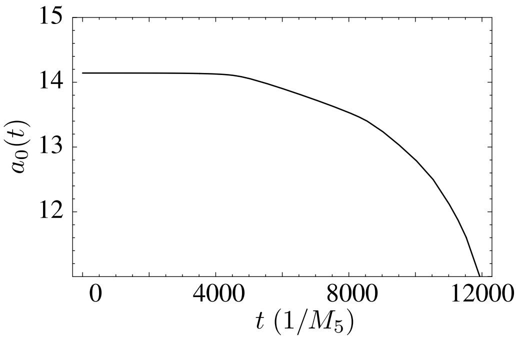
6.6 Stability

When a higher dimensional theory is compactified or a brane-world scenario is introduced to lower the observed number of dimensions, the fields in the effective four-dimensional theory describing the geometric data in the higher dimensional theory are called moduli fields. These could also be degrees of freedom associated to other bulk degrees of freedom. Changes in these moduli degrees of freedom have significant observable effects in the effective theory, e.g. as variations in coupling constants like the Newtonian gravitational constant. From equation (6.44) we see that the four-dimensional Planck mass depends on the values of moduli fields. These effects are highly constrained by experiments, so the

Figure 6.11: Visible brane scale factors, negative tension. They correspond to sections 6.5.2 and 6.5.4. It decreases in both cases.

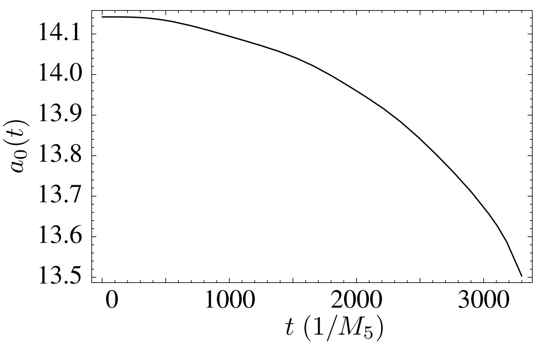


(a) The visible universe contracts as the bulk brane moves across the bulk.

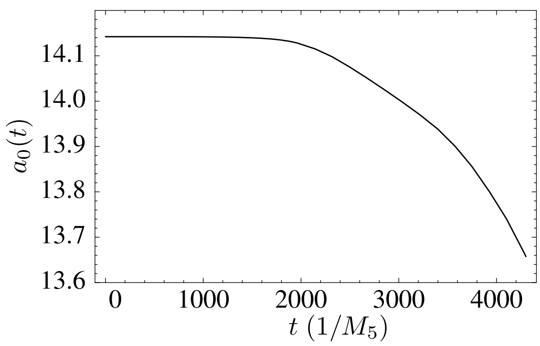


(b) The visible universe contracts like a smoothed out stepwise linear function.

Figure 6.12: Visible brane scale factors, positive tension. They correspond to sections 6.5.5 and 6.5.6. It decreases in both cases.



(a) The visible universe contracts.



(b) The visible universe contracts like a smoothed out stepwise linear function.

values of the moduli fields should be stabilized to some constant values, at least after the ekpyrosis.

From all the solutions we have seen, we notice that the functions A, B and C show no sign of stability. Physically, this instability can be seen in the behavior of the physical orbifold length in figures 6.9 and 6.10. After an eventual ekpyrosis in a modified version of the ekpyrotic universe, these moduli fields must be stabilized after ekpyrosis to not conflict with observations, so a post-ekpyrosis model would have to include either some extra moduli field potentials, or some dynamical stabilization mechanism to solve this problem.

6.7 Dimensional bounce scenario

We will now find a numerical solution for the dimensional bounce version of the ekpyrotic scenario. In this version, there is no bulk brane, and ekpyrosis is supposed to happen when the two boundary branes hit each other. Therefore we will look at what happens when we send the boundary branes towards each other at some speed. This we do by giving the orbifold scale factor B a finite negative derivative is an initial condition. First, we need to obtain the action describing this problem, by dropping terms from the action we used for the bulk brane scenario.

