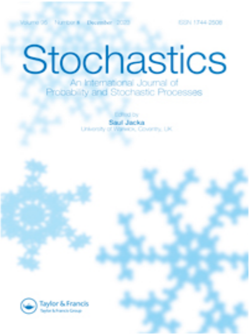

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The Donsker delta function and local time for McKean–Vlasov processes and applications

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ABSTRACT
The purpose of this paper is to establish a stochastic differential equation for the Donsker delta measure of the solution of a McKean–Vlasov (mean-field) stochastic differential equation.

If the Donsker delta measure is absolutely continuous with respect to Lebesgue measure, then its Radon–Nikodym derivative is called the Donsker delta function. In that case it can be proved that the local time of such a process is simply the integral with respect to time of the Donsker delta function. Therefore we also get an equation for the local time of such a process.

For some particular McKean–Vlasov processes, we find explicit expressions for their Donsker delta functions and hence for their local times.

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1. Introduction

The Donsker delta function of a random variable or a stochastic process arises in many studies, including quantum mechanical particles on a circle [7], financial markets with insider trading as in [10] and in [3] for financial markets with singular drift. It has also been used as a tool to determine explicit formulae for replicating portfolios in complete and incomplete markets [9].

Moreover, the Donsker delta function is also of interest because it can be regarded as a time derivative of the local time. Therefore, explicit expressions for the Donsker delta function lead to explicit formulae of the local time.

For example, if we let B be a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, P)$, then the Donsker delta function $\delta_{B(t)}(x)$ of a Brownian motion B at the point x can be regarded as the time derivative of the local time $L_t(x)$ of B. More precisely, we have

$$L_t(x) = \int_0^t \delta_{B(s)}(x)\,\mathrm{d}s.$$

Such an integral exists as an element of the Hida space $(\mathcal{S}^*)$ of stochastic distributions, see Section 2.3.

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In [6], the authors use white noise theory to obtain an explicit solution formula for a general stochastic differential equation (SDE), and this is used to find an expression for the Donsker delta function for the solution of an SDE. Subsequently this was also extended to SDEs driven by Lévy noise in [8].

The main result of the current paper is that the Donsker delta measure of a McKean–Vlasov process (see below) always satisfies a certain Fokker–Planck type SPDE in the sense of distributions. Moreover, we use this to find explicit formulae for the Donsker delta functions for McKean–Vlasov processes, and hence their local times, in specific cases.

Let $X(t) = X_t \in \mathbb{R}$ be the solution of a McKean–Vlasov SDE, i.e. a mean-field stochastic differential equation, of the form (using matrix notation),

$$\mathrm{d}X(t) = \alpha(t, X(t), \mu_t^X)\,\mathrm{d}t + \beta(t, X(t), \mu_t^X)\,\mathrm{d}B(t); \quad X(0) = Z \in \mathbb{R}.$$

We call X a *McKean–Vlasov process*.

Here the σ-algebra $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ denotes the filtration generated by Z and $B(\cdot)$, Z is a random variable which is independent of the σ-algebra generated by $B(\cdot)$ and such that $\mathbb{E}[|Z|^2] < \infty$.

Definition 1.1: Define $\mu_t^X(\mathrm{d}x) = \mu_t^X(\mathrm{d}x, \omega)$ to be regular conditional distribution of $X(t)$ given $\mathcal{F}_t$ generated by the Brownian motion B. This means that $\mu_t^X(\mathrm{d}x, \omega)$ is a Borel probability measure on $\mathbb{R}$ for all $t \in [0, T]$, $\omega \in \Omega$ and

$$\int_{\mathbb{R}} g(x)\mu_t^X(\mathrm{d}x, \omega) = \mathbb{E}[g(X(t))|\mathcal{F}_t](\omega),$$

for all functions g such that $\mathbb{E}[|g(X(t))|] < \infty$.

Since we consider only a one-dimensional Brownian motion $B(t) \in \mathbb{R}$, we will show that the regular conditional distribution of $X(t)$ given the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ can be identified with the Donsker delta measure in the sense of distribution. See details in Section 3.1

2. Preliminaries

In this section, we review some basic notions and results that will be used throughout this work.

2.1. *Radon measures*

A Radon measure on $\mathbb{R}^d$ is a Borel measure which is finite on compact sets, outer regular on all Borel sets and inner regular on all open sets. In particular, all Borel probability measures on $\mathbb{R}^d$ are Radon measures.

In the following, we let

- $\mathbb{M}_0$ be the set of deterministic Radon measures.
- $C_0(\mathbb{R}^d)$ be the uniform closure of the space $C_c(\mathbb{R}^d)$ of continuous functions with compact support.

If we equip $\mathbb{M}_0$ with the total variation norm $||\mu|| := |\mu|(\mathbb{R}^d)$, then $\mathbb{M}_0$ becomes a Banach space, and it is the dual of $C_0(\mathbb{R}^d)$. See Chapter 7 in Folland [5] for more information.

If $\mu \in \mathbb{M}_0$ is a finite measure, we define

$$\widehat{\mu}(y) := F[\mu](y) := \int_{\mathbb{R}^d} e^{-ixy}\mu\,(\mathrm{d}x); \quad y \in \mathbb{R}^d \tag{1}$$

to be the Fourier transform of μ at y.

In particular, if $\mu\,(\mathrm{d}x)$ is absolutely continuous with respect to Lebesgue measure $\mathrm{d}x$ with Radon–Nikodym derivative $m(x) = \frac{\mu\,(\mathrm{d}x)}{\mathrm{d}x}$, so that $\mu\,(\mathrm{d}x) = m(x)\,\mathrm{d}x$ with $m \in L^1(\mathbb{R}^d)$, we define the Fourier transform of m at y, denoted by $\widehat{m}(y)$ or $F[m](y)$, by

$$F[m](y) = \widehat{m}(y) = \int_{\mathbb{R}^d} e^{-ixy}m(x)\,\mathrm{d}x; \quad y \in \mathbb{R}^d.$$

We let $\mathbb{M}$ denote the set of all random measures $\mu\,(\mathrm{d}x, \omega); \omega \in \Omega$ such that $\mu\,(\mathrm{d}x, \omega) \in \mathbb{M}_0$ for each given $\omega \in \Omega$.

2.2. The Schwartz space of tempered distributions

We recall now some notions from white noise analysis.

