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Endomorphisms of $\mathbb{P}^1$ and $\mathbb{A}^n$

Motivic homotopy classes and open images

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Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* at the University of Oslo. The research presented here was conducted at the University of Oslo, under the supervision of associate professor Tuyen Trung Truong, and cosupervisors professor Erlend Fornæss Wold and professor Paul Arne Østvær. This work was supported by the Norwegian Research Council through the Young Research Talents grant 300814.

The thesis is a collection of three papers, presented in order of when the main results were obtained. The papers are preceded by an introductory chapter that relates them to each other and provides background information and motivation for the work. The first paper is joint work with William Hornslien, Gereon Quick, and Glen Matthew Wilson. I am the sole author of the second paper, and the third paper is joint work with Tuyen Trung Truong.

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Finally I would like to thank my wife Åsil for her endless love and support.

Viktor Balch Barth
Oslo, January 2024

Abstract

This thesis consists of three papers concerned with endomorphisms of the projective line $\mathbb{P}^1$ and of affine space $\mathbb{A}^n$. In the first paper we give an explicit geometric model for the group of $\mathbb{A}^1$-homotopy classes of endomorphisms of $\mathbb{P}^1$. In the two last papers we explicitly construct, for a large class of open subvarieties $Y \subset \mathbb{A}^n$, a surjective morphism $f\colon \mathbb{A}^n \to Y$, thereby proving the existence of such morphisms.

Sammendrag

Denne avhandlingen består av tre forskningsartikler som omhandler endomorfier av den projektive linjen $\mathbb{P}^1$ og av affint rom $\mathbb{A}^n$. I den første artikkelen gir vi en eksplisitt geometrisk modell for gruppen av $\mathbb{A}^1$-homotopiklasser av endomorfier av $\mathbb{P}^1$. I de to siste artiklene konstruerer vi eksplisitt, for en stor klasse av åpne undervarieteter $Y \subset \mathbb{A}^n$, en surjektiv morfi $f\colon \mathbb{A}^n \to Y$, og beviser dermed at slike morfier eksisterer.

List of Papers

Paper I

Viktor Balch Barth, William Hornslien, Gereon Quick, Glen Matthew Wilson, "Making the motivic group structure on the endomorphisms of the projective line explicit". Preprint, arXiv: 2306.00628.

Paper II

Viktor Balch Barth "Surjective morphisms from affine space to its Zariski open subsets". In: *International Journal of Mathematics*. Vol. 34, no. 12 (2023), Paper No. 2350075. DOI: 10.1142/S0129167X23500751.

Paper III

Viktor Balch Barth, Tuyen Trung Truong "Images of dominant endomorphisms of affine space". Preprint, arXiv: 2311.08238.

Contents

Chapter 1

Introduction

This thesis is concerned with endomorphisms of the projective line $\mathbb{P}^1$ and of affine space $\mathbb{A}^n$. We study one problem arising in $\mathbb{A}^1$-homotopy theory, and another inspired by algebraic Oka theory.

Our first problem is about $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$, the group of $\mathbb{A}^1$-homotopy classes of morphisms $\mathbb{P}^1 \to \mathbb{P}^1$. This object plays an analogous role in $\mathbb{A}^1$-homotopy theory to the fundamental group of the circle $\pi_1(S^1)$ in algebraic topology. The fundamental group of the circle has a simple, explicit group operation (see figure 1.1). However, such a straightforward description is not available for $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$. This is the problem we tackle in Paper I, entitled *Making the motivic group structure on the endomorphisms of the projective line explicit*.

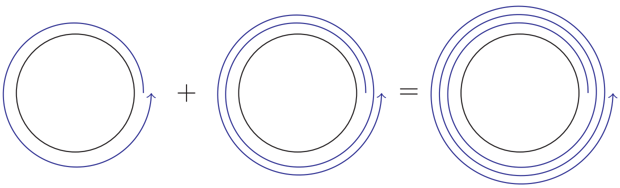

Figure 1.1: The fundamental group of the circle $\pi_1(S^1)$ is isomorphic to the integers $\mathbb{Z}$. The group operation is concatenation of loops.

Our second problem has to do with morphisms $\mathbb{A}^n \to \mathbb{A}^n$. We ask which open subvarieties $U \subset \mathbb{A}^n$ are the image of an $\mathbb{A}^n$-endomorphism. Given such an open subvariety U, we are interested in explicitly constructing a surjective map $\mathbb{A}^n \to U$. This is the topic of Paper II, *Surjective morphisms from affine space to its Zariski open subsets*, as well as Paper III, *Images of dominant endomorphisms of affine space*. For a large class of open subvarieties, we are able to give explicit constructions of surjective morphisms.

If we instead studied open images of $\mathbb{P}^1$-endomorphisms, or homotopy classes $[\mathbb{A}^n, \mathbb{A}^n]^{\mathbb{A}^1}$, then this thesis would have been much shorter. (The only open image is $\mathbb{P}^1$, and $\mathbb{A}^n$ is $\mathbb{A}^1$-contractible.)

This introduction begins with a look at $\mathbb{A}^n$-endomorphisms and surjectivity results from algebraic Oka theory. We then proceed to describe the Grothendieck–Witt ring, the $\mathbb{A}^1$-homotopy category and $\mathbb{P}^1$-endomorphisms. Along the way we highlight relevant definitions and theorems, hopefully preparing the reader for the paper summaries.

1.1 Endomorphisms of affine space

One of the fundamental objects in algebraic geometry is affine space. Working over some field k, we define n-dimensional affine space as $\mathbb{A}^n := \operatorname{Spec}(k[x_1, \ldots, x_n])$. The k-valued points of $\mathbb{A}^n$ can be identified with the points of k^n. Endomorphisms $\mathbb{A}^n \to \mathbb{A}^n$ correspond one-to-one to ring endomorphisms of $k[x_1, \ldots, x_n]$, and are completely characterized by n polynomials f_i, and the ring map $x_i \mapsto f_i$. It is perhaps surprising that there are still many unanswered questions about the images of maps from $\mathbb{A}^n$, and even about images of maps from $\mathbb{A}^n$ to $\mathbb{A}^n$. This is the focus of Paper II and Paper III, where we investigate which open subvarieties of $\mathbb{A}^n$ are the (set-theoretic) image of an $\mathbb{A}^n$-endomorphism. This is done in the setting of k being an algebraically closed field of characteristic 0, like the complex numbers $\mathbb{C}$. (Note however that the explicit constructions in Paper II and Paper III work in any characteristic.)

Definition 1.1.1 (Algebraically closed field)**.** A field k is *algebraically closed* if every non-constant polynomial in $k[x]$ has a root in k.

An immediate consequence of k being algebraically closed, is that the image of a polynomial map $f\colon \mathbb{A}^1 \to \mathbb{A}^1$ is either a point or all of $\mathbb{A}^1$. In higher dimensions however, different behavior might be possible. We ask the following question.

Question 1.1.2. Which open subvarieties of $\mathbb{A}^n$ are the image of an $\mathbb{A}^n$-endomorphism?

This is a special case of the question in [LT19, Remark 2 (d)] or in [Arz23, Problem 2]. It is a simple question about fundamental mathematical objects, but turns out to be difficult to answer.

Different approaches have been taken to study this and related problems, particularly in the case of $k = \mathbb{C}$. While there are no nontrivial examples in dimension one, they appear already in dimension two. This has been explored by using the idea of non-properness sets introduced by Jelonek in [Jel93]. The following example is an altered version of [Jel99, Example 9].

Example 1.1.3. The image of $f(x, y) = (x + y(xy + 1), xy + 1)$ is $\mathbb{A}^2 \setminus \{(0, 0)\}$.

El Hilany shows in [Hil22] the existence of maps $f\colon \mathbb{A}^2 \to \mathbb{A}^2$, where $A = \mathbb{A}^2 \setminus f(\mathbb{A}^2)$ is a finite set of points, and gives bounds on the number of points in A in terms of the degree of f.

1.1.1 Chevalley's theorem

The following definition is needed to describe the structure of images of algebraic morphisms.

Definition 1.1.4. A subset A of a topological space X is *locally closed* if it is the intersection of an open set and a closed set. A subset $A \subseteq X$ is *constructible* if it is a finite union of locally closed sets.

Equivalently, a locally closed set is the difference of two closed or of two open sets. With this definition, we state Chevalley's theorem [Gro64, Théorème 1.8.4].

Theorem 1.1.5 (Chevalley)**.** *Let $f\colon X \to Y$ be a finitely presented morphism of Noetherian schemes. For any constructible subset $C \subseteq X$, the image $f(C)$ is a constructible subset of Y.*

This theorem can be proved using recursion and the following idea. Given a constructible set C, we can find an open subset $U \subseteq \overline{f(C)}$ such that $U \subseteq f(C) \subseteq \overline{f(C)}$. The intersection $D = C \cap (X \setminus f^{-1}(U))$ has strictly lower dimension, and observe that $f(C)$ is constructible only if $f(D)$ is.