6.8 Action with no bulk brane

We can obtain the moduli space Lagrangian for the case with no bulk brane by letting $Y \to R$ and $\beta \to 0$ in the bulk space action (6.34). As before, the boundary branes does not contribute to the moduli space action. This leads to the Lagrangian

$$\mathcal{L} = -\frac{3M_5^3 A^3 B}{N}\left[I_3\left(\frac{\dot{A}}{A}\right)^2 + \frac{\dot{A}}{A}\left(I_3\left(\frac{\dot{B}}{B}\right) + 3I_2\dot{C}\right) - \frac{1}{12}I_3\left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{2}I_1\dot{C}^2\right], \quad (6.73)$$

where the integrals over the orbifold dimension I_m are now simpler than in the bulk brane case. We deduce them from the expressions (6.23) by letting $Y \to R$ and $\beta \to 0$. Since we take the limit $Y \to R$, the $I_{m,a}$ can be replaced by I_m, and the $I_{m,b}$ go to zero, since they are integrals from Y to R. The first few I_m functions are

$$I_0 = 2R \quad (6.74)$$

$$I_1 = \frac{1}{\alpha}\left((\alpha R + C)^2 - C^2\right) \quad (6.75)$$

$$I_2 = \frac{2}{3\alpha}\left((\alpha R + C)^3 - C^3\right) \quad (6.76)$$

$$I_3 = \frac{1}{2\alpha}\left((\alpha R + C)^4 - C^4\right). \quad (6.77)$$

6.8.1 Equation of motion for $N(t)$

The equation of motion for $N(t)$ is the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{N}}\right) - \frac{d\mathcal{L}}{dN} = 0 \tag{6.78}$$

The action contains no time derivatives of $N(t)$. From equation (6.73), we get

$$\frac{3M_5^3 A^3 B}{N^2}\left[I_3\left(\frac{\dot{A}}{A}\right)^2 + \frac{\dot{A}}{A}\left(I_3\frac{\dot{B}}{B} + 3I_2\dot{C}\right) - \frac{1}{12}I_3\left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{2}I_1\dot{C}^2\right] = 0 \tag{6.79}$$

We see that the only occurrence of $N(t)$ is in the prefactor common to all the terms. Dividing by this prefactor, we see that the equation of motion for $N(t)$ is independent of $N(t)$, so it is independent of our gauge choice $N(t) = 1$. The result is

$$I_3\left(\frac{\dot{A}}{A}\right)^2 + \frac{\dot{A}}{A}\left(I_3\frac{\dot{B}}{B} + 3I_2\dot{C}\right) - \frac{1}{12}I_3\left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{2}I_1\dot{C}^2 = 0 \tag{6.80}$$

6.8.2 Equation of motion for $A(t)$

In the gauge $N(t) = 1$, the equation of motion for $A(t)$ is

$$\begin{aligned}&\frac{d}{dt}\left[-A^2B\left(2I_3\frac{\dot{A}}{A} + I_3\frac{\dot{B}}{B} + 3I_2\dot{C}\right)\right]\\ &+A^2B\left[I_3\left(\frac{\dot{A}}{A}\right)^2 + 2\frac{\dot{A}}{A}\left(I_3\frac{\dot{B}}{B} + 3I_2\right) - \frac{1}{4}I_3\left(\frac{\dot{B}}{B}\right)^2 + \frac{3}{2}I_1\dot{C}^2\right] = 0\end{aligned} \tag{6.81}$$

6.8.3 Equation of motion for $B(t)$

The equation of motion for $B(t)$ in the same gauge is

$$\begin{aligned}&\frac{d}{dt}\left[-A^3I_3\left(\frac{\dot{A}}{A} + \frac{1}{6}\frac{\dot{B}}{B}\right)\right]\\ &+A^3\left[I_3\left(\frac{\dot{A}}{A}\right)^2 + 3I_2\frac{\dot{A}}{A}\dot{C} + \frac{1}{12}I_3\left(\frac{\dot{B}}{B}\right)^2 + \frac{1}{2}I_1\dot{C}^2\right] = 0\end{aligned} \tag{6.82}$$