- $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing smooth real functions on $\mathbb{R}^d$. It is a Fréchet space with respect to the family of seminorms:

$$\|f\|_{k,\alpha} := \sup_{x\in\mathbb{R}^d} \left\{(1 + |x|^k)|\partial^\alpha f(x)|\right\},$$

  where $k = 0, 1, \ldots, \alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index with $\alpha_j = 0, 1, \ldots$ $(j = 1, \ldots, d)$ and

$$\partial^\alpha f := \frac{\partial^{|\alpha|}f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \quad \text{for } |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

- $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions. It is the dual of $\mathcal{S}$.

2.3. The Hida space $(\mathcal{S})^*$ of stochastic distributions

We restrict ourselves to the white noise probability space $(\Omega = \mathcal{S}', \mathcal{F} = \mathcal{B}, P)$, where $\mathcal{B}$ is the Borel σ-algebra and the probability P is the probability measure on $\mathcal{S}'$ defined in virtue of the Bochner–Minlos–Sazonov theorem.

Let $\mathcal{J}$ denote the set of all finite multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $m = 1, 2, \ldots$, of non-negative integers α_i.

$$(2\mathbb{N})^\alpha = \prod_{j=1}^{m}(2j)^{\alpha_j} = (2\cdot 1)^{\alpha_1}(2\cdot 2)^{\alpha_2}(2\cdot 3)^{\alpha_3}\cdots(2m)^{\alpha_m}. \tag{2}$$

If $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathcal{J}$, we put

$$H_\alpha(\omega) := \prod_{j=1}^{m} h_{\alpha_j}(\theta_j(\omega)) = h_{\alpha_1}(\theta_1) h_{\alpha_2}(\theta_2) \cdots h_{\alpha_m}(\theta_m), \quad \omega \in \Omega. \tag{3}$$

The family $\{H_\alpha\}_{\alpha \in \mathcal{J}}$ constitutes an orthogonal basis of $L^2(P)$.

- $((\mathcal{S})_k)_{k \in \mathbb{R}}$ is the Hilbert space consisting of all $f = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \in L^2(P)$ such that $\|f\|_k^2 := \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2 (2\mathbb{N})^{\alpha k} < \infty$, for numbers $c_\alpha \in \mathbb{R}$.
- The space $(\mathcal{S}) = \bigcap_{k \in \mathbb{R}} (\mathcal{S})_k$ equipped with the projective topology is the Hida space of stochastic test functions.
- $((\mathcal{S})_{-k})_{k \in \mathbb{R}}$ is the Hilbert space consisting of all formal sums $F = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha$ equipped with the norm

$$\|F\|_{-k}^2 := \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2 (2\mathbb{N})^{-\alpha k} < \infty.$$

- The space $(\mathcal{S})^* = \bigcup_{k \in \mathbb{R}} (\mathcal{S})_{-k}$ equipped with the inductive topology is the Hida space of stochastic distributions. It can be regarded as the dual of $(\mathcal{S})$.

2.4. The Donsker delta function

We now recall some basic definitions:

Definition 2.1: Let $Y : \Omega \to \mathbb{R}$ be a random variable which also belongs to the Hida space $(\mathcal{S})^*$ of stochastic distributions. Then a continuous function

$$\delta_Y(\cdot) : \mathbb{R} \to (\mathcal{S})^* \tag{4}$$

is called a Donsker delta function of Y if it has the property that

$$\int_{\mathbb{R}} g(y) \delta_Y(y) \, \mathrm{d}y = g(Y) \quad a.s. \tag{5}$$

for all (measurable) $g : \mathbb{R} \to \mathbb{R}$ such that the integral converges in $(\mathcal{S})^*$.

The Donsker delta function is related to the *regular conditional distribution*. The connection is the following: the *regular conditional distribution* with respect to the σ-algebra $\mathcal{F}$ of a given real random variable Y, denoted by $\mu^Y(\mathrm{d}y) = \mu^Y(\mathrm{d}y, \omega); \omega \in \Omega$, is defined by the following properties:

- For any Borel set $\Lambda \subseteq \mathbb{R}$, $\mu^Y(\Lambda, \cdot)$ is a version of $\mathbb{E}[\mathbf{1}_{Y \in \Lambda} | \mathcal{F}]$.
- For each fixed $\omega \in \Omega$, $\mu^Y(\mathrm{d}y, \omega)$ is a probability measure on the Borel subsets of $\mathbb{R}$.

It is well known that such a regular conditional distribution always exists. See, e.g. [4, p. 79].

From the required properties of $\mu^Y(\mathrm{d}y, \omega)$, we get the following formula:

$$\int_{\mathbb{R}} f(y)\mu^Y(\mathrm{d}y, \omega) = \mathbb{E}[f(Y)|\mathcal{F}]. \tag{6}$$

Definition 2.2: We call $\mu^Y(\mathrm{d}y, \omega)$ the **Donsker delta measure** of the random variable Y and denote it by $\delta_Y(\mathrm{d}y, \omega)$.

Comparing this with the definition of the Donsker delta function, we obtain the following representation of the regular conditional distribution:

Lemma 2.3: *Suppose $\mu^Y(\mathrm{d}y, \omega)$ is absolutely continuous with respect to Lebesgue measure $\mathrm{d}y$ on $\mathbb{R}$ and that Y is measurable with respect to $\mathcal{F}$. Then the* ***Donsker delta function*** *of Y, $\delta_Y(y, \omega)$, is the Radon–Nikodym derivative of $\mu^Y(\mathrm{d}y, \omega)$ with respect to Lebesgue measure $\mathrm{d}y$, i.e.*

$$\delta_Y(y, \omega) = \frac{\mu^Y(\mathrm{d}y, \omega)}{\mathrm{d}y}. \tag{7}$$

We will prove in Theorem 3.3 that the Donsker delta function can be regarded as a stochastic distribution in $\mathcal{S}'$, satisfying a Fokker–Planck type SPDE in the sense of distributions. It can also be represented as an element of the Hida stochastic distribution space $(\mathcal{S})^*$, and as such it can in some cases be expressed explicitly in terms of Wick calculus. For example, if $Y(t) = B(t)$, we have

$$\delta_{B(t)}(x) = (2\pi t)^{-\frac{1}{2}} \exp^{\diamond}\left(-\frac{(B(t) - x)^{\diamond 2}}{2t}\right) \in (\mathcal{S})^*, \tag{8}$$

where $\diamond$ denotes Wick multiplication and $\exp^{\diamond}$ denotes Wick exponential. Note that even though the Donsker delta function can only be represented as a distribution, its conditional expectation can be a real-valued stochastic process. For example, for $t < T$ we have

$$\mathbb{E}[\delta_{B(T)}(x)|\mathcal{F}_t] = (2\pi(T - t))^{-\frac{1}{2}} \exp\left[-\frac{(B(t) - x)^2}{2(T - t)}\right]. \tag{9}$$

For more examples, we refer to e.g. [1] or [9].