Variants this algorithm can be implemented in software, as is done in [HMS19], and described in [BL21]. This is very useful in Paper III, where we use an algorithmic approach to compute images.

In particular, the image of any morphism $f\colon \mathbb{A}^n \to \mathbb{A}^n$ is constructible.

Example 1.1.6. The image of the map $f(x, y) = (x, xy)$ is constructible, but neither open, nor closed, nor locally closed.

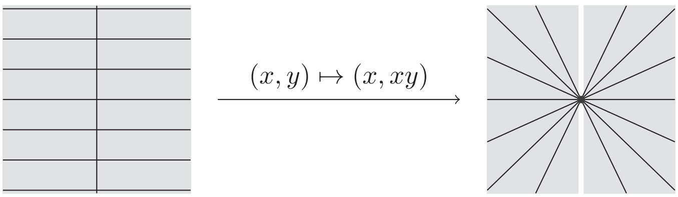

Figure 1.2: The vertical line $x = 0$ is sent to the origin, and the horizontal lines are sent to lines through the origin. The image of f is the union of two locally closed sets, the distinguished open $D(x)$ and the closed set $V(x, y)$.

1.1.2 Closed sets in the complement of the image

The following quick argument about the complement of dominant $\mathbb{A}^n$-endomorphisms is useful [Arz23; Kus22].

Lemma 1.1.7. *If $f\colon \mathbb{A}^n \to \mathbb{A}^n$ is dominant map, then $\mathbb{A}^n \setminus f(\mathbb{A}^n)$ does not contain a closed subvariety $Z \subset \mathbb{A}^n$ of codimension 1.*

Proof. If Z is closed and of codimension 1, then there exist nonconstant regular functions on $\mathbb{A}^n$ with zero set contained in Z. Choose one such function, say g, and notice that it is an invertible function on $\mathbb{A}^n \setminus Z$. The pullback of g along f would have to be a nonconstant invertible regular function $f^*(g)$ on $\mathbb{A}^n$. Since $\Gamma(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})^\times = k^\times$, this is a contradiction. ■

The algebraic Hartog's lemma [Vak17, Lemma 11.3.11] states that a rational function on a Noetherian normal scheme with no poles is in fact regular. This entails that $\mathbb{A}^n \setminus f(\mathbb{A}^n)$ can contain closed subvarieties $Z \subset \mathbb{A}^n$ of codimension at least 2, which we have already seen in examples 1.1.3 and 1.1.6.

1.2 Surjectivity results in Oka theory

The motivation for studying open images of $\mathbb{A}^n$-endomorphisms comes from results in Oka theory and on flexible varieties. We give a brief account of these fields, and refer to Forstnerič's book [For17a], which is a great comprehensive resource for Oka theory and algebraic Oka theory.

1.2.1 Oka theory

Stein manifolds are complex manifolds which admit the following characterization (see [Rem56]): A complex manifold is a *Stein manifold* if and only if it is biholomorphic to a closed complex submanifold of a euclidean space $\mathbb{C}^N$. Stein manifolds follow the Oka principle, coined by Serre in [Ser51, VII. (h)] as: *On a Stein manifold one can do with holomorphic functions what one can do with continuous functions.*[1]

In the seminal paper [Gro89], Gromov asked when all continuous maps $X \to Y$ are homotopic to holomorphic maps $X \to Y$. Furthermore, for which complex manifolds Y is the natural inclusion $\mathcal{O}(X,Y) \to \mathcal{C}(X,Y)$ a weak homotopy equivalence for all Stein manifolds X? Here, $\mathcal{O}(X,Y)$ and $\mathcal{C}(X,Y)$ denote respectively the space of holomorphic maps and the space of continuous maps, each equipped with the compact-open topology.

If Y is an Oka manifold, then for all Stein manifolds X, the inclusion $\mathcal{O}(X,Y) \to \mathcal{C}(X,Y)$ a weak homotopy equivalence. The class of Oka manifolds admit several different equivalent characterizations, one of which is the following Runge approximation property [For17a, Definition 5.4.1].

Definition 1.2.1. A complex manifold Y is an *Oka manifold* if every holomorphic map $f\colon K \to Y$ from a neighborhood of a compact convex set $K \subset \mathbb{C}^n$ to Y can be approximated uniformly on K by entire maps $\mathbb{C}^n \to Y$.

A (necessarily surjective) holomorphic map $F\colon S \to X$ is called *strongly dominating* if every point $x \in X$ has a preimage $p \in S$ such that F is a submersion at p. Forstnerič proves the existence of strongly dominating maps from $\mathbb{C}^n$ onto Oka manifolds of dimension n in the following theorem [For17b, Theorem 1.1].

Theorem 1.2.2. *Let X be a connected Oka manifold. If S is a Stein manifold and $\dim S \geq \dim X$, then every continuous map $f : S \to X$ is homotopic to a strongly dominating (surjective) holomorphic map $F : S \to X$. In particular, there exists a strongly dominating holomorphic map $F : \mathbb{C}^n \to X$ for $n = \dim X$.*

[1] My translation. Original: "Sur une variété de Stein on peut faire avec les fonctions holomorphes ce qu'on peut faire avec les fonctions continues."

Before moving on to algebraic Oka theory, we mention a connection to $\mathbb{A}^1$-homotopy theory. Lárusson constructed a model category for complex manifolds in [Lár03; Lár04], developing a holomorphic homotopy theory, analogous to the $\mathbb{A}^1$-homotopy category for schemes in [MV99].

1.2.2 Algebraic Oka theory

A smooth algebraic variety over $\mathbb{C}$ can be viewed as a manifold, and we call it an *algebraic manifold*. Certain techniques and results carry over from the setting of complex manifolds with holomorphic maps to algebraic manifolds with algebraic maps. A sufficient condition for a complex manifold to be Oka, is that it is subelliptic, a notion implicit in [Gro89], but first defined in [For02]. The algebraic analog is algebraic subellipticity, which we recount from [For17a, Definition 5.6.13].

Definition 1.2.3. Let Y be a smooth algebraic variety. An *algebraic spray* on Y is a triple (E, π, s) consisting of an algebraic vector bundle $\pi\colon E \to Y$ and an algebraic map $s\colon E \to Y$ (a spray map) such that for each $y \in Y$ we have $s(0_y) = y$. A spray (E, π, s) on Y is *dominating* on a subset $U \subset Y$ if the differential

$$ds_{0y}\colon T_{0y}E \to T_yY$$

maps the vertical subspace E_y of $T_{0y}E$ surjectively onto T_yY for every $y \in U$; s is dominating if this holds for all $y \in Y$. A smooth variety Y is *algebraically elliptic* if it admits a dominating algebraic spray.

One may further define the notion of *algebraic subellipticity*, which holds if Y admits a finite family of sprays that together dominate at each $y \in Y$, that is,

$$(ds_1)_{0_y}(E_{1,y}) + \ldots + (ds_m)_{0_y}(E_{m,y}) = T_yY.$$

It was recently shown that for smooth varieties, algebraic ellipticity is equivalent to algebraic subellipticity and to local ellipticity [KZ23, Theorem 0.1]. This is an algebraic counterpart of Kusakabe's result that a complex manifold Y is Oka if each point in Y has a Zariski open Oka neighborhood [Kus21, Theorem 1.4].

Forstnerič's algebraic analog of theorem 1.2.2 has the additional requirement that X must be compact manifold [For17b, Theorem 1.6].

Theorem 1.2.4. *Assume that X is a compact algebraically subelliptic manifold and S is an affine algebraic manifold such that $\dim S \geq \dim X$. Then, every algebraic map $S \to X$ is homotopic (through algebraic maps) to a surjective strongly dominating algebraic map $S \to X$. In particular, X admits a surjective strongly dominating algebraic map $F\colon \mathbb{C}^n \to X$ with $n = \dim X$.*

It is natural to ask whether the compactness condition can be omitted. Kusakabe showed the following in [Kus22, Theorem 1.2].

Theorem 1.2.5. *For any smooth subelliptic variety Y, there exists a surjective morphism $f\colon \mathbb{A}^{\dim Y+1} \to Y$ such that $f(\mathbb{A}^{\dim Y+1} \setminus \operatorname{Sing}(f)) = Y$.*

Here, $\mathrm{Sing}(f) = \{x \in \mathbb{A}^{\dim Y+1} \colon f \text{ is not smooth at } x\}$ is the singular locus of f, and the condition $f(\mathbb{A}^{\dim Y+1} \setminus \mathrm{Sing}(f)) = Y$ corresponds to being strongly dominating. Kusakabe's proof is purely algebro-geometric, formulated in terms of smooth subelliptic varieties rather than algebraically subelliptic manifolds, and the result holds more generally than just over $\mathbb{C}$.

