Comparison of qubit decoherence by quantum and classical telegraph noise

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Contents

1 Introduction

1.1 Outline

In this thesis we study the decoherence of a qubit coupled to a single two-level fluctuator. The fluctuator is coupled to an exterior environment of which we have created a simple model. We are interested in the entanglement between the qubit and fluctuator, and when we can make a classical approximation of the fluctuator. We expect that if the coupling to the environment is strong enough then the fluctuator will quickly dephase and behave like a classical system. In this case the fluctuator can be treated as a random telegraph noise signal.

We start by explaining the motivation and background for this study, beginning with a general introduction to quantum information and decoherence. Previous results are discussed and used to explain the motivation behind this study. In the introduction we have also include a theory section. Here we present relevant theory on the Bloch ball representation, decoherence, mutual information and generalized measurements. We conclude chapter 1 with a detailed description of the system we are analyzing, including the model of the environment. In chapter 2 we study a qubit coupled to a classical fluctuator producing telegraph noise. We look at the decoherence of the qubit and the mutual information between the qubit and fluctuator. In chapter 3 we look at a qubit coupled to a quantum fluctuator. We compare decoherence and mutual information with the case in chapter 2. We also look at the temperature-dependence of the system, which is where we have our main results. In chapter 4 we summarize the results and conclude with suggestions for further study. We have in addition an appendix where we have done some analytical calculations on a simplified version of our model. The results here are not very relevant for the main project, but can be of interest from a purely theoretical point of view. Among the topics touched upon in the appendix are the Bloch ball for n-dimensional systems and decoherence-free subspaces.

1.2 Background

A quantum mechanical system can never be completely isolated from the outside environment. The interactions between the quantum system and the environment give rise to a loss of information, called decoherence or dephasing. The information that is lost due to decoherence is the relative phase between states. If we look at a two-level system we have the wave-function

$$\psi = r_1|0\rangle + r_2 e^{i\theta}|1\rangle \tag{1.1}$$

where $|0\rangle$ and $|1\rangle$ are the basis states and we ignore a global phase factor. The measurement statistics are given by r_1 and r_2. This can be called the "classical" information in the system. The quantum information is given by the relative phase $e^{i\theta}$. Decoherence will cause this factor to decay, as explained later in the introduction.

Understanding decoherence is important from a theoretical point of view, as it can shed light on the quantum to classical transition [1]. This is known as the measurement problem. The theory of quantum mechanics does not currently have a proper explanation of measurements. Measurements are described by postulates stating that the wave-function will collapse to one of the eigenfunctions of the observable that is being measured. The probabilities are given by the eigenvalues. The details of this collapse are not fully understood, and many believe that decoherence is the key to this understanding.

An understanding of decoherence is also crucial to any technological application that makes use of quantum mechanical phenomena. An example of this is the quantum computer where prepared states are manipulated and used to store information and perform algorithms. These states will always decohere due to environmental noise, and a thorough understanding of this process is necessary in order to achieve the long dephasing times that are needed.

Much of the research done on decoherence is in the field of quantum information. This includes quantum computing, communication, cryptography and other applications. In this field quantum phenomena, such as superposition and entanglement, are exploited to create technology that would not be possible with purely classical systems. Examples of this are certain quantum algorithms that are faster than their classical counterparts [2]-[3], and cryptography schemes that are theoretically impossible to hack under ideal conditions [4]. The quantum mechanical system used to store information in most cases is called the quantum bit, or qubit. Like a classical bit the qubit has two possible states, normally called $|0\rangle$ and $|1\rangle$. The difference is that a qubit can also be in a superposition of the two states, allowing entanglement with other qubits. However, the qubit will also become entangled with the environment causing decoherence.

The main challenge when studying the decoherence of qubits is constructing a model of the environment. The model has to be realistic but still simple enough to do calculations. The key is to find the parts of the environment that are the biggest cause of decoherence, and ignore the smaller contributions. In many cases, the main contributions to decoherence are found to be electrons fluctuating between impurities in an insulator. For example, decoherence in Josephson qubits is caused by fluctuations in the insulator used as the tunnel barrier [5]. The electrons usually fluctuate between two impurities making them essentially two-level systems. We call these systems fluctuators. It is often the noise produced by a few fluctuators with low frequency that is the main cause of decoherene. This noise source is called $1/f$ noise due to the inverse frequency-dependence.

When analyzing the decoherence of a qubit coupled to a set of fluctuators it is normal to make a classical approximation. It is assumed that the fluctuators are so strongly coupled to the exterior environment that they will not become entangled with each other or the qubit. We can then treat the fluctuators as classical systems producing random telgraph noise (RTN). Qubit decoherence due to random telegraph noise has been studied extensively in recent years, both analytically and numerically [6]-[9]. The subject of interest in this thesis is when the approximation is valid. This has been analyzed by Grishin *et al.* [10], where they study a model of a qubit coupled to a set of fluctuating background charges. In this model the fluctuators are described as a set of impurities tunnel-coupled to the conduction band. Electrons can hop between the conduction band and the impurities creating the fluctuating background charges. With this model they derive the long-time decoherence rate of the qubit as a function of the temperature and coupling strength between the qubit and fluctuators. They find that, for high temperatures, the dependence of the decoherence rate on the coupling strength is the same as for the classical case. In a later study done by Abel and Marquardt [11], the same model is used but with only one fluctuator. Here they find the full time-dependence of the qubit decoherence. They characterize the strong-coupling regime by the critical coupling strength where visibility oscillations start to occur. They find that the critical coupling strength converges to the classical value at high temperatures. Both these papers focus mainly on deriving the time-dependence of the qubit decoherence, analyzing the quantum effects at low temperatures. In the chosen model, the temperature is the only parameter that decides how classically the fluctuator behaves. In this thesis we use a different model where we can also control the coupling between the fluctuator and environment. Here we hope to show that it is the combination of temperature and this coupling strength that decides the fluctuator dephasing rate and thereby controls the quantum to classical transition. This will validate the use of the classical approximation also at low temperatures.

1.3 Theory

1.3.1 Bloch Ball

Quantum states can be represented by density matrices. A general density matrix is given by

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{1.2}$$

where $|\psi_i\rangle$ are pure states and p_i is the probability of finding the system in the state $|\psi_i\rangle$. The density matrix has three conditions: It is hermitian, the trace is equal to one and all eigenvalues are greater than or equal to zero. For 2-level systems we can represent each density matrix as a point in a 3-dimensional space. This is because the matrix has three parameters. If we have the matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \tag{1.3}$$

then the parameters are ρ_{11}, $\mathrm{Re}\,\rho_{12}$ and $\mathrm{Im}\,\rho_{12}$. The third condition confines the points to a ball with a radius equal to one. The pure states are on the surface of the ball while points inside the ball represent mixed states. We can express the density matrix as

$$\rho = \frac{1}{2}(I + m_i \sigma_i) \tag{1.4}$$

where σ_i are the Pauli spin matrices, and summation over repeated indices is implied. The coefficients $\{m_i\}$ give us the coordinates of the Bloch vector. It is easy to show the following relations between the Bloch vector and density matrix elements.

$$m_x = \rho_{12} + \rho_{21} \tag{1.5}$$

$$m_y = i(\rho_{12} - \rho_{21}) \tag{1.6}$$

$$m_z = \rho_{11} - \rho_{22} \tag{1.7}$$

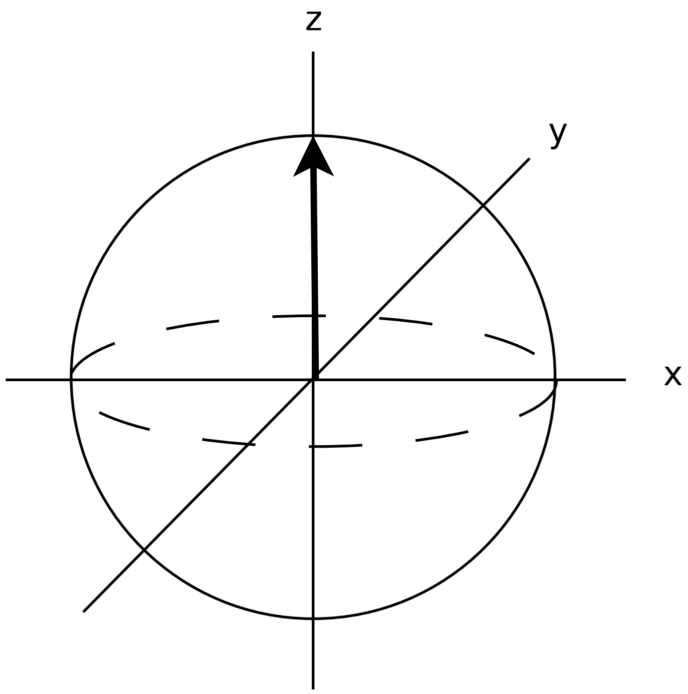

Figure 1.1: Bloch ball representation of the state $|0\rangle$

As an example, the state $|0\rangle$ has the density matrix $\rho = |0\rangle\langle 0|$ which can be written as $\rho = \frac{1}{2}(I + \sigma_z)$. The Bloch vector is then

$$\vec{m} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.8}$$

This vector is shown on the Bloch sphere in figure 1.1. The Bloch ball is useful for representing spin states because spin up and down are given by $m_z = 1$ and $m_z = -1$ respectively. The phase information is then given by the orientation in the xy-plane. Due to this useful representation it is normal to use spin systems as an example of a two-state qubit. It has also become standard due to magnetic resonance experiments using spin systems and external fields.

If an external field is present, we can analyze how it affects the Bloch vector. We continue our example with the state $|0\rangle$ but now add a field in the y-direction. This gives us the hamiltonian:

$$H = \frac{1}{2}\beta\sigma_y \tag{1.9}$$

It is normal to have a factor 1/2. β is then the energy splitting between the two eigenstates of the hamiltonian. The effect of the external field on the system is given by the time evolution operator:

$$U(t) = e^{-iHt} = e^{-\frac{i}{2}\beta\sigma_y t} \tag{1.10}$$

Using a series expansion and the fact that $\sigma_y^2 = I$ one can show that this becomes

$$U(t) = \cos(\frac{1}{2}\beta t)I - i\sin(\frac{1}{2}\beta t)\sigma_y \tag{1.11}$$

In matrix form we have

$$U(t) = \begin{pmatrix} \cos(\frac{1}{2}\beta t) & -\sin(\frac{1}{2}\beta t) \\ \sin(\frac{1}{2}\beta t) & \cos(\frac{1}{2}\beta t) \end{pmatrix} \tag{1.12}$$

If the system starts in the state $\rho = |0\rangle\langle 0|$ then after a time t the system will be in the state

$$\rho(t) = U(t)\rho U(t)^\dagger = \begin{pmatrix} \cos^2(\frac{1}{2}\beta t) & \cos(\frac{1}{2}\beta t)\sin(\frac{1}{2}\beta t) \\ \cos(\frac{1}{2}\beta t)\sin(\frac{1}{2}\beta t) & \sin^2(\frac{1}{2}\beta t) \end{pmatrix} \tag{1.13}$$

We can then find the Bloch vector elements:

$$m_x(t) = 2\cos(\frac{1}{2}\beta t)\sin(\frac{1}{2}\beta t) = \sin(\beta t) \tag{1.14}$$

$$m_y(t) = 0 \tag{1.15}$$

$$m_z(t) = \cos^2(\frac{1}{2}\beta t) - \sin^2(\frac{1}{2}\beta t) = \cos(\beta t) \tag{1.16}$$

We see that after a time t the Bloch vector rotates an angle βt towards the x-axis. This is shown in figure 1.2. In general an external field will cause the Bloch vector to precess in the plane normal to the field. The direction is given by the right-hand rule.

1.3.2 Decoherence

Quantum decoherence, or dephasing, denotes the loss of phase-information for a quantum state. This means that a pure quantum state will, after sufficiant dephasing, become a mixed state. A system in a pure state can be in a superposition of basis states. An example is the equal superposition state for a qubit:

$$\psi = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{1.17}$$

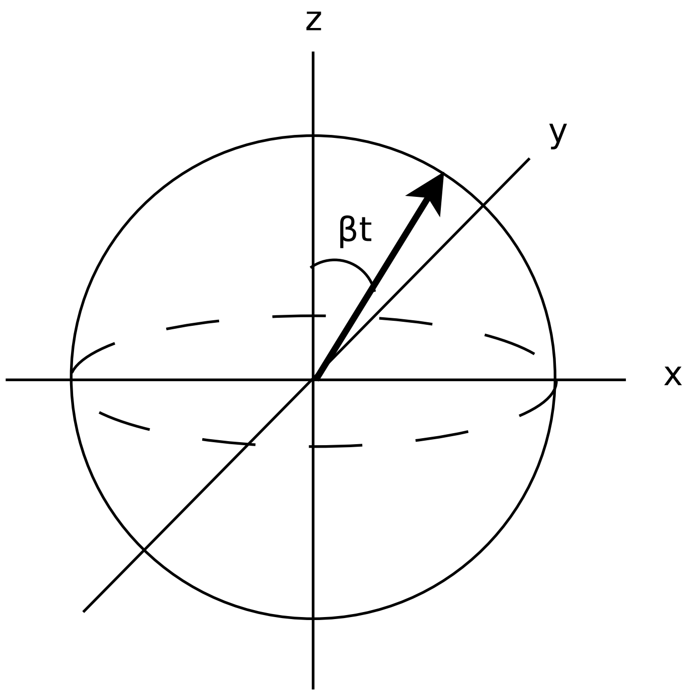

Figure 1.2: Rotation of Bloch vector due to external field

This state has the following density matrix

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{1.18}$$

The Bloch vector is

$$\vec{m} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{1.19}$$

shown in figure 1.3.

If we now add a field in the z-direction the Bloch vector will precess in the xy-plane. This will not cause any decoherence as we still have full information about the system. Instead of a constant field we can have a field that fluctuates between up and down in the z-direction. The precession of the Bloch vector will then change direction when the field does. If we don't know when the field changes direction we start to lose information about the qubit. In this example we can assume that the field has a 50% chance of changing direction every second. After the first second the state of the system is a mixture of the two pure states presessing in opposite directions.

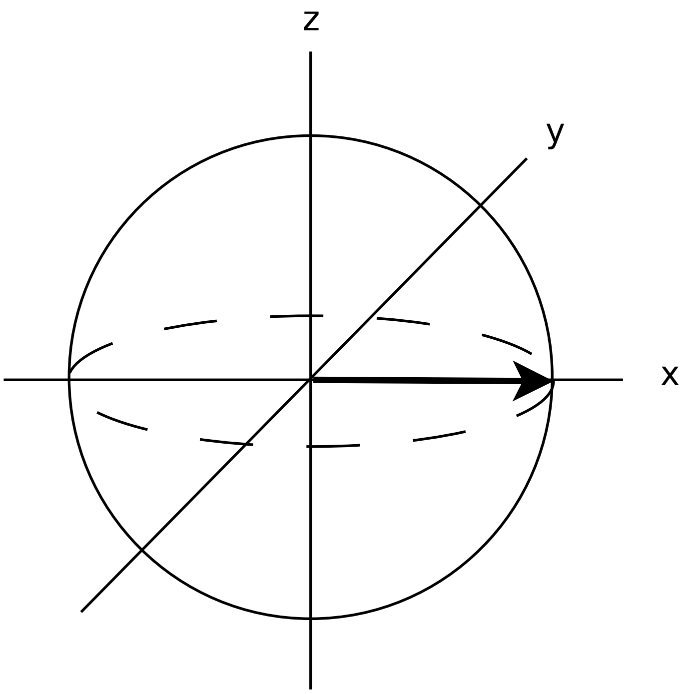

Figure 1.3: Bloch ball representation of the equal superposition state

The Bloch vector is then the sum of the Bloch vectors corresponding to the two states. Since the two vectors point in different directions, the sum will be a vector inside the Bloch sphere. After each subsequent second the number of states in the mixture will be doubled. The Bloch vector will always point in the x-direction but the length will go to zero. We then end up with a completely mixed state:

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.20}$$

1.3.3 Mutual Information

In classical information theory the Shannon entropy is a measure of the uncertainty about a random variable. It quantifies the amount of information we gain if we measure the variable. If the variable X has a set of values x_i with corresponding probabilities p_i then the Shannon entropy is defined as

$$H(X) = -\sum_i p_i \log p_i \tag{1.21}$$

It is easy to show that the entropy is greatest when all the probabilities are equal. This is when we gain the most information on average by measuring the variable. If one of the probabilities equals one and the rest are zero, we see that the entropy is zero. In this case we are certain of the outcome before we measure so we do not gain any new information.

If we have two random variables X and Y then we define the joint entropy

$$H(X,Y) = -\sum_{x,y} p_{xy} \log p_{xy} \tag{1.22}$$

We can now define mutual information $S(X:Y)$. Mutual information is a measure of how much information the two variables have in common, or how strong the correlation is between them. It tells us how much new information we gain about Y if we measure X. To calculate the mutual information we add the entropies of the to variables and subtract the joint entropy:

$$S(X:Y) = H(X) + H(Y) - H(X,Y) \tag{1.23}$$

For quantum mechanical systems we replace the Shannon entropy with the Von Neumann entropy

$$S(\rho_x) = \mathrm{Tr}\ \rho_x \log \rho_x \tag{1.24}$$

where ρ_x is the density matrix. The quantum mutual information is then given by

$$S(\rho_x : \rho_y) = S(\rho_x) + S(\rho_y) - S(\rho_{xy}) \tag{1.25}$$

where ρ_{xy} is the density matrix for the composite system. We see that the mutual information is zero when $S(\rho_{xy}) = S(\rho_x) + S(\rho_y)$. This is the case when the system is in a product state $\rho_{xy} = \rho_x \otimes \rho_y$. We then have no correlations between the systems. The mutual information is at a maximum when both systems have maximum entropy but the entropy of the composite system is zero. In this case all the uncertainty about one system is due to the correlations with the other system. Quantum mutual information is much used in this thesis as it quantifies both the classical correlations and quantum entanglement between the two systems.