3. The Donsker delta equation for McKean–Vlasov processes

3.1. The general multidimensional Fokker–Planck equation

To explain the background for this section, let us recall the general multidimensional situation studied in [2], where $X(t) \in \mathbb{R}^d$ is a McKean–Vlasov diffusion, of the form (using matrix notation),

$$\mathrm{d}X(t) = b(t, X(t), \mu_t^X)\,\mathrm{d}t + \sigma(t, X(t), \mu_t^X)\,\mathrm{d}B(t), \quad X(0) = Z, \tag{10}$$

where B is a multi-dimensional Brownian motion.

Here Z is a random variable which is independent of the σ-algebra generated by $B(\cdot)$ and such that

$$\mathbb{E}[|Z|^2] < \infty.$$

Define the σ-algebra $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ to be the filtration generated by Z and $B(\cdot)$.

Let $\mathbb{M}$ denote the set of all Borel measures on $\mathbb{R}^d$. We assume that the coefficients $b(t,x,\mu) : [0,T] \times \mathbb{R}^d \times \mathbb{M} \to \mathbb{R}^d$ and $\sigma(t,x,\mu) : [0,T] \times \mathbb{R}^d \times \mathbb{M} \to \mathbb{R}^d$ are bounded and $\mathbb{F}$-predictable processes for all x, μ, and that b and σ are continuous with respect to t and x for all μ.

One can check that under some assumptions, such as Lipschitz and linear growth conditions, there exists a unique solution of Equation (10).

Definition 3.1: Fix one of the Brownian motions, say $B_1 = B_1(t,\omega)$, with filtration $\{\mathcal{F}_t^{(1)}\}_{t\geq 0}$. We define $\mu_t^X = \mu_t^X(\mathrm{d}x,\omega)$ to be regular conditional distribution of $X(t)$ given $\mathcal{F}_t^{(1)}$. This means that $\mu_t^X(\omega,\ \mathrm{d}x)$ is a Borel probability measure on $\mathbb{R}^d$ for all $t \in [0,T], \omega \in \Omega$ and

$$\int_{\mathbb{R}^d} g(x)\mu_t^X(\mathrm{d}x,\omega) = \mathbb{E}[g(X(t))|\mathcal{F}_t^{(1)}](\omega) \tag{11}$$

for all functions g such that $\mathbb{E}[|g(X(t))|] < \infty$.

The following version of the stochastic Fokker–Planck integro-differential equation for the conditional law for McKean–Vlasov jump diffusions was proved by Agram and Øksendal [2]. For simplicity we consider only the case without jumps here.

Theorem 3.2 (Conditional stochastic Fokker–Planck equation [2]): *Let $X(t)$ be as in (10) with $d \geq 2$ and let $\mu_t^X := \mu_t^X(\mathrm{d}x,\omega)$ be the regular conditional distribution of $X(t)$ given $\mathcal{F}_t^{(1)}$.*

Then for a.a. $\omega \in \Omega$ the conditional law $\mu_t^X \in \mathcal{S}'$ and it satisfies the following SPDE (in the sense of distributions):

$$\mathrm{d}\mu_t^X = A_0^*\mu_t^X\,\mathrm{d}t + A_1^*\mu_t^X\,\mathrm{d}B_1(t), \quad \mu_0 = \mathcal{L}(X(0)). \tag{12}$$

Here A_0^*, A_1^* are the integro-differential operator and the differential operator which are given respectively by

$$A_0^*\mu = -\sum_{j=1}^{d} D_j[b_j\mu] + \frac{1}{2}\sum_{n,j=1}^{d} D_{n,j}[(\sigma\sigma^T)_{n,j}\mu] \tag{13}$$

and

$$A_1^*\mu = -\sum_{j=1}^{d} D_j[b_{1,j}\mu]. \tag{14}$$

In the above $D_j, D_{n,j}$ denote $\frac{\partial}{\partial x_j}$ and $\frac{\partial^2}{\partial x_n \partial x_j}$ respectively, in the sense of distributions, and $D_j[b_j\mu] = \frac{\partial}{\partial x_j}[b_j(t,x,\mu)\mu_t^X(\mathrm{d}x)]|_{\mu=\mu_t^X}$, and similarly with the other terms.

3.2. The Fokker–Planck equation for the Donsker measure

In [2], the theorem above was proved under the assumption that $d \geq 2$. However, the proof also works if $d = 1$ and $\mathcal{F}_t^{(1)} = \mathcal{F}_t$. Note that in this case, since $X(t)$ is $\mathcal{F}_t$-measurable, the identity (11) states that

$$\int_{\mathbb{R}^d} g(x)\mu_t^X(\mathrm{d}x,\omega) = g(X(t)) \tag{15}$$

for all functions g such that $\int_{\mathbb{R}^d} |g(x)|\mu_t^X(\mathrm{d}x,\omega) < \infty$.

In particular, if we choose $d = 1$ in the above we get that the conditional law coincides with the Donsker measure, i.e.

$$\mu_t^X(x,\omega) = \delta_{X(t)}(\mathrm{d}x,\omega). \tag{16}$$

Therefore we get the following Fokker–Planck equation for the Donsker measure:

Theorem 3.3: *Assume that $X(t)$ is as in (10), but with $d = 1$.*

Then the Donsker delta measure $\mu_t^X = \delta_{X(t)}(\mathrm{d}x,\omega)$ satisfies the following equation (in the sense of distribution):

$$\begin{aligned}
\mathrm{d}\mu_t^X &= \left\{ -D[b(t,x,\mu)\mu_t^X]|_{\mu=\mu_t^X} + \frac{1}{2}D^2[\sigma^2(t,x,\mu)\mu_t^X]|_{\mu=\mu_t^X} \right\} \mathrm{d}t \\
&\quad - D[\sigma(t,x,\mu)\mu_t^X]|_{\mu=\mu_t^X}\,\mathrm{d}B(t); \\
\mu_0 &= \mathcal{L}(X(0)),
\end{aligned} \tag{17}$$

where $D = \frac{\partial}{\partial x}$ and $D^2 = \frac{\partial^2}{\partial x^2}$.