It remains an open question whether the dimension of the affine space in theorem 1.2.5 can be lowered to $\dim Y$. Even the problem of existence of surjective algebraic maps $f \colon \mathbb{A}^{\dim Y} \to Y$ (without the condition on $f(\mathbb{A}^{\dim Y} \setminus \mathrm{Sing}(f))$) is open.

A corollary of theorem 1.2.5 is [Kus22, Corollary 1.4], which also follows from [Arz23, Theorem 1.1].

Corollary 1.2.6. *For a Zariski open subset $U \subset \mathbb{A}^n$, the following are equivalent:*

(i) *U is the image of a morphism from an affine space.*

(ii) *The complement $\mathbb{A}^n \setminus U$ is of codimension at least two.*

This corollary hints at a possible strategy to resolve question 1.1.2. First, construct a surjective map $F \colon \mathbb{A}^N \to Y$, then find an embedding $i : \mathbb{A}^n \to \mathbb{A}^N$ such that $(F \circ i) \colon \mathbb{A}^n \to Y$ is surjective. Techniques from the theory of flexible varieties are useful to achieve this.

1.2.3 Flexible varieties

An important class of elliptic varieties is flexible varieties We begin by defining flexibility, following [Arz+13; Arz23]. For a smooth variety Y, we denote by $\mathrm{Aut}(Y)$ its group of automorphisms. A nontrivial regular action $\varphi \colon \mathbb{G}_a \times Y \to Y$ by the additive group $\mathbb{G}_a = (\mathbb{A}^1, +)$ defines a one-parameter unipotent subgroup G of $\mathrm{Aut}(X)$, which we call a *$\mathbb{G}_a$-subgroup.* The subgroup of $\mathrm{Aut}(Y)$ generated by the $\mathbb{G}_a$-subgroups is denoted $\mathrm{SAut}(Y)$.

Definition 1.2.7. A smooth variety Y is *flexible* if for each $y \in Y$, the tangent space T_yY is spanned by the tangent vectors to the orbits of $\mathbb{G}_a$-actions on Y through y. Furthermore, Y is called *very flexible* if $\mathrm{SAut}(Y)$ acts transitively on Y.

For a smooth flexible variety Y, the $\mathbb{G}_a$-actions $\mathbb{G}_a \times Y \to Y$ can be composed to form a spray map $s \colon \mathbb{A}^N \times Y \to Y$, showing that flexibility implies algebraic ellipticity, as in [Gro89, Section 0.5.B].

The following result was first observed by Gromov in [Gro86, p. 72, Exercise (b')], and shown by Winkelmann in [Win90, Proposition 1].

Theorem 1.2.8 (Gromov–Winkelmann)**.** *Let A be an algebraic subvariety in $\mathbb{C}^n$ with* $\mathrm{codim}(A) \geq 2$. *Let G denote the group of all algebraic automorphisms of $\mathbb{C}^n$ fixing A pointwise. Then G acts transitively on $X = \mathbb{C}^n \setminus A$.*

This theorem is generalized in [FKZ16, Theorem 0.1], where they show that if X is a smooth quasi-affine flexible variety of dimension ≥ 2 and Y is any closed subscheme $Y \subseteq X$ of codimension ≥ 2, then $X \setminus Y$ is also flexible.

For any very flexible variety Y, the $\mathbb{G}_a$-actions can be used to construct a surjective map $F\colon \mathbb{A}^N \to Y$ [Arz23, Proposition 2]. (See also [Arz+13, Corollary 1.11].) This is done by picking a point $y \in Y$, then composing (finitely many) $\mathbb{G}_a$-actions until the orbit of y under the composed action is all of Y.

In general N is greater than $\dim Y$, and the result is slightly weaker than theorem 1.2.5. The upshot is that it is feasible to find an embedding $i : \mathbb{A}^n \to \mathbb{A}^N$ such that $(F \circ i)\colon \mathbb{A}^n \to Y$ is surjective. This is exactly the strategy we follow in Paper III. We now illustrate it with a simple example.

Example 1.2.9. For $\mathbb{A}^2 \setminus \{0\}$, two $\mathbb{G}_a$-actions $\varphi_i\colon \mathbb{A}^1 \times \left(\mathbb{A}^2 \setminus \{0\}\right) \to \mathbb{A}^2 \setminus \{0\}$ are $\varphi_1\colon (t,x,y) \mapsto (x, y+tx)$ and $\varphi_2\colon (t,x,y) \mapsto (x+ty, y)$. Each action fixes an affine line, but only the origin is fixed by both actions.

If we act on the point $(0,1)$ first by φ_2, then φ_1 and then φ_2, we obtain a surjective map $F\colon \mathbb{A}^3 \to \mathbb{A}^2 \setminus \{0\}$, given by $F(a,b,c) = (a + c(ab+1), ab+1)$. The restriction $F|_{b=c}\colon \mathbb{A}^2 \to \mathbb{A}^2 \setminus \{0\}$ is still surjective. In fact, it is precisely the map from example 1.1.3.

1.3 Motivic and naive motivic homotopy theory

1.3.1 Motivic homotopy theory

We recall very briefly how the $\mathbb{A}^1$-homotopy category comes about, and refer to [Dun+07], [Lev08] and [WW20] for more thorough introductions.

Quillen introduced homotopical algebra in [Qui67]. Central is the idea of a model category, which is a category C with small limits and colimits, as well as three subcategories of weak equivalences, fibrations and cofibrations. From a model category, one can form its homotopy category by inverting the weak equivalences.

In their seminal work [MV99], Morel and Voevodsky define the $\mathbb{A}^1$-homotopy category over a scheme. For our purposes, the base scheme will be $\mathrm{Spec}(k)$, where k is a field. To begin, let Sm_k be the category of smooth schemes of finite type over k. The idea of motivic homotopy theory is to have a notion of homotopies in this setting, with the affine line $\mathbb{A}^1$ playing the role of the unit interval. The construction begins with a Yoneda embedding of Sm_k into the category of simplicial presheaves on Sm_k. The next step is performing localizations to impose the Nisnevich topology and add weak equivalences. The result is an appropriate model category of motivic spaces $\mathrm{Spc}(k)$. Inverting the weak equivalences then yields the $\mathbb{A}^1$-homotopy category. In this thesis, the main focus will be pointed motivic spaces $\mathrm{Spc}_*(k)$ and the pointed motivic homotopy category $\mathcal{H}_*(k)$.

This category $\mathrm{Spc}_*(k)$ of pointed motivic spaces contains simplicial sets in addition to the smooth k-schemes. Thanks to the model category structure, it has the nice properties that you expect from a homotopy theory. In particular it has weak equivalences $X \times \mathbb{A}^1 \simeq X$ for any smooth scheme X, it has limits and colimits, and one can form homotopy groups.

For a pair of smooth k-schemes X, Y, we write $[X, Y]^{\mathbb{A}^1}$ for the set of morphisms $X \to Y$ in the $\mathbb{A}^1$-homotopy category.

1.3.2 Naive motivic homotopy theory

We recall how pointed elementary homotopies and naive motivic homotopy classes are defined.

Definition 1.3.1. Let X and Y be smooth k-schemes, pointed at k-points x and y respectively. A pointed elementary homotopy between two pointed morphisms $f\colon X \to Y$ and $g\colon X \to Y$ is given by a morphism $H(T)\colon X \times \mathbb{A}^1 \to Y$ mapping the generic point of $\{x\} \times \mathbb{A}^1$ to y, and satisfying $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$. We say that f and g are pointed elementarily homotopic and write $f \sim g$.

The relation of being pointed elementarily homotopic is symmetric and reflexive, but not transitive. In other words, there are cases where $f \sim g$ and $g \sim h$, but $f \not\sim h$. Taking the transitive closure of $\sim$ solves this problem and results in an equivalence relation $\simeq$ on the set of pointed morphisms $\mathrm{Sm}_k(X, Y)_*$. We say that pointed morphisms $f, g \in \mathrm{Sm}_k(X, Y)_*$ are naively homotopic, and write $f \simeq g$, if there is a chain of pointed elementary homotopies from f to g. We write $[X, Y]^{\mathrm{N}} := \mathrm{Sm}_k(X, Y)_*/\simeq$ for the set of naive motivic homotopy classes.

1.3.3 Comparing homotopy classes

Naive motivic homotopies are easy to define and to explicitly compute. Since the homotopy classes are formed as a set of morphisms modulo an equivalence relation, each homotopy class admits some representative. This is not the case in the $\mathbb{A}^1$-homotopy category, but in return it does have all the benefits of being an actual homotopy category.

As each perspective has advantages, it would be useful to compare the two. This is very difficult in general. There is a canonical map from naive to motivic homotopy classes, and affine representability results of Asok, Hoyois and Wendt in [AHW17] and [AHW18] give sufficient conditions for this to be a bijection. They define the notion of being $\mathbb{A}^1$-*naive* in [AHW18, Definition 2.1.1]. If $X, Y \in \mathrm{Sm}_k$, X is affine, and Y is $\mathbb{A}^1$-*naive*, then the canonical map $[X, Y]^N \to [X, Y]^{\mathbb{A}^1}$ is a bijection. This is more thoroughly described in appendix I.A, where we verify that these unpointed results carry over to the pointed setting.