6.8.4 Equation of motion for $C(t)$

This case is more complicated because the D and the I_m functions (6.21) and (6.22) depend on $C(t)$. Using equations (6.25), (6.28) and (6.31), we get

$$\frac{d}{dt}\left[-A^3B\left(3I_2\frac{\dot{A}}{A}+I_1\dot{C}\right)\right]$$
$$+A^3B\left[3I_2\left(\frac{\dot{A}}{A}\right)^2+3I_2\frac{\dot{A}}{A}\frac{\dot{B}}{B}-\frac{1}{4}I_2\left(\frac{\dot{B}}{B}\right)^2+6I_1\frac{\dot{A}}{A}\dot{C}+R\dot{C}^2\right]=0 \qquad (6.83)$$

6.8.5 Numerical solution

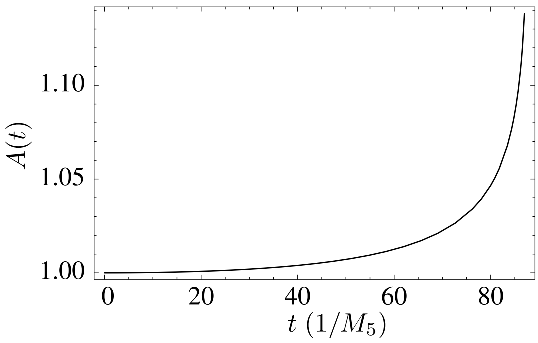
Here we solve the equations for the case of no bulk brane. These are the differential equations (6.81), (6.82) and (6.83) for the unknowns $A(t)$, $B(t)$ and $C(t)$. We must examine the behavior when we send the scale factor for the extra dimension towards zero, and see if it bounces back again or goes to zero, which would correspond to ekpyrosis. Therefore we use a negative time-derivative of the $B(t)$ function at $t = 0$. The equations are solved with Mathematica, using the program in appendix A. We have used parameter values $R = 1/M_5$, $\alpha = -100M_5$, $\beta = M_5$, and the initial conditions $A(0) = B(0) = 1$, $C(0) = 200$, $A'(0 = C'(0) = 0$, $B'(0) = -1/100M_5$.

Plots of the numerical solutions are given in figure 6.13. The plots show that $B(t)$ decreases during the evolution, but it suddenly turns around, so in this case there is no Big Bang on the visible brane. During the evolution the system, $A(t)$ increases and $C(t)$ decreases, their movements getting faster as $B(t)$ gets smaller. Even though the equations are only applicable when the fields change slowly, simulations with a large negative derivative for $B(t)$ at $t = 0$ also show that $B(t)$ don't reach zero, so the distance between the branes are always non-zero. This means that in the moduli space approximation, there is no ekpyrosis in the scenario with no bulk brane, without making some modifications to the action.

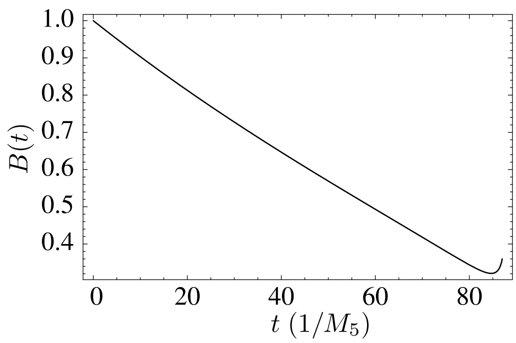
6.9 Brane matter

We end with a small note regarding the influence of brane matter. In our numerical work, we have used the action for the ekpyrotic universe as proposed originally, without allowing for matter/radiation creation on the branes during the pre-ekpyrosis evolution. We have argued that brane matter should be allowed for to satisfy local energy conservation in section 5.5. But it is clear that if brane matter is created during the bulk brane movement through e.g. brane interaction with the bulk scalar field, this would be at the expense of the bulk brane kinetic energy, so it would further prevent the possibility of a bulk brane collision with the visible brane. So we contend that this would not work in favor of the model.

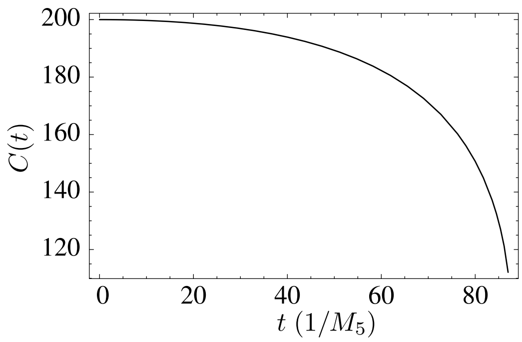
Figure 6.13: Dimensional bounce scenario. Evolution of $A(t)$, $B(t)$ and $C(t)$.



