

The Total Energy of Friedmann-Robertson-Walker Universes

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Chapter 1

Introduction

1.1 Mission statement

I will look at 'the total energy of the universe'. This is an interesting issue, because if the energy of the universe turns out not to be conserved, it will be in conflict with our common understanding of energy. So intuitively we expect the energy of the universe to be constant. Furthermore, if this total energy is constant and zero, it means that 'creating' a universe does not require any energy. Such a universe could then, in principle, just 'pop up' from nothing.

Our universe is dominated by a so-called cosmological constant, or vacuum energy. It has the property that the energy density is constant in volume, so when the universe expands, the total amount of vacuum energy increases. Where does this new energy come from? One might immediately think that it could be energy from other components in the universe that is converted into vacuum energy. But it turns out not to be that simple, since the vacuum energy increases also for universe models which contain vacuum energy only. For flat (Minkowski) spacetimes, a global energy conservation law can be set up without problems. But we know that we need the general theory of relativity to describe the universe more realistically. General relativity deals with curved spacetimes, and then it is in general not possible to set up a global law for conservation of energy.

Because of these problems, calculating the total energy of the universe is not trivial, but different attempts have been made in the past. I will in the following look at one particular method for calculating the energy of the universe, presented in [1] by Faraoni and Cooperstock. Faraoni and Cooperstock look at open and critically open Friedmann-Robertson-Walker (FRW) universes. They argue that the total energy of a universe of this type is 0. Their reasoning goes as follows: They consider a $k = 0$ FRW universe, and by deriving an equation for its total energy, they show that the energy here is constant. They argue that FRW universes have Minkowski space as their asymptotic state, since the density approaches 0 as $t \rightarrow \infty$ due to the expansion, and flat spacetime has zero energy. This last point is shown by setting $m = 0$ in the Schwarzschild metric, the metric then becomes the Minkowski metric. The Schwarzschild metric reads:

$$ds^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (1.1)$$

The energy of Schwarzschild spacetime is well defined, and equal to m . m is called the 'mass parameter', and is just the mass of the spherical object. Hence Faraoni and Cooperstock conclude that the constant energy has the value 0. By applying global conservation of energy, they argue that the energy of the universe is 0 at all times. They point out that one can just choose the value of the constant energy to be zero, since the zero level is arbitrary.

To sum up, Faraoni and Cooperstock's reasoning has two steps. First, they derive an equation for the total energy of the universe, and show that this energy is constant. Second, they argue that open and critically open Friedmann-Robertson-Walker (FRW) universes have zero energy, because these universes have Minkowski space as their asymptotic state, and Minkowski space has zero energy. I will look at both steps of their reasoning. Does their method for calculating the energy of the universe give results that are in agreement with other methods? And is their argument for zero energy water proof?

I will evaluate Faraoni and Cooperstock's results, and then carry out their calculations in a more general way. While their energy equation is valid only for models with zero spatial curvature, I will derive an energy equation that is valid for any value of k . Then, I will look at some specific universe models, and apply their/my results to these by calculating their energies. Is the energy really zero?

1.2 The expansion of the universe, vacuum energy, and energy conservation

The universe is expanding, and measurements done in the last ten years show that it is expanding at an ever increasing rate. (See [2].) The simplest explanation for this, is that the universe is now dominated by a cosmological constant (Lorentz invariant vacuum energy) that works as a repulsive gravitational force. This repulsive gravitation is what is needed to drive the accelerated expansion.

The cosmological constant's contribution to the energy density of the universe is constant in time. If we look at a region with volume $V(t)$ at time t , the cosmological constant will make a contribution

$$E_{\Lambda} = \rho_{\Lambda} c^2 V(t) \tag{1.2}$$

to the energy, where $\rho_{\Lambda} c^2$ is the energy density associated with the cosmological constant Λ . But $V(t)$ increases with time, due to the expansion of the universe. In other words, the energy of the universe seems to increase as time passes! Is this in agreement with energy conservation?

1.3 Newtonian cosmology

Our intuition is often Newtonian, so let us start by looking at the closest analogy in Newtonian physics. Let us start with an expanding universe filled with uniformly distributed non-relativistic matter with density ρ . We can model the universe as a sphere with radius R ,

where R is an increasing function of time due to the expansion. The total mass within this sphere, M , is constant, and we have

$$M = \frac{4\pi}{3} R^3 \rho = \text{constant} \quad (1.3)$$

$$\Rightarrow \rho = \frac{3M}{4\pi R^3} \quad (1.4)$$

so that the density decreases as the radius increases. Because of the expansion, the matter particles will move, and then again have kinetic energies. We can determine how the speed of the particles varies throughout the sphere by requiring that the number of particles within a given volume be constant (if not, particles would pile up in some areas of the sphere, and the density would then no longer be the same everywhere). The particle density is the same everywhere and is given by

$$n = \frac{n_0}{R^3} \quad (1.5)$$

where n_0 is a reference density of our choice.

The change in the number of particles within a small volume in a short time interval happens for two reasons. First, the density changes as a result of the expansion. This change can be written as

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial t} \frac{n_0}{R^3} = -3n_0 \frac{1}{R^4} \frac{dR}{dt} = -3 \frac{n_0}{R^3} \frac{\dot{R}}{R} \equiv -3nH \quad (1.6)$$

where $H = \frac{\dot{R}}{R}$ is the Hubble parameter. Second, particles may leave or enter the small volume. It can be shown that the net effect can be written as

$$n \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \quad (1.7)$$

where v_i is the component of the velocity in the i -direction. The sum of these two contributions to the change in particle density must equal zero:

$$-3nH + n \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0 \quad (1.8)$$

$$\Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 3H(t) \quad (1.9)$$

For this equation to be satisfied, we must have

$$\frac{\partial v_x}{\partial x} = \frac{\partial v_y}{\partial y} = \frac{\partial v_z}{\partial z} = H, \quad (1.10)$$

i.e.

$$v_x = Hx, \quad v_y = Hy, \quad v_z = Hz \quad (1.11)$$

or

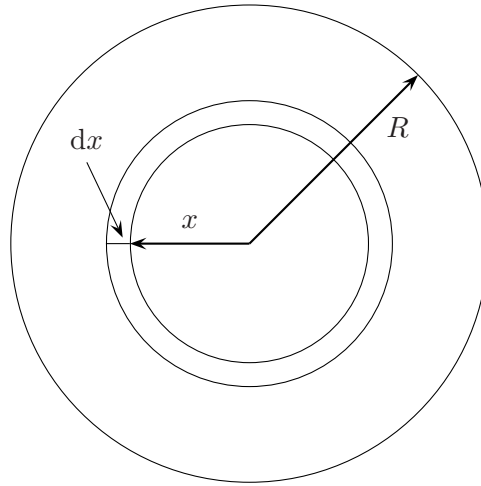
$$\vec{v} = H\vec{r}, \quad (1.12)$$

where

$$\vec{r} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z \quad (1.13)$$

and we have chosen the constant of integration so that $\vec{v}(\vec{r} = 0) = \vec{0}$.

Now, consider a shell of radius x and thickness dx .



This shell will make a contribution to the kinetic energy of the sphere given by

$$dE_k = \frac{1}{2} dm v^2(x) \quad (1.14)$$

where

$$dm = 4\pi x^2 dx \rho = 4\pi x^2 dx \frac{3M}{4\pi R^3} = \frac{3M}{R^3} x^2 dx \quad (1.15)$$

and

$$v^2(x) = (Hx)^2 = H^2 x^2 \quad (1.16)$$

so that

$$dE_k = \frac{3MH^2}{2R^3} x^4 dx. \quad (1.17)$$

The total kinetic energy of the sphere is then found by integration:

$$\begin{aligned}
E_k &= \int_0^R dE_k = \frac{3MH^2}{2R^3} \int_0^R x^4 dx \\
&= \frac{3}{10}MH^2R^2
\end{aligned} \tag{1.18}$$

The sphere also possesses gravitational potential energy: Imagine constructing the sphere layer by layer by bringing in spherical shells from infinitely far away. In the process, we work against gravity, and it is clear that it is only the mass within radius x that contributes to the potential energy of the shell located at $x + dx$.

The shell located at radius x with thickness dx therefore makes a contribution to the total potential energy of the sphere given by

$$dE_p = -G \frac{m(< x) dm}{x} \tag{1.19}$$

where $m(< x)$ is the mass within radius x :

$$m(< x) = \frac{4\pi}{3}x^3\rho = \frac{4\pi}{3}x^3 \frac{3M}{4\pi R^3} = \frac{M}{R^3}x^3. \tag{1.20}$$

As before,

$$dm = \frac{3M}{R^3}x^2 dx. \tag{1.21}$$

Hence, we get

$$\begin{aligned}
dE_p &= -G \frac{1}{x} \frac{M}{R^3} x^3 \frac{3M}{R^3} x^2 dx \\
&= -\frac{3GM^2}{R^6} x^4 dx.
\end{aligned} \tag{1.22}$$

Integrating this over the entire sphere gives the total potential energy of the sphere:

$$E_p = -\frac{3GM^2}{R^6} \int_0^R x^4 dx = -\frac{3GM^2}{5R}. \tag{1.23}$$

Now, we have what we need to write down an expression for the total mechanical energy of the sphere:

$$\begin{aligned}
E = E_k + E_p &= \frac{3}{10}MH^2R^2 - \frac{3}{5}\frac{GM^2}{R} \\
&= \frac{3M}{10}\left(H^2R^2 - \frac{2GM}{R}\right) \\
&= \frac{3M}{10}\left(H^2R^2 - \frac{2G}{R}\frac{4\pi}{3}R^3\rho\right) \\
&= \frac{3MR^2}{10}\left(H^2 - \frac{8\pi G}{3}\rho\right)
\end{aligned} \tag{1.24}$$

Requiring that the mechanical energy be conserved ($E = \text{constant}$), we get the equation

$$H^2 - \frac{8\pi G}{3}\rho = \frac{10E}{3M}\frac{1}{R^2} \tag{1.25}$$

$$\Rightarrow H^2 - \frac{10E}{3M}\frac{1}{R^2} = \frac{8\pi G}{3}\rho \tag{1.26}$$

where

$$H = \frac{1}{R}\frac{dR}{dt} \equiv \frac{\dot{R}}{R}. \tag{1.27}$$

According to the general theory of relativity (GR),

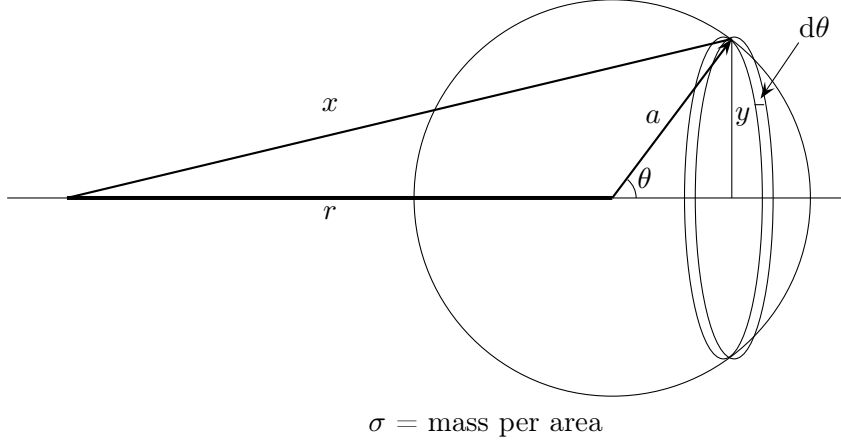
$$H^2 + \frac{kc^2}{R^2} = \frac{8\pi G}{3}\rho \tag{1.28}$$

where k is a constant, is the equation describing a homogeneous, isotropic universe. It is known as the first Friedmann equation. We see that by requiring that the energy of the Newtonian universe be conserved, this universe can be described by an equation that is on the same form as the equation we get from GR. As the sphere expands, the potential energy increases (gets less negative), and the kinetic energy will decrease. It is natural to expect the concept of potential energy to be a key to our understanding of the dynamics of expansion. Could this shed some light on what happens in the case with a cosmological constant?

1.4 A Newtonian cosmological constant

Does it make sense to introduce a cosmological constant within the framework of the Newtonian theory of gravitation? Yes, it does, and Newton himself considered this possibility! (See [3].) Newton's law of gravitation can be used successfully in calculations involving, say, planetary motion. One of the reasons for this is that the gravitational force exerted on a point mass by a spherical mass distribution is exactly the same as though all of the mass were concentrated at its center. The force can be expressed as the gradient of a potential, so the same must be true for the potential of the mass distribution. We know that we can think of the sphere as being made up by many thin spherical shells, so it is sufficient to consider the

potential outside one of these shells. We wish to find a potential that is consistent with the 'sphere theorem', and express it in its most general form.



We can calculate the potential in a location at a distance r from the center of a spherical shell with radius a and mass per unit area σ . The potential per mass due to a point particle is $\phi(x)$. All the points on the strip with width $d\theta$ in the figure are equidistant from our point of choice; we call this distance x . The mass of the strip is given by

$$dM = \sigma 2\pi y a d\theta = 2\pi\sigma a^2 \sin\theta d\theta. \quad (1.29)$$

The strip's contribution to the potential is then

$$d\Phi = \phi(x) dM = 2\pi\sigma a^2 \phi(x) \sin\theta d\theta \quad (1.30)$$

and the potential due to the spherical shell is

$$\Phi = \int d\Phi = 2\pi\sigma a^2 \int_0^\pi \phi(x) \sin\theta d\theta. \quad (1.31)$$

Using the law of cosines, we get

$$x^2 = r^2 + a^2 - 2ar \cos(\pi - \theta) = r^2 + a^2 + 2ar \cos\theta \quad (1.32)$$

so that

$$2x dx = -2ar \sin\theta d\theta \quad (1.33)$$

$$\Rightarrow \sin\theta d\theta = -\frac{x dx}{ar}. \quad (1.34)$$

Changing variables from θ to x , requires that we change the limits of integration as well. The new limits are

$$\theta = 0 \quad \Rightarrow \quad x^2 = r^2 + a^2 + 2ar = (r + a)^2 \quad \Rightarrow \quad x = r + a \quad (1.35)$$

$$\theta = \pi \quad \Rightarrow \quad x^2 = r^2 + a^2 - 2ar = (r - a)^2 \quad \Rightarrow \quad x = r - a \quad (1.36)$$

(Here, we are assuming that $r \geq a$, which corresponds to our being located outside the mass distribution.) Then,

$$\begin{aligned} \Phi &= -2\pi\sigma a^2 \frac{1}{ar} \int_{r+a}^{r-a} \phi(x) x \, dx \\ &= \frac{2\pi\sigma a}{r} \int_{r-a}^{r+a} x \phi(x) \, dx. \end{aligned} \quad (1.37)$$

The force is given by the gradient of Φ , i.e. the derivative of Φ with respect to r . Any constant terms in the potential will then vanish. So for the 'sphere theorem' to be satisfied, we see that the potential must be on the form

$$\begin{aligned} \Phi &= \text{the mass of the shell} \cdot \text{the potential due to a unit mass at the center of the shell} \\ &\quad + \text{constant}, \end{aligned} \quad (1.38)$$

that is

$$M(a) \phi(r) + 2\pi\sigma a \lambda(a) = \frac{2\pi\sigma a}{r} \int_{r-a}^{r+a} x \phi(x) \, dx. \quad (1.39)$$

Here, $M(a) = 4\pi\sigma a^2$ = the mass of the spherical shell. The only noteworthy thing about the second term on the left hand side is that it is independent of r , which means that it will not contribute to the force. It proves convenient to express it in this way.