1.4 The Grothendieck–Witt ring

An important invariant in $\mathbb{A}^1$-homotopy theory over a field k is the Grothendieck–Witt ring $\mathrm{GW}(k)$, the ring of isometry classes of symmetric bilinear forms over k. It is possible to define $\mathrm{GW}(k)$ over any commutative unital ring k, but we restrict this exposition to the case where k is a field, as in [EKM08]. (See [Lam05] for a treatement of fields of characteristic not 2, which slightly simplifies the situation.[2] See [MH73] for commutative unital rings.)

Let V be a k-vector space. A bilinear form is a map $B\colon V \times V \to k$ that is k-linear in both variables. A *symmetric bilinear form* satisfies $B(u, v) = B(v, u)$, and we call B *nondegenerate* if the map $V \to V^\vee$ sending $v \in V$ to $B(-, v)$ is an isomorphism of k-vector spaces. By picking a basis $\{e_1, \dots, e_n\}$ for V, we can define a *Gram matrix* $M_{i,j} := B(e_i, e_j)$ for B. The *discriminant* $\operatorname{disc}(B) := \det(M) \cdot k^{\times 2} \in k^\times / k^{\times 2}$ is independent of choice of basis.

If $\operatorname{char}(k) \neq 2$, every nondegenerate symmetric bilinear form is isometric to a diagonal form. We write diagonal forms as

$$\langle u_1, \dots, u_n \rangle \colon (x_1, \dots, x_n, y_1, \dots, y_n) \to \sum_i u_i x_i y_i.$$

The isometry classes of non-degenerate symmetric bilinear forms form a monoid, which we call the *Witt monoid*, and write as $\mathrm{WM}(k)$. The binary operation $\oplus$ is the direct sum. Equipped with the tensor product $\otimes$, it is a semiring. The Witt monoid is generated as a monoid in degree one, by bilinear forms $\langle u \rangle \colon (x, y) \mapsto uxy$, where $u \in k^\times$. To avoid problems in characteristic 2, we quotient out by the relation that $B' \simeq B''$ if $B \oplus B' \cong B \oplus B''$ to form the *stable Witt monoid* WM^s. We form the Grothendieck–Witt ring $\mathrm{GW}(k)$ as the Grothendieck group of the cancellative monoid $\mathrm{WM}^s(k)$. It admits the following presentation [Lam05, Theorem II.4.1]

Lemma 1.4.1. *As a group, $GW(k)$ is generated by generators $\langle u \rangle$, where $u \in k^\times$, and the following relations*

(i) $\langle uv^2 \rangle = \langle u \rangle$, *for* $u, v \in k^\times$.

(ii) $\langle u \rangle + \langle v \rangle = \langle u + v \rangle + \langle uv(u + v) \rangle$, *for* $u, v, u + v \in k^\times$.

(iii) $\langle u \rangle + \langle -u \rangle = \langle 1 \rangle + \langle -1 \rangle$, *for* $u \in k^\times$.

The same generators and relations generate the monoid $\mathrm{WM}^s(k)$. There are rank homomorphisms $\operatorname{rank}\colon \mathrm{GW}(k) \to \mathbb{Z}$ and $\operatorname{rank}\colon \mathrm{WM}^s(k) \to \mathbb{N}$. We now give some examples of $\mathrm{GW}(k)$ and $\mathrm{WM}^s(k)$ over different fields, using the convention $0 \in \mathbb{N}$.

Example 1.4.2. Let k be an algebraically closed field. For any $\langle u \rangle$ and $\langle a \rangle$, there is a unit $v \in k^\times$ such that $a = uv^2$, hence $\langle u \rangle = \langle uv^2 \rangle = \langle a \rangle$. Therefore $\mathrm{WM}(k) \cong \mathbb{N}$ and $\mathrm{GW}(k) \cong \mathbb{Z}$.

Example 1.4.3. Let k be the real numbers $\mathbb{R}$. Then $\langle 1 \rangle$ and $\langle -1 \rangle$ are the generators, and any symmetric bilinear form can be written as $p\langle 1 \rangle + q\langle -1 \rangle$. We call the difference $p - q$ the *signature* of the form, and the sum $p + q$ is the rank. Hence $\mathrm{WM}(\mathbb{R}) \cong \mathbb{N} \times \mathbb{N}$, and $\mathrm{GW}(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$.

[2] If $\operatorname{char}(k) \neq 2$, symmetric bilinear forms correspond one-to-one with quadratic forms.

Example 1.4.4. Let k be a finite field of odd characteristic $\mathbb{F}_q$. Then the generators are $\langle 1\rangle$ and $\langle u\rangle$, for a non-square unit u. We obtain $\mathrm{WM}(\mathbb{F}_q) = \mathbb{N}\times\mathbb{Z}/2$ and $\mathrm{GW}(\mathbb{F}_q) = \mathbb{Z}\times\mathbb{Z}/2$.

Another important object is Milnor–Witt K-theory $K_*^{MW}(k)$, due to Morel and Hopkins [Mor12, Definition 3.1]. There is a ring isomorphism $\mathrm{GW}(k) \to K_0^{MW}(k)$ relating the Grothendieck–Witt ring to the 0-th Milnor–Witt K-group. We omit the definition here, as $K_*^{MW}(k)$ will be defined where used in Paper I, in section I.6.3.

1.5 Morphisms to the projective line

We consider the projective line over a field $\mathbb{P}^1_k = \mathrm{Proj}(k[x_0, x_1])$ to be pointed at $\infty = [1 : 0]$. It admits a standard covering by two affine lines $\mathbb{A}^1$ intersecting along a punctured affine line $\mathbb{G}_m$. We will now describe morphisms to $\mathbb{P}^1$. By [Har77, Theorem II.7.1], for a smooth k-scheme X, the data needed to give a morphism $f\colon X \to \mathbb{P}^1$ are an invertible sheaf $\mathcal{L}$ over X and the choice of two global sections $s_0, s_1 \in \Gamma(X, \mathcal{L})$ that generate the invertible sheaf $\mathcal{L}$.

For the case of morphisms $f\colon \mathbb{P}^1 \to \mathbb{P}^1$, the Picard group is $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$. Hence any invertible sheaf on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(n)$ for some integer n. There only exist global sections of $\mathcal{O}(n)$ when n is nonnegative. Any scheme endomorphism of $\mathbb{P}^1$ corresponds to a pair of generating sections of $\mathcal{O}(n)$ for some nonnegative n.

1.5.1 Motivic circles and the group of $\mathbb{P}^1$-endomorphisms

Classically, the set of homotopy classes of continuous maps $[S^n, Y]$ from the n-sphere to a pointed topological space Y has a group structure, denoted $\pi_n(Y)$. This is due to the h-cogroup structure on S^n coming from the pinch map $\mu\colon S^n \to S^n \vee S^n$ collapsing the equator to a point.

The wedge sum $X \vee Y$ of two pointed topological spaces is the disjoint union modulo basepoints $(X \sqcup Y)/(x \sim y)$. Given maps $f\colon X \to Z$ and $g\colon Y \to Z$, there is a map $f \vee g\colon X \vee Y \to Z$ which restricted to X (resp. Y) is equal to f (resp. g).

The following formulation is more convenient for generalizations. The wedge sum $X \vee Y$ is the pushout of the diagram $X \leftarrow * \rightarrow Y$. With morphisms f, g as above, the universal property of wedge sums guarantees the existence of a morphism $f \vee g\colon X \vee Y \to Z$.

Two homotopy classes of maps $S^1 \to Y$, $\alpha = [f]$ and $\beta = [g]$, can be combined to form $\alpha\cdot\beta = [(f \vee g)\circ\mu]$. For maps $S^1 \to S^1$, this yields the group $\pi_1(S^1)$ previously illustrated in figure 1.1.

In the motivic world, there are two circles, the Tate circle $S^{1,1} = \mathbb{G}_m$, and the simplicial circle $S^{1,0} = S^1$. As in topology, one may form smash products $X \wedge Y = (X \times Y)/(X \vee Y)$. The smash products of circles form a bigraded family of motivic spheres $S^{p+q,q} := (S^1)^{\wedge p} \wedge \mathbb{G}_m^{\wedge q}$. In particular, $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m = \Sigma\mathbb{G}_m$.