(a) $A(t)$ grows slowly during the evolution.



(b) $B(t)$ decreases almost lineary, but is suddenly pushed away from the visible brane.



(c) $C(t)$ falls slowly, but decreases rapidly towards the end of the solution.

Chapter 7

Conclusion

In this thesis, in broad lines, we have gone from the smallest building blocks in physics, the tiny strings of string theory, to a cosmological theory trying to explain the nature of the universe as a whole and the big bang phenomenon. The bridge between the two is the existence of D-branes in string theory. We have analyzed one specific string based cosmology called the ekpyrotic universe and obtained numerical results that speak against it viability.

In chapter 2, we derived the fundamental properties of bosonic string theory. The possibility of compact dimensions led to the existence of the T-duality transformation, which ultimately led to the existence of higher-dimensional objects called D-branes. We saw how the branes are described mathematically by the Dirac-Born-Infeld action, and were endowed with Yang-Mills gauge theory on their world-volume. Observing that similar branes exist in the super-symmetric string theories and M-theory, this made it possible to envisage the possibility of brane-worlds, where the entire visible universe is located on a 3-brane.

In chapter 3, we saw that the conventional cosmological model contains a "Big Bang" singularity. The mathematical problem associated with this singularity is one of the motivating factors of the ekpyrotic universe, which contains no such singularity. Subsequently, we mentioned the different cosmological problems, and described how the phenomenon of inflation can solve these.

Chapter 4 was devoted to finding the Einstein equation governing the geometry on a brane-world, which is very different from the usual Einstein equation. The equation depends on the brane tension in a way that allowed us to conclude the the visible brane in the ekpyrotic universe should have positive brane tension.

In chapter 5, we introduced the ekpyrotic universe model mathematically by specifying its action functional. The equations of motion were then found. Subsequently we showed the validity of a time-independent solution, which is used as the basis for the moduli space approximation.

The moduli approximation was introduced in chapter 6, and the specific form of the action for the ekpyrotic universe was found in this approximation. The equations of motion derived from this action were solved numerically with the Mathematica program in appendix A. The solutions were plotted, and did not show any collision between the bulk brane and the visible brane universe. Different values of the brane tension signs and potential forms were tried, but none of the examples displayed ekpyrosis in their behavior, so we conclude that without modification, the ekpyrotic universe doesn't work. We also saw that also in the ekpyrotic scenario without a bulk brane, the boundary branes do not collide.

One of the theoretical motivating factors of the ekpyrotic universe is that it is based on M-theory. But since the implications of the ekpyrotic universe is of course very sensitive to the form of the inter-brane potential, this virtue is lost when only guessing the form of this potential. Calculating the inter-brane potential in M-theory is of utmost importance for further investigations in the ekpyrotic universe. One could of course say the same for inflationary models, which in the end also must have theoretical motivation for picking out a certain inflaton potential.

Our main conclusion is that in order to provide a higher-dimensional non-singular alternative to the big bang in the form of two branes colliding, the ekpyrotic universe must be modified. Since the model is based on simplified heterotic M-theory, one possibility could be to allow more fields to be non-zero, which could change the behavior of the model significantly. Furthermore, it is still possible that the model works if we don't restrict only to low speeds, as we do in the moduli space approximation. To investigate this, a different approximation scheme would have to be applied, or an exact numerical analysis could be attempted. In addition, stringy corrections to the action of the ekpyrotic universe could arise e.g. when the brane separation is small, and could be analyzed. These are still open possibilities that have yet to be explored, but the most important contribution to increase the understanding of the viability ekpyrotic universe is to learn more about the inter-brane potential for a pair of parallel 3-branes in the five-dimensional effective heterotic M-theory.