1.3.4 Generalized Measurements

In this project we will make us of generalized measurements as described by Nielsen and Chuang [12]. A quantum measurement is normally represented by a set of projection operators. If a qubit has two possible states, $|0\rangle$ and $|1\rangle$, then a measurement in this basis is given by the operators $P_0 = |0\rangle\langle 0|$

and $P_1 = |1\rangle\langle 1|$. If the qubit starts in the state $|\psi\rangle$ and the outcome of the measurement is 0 then the state after the measurement is

$$|\psi^0\rangle = \frac{P_0|\psi\rangle}{\langle 0|\psi\rangle} \tag{1.26}$$

If the outcome of the measurement is 1 we have

$$|\psi^1\rangle = \frac{P_1|\psi\rangle}{\langle 1|\psi\rangle} \tag{1.27}$$

Sometimes we want to describe measurements that aren't necessarily projective. These measurements can be combinations of projective measurements and unitary operations. This formalism of generalized measurements is never strictly necessary as all physical measurements are projective, but it can simplify some problems where we are interested in the state of a system after a series of operations and measurements. Also, if a projective measurement is done on a larger system the effect on a subsystem can be described by a generalized measurement. The generalized measurement is represented by a set of operators M_m where m denotes the measurement outcome. If the system is in the state $|\psi\rangle$ before the measurement and the measurement outcome is m, then the state after is given by

$$|\psi^m\rangle = \frac{M_m|\psi\rangle}{\sqrt{p(m)}} \tag{1.28}$$

where $p(m)$ is the probability of the measurement resulting in the outcome m:

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle \tag{1.29}$$

Requiring that the probabilities sum to one gives us the completeness relation for the measurement operators:

$$\sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle = 1 \tag{1.30}$$

$$\Leftrightarrow \sum_m M_m^\dagger M_m = I \tag{1.31}$$

Using this definition we can find how the measurement affects the density matrix. We assume that the system starts in a general mixed state $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$. After a measurement with the outcome m each pure state $|\psi_i\rangle$ will be transformed to the state $|\psi_i^m\rangle$ where

$$|\psi_i^m\rangle = \frac{M_m|\psi_i\rangle}{\sqrt{p(m|i)}} = \frac{M_m|\psi_i\rangle}{\sqrt{\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle}} \tag{1.32}$$

Here $p(m|i)$ denotes the conditional probability of measuring the outcome m if the system is in the state $|\psi_i\rangle$. The density matrix after measuring the outcome m can be written as

$$\rho^m = \sum_i p(i|m)|\psi_i^m\rangle\langle\psi_i^m| = \sum_i p(i|m)\frac{M_m|\psi_i^m\rangle\langle\psi_i^m|M_m^\dagger}{p(m|i)} \tag{1.33}$$

$p(i|m)$ is the probability that the system was in the state $|\psi_i\rangle$ if we have measured the outcome m. We now want to find an expression for $p(i|m)/p(m|i)$. Using the definition of joint probability we have

$$p(m,i) \equiv p_m p(i|m) = p_i p(m|i) \tag{1.34}$$

$$\Rightarrow \frac{p(i|m)}{p(m|i)} = \frac{p_i}{p_m} \tag{1.35}$$

p_m can be written as

$$p_m = \sum_i p_i p(m|i) \tag{1.36}$$

To express $p(m|i)$ in a different way we use the following relation:

$$\langle\psi_i|M_m^\dagger M_m|\psi_i\rangle = \mathrm{Tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|) \tag{1.37}$$

This can be shown by first noting that for a given set of basis vectors $|n\rangle$ we have

$$\mathrm{Tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|) = \sum_n \langle n|M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|n\rangle \tag{1.38}$$

Choosing the basis so that $|1\rangle = |\psi_i\rangle$ gives us 1.37. We now have

$$p(m|i) = \mathrm{Tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|) \tag{1.39}$$

Inserting into the expression for p_m we have

$$p_m = \sum_i p_i \mathrm{Tr}(M_m^\dagger M_m|\psi_i\rangle\langle\psi_i|) \tag{1.40}$$

$$= \mathrm{Tr}(M_m^\dagger M_m\rho) = \mathrm{Tr}(M_m\rho M_m^\dagger) \tag{1.41}$$

We now have

$$\frac{p(i|m)}{p(m|i)} = \frac{p_i}{\mathrm{Tr}(M_m\rho M_m^\dagger)} \tag{1.42}$$

Inserting into the expression for ρ^m we arrive at

$$\rho^m = \sum_i p(i|m) \frac{M_m |\psi_i^m\rangle\langle\psi_i^m| M_m^\dagger}{p(m|i)} \tag{1.43}$$

$$= \sum_i p_i \frac{M_m |\psi_i\rangle\langle\psi_i| M_m^\dagger}{\mathrm{Tr}(M_m \rho M_m^\dagger)} \tag{1.44}$$

$$= \frac{M_m \rho M_m^\dagger}{\mathrm{Tr}(M_m \rho M_m^\dagger)} \tag{1.45}$$

We see that the measurement operators act on the density matrix in the usual fashion but since they are generally non-unitary we have to divide by the trace of the new density matrix to ensure the restriction $\mathrm{Tr}\,\rho = 1$. If we have performed a measurement without knowing the outcome the density matrix is given by

$$\rho' = \sum_m p_m \rho^m = \sum_m M_m \rho M_m^\dagger \tag{1.46}$$

1.4 Model

1.4.1 Hamiltonian

In this project we study a qubit entangled with a single two-level fluctuator. The fluctuator has the hamiltonian:

$$H_f = \frac{1}{2}\Delta\sigma_z + \frac{1}{2}\Lambda\sigma_x \tag{1.47}$$

where σ_z and σ_x are the Pauli matrices, Δ is the energy splitting and Λ is the tunneling constant. This hamiltonian represents an electron fluctuating between two impurities. The two positions have different energies and the tunneling probability is given by Λ. Realistically the hamiltonian should represent a free particle in a double-well potential. However if the energy is low enough we can make an approximation saying that the electron can only occupy the lowest state in each well. We use these as the basis states and we can assume they are orthogonal as long as we include a tunneling probability in the hamiltonian.

For simplicity we assume the qubit is not affected by any other field than the one created by the fluctuator. We are only interested in the dephasing of the qubit state so the internal qubit hamiltonian is not important. If the qubit was affected by a field we could still make the analysis in a rotating coordinate system. The hamiltonian for the whole system is

$$H = \frac{1}{2}\Delta I \otimes \sigma_z + \frac{1}{2}\Lambda I \otimes \sigma_x + \frac{1}{2}v\sigma_z \otimes \sigma_z \tag{1.48}$$

where $\otimes$ denotes a tensor product between the qubit and fluctuator Hilbert spaces. When calculating in matrix form this becomes a kronecker product. v is the interaction strength between the qubit and fluctuator. The interaction term is in the z-direction and will cause the qubit and fluctuator to become entangled. A pure state of the composite system will be of the form

$$|\psi\rangle = c_1|0\rangle_q \otimes |0\rangle_f + c_2|0\rangle_q \otimes |1\rangle_f + c_3|1\rangle_q \otimes |0\rangle_f + c_4|1\rangle_q \otimes |1\rangle_f \tag{1.49}$$

$$= c_1|00\rangle + c_2|01\rangle + c_3|10\rangle + c_4|11\rangle \tag{1.50}$$

In vector form the basis states for the composite system are

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_q \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_f = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.51}$$

$$|01\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_q \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}_f = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{1.52}$$

$$|10\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_q \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_f = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{1.53}$$

$$|11\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_q \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}_f = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.54}$$

We let the qubit start in the equal superposition state

$$|\psi_q\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{1.55}$$

This gives us the density matrix

$$\rho_q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{1.56}$$

In quantum computing it is normal to prepare the qubit in this state. We assume the fluctuator starts in an equal ensemble of the two states

$$\rho_f = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.57}$$

This is because the fluctuator is not protected from the environment which causes it to quickly decohere to a mixed state. Later we will let the fluctuator start at thermal equilibrium at a finite temperature. The initial state of the composite system is the product state

$$\rho = \rho_q \otimes \rho_f = \frac{1}{4}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{1.58}$$

The evolution of the system is given by the time evolution operator $U(t) = e^{-iHt}$. After a time t the density matrix is given by

$$\rho(t) = U(t)\rho(0)U(t)^\dagger \tag{1.59}$$

We can then find the reduced density matrix for the qubit:

$$\rho_q = \mathrm{Tr}_f(\rho) = \langle 0|_f \rho |0\rangle_f + \langle 1|_f \rho |1\rangle_f \tag{1.60}$$

If the density matrix for the composite system is

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \tag{1.61}$$

then the reduced density matrix for the qubit is

$$\rho_q = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix} \tag{1.62}$$

1.4.2 Environment

We will now describe a simple model for the interaction of our system with the environment. In this model photons (or phonons) interact with the fluctuator at regualar intervals τ. After a photon has interacted with the fluctuator it leaves and we assume it does not interact with anything else during the time scale we are interested in. After an interaction the photon will be in one of two states, $|ph_0\rangle$ and $|ph_1\rangle$, corresponding to the fluctuator states $|0\rangle$ and $|1\rangle$. These states have an overlap given by

$$\alpha = \langle ph_0|ph_1\rangle \tag{1.63}$$

If we ignore the qubit, the state of the fluctuator before the interaction can be written as

$$\rho_f = f_{00}|0\rangle\langle 0| + f_{01}|0\rangle\langle 1| + f_{10}|1\rangle\langle 0| + f_{11}|1\rangle\langle 1| \tag{1.64}$$

Right after the interaction we have the composite state for the fluctuator and photon:

$$\begin{aligned}\rho_{f,ph} = {} & f_{00}|0\rangle\langle 0| \otimes |ph_0\rangle\langle ph_0| + f_{01}|0\rangle\langle 1| \otimes |ph_0\rangle\langle ph_1| \\ & + f_{10}|1\rangle\langle 0| \otimes |ph_1\rangle\langle ph_0| + f_{11}|1\rangle\langle 1| \otimes |ph_1\rangle\langle ph_1|\end{aligned} \tag{1.65}$$

To find the reduced density matrix for the fluctuator we have to trace over the photon states:

$$\rho_f' = \langle ph_0|\rho_f|ph_0\rangle + \langle ph_1|\rho_f|ph_1\rangle \tag{1.66}$$

The fluctuator state right after the photon interaction is then

$$\rho_f' = f_{00}|0\rangle\langle 0| + \alpha f_{01}|0\rangle\langle 1| + \alpha f_{10}|1\rangle\langle 0| + f_{11}|1\rangle\langle 1| \tag{1.67}$$

We see that the off-diagonal matrix elements are multiplied by α each time a photon interacts with the fluctuator:

$$\rho_f = \begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix} \xrightarrow{ph} \begin{pmatrix} f_{00} & \alpha f_{01} \\ \alpha f_{10} & f_{11} \end{pmatrix} \tag{1.68}$$

This effectively reduces the xy-component of the Bloch vector.

The qubit is isolated from the photons and is only affected indirectly through the fluctuator. If the qubit-fluctuator system is in a pure state then the state of the qubit-fluctuator-photon system right after an interaction is

$$\begin{aligned}|\psi\rangle = {} & c_1|0\rangle_q \otimes |0\rangle_f \otimes |ph_0\rangle + c_2|0\rangle_q \otimes |1\rangle_f \otimes |ph_1\rangle \\ & + c_3|1\rangle_q \otimes |0\rangle_f \otimes |ph_0\rangle + c_4|1\rangle_q \otimes |1\rangle_f \otimes |ph_1\rangle\end{aligned} \tag{1.69}$$

The density matrix is then given by

$$\rho = |\psi\rangle\langle\psi| \tag{1.70}$$

Tracing over the photon states gives the following change in the qubit-fluctuator density matrix:

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \xrightarrow{ph} \begin{pmatrix} \rho_{11} & \alpha\rho_{12} & \rho_{13} & \alpha\rho_{14} \\ \alpha\rho_{21} & \rho_{22} & \alpha\rho_{23} & \rho_{24} \\ \rho_{31} & \alpha\rho_{32} & \rho_{33} & \alpha\rho_{34} \\ \alpha\rho_{41} & \rho_{42} & \alpha\rho_{43} & \rho_{44} \end{pmatrix} \tag{1.71}$$

These same matrix elements are multiplied by α each time a photon interacts with the system. For a general mixed state we have the same result of the interaction since every mixed state can be written as a sum of pure states.

As we see the only way one of these photons interacts with the system is by changing its own state according to the state of the fluctuator. We assume this happens instantly and we are not interested in the details of the interaction. The important effect of this interaction is that the photon becomes entangled with the fluctuator and "steals" information. We can call this a type of measurement although it doesn't cause the fluctuator wave function to collapse. It does however reduce the the purity of the state, causing the fluctuator to behave more like a classical telegraph noise signal. With the parameter α we can adjust how strongly the fluctuator is coupled to the environment and thus control the entanglement between the fluctuator and qubit.

This model of the environment does not allow the photons to exchange energy with the fluctuator. This means that, even if we have an energy splitting Δ between the two fluctuator states, we cannot define a temperature-dependent thermal equilibrium. The equilibrium point will always be an equal ensemble of the two states, which is the high-temperature limit. To analyze the system for low temperatures we must allow the flucutuator to absorb and emit photons. We will do this after we have studied the high-temperature case.

2 Classical Fluctuator

2.1 Telegraph Noise

Telegraph noise is defined as a signal with two levels and two rates for switching between them. The probability of switching is proportional to time. This is different from white noise because there is a finite correlation time for the signal. For the fluctuator we have the two states $|0\rangle$ and $|1\rangle$ with corresponding switching rates Γ_{01} and Γ_{10}. If the fluctuator is in the state $|0\rangle$ then the probability of switching to state $|1\rangle$ during a time Δt is $\Gamma_{01}\Delta t$. Similarly if the fluctuator is in the state $|1\rangle$ then the chance of switching to $|0\rangle$ is $\Gamma_{10}\Delta t$. Here we assume that Δt is small so that the probability of switching more than once during this time frame is virtually zero. For a general fluctuator state at time t we call the probabilities for being in state $|0\rangle$ and $|1\rangle$ $p_0(t)$ and $p_1(t)$ respectively. After a time Δt these probabilities are given by

$$p_0(t + \Delta t) = p_0(t)(1 - \Gamma_{01}\Delta t) + p_1(t)\Gamma_{10}\Delta t \tag{2.1}$$

$$p_1(t + \Delta t) = p_0(t)\Gamma_{01}\Delta t + p_1(t)(1 - \Gamma_{10}\Delta t) \tag{2.2}$$

If we rearrange the terms we can recognize this as a taylor-expansion to first order:

$$p_0(t + \Delta t) = p_0(t) + [p_1(t)\Gamma_{10} - p_0(t)\Gamma_{01}]\Delta t \tag{2.3}$$

$$= p_0(t) + \dot{p}_0(t)\Delta t \tag{2.4}$$

$$p_1(t + \Delta t) = p_1(t) + [p_0(t)\Gamma_{01} - p_1(t)\Gamma_{10}]\Delta t \tag{2.5}$$

$$= p_1(t) + \dot{p}_1(t)\Delta t \tag{2.6}$$

This gives us the coupled differential equations:

$$\dot{p}_0 = \Gamma_{10}p_1 - \Gamma_{01}p_0 \tag{2.7}$$

$$\dot{p}_1 = \Gamma_{01}p_0 - \Gamma_{10}p_1 \tag{2.8}$$

Using the relation $p_0 + p_1 = 1$ we have

$$\dot{p}_0 = \Gamma_{10}(1 - p_0) - \Gamma_{10}p_0 \tag{2.9}$$

$$= -\Gamma p_0 + \Gamma_{10} \tag{2.10}$$

where $\Gamma = \Gamma_{01} + \Gamma_{10}$. If the fluctuator starts in the state $|0\rangle$ the solution of the differential equation is

$$p_0(t) = \frac{1}{\Gamma}(\Gamma_{10} + \Gamma_{01}e^{-\Gamma t}) \tag{2.11}$$

For p_1 we have

$$p_1(t) = \frac{1}{\Gamma}(\Gamma_{01} - \Gamma_{01}e^{-\Gamma t}) \tag{2.12}$$

When comparing with a quantum fluctuator we are interested in the difference $p_0 - p_1$ because this is the z-component of the Bloch vector. In this case we have

$$p_0 - p_1 = \frac{1}{\Gamma}(\Gamma_{10} - \Gamma_{01} + 2\Gamma_{01}e^{-\Gamma t}) \tag{2.13}$$

For $t = 0$ we have $p_0 - p_1 = 1$, as expected. If $t \to \infty$ then $p_0 - p_1 \to (\Gamma_{10} - \Gamma_{01})/\Gamma$. We see that if $\Gamma_{01} = \Gamma_{10} = \Gamma/2$ then $p_0 - p_1$ goes to zero. The fluctuator has a relaxation to an equilibrium level given by $t \to \infty$. Γ is the rate at which the fluctuator relaxes.