The h-cogroup structure of the simplicial circle makes $\mathbb{P}^1$ an h-cogroup. Given two maps $f, g\colon S^1 \wedge \mathbb{G}_m \to \mathbb{P}^1$ in the $\mathbb{A}^1$-homotopy category, the composition below represents the sum $f \oplus^{\mathbb{A}^1} g$ of the maps f and g.

$$S^1 \wedge \mathbb{G}_m \xrightarrow{\mu\wedge 1} (S^1 \vee S^1) \wedge \mathbb{G}_m \xrightarrow{\cong} (S^1 \wedge \mathbb{G}_m) \vee (S^1 \wedge \mathbb{G}_m) \xrightarrow{f\vee g} \mathbb{P}^1 \quad (1.1)$$

This operation gives $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}$ the structure of a group. Morel shows in [Mor12, Theorem 7.36] the isomorphism $[\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \cong \mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times$.

1.5.2 The monoid of naive homotopy classes

Cazanave's work in [Caz12] equips the set of pointed naive homotopy classes $[\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}$ with a monoid structure.

Any pointed $\mathbb{P}^1$-endomorphism given by two generating global sections s_0, s_1 of an invertible sheaf $\mathcal{O}(n)$ corresponds uniquely to a rational function $A(X)/B(X)$, where $A(X)$ is monic of degree n, and $B(X)$ has degree at most $n-1$, ensuring pointedness. Furthermore, the resultant $\mathrm{res}_{n,n}(A, B)$ must be a unit to guarantee that the sections generate $\mathcal{O}(n)$. We define the resultant of two univariate polynomials $A(X), B(X) \in S[X]$, where S an integral domain. Let $A = a_n X^n + a_{n-1}X^{n-1} + \ldots + a_0$ and $B = b_n X^n + b_{n-1}X^{n-1} + \ldots + b_0$, (with $b_n = 0$). The resultant is the determinant of a $2n \times 2n$-matrix,

$$\mathrm{res}_{n,n}(A, B) = \det\big(\mathrm{Syl}_{n,n}(A, B)\big) = \det \begin{pmatrix} a_n & 0 & 0 & b_n & 0 & 0 \\ \vdots & \ddots & 0 & \vdots & \ddots & 0 \\ a_0 & \ddots & a_n & b_0 & \ddots & b_n \\ 0 & \ddots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & a_0 & 0 & 0 & b_0 \end{pmatrix}.$$

As shorthand we write $\mathrm{res}(f) := \mathrm{res}(A, B) = \mathrm{res}_{n,n}(A, B)$ for the rational function $f = A/B$.

An equivalent condition to $\mathrm{res}(A, B) \in k^\times$ is that A and B satisfy a (neccessarily unique) Bézout relation, that is $AU + BV = 1$, for polynomials $U, V \in k[X]$, where $\deg(V) \leq n-1$ and $\deg(U) \leq n-2$ (unless $n = 0$). This establishes a correspondence between pointed rational functions and $\mathrm{SL}_2(k[X])$-matrices.

Cazanave uses this correspondence to define the following binary operation on rational functions. A pair of pointed rational functions $A_1/B_1, A_2/B_2$ correspond to a pair of $\mathrm{SL}_2(k[X])$-matrices, which are multiplied to from a third matrix,

$$\begin{bmatrix} A_3 & -V_3 \\ B_3 & U_3 \end{bmatrix} := \begin{bmatrix} A_1 & -V_1 \\ B_1 & U_1 \end{bmatrix} \cdot \begin{bmatrix} A_2 & -V_2 \\ B_2 & U_2 \end{bmatrix}.$$

The pointed rational function A_3/B_3 is called the *naive sum*, and we write

$$\frac{A_1}{B_1} \oplus^{\mathrm{N}} \frac{A_2}{B_2} := \frac{A_3}{B_3}.$$

The operation is still well-defined after passing to naive homotopy classes. This is how the set of naive homotopy classes of endomorphisms of the projective line obtains the structure of a graded commutative monoid $\left([\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}}\right)$.

To any rational function f of degree n, you can associate its Bézout matrix, a symmetric $n \times n$-matrix. A symmetric matrix can in turn be interpreted as a Gram matrix for a symmetric bilinear form. Cazanave uses this to show the following theorem.

Theorem 1.5.1 (Corollary 3.10 in [Caz12])**.** *There is a canonical isomorphism of graded monoids:*

$$\left([\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}}\right) \cong \left(\coprod_{n \geq 0} WM_n^s(k) \times_{k^\times/k^{\times 2}} k^\times, * \right).$$

Here, $\mathrm{WM}_n^s(k)$ denotes the rank n component of the stable Witt monoid, and the fiber product is with respect to the discriminant map $\mathrm{disc}\colon \mathrm{WM}_n^s(k) \to k^\times/k^{\times 2}$. The operation $*$ denotes the direct sum in the stable Witt monoid, and multiplication in $k^\times$.

Example 1.5.1. The monoid $\left([\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}}\right)$ is generated in degree 1 by homotopy classes of rational fuctions $[X/u]$, with $u \in k^\times$. The naive sum of two generators is

$$\frac{X}{u} \oplus^{\mathrm{N}} \frac{X}{v} = \frac{X^2 - \frac{v}{u}}{uX}.$$

The isomorphism of theorem 1.5.1 maps $[X/u] \mapsto (\langle u \rangle, u)$, and consequently the homotopy class of the naive sum of two generators to $(\langle u, v \rangle, uv)$.

Cazanave's main theorem [Caz12, Theorem 1.2] states that the canonical map

$$\nu_{\mathbb{P}^1} \colon \left([\mathbb{P}^1, \mathbb{P}^1]^{\mathrm{N}}, \oplus^{\mathrm{N}}\right) \to \left([\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1}, \oplus^{\mathbb{A}^1}\right)$$

from naive to motivic homotopy classes is a group completion.

1.5.3 The Jouanolou device of $\mathbb{P}^1$

Jouanolou showed in [Jou73, Lemme 1.5] the following result.

Theorem 1.5.2 (Jouanolou)**.** *Let X be a quasi-projective scheme. Then there exists a vector bundle E on X and a torsor $p\colon W \to X$ over E, with W affine. In particular, the fibers of W are vector spaces.*

This was generalized by Thomason [Wei89, Proposition 4.4] by weakening the conditions on X, and the result is often called the Jouanolou–Thomason homotopy lemma.

With modern terminology, the map $p\colon W \to X$ is an $\mathbb{A}^1$-weak equivalence. We call W the Jouanolou device of X.

Definition 1.5.3. We let R denote the ring

$$R = \frac{k[x,y,z,w]}{(x+w-1, xw-yz)}.$$

The Jouanolou device of $\mathbb{P}^1$ is the smooth affine k-scheme $\mathcal{J} = \operatorname{Spec} R$. There is a $\mathbb{A}^1$-homotopy equivalence $\pi\colon \mathcal{J} \to \mathbb{P}^1$ with fibers $\mathbb{A}^1$.

The Jouanolou device of $\mathbb{P}^1$ is a crucial object in Paper I, where we describe the ring R, and the projective R-modules of rank 1. These determine invertible sheaves on $\mathcal{J}$, which in turn are used to describe morphisms $\mathcal{J} \to \mathbb{P}^1$ and their naive homotopy classes.

1.6 Motivic Brouwer degree

In algebraic topology, for $n \geq 1$, the Brouwer degree is a complete homotopy invariant for endomorphisms of the n-sphere [Hat02]. A map $f\colon S^n \to S^n$ induces a map of homology groups $f_*\colon H_n(S^n) \to H_n(S^n)$. If we pick a generator α of $H_n(S^n)$, then since $H_n(S^n) \cong \mathbb{Z}$, it follows that $f_*(\alpha) = d\alpha$ for some $d \in \mathbb{Z}$ which depends on f. This is the (topological) Brouwer degree of f, which we write as $\deg^{\mathrm{top}} f = d$. For any positive integer n, it is an isomorphism

$$\deg^{\mathrm{top}}\colon [S^n, S^n] \to \mathbb{Z}.$$

In [Mor06], Morel describes the analog in $\mathbb{A}^1$-homotopy theory. For pointed endomorphisms of $(\mathbb{P}^1)^{\wedge n}$, the motivic Brouwer degree is a homomorphism

$$\deg^{\mathbb{A}^1}\colon [(\mathbb{P}^1)^{\wedge n}, (\mathbb{P}^1)^{\wedge n}]^{\mathbb{A}^1} \to \mathrm{GW}(k).$$

When $n \geq 2$, this is an isomorphism, and for $n = 1$, it is an epimorphism. For $u \in k^\times$, let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be given by $f\colon [x_0 : x_1] \to [x_0 : ux_1]$, i.e. the rational function X/u. Its $\mathbb{A}^1$-Brouwer degree is $\deg^{\mathbb{A}^1}(f) = \langle u \rangle$.