4. Local time

In this section, we first recall the definition of local time of a stochastic process $Y(\cdot)$:

Definition 4.1: The local time $L_t(y)$ of $Y(\cdot)$ at the point y and at time t is defined by

$$L_t(y) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}\lambda(\{s \in [0,t]; Y(s) \in (y-\epsilon, y+\epsilon)\}),$$

where λ denotes Lebesgue measure on $\mathbb{R}$ and the limit is in $L^2(P)$.

In the white noise context, the local time can be represented as the integral of the Donsker delta function. More precisely, we have the following result:

Theorem 4.2: *The local time $L_t(x)$ of X at the point x and the time t is given by*

$$L_t(x) = \int_0^t \delta_{X(s)}(x)\,\mathrm{d}s, \tag{18}$$

where the integration takes place in $(\mathcal{S})^*$ (or in $\mathcal{S}'$ for each ω).

Proof: For completeness, we give the proof.

By definition of the local time and the Donsker delta function, we have

$$
\begin{aligned}
L_t(z) &= \lim_{\epsilon\to 0}\int_0^t \chi_{(z-\epsilon,z+\epsilon)}(Y(s))\,\mathrm{d}s \\
&= \lim_{\epsilon\to 0}\int_0^t \left(\int_{\mathbb{R}} \chi_{(z-\epsilon,z+\epsilon)}(y)\delta_{Y(s)}(y)\,\mathrm{d}y\right)\mathrm{d}s \\
&= \lim_{\epsilon\to 0}\int_{\mathbb{R}} \chi_{(z-\epsilon,z+\epsilon)}(y)\left(\int_0^t \delta_{Y(s)}(y)\,\mathrm{d}s\right)\mathrm{d}y = \int_0^t \delta_{Y(s)}(z)\,\mathrm{d}s,
\end{aligned}
$$

because the function $y \mapsto \delta_{Y(s)}(y)$ is continuous in $(\mathcal{S})^*$ (and in $\mathcal{S}'$). ■

Remark 4.3: Note that even though we in general can only say that $\delta_{X(t)}(x) \in (\mathcal{S})^*$, $L_t(x)$ usually exists as a real-valued stochastic process.

5. Explicit solutions

In this section, we find explicitly the Donsker delta function for some particular McKean–Vlasov processes and accordingly their local time.

Suppose that μ_t^X is absolutely continuous, i.e.

$$
\mu_t^X(\mathrm{d}x) = m^X(t,x)\,\mathrm{d}x. \tag{19}
$$

Then (10) gets the form

$$
\mathrm{d}X(t) = b(t,X(t),m_t^X)\,\mathrm{d}t + \sigma(t,X(t),m_t^X)\,\mathrm{d}B(t); \quad X(0)=Z, \tag{20}
$$

where $m_t^X(x) = m^X(t,x)$ and (17) becomes a *stochastic partial differential equation (SPDE)*, as follows:

Theorem 5.1: *Suppose* (19) *holds. Then the Donsker delta function* $m^X(t,x) = \delta_{X(t)}(x)$ *is the solution in* $(\mathcal{S})^*$ *of the following SPDE:*

$$
\begin{aligned}
d_t m^X(t,x) &= \left\{-\frac{\partial}{\partial x}[b(t,x,m)m^X(t,x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(t,x,m)m^X(t,x)]\right\}\mathrm{d}t \qquad (21)\\
&\quad - \frac{\partial}{\partial x}[\sigma(t,x,m)m^X(t,x)]\,\mathrm{d}B(t); \quad t\ge 0,\\
m(0,x) &= \frac{\partial}{\partial x}\mathcal{L}(X(0)). \qquad (22)
\end{aligned}
$$

5.1. Brownian motion

Consider the special case when $X(t) = B(t); B(0) = Z$. Then $b = 0$ and $\sigma = 1$ and Equation (17) becomes

$$
\begin{aligned}
\frac{\partial m^X}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 m^X}{\partial x^2}(t,x) + \frac{\partial m^X}{\partial x}(t,x)\diamond \dot{B}(t); \quad t\ge 0, \qquad (23)\\
m(0,x) &= \frac{\partial}{\partial x}\mathcal{L}(X(0)). \qquad (24)
\end{aligned}
$$

We can easily verify by Wick calculus that a solution in $(\mathcal{S})^*$ of Equation (23) is

$$\delta_{B(t)}(x) = (2\pi t)^{-\frac{1}{2}} \exp^{\diamond}\left(-\frac{(B(t)-x)^{\diamond 2}}{2t}\right), \tag{25}$$

which is in agreement with (8). The details are as follows:

Try

$$m^X(t,x) = \frac{1}{\sqrt{2\pi t}} \exp^{\diamond}\left[-\frac{(x-B(t))^{\diamond 2}}{2t}\right].$$

Then

$$\begin{aligned}\frac{\partial m^X}{\partial t}(t,x) = &-\frac{1}{2}\frac{t^{-3/2}}{\sqrt{2\pi}} \exp^{\diamond}\left[-\frac{(x-B(t))^{\diamond 2}}{2t}\right] \\ &+ \frac{1}{\sqrt{2\pi t}} \exp^{\diamond}\left[-\frac{(x-B(t))^{\diamond 2}}{2t}\right] \diamond \left(-\frac{x-B(t)}{t}\right) \diamond \dot{B}(t) \\ &+ \frac{1}{\sqrt{2\pi t}} \exp^{\diamond}\left[-\frac{(x-B(t))^{\diamond 2}}{2t}\right] \frac{(x-B(t))^2}{2t^2},\end{aligned}$$

and

$$\frac{\partial m^X}{\partial x} = \frac{1}{\sqrt{2\pi t}} \exp^{\diamond}\left[-\frac{(x-B(t))^{\diamond 2}}{2t}\right] \diamond \left(-\frac{x-B(t)}{t}\right),$$

and

$$\begin{aligned}\frac{\partial^2 m^X}{\partial x^2}(t,x) = &\frac{1}{\sqrt{2\pi t}} \exp^{\diamond}\left[-\frac{(x-B(t))^{\diamond 2}}{2t}\right] \diamond \left(\frac{x-B(t)}{t}\right)^{\diamond 2} \\ &+ \frac{1}{\sqrt{2\pi t}} \exp^{\diamond}\left[-\frac{(x-B(t))^{\diamond 2}}{2t}\right]\left(-\frac{1}{t}\right).\end{aligned}$$

Collecting the terms we see that

$$m^X(t,x) = \frac{1}{\sqrt{2\pi t}} \exp^{\diamond}\left(-\frac{(x-B(t))^{\diamond 2}}{2t}\right),$$

satisfies the Fokker–Planck equation (23) for the conditional law of $B(t)$.