2.2 Transfer Matrix Solution

We will now analyze a qubit coupled with a telegraph signal using the transfer matrix method developed by Cheng *et al.* [9]. We assume the qubit has no internal dynamics. The Hamiltonian for the system is then

$$H = \frac{1}{2}v(t)\sigma_z \tag{2.14}$$

where $v(t)$ is the is either v for fluctuator state $|0\rangle$ or $-v$ for fluctuator state $|1\rangle$. The fluctuator has the switching rates Γ_{01} and Γ_{10}. For each time step Δt the system will evolve according to the time evolution operator corresponding to the fluctuator state:

$$U_0 = e^{-iv\sigma_z\Delta t} = \begin{pmatrix} c - is & 0 \\ 0 & c + is \end{pmatrix} \tag{2.15}$$

$$U_1 = e^{iv\sigma_z\Delta t} = \begin{pmatrix} c + is & 0 \\ 0 & c - is \end{pmatrix} \tag{2.16}$$

where $c = \cos(\frac{1}{2}v\Delta t)$ and $s = \sin(\frac{1}{2}v\Delta t)$. After n time steps, the density matrix for a given telegraph signal is

$$\rho(n\Delta t) = U_n \ldots U_1\rho_0 U_1^\dagger \ldots U_n^\dagger \tag{2.17}$$

where each U_j is either U_0 or U_1 corresponding to the fluctuator state in the time interval, and ρ_0 is the initial qubit density matrix. This evolution can be mapped to the Bloch sphere representation as 3×3 matrices operating on the Bloch vector:

$$\vec{m}(n\Delta t) = T_n \dots T_1 \vec{m}_0 \tag{2.18}$$

Since we don't know the precise telegraph signal, we have to average over all possible signals:

$$\vec{m}(n\Delta t) = \overline{T_n \dots T_1} \vec{m}_0 = T\vec{m}_0 \tag{2.19}$$

where T_j is either T_0 or T_1 and T is called the ensemble averaged transfer matrix. To calculate T_0 and T_1 we need to first look at the evolution of the density matrix. For an arbitrary density matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \tag{2.20}$$

the evolution during a time step Δt with the fluctuator in state $|0\rangle$ is given by

$$U_0 \rho U_0^\dagger = \begin{pmatrix} c - is & 0 \\ 0 & c + is \end{pmatrix} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \begin{pmatrix} c + is & 0 \\ 0 & c - is \end{pmatrix} \tag{2.21}$$

$$= \begin{pmatrix} \rho_{11} & (c^2 - s^2 - 2ics)\rho_{12} \\ (c^2 - s^2 + 2ics)\rho_{21} & \rho_{22} \end{pmatrix} \tag{2.22}$$

The Bloch vector corresponding to ρ is

$$\vec{m} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} = \begin{pmatrix} \rho_{12} + \rho_{21} \\ i(\rho_{12} - \rho_{21}) \\ \rho_{11} - \rho_{22} \end{pmatrix} \tag{2.23}$$

After the time evolution we then have

$$T_0 \vec{m} = \begin{pmatrix} (c^2 - s^2 - 2ics)\rho_{12} + (c^2 - s^2 + 2ics)\rho_{21} \\ i[(c^2 - s^2 - 2ics)\rho_{12} - (c^2 - s^2 + 2ics)\rho_{21}] \\ \rho_{11} - \rho_{22} \end{pmatrix} \tag{2.24}$$

$$= \begin{pmatrix} (c^2 - s^2)m_x - 2csm_y \\ 2csm_x + (c^2 - s^2)m_y \\ m_z \end{pmatrix} \tag{2.25}$$

This gives us T_0:

$$T_0 = \begin{pmatrix} c^2 - s^2 & -2cs & 0 \\ 2cs & c^2 - s^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos v\Delta t & -\sin v\Delta t & 0 \\ \sin v\Delta t & \cos v\Delta t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.26}$$

Using the same procedure we find T_1:

$$T_1 = \begin{pmatrix} \cos v\Delta t & \sin v\Delta t & 0 \\ -\sin v\Delta t & \cos v\Delta t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.27}$$

We can then express the general matrix as

$$T_r = \begin{pmatrix} \cos v\Delta t & -a\sin v\Delta t & 0 \\ a\sin v\Delta t & \cos v\Delta t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.28}$$

where $a = 1$ for $r = 0$ and $a = -1$ for $r = 1$. As expected, the z-componant of the Bloch vector remains unchanged. Since the noise is oriented in the z-direction, it only affects the direction of precession in the xy-plane. T_0 makes the Bloch vector precess counter-clockwise while T_1 causes a clockwise precession. We now want to find the ensemble averaged transfer matrix. We define $G_n^{r'r}$ to be the transfer matrix for n time steps corresponding to the fluctuator starting in state r' and ending in r. This transfer matrix is averaged over intermediate fluctuator states and is weighted by the probability of the fluctuator being in state r at the end of the time evolution. For one time step we have

$$G_1^{r'r} = W_{r'r}T_{r'} \tag{2.29}$$

where $W_{r'r}$ is the probability of the fluctuator switching from state r' to r. These are given by

$$W_{01} = \Gamma_{01}\Delta t \tag{2.30}$$
$$W_{00} = 1 - \Gamma_{01}\Delta t \tag{2.31}$$
$$W_{10} = \Gamma_{10}\Delta t \tag{2.32}$$
$$W_{11} = 1 - \Gamma_{10}\Delta t \tag{2.33}$$

For two time steps we have

$$G_2^{r'r} = \sum_{r''} W_{r''r}T_{r''}W_{r'r''}T_{r'} \tag{2.34}$$
$$= \sum_{r''} W_{r''r}T_{r''}G_1^{r'r''} \tag{2.35}$$

For n time steps we have

$$G_n^{r'r} = \sum_{r''} W_{r''r}T_{r''}G_{n-1}^{r'r''} \tag{2.36}$$

Before we proceed we wish to express T_r in a different form. We see that

$$T_r = \cos(v\Delta t)L_z^2 + ia\sin(v\Delta t)L_z + I - L_z^2 \tag{2.37}$$

where L_z is one of the generators of SO(3):

$$L_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.38}$$

Performing a series expansion we find

$$T_r = I + iav\Delta t L_z + \frac{1}{2}(iav\Delta t L_z)^2 + \dots \tag{2.39}$$
$$= e^{iav\Delta t L_z} \tag{2.40}$$

We can now define an operator

$$A = We^{iv\Delta t L_z \otimes \sigma_z} \tag{2.41}$$

where σ_z is the Pauli matrix acting on the fluctuator state space, giving ± 1 according to the fluctuator state. W is a matrix acting on the fluctuator state space giving the correct probabilities. The fluctuator states in vector form are

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.42}$$

The matrix W is then

$$W = \begin{pmatrix} W_{00} & W_{10} \\ W_{01} & W_{11} \end{pmatrix} = \begin{pmatrix} 1 - \Gamma_{01}\Delta t & \Gamma_{10}\Delta t \\ \Gamma_{01}\Delta t & 1 - \Gamma_{10}\Delta t \end{pmatrix} \tag{2.43}$$

The operator A has the property $\langle r|A|r'\rangle = W_{r'r}T_{r'}$. We then have

$$G_1^{r'r} = \langle r|A|r'\rangle \tag{2.44}$$
$$G_2^{r'r} = \sum_{r''} \langle r|A|r''\rangle\langle r''|A|r'\rangle \tag{2.45}$$
$$= \langle r|A \sum_{r''} |r''\rangle\langle r''|A|r'\rangle \tag{2.46}$$
$$= \langle r|A^2|r'\rangle \tag{2.47}$$
$$G_n^{r'r} = \langle r|A^n|r'\rangle \tag{2.48}$$

where we have used the completeness relation $\sum_r |r\rangle\langle r| = I$. The transfer matrix is then given by summing over the final fluctuator state. We can also to choose to average over the initial state. We then have

$$T = \sum_{r,r'} G_n^{r'r} p_{r'} = (\langle 0| + \langle 1|) A^n (p_0|0\rangle + p_1|1\rangle) \tag{2.49}$$

where $p_{r'}$ is the probability of the fluctuator starting in the state r'. We now take the continuum limit. Expanding to first order we have

$$A = W(I + iv\Delta t L_z \otimes \sigma_z) \tag{2.50}$$
$$= (I - V\Delta t)(I + iv\Delta t L_z \otimes \sigma_z) \tag{2.51}$$
$$= I - (I \otimes V - ivL_z \otimes \sigma_z)\Delta t \tag{2.52}$$
$$\tag{2.53}$$

where

$$V = \begin{pmatrix} \Gamma_{01} & -\Gamma_{10} \\ -\Gamma_{01} & \Gamma_{10} \end{pmatrix} \tag{2.54}$$

and we have excluded higher order terms. As Δt goes to zero this becomes a matrix exponential function:

$$A = I - (I \otimes V - ivL_z \otimes \sigma_z)\Delta t \tag{2.55}$$
$$= I - B\Delta t \tag{2.56}$$
$$= e^{-B\Delta t} \tag{2.57}$$

where $B = I \otimes V - ivL_z \otimes \sigma_z$. Finally we have

$$A^n = e^{-Bn\Delta t} = e^{-Bt} \tag{2.58}$$

Instead of performing the partial inner product in 2.49, we can represent A^n as a 6×6 matrix acting on the 6-dimensional vector given by the kronecker product of the initial qubit Bloch vector and fluctuator state. We call this vector $\vec{q}_i$:

$$\vec{q}_i = \vec{m}_i \otimes |f_i\rangle = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} \otimes \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \tag{2.59}$$

After a time t we can express the state of the system as

$$\vec{q}(t) = e^{-Bt}\vec{q}_i = p_0(t)\vec{m}_0(t) \otimes |0\rangle + p_1(t)\vec{m}_1(t) \otimes |1\rangle \tag{2.60}$$

where $p_0(t)$ and $p_1(t)$ are the probabilities of the fluctuator being in state $|0\rangle$ and $|1\rangle$ respectively after a time t, and $\vec{m}_0(t)$ and $\vec{m}_1(t)$ are the corresponding qubit Bloch vectors. We can then find the final qubit Bloch vector by taking the partial inner product with the end fluctuator state:

$$\vec{m}(t) = (\langle 0| + \langle 1|)(p_0(t)\vec{m}_0(t) \otimes |0\rangle + p_1(t)\vec{m}_1(t) \otimes |1\rangle) \tag{2.61}$$
$$= p_0(t)\vec{m}_0(t) + p_1(t)\vec{m}_1(t) \tag{2.62}$$

When solving this numerically we compute $\vec{q}(t)$:

$$\vec{q}(t) = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} \tag{2.63}$$

$\vec{m}(t)$ is then given by

$$\vec{m}(t) = \begin{pmatrix} q_1 + q_2 \\ q_3 + q_4 \\ q_5 + q_6 \end{pmatrix} \tag{2.64}$$

$\vec{m}_0(t)$ and $\vec{m}_1(t)$ are given by

$$\vec{m}_0 = \frac{1}{p_0(t)} \begin{pmatrix} q_1 \\ q_3 \\ q_5 \end{pmatrix}, \quad \vec{m}_1 = \frac{1}{p_1(t)} \begin{pmatrix} q_2 \\ q_4 \\ q_6 \end{pmatrix} \tag{2.65}$$

2.3 Qubit Decoherence

As a measure of the qubit coherence we use the length of the Bloch vector in the xy-direction. Using the transfer matrix method to compute the Bloch vector numerically, we can plot the coherence as a function of time. In this case we choose $\Gamma_0 1 = \Gamma_1 0$ and the fluctuator starts in an equal ensemble of $|0\rangle$ and $|1\rangle$. The qubit starts in the equal superposition state

$$|\psi_q\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{2.66}$$

In figure 2.1 the coherence is plotted for various coupling strengths. The result is well known and we can see the exponential long-time decay. For

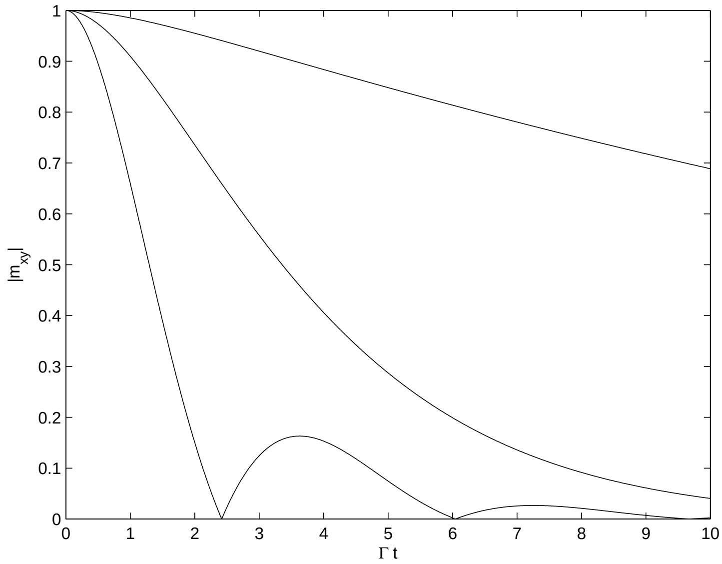

Figure 2.1: Qubit decoherence due to classical telegraph noise. Top to bottom: coupling strength $v/\Gamma = 0.2$, 0.5, 1.0.

stronger coupling we observe oscillations in the qubit Bloch vector length. This is due to the fact that the Bloch vector is a weighted sum over all possible telegraph signals. The two vectors corresponding to the two possible initial fluctuator states will precess in opposite directions and, if the coupling is strong enough, will reach opposite sides of the Bloch sphere while the probability of the fluctuator switching states is still low. This gives a premature loss of coherence which returns once there are more contributions from vectors precessing in other directions.

2.4 Mutual Information

After the time evolution the qubit-fluctuator system will be in a separable state. This means that there is no quantum entanglement, but there can still be classical correlations between the qubit and fluctuator. The density matrix for the composite system is given by

$$\rho_{qf} = p_0 \rho_q^0 \otimes \rho_f^0 + p_1 \rho_q^1 \otimes \rho_f^1 \tag{2.67}$$

where p_0 and p_1 are the probabilities for the fluctuator being in state $|0\rangle$ and $|1\rangle$, and ρ_q^0, ρ_f^0 and ρ_q^1, ρ_f^1 are the corresponding reduced density matrices for the qubit and fluctuator. We have

$$\rho_f^0 = |0\rangle\langle 0|, \quad \rho_f^1 = |1\rangle\langle 1| \tag{2.68}$$

and ρ_q^0, ρ_q^1 are given by $\vec{m}_0$, $\vec{m}_1$. The general reduced density matrix for the qubit is then

$$\rho_q = p_0\rho_q^0 + p_1\rho_q^1 \tag{2.69}$$

and for the fluctuator

$$\rho_f = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1| \tag{2.70}$$

The mutual information is then given by

$$S(q:f) = S(\rho_q) + S(\rho_f) - S(\rho_{qf}) \tag{2.71}$$

In figure 2.2 we have plotted the mutual information for various coupling strengths. As expected a stronger coupling between the qubit and fluctuator will give a higher peak in mutual information. We also see that a stronger coupling causes the mutual information to decay faster. This is because the entropy of the entire system reaches its maximum sooner.

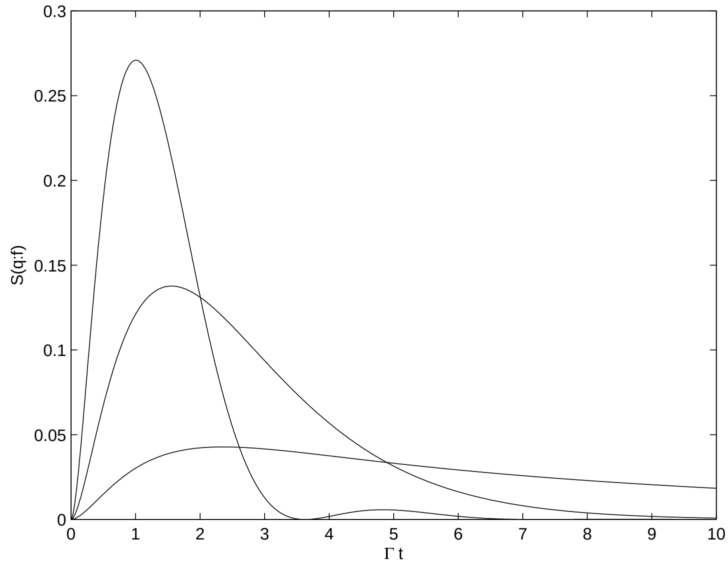

Figure 2.2: Mutual information between qubit and classical fluctuator. Top to bottom: coupling strength $v/\Gamma = 1.0,\ 0.5,\ 0.2$.

3 Quantum Fluctuator

3.1 Fluctuator Relaxation

We would like to study the relaxation of the fluctuator due to the photon interactions. We choose a simple case for the hamiltonian:

$$H = \frac{1}{2}\Lambda\sigma_x \tag{3.1}$$

We assume that the coupling to the qubit is weak compared to the internal hamiltonian. For this case we are able to study the relaxation analytically. Later when the hamiltonian is more complex we compute the relaxation rate numerically, so the aim of this section is merely to give an illustration of the fluctuator behavior. The time evolution between each photon interaction is given by

$$U = e^{-iH\tau} = \cos(\frac{1}{2}\Lambda\tau) - i\sigma_x \sin(\frac{1}{2}\Lambda\tau) \tag{3.2}$$

$$= \begin{pmatrix} u & -iv \\ -iv & u \end{pmatrix} \tag{3.3}$$

where $u = \cos(\frac{1}{2}\Lambda\tau)$, $v = \sin(\frac{1}{2}\Lambda\tau)$ and τ is the interval between photons. For a given density matrix

$$\rho_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \tag{3.4}$$

the density matrix after the time evolution and before the photon interaction is given by

$$\tilde{\rho}_{n+1} = U\rho_n U^\dagger \tag{3.5}$$

$$= \begin{pmatrix} u & -iv \\ -iv & u \end{pmatrix} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} u & iv \\ iv & u \end{pmatrix} \tag{3.6}$$

$$= \begin{pmatrix} u^2 a_n + iuvb_n - iuvc_n + v^2 d_n & iuva_n + u^2 b_n + v^2 c_n - iuvd_n \\ -iuva_n + v^2 b_n + u^2 c_n + iuvd_n & v^2 a_n - iuvb_n + iuvc_n + u^2 d_n \end{pmatrix} \tag{3.7}$$

After the photon interaction the off-diagonal elements are multiplied by α. We then have the following recursion relations:

$$a_{n+1} = u^2 a_n + iuvb_n - iuvc_n + v^2 d_n \tag{3.8}$$
$$b_{n+1} = (iuva_n + u^2 b_n + v^2 c_n - iuvd_n)\alpha \tag{3.9}$$
$$c_{n+1} = (-iuva_n + v^2 b_n + u^2 c_n + iuvd_n)\alpha \tag{3.10}$$
$$d_{n+1} = v^2 a_n - iuvb_n + iuvc_n + u^2 d_n \tag{3.11}$$

We can simplify the problem by switching to the Bloch ball representation. The Bloch vector elements are given by

$$x_n = b_n + c_n \tag{3.12}$$
$$y_n = i(b_n - c_n) \tag{3.13}$$
$$z_n = a_n - d_n \tag{3.14}$$

We then have the recursion relations for the Bloch vector elements:

$$x_{n+1} = (b_n + c_n)\alpha \tag{3.15}$$
$$= \alpha x_n \tag{3.16}$$
$$y_{n+1} = -2uv(a_n - d_n)\alpha + i(u^2 - v^2)(b_n - c_n)\alpha \tag{3.17}$$
$$= -\alpha \sin(\Lambda\tau) z_n + \alpha \cos(\Lambda\tau) y_n \tag{3.18}$$
$$z_{n+1} = (u^2 - v^2)(a_n - d_n) + 2iuv(b_n - c_n) \tag{3.19}$$
$$= \cos(\Lambda\tau) z_n + \sin(\Lambda\tau) y_n \tag{3.20}$$

The evolution of the system is then given by

$$\vec{m}_{n+1} = M\vec{m}_n \tag{3.21}$$

where $\vec{m}$ is the Bloch vector and

$$M = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha\cos(\Lambda\tau) & -\alpha\sin(\Lambda\tau) \\ 0 & \sin(\Lambda\tau) & \cos(\Lambda\tau) \end{pmatrix} \tag{3.22}$$