Appendix A

Moduli space approximation

The following Mathematica program was used to solve the ekpyrotic scenario moduli space approximation equations numerically, plot the solutions, and to export the plots to graphics files in EPS format.

```
(* Choose numeric values for the 5D Planck mass, and the coordinate
   length of the orbifold dimension. *)

m5 = 1; r = 1/m5;

(* Define the functions that are used in fomulating the action *)

Assuming[alpha*y[t] + c[t] > 0 && beta > 0 && alpha != beta &&
  c[t] > 0 && y[t] > 0 && r >= y[t],
  d0 = alpha*v - beta*(v - y[t])*UnitStep[v - y[t]] + c[t];
  i1b = 2*Integrate[d0, {v, y[t], r}];
  i2b = 2*Integrate[d0^2, {v, y[t], r}];
  i1 = 2*Integrate[d0, {v, 0, r}];
  i2 = 2*Integrate[d0^2, {v, 0, r}];
  i3 = 2*Integrate[d0^3, {v, 0, r}];
];

(* Define the function 'd' that is used to formulate the Lagrangian, and
   also other interesting functions like 'a1(t)' which describes the
   effective scale factor on the visible brane, 'length(t)' which gives
   the physical length of the orbifold direction as a function of time,
   and 'ash(t)' which gives the scale factor defined in Steinhardt et al.
   to calculate the motion of the bulk brane in the model with constant
   functions B(t) and C(t). *)

d = d0 /. v -> y[t];
```

```
a1 = a[t]*Sqrt[c[t]];
length = b[t]*i2/2;
ash = a[t]*Sqrt[b[t]*i3*m5];

(* The moduli space approximation Lagrangian for the bulk space and the
   bulk brane. Also define the Hamiltonian, which is not needed in the
   calculations, but it is interesting to plot it to check that the total
   energy indeed is zero as it should be. *)

lagrangianbulk = -3*m5^3*a[t]^2*b[t] * (i3*(D[a[t],t]/a[t])^2
  + D[a[t],t]/a[t] * (i3*D[b[t],t]/b[t] + 3*i2*D[c[t],t]
  + 3* beta*i2b*D[y,t]) - 1/12*i3*(D[b[t],t]/b[t])^2 + 1/2*i1*D[c[t],t]^2
  + i1b* D[c[t],t]*beta*D[y,t] + 1/2*beta^2*i1b*D[y,t]^2);

lagrangianbeta = 3*m5^3*beta*a[t]^2*b[t] * (d^2*D[y[t],t]^2/2
  - a[t]^2*p[y[t]]);

lagrangian = lagrangianbulk + lagrangianbeta;

hamiltonian = Simplify[a'[t]*D[lagrangian,a'[t]]
  + b'[t]*D[lagrangian,b'[t]] + c'[t]*D[lagrangian,c'[t]]
  + y'[t]*D[lagrangian,y'[t]]];

(* Calculate the equations of motion for the various functions, and put
   them together into a list. *)

eoma = Simplify[D[D[lagrangian,a'[t]],t] - D[lagrangian, a[t]]];
eomb = Simplify[D[D[lagrangian,b'[t]],t] - D[lagrangian, b[t]]];
eomc = Simplify[D[D[lagrangian,c'[t]],t] - D[lagrangian, c[t]]];
eomy = Simplify[D[D[lagrangian,y'[t]],t] - D[lagrangian, y[t]]];
eqs = {eoma == 0, eomb == 0, eomc == 0, eomy == 0};

(* Calculate the Lagrangian and Hamiltonian in the case where we put the
   functions B(t) and C(t) to constant values. We use 'bi' and 'ci' for
   the values of the functions. *)

lagrangianbc = lagrangian /. {b'[t] -> 0, b[t] -> bi, c'[t] -> 0,
  c[t] -> ci};

hamiltonianbc = Simplify[a'[t]*D[lagrangianbc, a'[t]]
  + y'[t]*D[lagrangianbc, y'[t]]];

(* Find the equations of motion in that case, too. *)
```

```
eombca = Simplify[D[D[lagrangianbc, a'[t]], t] - D[lagrangianbc, a[t]]];
eombcy = Simplify[D[D[lagrangianbc, y'[t]], t] - D[lagrangianbc, y[t]]];
eqsbc = {eombca == 0, eombcy == 0};

(* Brane tensions. *)

alpha = 100*m5; beta = 1*m5;

(* Time fra for the solution. *)

ti = 0; tf = 7450;

(* This is the bulk brane potential *)

p = Function[y, 1/10*y*(y - r)];

(* Initial field values. *)

ai = 1; bi = 1; ci = 200; yi = r;

(* Define the initial conditions in a list called 'ics'. *)

ics = {a'[ti] == 0, a[ti] == ai, b'[ti] == 0, b[ti] == bi, c'[ti] == 0,
  c[ti] == ci, y'[ti] == 0, y[ti] == yi};

(* Solve the equations of motion numerically. Output the solution into
   the list 'sol'. *)

sol = NDSolve[{eqs, ics}, {a, b, c, y}, {t, ti, tf}];

(* Find the time at which the bulk brane position Y(t), within the
   interval given in the argument to the function 'FindRoot', is
   minimised. *)

tmin = t /. FindRoot[0 == y'[t] /. sol, {t, 1800, 1600, 2000}]

(* The minimum value of the Y(t) function *)

y[tmin] /. sol

(* Plot the potential and numerical solutions for the various functions *)
```

```
fignegparp = Plot[p[y], {y, 0, r}];
fignegpara = Plot[Evaluate[a[t] /. sol], {t, ti, tf}];
fignegparb = Plot[Evaluate[b[t] /. sol], {t, ti, tf}];
fignegparc = Plot[Evaluate[c[t] /. sol], {t, ti, tf}];
fignegpary = Plot[Evaluate[y[t] /. sol], {t, ti, tf}, PlotRange -> {0, r}];
fignegpary2 = Plot[Evaluate[y[t] /. sol], {t, 1725, 1750}];

(* Export all the plots to EPS files *)

Export["~/fig_neg_par_p.eps", fignegparp, "eps"];
Export["~/fig_neg_par_a.eps", fignegpara, "eps"];
Export["~/fig_neg_par_b.eps", fignegparb, "eps"];
Export["~/fig_neg_par_c.eps", fignegparc, "eps"];
Export["~/fig_neg_par_y.eps", fignegpary, "eps"];
Export["~/fig_neg_par_y_2.eps", fignegpary2, "eps"];
```