Dividing (1.39) through by $M(a)$ gives

$$\phi(r) + \frac{\lambda(a)}{2a} = \frac{1}{2ar} \int_{r-a}^{r+a} x \phi(x) \, dx. \quad (1.40)$$

This is an integral equation for $\phi(r)$. Let us try a solution on the form

$$\phi(r) = \frac{A}{r} + Br^2 + F \quad (1.41)$$

where A , B , and F are constants. Inserting this solution into the integral on the right hand side of (1.40) gives

$$\begin{aligned}
\int_{r-a}^{r+a} x \phi(x) dx &= \int_{r-a}^{r+a} (A + Bx^3 + Fx) dx \\
&= \left[Ax + \frac{1}{4}Bx^4 + \frac{1}{2}Fx^2 \right]_{r-a}^{r+a} \\
&= A(r+a - r+a) + \frac{1}{4}B [(r+a)^4 - (r-a)^4] \\
&\quad + \frac{1}{2}F [(r+a)^2 - (r-a)^2] \\
&= 2aA + 2ar(Br^2 + a^2B + F),
\end{aligned} \tag{1.42}$$

so that the right hand side of (1.40) becomes

$$\frac{1}{2ar} [2aA + 2ar(Br^2 + a^2B + F)] = \frac{A}{r} + Br^2 + a^2B + F. \tag{1.43}$$

Substituting (1.41) into the left hand side of (1.40) gives

$$\phi(r) + \frac{\lambda(a)}{2a} = \frac{A}{r} + Br^2 + F + \frac{\lambda(a)}{2a}. \tag{1.44}$$

Equating (1.43) and (1.44), we see that by choosing

$$\frac{\lambda(a)}{2a} = a^2B \Rightarrow \lambda(a) = 2a^3B, \tag{1.45}$$

equation (1.40) is satisfied.

It can be shown that this is the most general solution of (1.40). We recognize the $\frac{A}{r}$ -term as the ordinary gravitational potential from the Newtonian theory of gravitation. We can express the potential energy of a point mass m sitting in the gravitational field set up by a spherical mass distribution of mass M as

$$E_p = -\frac{GMm}{r} - \frac{1}{6}\Lambda mr^2. \tag{1.46}$$

How should this be interpreted? First, we calculate $\nabla^2\phi$:

$$\nabla^2\phi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = \frac{1}{r^2} \frac{d}{dr} \left[r^2 \left(-\frac{A}{r^2} + 2Br \right) \right] = 6B = \text{constant} \tag{1.47}$$

The ordinary Newtonian theory of gravitation can be established if we start out with Poisson's equation

$$\nabla^2\phi = 4\pi G\rho(\vec{r}) \tag{1.48}$$

where $\rho(\vec{r})$ is the mass density at \vec{r} . For a point mass m at the origin of an empty space, the solution to this equation is the familiar $\frac{1}{r}$ -potential. For such a point mass, $\rho(\vec{r}) = m\delta(\vec{r})$

where $\delta(\vec{r})$ is the Dirac delta function. The Dirac delta function is zero where-ever $\vec{r} \neq 0$, so except for at the origin, Poisson's equation becomes

$$\nabla^2\phi = 0. \quad (\text{Laplace's equation}) \quad (1.49)$$

But in our case, the potential ϕ is on a more general form. Then, even in an empty space, we get

$$\nabla^2\phi = 6B = 4\pi G \frac{3B}{2\pi G} = 4\pi G \frac{\Lambda}{8\pi G} \quad (1.50)$$

where we have introduced $\Lambda = 12B$.

We no longer have $\nabla^2\phi = 0$ in the empty space surrounding the point mass. But we can still keep Poisson's equation in its general form if we express it as

$$\nabla^2\phi = 4\pi G\rho_{tot} \quad (1.51)$$

where

$$\rho_{tot} = \rho + \rho_\Lambda \quad (1.52)$$

and we interpret

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} \quad (1.53)$$

as the mass density of 'empty space'. In other words, we can view the more general form of ϕ as a result of empty space making a constant contribution to the mass density. We may call this a 'cosmological constant'.

1.5 Newtonian cosmology including a cosmological constant

Let us return to our 'spherical universe', but this time we will start out with the more general expression for the gravitational potential energy we just found:

$$E_p = -\frac{GMm}{r} - \frac{1}{6}\Lambda mr^2 \quad (1.54)$$

The kinetic energy and the contribution from the first term of E_p are found in the same way as before:

$$E_k = \frac{3}{10}MH^2R^2 \quad (1.55)$$

$$E_{p,1} = -\frac{3GM^2}{5R} \quad (1.56)$$

All we need to do now is to calculate the contribution from the second term of E_p : We have that

$$dE_{p,2} = -\frac{1}{6}\Lambda x^2 dm = -\frac{1}{6}\Lambda x^2 \frac{3M}{R^3} x^2 dx = -\frac{1}{2} \frac{\Lambda M}{R^3} x^4 dx \quad (1.57)$$

so that

$$E_{p,2} = -\frac{\Lambda M}{2R^3} \int_0^R x^4 dx = -\frac{\Lambda M}{10} R^2. \quad (1.58)$$

The total energy of the sphere is then

$$\begin{aligned} E &= E_k + E_{p,1} + E_{p,2} \\ &= \frac{3}{10} M H^2 R^2 - \frac{3}{5} \frac{GM^2}{R} - \frac{\Lambda M}{10} R^2 \\ &= \frac{3MR^2}{10} \left(H^2 - \frac{2GM}{R^3} - \frac{1}{3} \Lambda \right) \\ &= \frac{3MR^2}{10} \left(H^2 - \frac{2G}{R^3} \frac{4\pi}{3} \rho R^3 - \frac{1}{3} \Lambda \right) \\ &= \frac{3MR^2}{10} \left(H^2 - \frac{8\pi G}{3} \rho - \frac{1}{3} \Lambda \right). \end{aligned} \quad (1.59)$$

In this case, using conservation of energy, we get

$$H^2 - \frac{8\pi G}{3} \rho - \frac{1}{3} \Lambda = \frac{10E}{3M} \frac{1}{R^2} \equiv -\frac{kc^2}{R^2}, \quad (1.60)$$

which is on the same form as the Friedmann equation with a cosmological constant from general relativity. From our Newtonian viewpoint, we see that an accelerated expansion is completely consistent with energy conservation. When R is large, the term $-\frac{\Lambda M}{10} R^2$ dominates, and increasing kinetic energy will imply decreasing potential energy. Note that it is possible for the universe to start out with $E = 0$, and still expand!

1.6 The realistic case: General relativity

Looking at this from a Newtonian point of view suggests that vacuum energy and an accelerated expansion do not interfere with any fundamental principles. That being said, the analysis is not complete, because we know that the Newtonian theory of gravitation must be replaced by the general theory of relativity, and because the expressions we have used for the potential energy are not valid in an infinitely large universe with a uniform mass density.¹ For example, the expression $E_p = \frac{GMm}{r}$ is only valid if the mass M is finite, and if the density of the mass distribution approaches 0 when $r \rightarrow \infty$. The expression for the potential energy is not valid in a universe where the mass density is constant. In the Newtonian analysis, the universe is pictured as a sphere that is expanding in an empty space. What does general relativity have to say about the cosmological constant? Interpreted as vacuum energy, it obeys a local law of energy conservation that can be written

$$\frac{\partial \rho}{\partial t} + 3H \left(\rho + \frac{p}{c^2} \right) = 0, \quad (1.61)$$

¹In the following, I have made use of [4], [5], [6], and [7].

and furthermore, for vacuum energy $p = -\rho c^2$, so

$$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \rho = \text{constant}. \quad (1.62)$$

But in order for us to get from a local law of conservation of energy, valid for a small volume element, to a global conservation law, valid for the entire universe, we must integrate over all such volume elements. This will cause problems that can most easily be identified if we consider the general case of energy conservation. According to general relativity, the so-called energy-momentum tensor contains information about the energy. It is denoted by T . Let us have a look at special relativity first, where spacetime is flat. Consider a volume V at times t_0 and t_1 , and let S be the surface of the volume. The energy conservation law for this volume can then be written

$$\int_{V, t=t_0} T \, dV - \int_{V, t=t_1} T \, dV = \int_{t=t_0}^{t=t_1} T \, dt \, dS. \quad (1.63)$$

The first integral is an expression for the energy within V at t_0 , and the second integral expresses the energy within V at t_1 . Thus, the left hand side of the equation expresses how the energy within the volume V changes during the time that elapses from t_0 to t_1 . The right hand side describes the energy that flows through the surface S . Thus, the equation says that the change in energy within the volume V is caused by energy flowing into or out of the volume through the surface S .

The most important thing we need to know about T is that we can produce vectors in 4-dimensional $((t, x, y, z))$ Minkowski space by integrating it. This global conservation law can be re-written as a local conservation law by the use of Gauss' integral theorem:

$$\begin{aligned} \text{Coordinate divergence of } T &\equiv T_{;\nu}^{\mu\nu} \equiv \frac{\partial}{\partial x^\nu} T^{\mu\nu} \\ &= \frac{\partial}{\partial t} T^{\mu t} + \frac{\partial}{\partial x} T^{\mu x} + \frac{\partial}{\partial y} T^{\mu y} + \frac{\partial}{\partial z} T^{\mu z} \\ &= \frac{\partial V^0}{\partial t} + \nabla \cdot \vec{V} = 0 \end{aligned} \quad (1.64)$$

where we have defined $\vec{V} \equiv (T^{\mu t}, T^{\mu x}, T^{\mu y}, T^{\mu z})$.

Let us move on to general relativity. Now, a local conservation law for T follows from the field equation:

$$\text{Covariant divergence of } T \equiv T_{;\nu}^{\mu\nu} \equiv \nabla_\nu T^{\mu\nu} = 0 \quad (1.65)$$

This is a generalization of (1.64) for a curved spacetime. The trouble is that it is not possible to integrate it in order to establish a global conservation law like in special relativity. The reason for this is that the local law in GR involves a so-called covariant divergence, and because of this, Gauss' theorem can no longer be used in order to establish a global law of conservation.

For a given set of coordinates, the integral

$$\int T^{\mu\nu} \sqrt{-g} dS_\nu \quad (1.66)$$

is conserved only if

$$(\sqrt{-g} T^{\mu\nu})_{,\nu} = \frac{\partial(\sqrt{-g} T^{\mu\nu})}{\partial x^\nu} = 0 \quad (1.67)$$

by use of the integral theorem. In other words, the integral theorem involves the coordinate divergence, not the covariant divergence. Given (1.65), does not necessarily mean that (1.67) is satisfied. Hence, there is no bridge that can take us from local to global conservation laws. The underlying reason for this, is that the energy-momentum tensor does not include gravitational energy. Locally, we can neglect gravitational terms, because they will disappear if we work within a local free fall system. But globally, they must be taken into account. This is due to the fact that a curved space will appear flat (that is, as Minkowski space) if we look at a sufficiently small area. In other words, a curved space can locally be described as Minkowski space.

However, one can choose a coordinate system and manipulate the conservation law valid in GR so that it is written on the form

$$\text{Coordinate divergence of } T = 0, \quad (1.68)$$

and then one can establish a global conservation law. But this can only be done uniquely in special cases. One such case is the Schwarzschild spacetime surrounding a black hole. The reason that a unique global conservation law can be established in this case, is that this spacetime approaches Minkowski (flat) spacetime as we move away from the black hole. However, the Schwarzschild metric describes a space where the cosmological principle is not fulfilled. The cosmological principle, fundamental in cosmology, states that there is no preferred or special location in the universe. Once we place a black hole in an otherwise empty space, the location of the black hole will be just that, a 'special place' in the universe. For the types of spacetimes we deal with in cosmology, there is unfortunately no global, coordinate independent law for conservation of energy.

In the following section, I will look at one example where we can manipulate the GR conservation law and so get a global conservation law. I will present one suggestion to how it can be done. I stress that this is not a unique approach.

1.7 Energy in a homogeneous isotropic universe using pseudotensors

In this section, I will consider a homogeneous isotropic universe. In this case it is possible to calculate its energy, using so-called pseudotensors. A pseudotensor is a quantity that is really NOT a tensor, but acts as one: Pseudotensors are not invariant like tensors. In general, the form of a pseudotensor will depend on the frame of reference. Because of this, an equation

containing pseudotensors which holds in one frame will not necessarily hold in a different frame.

I will follow the procedure presented in [8], except I will allow for any spatial curvature k . Rosen only considers the case where $k = 1$.

The Einstein pseudotensor of the gravitational field is given by

$$t_{\mu}^{\nu} = \frac{1}{16\pi} \left(\mathcal{L} \delta_{\mu}^{\nu} - g_{\lambda\sigma, \mu} \frac{\partial \mathcal{L}}{\partial g_{\lambda\sigma, \nu}} \right), \quad (1.69)$$

where

$$\mathcal{L} = g^{\sigma\tau} \left(\Gamma_{\sigma\beta}^{\alpha} \Gamma_{\tau\alpha}^{\beta} - \Gamma_{\alpha\beta}^{\alpha} \Gamma_{\sigma\tau}^{\beta} \right). \quad (1.70)$$

If T_{μ}^{ν} is the energy-momentum density of the matter, then one has the Einstein conservation equation

$$[(-g)^{\frac{1}{2}}(T_{\mu}^{\nu} + t_{\mu}^{\nu})]_{,\nu} = 0. \quad (1.71)$$

In other words, by introducing this pseudotensor, we can write down a global conservation law. As mentioned earlier, the energy-momentum tensor does not include gravitational terms. We need the pseudotensor to make up for this.

The time component ($\mu = 0$) describes conservation of energy, and the space components ($\mu = 1, 2, 3$) conservation of momentum. Because we are assuming isotropy, T_0^j vanishes. (j is a spatial index.) The energy conservation equation then reads

$$[(-g)^{\frac{1}{2}}(T_0^0 + t_0^0)]_{,\nu} = 0. \quad (1.72)$$

We wish to calculate the total energy of this universe. Since we have a global conservation law in this case, we can find the total energy by integrating over the entire universe:

$$E = \int (-g)^{\frac{1}{2}} (T_0^0 + t_0^0) d^3x \quad (1.73)$$

The homogeneous isotropic universe can be described by the familiar Robertson-Walker line element

$$ds^2 = dt^2 - R^2(t) \left[\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2 \theta d\phi^2 \right] \quad (1.74)$$

where $R(t)$ is the scale factor and k is spatial curvature. In the following however, we will use the line element in the so-called isotropic form. We introduce a new radial coordinate defined by

$$\bar{r} \equiv \frac{r}{1 + \frac{kr^2}{4}} \quad (1.75)$$

so that

$$d\bar{r} = \frac{1 + \frac{kr^2}{4} - r\frac{kr}{2}}{\left(1 + \frac{kr^2}{4}\right)^2} dr = \frac{1 - \frac{kr^2}{4}}{\left(1 + \frac{kr^2}{4}\right)^2} dr. \quad (1.76)$$

Substituting this into the line element gives the line element in the isotropic form (See Appendix A for details):

$$ds^2 = dt^2 - \frac{R^2}{\left(1 + \frac{kr^2}{4}\right)^2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \quad (1.77)$$

Expressed in Cartesian coordinates, this becomes

$$ds^2 = dt^2 - \frac{R^2}{\left(1 + \frac{kr^2}{4}\right)^2} (dx^2 + dy^2 + dz^2) \quad (1.78)$$

with $r^2 = x^2 + y^2 + z^2$ and the scale factor $R = R(t)$. We need to work with Cartesian coordinates, since other coordinates may lead to non-physical values for t'_μ . The reason for this is that t'_μ transforms as a tensor ONLY when expressed in Cartesian coordinates.