We can decompose $\vec{m}_n$ in the eigenvectors $\vec{v}_i$ of M:

$$\vec{m}_n = \sum_i c_{n_i} \vec{v}_i \tag{3.23}$$

$\vec{m}_{n+1}$ is then given by

$$\vec{m}_{n+1} = M\vec{m}_n \tag{3.24}$$
$$= \sum_i c_{n_i} \lambda_i \vec{v}_i \tag{3.25}$$

where λ_i are the eigenvalues of M. The general solution is then

$$\vec{m}_n = M^n \vec{m}_0 \tag{3.26}$$
$$= \sum_i c_{0_i} \lambda_i^n \vec{v}_i \tag{3.27}$$

The eigenvalues of M are

$$\lambda_1 = \alpha \tag{3.28}$$
$$\lambda_2 = \frac{1}{2}[\cos(\Lambda\tau)(1+\alpha) + \sqrt{\cos^2(\Lambda\tau)(1+\alpha)^2 - 4\alpha}] \tag{3.29}$$
$$\lambda_3 = \frac{1}{2}[\cos(\Lambda\tau)(1+\alpha) - \sqrt{\cos^2(\Lambda\tau)(1+\alpha)^2 - 4\alpha}] \tag{3.30}$$

If $\tau \ll 1$ then we have $\cos(\Lambda\tau) \approx 1$. We can then approximate λ_2 and λ_3:

$$\lambda_2 \approx \frac{1}{2}(1+\alpha+\sqrt{(1+\alpha)^2 - 4\alpha}) \tag{3.31}$$
$$= 1 \tag{3.32}$$
$$\lambda_3 \approx \frac{1}{2}(1+\alpha-\sqrt{(1+\alpha)^2 - 4\alpha}) \tag{3.33}$$
$$= \alpha \tag{3.34}$$

We see that for small τ λ_2^n will dominate over the other terms so that

$$\vec{m}_n \approx c_{0_2} \lambda_2^n \vec{v}_2 \tag{3.35}$$

If the fluctuator starts in the state $|0\rangle$ then the z-component $m_{0_z} = 1 = \lambda_2^0$. After a time t the z-component can be approximated by

$$m_z(t) \approx \lambda_2^n \tag{3.36}$$
$$= \lambda_2^{t/\tau} \tag{3.37}$$
$$= e^{\frac{t}{\tau}\ln\lambda_2} \tag{3.38}$$
$$= e^{-\Gamma t} \tag{3.39}$$

The z-component of the Bloch vector has an exponential decay with the rate $\Gamma = -\frac{1}{\tau}\ln\lambda_2$. This is the rate usually associated with the relaxation time T_1. If we analyze λ_2 we see that it has an imaginary part when the term inside the square root is negative:

$$\cos^2(\Lambda\tau)(1+\alpha)^2 < 4\alpha \tag{3.40}$$

This happens when α approaches 1. When λ_2 is complex we can write it as

$$\lambda_2 = re^{i\phi} \tag{3.41}$$

where r is the modulus and ϕ is the phase. Taking the logarithm we find the relaxation rate:

$$\Gamma = -\frac{1}{\tau}(\ln r + i\phi) \tag{3.42}$$

The imaginary term gives rise to oscillations in the relaxation curve. If $\alpha = 1$ then Γ is completely imaginary. This can be shown by multiplying λ_2 with its complex conjugate. This gives us the absolute value 1. If $r = 1$ then $\ln r = 0$. We then have no decoherence and the z-component of the Bloch vector will simply oscillitate between -1 and 1.

Later we will add a σ_z term to the hamiltonian. This makes analytical calculations much more difficult, but we can show numerically that the decay is almost identical. However, for high energy splitting $\Delta \gg \Lambda$ we see a decrease in the relaxation rate.

3.2 Qubit Decoherence

We now study the decoherence of a qubit coupled with a quantum fluctuator. The hamiltonian is

$$H = \frac{1}{2}\Delta I \otimes \sigma_z + \frac{1}{2}\lambda I \otimes \sigma_z + \frac{1}{2}v\sigma_z \otimes \sigma_z \tag{3.43}$$

and the fluctuator interacts regularly with external photons as described in the introduction.

We are interested in comparing the qubit decoherence to that of the previous case where the qubit is coupled to classical telegraph noise. When the overlap α between the photon states is small then the fluctuator should not have time to become entangled with the qubit. We should then recover the same decoherence behavior as in the classical case.

We implement this model numerically. We let the qubit start in the equal superposition state

$$|\psi_q\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{3.44}$$

and the fluctuator starts in the mixed state

$$\rho_f = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \tag{3.45}$$

For each time step Δt we have the time evolution $\rho(t+\Delta t) = U(\Delta t)\rho(t)U(\Delta t)^\dagger$. If a time τ has passed we let the flucuator interact with a photon giving the following change to the density matrix:

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \xrightarrow{ph} \begin{pmatrix} \rho_{11} & \alpha\rho_{12} & \rho_{13} & \alpha\rho_{14} \\ \alpha\rho_{21} & \rho_{22} & \alpha\rho_{23} & \rho_{24} \\ \rho_{31} & \alpha\rho_{32} & \rho_{33} & \alpha\rho_{34} \\ \alpha\rho_{41} & \rho_{42} & \alpha\rho_{43} & \rho_{44} \end{pmatrix} \tag{3.46}$$

We then find the reduced density matrix for the qubit:

$$\rho_q = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix} \tag{3.47}$$

Finally we want to find the Bloch vector component in the xy-plane. The components m_x and m_y are given by

$$m_x = \rho_{q_{12}} + \rho_{q_{21}} \tag{3.48}$$

$$m_y = i(\rho_{q_{12}} - \rho_{q_{21}}) \tag{3.49}$$

The xy-component is then

$$m_{xy} = \sqrt{m_x^2 + m_y^2} \tag{3.50}$$

To compare the decoherence to the case with telegraph noise we have to find the relaxation rate Γ. To do this we have a seperate program that calculates the z-component of the flucuator Bloch vector. The fluctuator starts in the state $|0\rangle$ and we let it evolve in time according to the hamiltonian

$$H = \frac{1}{2}\Delta\sigma_z + \frac{1}{2}\Lambda\sigma_x \tag{3.51}$$

We neglect the coupling to the qubit, assuming the internal hamiltonian is much stronger. In addition to the time evolution we have the photon interactions after every interval τ. After a sufficient amount of time has passed the z-component of the Bloch vector will decrease exponentially from 1 to 0 with the rate Γ:

$$m_z = e^{-\Gamma t} \tag{3.52}$$

To find Γ we can plot the logarithm

$$\log m_z = -\Gamma t \tag{3.53}$$

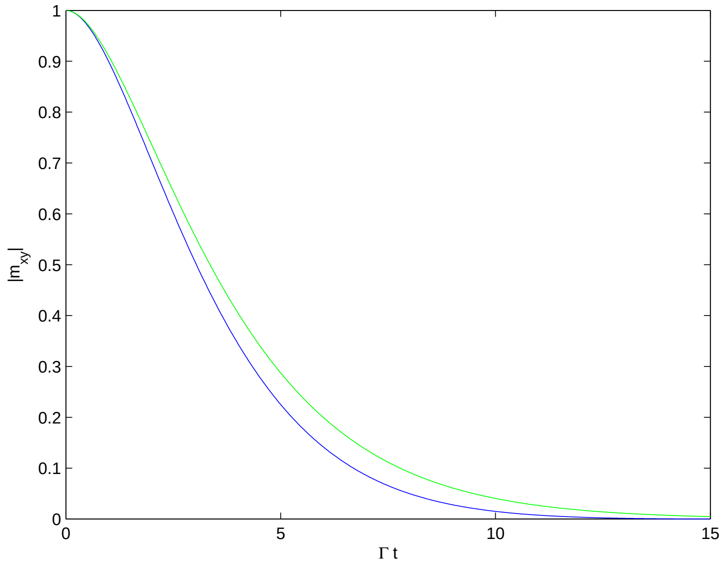

Figure 3.1: Decoherence of qubit due to quantum (blue) and classical (green) telegraph noise for $\alpha = 0.9$, $v/\Gamma = 0.5$, $\Lambda\tau = 0.05$, $\Lambda = \Delta$.

and make a linear fit. The switching rate for the telegraph signal we want to compare with is then $\Gamma/2$.

We can now plot the xy-component of the qubit Bloch vector as a function of time. In figure 3.1 we compare decoherence due to quantum and classical telegraph noise. Here α is close to 1 so we see that the curves do not match completely. We see a larger rate of decoherence for the qubit coupled with the quantum fluctuator. This is due to the entanglement of the two systems in addition to classical correlations. If we reduce α the decoherence curve from quantum telegraph noise is closer to the classical limit. This is shown in figure 3.2.

3.3 Mutual Information

In figures 3.3 and 3.4 we plot the mutual information between the qubit and fluctuator given by

$$S(q:f) = S(\rho_q) + S(\rho_f) - S(\rho_{qf}) \tag{3.54}$$

We see the same trend as in the curves for decoherence. A larger α increases the difference between the curves. In this case it is easier to understand.

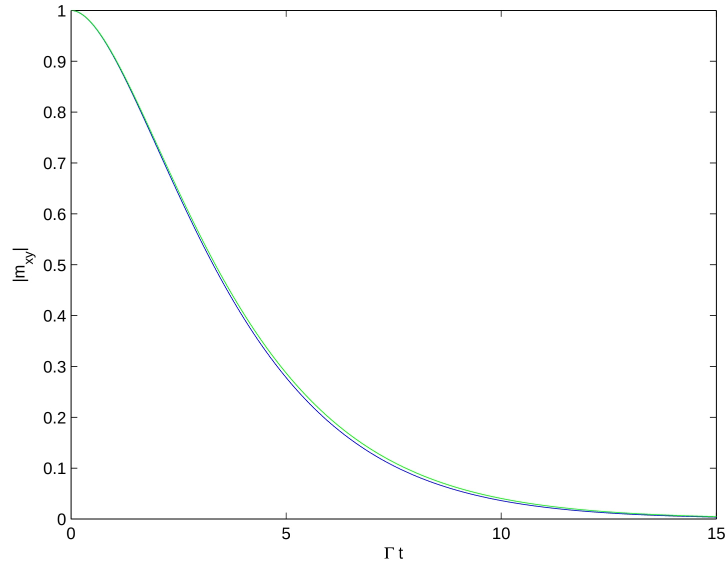

Figure 3.2: Decoherence of qubit due to quantum (blue) and classical (green) telegraph noise for $\alpha = 0.7$, $v/\Gamma = 0.5$, $\Lambda\tau = 0.05$, $\Lambda = \Delta$.

When α is close to 1 the photon interactions do not cause as much decoherence in the fluctuator. The fluctuator will have the same classical correlations with the qubit as a telegraph noise signal, but there will also be quantum entanglement. Quantum mutual information includes both these effects so we see that the mutual information between the qubit and fluctuator is greater when α is closer to 1.

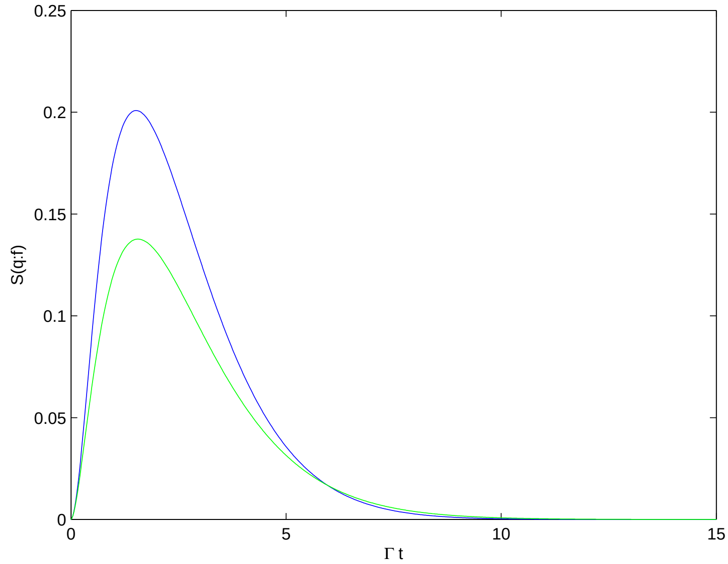

Figure 3.3: Mutual information between qubit and fluctuator (quantum: blue, classical: green) for $\alpha = 0.9$, $v/\Gamma = 0.5$, $\Lambda\tau = 0.05$, $\Lambda = \Delta$.

3.4 Finite Temperature

We will now use a simple model for absorption and emission of photons, which can be implemented using the formalism of generalized measurements. First we assume the fluctuator is coupled to a thermal bath with a constant temperature T. The fluctuator can exchange energy with the thermal bath by emitting and absorbing photons. Note that these photons are in addition to the interacting photons. For now we ignore the qubit and just analyze the fluctuator and environment. We have the hamiltonian

$$H_f = \frac{1}{2}\Delta\sigma_z + \frac{1}{2}\Lambda\sigma_x \tag{3.55}$$

We denote the energy eigenstates $|\psi_g\rangle$ and $|\psi_e\rangle$, where $|\psi_g\rangle$ is the ground state and $|\psi_e\rangle$ is the excited state. These are given by the "down" and "up" states on a rotated Bloch sphere, where the z-axis is rotated an angle $\theta = \arctan(\Lambda/\Delta)$ towards the x-axis. The two states have an energy splitting $E = \sqrt{\Delta^2 + \Lambda^2}$. When the fluctuator absorbs a photon it goes from the ground state to the excited state. The absorption rate β_{ab} is proportional to the number of photons with energy E. This number is given by the Bose-

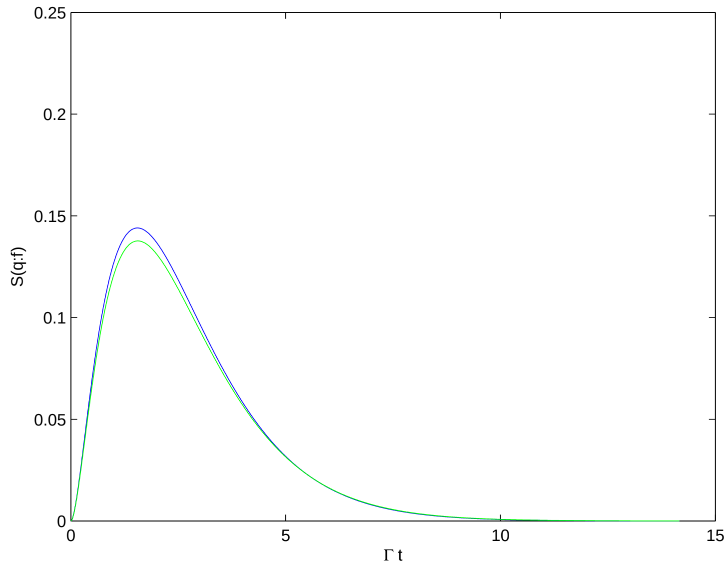

Figure 3.4: Mutual information between qubit and fluctuator (quantum: blue, classical: green) for $\alpha = 0.7$, $v/\Gamma = 0.5$, $\Lambda\tau = 0.05$, $\Lambda = \Delta$.

Einstein distribution so we have

$$\beta_{ab} \propto \frac{1}{e^{E/T} - 1} \tag{3.56}$$

The emission rate β_{em} is then given by

$$\beta_{em} = e^{E/T}\beta_{ab} \propto \frac{e^{E/t}}{e^{E/T} - 1} \tag{3.57}$$

We now want an algorithm that, for each time step Δt, gives us the density matrix that is a mixture of the three states where either a photon has been absorbed, a photon has been emitted, or nothing has happened. What we do is essentially a measurement of whether or not a photon has been emitted or absorbed. If a photon has been emitted then we know that the fluctuator is in the ground state. Similarly if a photon has been absorbed the fluctuator is in the excited state. The effect this measurement has on the fluctuator can be described by a set of generalized measurement operators. In the rotated coordinate system these operators are

$$M_1 = \sqrt{\beta_{ab}\Delta t}\sigma_x|\psi_g\rangle\langle\psi_g| \tag{3.58}$$

$$M_2 = \sqrt{\beta_{em}\Delta t}\sigma_x|\psi_e\rangle\langle\psi_e| \tag{3.59}$$

$$M_3 = \sqrt{I - M_1^\dagger M_1 - M_2^\dagger M_2} \tag{3.60}$$

In the case of M_1 and M_2 we first perform a projective measurement on the fluctuator. The state is then flipped by the σ_x operation. These two operators represent absorption and emission. The third operator M_3 represents the outcome where nothing has happened. If we first look at M_1 we can see how the operator affects the density matrix. As described in the introduction we have

$$\rho^m = \frac{M_m \rho M_m^\dagger}{\mathrm{Tr}(M_m \rho M_m^\dagger)} \tag{3.61}$$

where m is the measurement outcome and $\mathrm{Tr}(M_m \rho M_m^\dagger)$ is the probability of the measurement resulting in the outcome m. If the fluctuator in the rotated system has the density matrix

$$\rho_f = \begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix} \tag{3.62}$$

then we get the following matrix after absorbing a photon:

$$\rho_f^1 = \frac{M_1 \rho_f M_1^\dagger}{\mathrm{Tr}(M_1 \rho_f M_1^\dagger)} = \frac{\beta_{ab} \Delta t f_{11} |\psi_e\rangle\langle\psi_e|}{\beta_{ab} \Delta t f_{11}} = |\psi_e\rangle\langle\psi_e| \tag{3.63}$$

We see that after aborbing a photon the system is in the state $|\psi_e\rangle$ and the probability for this happening is $\beta_{ab}\Delta t f_{11}$, where f_{11} is the probability of the fluctuator being in the state $|\psi_g\rangle$. The operator M_2 behaves in the same way:

$$\rho_f^2 = \frac{M_2 \rho_f M_2^\dagger}{\mathrm{Tr}(M_2 \rho_f M_2^\dagger)} = \frac{\beta_{em} \Delta t f_{00} |\psi_g\rangle\langle\psi_g|}{\beta_{em} \Delta t f_{00}} = |\psi_g\rangle\langle\psi_g| \tag{3.64}$$

We see that M_1 and M_2 give us the desired behavior for absorption and emission. M_3 is constructed to ensure

$$\sum_m M_m^\dagger M_m = I \tag{3.65}$$

We can still calculate M_3 and give an interpretation. We have

$$M_1^\dagger M_1 = \beta_{ab}\Delta t |\psi_g\rangle\langle\psi_g|\sigma_x\sigma_x|\psi_g\rangle\langle\psi_g| = \beta_{ab}\Delta t |\psi_g\rangle\langle\psi_g| \tag{3.66}$$

$$M_2^\dagger M_2 = \beta_{em}\Delta t |\psi_e\rangle\langle\psi_e|\sigma_x\sigma_x|\psi_e\rangle\langle\psi_e| = \beta_{em}\Delta t |\psi_e\rangle\langle\psi_e| \tag{3.67}$$

M_3 is then

$$M_3 = \sqrt{I - M_1^\dagger M_1 - M_2^\dagger M_2} \tag{3.68}$$

$$= \sqrt{1 - \beta_{ab}\Delta t} |\psi_g\rangle\langle\psi_g| + \sqrt{1 - \beta_{em}\Delta t} |\psi_e\rangle\langle\psi_e| \tag{3.69}$$

We see that the operator includes the probabilities for not absorbing $1-\beta_{ab}\Delta t$ and not emitting $1-\beta_{em}\Delta t$. When this is the result of the measurement every element in the density matrix is multiplied with the corresponding probabilities of not changing.