Combining the $\mathbb{A}^1$-Brouwer degree and the resultant gives a map

$$f \mapsto \left(\deg^{\mathbb{A}^1}(f), \operatorname{res}(f)\right),$$

which, by [Caz12, Corollary 3.10] and [Mor12, Theorem 7.36] induces an isomorphism of groups

$$\rho\colon [\mathbb{P}^1, \mathbb{P}^1]^{\mathbb{A}^1} \xrightarrow{\cong} \mathrm{GW}(k) \times_{k^\times/k^{\times 2}} k^\times. \tag{1.2}$$

Using complex and real realization functors, the $\mathbb{A}^1$-Brouwer degree can be related to the topological Brouwer degree [Mor06]. For any subfield $k \subseteq \mathbb{R}$, sending a smooth k-scheme X to the topological space $X(\mathbb{R})$ equipped with its usual real manifold structure extends to a functor $\Re\colon \mathcal{H}_*(k) \to \mathcal{H}_*$ [AFW20; DI04; MV99]. Here $\mathcal{H}_*$ denotes the homotopy category of pointed topological spaces, and $\mathcal{H}_*(k)$ is the pointed $\mathbb{A}^1$-homotopy category over k.

Let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism and $\Re(f)\colon \mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ be its real realization. The signature (recall example 1.4.3), denoted sgn, of the symmetric bilinear form given by the $\mathbb{A}^1$-Brouwer degree of f equals the topological Brouwer degree of $\Re(f)$, i.e.,

$$\operatorname{sgn}\left(\deg^{\mathbb{A}^1}(f)\right) = \deg^{\mathrm{top}}(\Re(f)). \tag{1.3}$$

The real points of $\mathbb{P}^1$ and $\mathcal{J}$ are homeomorphic to respectively a circle and a cylinder. In appendix I.C, we use real realization and equation (1.3) to determine the $\mathbb{A}^1$-Brouwer degree of several morphisms $\mathcal{J} \to \mathbb{P}^1$.

1.7 The Eckmann–Hilton argument

Given a set with two unital magma structures that are mutually distributive, the two structures actually coincide and are commutative and associative. This powerful result is known as the Eckmann–Hilton argument.

The argument goes as follows. Let S be a set with magma operations $*$ and $\circ$ with respective neutral elements e_* and $e_\circ$. The binary operations are required to be mutually distributive in the sense that for any $\alpha, \beta, \gamma, \delta \in S$, the following equation holds:

$$(\alpha \circ \beta) * (\gamma \circ \delta) = (\alpha * \gamma) \circ (\beta * \delta). \tag{1.4}$$

By setting $\alpha = \delta = e_\circ$ and $\beta = \gamma = e_*$, the equation simplifies to $e_* = e_\circ$. This shows that the neutral elements coincide, and from here we just write e. By setting $\beta = \gamma = e$, equation 1.4 simplifies to $\alpha * \delta = \alpha \circ \delta$, showing that the two operations coincide. Similarly, setting $\alpha = \delta = e$ shows commutativity, and $\beta = e$ shows associativity.

In the category of pointed topological spaces, an h-group is a space G which is a group object in the homotopy category $\mathcal{H}_*$. This provides the hom-sets in the homotopy category $[X, G]$ with a group structure. Dually, an h-cogroup C gives the hom-sets $[C, Y]$ a group structure. The set of morphisms in the homotopy category from an h-cogroup to an h-group $[C, G]$ is then equipped with two group structures. In this case, one can show that the conditions for the Eckmann–Hilton argument hold (see [Swi75, Proposition 2.25] or [Ark11, Proposition 2.2.12]). This argument carries over to the $\mathbb{A}^1$-homotopy category.

The Eckmann–Hilton argument plays an important role in Paper I, where it is used in the proof of theorem I.3.7. In the $\mathbb{A}^1$-homotopy category, $\mathbb{P}^1$ is a cogroup object, and SL_2 is a group object, and by the argument above, the two group structures on $[\mathbb{P}^1, \mathrm{SL}_2]^{\mathbb{A}^1}$ coincide.

1.8 Paper summaries

In this section we provide a concise summary of each of the three papers, emphasizing the main results.

1.8.1 Summary of Paper I

We construct a group structure on the set of pointed naive homotopy classes of scheme morphisms from the Jouanolou device to the projective line, and write this as $\big([\mathcal{J},\mathbb{P}^1]^{\mathrm{N}},\oplus\big)$. The group operation is defined via matrix multiplication on generating sections of line bundles and only requires basic algebraic geometry. In particular, it is completely independent of the construction of the motivic homotopy category. The scheme morphism $\pi\colon \mathcal{J}\to\mathbb{P}^1$, exhibits the Jouanolou device as an affine torsor bundle over the projective line. The set of naive homotopy classes of $\mathbb{P}^1$-endomorphisms admits a commutative monoid structure, $\big([\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}},\oplus^{\mathrm{N}}\big)$, defined in [Caz12]. We prove in Theorem I.1.1 that π induces a morphism of commutative monoids,

$$\pi_{\mathrm{N}}^*\colon\ \big([\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}},\oplus^{\mathrm{N}}\big)\to\big([\mathcal{J},\mathbb{P}^1]^{\mathrm{N}},\oplus\big)\,.$$

We then show that the image of π_{N}^* is contained in, and group completes to, a concrete (a priori) subgroup $\mathbf{G}$. For finite fields, we show equality $\mathbf{G}=\big([\mathcal{J},\mathbb{P}^1]^{\mathrm{N}},\oplus\big)$. The group $\mathbf{G}$ is generated by a set of morphisms that are simple to describe. We are able to prove the following in Theorem I.1.2.

Theorem 1.8.1. *The monoid morphism $\pi_{\mathrm{N}}^*\colon[\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}\to\mathbf{G}$ is a group completion. There is a unique isomorphism $\chi\colon\mathbf{G}\to[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$ such that the diagram below commutes, where χ and ψ are mutual inverses to each other.*

$$\begin{tikzcd}
\mathbf{G} \arrow[dr, bend left, "\chi"] & \\
{[\mathbb{P}^1,\mathbb{P}^1]^{\mathrm{N}}} \arrow[u, "\pi_{\mathrm{N}}^*"] \arrow[r, "\nu_{\mathbb{P}^1}"'] & {[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}} \arrow[ul, dotted, "\psi"]
\end{tikzcd}$$

Theorem 1.8.1 gives a concrete description of all pointed endomorphisms of $\mathbb{P}^1$ in the unstable $\mathbb{A}^1$-homotopy category in the following sense: the group $\mathbf{G}$ is given by a simple set of generating morphisms, and the group operation $\oplus$ in $\mathbf{G}$ inherited from $[\mathcal{J},\mathbb{P}^1]^{\mathrm{N}}$ is defined in basic algebro-geometric terms.

The group $\mathbf{G}$ is an explicit geometric model for $[\mathbb{P}^1,\mathbb{P}^1]^{\mathbb{A}^1}$, the motivic analog of the fundamental group of the circle.

1.8.2 Summary of Paper II

In this paper, we constructively prove the existence of surjective morphisms from affine space $\mathbb{A}^n$ onto certain open subvarieties of $\mathbb{A}^n$. This is done over an algebraically closed field k of characteristic 0. For any irreducible algebraic set $Z\subset\mathbb{A}^{n-2}\subset\mathbb{A}^n$, we explicitly construct an $\mathbb{A}^n$-endomorphism having $\mathbb{A}^n\setminus Z$ as its image, and with $\deg(f)\le n+\deg(Z)$.

The construction is based off an example by Jelonek [Jel99] (see example 1.1.3), which we generalize to higher dimensions.

We are also able to explicitly construct, for $n\ge 2$ and any finite set of points $F\subset\mathbb{A}^n$, a morphism $f\colon\mathbb{A}^n\to\mathbb{A}^n$ such that $f(\mathbb{A}^n)=\mathbb{A}^n\setminus F$.

The constructions provide a big class of examples of maps $f\colon \mathbb{A}^n \to \mathbb{A}^n \setminus Z$, for different closed sets $Z \subset \mathbb{A}^n$. Previous examples in the literature have been of dimension two, with Z being a finite set of points.

1.8.3 Summary of Paper III

In this paper, we generalize and improve the results of Paper II. In Theorem III.1.1, we show the following main result.

Theorem 1.8.2. *Let $Z \subset \{0\} \times \mathbb{A}^{n-1} \subset \mathbb{A}^n$ be an algebraic subvariety. Then there exists a surjective algebraic map $F\colon \mathbb{A}^n \to \mathbb{A}^n \setminus Z$.*

A corollary of this is that for any affine algebraic variety Z of dimension $\leq n-2$, there is an algebraic variety $W \subset \mathbb{A}^n$ birational to Z and a surjective algebraic morphism $\mathbb{A}^n \to \mathbb{A}^n \setminus W$.

Based on the Gromov–Winkelmann theorem [Win90, Proposition 1] and results on flexible varieties [Arz+13; Arz23], the main idea is to utilize $\mathbb{G}_a$-actions to construct a morphism $\mathbb{A}^N \to \mathbb{A}^n \setminus Z$, then find an appropriate subspace isomorphic to $\mathbb{A}^n$ such that the restriction is still surjective. We also describe computational methods used to assist in finding apropriate subspaces.