As stated earlier, the Einstein pseudotensor is given by

$$t'_\mu{}^\nu = \frac{1}{16\pi} \left(\mathcal{L} \delta_\mu^\nu - g_{\lambda\sigma, \mu} \frac{\partial \mathcal{L}}{\partial g_{\lambda\sigma, \nu}} \right), \quad (1.79)$$

where

$$\mathcal{L} = g^{\sigma\tau} \left(\Gamma_{\sigma\beta}^\alpha \Gamma_{\tau\alpha}^\beta - \Gamma_{\alpha\beta}^\alpha \Gamma_{\sigma\tau}^\beta \right). \quad (1.80)$$

In the following, however, we will use that we can write

$$(-g)^{\frac{1}{2}} (T_\mu^\nu + t'_\mu{}^\nu) = \frac{1}{16\pi} H_{\mu, \sigma}^{\nu\sigma}, \quad (1.81)$$

where

$$H_{\mu}^{\nu\sigma} = (-g)^{-\frac{1}{2}} g_{\mu\lambda} \left[-g \left(g^{\nu\lambda} g^{\sigma\tau} - g^{\sigma\lambda} g^{\nu\tau} \right) \right]_{, \tau} \quad (1.82)$$

so that the Einstein conservation equation follows from the fact that

$$H_{\mu}^{\nu\sigma} = -H_{\mu}^{\sigma\nu}. \quad (1.83)$$

We can write

$$(x^0, x^1, x^2, x^3) = (t, x, y, z). \quad (1.84)$$

Latin letters will denote spatial indices. The metric tensor has the non-vanishing covariant components

$$g_{00} = 1, \quad g_{ij} = -\frac{R^2}{\left(1 + \frac{kr^2}{4}\right)^2} \delta_i^j. \quad (1.85)$$

For clarity, I write out the covariant metric tensor:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{R^2}{\left(1 + \frac{kr^2}{4}\right)^2} & 0 & 0 \\ 0 & 0 & -\frac{R^2}{\left(1 + \frac{kr^2}{4}\right)^2} & 0 \\ 0 & 0 & 0 & -\frac{R^2}{\left(1 + \frac{kr^2}{4}\right)^2} \end{bmatrix} \quad (1.86)$$

Since $g_{\mu\nu}$ is diagonal, g is just the product of the diagonal components of $g_{\mu\nu}$:

$$g = \det g_{\mu\nu} = -\frac{R^6}{\left(1 + \frac{kr^2}{4}\right)^6} \quad (1.87)$$

$$-g = \frac{R^6}{\left(1 + \frac{kr^2}{4}\right)^6} \quad (1.88)$$

$$(-g)^{-\frac{1}{2}} = \frac{\left(1 + \frac{kr^2}{4}\right)^3}{R^3} \quad (1.89)$$

To make use of (1.82) and find the components of H , we need the contravariant components of the metric tensor. Luckily, these are easy to find, since $g_{\mu\nu}$ is a diagonal matrix: The diagonal components of $g^{\mu\nu}$ are just the inverses of the corresponding components of $g_{\mu\nu}$, and all other components are zero:

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\left(1 + \frac{kr^2}{4}\right)^2}{R^2} & 0 & 0 \\ 0 & 0 & -\frac{\left(1 + \frac{kr^2}{4}\right)^2}{R^2} & 0 \\ 0 & 0 & 0 & -\frac{\left(1 + \frac{kr^2}{4}\right)^2}{R^2} \end{bmatrix} \quad (1.90)$$

We find from (1.82) that

$$H_0^{0j} = (-g)^{-\frac{1}{2}} g_{0\lambda} \left[-g \left(g^{0\lambda} g^{j\tau} - g^{j\lambda} g^{0\tau} \right) \right]_{,\tau}. \quad (1.91)$$

This is a sum over all λ , and we see that only $\lambda = 0$ survives:

$$\begin{aligned}
 H_0^{0j} &= (-g)^{-\frac{1}{2}} g_{00} [-g (g^{00} g^{j\tau} - g^{j0} g^{0\tau})]_{,\tau} \\
 &= (-g)^{-\frac{1}{2}} [-g (g^{j\tau})]_{,\tau}
 \end{aligned} \tag{1.92}$$

Here, we sum over τ , and only the spatial index $\tau = j$ will contribute:

$$H_0^{0j} = (-g)^{-\frac{1}{2}} [-g (g^{jj})]_{,j} \tag{1.93}$$

$$= \frac{\left(1 + \frac{kr^2}{4}\right)^3}{R^3} \left[\frac{R^6}{\left(1 + \frac{kr^2}{4}\right)^6} \left(-\frac{\left(1 + \frac{kr^2}{4}\right)^2}{R^2} \right) \right]_{,j} \tag{1.94}$$

$$\begin{aligned}
 &= \frac{\left(1 + \frac{kr^2}{4}\right)^3}{R^3} \left[\frac{-R^4}{\left(1 + \frac{kr^2}{4}\right)^4} \right]_{,j} \\
 &= -R \left(1 + \frac{kr^2}{4}\right)^3 \left[\left(1 + \frac{kr^2}{4}\right)^{-4} \right]_{,j}
 \end{aligned} \tag{1.95}$$

Using that $r^2 = x^2 + y^2 + z^2$ gives (remember that x^j can be either x, y or z)

$$\begin{aligned}
 H_0^{0j} &= -R \left(1 + \frac{kr^2}{4}\right)^3 \left[\left(1 + \frac{k}{4} (x^2 + y^2 + z^2)\right)^{-4} \right]_{,j} \\
 &= -R \left(1 + \frac{kr^2}{4}\right)^3 (-4) \left(1 + \frac{k}{4} (x^2 + y^2 + z^2)\right)^{-5} \cdot \frac{k}{4} \cdot 2x^j \\
 &= 2R \left(1 + \frac{kr^2}{4}\right)^{-2} k \cdot x^j
 \end{aligned} \tag{1.96}$$

Finally, we get

$$H_0^{0j} = \frac{2kRx^j}{\left(1 + \frac{kr^2}{4}\right)^2} = -H_0^{j0} \tag{1.97}$$

where the last equality holds by the use of (1.83).

All other components $H_\mu^{\nu\sigma}$ vanish.

Starting out with (1.81) we get

$$(-g)^{\frac{1}{2}} (T_0^0 + t_0^0) = \frac{1}{16\pi} H_{0,\sigma}^{0\sigma} \tag{1.98}$$

$$= \frac{1}{16\pi} [H_{0,0}^{00} + H_{0,1}^{01} + H_{0,2}^{02} + H_{0,3}^{03}]. \tag{1.99}$$

We calculate the four terms one at a time:

$$H_{0,0}^{00} = 0 \quad (1.100)$$

$$\begin{aligned} H_{0,1}^{01} &= \left[\frac{2kR(x^1)}{\left(1 + \frac{kr^2}{4}\right)^2} \right]_{,1} = \left[\frac{2kRx}{\left(1 + \frac{kr^2}{4}\right)^2} \right]_{,x} \\ &= 2kR \left[\frac{1}{\left(1 + \frac{kr^2}{4}\right)^2} + x(-2) \left(1 + \frac{k(x^2 + y^2 + z^2)}{4}\right)^{-3} \cdot \frac{1}{4} \cdot 2x \right] \\ &= 2kR \left[\frac{1}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{x^2}{\left(1 + \frac{kr^2}{4}\right)^3} \right] \end{aligned} \quad (1.101)$$

Because of symmetry, we quickly see that

$$H_{0,2}^{02} = 2kR \left[\frac{1}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{y^2}{\left(1 + \frac{kr^2}{4}\right)^3} \right] \quad (1.102)$$

and

$$H_{0,3}^{03} = 2kR \left[\frac{1}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{z^2}{\left(1 + \frac{kr^2}{4}\right)^3} \right]. \quad (1.103)$$

Now, we have

$$\begin{aligned}
 H_{0,0}^{00} + H_{0,1}^{01} + H_{0,2}^{02} + H_{0,3}^{03} &= 0 + 2kR \left[\frac{1}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{x^2}{\left(1 + \frac{kr^2}{4}\right)^3} \right] \\
 &+ 2kR \left[\frac{1}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{y^2}{\left(1 + \frac{kr^2}{4}\right)^3} \right] \\
 &+ 2kR \left[\frac{1}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{z^2}{\left(1 + \frac{kr^2}{4}\right)^3} \right] \\
 &= 2kR \left[\frac{3}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{x^2}{\left(1 + \frac{kr^2}{4}\right)^3} - \frac{y^2}{\left(1 + \frac{kr^2}{4}\right)^3} - \frac{z^2}{\left(1 + \frac{kr^2}{4}\right)^3} \right] \\
 &= 2kR \left[\frac{3}{\left(1 + \frac{r^2}{4}\right)^2} - \frac{x^2 + y^2 + z^2}{\left(1 + \frac{kr^2}{4}\right)^3} \right] \\
 &= 2kR \left[\frac{3}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{r^2}{\left(1 + \frac{kr^2}{4}\right)^3} \right], \tag{1.104}
 \end{aligned}$$

so finally we get

$$\begin{aligned}
 (-g)^{\frac{1}{2}} (T_0^0 + t_0^0) &= \frac{1}{16\pi} 2kR \left[\frac{3}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{r^2}{\left(1 + \frac{kr^2}{4}\right)^3} \right] \\
 &= \frac{kR}{8\pi} \left[\frac{3}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{r^2}{\left(1 + \frac{kr^2}{4}\right)^3} \right]. \tag{1.105}
 \end{aligned}$$

We see that this expression involves position. That means that it cannot have any physical significance, since the universe is homogeneous. This illustrates the point that gravitational energy is not localizable. But the integral over all space,

$$E = \int (-g)^{\frac{1}{2}} (T_0^0 + t_0^0) d^3x \tag{1.106}$$

has a physical meaning and gives the total energy of the universe. Expressed in polar coordinates, the integral becomes

$$E = 4\pi \int_0^\infty (-g)^{\frac{1}{2}} (T_0^0 + t_0^0) r^2 dr \quad (1.107)$$

$$= 4\pi \frac{kR}{8\pi} \int_0^\infty \left[\frac{3}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{r^2}{\left(1 + \frac{kr^2}{4}\right)^3} \right] r^2 dr \quad (1.108)$$

$$= \frac{kR}{2} \int_0^\infty \left[\frac{3r^2}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{r^4}{\left(1 + \frac{kr^2}{4}\right)^3} \right] dr. \quad (1.109)$$

First, we look at the case where the universe is closed, that is, $k = +1$. In this case, the energy integral becomes

$$\begin{aligned} E &= 4\pi \int_0^\infty (-g)^{\frac{1}{2}} (T_0^0 + t_0^0) r^2 dr \\ &= \frac{kR}{2} \int_0^\infty \left[\frac{3r^2}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{r^4}{\left(1 + \frac{kr^2}{4}\right)^3} \right] dr \\ &= \frac{3R}{2} \int_0^\infty \frac{r^2 dr}{\left(1 + \frac{r^2}{4}\right)^2} - \frac{R}{2} \int_0^\infty \frac{r^4 dr}{\left(1 + \frac{r^2}{4}\right)^3}. \end{aligned} \quad (1.110)$$

We find that (the detailed calculation is presented in Appendix B)

$$\int_0^\infty \frac{r^2 dr}{\left(1 + \frac{r^2}{4}\right)^2} = 2\pi \quad (1.111)$$

and

$$\int_0^\infty \frac{r^4 dr}{\left(1 + \frac{r^2}{4}\right)^3} = 6\pi \quad (1.112)$$

so that

$$E = \left(\frac{3R}{2} \cdot 2\pi \right) - \left(\frac{R}{2} \cdot 6\pi \right) = 0. \quad (1.113)$$

Next, we consider the case where $k = 0$. Now, the energy integral is

$$\begin{aligned} E &= 4\pi \int_0^\infty (-g)^{\frac{1}{2}} (T_0^0 + t_0^0) r^2 dr \\ &= \frac{kR}{2} \int_0^\infty \left[\frac{3r^2}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{r^4}{\left(1 + \frac{kr^2}{4}\right)^3} \right] dr \\ &= 0. \end{aligned} \quad (1.114)$$

Finally, for an open universe, $k = -1$, we get

$$\begin{aligned}
 E &= 4\pi \int_0^\infty (-g)^{\frac{1}{2}} (T_0^0 + t_0^0) r^2 dr \\
 &= \frac{kR}{2} \int_0^\infty \left[\frac{3r^2}{\left(1 + \frac{kr^2}{4}\right)^2} - \frac{r^4}{\left(1 + \frac{kr^2}{4}\right)^3} \right] dr \\
 &= -\frac{R}{2} \int_0^\infty \left[\frac{3r^2}{\left(1 - \frac{r^2}{4}\right)^2} - \frac{r^4}{\left(1 - \frac{r^2}{4}\right)^3} \right] dr \\
 &= -\frac{R}{2} \int_0^\infty \left[\frac{3r^2 \left(1 - \frac{r^2}{4}\right)}{\left(1 - \frac{r^2}{4}\right)^3} - \frac{r^4}{\left(1 - \frac{r^2}{4}\right)^3} \right] dr \\
 &= -\frac{R}{2} \int_0^\infty \frac{3r^2 - \frac{3}{4}r^4 - r^4}{\left(1 - \frac{r^2}{4}\right)^3} dr \\
 &= -\frac{R}{2} \int_0^\infty \frac{3r^2 - \frac{7}{4}r^4}{\left(1 - \frac{r^2}{4}\right)^3} dr \\
 &= \frac{R}{2} \int_0^\infty \frac{\frac{7}{4}r^4 - 3r^2}{\left(1 - \frac{r^2}{4}\right)^3} dr. \tag{1.115}
 \end{aligned}$$

Here, we have a singularity at $r = 2$. We solve this problem by splitting the interval of integration in two, at the singularity:

$$E = \frac{R}{2} \int_0^2 \frac{\frac{7}{4}r^4 - 3r^2}{\left(1 - \frac{r^2}{4}\right)^3} dr + \frac{R}{2} \int_2^\infty \frac{\frac{7}{4}r^4 - 3r^2}{\left(1 - \frac{r^2}{4}\right)^3} dr \tag{1.116}$$

The second integral converges (to see why, compare limits with the function $\frac{1}{r^2}$ when $r \rightarrow \infty$). But the first integral diverges, because the integrand approaches infinity faster than $\frac{1}{(r-2)^2}$, and we know that the integral of this function diverges. This means that our energy integral diverges in this case. So for $k = 0, +1$, the energy of the universe is zero. For $k = -1$ the energy is infinite.

These results intuitively make sense. For $k = 0$, the volume of the universe is finite, and it is not hard to imagine that the energy in this case should be constant and zero. For $k = -1$, the universe is infinitely large, and then it makes sense that the energy should be infinite too. When $k = 0$, it is not so clear what we should expect. This universe is also infinite, and in that sense we might expect the energy to be infinite. On the other hand, if we go back to the Newtonian interpretation (equation (1.60)), we see that setting $k = 0$ gives $E = 0$. This is in agreement with what we get using this pseudotensor approach.

As mentioned earlier, what I've done here is exactly the same as Rosen does in [8], except I've carried out the analysis in a more general way, allowing for any spatial curvature. Rosen only considers the case $k = 0$.

In this chapter I have shown that vacuum energy and accelerated expansion of the universe is possible in a Newtonian universe. We have seen why a Newtonian analysis is not sufficient when attempting to calculate the energy of the universe. Last, I have presented one way of setting up a global law for conservation of energy, consistent with general relativity. This method involves pseudotensors, and is generally NOT a unique approach. Another 'flaw' of this approach, is that we need to work with Cartesian coordinates. The results of this pseudotensor analysis suggest that closed and critically open universes contain zero energy, and open universes contain infinite energy.

In the next chapter I will look at an article by Faraoni and Cooperstock [1]. Here, they develop a method for calculating the energy of the universe. Their work restricts to the case where $k = 0$. I will go through their calculations in detail, showing how they arrive at their expression for the total energy of the universe.

Chapter 2

Deriving the Basic Equations

In this chapter I will look at an article by Faraoni and Cooperstock [1]. In this article, they present a method for calculating the energy of a spatially flat universe. I will go over the derivation of their energy equation in detail. All results in this chapter are Faraoni and Cooperstock's, I only show how they arrive at these results. (From now on, when referring to this article, I will use the shorthand F&C).