From white noise theory, we know that

- $E[X \diamond Y] = E[X]E[Y]$
- $E[\exp^{\diamond} Y] := E[\sum_{n=0}^{\infty} \frac{1}{n!} Y^{\diamond n}] = \sum_{n=0}^{\infty} \frac{1}{n!} E[Y^{\diamond n}] = \sum_{n=0}^{\infty} \frac{1}{n!} E[Y]^n = \exp(E[Y])$

for all random variables X, Y with a finite expectation (independent or not). From this we see that

$$E[\delta_{B(t)}(x)] = \frac{1}{\sqrt{2\pi t}} \exp\left(E\left[-\frac{(x-B(t))^{\diamond 2}}{2t}\right]\right) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-E[Z])^2}{2t}\right). \quad (26)$$

In particular, if $X(0) = Z = z$ (constant) $\in \mathbb{R}$ a.e., then

$$E[\delta_{B(t)}(x)] = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-z)^2}{2t}\right), \quad (27)$$

which has a singularity at $x = z$.

5.2. Coefficients not depending on x

The next result shows that, under some conditions, the Donsker delta function can be an ordinary function if the initial value $X(0)$ has a density.

Theorem 5.2: *Assume that $X(t)$ is the solution of the following McKean–Vlasov equation:*

$$\mathrm{d}X(t) = \alpha(t,\mu_t^X)\,\mathrm{d}t + \beta(t,\mu_t^X)\,\mathrm{d}B(t); \quad X(0) = Z, \quad (28)$$

where the coefficients $\alpha(t,x,\mu) = \alpha(t,\mu)$ and $\beta(t,x,\mu) = \beta(t,\mu)$ do not depend on x. Suppose that $X(0) = Z$ is a random variable (independent of B) with density

$$h(z) = \frac{\partial}{\partial z}\mathcal{L}(Z)(z); \quad z \in \mathbb{R}. \quad (29)$$

(1) *Define*

$$\begin{aligned} Y_t(x) &= h\left(x - \int_0^t \alpha(s,\mu_s^X)\,\mathrm{d}s - \int_0^t \beta(s,\mu_s^X)\,\mathrm{d}B(s)\right) \\ &= h(K(t,x)), \end{aligned} \quad (30)$$

where

$$K(t,x) = x - \int_0^t \alpha(s,\mu_s^X)\,\mathrm{d}s - \int_0^t \beta(s,\mu_s^X)\,\mathrm{d}B(s).$$

Then $Y(t,x)$ is the Donsker delta function of $X(t)$.

(2) *The solution $X(t)$ of (28) is given by*

$$X(t) = \int_{\mathbb{R}} x Y_t(x)\,\mathrm{d}x. \quad (31)$$

Proof: (1) We show that $Y(t,x)$ satisfies Equation (21).

By the Ito formula, we have

$$\begin{aligned} d_tY(t,x) &= h'(K(t,x))d_tK(t,x) + \frac{1}{2}h''(K(t,x))\beta^2(t)\,\mathrm{d}t \\ &= \left\{-\alpha(t,\mu_t^X)h'(K(t,x)) + \frac{1}{2}\beta^2(t,\mu_t^X)h''(K(t,x))\right\}\mathrm{d}t \\ &\quad - \beta(t,\mu_t^X)h'(K(t,x))\,\mathrm{d}B(t). \end{aligned} \tag{32}$$

Since

$$h'(K(t,x)) = \frac{\mathrm{d}}{\mathrm{d}z}h(z)_{z=K(t,x)} = \frac{\partial}{\partial x}Y(t,x), \tag{33}$$

we see that Equation (32) can be written as

$$\begin{aligned} d_tY(t,x) &= \left[-\alpha(t,\mu_t^X)\frac{\partial}{\partial x}Y(t,x) + \frac{1}{2}\beta^2(t,\mu_t^X)\frac{\partial^2}{\partial x^2}Y(t,x)\right]\mathrm{d}t \\ &\quad - \beta(t,\mu_t^X)\frac{\partial}{\partial x}Y(t,x)\,\mathrm{d}B(t), \end{aligned} \tag{34}$$

which is the same as Equation (21).

Since $Y(0,x) = h(x) = m(0,x)$, we conclude by uniqueness that $Y(t,x) = m(t,x)$ for all t.

(2) This follows from the definition of the Donsker delta function.

■

5.2.1. Constant coefficients

As a special case of the case above, suppose that

$$\mathrm{d}X(t) = \alpha\,\mathrm{d}t + \beta\,\mathrm{d}B(t), \quad X(0) = Z, \tag{35}$$

where α and β are constants. Then by Theorem 5.2, the Donsker delta function is

$$\delta_{X(t)}(x) = h(x - \alpha t - \beta B(t)). \tag{36}$$

5.3. Mean-field geometric Brownian motion

Suppose that $X(t)$ is a McKean–Vlasov process of the form

$$\mathrm{d}X_t = \alpha(t,\mu_t^X)X_t\,\mathrm{d}t + \beta(t,\mu_t^X)X_t\,\mathrm{d}B_t; \quad X_0 = Z > 0. \tag{37}$$

We call this a *mean-field geometric Brownian motion*. For such processes, we have:

Theorem 5.3: (i) *The Donsker delta function $m_t^X(x)$ for the mean-field geometric Brownian motion $X(t)$ is*

$$m_t^X(x) = \delta_{X_t}(x) = \frac{1}{x}H\left(\ln x - \int_0^t \alpha(s,\mu_s^X)\,\mathrm{d}s - \int_0^t \beta(s,\mu_s^X)\,\mathrm{d}B(s)\right), \tag{38}$$

where

$$H(z) = \frac{\partial}{\partial z}\mathcal{L}(\ln Z)(z); \quad z \in \mathbb{R}. \tag{39}$$