We propose a general strategy to resolve the remaining cases, and conjecture that this can be used to construct surjective maps $\mathbb{A}^n \to \mathbb{A}^n \setminus Z$ for any closed subvariety Z of codimension at least 2.

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Papers

Paper II

Surjective morphisms from affine space to its Zariski open subsets

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II

Contents

Abstract

We prove constructively the existence of surjective morphisms from affine space onto certain open subvarieties of affine space of the same dimension. For any algebraic set $Z \subset \mathbb{A}^{n-2} \subset \mathbb{A}^n$, we construct an endomorphism of $\mathbb{A}^n$ with $\mathbb{A}^n \setminus Z$ as its image. By Noether's normalization lemma, these results extend to give surjective maps from any n-dimensional affine variety X to $\mathbb{A}^n \setminus Z$.

II.1 Introduction

In the context of Oka theory, Forstnerič [For17b] showed that every connected Oka manifold Y admits a surjective holomorphic map from an affine space $\mathbb{A}^N$. Moreover, this affine space can be taken to be $\mathbb{A}^{\dim Y}$. Motivated by this, Arzhantsev and Kusakabe [Arz23; Kus22] recently proved analogous results in the algebraic setting. For this paper, we consider algebraic varieties over an algebraically closed field $\mathbb{K}$ of characteristic zero.

Definition II.1.1. An algebraic variety Y is called an *A-image* if for some positive integer N there is a surjective morphism $f : \mathbb{A}^N \to Y$.

Arzhantsev proved that every very flexible variety is an A-image [Arz23]. As shown in [Arz+13], every very flexible variety is smooth and subelliptic. Kusakabe extended Arzhantsev's result by showing that any smooth subelliptic variety Y admits a surjective morphism $f : \mathbb{A}^{\dim Y+1} \to Y$ [Kus22]. For a definition of algebraic subellipticity, see [For17a, Definition 5.6.13 (e)], and note that it was recently shown to be equivalent to algebraic ellipticity for smooth varieties [KZ23]. Whether any A-image Y also admits surjective morphisms $f : \mathbb{A}^{\dim Y} \to Y$ is an open question [Arz23; For17b; For23; LT19]. When Y is a smooth *proper* subelliptic variety, Forstnerič proved over the complex numbers that such a surjective map from $\mathbb{A}^{\dim Y}$ does indeed exist [For17b, Theorem 1.6]. It is worth pointing out that Forstnerič and Kusakabe's results give maps which are surjective even when restricted to the smooth locus $\mathbb{A}^n \setminus (\mathrm{Sing}(f))$.

In this paper, we ask whether there are surjective morphisms $f : \mathbb{A}^{\dim Y} \to Y$ for open subvarieties $Y \subset \mathbb{A}^n$. For certain families of cases we give a positive answer. First of all, note that any A-image Y must have complement $\mathbb{A}^n \setminus Y$ of codimension at least two. If not, then Y admits a nonconstant invertible regular function g which pulls back to a nonconstant invertible regular function $f^*(g)$ on $\mathbb{A}^N$, giving a contradiction. However, as long as $\mathbb{A}^n \setminus Y$ has codimension at least two, Y is an A-image [Arz23; Kus22].

In the two-dimensional case, Jelonek gave the example of a surjective map $f : \mathbb{A}^2 \to \mathbb{A}^2 \setminus \{(0, -1)\}$ defined by $f(z_1, z_2) = (z_1(z_1 z_2 + 1) - z_2, z_1 z_2)$ [Jel99]. He also proved that no such map of lower degree exists. Modifying and generalizing this example yields the constructions used in Theorem II.1.2 and Proposition II.1.5. Note that Jelonek's map answered the question of the existence of a surjective morphism $f : \mathbb{A}^2 \to \mathbb{A}^2 \setminus \{(0, 0)\}$ even before it was asked in [LT19] (and then answered again in [Arz23]).

In what follows we state our main results and their corollaries, postponing the proofs until Section II.2.

Theorem II.1.2. *Let $Z \subset \mathbb{A}^n$ be an algebraic set of the form $Z = F \times W \subset \mathbb{A}^2 \times \mathbb{A}^{n-2}$, where $F \subset \mathbb{A}^2$ is a finite set of l points, and $W \subset \mathbb{A}^{n-2}$ is the zero set of m polynomials q_i of degree at most d. Then there exists a surjective map $f : \mathbb{A}^n \to \mathbb{A}^n \setminus Z$ of degree* $\deg(f) \leq \max(1, l-1) \cdot \max(l+2, m+d+1)$.

Given a closed algebraic set $Z \subset \mathbb{A}^k$ satisfying some mild conditions, there are bounds on the number m of polynomials required to specify Z as a set, and on the maximal degree d of these polynomials. For Z an irreducible closed subset of $\mathbb{A}^k$, [Hei83, Proposition 3] states that it is described by $k+1$ polynomials of degree bounded by $\deg Z$. If Z is a smooth equidimensional variety, then by [BJS04, Theorem I] it is described by $m = (k - \dim Z)(1 + \dim Z)$ polynomials, of degree bounded by $\deg Z$. As a consequence, we have the following corollary.

Corollary II.1.3. *If $Z \subset \mathbb{A}^{n-2} \subset \mathbb{A}^n$ is an irreducible closed algebraic set, then there exists a surjective map $f : \mathbb{A}^n \to \mathbb{A}^n \setminus Z$ with* $\deg(f) \leq n + \deg(Z)$. *If $Z \subset \mathbb{A}^{n-2} \subset \mathbb{A}^n$ is a smooth equidimensional variety, then there exists a surjective map $f : \mathbb{A}^n \to \mathbb{A}^n \setminus Z$ with* $\deg(f) \leq (n-2-\dim(Z))(1+\dim(Z))+\deg(Z)+1$.

Srinivas and Kaliman [Kal91; Sri91] showed that if $n \geq \max(2 \dim Z_1 + 1, \dim TZ_1) + 1$, and $\varphi : Z_1 \to Z_2$ is an isomorphism of closed varieties in $\mathbb{A}^n$, then φ extends to an $\mathbb{A}^n$-isomorphism. We obtain the following corollary.

Corollary II.1.4. *Let $Z_1 \subset \mathbb{A}^{n-2} \subset \mathbb{A}^n$ be a closed variety such that $n \geq \max(2\dim Z_1 + 1, \dim TZ_1) + 1$. Then for any $Z_2 \subset \mathbb{A}^n$ isomorphic to Z_1, there exists a surjective map $f : \mathbb{A}^n \to \mathbb{A}^n \setminus Z_2$.*

Over the real numbers, Fernando and Gamboa [FG03] showed the existence of a surjective morphism $f : \mathbb{R}^n \to \mathbb{R}^n \setminus F$ for any finite set F of l points, and $n \geq 2$. For F a subset of a line, their map f has degree $2(l+2)$. Morphisms of lower degree, namely $\deg(f) = l + 2$, were obtained by El Hilany [Hil22] in the setting where F is a subset of a line, l is even, in dimension 2 over $\mathbb{K}$ an algebraically closed field of characteristic zero. In the following proposition, by use of a different construction, we extend El Hilany's degree bound to also hold for l odd, and for any $n \geq 2$.

Proposition II.1.5. *For $n \geq 2$, and any set of l points $F \subset \mathbb{A}^n$, there exists a morphism $f : \mathbb{A}^n \to \mathbb{A}^n$ such that $f(\mathbb{A}^n) = \mathbb{A}^n \setminus F$. In general, if $l > 2$, then there exists an f with $\deg(f) \leq (l-1) \cdot (l+2)$. If F is a subset of a line, then there exists f with $\deg(f) = l + 2$.*

In particular, if $l < 3$, then there is a map f with $\deg(f) = l + 2$. It is worth noting that for Y open in $\mathbb{A}^2$, Y is an A-image precisely if $\mathbb{A}^2 \setminus Y$ has codimension two. Hence any two-dimensional A-image $Y \subset \mathbb{A}^2$ admits a surjective morphism from $\mathbb{A}^2$.

Finally, recall that the Noether normalization lemma [Eis95, Theorem 13.3] states that for any n-dimensional affine variety X, there exists a finite, hence surjective, morphism $X \to \mathbb{A}^n$. As a composition of surjections is surjective, we have the following proposition.

Proposition II.1.6. *For any n-dimensional affine variety X, and any Y admitting a surjective map $f : \mathbb{A}^n \to Y$, there exists a surjective map*

$$\varphi : X \to \mathbb{A}^n \to Y.$$

In particular, this shows that from any n-dimensional affine variety X, there are surjective maps onto varieties Y as in Theorem II.1.2 and Proposition II.1.5. By combining Noether normalization with Kusakabe's result [Kus22], we conclude that any smooth subelliptic variety Y of dimension n admits a surjective map from any $(n+1)$-dimensional affine variety X.

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II.2 Proofs and explicit constructions

We first construct a map g in Lemma II.2.1, which we then use to prove Theorem II.1.2. We then prove Proposition II.1.5.