2.1 Fundamental assumptions

We start out with the action integral

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \left(\frac{1}{\kappa} - \xi \phi^2 \right) R - \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - V(\phi) \right] \quad (2.1)$$

where $\kappa = 8\pi G$ is Einstein's gravitational constant, ξ is a dimensionless coupling constant and $V(\phi)$ is the scalar field potential. If the quantities $H \equiv \frac{\dot{a}}{a}$ and ϕ are chosen as dynamical variables, the relevant dynamical system derived from this action integral is

$$[1 - \xi(1 - 6\xi)\kappa\phi^2] R - \kappa(6\xi - 1)\dot{\phi}^2 - 4\kappa V + 6\kappa\xi\phi \frac{dV}{d\phi} = 0 \quad (2.2)$$

$$\ddot{\phi} + 3H\dot{\phi} + \xi R\phi + \frac{dV}{d\phi} = 0. \quad (2.3)$$

From this point on, the article authors assume a conformally coupled scalar field, obtained by setting $\xi = \frac{1}{6}$ in the action from equation (2.1). This means that if we let the metric undergo a transform

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \quad (2.4)$$

the transform is said to be conformal if the action is invariant under the transform. Choosing $\xi = \frac{1}{6}$ ensures that the action is invariant. We allow the scalar field to acquire a mass and

include in the picture a quartic self-interaction plus the possibility of a cosmological constant, as described by the potential

$$V(\phi) = \frac{m^2\phi^2}{2} + \lambda\phi^4 + V_0. \quad (2.5)$$

Differentiating the potential with respect to ϕ gives

$$\frac{dV}{d\phi} = m^2\phi + 4\lambda\phi^3. \quad (2.6)$$

In the following, we will also need the expression for the Ricci scalar. For Friedmann-Robertson-Walker universes, it reads

$$R = 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{\kappa c^2}{a^2} \right] \quad (2.7)$$

where a is the scale factor and κ is spatial curvature.

2.2 The field equations in the case with zero spatial curvature

Now, we restrict ourselves to the case where there is no spatial curvature; $\kappa = 0$. We wish to rewrite the field equations (2.2) and (2.3), and introduce the new variables

$$\psi \equiv \sqrt{\frac{\kappa}{6}} a \quad (2.8)$$

and

$$\varphi \equiv a\phi \quad \Rightarrow \quad \phi = \frac{\varphi}{a}. \quad (2.9)$$

Instead of t , we will use the conformal time η as the time coordinate. It is defined by

$$dt \equiv a d\eta. \quad (2.10)$$

We also use that $\xi = 1/6$.

The 1st field equation

Equation (2.2) reads

$$\left[1 - \xi (1 - 6\xi) \kappa \phi^2 \right] R - \kappa (6\xi - 1) \dot{\phi}^2 - 4\kappa V + 6\kappa\xi\phi \frac{dV}{d\phi} = 0. \quad (2.11)$$

Setting $\xi = \frac{1}{6}$ gives

$$\Rightarrow \left[1 - \frac{1}{6} \left(1 - 6 \cdot \frac{1}{6} \right) \kappa \phi^2 \right] R - \kappa \left(6 \cdot \frac{1}{6} - 1 \right) \dot{\phi}^2 - 4\kappa V + 6\kappa \frac{1}{6} \phi \frac{dV}{d\phi} = 0 \quad (2.12)$$

$$\Rightarrow R - 4\kappa V + \kappa\phi \frac{dV}{d\phi} = 0 \quad (2.13)$$

When $k = 0$, the Ricci scalar reads

$$R = 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right] = 6 \left(\dot{H} + 2H^2 \right). \quad (2.14)$$

Our equation becomes

$$6 \left(\dot{H} + 2H^2 \right) - 4\kappa V + \kappa\phi \frac{dV}{d\phi} = 0 \quad (2.15)$$

Inserting for V (equation (2.5)) and $\frac{dV}{d\phi}$ (equation (2.6)) gives

$$6 \left(\dot{H} + 2H^2 \right) - 4\kappa \left[\frac{m^2\phi^2}{2} + \lambda\phi^4 + V_0 \right] + \kappa\phi [m^2\phi + 4\lambda\phi^3] = 0 \quad (2.16)$$

$$\Rightarrow 6 \left(\dot{H} + 2H^2 \right) - 2\kappa m^2\phi^2 - 4\kappa\lambda\phi^4 - 4\kappa V_0 + \kappa\phi^2 m^2 + 4\lambda\kappa\phi^4 = 0 \quad (2.17)$$

$$\Rightarrow 6\dot{H} + 12H^2 - \kappa m^2\phi^2 - 4\kappa V_0 = 0 \quad (2.18)$$

Now, use $H = \frac{\dot{a}}{a}$ and $\phi = \frac{\varphi}{a}$:

$$\Rightarrow 6 \left[\frac{d}{dt} \frac{\dot{a}}{a} \right] + 12 \frac{\dot{a}^2}{a^2} - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (2.19)$$

$$\Rightarrow 6 \left[\dot{a} \frac{d}{dt} (a^{-1}) + \frac{1}{a} \frac{d}{dt} \dot{a} \right] + 12 \frac{\dot{a}^2}{a^2} - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (2.20)$$

$$\Rightarrow 6 \left[\dot{a} \left(-\frac{1}{a^2} \dot{a} \right) + \frac{\ddot{a}}{a} \right] + 12 \frac{\dot{a}^2}{a^2} - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (2.21)$$

$$\Rightarrow 6 \left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right] + 12 \frac{\dot{a}^2}{a^2} - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (2.22)$$

$$\Rightarrow 6 \frac{\ddot{a}}{a} + 6 \frac{\dot{a}^2}{a^2} - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (2.23)$$

$$\Rightarrow 6 \frac{1}{a} \frac{d}{dt} \left[\frac{da}{dt} \right] + 6 \frac{1}{a^2} \left[\frac{da}{dt} \right]^2 - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (2.24)$$

Replace dt with $a d\eta$:

$$\Rightarrow 6 \frac{1}{a} \frac{d}{d\eta} \left[\frac{da}{a d\eta} \right] + 6 \frac{1}{a^2} \left[\frac{da}{a d\eta} \right]^2 - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (2.25)$$

$$\Rightarrow 6 \frac{1}{a^2} \frac{d}{d\eta} \left[\frac{1 da}{a d\eta} \right] + 6 \frac{1}{a^2} \frac{1}{a^2} \left[\frac{da}{d\eta} \right]^2 - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (2.26)$$

Now, use equation (2.8): $\psi \equiv \sqrt{\frac{\kappa}{6}}a \quad \Rightarrow \frac{1}{a} = \frac{1}{\psi}\sqrt{\frac{\kappa}{6}}$

$$\Rightarrow 6 \frac{1}{\psi^2} \frac{\kappa}{6} \frac{d}{d\eta} \left[\frac{1}{\psi} \sqrt{\frac{\kappa}{6}} \cdot \frac{d}{d\eta} \left(\sqrt{\frac{6}{\kappa}} \psi \right) \right] + 6 \frac{1}{\psi^4} \frac{\kappa^2}{6^2} \left[\frac{d}{d\eta} \left(\sqrt{\frac{6}{\kappa}} \right) \right]^2 - \kappa m^2 \varphi^2 \frac{1}{\psi^2} \frac{\kappa}{6} - 4\kappa V_0 = 0 \quad (2.27)$$

Prime denotes differentiation with respect to the conformal time η :

$$\Rightarrow \kappa \frac{1}{\psi^2} \frac{d}{d\eta} \left[\frac{1}{\psi} \cdot \psi' \right] + 6 \frac{1}{\psi^4} \frac{\kappa^2}{6^2} \frac{6}{\kappa} \psi'^2 - \kappa m^2 \varphi^2 \frac{1}{\psi^2} \frac{\kappa}{6} - 4\kappa V_0 = 0 \quad (2.28)$$

$$\Rightarrow \kappa \frac{1}{\psi^2} \left[\frac{1}{\psi} \psi'' - \psi'^2 \frac{1}{\psi^2} \right] + \kappa \frac{1}{\psi^4} \psi'^2 - \kappa \frac{\kappa}{6} m^2 \varphi^2 \frac{1}{\psi^2} - 4\kappa V_0 = 0 \quad (2.29)$$

$$\Rightarrow \frac{1}{\psi^3} \psi'' - \frac{1}{\psi^4} \psi'^2 + \frac{1}{\psi^4} \psi'^2 - \frac{\kappa}{6} m^2 \varphi^2 \frac{1}{\psi^2} - 4V_0 = 0 \quad (2.30)$$

Multiplying through by ψ^3 gives the following form of the first field equation:

$$\Rightarrow \underline{\underline{\psi'' - \frac{\kappa}{6} m^2 \varphi^2 \psi - 4V_0 \psi^3 = 0}} \quad (2.31)$$

The 2nd field equation

Equation (2.3) reads

$$\ddot{\phi} + 3H\dot{\phi} + \xi R\phi + \frac{dV}{d\phi} = 0. \quad (2.32)$$

Use that here, the Ricci scalar $R = 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right]$, the Hubble constant $H = \frac{\dot{a}}{a}$, $\xi = \frac{1}{6}$, and equation (2.6) above:

$$\Rightarrow \ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} + \frac{1}{6} \cdot 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right] \phi + m^2 \phi + 4\lambda \phi^3 = 0. \quad (2.33)$$

Using $\phi = \frac{\varphi}{a}$ gives

$$\Rightarrow \frac{d}{dt} \left[\frac{d}{dt} \left(\frac{\varphi}{a} \right) \right] + 3 \frac{\dot{a}}{a} \frac{d}{dt} \left(\frac{\varphi}{a} \right) + \frac{\ddot{a}}{a} \frac{\varphi}{a} + \frac{\dot{a}^2}{a^2} \frac{\varphi}{a} + m^2 \frac{\varphi}{a} + 4\lambda \frac{\varphi^3}{a^3} = 0. \quad (2.34)$$

Carrying out the differentiation gives

$$\Rightarrow \frac{d}{dt} \left[\frac{1}{a} \dot{\varphi} - \varphi \frac{1}{a^2} \dot{a} \right] + 3 \frac{\dot{a}}{a} \left[\frac{1}{a} \dot{\varphi} - \varphi \frac{1}{a^2} \dot{a} \right] + \frac{\ddot{a}\varphi}{a^2} + \frac{\dot{a}^2 \varphi}{a^3} + m^2 \frac{\varphi}{a} + 4\lambda \frac{\varphi^3}{a^3} = 0 \quad (2.35)$$

$$\begin{aligned} \Rightarrow \left[\frac{1}{a} \ddot{\varphi} - \dot{\varphi} \frac{1}{a^2} \dot{a} \right] - \left[\frac{1}{a^2} (\varphi \ddot{a} + \dot{\varphi} \dot{a}) - 2\varphi \dot{a}^2 \frac{1}{a^3} \right] + 3 \frac{1}{a^2} \dot{a} \dot{\varphi} - 3 \frac{1}{a^3} \dot{a}^2 \varphi \\ + \frac{1}{a^2} \ddot{a} \varphi + \frac{1}{a^3} \dot{a}^2 \varphi + m^2 \frac{\varphi}{a} + 4\lambda \frac{\varphi^3}{a^3} = 0 \end{aligned} \quad (2.36)$$

$$\begin{aligned} \Rightarrow \frac{1}{a} \ddot{\varphi} - \frac{1}{a^2} \dot{a} \dot{\varphi} - \frac{1}{a^2} \varphi \ddot{a} - \frac{1}{a^2} \dot{a} \dot{\varphi} + 2 \frac{1}{a^3} \dot{a}^2 \varphi + 3 \frac{1}{a^2} \dot{a} \dot{\varphi} - 3 \frac{1}{a^3} \dot{a}^2 \varphi \\ + \frac{1}{a^2} \ddot{a} \varphi + \frac{1}{a^3} \dot{a}^2 \varphi + m^2 \frac{\varphi}{a} + 4\lambda \frac{\varphi^3}{a^3} = 0 \end{aligned} \quad (2.37)$$

$$\Rightarrow \frac{1}{a} \ddot{\varphi} + \frac{1}{a^2} \dot{a} \dot{\varphi} + m^2 \frac{\varphi}{a} + 4\lambda \frac{\varphi^3}{a^3} = 0 \quad (2.38)$$

Multiply through by a :

$$\Rightarrow \ddot{\varphi} + \frac{1}{a} \dot{a} \dot{\varphi} + m^2 \varphi + 4\lambda \frac{1}{a^2} \varphi^3 = 0 \quad (2.39)$$

Write out the differentiations explicitly:

$$\Rightarrow \frac{d}{dt} \left[\frac{d\varphi}{dt} \right] + \frac{1}{a} \frac{da}{dt} \frac{d\varphi}{dt} + m^2 \varphi + 4\lambda \frac{1}{a^2} \varphi^3 = 0 \quad (2.40)$$

$dt \equiv a d\eta$:

$$\Rightarrow \frac{d}{a d\eta} \left[\frac{d\varphi}{a d\eta} \right] + \frac{1}{a} \frac{da}{a d\eta} \frac{d\varphi}{a d\eta} + m^2 \varphi + 4\lambda \frac{1}{a^2} \varphi^3 = 0 \quad (2.41)$$

$$\Rightarrow \frac{1}{a} \frac{d}{d\eta} \left[\frac{1}{a} \varphi' \right] + \frac{1}{a^3} \left[\frac{da}{d\eta} \right] \varphi' + m^2 \varphi + 4\lambda \frac{1}{a^2} \varphi^3 = 0 \quad (2.42)$$

Now use (2.8): $\psi \equiv \sqrt{\frac{\kappa}{6}} a \quad \Rightarrow \frac{1}{a} = \frac{1}{\psi} \sqrt{\frac{\kappa}{6}}$

$$\Rightarrow \sqrt{\frac{\kappa}{6}} \frac{1}{\psi} \frac{d}{d\eta} \left[\sqrt{\frac{\kappa}{6}} \frac{1}{\psi} \varphi' \right] + \frac{1}{\psi^3} \left[\frac{\kappa}{6} \right]^{\frac{3}{2}} \varphi' \frac{d}{d\eta} \left[\sqrt{\frac{6}{\kappa}} \psi \right] + m^2 \varphi + 4\lambda \frac{1}{\psi^2} \frac{\kappa}{6} \varphi^3 = 0 \quad (2.43)$$

$$\Rightarrow \frac{\kappa}{6} \frac{1}{\psi} \frac{d}{d\eta} \left[\frac{1}{\psi} \varphi' \right] + \frac{\kappa}{6} \frac{1}{\psi^3} \varphi' \psi' + m^2 \varphi + 4 \frac{\kappa}{6} \lambda \frac{1}{\psi^2} \varphi^3 = 0 \quad (2.44)$$

Differentiate with respect to the conformal time η :

$$\Rightarrow \frac{\kappa}{6} \frac{1}{\psi} \left(\frac{1}{\psi} \varphi'' - \varphi' \frac{1}{\psi^2} \psi' \right) + \frac{\kappa}{6} \frac{1}{\psi^3} \varphi' \psi' + m^2 \varphi + 4 \frac{\kappa}{6} \lambda \frac{1}{\psi^2} \varphi^3 = 0 \quad (2.45)$$

$$\Rightarrow \frac{\kappa}{6} \frac{1}{\psi^2} \varphi'' - \frac{\kappa}{6} \frac{1}{\psi^3} \varphi' \psi' + \frac{\kappa}{6} \frac{1}{\psi^3} \varphi' \psi' + m^2 \varphi + 4 \frac{\kappa}{6} \lambda \frac{1}{\psi^2} \varphi^3 = 0 \quad (2.46)$$

Multiplying through by $\frac{6}{\kappa}\psi^2$ gives the following form of the second field equation:

$$\Rightarrow \underline{\underline{\varphi'' - \frac{6m^2}{\kappa}\psi^2\varphi + 4\lambda\varphi^3 = 0.}} \quad (2.47)$$