Appendix B

Einstein tensor

This Mathematica program calculates the Einstein tensor in a coordinate basis from a general metric.

```
(* An array containing the names of the coordinates, in this case with
   five components *)

q = {t, x1, x2, x3, y};

(* The covariant metric components, in this case a diagonal 5 by 5
   array *)

g = {{ -n[t, y]^2, 0, 0, 0, 0}, { 0, a[t, y]^2, 0, 0, 0},
     { 0, 0, a[t, y]^2, 0, 0}, { 0, 0, 0, a[t, y]^2, 0},
     { 0, 0, 0, 0, b[t, y]^2}}

(* Calculate the number of dimensions and the inverse metric
   (contravariant components). *)

dim = Length[q]; ig = Inverse[g];

(* We define a three dimensional array 'christoffel' defined to be the
   Cristoffel symbols with index positions (up,down,down) from the
   metric 'g'. The calculation utilises the symmetry in the two lower
   indices. We use the standard formula for the christoffel symbols in
   terms of the metric *)

christoffel = Table[0, {i, 1, dim}, {j, 1, dim}, {k, 1, dim}];

(* Components with k >= j *)
```

```
For[i = 1, i <= dim, i++,
  For[j = 1, j <= dim, j++,
    For[k = j, k <= dim, k++,
      christoffel[[i, j, k]] =
        1/2*Sum[ig[[i, l]]*(D[g[[l, j]], q[[k]]] + D[g[[l, k]], q[[j]]]
- D[g[[j, k]], q[[l]]]), {l, 1, dim}];
    ];
  ];
];

(* Components with j > k *)

For[i = 1, i <= dim, i++,
  For[k = 1, k <= dim, k++,
    For[j = k, j <= dim, j++,
      christoffel[[i, j, k]] = christoffel[[i, k, j]];
    ];
  ];
];

(* Define a rank four array 'riemann'. It is the Riemann tensor with index
   positions (up,down,down,down). We utilise the symmetry in the two last
   lower indices to speed up the calculation. We use the standard formula
   in terms of the christoffel symbols *)

riemann = Table[0, {i, 1, dim}, {j, 1, dim}, {k, 1, dim}, {l, 1, dim}];

(* Components with l > k *)

For[i = 1, i <= dim, i++,
  For[j = 1, j <= dim, j++,
    For[k = 1, k <= dim, k++,
      For[l = k + 1, l <= dim, l++,
        riemann[[i, j, k, l]] =
        D[christoffel[[i, j, l]], q[[k]]]
- D[christoffel[[i, j, k]], q[[l]]]
+ Sum[christoffel[[m, j, l]]*christoffel[[i, m, k]]
        - christoffel[[m, j, k]]*christoffel[[i, m, l]], {m, 1, dim}];
      ];
    ];
  ];
];
```

```
(* Components with k > l *)

For[i = 1, i <= dim, i++,
  For[j = 1, j <= dim, j++,
    For[l = 1, l <= dim, l++,
      For[k = l + 1, k <= dim, k++,
        riemann[[i, j, k, l]] = -riemann[[i, j, l, k]];
      ];
    ];
  ];
];

(* Define a two dimensional array 'riccicovariant' which will contain
   the covariant components of the ricci tensor. We use the reflection