(ii) *The solution* $X(t)$ *of the mean-field geometric Brownian motion equation* (37) *can be written as*

$$X(t) = \int_0^{\infty} H\left(\ln x - \int_0^t \alpha(s,\mu_s^X)\,\mathrm{d}s - \int_0^t \beta(s,\mu_s^X)\,\mathrm{d}B(s)\right)\mathrm{d}x$$
$$= \int_{\mathbb{R}} e^u H\left(u - \int_0^t \alpha(s,\mu_s^X)\,\mathrm{d}s - \int_0^t \beta(s,\mu_s^X)\,\mathrm{d}B(s)\right)\mathrm{d}u. \qquad (40)$$

Proof: (i) The corresponding Fokker–Planck equation for the Donsker delta function $m_t = m_t^X(x) = \delta_{X(t)}(x)$ is

$$\mathrm{d}m_t(x) = \left\{-\frac{\partial}{\partial x}[\alpha(t,m)xm_t(x)] + \frac{1}{2}\beta^2(t,m)\frac{\partial^2}{\partial x^2}[x^2 m_t(x)]\right\}\mathrm{d}t$$
$$- \beta(t,m)\frac{\partial}{\partial x}[xm_t(x)]\,\mathrm{d}B_t$$
$$= \Big\{-\alpha(t,m)m_t(x) - \alpha(t,m)xm_t'(x)$$
$$+ \frac{1}{2}\beta^2(t,m)[2m_t(x) + 4xm_t'(x) + x^2 m_t''(x)]\Big\}\,\mathrm{d}t$$
$$- \beta(t,m)[m_t(x) + xm_t'(x)]\,\mathrm{d}B_t; \quad m_0(x) = \frac{\partial}{\partial x}\mathcal{L}(Z)(x). \qquad (41)$$

This is a stochastic partial differential equation in $m_t(x)$. It seems difficult to find directly an explicit solution of this equation. However, we can find the solution $m_t(x) = \delta_{X(t)}(x)$ by proceeding as follows:

The solution of (37) is

$$X_t = Z\exp\left(\int_0^t \beta(s,m_s)\,\mathrm{d}B(s) + \int_0^t \{\alpha(s,m_s) - \frac{1}{2}\beta^2(s,m_s)\}\,\mathrm{d}s\right) = \exp(Y_t),$$

where

$$Y_t = \ln Z + \int_0^t \beta(s,m_s)\,\mathrm{d}B(s) + \int_0^t \{\alpha(s,m_s) - \frac{1}{2}\beta^2(s,m_s)\}\,\mathrm{d}s.$$

By Theorem 5.2, we know that

$$\delta_{Y_t}(x) = H\left(x - \int_0^t \alpha(s,\mu_s)\,\mathrm{d}s - \int_0^t \beta(s,\mu_s)\,\mathrm{d}B(s)\right),$$

where

$$H(z) = \frac{\partial}{\partial z}\mathcal{L}(\ln Z)(z).$$

By definition we have

$$\int_{\mathbb{R}^+} g(y)\delta_{Y_t}(y)\,\mathrm{d}y = g(Y_t).$$

With $g(y) = \exp(y)$, this gives

$$\int_{\mathbb{R}} \exp(y)\delta_{Y_t}(y)\,\mathrm{d}y = \exp(Y_t) = X_t.$$

Hence, substituting $\exp(y) = x$,

$$X_t = \int_{\mathbb{R}} \exp(y)\delta_{Y_t}(y)\,\mathrm{d}y = \int_{\mathbb{R}+} x\delta_{Y_t}(\ln(x))\frac{\mathrm{d}x}{x}.$$

From this we deduce that

$$m_t(x) = \delta_{X_t}(x) = \frac{\delta_{Y_t}(\ln(x))}{x} = \frac{1}{x}H\left(\ln x - \int_0^t \alpha(s,\mu_s)\,\mathrm{d}s - \int_0^t \beta(s,\mu_s)\,\mathrm{d}B(s)\right) \tag{42}$$

is the Donsker delta function of X_t.

(ii) This part follows by the definition of the Donsker delta function.

■

5.4. An example related to the Burgers equation

Suppose the McKean–Vlasov equation has the form

$$\mathrm{d}X(t) = \alpha m(t, X(t))\,\mathrm{d}t + \beta\,\mathrm{d}B(t); \quad X(0) = Z, \tag{43}$$

where $m(t,x) = \frac{\partial}{\partial x}\mu^X(t,x) = \frac{\partial}{\partial x}\mathcal{L}(X(t))(x) \in L^2([0,T]\times\mathbb{R})$.

Then the corresponding FP equation for the Donsker function $m(t,x)$ is

$$\mathrm{d}m(t,x) = \left\{-\alpha\frac{\partial}{\partial x}(m^2(t,x)) + \frac{1}{2}\beta^2\frac{\partial^2}{\partial x^2}m(t,x)\right\}\mathrm{d}t - \beta\frac{\partial}{\partial x}m(t,x)\,\mathrm{d}B(t), \tag{44}$$

$$m(0,x) = h(x) = \frac{\partial}{\partial x}Z(x). \tag{45}$$

This is a stochastic Burgers equation. It is well known that by using the Cole–Hopf transformation the equation can be transformed into the classical heat equation. The details are as follows: if we introduce a new function $\psi = \psi(t,x)$ such that

$$m := \psi_x := \frac{\partial}{\partial x}\psi, \tag{46}$$

then we see that the Burgers equation (44) becomes the following equation in ψ:

$$(\psi_x)_t = -2\alpha\psi_x(\psi_x)_x + \frac{1}{2}\beta^2(\psi_x)_{xx} - \beta\psi_{xx} \diamond \dot{B}(t). \tag{47}$$

Integrating with respect to x this gives

$$\psi_t = -\alpha(\psi_x)^2 + \frac{1}{2}\beta^2\psi_{xx} - \beta\psi_x \diamond \dot{B}(t). \tag{48}$$

Now define the function $\varphi = \varphi(t,x)$ by

$$\psi = \gamma \ln \varphi, \tag{49}$$

for some constant γ. Then in terms of φ the above equation gets the form

$$\begin{aligned}
\gamma \frac{\varphi_t}{\varphi} &= -\alpha\gamma^2 \left(\frac{\varphi_x}{\varphi}\right)^2 + \frac{1}{2}\gamma\beta^2 \left(\frac{\varphi_x}{\varphi}\right)_x - \beta\gamma \frac{\varphi_x}{\varphi} \diamond \dot{B}(t) \\
&= -\alpha\gamma^2 \left(\frac{\varphi_x}{\varphi}\right)^2 + \frac{1}{2}\gamma\beta^2 \left(\frac{\varphi\varphi_{xx} - (\varphi_x)^2}{\varphi^2}\right)_x - \beta\gamma \frac{\varphi_x}{\varphi} \diamond \dot{B}(t) \\
&= -\alpha\gamma^2 \left(\frac{\varphi_x}{\varphi}\right)^2 + \frac{1}{2}\gamma\beta^2 \left(\frac{\varphi_{xx}}{\varphi}\right) - \frac{1}{2}\gamma\beta^2 \left(\frac{\varphi_x}{\varphi}\right)^2 - \beta\gamma \frac{\varphi_x}{\varphi} \diamond \dot{B}(t).
\end{aligned} \tag{50}$$