2.3 Derivation of the field equations by variation of the action integral

In this section I show how the field equations can be derived from the action integral using the principle of least action. First, we set up the action integral as given in F&C's article:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} \left(\frac{1}{\kappa} - \xi\phi^2 \right) R - \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - V(\phi) \right]. \quad (2.48)$$

The integral can be written as

$$S = \int \mathcal{L} d^4x \quad (2.49)$$

where

$$\mathcal{L} = \sqrt{-g} \left[\frac{1}{2} \left(\frac{1}{\kappa} - \xi\phi^2 \right) R - \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - V(\phi) \right] \quad (2.50)$$

is the Lagrangian density and

$$g = \det g_{\mu\nu}. \quad (2.51)$$

$g_{\mu\nu}$ is the Friedmann-Robertson-Walker metric, and is given by

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{bmatrix} \quad (2.52)$$

so that

$$g = \det g_{\mu\nu} = -(a^2(t))^3 = -a^6(t) \quad (2.53)$$

$$\Rightarrow -g = a^6 \quad \Rightarrow \sqrt{-g} = a^3. \quad (2.54)$$

For scalars, the covariant derivatives are the same as partial derivatives. a and ϕ are scalars, so

$$\nabla_\mu j = \partial_\mu j \quad \text{and} \quad \nabla^\mu j = \partial^\mu j \quad (2.55)$$

where $j = a, \phi$. We also use that the Ricci scalar $R = 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right]$.

$$\nabla^\mu \phi = -\nabla_\mu \phi \Rightarrow \nabla^\mu \phi \nabla_\mu \phi = -(\nabla_\mu \phi)^2 \quad (2.56)$$

ϕ is constant in the spatial coordinates, and varies only in time:

$$\Rightarrow \nabla^\mu \phi = \frac{d\phi}{dt} \equiv \dot{\phi} \quad (2.57)$$

$$\Rightarrow \nabla^\mu \phi \nabla_\mu \phi = -\dot{\phi}^2 \quad (2.58)$$

The Lagrangian density now becomes

$$\mathcal{L} = a^3 \left[\frac{1}{2} \left(\frac{1}{\kappa} - \xi \phi^2 \right) 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) + \frac{1}{2} \dot{\phi}^2 - V(\phi) \right] \quad (2.59)$$

$$\mathcal{L} = 3 \left(\frac{1}{\kappa} - \xi \phi^2 \right) [a^2 \ddot{a} + a \dot{a}^2] + \frac{1}{2} a^3 \dot{\phi}^2 - a^3 V(\phi). \quad (2.60)$$

Variation of a :

$$a \rightarrow a' = a + \delta a \quad (2.61)$$

$$\dot{a} \rightarrow \dot{a}' = \dot{a} + \delta \dot{a} \quad (2.62)$$

$$\ddot{a} \rightarrow \ddot{a}' = \ddot{a} + \delta \ddot{a} \quad (2.63)$$

$$\delta S = \int d^4x \delta \mathcal{L} \quad (2.64)$$

We find $\mathcal{L} + \delta \mathcal{L}$ by replacing a , \dot{a} , and \ddot{a} in the expression for \mathcal{L} with their respective primes:

$$\begin{aligned} \mathcal{L} + \delta \mathcal{L} &= 3 \left(\frac{1}{\kappa} - \xi \phi^2 \right) [(a + \delta a)^2 (\ddot{a} + \delta \ddot{a}) + (a + \delta a) (\dot{a} + \delta \dot{a})^2] \\ &\quad + \frac{1}{2} (a + \delta a)^3 \dot{\phi}^2 - (a + \delta a)^3 V(\phi) \end{aligned} \quad (2.65)$$

To first order, we have that

$$(a + \delta a)^2 = a^2 + 2a \delta a$$

$$(a + \delta a)^3 = (a + \delta a)^2 (a + \delta a) = (a^2 + 2a \delta a)(a + \delta a) = a^3 + a^2 \delta a + 2a \delta a = a^3 + 3a^2 \delta a$$

Inserting this, we get (to first order)

$$\begin{aligned} \mathcal{L} + \delta \mathcal{L} &= 3 \left(\frac{1}{\kappa} - \xi \phi^2 \right) [(a^2 + 2a \delta a) (\ddot{a} + \delta \ddot{a}) + (a + \delta a) (\dot{a}^2 + 2\dot{a} \delta \dot{a})] \\ &\quad + \frac{1}{2} (a^3 + 3a^2 \delta a) \dot{\phi}^2 - (a^3 + 3a^2 \delta a) V(\phi) \\ &= 3 \left(\frac{1}{\kappa} - \xi \phi^2 \right) [a^2 \ddot{a} + a^2 \delta \ddot{a} + 2a \dot{a} \delta \dot{a} + a \dot{a}^2 + \dot{a}^2 \delta a + 2a \dot{a} \delta \dot{a}] \\ &\quad + \frac{1}{2} a^3 \dot{\phi}^2 + \frac{3}{2} a^2 \delta a \dot{\phi}^2 - a^3 V(\phi) - 3a^2 \delta a V(\phi) \end{aligned} \quad (2.66)$$

By comparing this with our expression for \mathcal{L} , we identify $\delta\mathcal{L}$ as

$$\delta\mathcal{L} = 3 \left(\frac{1}{\kappa} - \xi\phi^2 \right) [a^2 \delta\ddot{a} + 2a\ddot{a} \delta a + \dot{a}^2 \delta a + 2a\dot{a} \delta\dot{a}] + \frac{3}{2} a^2 \delta a \dot{\phi}^2 - 3a^2 \delta a V(\phi). \quad (2.67)$$

Now, our expression for δS becomes

$$\begin{aligned} \delta S &= \int_{\mathcal{R}} d^4x \delta\mathcal{L} \\ &= \int_{\mathcal{R}} d^4x \left[3 \left(\frac{1}{\kappa} - \xi\phi^2 \right) [a^2 \delta\ddot{a} + 2a\dot{a} \delta\dot{a} + (2a\ddot{a} + \dot{a}^2)\delta a] + \frac{3}{2} \dot{\phi}^2 a^2 \delta a - 3V(\phi)a^2 \delta a \right]. \end{aligned} \quad (2.68)$$

Next, we wish to factor out the variation δa . First we observe that

$$\frac{d}{dt}(a^2 \delta\dot{a}) = 2a\dot{a} \delta\dot{a} + a^2 \delta\ddot{a} \quad (2.69)$$

so that

$$\delta S = \int_{\mathcal{R}} d^4x \left[3 \left(\frac{1}{\kappa} - \xi\phi^2 \right) \left[\frac{d}{dt}(a^2 \delta\dot{a}) + (2a\ddot{a} + \dot{a}^2)\delta a \right] + \frac{3}{2} \dot{\phi}^2 a^2 \delta a - 3V(\phi)a^2 \delta a \right]. \quad (2.70)$$

Here, $\frac{d}{dt}(a^2 \delta\dot{a}) = 0$ because it is a total derivative. (We assume that $\int \frac{d}{dt}(a^2 \delta\dot{a}) dt$ vanishes on the boundary of \mathcal{R}). Finally, since we require that $\delta S = 0$, the integrand must equal 0:

$$3 \left(\frac{1}{\kappa} - \xi\phi^2 \right) (2a\ddot{a} + \dot{a}^2)\delta a + \frac{3}{2} \dot{\phi}^2 a^2 \delta a - 3V(\phi)a^2 \delta a = 0 \quad (2.71)$$

$$\Rightarrow 3 \left(\frac{1}{\kappa} - \xi\phi^2 \right) [2a\ddot{a} + \dot{a}^2] + \frac{3}{2} \dot{\phi}^2 a^2 - 3V(\phi)a^2 = 0 \quad (2.72)$$

$$\Rightarrow \left(\frac{1}{\kappa} - \xi\phi^2 \right) [2a\ddot{a} + \dot{a}^2] = a^2 \left[-\frac{1}{2} \dot{\phi}^2 + V(\phi) \right] \quad (2.73)$$

$$\Rightarrow \left(\frac{1}{\kappa} - \xi\phi^2 \right) \left[2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right] = V(\phi) - \frac{1}{2} \dot{\phi}^2 \quad (2.74)$$

Hence the first field equation can be written

$$\Rightarrow \underline{\underline{\left(\frac{1}{\kappa} - \xi\phi^2 \right) [2\dot{H} + 3H^2] = V(\phi) - \frac{1}{2} \dot{\phi}^2.}} \quad (2.75)$$

Note that this looks different from equation (2.2). The reason for this is that equation (2.2) is found by variation of H , not a .¹

¹In the case that $\xi = 0$, $\phi = 0$, and $V(\phi)$ is a constant, we can easily check that $a(t) = e^{\text{constant} \cdot t}$ is a solution to the equation above, as well as to equation (2.2).

Variation of ϕ :

We start out by setting up the Lagrangian density:

$$\mathcal{L} = 3 \left(\frac{1}{\kappa} - \xi \phi^2 \right) [a^2 \ddot{a} + a \dot{a}^2] + \frac{1}{2} a^3 \dot{\phi}^2 - a^3 V(\phi) \quad (2.76)$$

The variations are (we don't need the second derivative here):

$$\begin{aligned} \phi &\rightarrow \phi' = \phi + \delta\phi \\ \dot{\phi} &\rightarrow \dot{\phi}' = \dot{\phi} + \delta\dot{\phi} \end{aligned} \quad (2.77)$$

In the last line we have used that the δ -operator commutes with derivatives. Varying ϕ leads to a variation in \mathcal{L} :

$$\mathcal{L} + \delta\mathcal{L} = 3 \left[\frac{1}{\kappa} - \xi(\phi + \delta\phi)^2 \right] [a^2 \ddot{a} + a \dot{a}^2] + \frac{1}{2} a^3 (\dot{\phi} + \delta\dot{\phi})^2 - a^3 V(\phi + \delta\phi). \quad (2.78)$$

We assume that $\delta\phi$ is small, hence we can treat the change in ϕ as linear. This allows us to write

$$V(\phi + \delta\phi) = V(\phi) + V(\delta\phi) = V(\phi) + \delta V(\phi) = V(\phi) + \frac{dV}{d\phi} \delta\phi. \quad (2.79)$$

Then we get, to first order

$$\mathcal{L} + \delta\mathcal{L} = 3 \left[\frac{1}{\kappa} - \xi(\phi^2 + 2\phi \delta\phi) \right] [a^2 \ddot{a} + a \dot{a}^2] + \frac{1}{2} a^3 (\dot{\phi}^2 + 2\dot{\phi} \delta\dot{\phi}) - a^3 V(\phi) - a^3 \frac{dV}{d\phi} \delta\phi \quad (2.80)$$

$$\mathcal{L} + \delta\mathcal{L} = 3 \left[\frac{1}{\kappa} - \xi\phi^2 - 2\xi\phi \delta\phi \right] [a^2 \ddot{a} + a \dot{a}^2] + \frac{1}{2} a^3 \dot{\phi}^2 + a^3 \dot{\phi} \delta\dot{\phi} - a^3 V(\phi) - a^3 \frac{dV}{d\phi} \delta\phi \quad (2.81)$$

Comparing this with our expression for \mathcal{L} , we see that

$$\delta\mathcal{L} = -6\xi\phi \delta\phi [a^2 \ddot{a} + a \dot{a}^2] + a^3 \dot{\phi} \delta\dot{\phi} - a^3 \frac{dV}{d\phi} \delta\phi. \quad (2.82)$$

Now, we get

$$\delta S = \int_{\mathcal{R}} d^4x \delta\mathcal{L} \quad (2.83)$$

$$= \int_{\mathcal{R}} d^4x \left(-6\xi\phi \delta\phi [a^2 \ddot{a} + a \dot{a}^2] + a^3 \dot{\phi} \delta\dot{\phi} - a^3 \frac{dV}{d\phi} \delta\phi \right). \quad (2.84)$$

The next step is to factor out the variation $\delta\phi$. First we rewrite $a^3\dot{\phi}\delta\dot{\phi}$:

$$\frac{d}{dt} \left[a^3\dot{\phi}\delta\phi \right] = a^3\dot{\phi}\delta\dot{\phi} + a^3\ddot{\phi}\delta\phi + 3a^2\dot{a}\dot{\phi}\delta\phi \quad (2.85)$$

$$\Rightarrow a^3\dot{\phi}\delta\dot{\phi} = \frac{d}{dt} \left[a^3\dot{\phi}\delta\phi \right] - a^3\ddot{\phi}\delta\phi - 3a^2\dot{a}\dot{\phi}\delta\phi. \quad (2.86)$$

Now, we have

$$\delta S = \int_{\mathcal{R}} d^4x \left(-6\xi\phi\delta\phi [a^2\ddot{a} + a\dot{a}^2] + \frac{d}{dt} \left[a^3\dot{\phi}\delta\phi \right] - a^3\ddot{\phi}\delta\phi - 3a^2\dot{a}\dot{\phi}\delta\phi - a^3\frac{dV}{d\phi}\delta\phi \right). \quad (2.87)$$

We have that

$$\frac{d}{dt} \left[a^3\dot{\phi}\delta\phi \right] = 0 \quad (2.88)$$

because of the total derivative.

Since $\delta S = 0$, the integrand must vanish:

$$\Rightarrow -6\xi\phi [a^2\ddot{a} + a\dot{a}^2] - a^3\ddot{\phi} - 3a^2\dot{a}\dot{\phi} - a^3\frac{dV}{d\phi} = 0. \quad (2.89)$$

Dividing through by a^3 gives

$$\Rightarrow -6\xi\phi \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right] - \ddot{\phi} - 3\frac{\dot{a}}{a}\dot{\phi} - \frac{dV}{d\phi} = 0, \quad (2.90)$$

and we finally arrive at the second field equation

$$\Rightarrow \ddot{\phi} + 3H\dot{\phi} + \xi R\phi + \frac{dV}{d\phi} = 0 \quad (2.91)$$

which is just equation (2.3).

2.4 The new field equations derived from the Lagrangian

Here I will show how the new field equations (2.31) and (2.47) can be derived from the Lagrangian function given in Faraoni and Cooperstock's article. Note that I still assume that $k = 0$.

First we set up the Lagrangian:

$$L = \frac{1}{2} (\varphi')^2 - \frac{18}{\kappa^2} (\psi')^2 - \frac{3m^2}{\kappa} \varphi^2 \psi^2 - \lambda \varphi^4 - \frac{36}{\kappa^2} V_0 \psi^4. \quad (2.92)$$

In our case, the Lagrange equations can be written:

$$\frac{d}{d\eta} \left[\frac{\partial L}{\partial q'_j} \right] - \frac{\partial L}{\partial q_j} = 0 \quad (2.93)$$

where $q_j = \varphi, \psi$.

The 1st field equation

Lagrange's equation for ψ reads

$$\frac{d}{d\eta} \left[\frac{\partial L}{\partial \psi'} \right] - \frac{\partial L}{\partial \psi} = 0. \quad (2.94)$$

We find

$$\frac{\partial L}{\partial \psi'} = -\frac{36}{\kappa^2} \psi', \quad \frac{d}{d\eta} \left[\frac{\partial L}{\partial \psi'} \right] = -\frac{36}{\kappa^2} \psi'' \quad (2.95)$$

and

$$\frac{\partial L}{\partial \psi} = -\frac{6m^2}{\kappa} \varphi^2 \psi - 4 \frac{36}{\kappa^2} V_0 \psi^3. \quad (2.96)$$

Lagrange's equation for ψ now becomes

$$\Rightarrow -\frac{36}{\kappa^2} \psi'' + \frac{6m^2}{\kappa} \varphi^2 \psi + 4 \cdot \frac{36}{\kappa^2} V_0 \psi^3 = 0. \quad (2.97)$$

Finally, multiplying through by $-\frac{\kappa^2}{36}$ gives

$$\underline{\underline{\psi'' - \frac{\kappa m^2}{6} \varphi^2 \psi - 4V_0 \psi^3 = 0}}, \quad (2.98)$$

which is exactly the same as equation (2.31).