Lemma II.2.1. *Let $g : \mathbb{A}^n_z \to \mathbb{A}^n_w$ be given by*

$$g = \big(z_1 + z_2(r(z_1)z_2 + 1) + \sum_{i=1}^{m} z_2^{i+1} q_i,\ r(z_1)z_2 + 1,\ z_3,\ \ldots,\ z_n\big), \tag{II.1}$$

where $r(z_1)$ has l simple roots $\{\beta_j\}_j$, and the m polynomials $q_i(z_3, z_4, \ldots, z_n)$ have degree at most d. Then g has image $\mathbb{A}^n \setminus Z$, where $Z = \{\beta_j\}_j \times \{0\} \times W$, where $W \subset \mathbb{A}^{n-2}$ is the set of common zeros of the q_i.

Proof. Fix $w \in \mathbb{A}^n_w$, and notice that any point z in the preimage $g^{-1}(w)$ must satisfy $z_3 = w_3, \ldots, z_n = w_n$. Fixing the values of $z_3, \ldots, z_n$ will in turn determine the values $q_i(z_3, \ldots, z_n) = \alpha_i$. Thus whether $g^{-1}(w)$ is empty only depends on the values of $w_1, w_2, \alpha_1, \ldots, \alpha_m$, and it reduces to solving the following system of equations for z_1 and z_2.

$$w_1 = z_1 + z_2(r(z_1)z_2 + 1) + \sum_{i=1}^{m} z_2^{i+1}\alpha_i \tag{II.2}$$

$$w_2 = r(z_1)z_2 + 1. \tag{II.3}$$

Solve Equation (II.2) for z_1, then substitute for z_1 in Equation (II.3) and simplify to obtain

$$z_1 = w_1 - z_2 w_2 - \sum_{i=1}^{m} \alpha_i z_2^{i+1} \tag{II.4}$$

$$0 = 1 - w_2 + z_2 \cdot r\left(w_1 - w_2 z_2 - \sum_{i=1}^{m} \alpha_i z_2^{i+1}\right). \tag{II.5}$$

If $w \in (\{\beta_j\}_j \times \{0\} \times W)$, then $(w_1, w_2, \alpha_1, \ldots, \alpha_m) = (\beta_j, 0, \ldots, 0)$ for some j, and Equation (II.5) reduces to $0 = 1$, which is clearly unsolvable.

If $w \notin (\{\beta_j\}_j \times \{0\} \times W)$, then $(w_1, w_2, \alpha_1, \ldots, \alpha_m) \neq (\beta_j, 0, \ldots, 0)$ for any j, and one may solve Equation (II.5) for z_2, and then solve Equation (II.4) for z_1. Hence, $g(\mathbb{A}^n) = \mathbb{A}^n \setminus Z$. ■

We may now prove the main theorem.

Proof of Theorem II.1.2. Given $Z = F \times W$ as in the theorem statement, we modify the function g in Equation (II.1) by composing with suitable automorphisms. For any $F \subset \mathbb{A}^2$, first pick a convenient coordinate system by applying an affine $\mathbb{A}^2$-automorphism t such that $t(F) = \{p_j\}_j$, where the points p_j have pairwise distinct first coordinates $\beta_j = t(p_j)^{(1)}$. To simplify the exposition, we assume $F = \{p_j\}_j$. Define

$$r(z_1) = \prod_{j=1}^{m} (z_1 - \beta_j)\,, \tag{II.6}$$

and use Lemma II.2.1 to construct a surjective map $g : \mathbb{A}^n \to \mathbb{A}^n \setminus (\{\beta_j\}_j \times \{0\} \times W)$. Let $L(z_1)$ be a Lagrange polynomial such that $L(\beta_j) = p_j^{(2)}$ for all j, and $h = (z_1, z_2 + L(z_1), z_3, \ldots, z_n)$, and set $f = h \circ g$. Then $f(\mathbb{A}^n) = \mathbb{A}^n \setminus Z$. The degree of h is at most $\max(1, l-1)$, and g has degree at most $\max(l+2, m+d+1)$. Hence the degree of f is bounded by $\max(1, l-1) \cdot \max(l+2, m+d+1)$. ■

Example II.2.2. Let $Z \subset \mathbb{A}^4$ be two parallel copies (contained in $\{(1,0)\} \times \mathbb{A}^2$ and $\{(-1,0)\} \times \mathbb{A}^2$ respectively) of a nodal cubic curve $W = V(w_3^2 - w_4^3 - w_4^2) \subset \mathbb{A}^2$. Then

$$f = \big(z_1 + z_2((z_1^2-1)z_2+1) + z_2^2(z_3^2 - z_4^3 - z_4^2),\, (z_1^2-1)z_2+1,\, z_3,\, z_4\big) \tag{II.7}$$

is a surjective map $f : \mathbb{A}^4 \to \mathbb{A}^4 \setminus Z$.

Example II.2.3. Let $d \geq n > 2$ be natural numbers, and for $i = 1, \ldots, n-2$, let $q_i(z_{i+2}) = \prod_{j=1}^{j=d-i-1}(z_{i+2} - j)$. Similarly, let $r(z_1) = \prod_{j=1}^{j=d-2}(z_1 - j)$. Then $f : \mathbb{A}^n \to \mathbb{A}^n$ given by

$$f = \Big(z_1 + z_2(r(z_1)z_2+1) + \sum_{i=1}^{n-2} z_2^{i+1} q_i,\, r(z_1)z_2+1,\, z_3,\, \ldots,\, z_n\Big), \tag{II.8}$$

has degree d, and the complement of the image of f is the set $\{1,2,\ldots,d-2\} \times \{0\} \times \{1,\ldots,d-2\} \times \{1,\ldots,d-3\} \times \ldots \times \{1,2,\ldots,d-n+1\}$, consisting of $(d-2)\cdot(d-2)!/(d-n)!$ points.

The above example gives a function with superexponentially many points in the complement of the image as n and d grow. We now construct f in Proposition II.1.5, which only avoids $\deg(f) - 2$ points. However, for any n, the proposition gives a surjective map $f : \mathbb{A}^n \to \mathbb{A}^n \setminus \{0\}$ of degree only $\deg(f) = 3$.

Proof of Proposition II.1.5. Given l points $F = \{p_j\}_j \subset \mathbb{A}^n$, pick a coordinate system in which $p_j^{(1)}$ are all pairwise distinct, as in the proof of Theorem II.1.2. Define the polynomial $r(z_1) = \prod_{j=1}^{l}\left(z_1 - p_j^{(1)}\right)$. Then define $\sigma_{2,r} : \mathbb{A}_z^2 \to \mathbb{A}_w^2$ to be

$$\sigma_{2,r} = \big(z_1 + z_2(r(z_1)z_2+1),\, r(z_1)z_2+1\big). \tag{II.9}$$

Note that $\sigma_{2,r}$ is equal to g defined in Equation (II.1), and hence $\deg(\sigma_{2,r}) = l+2$. For $n > 2$, define $\sigma_{n,r} : \mathbb{A}_z^n \to \mathbb{A}_w^n$ to be

$$\sigma_{n,r} = \big(z_1 + z_2(r(z_1)z_2+1),\, r(z_1)z_2+1+z_3,\, z_3^2+z_4,\, \ldots,\, z_{n-1}^2+z_n,\, z_n^2\big). \tag{II.10}$$

Let $n > 2$ and fix $w \in \mathbb{A}^n_w$. The same calculation as in the proof of Lemma II.2.1 leads to an equation similar to Equation (II.5), namely

$$0 = 1 - (w_2 - z_3) + z_2 \cdot r\,(w_1 - (w_2 - z_3)z_2)\,. \tag{II.11}$$

This is unsolvable for z_2 if and only if $w_1 \in \{p_j^{(1)}\}_j$ and $w_2 = z_3$. However, $z_3 = \pm\sqrt{w_3 \mp \sqrt{w_4 \pm \sqrt{\dots}}}$, which has multiple solutions unless $w_3 = 0, \dots, w_n = 0$. Hence $\sigma_{n,r}(\mathbb{A}^n) = \mathbb{A}^n \setminus \left(\{p_1^{(1)}, p_2^{(1)}, \dots, p_l^{(1)}\} \times \{0\} \times \dots \times \{0\}\right)$. Similarly to the proof of Theorem II.1.2, we may construct an $\mathbb{A}^n$-automorphism h by use of interpolation polynomials $L_i(z_1)$ such that $L_i\left(p_j^{(1)}\right) = p_j^{(i)}$ for all j. By composing, we obtain $f = h \circ \sigma_{n,r}$, which satisfies $f(\mathbb{A}^n) = \mathbb{A}^n \setminus F$, and has degree bounded by $\deg(f) \leq (l-1)(l+2)$. If F is a subset of a line, then h can be an affine automorphism, and we obtain $\deg(f) = l + 2$. ■

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