   symmetry to speed up the calculation. *)

riccicovariant = Table[0, {i, 1, dim}, {j, 1, dim}];

(* Components with j > i *)

For[i = 1, i <= dim, i++,
  For[j = i, j <= dim, j++,
    riccicovariant[[i, j]] = Sum[riemann[[k, i, k, j]], {k, 1, dim}];
  ];
];

(* Components with i > j *)

For[j = 1, j <= dim, j++,
  For[i = j + 1, i <= dim, i++,
    riccicovariant[[i, j]] = riccicovariant[[j, i]];
  ];
];

(* Calculate the (up,down) components of the Ricci tensor by muliplying
   with the inverse metric. *)

ricci = Inverse[g].riccicovariant;

(* Contract the Ricci tensor to find the Ricci scalar. We call it
   'ricciscalar'. *)

ricciscalar = Sum[ricci[[i, i]], {i, 1, dim}];
```

```
(* Define a two dimensional array 'einstein', which will be the (up,down)
   components of the einstein tensor. *)

einstein = Table[0, {i, 1, dim}, {j, 1, dim}];

(* Calculate the Einstein tensor with index positions (up,down) by the
   standard formula, and use the mathematica function 'Simplify' to
   simplify the result. *)

einstein = Simplify[ricci - 1/2*ricciscalar*IdentityMatrix[dim]]
```

Bibliography

[1] Edward Witten. Phases of N = 2 Theories in Two Dimensions. *Nucl. Phys.*, B403:159–222, 1993, hep-th/9301042.

[2] Paul S. Aspinwall, Brian R. Greene, and David R. Morrison. Calabi-Yau Moduli Space, Mirror Manifolds and Spacetime Topology Change in String Theory. *Nucl. Phys.*, B416:414–480, 1994, hep-th/9309097.

[3] Brian R. Greene, David R. Morrison, and Andrew Strominger. Black Hole Condensation and the Unification of String Vacua. *Nucl. Phys.*, B451:109–120, 1995, hep-th/9504145.

[4] V.A Rubakov and M.R. Shaposhnikov. Do we live inside a domain wall? *Phys. Lett.*, B125:136, 1983.

[5] K. Akama. An Early Proposal of 'Brane World'. *Lect. Notes Phys.*, 176:267–271, 1982, hep-th/0001113.

[6] Petr Horava and Edward Witten. Heterotic and Type I String Dynamics from Eleven Dimensions. *Nucl. Phys.*, B460:506–524, 1996, hep-th/9510209.

[7] Petr Horava and Edward Witten. Eleven-Dimensional Supergravity on a Manifold with Boundary. *Nucl. Phys.*, B475:94–114, 1996, hep-th/9603142.

[8] Justin Khoury, Burt A. Ovrut, Paul J. Steinhardt, and Neil Turok. The Ekpyrotic Universe: Colliding Branes and the Origin of the Hot Big Bang. *Phys. Rev.*, D64:123522, 2001, hep-th/0103239.

[9] Torquil MacDonald Sørensen. Numerical solution of the ekpyrotic scenario in the moduli space approximation. *Phys. Rev.*, D71:107302, 2005, hep-th/0502136.

[10] Michael B. Green, John H. Schwarz, and Edward Witten. *Superstring Theory Vols. 1 and 2.* Cambridge University Press, 1987.