In the algorithm we first switch to the rotated basis. This is done by acting on the density matrix with the operator

$$R(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{3.70}$$

The density matrix in the rotated basis is then

$$\rho_{f\theta} = R(\theta)\rho_f R(\theta)^\dagger \tag{3.71}$$

In this basis we find the new density matrix $\rho'_{f\theta}$ after the measurement:

$$\rho'_{f\theta} = M_1\rho_{f\theta}M_1^\dagger + M_2\rho_{f\theta}M_2^\dagger + M_3\rho_{f\theta}M_3^\dagger \tag{3.72}$$

Finally we switch back to the original basis:

$$\rho'_f = R(\theta)^\dagger\rho'_{f\theta}R(\theta) \tag{3.73}$$

When the fluctuator is repetedly subject to this measurement it will relax to a temperature-dependent equilibrium state. Generally the equilibrium state will also depend on the angle θ and the coupling to the environment α. We can first look at the case where $\theta = 0$ and the fluctuator starts in the excited state $|\psi_e\rangle = |0\rangle$. The interaction photons will then have no effect and the relaxation will only depend on the emission and absorption rates and the energy splitting Δ between the states $|0\rangle$ and $|1\rangle$. In this case the fluctuator should be completely classical and behave exactly as a telegraph noise signal with switching rates $\Gamma_{10} = \beta_{ab}$ and $\Gamma_{01} = \beta_{em}$. It is then simple to find how the absorption/emission should effect the density matrix. If the fluctuator after a time t is in the state

$$\rho_f(t) = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1| \tag{3.74}$$

where p_0 and p_1 are probabilities corresponding to state $|0\rangle$ and $|1\rangle$ respectively, then the density matrix after a time Δt is a mixture of the four possibilites: absorption, no absorption, emission, and no emission. This is given by

$$\begin{aligned} \rho_f(t+\Delta t) = {} & p_0[\beta_{em}\Delta t|1\rangle\langle 1| + (1-\beta_{em}\Delta t)|0\rangle\langle 0|] \\ & + p_1[\beta_{ab}\Delta t|0\rangle\langle 0| + (1-\beta_{ab}\Delta t)|1\rangle\langle 1|] \end{aligned} \tag{3.75}$$

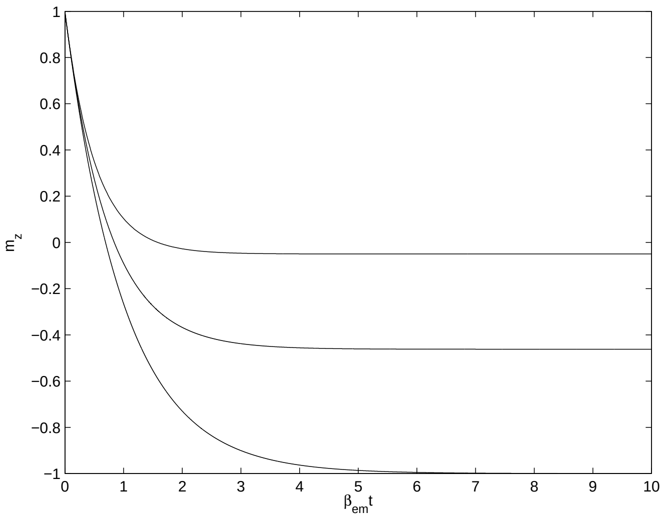

Figure 3.5: Fluctuator relaxation for $\Lambda = 0$: z-component of the fluctuator Bloch vector as a function of time for three different temperatures. Top to bottom: $T = 10\Delta$, $T = \Delta$, $T = 0.1\Delta$.

It is easy to check that the measurement operators give the same result. In figure 3.5 we have plottet the z-component of the fluctuator Bloch vector as a function of time for different temperatures. We see that for high temperatures the equilibrium state is close to an equal ensemble. This is because the absorption rate almost equals the emission rate. For temperatures close to zero we see that the fluctuator relaxes to the ground state $|1\rangle$. Since the fluctuator behaves classically the probabilities corresponding to the two states in thermal equilibrium are given by the Maxwell-Boltzmann distribution. This gives us the the z-component of the Bloch vector at equilibrium:

$$\tilde{m}_z = -\tanh\left(\frac{\Delta}{2T}\right) \tag{3.76}$$

We can now look at the case where $\Lambda \neq 0$. We now have three processes competing with eachother. First we have thermalization along the rotated axis in the Bloch ball. Second we have the hamiltonian which causes the Bloch vector to precess around the same axis. Finally we have the interaction photons reducing the xy-component of the Bloch vector. These three processes decide the relaxation rate and the equilibrium state. Note that we are still looking at the z-component of the Bloch vector as this is the component that affects the qubit. In figure 3.6 we have plottet the relaxation for the same temperatures as in figure 3.5, but with $\theta = \pi/4$ and $\alpha = 0.6$, $E\tau = 0.1$. We

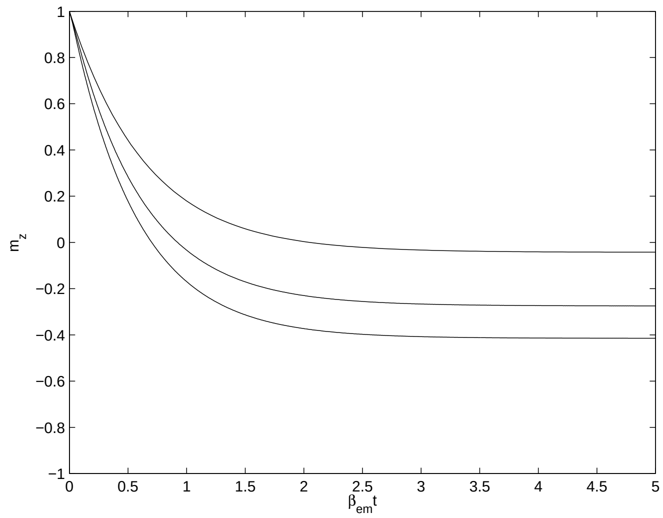

Figure 3.6: Fluctuator relaxation for $\Lambda \neq 0$: z-component of the fluctuator Bloch vector as a function of time for three different temperatures. Top to bottom: $T = 10E$, $T = E$, $T = 0.1E$. ($\alpha = 0.6$, $E\tau = 0.1$, $\theta = \pi/4$)

see that the rate of relaxation has increased and the equilibrium level is closer to the excited state. In figure 3.7 we plot the relaxation for $\alpha = 0.1$. Here we see that the effect of the σ_x term is decreased due to a stronger coupling to the environment.

We want to compare the quantum fluctuator with classical telegraph noise. To do this we have to find the correct switching rates for the telegraph signal. These rates should reproduce the relaxation curve for the quantum fluctuator. If we have the switching rates Γ_{01} and Γ_{10} then the z-component of the Bloch vector is given by

$$p_0 - p_1 = \frac{2\Gamma_{01}}{\Gamma} e^{\Gamma t} + \frac{\Gamma_{10} - \Gamma_{01}}{\Gamma} \tag{3.77}$$

where the system starts in the excited state and $\Gamma = \Gamma_{01} + \Gamma_{10}$ is the relaxation rate. To find Γ_{01} and Γ_{10} we just need to analyze the relaxation curve for the quantum fluctuator. The equilibrium level $\tilde{m}_z = m_z(t \to \infty)$ gives us $(\Gamma_{10} - \Gamma_{01})/\Gamma$. To find the rate Γ we can subtract $\tilde{m}_z$ from the curve. We can then take the logarithm and make a linear fit. Γ is then given by the slope. This gives us two equations to find the two switching rates.

We are now ready to analyze the qubit-fluctuator system. In this case

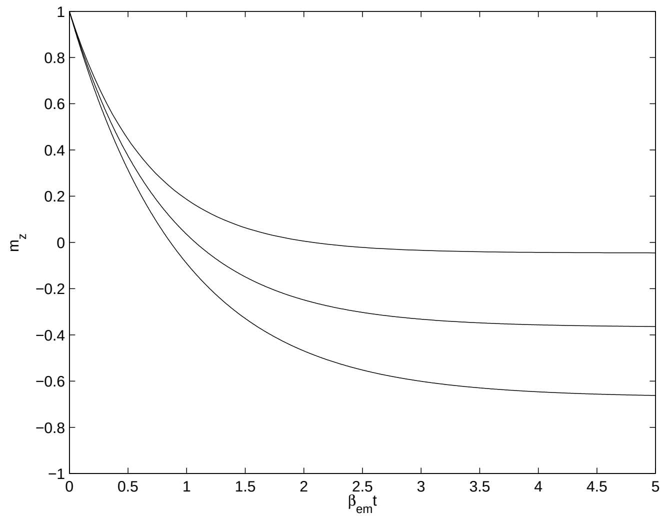

Figure 3.7: Fluctuator relaxation for $\Lambda \neq 0$: z-component of the fluctuator Bloch vector as a function of time for three different temperatures. Top to bottom: $T = 10E$, $T = E$, $T = 0.1E$. ($\alpha = 0.1$, $E\tau = 0.1$, $\theta = \pi/4$)

the measurement operators act on the density matrix for the whole system:

$$M_1 = \sqrt{\beta_{ab}\Delta t} I \otimes (\sigma_x|\psi_g\rangle\langle\psi_g|) \tag{3.78}$$

$$M_2 = \sqrt{\beta_{em}\Delta t} I \otimes (\sigma_x|\psi_e\rangle\langle\psi_e|) \tag{3.79}$$

$$M_3 = \sqrt{I - M_1^\dagger M_1 - M_2^\dagger M_2} \tag{3.80}$$

We let the qubit start in the equal superposition state and the fluctuator in the state

$$\rho_f = \tilde{p}_0|0\rangle\langle 0| + \tilde{p}_1|1\rangle\langle 1| \tag{3.81}$$

where $\tilde{p}_0$ and $\tilde{p}_1$ are the probabilities at thermal equilibrium. We can now once again compare the qubit decoherence for quantum and classical telegraph noise. In figures 3.8-3.11 we have plotted the decoherence and mutual information for low temperature. We see that the results are almost identical to what we have seen before. When α decreases we have less entanglement and the curves approach the ones for the classical case.

We now want to analyze the temperature-dependence of the system. To do this we need a variable that quantifies how "classical" the system is. We see that the qubit has an exponential dephasing rate when subject to both

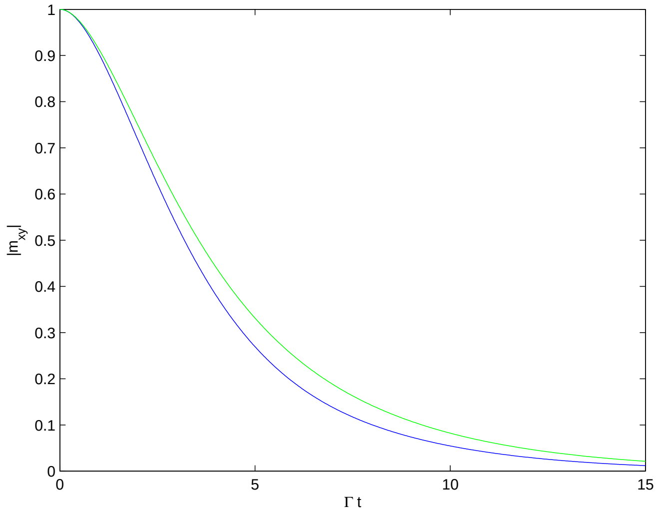

Figure 3.8: Qubit decoherence due to classical (green) and quatum (blue) telegraph noise for $\alpha = 0.9$, $T = 0.1E$, $\theta = \pi/4$, $v/\Gamma = 0.5$, $E\tau = 0.1$

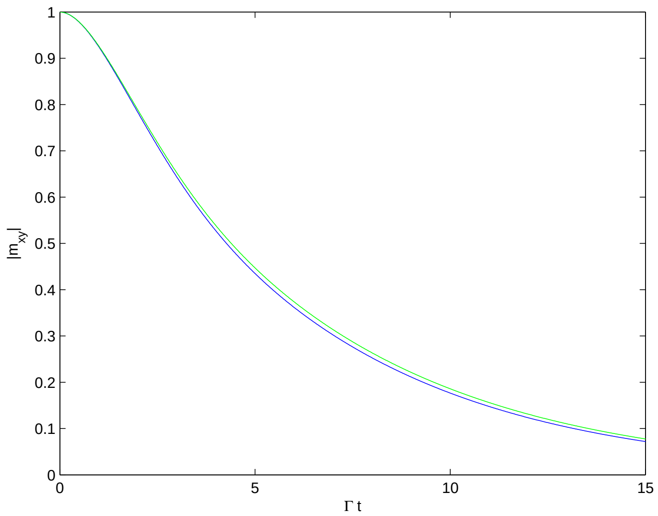

Figure 3.9: Qubit decoherence due to classical (green) and quatum (blue) telegraph noise for $\alpha = 0.7$, $T = 0.1E$, $\theta = \pi/4$, $v/\Gamma = 0.5$, $E\tau = 0.1$

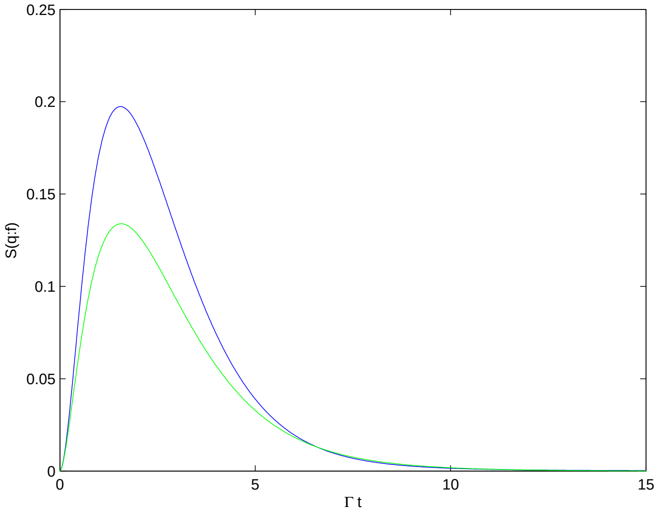

Figure 3.10: Mutual information between qubit and fluctuator (quantum: blue, classical: green) for $\alpha = 0.9$, $T = 0.1E$, $\theta = \pi/4$, $v/\Gamma = 0.5$, $E\tau = 0.1$

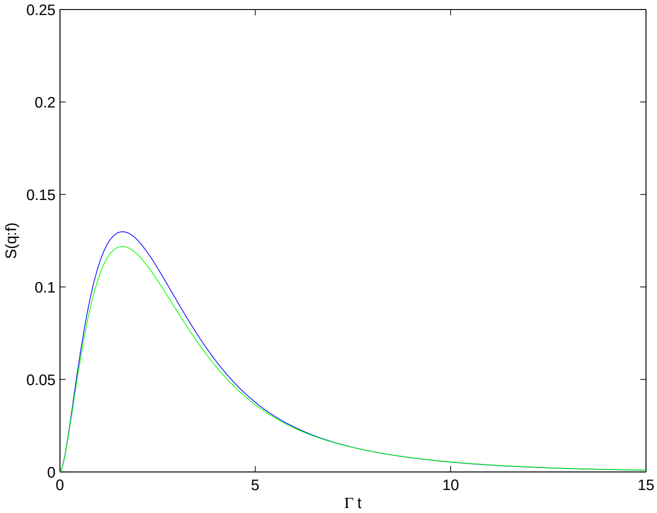

Figure 3.11: Mutual information between qubit and fluctuator (quantum: blue, classical: green) for $\alpha = 0.7$, $T = 0.1E$, $\theta = \pi/4$, $v/\Gamma = 0.5$, $E\tau = 0.1$

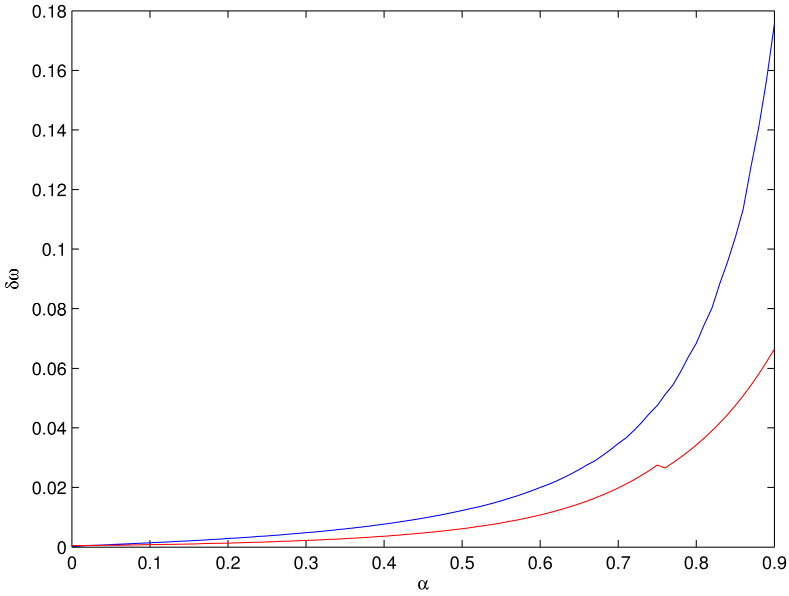

Figure 3.12: Relative difference in qubit decoherence rates $\delta\omega$ as a function of α for $T = 0.1E$ (blue) and $T = 10E$ (red). ($\theta = \pi/4$, $v/E = 0.05$, $E\tau = 0.1$)

quantum and classical telegraph noise. We call these rates ω_q and ω_c. The variable we are after is the relative difference between the two rates:

$$\delta\omega = \frac{\omega_q - \omega_c}{\omega_c} \tag{3.82}$$

When the fluctuator approaches the classical limit these rates become identical and $\delta\omega$ goes to zero. We can now compare the dependence on α for low and high temperature. This is done in figure 3.12. We see that $\delta\omega$ goes to zero for small α regardless of the temperature. For larger values of α we see a significant temperature-dependence. For high temperature the fluctuator behaves more classically and $\delta\omega$ is smaller. We do not present the results for $\alpha > 0.9$. In this domain quantum effects are strong and numerical issues prevent us from comparing the fluctuator with telegraph noise. To illustrate the general dependence on α and temperature we have made a contour-plot of $\delta\omega$ shown in figure 3.13. Here we clearly see that $\delta\omega$ is smallest for high temperatures and small α, and largest for low temperatures and large α. As in figure 3.12 we see that for small values of α there is little temperature-dependence. We also see the opposite case: for high temperatures there is little dependence on α.