The 2nd field equation

Lagrange's equation for φ reads

$$\frac{d}{d\eta} \left[\frac{\partial L}{\partial \varphi'} \right] - \frac{\partial L}{\partial \varphi} = 0. \quad (2.99)$$

We find

$$\frac{\partial L}{\partial \varphi'} = \varphi', \quad \frac{d}{d\eta} \left[\frac{\partial L}{\partial \varphi'} \right] = \varphi'' \quad (2.100)$$

and

$$\frac{\partial L}{\partial \varphi} = -\frac{3m^2}{\kappa} \psi^2 \cdot 2\varphi - 4\lambda \varphi^3. \quad (2.101)$$

Lagrange's equation for φ now becomes

$$\underline{\underline{\varphi'' + \frac{6m^2}{\kappa} \psi^2 \varphi + 4\lambda \varphi^3 = 0}}. \quad (2.102)$$

This is just equation (2.47).

2.5 Derivation of an energy equation

In this section I will show how the equation for a quantity that can be interpreted as the energy of the universe can be derived.

Multiplying the 1st field equation (2.98) by $\frac{36}{\kappa^2}\psi'$ gives the result

$$\frac{36}{\kappa^2}\psi'\psi'' - \frac{36}{\kappa^2}\psi'\frac{\kappa m^2}{6}\varphi^2\psi - \frac{36}{\kappa^2}\psi' \cdot 4V_0\psi^3 = 0 \quad (2.103)$$

$$\Rightarrow \frac{36}{\kappa^2}\psi'\psi'' - \frac{6m^2}{\kappa}\varphi^2\psi\psi' - \frac{36}{\kappa^2} \cdot 4V_0\psi^3\psi' = 0. \quad (2.104)$$

Multiplying the 2nd field equation (2.102) by φ' gives

$$\varphi'\varphi'' + \varphi'\frac{6m^2}{\kappa}\psi^2\varphi + \varphi'4\lambda\varphi^3 = 0. \quad (2.105)$$

Equating the two equations (2.104) and (2.105), we get

$$\varphi'\varphi'' + \frac{6m^2}{\kappa}\varphi\psi^2\varphi' + 4\lambda\varphi^3\varphi' = \frac{36}{\kappa^2}\psi'\psi'' - \frac{6m^2}{\kappa}\varphi^2\psi\psi' - \frac{36}{\kappa^2} \cdot 4V_0\psi^3\psi' \quad (2.106)$$

$$\Rightarrow \varphi'\varphi'' - \frac{36}{\kappa^2}\psi'\psi'' + \frac{6m^2}{\kappa}\varphi^2\psi\psi' + \frac{6m^2}{\kappa}\psi^2\varphi\varphi' + 4\lambda\varphi^3\varphi' + \frac{36}{\kappa^2} \cdot 4V_0\psi^3\psi' = 0. \quad (2.107)$$

Integrating this equation with respect to the conformal time η gives

$$\begin{aligned} &\Rightarrow \int \varphi'\varphi'' d\eta - \frac{36}{\kappa^2} \int \psi'\psi'' d\eta + \frac{6m^2}{\kappa} \int \varphi^2\psi\psi' d\eta \\ &+ \frac{6m^2}{\kappa} \int \psi^2\varphi\varphi' d\eta + 4\lambda \int \varphi^3\varphi' d\eta + \frac{36}{\kappa^2} \cdot 4V_0 \int \psi^3\psi' d\eta = \text{constant} \end{aligned} \quad (2.108)$$

$$\begin{aligned} &\Rightarrow \int \varphi' \frac{d}{d\eta} [\varphi'] d\eta - \frac{36}{\kappa^2} \int \psi' \frac{d}{d\eta} [\psi'] d\eta + \frac{3m^2}{\kappa} \int 2\varphi^2\psi\psi' d\eta \\ &+ \frac{3m^2}{\kappa} \int 2\psi^2\varphi\varphi' d\eta + 4\lambda \int \varphi^3 \frac{d\varphi}{d\eta} d\eta + \frac{36}{\kappa^2} \cdot 4V_0 \int \psi^3 \frac{d\psi}{d\eta} d\eta = \text{constant} \end{aligned} \quad (2.109)$$

$$\begin{aligned} &\Rightarrow \int \varphi' d\varphi - \frac{36}{\kappa^2} \int \psi' d\psi + \frac{3m^2}{\kappa} \int 2\varphi^2\psi\psi' d\eta \\ &+ \frac{3m^2}{\kappa} \int 2\psi^2\varphi\varphi' d\eta + 4\lambda \int \varphi^3 d\varphi + \frac{36}{\kappa^2} \cdot 4V_0 \int \psi^3 d\psi = \text{constant} \end{aligned} \quad (2.110)$$

Now, look at the third integral:

$$\frac{3m^2}{\kappa} \int 2\varphi^2\psi\psi' d\eta \quad (2.111)$$

We can solve this by integrating by parts. We choose

$$u' = 2\psi\psi' \quad \text{and} \quad v = \varphi^2 \quad (2.112)$$

so that

$$u = \psi^2 \quad \text{and} \quad v' = 2\varphi\varphi'. \quad (2.113)$$

The integral can now be written

$$\frac{3m^2}{\kappa} \int 2\varphi^2\psi\psi' d\eta = \frac{3m^2}{\kappa} \left(\psi^2\varphi^2 - \int 2\psi^2\varphi\varphi' d\eta \right). \quad (2.114)$$

Substituting this into equation (2.110) gives the result

$$\begin{aligned} & \int \varphi' d\varphi' - \frac{36}{\kappa^2} \int \psi' d\psi' + \frac{3m^2}{\kappa} \psi^2\varphi^2 - \frac{3m^2}{\kappa} \int 2\psi^2\varphi\varphi' d\eta \\ & + \frac{3m^2}{\kappa} \int 2\psi^2\varphi\varphi' d\eta + 4\lambda \int \varphi^3 d\varphi + \frac{36}{\kappa^2} \cdot 4V_0 \int \psi^3 d\psi = \text{constant}. \end{aligned} \quad (2.115)$$

Finally, we carry out the integrations and our equation becomes

$$\frac{1}{2} (\varphi')^2 - \frac{36}{\kappa^2} \frac{1}{2} (\psi')^2 + \frac{3m^2}{\kappa} \varphi^2\psi^2 + 4\lambda \frac{1}{4} \varphi^4 + \frac{36}{\kappa^2} \cdot 4V_0 \frac{1}{4} \psi^4 = \text{constant} \quad (2.116)$$

$$\underline{\underline{\Rightarrow \frac{1}{2} (\varphi')^2 - \frac{18}{\kappa^2} (\psi')^2 + \frac{3m^2}{\kappa} \varphi^2\psi^2 + \lambda\varphi^4 + \frac{36}{\kappa^2} V_0\psi^4 = \text{constant}}} \quad (2.117)$$

The Hamiltonian of the system is defined as

$$H = \sum_i p_i \dot{q}_i - L, \quad p_i = \frac{\partial L}{\partial \dot{q}_i}. \quad (2.118)$$

In our case this becomes

$$H = p_\varphi \varphi' + p_\psi \psi' - L \quad (2.119)$$

where

$$p_\varphi = \frac{\partial L}{\partial \varphi'} = \varphi' \quad \text{and} \quad p_\psi = \frac{\partial L}{\partial \psi'} = -\frac{36}{\kappa^2} \psi', \quad (2.120)$$

so

$$H = \varphi'^2 - \frac{36}{\kappa^2} \psi'^2 - \frac{1}{2} \varphi'^2 + \frac{18}{\kappa^2} \psi'^2 + \frac{3m^2}{\kappa} \varphi^2\psi^2 + \lambda\varphi^4 + \frac{36}{\kappa^2} V_0\psi^4 \quad (2.121)$$

$$\Rightarrow H = \frac{1}{2}\varphi'^2 - \frac{18}{\kappa^2}\psi'^2 + \frac{3m^2}{\kappa}\psi^2\varphi^2 + \lambda\varphi^4 + \frac{36}{\kappa^2}V_0\psi^4. \quad (2.122)$$

It is straightforward to check that the Hamilton equations

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (2.123)$$

which in our case become

$$p'_\varphi = -\frac{\partial H}{\partial \varphi} \quad \text{and} \quad p'_\psi = -\frac{\partial H}{\partial \psi}, \quad (2.124)$$

reproduce the field equations:

$$\begin{aligned} p'_\psi &= -\frac{\partial H}{\partial \psi} \\ \Rightarrow -\frac{36}{\kappa^2}\psi'' &= -\left[\frac{6m^2}{\kappa}\psi\varphi^2 + \frac{36}{\kappa^2} \cdot 4V_0\psi^3\right] \\ \Rightarrow \psi'' - \frac{\kappa m^2}{6}\varphi^2\psi - 4V_0\psi^3 &= 0 \end{aligned} \quad (2.125)$$

and

$$\begin{aligned} p'_\varphi &= -\frac{\partial H}{\partial \varphi} \\ \Rightarrow \varphi'' &= -\left[\frac{6m^2}{\kappa} + 4\lambda\varphi^3\right] \\ \Rightarrow \varphi'' + \frac{6m^2}{\kappa}\psi^2\varphi + 4\lambda\varphi^3 &= 0. \end{aligned} \quad (2.126)$$

We can identify the Hamiltonian as the energy of the system. This enables us to write

$$E = \frac{1}{2}\varphi'^2 - \frac{18}{\kappa^2}\psi'^2 + \frac{3m^2}{\kappa}\psi^2\varphi^2 + \lambda\varphi^4 + \frac{36}{\kappa^2}V_0\psi^4, \quad (2.127)$$

and we have an expression for the total energy of the universe.

Using eq. (2.117), we get

$$E = \text{constant}. \quad (2.128)$$

In this chapter I have gone through Faraoni and Cooperstock's derivation of an energy equation, and shown that their method gives the result that this energy is constant. Like the authors of the article, I have assumed all the way that $k = 0$, so this result is only valid for universes with no spatial curvature. This concludes the first step of Faraoni and Cooperstock's

argument. Next, they argue that this constant energy should be zero. They support this in two different and independent ways: They claim that 'open or critically open FRW cosmologies have Minkowski space as their asymptotic state'. They support this by saying that in an expanding universe filled with matter, the density will in the infinitely distant future reach zero, and a universe with zero density has flat (Minkowski) spacetime. They show that this implies zero energy by setting up the Schwarzschild metric, and setting the mass parameter equal to zero. The Schwarzschild metric then becomes the Minkowski metric. The Minkowski metric is a special case of the Schwarzschild metric, in the absence of energy. Furthermore, they argue that since the total energy of the universe is zero at one point in time (the infinite future), then it must be zero at all times due to conservation of energy. Their alternative way of reasoning is amazingly simple: Since they have shown that the energy is constant, they argue that one can just choose this constant to be zero. They support this by saying that the zero level is arbitrary.

In the next chapter I will follow the same procedure as in this chapter, only this time I will allow for any spatial curvature. The result will be an energy equation that is valid for any k .

Chapter 3

FRW Universes with Curvature

3.1 The general field equations

So far I have only considered the case where we have no spatial curvature; $k = 0$. The results presented in chapter 2 are the results of Faraoni and Cooperstock. Now, I wish to generalize all the equations so that they are valid for $k = -1$, negative curvature, and $k = 1$, positive curvature, also.

We start out with the field equations (2.2) and (2.3) as before, and remember that for FRW universes, the Ricci scalar is given by

$$R = 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} \right] \quad (3.1)$$

where k is spatial curvature.

From this point on I will set $c = 1$. The Ricci scalar is then written

$$R = 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right]. \quad (3.2)$$

Equations (2.2) and (2.3) are

$$[1 - \xi(1 - 6\xi)\kappa\phi^2] R - \kappa(6\xi - 1)\dot{\phi}^2 - 4\kappa V + 6\kappa\xi\phi \frac{dV}{d\phi} = 0 \quad (3.3)$$

and

$$\ddot{\phi} + 3H\dot{\phi} + \xi R\phi + \frac{dV}{d\phi} = 0. \quad (3.4)$$

Substituting the general expression for R into (2.2) and (2.3) gives the field equations in their general form:

$$6 [1 - \xi(1 - 6\xi)\kappa\phi^2] \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] - \kappa(6\xi - 1)\dot{\phi}^2 - 4\kappa V + 6\kappa\xi\phi \frac{dV}{d\phi} = 0 \quad (3.5)$$

and

$$\ddot{\phi} + 3H\dot{\phi} + \xi 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] \phi + \frac{dV}{d\phi} = 0. \quad (3.6)$$

These equations can, of course, be derived by variation of the action integral (2.1), using the general FRW form of R allowing for any k .

Next, I wish to derive 'new' field equations equivalent to (2.31) and (2.47) with curvature taken into account.

We start out with (3.5):

$$6 [1 - \xi (1 - 6\xi) \kappa \phi^2] \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] - \kappa (6\xi - 1) \dot{\phi}^2 - 4\kappa V + 6\kappa \xi \phi \frac{dV}{d\phi} = 0 \quad (3.7)$$

We set $\xi = \frac{1}{6}$ as before, and the potential $V(\phi)$ is also the same. We get

$$6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] - 4\kappa \left[\frac{m^2 \phi^2}{2} + \lambda \phi^4 + V_0 \right] + \kappa \phi [m^2 \phi + 4\lambda \phi^3] = 0 \quad (3.8)$$

$$\Rightarrow 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] - 2\kappa m^2 \phi^2 - 4\kappa \lambda \phi^4 - 4\kappa V_0 + \kappa m^2 \phi^2 + 4\kappa \lambda \phi^4 = 0 \quad (3.9)$$

$$\Rightarrow 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] - \kappa m^2 \phi^2 - 4\kappa V_0 = 0. \quad (3.10)$$

Next, we substitute $\phi = \frac{\varphi}{a}$:

$$6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (3.11)$$

$$\Rightarrow 6 \frac{\ddot{a}}{a} + 6 \frac{\dot{a}^2}{a^2} + 6 \frac{k}{a^2} - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (3.12)$$

We write out the derivatives explicitly:

$$6 \frac{1}{a} \frac{d}{dt} \left[\frac{da}{dt} \right] + 6 \frac{1}{a^2} \left[\frac{da}{dt} \right]^2 + 6 \frac{k}{a^2} - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (3.13)$$

If we make the substitution $dt = a d\eta$, we get

$$6 \frac{1}{a} \frac{d}{a d\eta} \left[\frac{da}{a d\eta} \right] + 6 \frac{1}{a^2} \left[\frac{da}{a d\eta} \right]^2 + 6 \frac{k}{a^2} - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (3.14)$$

$$\Rightarrow 6 \frac{1}{a^2} \frac{d}{d\eta} \left[\frac{1}{a} \frac{da}{d\eta} \right] + 6 \frac{1}{a^4} \left[\frac{da}{d\eta} \right]^2 + 6 \frac{k}{a^2} - \kappa m^2 \frac{\varphi^2}{a^2} - 4\kappa V_0 = 0 \quad (3.15)$$

As before, we put

$$\psi \equiv \sqrt{\frac{\kappa}{6}} a \quad \Rightarrow a = \sqrt{\frac{6}{\kappa}} \psi \quad \Rightarrow \frac{1}{a} = \frac{1}{\psi} \sqrt{\frac{\kappa}{6}} \quad (3.16)$$

and get

$$6 \frac{1}{\psi^2} \frac{\kappa}{6} \frac{d}{d\eta} \left[\frac{1}{\psi} \sqrt{\frac{\kappa}{6}} \frac{d}{d\eta} \left(\sqrt{\frac{6}{\kappa}} \psi \right) \right] + 6 \frac{1}{\psi^4} \frac{\kappa^2}{6^2} \left[\frac{d}{d\eta} \left(\sqrt{\frac{6}{\kappa}} \psi \right) \right]^2 + 6 \frac{k}{\psi^2} \frac{\kappa}{6} - \kappa m^2 \frac{\varphi^2}{\psi^2} \frac{\kappa}{6} - 4\kappa V_0 = 0 \quad (3.17)$$

$$\Rightarrow \kappa \frac{1}{\psi^2} \frac{d}{d\eta} \left[\frac{1}{\psi} \psi' \right] + \frac{1}{\psi^4} \frac{\kappa^2}{6} \frac{6}{\kappa} \psi'^2 + \kappa k \frac{1}{\psi^2} - \kappa^2 \frac{1}{6} m^2 \varphi^2 \frac{1}{\psi^2} - 4\kappa V_0 = 0 \quad (3.18)$$

$$\Rightarrow \kappa \frac{1}{\psi^2} \left[\frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2} \right] + \kappa \frac{1}{\psi^4} \psi'^2 + \kappa k \frac{1}{\psi^2} - \kappa^2 \frac{1}{6} m^2 \varphi^2 \frac{1}{\psi^2} - 4\kappa V_0 = 0 \quad (3.19)$$

$$\Rightarrow \kappa \frac{\psi''}{\psi^3} - \kappa \frac{\psi'^2}{\psi^4} + \kappa \frac{\psi'^2}{\psi^4} + \kappa k \frac{1}{\psi^2} - \kappa^2 \frac{1}{6} m^2 \varphi^2 \frac{1}{\psi^2} - 4\kappa V_0 = 0 \quad (3.20)$$

Now, we multiply through by $\frac{1}{\kappa} \psi^3$:

$$\Rightarrow \underline{\underline{\psi'' + k\psi - \frac{\kappa m^2}{6} \varphi^2 \psi - 4V_0 \psi^3 = 0}} \quad (3.21)$$

This is the general version of the first field equation (2.31).