This simplifies to

$$\varphi_t = -(\gamma\alpha + \frac{1}{2}\beta^2) \left(\frac{\varphi_x}{\varphi}\right)^2 + \frac{1}{2}\beta^2 \varphi_{xx} - \beta\varphi_x \diamond \dot{B}(t). \tag{51}$$

If we choose

$$\gamma = -\frac{\beta^2}{2\alpha}, \tag{52}$$

the equation for φ reduces to the (linear) stochastic heat equation

$$\varphi_t = \frac{1}{2}\beta^2 \varphi_{xx} - \beta\varphi_x \diamond \dot{B}(t), \tag{53}$$

$$\varphi(0,x) = k(x) (\text{to be determined}), \tag{54}$$

or, using Ito differential notation,

$$\mathrm{d}\varphi(t,x) = \frac{1}{2}\beta^2 \varphi_{xx}(t,x)\,\mathrm{d}t - \beta\varphi_x(t,x)\,\mathrm{d}B(t); \quad t \geq 0, \tag{55}$$

$$\varphi(0,x) = k(x). \tag{56}$$

To find an expression for the solution of (53), define an auxiliary process $R(t) = R^{(x)}(t)$ by

$$R(t) = x - \beta B(t) + \beta \tilde{B}(t); \quad t \geq 0, \tag{57}$$

where $\tilde{B}$ is an auxiliary Brownian motion with law $\tilde{P}$ and independent of B. Then by the Feynman–Kac formula

$$\varphi(t,x) := \tilde{E}[k(R^{(x)}(t))] = \tilde{E}[k(x - \beta B(t) + \beta\tilde{B}(t))], \tag{58}$$

where $\tilde{E}$ denotes expectation with respect to $\tilde{P}$ and $k(z) = \varphi(0,z)$, solves Equation (53). Going back to m, we get

$$m(t,x) = \psi_x(t,x) = \gamma \frac{\partial}{\partial x} \ln \varphi(t,x)) = \gamma \frac{\varphi_x(t,x)}{\varphi(t,x)} = \gamma \frac{\tilde{E}[k_x(x - \beta B(t) + \beta\tilde{B}(t))]}{\tilde{E}[k(x - \beta B(t) + \beta\tilde{B}(t))]}. \tag{59}$$

In particular, setting $t = 0$ we get

$$h(x) := m(0,x) = \gamma \frac{\tilde{E}[k_x(x)]}{\tilde{E}[k(x)]} = \gamma \frac{k_x(x)}{k(x)}, \tag{60}$$

from which we deduce that

$$k(x) = \exp\left(\frac{1}{\gamma}\int_0^x h(y)\,\mathrm{d}y\right). \tag{61}$$

We summarize what we have proved as follows:

Theorem 5.4: (1) *The Donsker delta function* $m(t,x) = \delta_{X(t)}(x)$ *for the solution* $X(t)$ *of the McKean–Vlasov equation* (43) *is given by*

$$m(t,x) = \gamma \frac{\varphi_x(t,x)}{\varphi(t,x)} = \gamma \frac{\tilde{E}[k_x(x - \beta B(t) + \beta \tilde{B}(t))]}{\tilde{E}[k(x - \beta B(t) + \beta \tilde{B}(t))]}, \tag{62}$$

where

$$k(x) = \exp\left(\frac{1}{\gamma}\int_0^x m(0,y)\,\mathrm{d}y\right) = \exp\left(\frac{1}{\gamma}\int_0^x \mathcal{L}(Z)(y)\,\mathrm{d}y\right) \tag{63}$$

and

$$\gamma = -\frac{\beta^2}{2\alpha}. \tag{64}$$

(2) *The solution* $X(t)$ *of* (43) *is given by*

$$X(t) = \int_{\mathbb{R}} x m(t,x)\,\mathrm{d}x \tag{65}$$

with $m(t,x)$ *as in part* 1.

5.5. A solution approach based on Laplace and Fourier transforms

Consider the Fokker–Planck equation, with $\mu = \mu_t^X$,

$$\mathrm{d}\mu_t = \left\{-D[\alpha\mu] + \frac{1}{2}D^2[\beta^2\mu]\right\}\mathrm{d}t - -D[\beta\mu]\,\mathrm{d}B(t); \quad \mu_0 = \delta_{x_0}, \tag{66}$$

for the McKean–Vlasov equation (35). If α, β are constants, this becomes

$$\mathrm{d}\mu_t = \left\{-\alpha D[\mu] + \frac{1}{2}\beta^2 D^2[\mu]\right\}\mathrm{d}t - \beta D[\mu]\,\mathrm{d}B(t); \quad \mu_0 = \delta_{x_0}. \tag{67}$$

If $\mathrm{d}\mu_t = m(t,x)\,\mathrm{d}x$, the equation can be written as

$$\frac{\partial}{\partial t}m(t,x) = -\alpha\frac{\partial}{\partial x}m(t,x) + \frac{1}{2}\beta^2\frac{\partial^2}{\partial x^2}m(t,x) + \beta\frac{\partial}{\partial x}m(t,x) \diamond \overset{\circ}{B}(t). \tag{68}$$

Let

$$\widetilde{f}(s) = Lf(s) = \int_0^s e^{-st} f(t)\,\mathrm{d}t \text{ denote the Laplace transform} \tag{69}$$

and

$$\widehat{f}(y) = Ff(y) = \int_{\mathbb{R}} e^{-ixy} f(x)\,\mathrm{d}x \text{ denote the Fourier transform.} \tag{70}$$