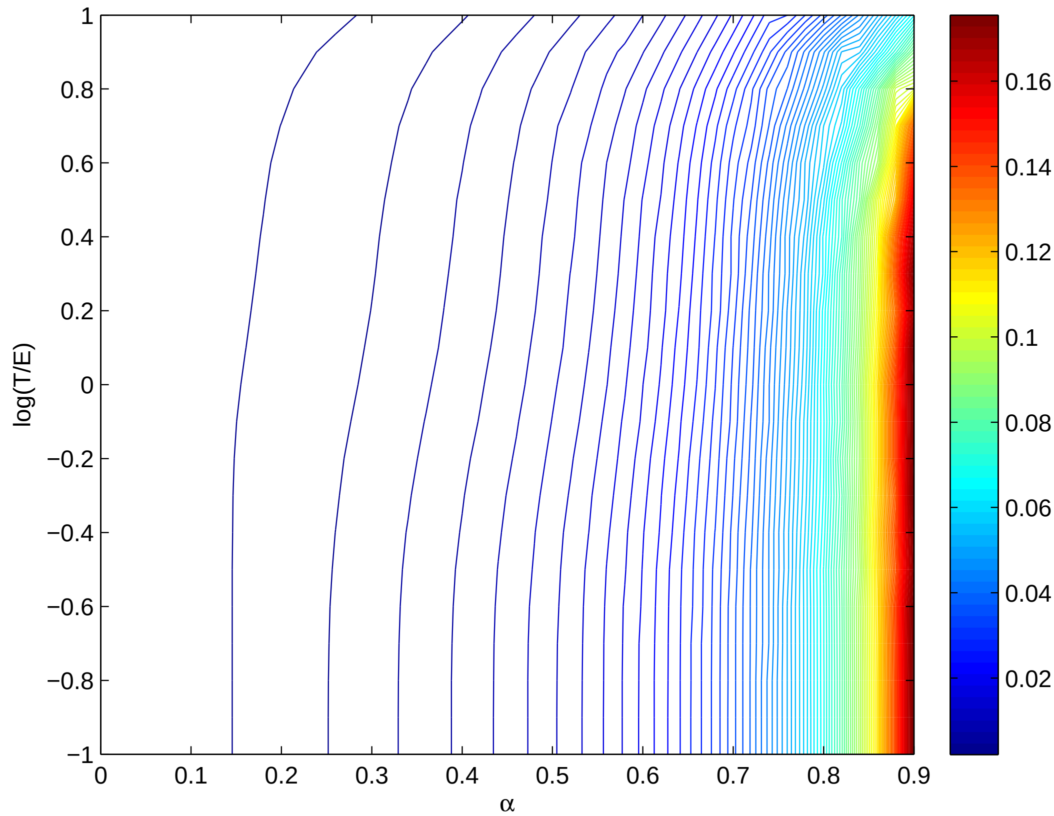

Figure 3.13: Contour-plot of $\delta\omega$ as a function of α and T. ($\theta = \pi/4$, $v/E = 0.05$, $E\tau = 0.1$)

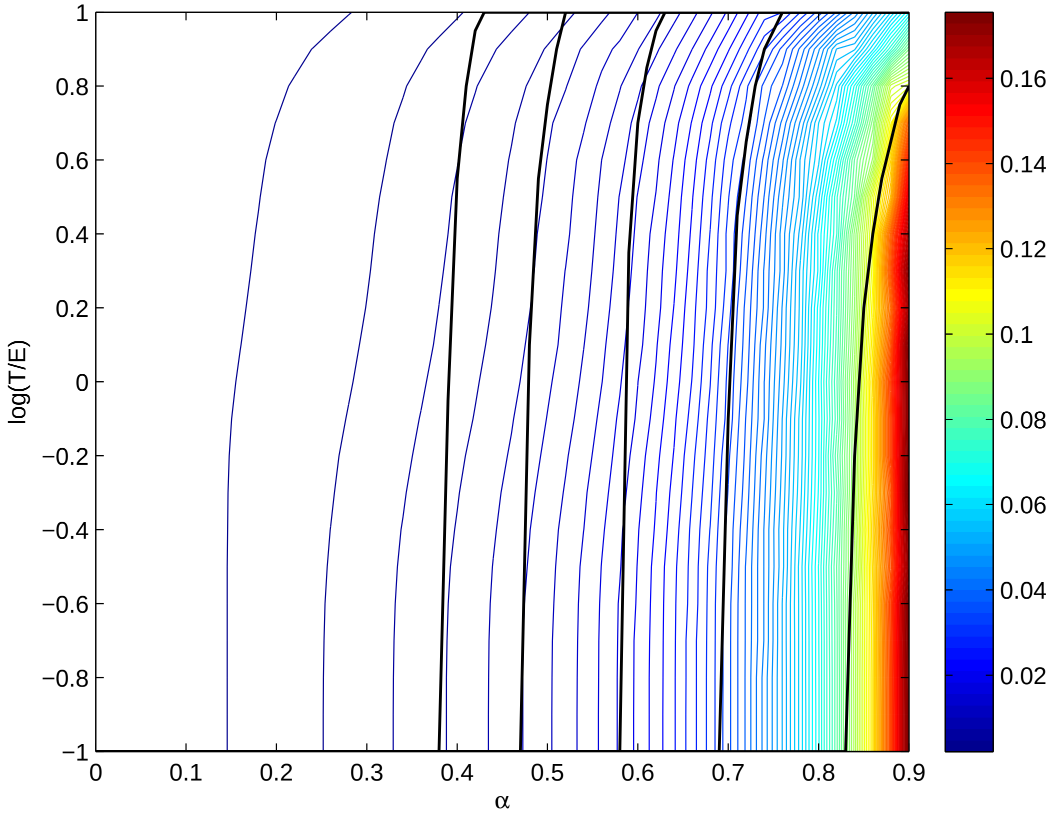

Figure 3.14: Contour-plot of $\delta\omega$ as a function of α and T with superimposed contours of constant γ (black curves). Left to right: $\gamma = 200v$, $\gamma = 160v$, $\gamma = 120v$, $\gamma = 80v$, $\gamma = 40v$. ($\theta = \pi/4$, $v/E = 0.05$, $E\tau = 0.1$)

The temperature and the coupling to the environment α both determine how fast the fluctuator decoheres and shows classical behavior. This leads us to believe that it is the decoherence rate of the fluctuator that decides the behavior of the qubit-fluctuator system. We can compute this rate, which we call γ, and find curves where $\gamma(T, \alpha)$ is constant. In figure 3.14 we have taken the contour-plot from figure 3.13 and superimposed a few of these curves. We see that the curves for constant γ roughly follow the contours for constant $\delta\omega$, indicating that the decoherence rate of the fluctuator is indeed the most important factor in determining the behavior of the system. It seems that for this model we start observing significant quantum effects when $\gamma < 80v$.

4 Conclusion

With the model used in this thesis we have divided the external environment into two parts. The photon emission/absorption represents the thermalizing part of the environment. This causes both dephasing and energy relaxation of the fluctuator. In addition to thermalization we have the entangling photons. These only exchange information with the fluctuator and have a purely dephasing effect. Using the curve for the fluctuator relaxation in the z-direction we are able to construct a telegraph noise signal with the same curve, given by the switching rates and equilibrium state. This has allowed us to compare the decoherence of a qubit coupled to the fluctuator with a qubit coupled to a telegraph noise signal. The aim of this comparison is to determine when the fluctuator can be treated as a classical system.

Had we only included the thermalizing part of the environment then we would have obtained the same high-temperature limit as in previous results. The entangling photons give us another parameter we can tune to affect the fluctuator decoherence rate. This rate now depends on the amount of entanglement between the fluctuator and environment, in addition to temperature. We have quantified the amount of entanglement by the overlap between the photon states (and also the time between photons which we have chosen to hold constant). This variable would not be very easy to identify in a physical system. One can however assume that there will often be elements in the environment that entangle with the fluctuator, causing decoherence also at low temperatures. In a realistic system the decoherence rate will depend on many unknown variables. What we have been able to show is that the amount of quantum behavior the fluctuator demonstrates is decided by the decoherence rate, regardless of the cause of decoherence.

Areas of interest for further study could be the qubit decoherence when coupled to more than one fluctuator, and the effects of entanglement between the fluctuators. It could also of interest to compare the results in this thesis with results obtained using the Caldeira-Leggett model [13] for the environment. In this model the environment is approximated by a bath of harmonic oscillators. In our model we assume short correlation times in the environment, allowing us to trace over the entangling photons and forget about them. For the Caldeira-Leggett model this assumption would allow us to make a Markov-approximation. One might then obtain analytical results similar to the numercial ones in this thesis.

A Analytical Calculations

A.1 Introduction

In this appendix we study a simple case of our model, with the hope of achieving some analytical insight. Here we look only at the interaction term of the hamiltonian, neglecting the internal dynamics of both the qubit and fluctuator. The environment consists only of the entangling photons, and we assume the overlap between photon states to be zero. This model may seem artificial but it has some of the properties we are interested in studying. A more complex hamiltonian would be too dfficult in these analytical calculations. We will also see that this system has some surprising properties and we are able to make use of some interesting techniques for calculation. Concerning are main goal of determining the quantum to classical transition for the fluctuator, this appendix does not give us any significant insight. It is included in this thesis because of the amount of time spent working on it, and because it has some interesting results from a fundamental point of view.

A.2 Evolution Matrix

We start with the hamiltonian

$$H = \frac{1}{2}v\sigma_x \otimes \sigma_x \tag{A.1}$$

We want to find the evolution of the system including the interactions with the entangling photons. The time evolution between photon interactions is

$$U = e^{-\frac{i}{2}\sigma_x\otimes\sigma_x\tau} = \cos(\frac{1}{2}v\tau)I - i\sin(\frac{1}{2}v\tau)\sigma_x \otimes \sigma_x = cI - is\sigma_x \otimes \sigma_x, \tag{A.2}$$

where $c = \cos(\frac{1}{2}v\tau)$, $s = \sin(\frac{1}{2}v\tau)$ and τ is the time between photons. If the system is in a state ρ_n then we define ρ_{n+1} as the system after a period τ of free evolution and a subsequent photon interaction:

$$\rho_n \longrightarrow U\rho_n U^\dagger \xrightarrow{ph} \rho_{n+1} \tag{A.3}$$

The matrix form of U is

$$U = \begin{pmatrix} c & 0 & 0 & -is \\ 0 & c & -is & 0 \\ 0 & -is & c & 0 \\ -is & 0 & 0 & c \end{pmatrix} \tag{A.4}$$

and we have

$$\rho_n = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \tag{A.5}$$

We can then calculate the time evolution for $t = \tau$:

$$U\rho_n U^\dagger = \tag{A.6}$$

$$\begin{pmatrix} c^2a_{11} + icsa_{14} - icsa_{41} + s^2a_{44} & c^2a_{12} + icsa_{13} - icsa_{42} + s^2a_{43} & icsa_{12} + c^2a_{13} + s^2a_{42} - icsa_{43} & icsa_{11} + c^2a_{14} + s^2a_{41} - icsa_{44} \\ c^2a_{21} + icsa_{24} - icsa_{31} + s^2a_{34} & c^2a_{22} + icsa_{23} - icsa_{32} + s^2a_{33} & icsa_{22} + c^2a_{23} + s^2a_{32} - icsa_{33} & icsa_{21} + c^2a_{24} + s^2a_{31} - icsa_{34} \\ -icsa_{21} + s^2a_{24} + c^2a_{31} + icsa_{34} & -icsa_{22} + s^2a_{23} + c^2a_{32} + icsa_{33} & s^2a_{22} - icsa_{23} + icsa_{32} + c^2a_{33} & s^2a_{21} - icsa_{24} + icsa_{31} + c^2a_{34} \\ -icsa_{11} + s^2a_{14} + c^2a_{41} + icsa_{44} & -icsa_{12} + s^2a_{13} + c^2a_{42} + icsa_{43} & s^2a_{12} - icsa_{13} + icsa_{42} + c^2a_{43} & s^2a_{11} - icsa_{14} + icsa_{41} + c^2a_{44} \end{pmatrix}$$

After tracing over photon states we have

$$\rho_{n+1} = \tag{A.7}$$

$$\begin{pmatrix} c^2a_{11} + icsa_{14} - icsa_{41} + s^2a_{44} & 0 & icsa_{12} + c^2a_{13} + s^2a_{42} - icsa_{43} & 0 \\ 0 & c^2a_{22} + icsa_{23} - icsa_{32} + s^2a_{33} & 0 & icsa_{21} + c^2a_{24} + s^2a_{31} - icsa_{34} \\ -icsa_{21} + s^2a_{24} + c^2a_{31} + icsa_{34} & 0 & s^2a_{22} - icsa_{23} + icsa_{32} + c^2a_{33} & 0 \\ 0 & -icsa_{12} + s^2a_{13} + c^2a_{42} + icsa_{43} & 0 & s^2a_{11} - icsa_{14} + icsa_{41} + c^2a_{44} \end{pmatrix}$$

We see here that each element in ρ_{n+1} is a linear combination of the elements in ρ_n. If we define $\vec{\rho}_n$ to be the vector

$$\vec{\rho}_n = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{21} \\ a_{22} \\ \vdots \\ a_{44} \end{pmatrix} \tag{A.8}$$

then there exists a matrix M so that

$$\vec{\rho}_{n+1} = M\vec{\rho}_n \tag{A.9}$$

or genrally

$$\vec{\rho}_n = M^n\vec{\rho}_0, \tag{A.10}$$

where $\vec{\rho}_0$ is the initial state. This matrix is

$$M = \tag{A.11}$$

$$\begin{pmatrix}
c^2 & 0 & 0 & ics & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -ics & 0 & 0 & s^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & ics & c^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s^2 & -ics & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c^2 & ics & 0 & 0 & -ics & s^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & ics & 0 & 0 & c^2 & s^2 & 0 & 0 & -ics & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -ics & 0 & 0 & s^2 & c^2 & 0 & 0 & ics & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & s^2 & -ics & 0 & 0 & ics & c^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -ics & s^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c^2 & ics & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s^2 & 0 & 0 & -ics & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & ics & 0 & 0 & c^2
\end{pmatrix}$$

We can perform en eigendecomposition on M and we get

$$M = SDS^{-1} \tag{A.12}$$

where S is the matrix with the eigenvectors of M as its columns, and D is a diagonal matrix with the eigenvalues of M as its elements. We then have

$$\vec{\rho}_n = SD^n S^{-1}\vec{\rho}_0 \tag{A.13}$$

We can plot the von Neumann entropy for the whole system as a function of time and we see that it does not generally converge towards the maximum entropy for the system, $S = \ln 4$, for large times. It seems rather that the final entropy is between $\ln 2$ and $\ln 4$, depending on the initial state. This motivates further analysis.

A.3 Convergence

We start by finding the generall form of the density matrix after convergence. Studying the matrices $U\rho_n U^\dagger$ and M we see that the new elements after time evolution and tracing, $\tilde{a}_{11}, \tilde{a}_{14}, \tilde{a}_{41}, \tilde{a}_{44}$ are all linear combinations of the old elements $a_{11}, a_{14}, a_{41}, a_{44}$. The same applies to the groups $a_{12}, a_{13}, a_{42}, a_{43}$; $a_{21}, a_{24}, a_{31}, a_{34}$; $a_{22}, a_{23}, a_{32}, a_{33}$. Therefore it is only necessary to analyze one group. After the initial trace, $a_{14} = a_{41} = 0$. Each subsequent multiplication with M gives us

$$\tilde{a}_{11} = c^2 a_{11} + s^2 a_{44} \tag{A.14}$$

$$\tilde{a}_{44} = s^2 a_{11} + c^2 a_{44} \tag{A.15}$$

For tidiness we rename the elements $x_n \equiv a_{11}$, $y_n \equiv a_{44}$. We then have

$$x_{n+1} = c^2 x_n + s^2 y_n \tag{A.16}$$

$$y_{n+1} = s^2 x_n + c^2 y_n \tag{A.17}$$

We can guess that the two elements will converge towards the average of the initial ones. To prove this we start by re-writing the equations:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} c^2 & s^2 \\ s^2 & c^2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{A.18}$$

If x_1 and y_1 are the elements after the first trace, then we have

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c^2 & s^2 \\ s^2 & c^2 \end{pmatrix}^n \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{A.19}$$

Performing an eigendecomposition gives us

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (c^2 - s^2)^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{A.20}$$

$$= \frac{1}{2} \begin{pmatrix} x_1 + y_1 + (c^2 - s^2)^n (x_1 - y_1) \\ x_1 + y_1 - (c^2 - s^2)^n (x_1 - y_1) \end{pmatrix} \tag{A.21}$$

For large n this gives us

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_1 + y_1 \\ x_1 + y_1 \end{pmatrix} \tag{A.22}$$

except for the special cases $c^2 - s^2 = 1$, and $c^2 - s^2 = -1$.

We can use the same argument for the other groups of elements and we arrive finally to the density matrix

$$\rho_f = \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \gamma & 0 & \delta \\ \delta & 0 & \gamma & 0 \\ 0 & \beta & 0 & \alpha \end{pmatrix}, \tag{A.23}$$

where $\alpha, \beta, \gamma, \delta$ depend on the initial state. This matrix has the property $M\rho_f = \rho_f$. When the system has reached this state, then it will stay in the same state indefinitely. The entropy will also be constant.

We have three conditions for ρ to be an allowed density matrix:

(1) The trace must equal one: $2\alpha + 2\gamma = 1$
(2) ρ must be hermitian: $\delta = \beta^*$
(3) All of the eigenvalues must be greater than or equal to zero

These three conditions give us the allowed values of $\alpha, \beta, \gamma, \delta$. Calculating the eigenvalues in closed form is messy. We can however formulate the third condition using the generalized Bloch ball for 4-level systems.