[11] Joseph Polchinski. *String Theory, Volumes 1 and 2.* Cambridge University Press, 1998.

[12] Clifford Johnson. *D-Branes.* Cambridge University Press, 2002.

[13] Barton Zwiebach. *A First Course in String Theory.* Cambridge University Press, 2004.

[14] J. E. Paton and Hong-Mo Chan. Generalized Veneziano Model with Isospin. *Nucl. Phys.*, B10:516–520, 1969.

[15] Tetsuya Shiromizu, Takashi Torii, and Tomoko Uesugi. D-Braneworld Cosmology. *Phys. Rev.*, D67:123517, 2003, hep-th/0302223.

[16] Tomoko Uesugi, Tetsuya Shiromizu, Takashi Torii, and Keitaro Takahashi. D-Braneworld Cosmology II: Higher Order Corrections. *Phys. Rev.*, D69:043511, 2004, hep-th/0310059.

[17] Alan Guth. The Inflationary Universe: A Possible Solution To The Horizon And Flatness Problems. *Phys. Rev*, D23:347, 1981.

[18] Lisa Randall and Raman Sundrum. An Alternative to Compactification. *Phys. Rev. Lett.*, 83:4690–4693, 1999, hep-th/9906064.

[19] Tetsuya Shiromizu, Kei-ichi Maeda, and Misao Sasaki. The Einstein Equations on the 3-Brane World. *Phys. Rev.*, D62:024012, 2000, gr-qc/9910076.

[20] Roy Maartens. Cosmological Dynamics on the Brane. *Phys. Rev.*, D62:084023, 2000, hep-th/0004166.

[21] Øyvind Grøn and Sigbjørn Hervik. *Einstein's General Theory of Relativity.*

[22] W. Israel. Singular Hypersurfaces and Thin Shells in General Relativity. *Il Nouvo Cimento*, 44 B:1, 1966.

[23] Justin Khoury, Burt A. Ovrut, Nathan Seiberg, Paul J. Steinhardt, and Neil Turok. From Big Crunch to Big Bang. *Phys. Rev.*, D65:086007, 2002, hep-th/0108187.

[24] Paul J. Steinhardt and Neil Turok. Cosmic Evolution in a Cyclic Universe. *Phys. Rev.*, D65:126003, 2002, hep-th/0111098.

[25] Renata Kallosh, Lev Kofman, and Andrei D. Linde. Pyrotechnic Universe. *Phys. Rev.*, D64:123523, 2001, hep-th/0104073.

[26] Burt A. Ovrut, Tony Pantev, and Jaemo Park. Small Instanton Transitions in Heterotic M-theory. *JHEP*, 05:045, 2000, hep-th/0001133.

[27] Andre Lukas, Burt A. Ovrut, K. S. Stelle, and Daniel Waldram. Heterotic M-Theory in Five Dimensions. *Nucl. Phys.*, B552:246–290, 1999, hep-th/9806051.

[28] Andre Lukas, Burt A. Ovrut, K. S. Stelle, and Daniel Waldram. The Universe as a Domain Wall. *Phys. Rev.*, D59:086001, 1999, hep-th/9803235.

[29] Renata Kallosh, Lev Kofman, Andrei D. Linde, and Arkady A. Tseytlin. BPS Branes in Cosmology. *Phys. Rev.*, D64:123524, 2001, hep-th/0106241.

[30] Ron Y. Donagi, Justin Khoury, Burt A. Ovrut, Paul J. Steinhardt, and Neil Turok. Visible Branes with Negative Tension in Heterotic M-theory. *JHEP*, 11:41, 2001, hep-th/0105199.

[31] Syksy Rasanen. A Primer on the Ekpyrotic Scenario. 2002, astro-ph/0208282.

[32] Nima Arkani-Hamed, Savas Dimopoulos, and G. R. Dvali. The hierarchy problem and new dimensions at a millimeter. *Phys. Lett.*, B429:263–272, 1998, hep-ph/9803315.

[33] Gary T. Horowitz and Joseph Polchinski. Instability of Spacelike and Null Orbifold Singularities. *Phys. Rev.*, D66:103512, 2002, hep-th/0206228.