Then

$$L\left(\frac{\partial}{\partial t} f(t)\right)(s) = s(Lf)(s) - f(0) \tag{71}$$

and

$$L(\exp(bt))(s) = \frac{1}{s-b} \tag{72}$$

and

$$F[D^n w](y) = (iy)^n F[w](y). \tag{73}$$

Hence, applying the Laplace and Fourier transform to (68), we get

$$s\widetilde{\widehat{m}}(s,y) - \widehat{m}(0,x_0) = -i\alpha y \widetilde{\widehat{m}}(s,y) + \frac{1}{2}\beta^2 (iy)^2 \widetilde{\widehat{m}}(s,y) + \beta iy (\widehat{m}(.,y) \diamond \mathring{B}(.))(s)$$

or

$$\widetilde{\widehat{m}}(s,y)[s + i\alpha y + \frac{1}{2}\beta^2 y^2] = \widehat{m}(0,x_0) + \beta iy \widetilde{(\widehat{m}(.,y) \diamond \mathring{B}(.))}(s)$$

or

$$\begin{aligned} \widetilde{\widehat{m}}(s,y) &= \frac{\widehat{m}(0,x_0)}{s + i\alpha y + \frac{1}{2}\beta^2 y^2} + \frac{\beta iy}{s + i\alpha y + \frac{1}{2}\beta^2 y^2} \widetilde{(\widehat{m}(.,y) \diamond \mathring{B}(.))}(s) \\ &= \frac{\widehat{m}(0,x_0)}{s + i\alpha y + \frac{1}{2}\beta^2 y^2} + \beta iy L(e^{(-i\alpha y - \frac{1}{2}\beta^2 y^2)t}(s) \widetilde{\widehat{m}(.,y) \diamond \mathring{B}(.))}(s). \end{aligned} \tag{74}$$

Put $g(t) = e^{(-i\alpha y - \frac{1}{2}\beta^2 y^2)t}$ and $h(t) = \widehat{m}(t,y) \diamond \mathring{B}(t))$.

Taking inverse Laplace transform, we get

$$\begin{aligned} \widehat{m}(t,y) &= \widehat{m}(0,x_0)\exp((i\alpha y - \frac{1}{2}\beta^2 y^2)t) + \beta iy L^{-1}(Lg \cdot Lh)(t,y) \\ &= \widehat{m}(0,x_0)\exp((i\alpha y - \frac{1}{2}\beta^2 y^2)t) + \beta iy (g * h)(t,y), \end{aligned}$$

where

$$(g * h)(t) = \int_0^t g(s)h(t-s)\,\mathrm{d}s.$$

Recall that

$$\int_{\mathbb{R}} e^{-ay^2 - 2by}\,\mathrm{d}y = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{a}}; \quad a > 0. \tag{75}$$

Hence

$$
\begin{aligned}
F^{-1}(g) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\alpha yt - \frac{1}{2}\beta^2 y^2 t + iyx}\,\mathrm{d}y = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}\beta^2 t y^2 - 2yi\left(\frac{1}{2}\alpha t - \frac{1}{2}x\right)}\,\mathrm{d}y \\
&= \frac{1}{2\pi}\sqrt{\frac{\pi}{\frac{1}{2}\beta^2 t}}\exp\left(\frac{i^2(\frac{1}{2}\alpha t + \frac{1}{2}x)^2}{\frac{1}{2}\beta^2 t}\right) \\
&= \frac{1}{\sqrt{2\pi\beta^2 t}}\exp\left(-\frac{(\alpha t + x)^2}{2\beta^2 t}\right) \\
&= \frac{1}{\sqrt{2\pi\beta^2 t}}\exp\left(-\frac{x^2}{2\beta^2 t} - \frac{\alpha x}{\beta^2}\right)\exp\left(-\frac{\alpha^2 t}{2\beta^2}\right) =: k(t,x).
\end{aligned}
$$

Therefore $g(t,x) = F[k(t,.)](y)]$ and (75) can be written as

$$
\begin{aligned}
\widehat{m}(t,y) &= \widehat{m}(0,x_0)F[k](t,y) \\
&\quad + \beta i y\int_0^t F[k(t-s,.)](y)F[m(s,y)\diamond \overset{\circ}{B}(s))]\,\mathrm{d}s.
\end{aligned}
$$

Taking inverse Fourier transform we get, with $k' = \frac{\mathrm{d}}{\mathrm{d}x}k(t,x)$

$$
\begin{aligned}
m(t,x) &= m(0,x_0) * k)(t,x) \\
&\quad + F^{-1}\left[\beta\int_0^t F[k'(t-s,\cdot)](y)F[m(s,y)\diamond \overset{\circ}{B}(s))\Big]\right](t,x) \\
&= \int_{\mathbb{R}} \delta_{x_0}(x-y)k(t,y)\,\mathrm{d}y \\
&\quad + \beta\int_0^t \left(\int_{\mathbb{R}} k'(t-s,x-y)m(s,y)\diamond \overset{\circ}{B}(s)\,\mathrm{d}s\,\mathrm{d}y\right. \\
&= k(t,x-x_0) + \beta\int_{\mathbb{R}}\left(\int_0^t k'(t-s,x-y)m(s,y)\,\mathrm{d}B(s)\right)\mathrm{d}y.
\end{aligned}
$$

We have proved the following:

Theorem 5.5: *Suppose α and β are constants and that the Donsker delta measure is absolutely continuous with respect to Lebesgue measure. Then the Donsker delta function $m(t,x) = \delta_{X(t)}(x)$ of the corresponding McKean–Vlasov process is a solution in $(\mathcal{S})^*$ of the following stochastic Volterra equation:*

$$
m(t,x) = k(t,x-x_0) + \beta\int_{\mathbb{R}}\left(\int_0^t k'(t-s,x-y)m(s,y)\,\mathrm{d}B(s)\right)\mathrm{d}y,
$$

where

$$
k(t,z) = \frac{1}{\sqrt{2\pi\beta^2 t}}\exp\left(-\frac{z^2}{2\beta^2 t} - \frac{\alpha z}{\beta^2}\right)\exp\left(-\frac{\alpha^2 t}{2\beta^2}\right); \quad k'(u,z) = \frac{\mathrm{d}}{\mathrm{d}z}k(u,z).
$$

Remark 5.6: If $\alpha = 0, \beta = 1$, we get

$$k(t,z) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{z^2}{2t}\right)$$
$$k'(u,z) = -\frac{z}{u}\frac{1}{\sqrt{2\pi u}} \exp\left(-\frac{z^2}{2u}\right).$$

For comparison, recall that the density of Brownian motion at t, x (when starting at x_0) is

$$p(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-x_0)^2}{2t}\right).$$

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