A.3.1 Bloch Ball

For N-level systems we have $N^2 - 1$ parameters: one for each diagonal element except for the last one, which is decided by (1), and two for each element above the diagonal. For a 4-level system we then have 15 parameters. This means that the density matrix can be represented by a point in a 15-dimensional space. We can now express the density matrix as

$$\rho = \frac{1}{4} I + m_i \lambda_i \tag{A.24}$$

where $\{\lambda_i\}$ are the generators of $SU(4)$ [14], which are all hermitian and have zero trace. For our system ρ_f we only need 7 of the 15 matrices:

$$\lambda_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{A.25}$$

$$\lambda_a = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_{ai} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.26}$$

$$\lambda_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \lambda_{bi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \tag{A.27}$$

We can now find the coefficients $\{m_i\}$ expressed in terms of α, $\mathrm{Re}\beta$ and $\mathrm{Im}\beta$. We start with the diagonal elements:

$$\frac{1}{4} + m_1 + m_2 + m_3 = \alpha \tag{A.28}$$

$$\frac{1}{4} - 2m_2 + m_3 = \gamma \tag{A.29}$$

$$\frac{1}{4} - 3m_3 = \alpha \tag{A.30}$$

Noting that $\gamma = \frac{1}{2} - \alpha$ we have

$$m_3 = -\frac{1}{3}(\alpha - \frac{1}{4}) \tag{A.31}$$

$$\frac{1}{4} - 2m_2 + m_3 = \gamma \tag{A.32}$$

$$\frac{1}{4} - 2m_2 - \frac{1}{3}(\alpha - \frac{1}{4}) = \frac{1}{2} - \alpha \tag{A.33}$$

$$m_2 = \frac{1}{3}(\alpha - \frac{1}{4}) \tag{A.34}$$

$$\frac{1}{4} + m_1 + m_2 + m_3 = \alpha \tag{A.35}$$

$$\frac{1}{4} + m_1 + \frac{1}{3}(\alpha - \frac{1}{4}) - \frac{1}{3}(\alpha - \frac{1}{4}) = \alpha \tag{A.36}$$

$$m_1 = \alpha - \frac{1}{4} \tag{A.37}$$

Next we find the off-diagonal elements:

$$m_a - im_{ai} = \beta \qquad m_b - im_{bi} = \delta \tag{A.38}$$
$$m_a + im_{ai} = \delta \qquad m_b + im_{bi} = \beta \tag{A.39}$$

Since $\delta = \beta^*$ have have

$$m_a = m_b = \mathrm{Re}\,\beta \tag{A.40}$$
$$-m_{ai} = m_{bi} = \mathrm{Im}\,\beta \tag{A.41}$$

We can now re-write our matrix ρ_f in the following way:

$$\begin{aligned}\rho_f &= \frac{1}{4}I + (\alpha - \frac{1}{4})(\lambda_1 + \frac{1}{3}\lambda_2 - \frac{1}{3}\lambda_3) \\ &\quad + \mathrm{Re}\,\beta(\lambda_a + \lambda_b) + \mathrm{Im}\,\beta(-\lambda_{ai} + \lambda_{bi}) \qquad (A.42)\\ &= \frac{1}{4}I + \tilde{m}_i\tilde{\lambda}_i \qquad (A.43)\end{aligned}$$

where

$$\tilde{m}_1 = \alpha - \frac{1}{4},\ \tilde{m}_2 = \mathrm{Re}\,\beta,\ \tilde{m}_3 = \mathrm{Im}\,\beta \tag{A.44}$$

and

$$\tilde{\lambda}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \tilde{\lambda}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ \tilde{\lambda}_3 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \tag{A.45}$$

We have now managed to represent the density matrix ρ_f as a point in a 3-dimensional space, where the coordinates are given by $\{\tilde{m}_i\}$. We still need to impose condition (3) to find the constraints. Let $\{x_i\}$ be the eigenvalues of ρ_f, and $\{a_i\}$ be the coefficients of the characteristic polynomial $\det(\rho_f - xI)$. It can then be shown that all the eigenvalues x_i are greater than or equal to 0 if and only if all the coefficients a_i are greater than or equal to 0:

$$x_i \geq 0,\ \forall i \ \Leftrightarrow\ a_i \geq 0,\ \forall i \tag{A.46}$$

It can also be shown that

$$1!a_1 = 1, \tag{A.47}$$
$$2!a_2 = 1 - \mathrm{Tr}\,\rho_f^2, \tag{A.48}$$
$$3!a_3 = 1 - 3\mathrm{Tr}\,\rho_f^2 + 2\mathrm{Tr}\,\rho_f^3, \tag{A.49}$$
$$4!a_4 = 1 - 6\mathrm{Tr}\,\rho_f^2 + 8\mathrm{Tr}\,\rho_f^3 + 3\mathrm{Tr}\,\rho_f^2 - 6\mathrm{Tr}\,\rho_f^4 \tag{A.50}$$

This is explained in detail in [14]. Now we just need to calculate $\mathrm{Tr}\rho_f^2$, $\mathrm{Tr}\rho_f^3$ and $\mathrm{Tr}\rho_f^4$. Then we can impose the conditions on $\{a_i\}$ which in turn gives us the the constraints for $\{m_i\}$. To help in the calculations one can show the following indentities for $\{\tilde{\lambda}_i\}$:

$$\mathrm{Tr}\,\tilde{\lambda}_i = 0,\ \forall i \tag{A.51}$$

$$\tilde{\lambda}_i^2 = I,\ \forall i \tag{A.52}$$

$$\mathrm{Tr}\,(\tilde{\lambda}_i\tilde{\lambda}_j) = 4\delta_{ij} \tag{A.53}$$

$$\{\tilde{\lambda}_i, \tilde{\lambda}_j\} = 0,\ i \neq j \tag{A.54}$$

$$\mathrm{Tr}\,(\tilde{\lambda}_i, \tilde{\lambda}_j, \tilde{\lambda}_k) = -4i\epsilon_{ijk} \tag{A.55}$$

where δ_{ij} is the Kronecker-delta, ϵ_{ijk} is the Levi-Civita symbol and $\{\tilde{\lambda}_i, \tilde{\lambda}_j\}$ is the anti-commutator. We can then calculate the traces (note that $\sum_{i,j,k} \mathrm{Tr}\,(\tilde{\lambda}_i\tilde{\lambda}_j\tilde{\lambda}_k) = 0$):

$$\mathrm{Tr}\,\rho_f^2 = \mathrm{Tr}\,(\frac{1}{4^2}I + \tilde{m}_i\tilde{m}_j\tilde{\lambda}_i\tilde{\lambda}_j) \tag{A.56}$$

$$= \frac{1}{4} + 4\tilde{m}_i\tilde{m}_i \tag{A.57}$$

$$= \frac{1}{4} + 4|\tilde{m}_i|^2 \tag{A.58}$$

$$\tag{A.59}$$

$$\mathrm{Tr}\,\rho_f^3 = \mathrm{Tr}\,(\frac{1}{4^3}I + \frac{3}{4}\tilde{m}_i\tilde{m}_j\tilde{\lambda}_i\tilde{\lambda}_j + \tilde{m}_i\tilde{m}_j\tilde{m}_k\tilde{\lambda}_i\tilde{\lambda}_j\tilde{\lambda}_k) \tag{A.60}$$

$$= \frac{1}{16} + 3|\tilde{m}|^2 \tag{A.61}$$

$$\tag{A.62}$$

$$\mathrm{Tr}\,\rho_f^4 = \mathrm{Tr}\,(\frac{1}{4^4}I + \frac{6}{4^2}\tilde{m}_i\tilde{m}_j\tilde{\lambda}_i\tilde{\lambda}_j + \tilde{m}_i\tilde{m}_j\tilde{m}_k\tilde{m}_l\tilde{\lambda}_i\tilde{\lambda}_j\tilde{\lambda}_k\tilde{\lambda}_l) \tag{A.63}$$

$$= \frac{1}{64} + \frac{3}{2}|\tilde{m}|^2 + 4(\tilde{m}_1^4 + \tilde{m}_2^4 + \tilde{m}_3^4) + 8(\tilde{m}_1^2\tilde{m}_2^2 + \tilde{m}_1^2\tilde{m}_3^2 + \tilde{m}_2^2\tilde{m}_3^2) \tag{A.64}$$

$$= \frac{1}{64} + \frac{3}{2}|\tilde{m}|^2 + 4|\tilde{m}|^4 \tag{A.65}$$

Inserting into the expressions for a_i gives us the constraints on $|\tilde{m}|$:

$$1 - \mathrm{Tr}\,\rho_f^2 \geq 0 \tag{A.66}$$

$$1 - \frac{1}{4} - 4|\tilde{m}|^2 \geq 0 \tag{A.67}$$

$$|\tilde{m}| \leq \frac{3}{4} \tag{A.68}$$

$$1 - 3\mathrm{Tr}\,\rho_f^2 + 2\mathrm{Tr}\,\rho_f^3 \geq 0 \tag{A.69}$$

$$1 - \frac{3}{4} - 12|\tilde{m}|^2 + \frac{1}{8} + 6|\tilde{m}|^2 \geq 0 \tag{A.70}$$

$$|\tilde{m}| \leq \frac{1}{4} \tag{A.71}$$

$$1 - 6\mathrm{Tr}\,\rho_f^2 + 8\mathrm{Tr}\,\rho_f^3 + 3(\mathrm{Tr}\,\rho_f^2)^2 - 6\mathrm{Tr}\,\rho_f^4 \geq 0 \tag{A.72}$$

$$3(\frac{1}{4} + 4|\tilde{m}|^2)^2 - \frac{3}{32} - 9|\tilde{m}|^2 - 24|\tilde{m}|^4 \geq 0 \tag{A.73}$$

$$|\tilde{m}|^2 - 8|\tilde{m}|^4 \leq \frac{1}{32} \tag{A.74}$$

$$|\tilde{m}| \leq \frac{1}{4} \tag{A.75}$$

We see that our only constraint on the coefficients $\{\tilde{m}_i\}$ is $|\tilde{m}| \leq \frac{1}{4}$. This means that the allowed density matrices for our system our contrained to a ball with radius 1/4 in the 3-dimensional space spanned by $\{\tilde{\lambda}_i\}$. Expressing in terms of α and β we have:

$$|\tilde{m}|^2 = (\alpha - \frac{1}{2})^2 + \mathrm{Re}\,(\beta)^2 + \mathrm{Im}\,(\beta)^2 \leq \frac{1}{16} \tag{A.76}$$

$$(\alpha - \frac{1}{2})^2 + |\beta|^2 \leq \frac{1}{16} \tag{A.77}$$

Remebering that $\gamma = \frac{1}{2} - \alpha$ and $\delta = \beta^*$, we now know the allowed matrix elements of ρ_f.

A.3.2 Entropy

We now wish to find the von Neumann entropy of the system after convergence in terms of α and β. We start with the expression

$$\rho_f = \frac{1}{4}I + \tilde{m}_i\tilde{\lambda}_i \tag{A.78}$$

$$= \frac{1}{4}(I + 4\tilde{m}_i\tilde{\lambda}_i) \tag{A.79}$$

This gives us

$$\ln\rho_f = \ln(\frac{1}{4}I) + \ln(I + 4\tilde{m}_i\tilde{\lambda}_i) \tag{A.80}$$

Performing a series expansion on $\ln(I + 4\tilde{m}_i\tilde{\lambda}_i)$ we get

$$\ln \rho_f = \ln \frac{1}{4} I + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(4\tilde{m}_i\tilde{\lambda}_i)^n}{n} \tag{A.81}$$

The entropy is then given by

$$S = -\mathrm{Tr}(\rho_f \ln \rho_f) \tag{A.82}$$

$$= \ln 4 - \mathrm{Tr}\left[\frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4^n}{n}(\tilde{m}_i\tilde{\lambda}_i)^n\right] - \mathrm{Tr}\left[\sum_{n=1}^{\infty}(-1)^{n-1}\frac{4^n}{n}(\tilde{m}_i\tilde{\lambda}_i)^{n+1}\right] \tag{A.83}$$

It is straight-forward to show that $\mathrm{Tr}[(\tilde{m}_i\tilde{\lambda}_i)^n] = 4|\tilde{m}|^n$ for even n and 0 for odd n. This gives us

$$S = \ln 4 - \left(-\frac{4^2}{2}|\tilde{m}|^2 - \frac{4^4}{4}|\tilde{m}|^4 - \frac{4^6}{6}|\tilde{m}|^6 \ldots\right) - \left(4^2|\tilde{m}|^2 + \frac{4^3}{3}|\tilde{m}|^4 + \frac{4^6}{5}|\tilde{m}|^6 \ldots\right) \tag{A.84}$$

$$= \ln 4 - \frac{1}{2}\left[-(4|\tilde{m}|)^2 - \frac{1}{2}(4|\tilde{m}|)^4 - \frac{1}{3}(4|\tilde{m}|)^6 \ldots\right] - 4|\tilde{m}|\left[4|\tilde{m}| + \frac{1}{3}(4|\tilde{m}|^3) + \frac{1}{5}(4|\tilde{m}|)^6 \ldots\right] \tag{A.85}$$

$$= \ln 4 - \frac{1}{2}\ln[1 - (4|\tilde{m}|)^2] - 4|\tilde{m}|\mathrm{artanh}(4|\tilde{m}|) \tag{A.86}$$

This gives us the maximum entropy $S_{max} = \ln 4$ for $|\tilde{m}| = 0$ and the minimum $S_{min} = \ln 2$ for $|\tilde{m}| = \frac{1}{4}$.

A.3.3 Decoherence-Free Subspaces

The decoherence in our system is caused by the photon interactions. Because of the nature of these interactions we can see that a density matrix of the form

$$\rho = \begin{pmatrix} \rho_{11} & 0 & \rho_{13} & 0 \\ 0 & \rho_{22} & 0 & \rho_{24} \\ \rho_{31} & 0 & \rho_{33} & 0 \\ 0 & \rho_{42} & 0 & \rho_{44} \end{pmatrix} \tag{A.87}$$

will not be affected. The elements that are changed by the photons are already zero so there will be no decoherence. There exist two subspaces of pure states which give density matrices with this form. The state

$$|\phi_1\rangle = c_1|00\rangle + c_2|10\rangle \tag{A.88}$$

gives us the density matrix

$$\rho_1 = \begin{pmatrix} |c_1|^2 & 0 & c_1c_2^* & 0 \\ 0 & 0 & 0 & 0 \\ c_1^*c_2 & 0 & |c_2|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.89}$$

and the state

$$|\phi_2\rangle = d_1|01\rangle + d_2|11\rangle \tag{A.90}$$

gives us the density matrix

$$\rho_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |d_1|^2 & 0 & d_1d_2^* \\ 0 & 0 & 0 & 0 \\ 0 & d_1^*d_2 & 0 & |d_2|^2 \end{pmatrix} \tag{A.91}$$

We then have two 2-dimensional decoherence-free subspaces (DFS):

$$\mathrm{DFS}_1 = \mathrm{span}(|00\rangle, |10\rangle) \tag{A.92}$$

$$\mathrm{DFS}_2 = \mathrm{span}(|01\rangle, |11\rangle) \tag{A.93}$$

Our matrix ρ_f cannot be in either of these pure states. It can however be in a mixture of the two:

$$\rho_f = b_1\rho_1 + b_2\rho_2 \tag{A.94}$$

This gives us the following conditions on the coefficients c_1, c_2, d_1, d_2:

$$b_1|c_1|^2 = b_2|d_2|^2 = \alpha \tag{A.95}$$
$$b_1c_1c_2^* = b_2d_1^*d_2 = \beta \tag{A.96}$$
$$b_1c_1^*c_2 = b_2d_1d_2^* = \delta \tag{A.97}$$
$$b_1|c_2|^2 = b_2|d_1|^2 = \gamma \tag{A.98}$$

In addition we have $2\alpha + 2\gamma = 1$ which gives us

$$2b_1|c_1|^2 + 2b_1|c_2|^2 = 1 \tag{A.99}$$
$$b_1 = \frac{1}{2} \tag{A.100}$$

and

$$2b_2|d_2|^2 + 2b_2|d_1|^2 = 1 \tag{A.101}$$
$$b_2 = \frac{1}{2} \tag{A.102}$$

We then must have $|c_1|^2 = |d_2|^2$, $|c_2|^2 = |d_1|^2$ and $c_1c_2^* = d_1^*d_2$. This gives us the following density matrix which we call ρ_{DF}:

$$\rho_{DF} = \frac{1}{2}\begin{pmatrix} |c_1|^2 & 0 & c_1c_2^* & 0 \\ 0 & |c_2|^2 & 0 & c_1^*c_2 \\ c_1^*c_2 & 0 & |c_2|^2 & 0 \\ 0 & c_1c_2^* & 0 & |c_1|^2 \end{pmatrix} \tag{A.103}$$

We would like to find where these states lie in the Bloch ball. We have

$$|\tilde{m}|^2 = (\alpha - \frac{1}{4})^2 + |\beta|^2 \tag{A.104}$$

Inserting for α and β gives us

$$(\frac{1}{2}|c_1|^2 - \frac{1}{4})^2 + \frac{1}{4}|c_1c_2^*|^2 = \frac{1}{4}|c_1|^4 - \frac{1}{4}|c_1|^2 + \frac{1}{16} + \frac{1}{4}|c_1|^2|c_2|^2 \tag{A.105}$$
$$= \frac{1}{4}|c_1|^2(|c_1|^2 + |c_2|^2 - 1) + \frac{1}{16} \tag{A.106}$$
$$= \frac{1}{16} \tag{A.107}$$

This state is then on the surface of the Bloch ball: $|\tilde{m}| = \frac{1}{4}$. In fact, it can be shown that every point on the surface of the Bloch ball can be represented by a state of the form ρ_{DF}. The proof is as follows: For the surface of the Bloch ball we have

$$(\alpha - \frac{1}{4})^2 + |\beta|^2 = \frac{1}{16} \tag{A.108}$$

We see that the allowed values for $|\beta|$ are $0 \leq |\beta| \leq \frac{1}{4}$. Given two complex numbers z_1 and z_2 where $|z_1|^2 + |z_2|^2 = 1$, we can write β as $\beta = \frac{1}{2}z_1z_2$. Any

β in the allowed interval can be generated by a suitable choice of z_1 and z_2. We can then find α:

$$\alpha^2 - \frac{1}{2}\alpha + |\beta|^2 = 0 \tag{A.109}$$

$$\alpha = \frac{1}{2}\left[\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4|\beta|^2}\right] \tag{A.110}$$

$$= \frac{1}{2}\left[\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2|z_2|^2}\right] \tag{A.111}$$

$$= \frac{1}{2}\left[\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2(1 - |z_1|^2)}\right] \tag{A.112}$$

$$= \frac{1}{2}\left[\frac{1}{2} \pm \sqrt{\frac{1}{4} - |z_1|^2 + |z_1|^4}\right] \tag{A.113}$$

$$= \frac{1}{2}\left[\frac{1}{2} \pm \sqrt{\left(\frac{1}{2} - |z_1|^2\right)^2}\right] \tag{A.114}$$

For $|z_1|^2 < \frac{1}{2}$ we have

$$\alpha = \frac{1}{2}\left[\frac{1}{2} \pm \left(\frac{1}{2} - |z_1|^2\right)^2\right] \tag{A.115}$$

$$\alpha = \frac{1}{2}|z_1|^2 \ \vee \ \alpha = \frac{1}{2}|z_2|^2 \tag{A.116}$$

For $|z_1|^2 > \frac{1}{2}$ we have

$$\alpha = \frac{1}{2}\left[\frac{1}{2} \pm \left(|z_1|^2 - \frac{1}{2}\right)^2\right] \tag{A.117}$$

$$\alpha = \frac{1}{2}|z_1|^2 \ \vee \ \alpha = \frac{1}{2}|z_2|^2 \tag{A.118}$$

The two solutions give us the matrices

$$\rho_{\alpha_1} = \frac{1}{2}\begin{pmatrix} |z_1|^2 & 0 & z_1 z_2 & 0 \\ 0 & |z_2|^2 & 0 & z_1^* z_2^* \\ z_1^* z_2^* & 0 & |z_2|^2 & 0 \\ 0 & z_1 z_2 & 0 & |z_1|^2 \end{pmatrix} \tag{A.119}$$

$$\rho_{\alpha_2} = \frac{1}{2}\begin{pmatrix} |z_2|^2 & 0 & z_1 z_2 & 0 \\ 0 & |z_1|^2 & 0 & z_1^* z_2^* \\ z_1^* z_2^* & 0 & |z_1|^2 & 0 \\ 0 & z_1 z_2 & 0 & |z_2|^2 \end{pmatrix} \tag{A.120}$$

Finally we have $\rho_{DF} = \rho_{\alpha_1}$ for $c_1 = z_1$, $c_2 = z_2^*$ and $\rho_{DF} = \rho_{\alpha_2}$ for $c_1 = z_2$, $c_2 = z_1^*$. This means that any state with $|\tilde{m}| = \frac{1}{4}$ has the form of ρ_{DF}.