Next, we look at (3.6):

First, we set $\xi = \frac{1}{6}$, $H = \frac{\dot{a}}{a}$ and $\frac{dV}{d\phi}$ into the equation:

$$\ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} + \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] \phi + m^2 \phi + 4\lambda \phi^3 = 0 \quad (3.22)$$

Using $\phi = \frac{\varphi}{a}$ gives

$$\frac{d}{dt} \left[\frac{d}{dt} \left(\frac{\varphi}{a} \right) \right] + 3 \frac{\dot{a}}{a} \frac{d}{dt} \left(\frac{\varphi}{a} \right) + \frac{\ddot{a}}{a} \frac{\varphi}{a} + \frac{\dot{a}^2}{a^2} \frac{\varphi}{a} + \frac{k}{a^2} \frac{\varphi}{a} + m^2 \frac{\varphi}{a} + 4\lambda \frac{\varphi^3}{a^3} = 0 \quad (3.23)$$

Carrying out the differentiation gives

$$\frac{d}{dt} \left[\frac{1}{a} \dot{\varphi} - \varphi \frac{1}{a^2} \dot{a} \right] + 3 \frac{\dot{a}}{a} \left[\frac{1}{a} \dot{\varphi} - \varphi \frac{1}{a^2} \dot{a} \right] + \frac{\ddot{a}}{a^2} \varphi + \frac{\dot{a}^2}{a^3} \varphi + \frac{k}{a^3} \varphi + m^2 \frac{\varphi}{a} + 4\lambda \frac{\varphi^3}{a^3} = 0 \quad (3.24)$$

$$\begin{aligned} \Rightarrow \left[\frac{1}{a} \ddot{\varphi} - \dot{\varphi} \frac{1}{a^2} \dot{a} \right] - \left[\frac{1}{a^2} (\varphi \ddot{a} + \dot{\varphi} \dot{a}) - 2\varphi \dot{a}^2 \frac{1}{a^3} \right] + 3 \frac{1}{a^2} \dot{a} \dot{\varphi} - 3 \frac{1}{a^3} \dot{a}^2 \varphi \\ + \frac{1}{a^2} \ddot{a} \varphi + \frac{1}{a^3} \dot{a}^2 \varphi + k \frac{1}{a^3} \varphi + m^2 \frac{\varphi}{a} + 4\lambda \frac{\varphi^3}{a^3} = 0 \end{aligned} \quad (3.25)$$

$$\begin{aligned} \Rightarrow \frac{1}{a}\ddot{\varphi} - \frac{1}{a^2}\dot{a}\dot{\varphi} - \frac{1}{a^2}\varphi\ddot{a} - \frac{1}{a^2}\dot{a}\dot{\varphi} + 2\frac{1}{a^3}\dot{a}^2\varphi + 3\frac{1}{a^2}\dot{a}\dot{\varphi} - 3\frac{1}{a^3}\dot{a}^2\varphi \\ + \frac{1}{a^2}\ddot{a}\varphi + \frac{1}{a^3}\dot{a}^2\varphi + k\frac{1}{a^3}\varphi + m^2\frac{1}{a}\varphi + 4\lambda\frac{1}{a^3}\varphi^3 = 0 \end{aligned} \quad (3.26)$$

$$\Rightarrow \frac{1}{a}\ddot{\varphi} + \frac{1}{a^2}\dot{a}\dot{\varphi} + k\frac{1}{a^3}\varphi + m^2\frac{1}{a}\varphi + 4\lambda\frac{1}{a^3}\varphi^3 = 0 \quad (3.27)$$

Multiply through by a :

$$\ddot{\varphi} + \frac{1}{a}\dot{a}\dot{\varphi} + k\frac{1}{a^2}\varphi + m^2\varphi + 4\lambda\frac{1}{a^2}\varphi^3 = 0 \quad (3.28)$$

Write out the differentiations explicitly:

$$\frac{d}{dt} \left[\frac{d\varphi}{dt} \right] + \frac{1}{a} \frac{da}{dt} \frac{d\varphi}{dt} + k\frac{1}{a^2}\varphi + m^2\varphi + 4\lambda\frac{1}{a^2}\varphi^3 = 0 \quad (3.29)$$

Use $dt \equiv a d\eta$:

$$\frac{d}{a d\eta} \left[\frac{d\varphi}{a d\eta} \right] + \frac{1}{a} \frac{da}{a d\eta} \frac{d\varphi}{a d\eta} + k\frac{1}{a^2}\varphi + m^2\varphi + 4\lambda\frac{1}{a^2}\varphi^3 = 0 \quad (3.30)$$

$$\Rightarrow \frac{1}{a} \frac{d}{d\eta} \left[\frac{1}{a} \varphi' \right] + \frac{1}{a^3} \left[\frac{da}{d\eta} \right] \varphi' + k\frac{1}{a^2}\varphi + m^2\varphi + 4\lambda\frac{1}{a^2}\varphi^3 = 0 \quad (3.31)$$

Using $\psi \equiv \sqrt{\frac{\kappa}{6}}a \quad \Rightarrow \frac{1}{a} = \frac{1}{\psi} \sqrt{\frac{\kappa}{6}}$ we get

$$\sqrt{\frac{\kappa}{6}} \frac{1}{\psi} \frac{d}{d\eta} \left[\sqrt{\frac{\kappa}{6}} \frac{1}{\psi} \varphi' \right] + \frac{1}{\psi^3} \left[\frac{\kappa}{6} \right]^{\frac{3}{2}} \varphi' \frac{d}{d\eta} \left[\sqrt{\frac{6}{\kappa}} \psi \right] + k\frac{1}{\psi^2} \frac{\kappa}{6} \varphi + m^2\varphi + 4\lambda\frac{1}{\psi^2} \frac{\kappa}{6} \varphi^3 = 0 \quad (3.32)$$

$$\Rightarrow \frac{\kappa}{6} \frac{1}{\psi} \frac{d}{d\eta} \left[\frac{1}{\psi} \varphi' \right] + \frac{\kappa}{6} \frac{1}{\psi^3} \varphi' \psi' + \frac{\kappa}{6} k \frac{1}{\psi^2} \varphi + m^2\varphi + 4\frac{\kappa}{6} \lambda \frac{1}{\psi^2} \varphi^3 = 0 \quad (3.33)$$

Differentiate with respect to the conformal time η :

$$\frac{\kappa}{6} \frac{1}{\psi} \left(\frac{1}{\psi} \varphi'' - \varphi' \frac{1}{\psi^2} \psi' \right) + \frac{\kappa}{6} \frac{1}{\psi^3} \varphi' \psi' + \frac{\kappa}{6} k \frac{\varphi}{\psi^2} + m^2\varphi + 4\frac{\kappa}{6} \lambda \frac{1}{\psi^2} \varphi^3 = 0 \quad (3.34)$$

$$\Rightarrow \frac{\kappa}{6} \frac{1}{\psi^2} \varphi'' - \frac{\kappa}{6} \frac{1}{\psi^3} \varphi' \psi' + \frac{\kappa}{6} \frac{1}{\psi^3} \varphi' \psi' + \frac{\kappa}{6} k \frac{\varphi}{\psi^2} + m^2\varphi + 4\frac{\kappa}{6} \lambda \frac{1}{\psi^2} \varphi^3 = 0 \quad (3.35)$$

Finally, we multiply through by $\frac{6}{\kappa}\psi^2$ and get

$$\underline{\underline{\varphi'' + k\varphi + \frac{6m^2}{\kappa}\psi^2\varphi + 4\lambda\varphi^3 = 0}} \quad (3.36)$$

This is the general version of the second field equation (2.47).

To summarize, our generalized field equations are:

$$\underline{\underline{\psi'' - \frac{\kappa m^2}{6} \varphi^2 \psi - 4V_0 \psi^3 + k\psi = 0}} \quad (3.37)$$

and

$$\underline{\underline{\varphi'' + \frac{6m^2}{\kappa} \psi^2 \varphi + 4\lambda \varphi^3 + k\varphi = 0}} \quad (3.38)$$

3.2 The general Lagrangian

In the case where k can take on any of the values $-1, 0$, or 1 , the Lagrangian of the system will contain terms containing k . I propose that the generalized Lagrangian will now be

$$L = \frac{1}{2} (\varphi')^2 - \frac{18}{\kappa^2} (\psi')^2 - \frac{3m^2}{\kappa} \psi^2 \varphi^2 - \lambda \varphi^4 - \frac{36}{\kappa^2} V_0 \psi^4 - \frac{1}{2} k \varphi^2 + \frac{18}{\kappa^2} k \psi^2. \quad (3.39)$$

I will now show that this indeed is a correct expression for the Lagrangian, by showing that it reproduces the generalized field equations by use of Lagrange's equations:

Lagrange's equation for φ :

$$\frac{\partial L}{\partial \varphi'} = \varphi', \quad \frac{d}{d\eta} \left[\frac{\partial L}{\partial \varphi'} \right] = \frac{d}{d\eta} \varphi' = \varphi'' \quad (3.40)$$

$$\frac{\partial L}{\partial \varphi} = -\frac{6m^2}{\kappa} \psi^2 \varphi - 4\lambda \varphi^3 - k\varphi \quad (3.41)$$

Lagrange's equation states

$$\frac{d}{d\eta} \left[\frac{\partial L}{\partial \varphi'} \right] - \frac{\partial L}{\partial \varphi} = 0 \quad (3.42)$$

and our equation for φ becomes

$$\underline{\underline{\varphi'' + \frac{6m^2}{\kappa} \psi^2 \varphi + 4\lambda \varphi^3 + k\varphi = 0}} \quad (3.43)$$

as before.

Lagrange's equation for ψ :

$$\frac{\partial L}{\partial \psi'} = -\frac{36}{\kappa^2} \psi', \quad \frac{d}{d\eta} \left[\frac{\partial L}{\partial \psi'} \right] = \frac{d}{d\eta} \left[-\frac{36}{\kappa^2} \psi'' \right] = -\frac{36}{\kappa^2} \psi'' \quad (3.44)$$

$$\frac{\partial L}{\partial \psi} = -\frac{6m^2}{\kappa} \psi \varphi^2 - \frac{36}{\kappa^2} 4V_0 \psi^3 + \frac{36}{\kappa^2} k\psi \quad (3.45)$$

Lagrange's equation states

$$\frac{d}{d\eta} \left[\frac{\partial L}{\partial \psi'} \right] - \frac{\partial L}{\partial \psi} = 0 \quad (3.46)$$

and our equation for ψ becomes

$$-\frac{36}{\kappa^2} \psi'' + \frac{6m^2}{\kappa} \psi \varphi^2 + \frac{36}{\kappa^2} 4V_0 \psi^3 - \frac{36}{\kappa^2} k \psi = 0 \quad (3.47)$$

Multiplying through by $-\frac{\kappa^2}{36}$ gives

$$\psi'' - \frac{\kappa m^2}{6} \psi \varphi^2 - 4V_0 \psi^3 + k \psi = 0 \quad (3.48)$$

as before.

This shows that equation (3.39) is the generalized Lagrangian, since it reproduces the generalized field equations.

3.3 Derivation of a generalized energy equation

The derivation of an energy equation allowing for non-zero spatial curvature, is completely analogous to the case where $k = 0$:

Multiplying (3.48) by $\frac{36}{\kappa^2} \psi'$ gives the result

$$\frac{36}{\kappa^2} \psi' \psi'' - \frac{36}{\kappa^2} \psi' \frac{\kappa m^2}{6} \varphi^2 \psi - \frac{36}{\kappa^2} \psi' \cdot 4V_0 \psi^3 + \frac{36}{\kappa^2} \psi' k \psi = 0 \quad (3.49)$$

$$\Rightarrow \frac{36}{\kappa^2} \psi' \psi'' - \frac{6m^2}{\kappa} \varphi^2 \psi \psi' - \frac{36}{\kappa^2} \cdot 4V_0 \psi^3 \psi' + \frac{36}{\kappa^2} k \psi \psi' = 0 \quad (3.50)$$

Multiplying equation (3.43) by φ' gives

$$\varphi' \varphi'' + \varphi' \frac{6m^2}{\kappa} \psi^2 \varphi + \varphi' 4\lambda \varphi^3 + \varphi' k \varphi = 0 \quad (3.51)$$

Equating the two equations above ((3.50) and (3.51)) gives the result

$$\varphi' \varphi'' + \frac{6m^2}{\kappa} \varphi \psi^2 \varphi' + 4\lambda \varphi^3 \varphi' + k \varphi \varphi' = \frac{36}{\kappa^2} \psi' \psi'' - \frac{6m^2}{\kappa} \varphi^2 \psi \psi' - \frac{36}{\kappa^2} \cdot 4V_0 \psi^3 \psi' + \frac{36}{\kappa^2} k \psi \psi' \quad (3.52)$$

$$\begin{aligned} \Rightarrow \varphi' \varphi'' - \frac{36}{\kappa^2} \psi' \psi'' + \frac{6m^2}{\kappa} \varphi^2 \psi \psi' + \frac{6m^2}{\kappa} \psi^2 \varphi \varphi' \\ + 4\lambda \varphi^3 \varphi' + \frac{36}{\kappa^2} \cdot 4V_0 \psi^3 \psi' + k \varphi \varphi' - \frac{36}{\kappa^2} k \psi \psi' = 0 \end{aligned} \quad (3.53)$$

Integrating this equation with respect to the conformal time η gives

$$\begin{aligned}
& \Rightarrow \int \varphi' \varphi'' d\eta - \frac{36}{\kappa^2} \int \psi' \psi'' d\eta + \frac{6m^2}{\kappa} \int \varphi^2 \psi \psi' d\eta \\
& + \frac{6m^2}{\kappa} \int \psi^2 \varphi \varphi' d\eta + 4\lambda \int \varphi^3 \varphi' d\eta + \frac{36}{\kappa^2} \cdot 4V_0 \int \psi^3 \psi' d\eta \\
& + k \int \varphi \varphi' d\eta - \frac{36}{\kappa^2} k \int \psi \psi' d\eta = \text{constant} \quad (3.54)
\end{aligned}$$

$$\begin{aligned}
& \Rightarrow \int \varphi' \frac{d}{d\eta} [\varphi'] d\eta - \frac{36}{\kappa^2} \int \psi' \frac{d}{d\eta} [\psi'] d\eta + \frac{3m^2}{\kappa} \int 2\varphi^2 \psi \psi' d\eta \\
& + \frac{3m^2}{\kappa} \int 2\psi^2 \varphi \varphi' d\eta + 4\lambda \int \varphi^3 \frac{d\varphi}{d\eta} d\eta + \frac{36}{\kappa^2} \cdot 4V_0 \int \psi^3 \frac{d\psi}{d\eta} d\eta \\
& + k \int \varphi \frac{d\varphi}{d\eta} d\eta - \frac{36}{\kappa^2} k \int \psi \frac{d\psi}{d\eta} d\eta = \text{constant} \quad (3.55)
\end{aligned}$$

Everything is the same as in the case with no curvature, except for the last two terms:

$$k \int \varphi d\varphi - \frac{36}{\kappa^2} k \int \psi d\psi = k \left(\frac{1}{2} \varphi^2 \right) - \frac{36}{\kappa^2} k \left(\frac{1}{2} \psi^2 \right) \quad (3.56)$$

$$= \frac{1}{2} k \varphi^2 - \frac{18}{\kappa^2} k \psi^2 \quad (3.57)$$

Adding this to equation (2.117), we get:

$$\underline{\underline{\frac{1}{2} (\varphi')^2 - \frac{18}{\kappa^2} (\psi')^2 + \frac{3m^2}{\kappa} \varphi^2 \psi^2 + \lambda \varphi^4 + \frac{36}{\kappa^2} V_0 \psi^4 + \frac{1}{2} k \varphi^2 - \frac{18}{\kappa^2} k \psi^2 = \text{constant}}} \quad (3.58)$$

This is the generalized version of equation (2.117).