For a 2-level system, with the pure states on the surface of the Bloch ball, one can draw a line connecting two points on the surface. All the points on this line are then mixtures of the two pure states. Each point in the Bloch ball can be represented by a mixture of two states on the surface. For our system each state can be represented by a mixture of two states of the form ρ_{DF}. In other words, any state ρ_f can be written as

$$\rho_f = p^{(1)}\rho_{DF}^{(1)} + p^{(2)}\rho_{DF}^{(2)}, \tag{A.121}$$

where $p^{(1)} + p^{(2)} = 1$.

A.4 Entropy of Subsystems

We now go back to the system before convergence. We would like to find the entropies of the qubit and fluctuator as a function of time. Starting with the qubit, we denote the density matrix

$$\rho_q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \tag{A.122}$$

which can be decomposed using the pauli-matrices:

$$\rho_q = \frac{1}{2}I + \frac{1}{2}m_i\sigma_i \tag{A.123}$$

We can then find the coefficients m_i in terms of the matrix elements q_{ij}

$$m_1 = 2q_{11} - 1 \tag{A.124}$$
$$m_2 = 2\mathrm{Re}\, q_{12} \tag{A.125}$$
$$m_3 = -2\mathrm{Im}\, q_{12} \tag{A.126}$$

The length of the Bloch vector is then

$$|m|^2 = (2q_{11} - 1)^2 + 4|q_{12}|^2 \tag{A.127}$$

We now need to find expressions for q_{11} and q_{12}. Since $q_{11} = \rho_{11} + \rho_{22}$ and $q_{12} = \rho_{13} + \rho_{24}$ we need the expressions for the matrix elements of the whole system. We assume the we start the evolution of the system with a photon measurement. We can then write the initial density matrix of the whole system as

$$\rho_0 = \begin{pmatrix} \rho_{11}^0 & 0 & \rho_{13}^0 & 0 \\ 0 & \rho_{22}^0 & 0 & \rho_{24}^0 \\ \rho_{31}^0 & 0 & \rho_{33}^0 & 0 \\ 0 & \rho_{42}^0 & 0 & \rho_{44}^0 \end{pmatrix} \tag{A.128}$$

Using the evolution matrix we then find elements after a time, t:

$$\rho_{11} = \frac{1}{2}[1+(c^2-s^2)^n]\rho_{11}^0 + \frac{1}{2}[1-(c^2-s^2)^n]\rho_{44}^0 \tag{A.129}$$

$$\rho_{22} = \frac{1}{2}[1+(c^2-s^2)^n]\rho_{22}^0 + \frac{1}{2}[1-(c^2-s^2)^n]\rho_{33}^0 \tag{A.130}$$

$$\rho_{13} = \frac{1}{2}[1+(c^2-s^2)^n]\rho_{13}^0 + \frac{1}{2}[1-(c^2-s^2)^n]\rho_{42}^0 \tag{A.131}$$

$$\rho_{24} = \frac{1}{2}[1+(c^2-s^2)^n]\rho_{24}^0 + \frac{1}{2}[1-(c^2-s^2)^n]\rho_{31}^0 \tag{A.132}$$

We then have

$$q_{11} = \frac{1}{2}[1+(c^2-s^2)^n]q_{11}^0 + \frac{1}{2}[1-(c^2-s^2)^n]q_{22}^0 \tag{A.133}$$

$$q_{12} = \frac{1}{2}[1+(c^2-s^2)^n]q_{12}^0 + \frac{1}{2}[1-(c^2-s^2)^n]q_{21}^0 \tag{A.134}$$

where q_{ij}^0 are the elements of the initial reduced density matrix for the qubit. We can now find the length of the Bloch vector:

$$\begin{aligned} |m_q|^2 &= (2q_{11}-1)^2 + 4|q_{12}|^2 \qquad \text{(A.135)} \\ &= \big([1+(c^2-s^2)^n]q_{11}^0 + [1-(c^2-s^2)^n]q_{22}^0 - 1\big)^2 \\ &\quad + \big([1+(c^2-s^2)^n[q_{12}^0 + [1-(c^2-s^2)^n]q_{21}^0\big) \\ &\quad \cdot \big([1+(c^s-s^2)^n]q_{21}^0 + [1-(c^2-s^2)^n]q_{12}^0\big) \qquad \text{(A.136)} \end{aligned}$$

We can find the Bloch vector for the fluctuator in a similar manner (remembering that we begin and end the evolution with a photon measurement):

$$\rho_f = \begin{pmatrix} \rho_{11}+\rho_{33} & \rho_{12}+\rho_{34} \\ \rho_{21}+\rho_{43} & \rho_{22}+\rho_{44} \end{pmatrix} = \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix} \tag{A.137}$$

$$\rho_{33} = \frac{1}{2}[1-(c^2-s^2)^n]\rho_{22}^0 + \frac{1}{2}[1+(c^2-s^2)^n]\rho_{33}^0 \tag{A.138}$$

$$f_{11} = \rho_{11} - \rho_{33} = \frac{1}{2}[1 - (c^2 - s^2)^n]f_{11}^0 + \frac{1}{2}[1 + (c^2 - s^2)^n]f_{22}^0 \tag{A.139}$$

$$|m_f|^2 = (2f_{11} - 1)^2 \tag{A.140}$$
$$= \left([1 - (c^2 - s^2)^n]f_{11}^0 + [1 + (c^2 - s^2)^n]f_{22}^0 - 1\right)^2 \tag{A.141}$$

To simplify the calculations we assume that the time τ between photons is much smaller than the characteristic period for oscillations of the system. This means that $v\tau \ll 1$. We can use this to find a simplified expression for $(c^2 - s^2)^n$. Expanding to second order in $v\tau$ we have

$$\cos(\frac{1}{2}v\tau) \approx 1 - \frac{1}{2}(\frac{1}{2}v\tau)^2 \tag{A.142}$$
$$\sin(\frac{1}{2}v\tau) \approx \frac{1}{2}v\tau \tag{A.143}$$

which gives us

$$\cos^2(\frac{1}{2}v\tau) - \sin^2(\frac{1}{2}v\tau) \approx 1 - \frac{1}{2}v^2\tau^2 \tag{A.144}$$

Next we want to find $(1 - \frac{1}{2}v^2\tau^2)^n$. Noticing that

$$(1 - \frac{1}{2}v^2\tau^2)^2 \approx 1 - v^2\tau^2 \tag{A.145}$$
$$(1 - \frac{1}{2}v^2\tau^2)^3 \approx 1 - \frac{3}{2}v^2\tau^2 \tag{A.146}$$
$$(1 - \frac{1}{2}v^2\tau^2)^4 \approx 1 - 2v^2\tau^2 \tag{A.147}$$

we conclude that

$$(1 - \frac{1}{2}v^2\tau^2)^n \approx 1 - \frac{n}{2}v^2\tau^2 \tag{A.148}$$

Finally, inserting for $n = t/\tau$, we have

$$[\cos^2(\frac{1}{2}v\tau) - \sin^2(\frac{1}{2}v\tau)]^n \approx 1 - \frac{1}{2}v^2\tau t \tag{A.149}$$

We can now insert this into the expression for $|m_q|^2$:

$$\begin{aligned} |m_q|^2 \approx {} & [(2 - \frac{1}{2}v^2\tau t)q_{11}^0 + \frac{1}{2}v^2\tau t q_{22}^0 - 1]^2 \\ & + [(2 - \frac{1}{2}v^2\tau t)q_{12}^0 + \frac{1}{2}v^2\tau t q_{21}^0] \\ & \cdot [(2 - \frac{1}{2}v^2\tau t)q_{21}^0 + \frac{1}{2}v^2\tau t q_{12}^0] \end{aligned} \tag{A.150}$$

Mulitplying and excluding higher order terms we arrive at

$$|m_q|^2 \approx 4(q_{11}^0)^2 + 1 - 4q_{11}^0 + 4|q_{12}^0|^2 + v^2\tau t[2q_{11}^0 q_{22}^0 - 2(q_{11}^0)^2 + q_{11}^0 - q_{22}^0 - 2|q_{12}^0|^2 + (q_{12}^0)^2 + (q_{21}^0)^2] \tag{A.151}$$

$$= (2q_{11}^0 - 1)^2 + 4|q_{12}^0|^2 + v^2\tau t[2q_{11}^0 q_{22}^0 - 2(q_{11}^0)^2 + q_{11}^0 - q_{22}^0 - 2|q_{12}^0|^2 + (q_{12}^0)^2 + (q_{21}^0)^2] \tag{A.152}$$

We identify $(2q_{11}^0 - 1)^2 + 4|q_{12}^0|^2$ as the length of the Bloch vector $|m_{q^0}|^2$ of the initial reduced density matrix of the qubit. Since the qubit starts in a pure state this length is equal to 1. This gives us

$$|m_q|^2 \approx 1 - v^2\tau t[2(q_{11}^0)^2 - 2q_{11}^0 q_{22}^0 - q_{11}^0 + q_{22}^0 + 2|q_{12}^0|^2 - (q_{12}^0)^2 - (q_{21}^0)^2] \tag{A.153}$$

Next we use the fact that $\mathrm{Tr}\,\rho_{q^0} = q_{11}^0 + q_{22}^0 = 1$ to find a simplified expression for $2(q_{11}^0)^2 - 2q_{11}^0 q_{22}^0 - q_{11}^0 + q_{22}^0$. Inserting for q_{22}^0 we have

$$2(q_{11}^0)^2 - 2q_{11}^0 q_{22}^0 - q_{11}^0 + q_{22}^0$$

$$= 2(q_{11}^0)^2 - 2q_{11}^0(1 - q_{11}^0) - q_{11}^0 + 1 - q_{11}^0 \tag{A.154}$$

$$= 4(q_{11}^0)^2 - 4q_{11}^0 + 1 \tag{A.155}$$

$$= (2q_{11}^0 - 1)^2 \tag{A.156}$$

We can also find an expression for $2|q_{12}^0|^2 - (q_{12}^0)^2 - (q_{21}^0)^2$:

$$2|q_{12}^0|^2 - (q_{12}^0)^2 - (q_{21}^0)^2$$

$$= 4|q_{12}^0|^2 - [2|q_{12}^0|^2 + (q_{12}^0)^2 + (q_{21}^0)^2] \tag{A.157}$$

$$= 4|q_{12}^0|^2 - (q_{12}^0 + q_{21}^0)^2 \tag{A.158}$$

We now have

$$|m_q|^2 \approx 1 - v^2\tau t[(2q_{11}^0 - 1)^2 + 4|q_{12}^0|^2 - (q_{12}^0 + q_{21}^0)^2] \tag{A.159}$$

We can again identify $(2q_{11}^0 - 1)^2 + 4|q_{12}^0|^2$ as 1. Also, since $q_{21}^0 = (q_{12}^0)^*$, we have $(q_{12}^0 + q_{21}^0)^2 = 4\mathrm{Re}(q_{12}^0)^2$. Finally we arrive at

$$|m_q| \approx \sqrt{1 - v^2\tau t[1 - 4\mathrm{Re}(q_{12}^0)^2]} \tag{A.160}$$

$$= \sqrt{1 - \theta v^2\tau t} \tag{A.161}$$

where $\theta = 1 - 4\mathrm{Re}(q_{12}^0)^2$ is a constant which depends on the initial state of the qubit. We see that the length starts at one at $t = 0$ and goes to zero as t

increases. For large interaction strength v the length decreases faster. This is because the qubit becomes more entangled with the fluctuator between measurements. This is also the case for large τ as the qubit has more time to become entangled.

We now use the same approximation on the fluctuator. Inserting the series expansion we have

$$|m_f|^2 \approx (2f_{11}^0 - \frac{1}{2}v^\tau t f_{11}^0 + \frac{1}{2}v^2\tau t f_{22}^0 - 1)^2 \tag{A.162}$$

$$\approx 4(f_{11}^0)^2 - 4f_{11}^0 + 1 - v^2\tau t[2(f_{11}^0)^2 - 2f_{11}^0 f_{22}^0 - f_{11}^0 + f_{11}^0] \tag{A.163}$$

Using that $f_{11}^0 + f_{22}^0 = 1$ we have

$$|m_f|^2 \approx (2f_{11}^0 - 1)^2(1 - v^2\tau t) \tag{A.164}$$

We can identify $(2f_{11}^0 - 1)^2$ as the initial length of the Bloch vector $|m_{f^0}|^2$ for the fluctuator. This is not necessarily equal to 1 since we start the evolution with a photon measurement. It is however easy to calculate. We then have our final expression:

$$|m_f| \approx |m_{f^0}|\sqrt{1 - v^2\tau t} \tag{A.165}$$

We now need to find an expression for the von Neuman entropy as a function of the length of the Bloch vector $|m|$. This is equivalent to the procedure in section (). We have

$$\rho_q = \frac{1}{2}\mathbb{1} + \frac{1}{2}m_i\sigma_i \tag{A.166}$$

Taking the logarithm we have

$$\ln \rho_q = \ln(\frac{1}{2}\mathbb{1}) + \ln(\mathbb{1} + m_i\sigma_i) \tag{A.167}$$

$$= \ln(\frac{1}{2}\mathbb{1}) + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(m_i\sigma_i)^n}{n} \tag{A.168}$$

The entropy is then given by

$$S = -\mathrm{Tr}(\rho_q \ln \rho_q) \tag{A.169}$$

$$= -\ln\frac{1}{2} - \frac{1}{2}\mathrm{Tr}\left[\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(m_i\sigma_i)^n}{n}\right] - \frac{1}{2}\mathrm{Tr}\left[\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(m_i\sigma_i)^{n+1}}{n}\right] \tag{A.170}$$

One can show that $\mathrm{Tr}[(m_i\sigma_i)^n]$ is $2|m|^n$ for even n and 0 for odd n. We then have

$$\begin{aligned} S = &-\ln\frac{1}{2} - \left(-\frac{|m|^2}{2} - \frac{|m|^4}{4} - \frac{|m|^6}{6} - \dots\right) \\ &- \left(|m|^2 + \frac{|m|^4}{3} + \frac{|m|^6}{5} + \dots\right) \end{aligned} \tag{A.171}$$

$$\begin{aligned} = &-\ln\frac{1}{2} - \frac{1}{2}\left(-|m|^2 - \frac{|m|^4}{2} - \frac{|m|^6}{3} - \dots\right) \\ &- |m|\left(|m| + \frac{|m|^3}{3} + \frac{|m|^5}{5} + \dots\right) \end{aligned} \tag{A.172}$$

$$= -\ln\frac{1}{2} - \frac{1}{2}\ln(1-|m|^2) - \frac{1}{2}|m|\ln\frac{1+|m|}{1-|m|} \tag{A.173}$$

$$= -\frac{1}{2}(1+|m|)\ln\frac{1}{2}(1+|m|) - \frac{1}{2}(1-|m|)\ln\frac{1}{2}(1-|m|) \tag{A.174}$$

We can now insert the approximations for $|m_q|$ and $|m_f|$ and we arrive at

$$\begin{aligned} S_q \approx &-\frac{1}{2}(1-\sqrt{1-\theta v^2\tau t})\ln\frac{1}{2}(1-\sqrt{1-\theta v^2\tau t}) \\ &-\frac{1}{2}(1+\sqrt{1-\theta v^2\tau t})\ln\frac{1}{2}(1+\sqrt{1-\theta v^2\tau t}) \end{aligned} \tag{A.175}$$

$$\begin{aligned} S_f \approx &-\frac{1}{2}(1-|m_{f^0}|\sqrt{1-v^2\tau t})\ln\frac{1}{2}(1-|m_{f^0}|\sqrt{1-v^2\tau t}) \\ &-\frac{1}{2}(1+|m_{f^0}|\sqrt{1-v^2\tau t})\ln\frac{1}{2}(1+|m_{f^0}|\sqrt{1-v^2\tau t}) \end{aligned} \tag{A.176}$$

By calculating these entropies our hope was to analyze the mutual information between the qubit and fluctuator. It turns out that the time-dependent entropy of the whole system is too messy to calculate analytically. Since even this simple, unrealistic model was too difficult for analytical calculations we chose to concentrate on the numerical solution.

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