Using once again the definition of the Hamiltonian of the system, and identifying the Hamiltonian as its energy, we arrive at the following energy equation:

$$\underline{\underline{E = \frac{1}{2} \varphi'^2 - \frac{18}{\kappa^2} \psi'^2 + \frac{3m^2}{\kappa} \varphi^2 \psi^2 + \lambda \varphi^4 + \frac{36}{\kappa^2} V_0 \psi^4 + \frac{1}{2} k \varphi^2 - \frac{18}{\kappa^2} k \psi^2.}} \quad (3.59)$$

Using eq. (3.58), we see that $E = \text{constant}$ still holds, even if $k \neq 0$.

In this chapter I have, using F&C's approach, derived an equation for the total energy of the universe allowing for any spatial curvature. I have shown that the energy should still be constant, and that the energy equation found by Faraoni and Cooperstock is obtained by setting $k = 0$.

In the following chapters, I will look at three different cosmological models; the de Sitter model, Einstein's static universe, and the Milne model. I will use the equations I have derived in this chapter in attempt to calculate the energy of these universe models.

From now on, I will always use that $\xi = \frac{1}{6}$. The equations of motion then become

$$R - 4\kappa V + \kappa\phi \frac{dV}{d\phi} = 0 \quad (3.60)$$

and

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{6}R\phi + \frac{dV}{d\phi} = 0. \quad (3.61)$$

First, I look at the de Sitter model.

Chapter 4

The de Sitter Universe

In this chapter, I will consider the de Sitter model. I will use the equations from chapter 3 in attempt to calculate the total energy of this model. This is an interesting model to look at because other methods for calculating the energy of this universe give the result of zero energy. For example, in quantum cosmology the $k = +1$ de Sitter model is a solution of the Wheeler-DeWitt equation, the analog of the Schrödinger equation, with energy eigenvalue equal to zero. (See [9]). Is Faraoni and Cooperstock's method in agreement with this?

4.1 The solution with $k = +1$

In this section, I will consider the De Sitter universe with positive spatial curvature. For this model, we have $m = 0$, $k = +1$ and the scale factor $a = a_\Lambda \cosh\left(\frac{t}{a_\Lambda}\right)$ where $a_\Lambda = \sqrt{\frac{3}{\Lambda}}$.

First, we calculate the Ricci scalar R :

$$R = 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] \quad (4.1)$$

From the expression for a , we find

$$\begin{aligned} \dot{a} &= \frac{d}{dt} \left[a_\Lambda \cosh\left(\frac{t}{a_\Lambda}\right) \right] \\ &= a_\Lambda \sinh\left(\frac{t}{a_\Lambda}\right) \cdot \frac{1}{a_\Lambda} \\ &= \sinh\left(\frac{t}{a_\Lambda}\right) \end{aligned} \quad (4.2)$$

$$\begin{aligned} \ddot{a} &= \frac{d}{dt} \left[\sinh\left(\frac{t}{a_\Lambda}\right) \right] \\ &= \cosh\left(\frac{t}{a_\Lambda}\right) \cdot \frac{1}{a_\Lambda} \\ &= \frac{1}{a_\Lambda} \cosh\left(\frac{t}{a_\Lambda}\right) \end{aligned} \quad (4.3)$$

Inserting this into the expression for R gives

$$\begin{aligned}
R &= 6 \left[\frac{\cosh\left(\frac{t}{a_\Lambda}\right)}{a_\Lambda \cdot a_\Lambda \cosh\frac{t}{a_\Lambda}} + \frac{\sinh^2\left(\frac{t}{a_\Lambda}\right)}{a_\Lambda^2 \cosh^2\left(\frac{t}{a_\Lambda}\right)} + \frac{1}{a_\Lambda^2 \cosh^2\left(\frac{t}{a_\Lambda}\right)} \right] \\
&= 6 \left[\frac{1}{a_\Lambda^2} + \frac{\sinh^2\left(\frac{t}{a_\Lambda}\right)}{a_\Lambda^2 \cosh^2\left(\frac{t}{a_\Lambda}\right)} + \frac{1}{a_\Lambda^2 \cosh^2\left(\frac{t}{a_\Lambda}\right)} \right] \\
&= 6 \left[\frac{\cosh^2\left(\frac{t}{a_\Lambda}\right) + \sinh^2\left(\frac{t}{a_\Lambda}\right) + 1}{a_\Lambda^2 \cosh^2\left(\frac{t}{a_\Lambda}\right)} \right] \\
&= 6 \left[\frac{2 \cosh^2\left(\frac{t}{a_\Lambda}\right)}{a_\Lambda^2 \cosh^2\left(\frac{t}{a_\Lambda}\right)} \right] \\
&= \frac{12}{a_\Lambda^2},
\end{aligned} \tag{4.4}$$

Where I have used that

$$\cosh^2 x - \sinh^2 x = 1. \tag{4.5}$$

Using that $a_\Lambda = \sqrt{\frac{3}{\Lambda}}$ gives the result

$$R = 4\Lambda. \tag{4.6}$$

Equation (3.60) with $R = 4\Lambda$ gives

$$\begin{aligned}
4\Lambda - 4\kappa V + \kappa\phi \frac{dV}{d\phi} &= 0 \\
\Rightarrow \phi \frac{dV}{d\phi} - 4V &= -\frac{4\Lambda}{\kappa}.
\end{aligned} \tag{4.7}$$

This is a differential equation for V . Let us try a solution

$$V(\phi) = \lambda\phi^4 + V_0 \tag{4.8}$$

$$\Rightarrow \frac{dV}{d\phi} = 4\lambda\phi^3. \tag{4.9}$$

Our differential equation then becomes

$$\phi \cdot 4\lambda\phi^3 - 4(\lambda\phi^4 + V_0) = -\frac{4\Lambda}{\kappa} \quad (4.10)$$

$$\Rightarrow 4\lambda\phi^4 - 4\lambda\phi^4 - 4V_0 = -\frac{4\Lambda}{\kappa} \quad (4.11)$$

$$\Rightarrow V_0 = \frac{\Lambda}{\kappa} \quad (4.12)$$

$$\Rightarrow V(\phi) = \lambda\phi^4 + \frac{\Lambda}{\kappa}. \quad (4.13)$$

Next, we look at equation (3.61):

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{6}R\phi + \frac{dV}{d\phi} = 0 \quad (4.14)$$

Assuming that ϕ is a constant, so that $\dot{\phi} = \ddot{\phi} = 0$, and using our expressions for R and $\frac{dV}{d\phi}$, we get

$$\frac{1}{6} \cdot 4\Lambda\phi + 4\lambda\phi^3 = 0 \quad (4.15)$$

$$\Rightarrow \frac{2\Lambda}{3} + 4\lambda\phi^2 = 0 \quad (4.16)$$

$$\Rightarrow \phi = \sqrt{-\frac{\Lambda}{6\lambda}}. \quad (4.17)$$

This has a real solution if $\lambda < 0$.

The energy equation is

$$E = \frac{1}{2}\varphi'^2 - \frac{18}{\kappa^2}\psi'^2 + \frac{3m^2}{\kappa}\psi^2\varphi^2 + \lambda\varphi^4 + \frac{36}{\kappa^2}V_0\psi^4 + \frac{1}{2}k\varphi^2 - \frac{18}{\kappa^2}k\psi^2. \quad (4.18)$$

Here, $m = 0$, $k = 1$ and $V_0 = \frac{\Lambda}{\kappa}$, so

$$E = \frac{1}{2}\varphi'^2 - \frac{18}{\kappa^2}\psi'^2 + \lambda\varphi^4 + \frac{36\Lambda}{\kappa^2\kappa}\psi^4 + \frac{1}{2}\varphi^2 - \frac{18}{\kappa^2}\psi^2. \quad (4.19)$$

Setting

$$\psi = \sqrt{\frac{\kappa}{6}} \quad \text{and} \quad \varphi = a\phi \quad (4.20)$$

this becomes

$$E = \frac{1}{2} \left[\frac{d}{d\eta} (a\phi) \right]^2 - \frac{18}{\kappa^2} \left[\frac{d}{d\eta} \left(\sqrt{\frac{\kappa}{6}} a \right) \right]^2 + \lambda a^4 \phi^4 + \frac{36\Lambda}{\kappa^3} \frac{\kappa^2}{36} a^4 + \frac{1}{2} a^2 \phi^2 - \frac{18}{\kappa^2} \frac{\kappa}{6} a^2 \quad (4.21)$$

$$\Rightarrow E = \frac{1}{2} \left[a \frac{d}{dt} (a\phi) \right]^2 - \frac{18}{\kappa^2} \left[a \frac{d}{dt} \left(\sqrt{\frac{\kappa}{6}} a \right) \right]^2 + \lambda a^4 \phi^4 + \frac{\Lambda}{\kappa} a^4 + \frac{1}{2} a^2 \phi^2 - \frac{3}{\kappa} a^2. \quad (4.22)$$

We have assumed that ϕ is constant, so

$$E = \frac{1}{2}a^2\phi^2 \left[\frac{da}{dt} \right]^2 - \frac{18}{\kappa^2}a^2\frac{\kappa}{6} \left[\frac{da}{dt} \right]^2 + \lambda a^4\phi^4 + \frac{\Lambda}{\kappa}a^4 + \frac{1}{2}a^2\phi^2 - \frac{3}{\kappa}a^2 \quad (4.23)$$

$$\Rightarrow E = \frac{1}{2}a^2\phi^2 \left[\frac{da}{dt} \right]^2 + \frac{1}{2}a^2\phi^2 - \frac{3}{\kappa}a^2 \left[\frac{da}{dt} \right]^2 - \frac{3}{\kappa}a^2 + \lambda a^4\phi^4 + \frac{\Lambda}{\kappa}a^4. \quad (4.24)$$

Next, we use that $a = a_\Lambda \cosh\left(\frac{t}{a_\Lambda}\right)$:

$$\begin{aligned} E &= \frac{1}{2}a_\Lambda^2 \cosh^2\left(\frac{t}{a_\Lambda}\right) \cdot \left(-\frac{\Lambda}{6\lambda}\right) \left[\sinh^2\left(\frac{t}{a_\Lambda}\right) + 1 \right] \\ &\quad - \frac{3}{\kappa}a_\Lambda^2 \cosh^2\left(\frac{t}{a_\Lambda}\right) \left[\sinh^2\left(\frac{t}{a_\Lambda}\right) + 1 \right] \\ &\quad + \lambda a_\Lambda^4 \cosh^4\left(\frac{t}{a_\Lambda}\right) \frac{\Lambda^2}{36\lambda^2} + \frac{\Lambda}{\kappa}a_\Lambda^4 \cosh^4\left(\frac{t}{a_\Lambda}\right) \end{aligned} \quad (4.25)$$

$$\begin{aligned} \Rightarrow E &= -\frac{1}{2}\frac{3}{\Lambda}\frac{\Lambda}{6\lambda} \cosh^4\left(\frac{t}{a_\Lambda}\right) - \frac{3}{\kappa}\frac{3}{\Lambda} \cosh^4\left(\frac{t}{a_\Lambda}\right) \\ &\quad + \lambda \frac{9}{\Lambda^2} \cosh^4\left(\frac{t}{a_\Lambda}\right) \frac{\Lambda^2}{36\lambda^2} + \frac{\Lambda}{\kappa} \frac{9}{\Lambda^2} \cosh^4\left(\frac{t}{a_\Lambda}\right) \end{aligned} \quad (4.26)$$

$$\Rightarrow E = \left[-\frac{1}{4\lambda} - \frac{9}{\kappa\Lambda} + \frac{1}{4\lambda} + \frac{9}{\kappa\Lambda} \right] \cosh^4\left(\frac{t}{a_\Lambda}\right). \quad (4.27)$$

All terms cancel out, so we end up with

$$E = 0. \quad (4.28)$$

According to quantum cosmology, the energy of the de Sitter universe should be zero. So in this case, F&C's method for calculating the energy of the universe is in agreement with other ways of reasoning.

4.2 The solution with $k = 0$

Here, I will look at the de Sitter universe with zero spatial curvature.

We have $a(t) = e^{H_0 t}$ where $H_0 = \sqrt{\frac{\Lambda}{3}}$, so that

$$\dot{a} = H_0 e^{H_0 t} \quad (4.29)$$

and

$$\ddot{a} = H_0^2 e^{H_0 t}. \quad (4.30)$$

From this we calculate the Ricci scalar:

$$\begin{aligned}
R &= 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] \\
&= 6 \left[\frac{H_0^2 e^{H_0 t}}{e^{H_0 t}} + \frac{H_0^2 e^{2H_0 t}}{e^{2H_0 t}} \right] \\
&= 6 \cdot 2H_0^2 \\
&= \frac{12\Lambda}{3} \\
&= 4\Lambda.
\end{aligned} \tag{4.31}$$

Note that this is the same R as for the de Sitter model with $k = 1$. That means that the scalar field also is the same as before, namely

$$\phi = \sqrt{-\frac{\Lambda}{6\lambda}}. \tag{4.32}$$

The energy equation reads

$$E = \frac{1}{2}\varphi'^2 - \frac{18}{\kappa^2}\psi'^2 + \frac{3m^2}{\kappa}\psi^2\varphi^2 + \lambda\varphi^4 + \frac{36}{\kappa^2}V_0\psi^4 + \frac{1}{2}kc^2\varphi^2 - \frac{18}{\kappa^2}kc\psi^2. \tag{4.33}$$

We have $k = 0$, $V_0 = \frac{\Lambda}{\kappa}$ and $m = 0$, so this becomes

$$E = \frac{1}{2}\varphi'^2 - \frac{18}{\kappa^2}\psi'^2 + \lambda\varphi^4 + \frac{36\Lambda}{\kappa^2\kappa}\psi^4 \tag{4.34}$$

$$\Rightarrow E = \frac{1}{2} \left[\frac{d\varphi}{d\eta} \right]^2 - \frac{18}{\kappa^2} \left[\frac{d\psi}{d\eta} \right]^2 + \lambda\varphi^4 + \frac{36\Lambda}{\kappa^3}\psi^4 \tag{4.35}$$

$$\Rightarrow E = \frac{1}{2} \left[a \frac{d\varphi}{dt} \right]^2 - \frac{18}{\kappa^2} \left[a \frac{d\psi}{dt} \right]^2 + \lambda\varphi^4 + \frac{36\Lambda}{\kappa^3}\psi^4. \tag{4.36}$$

We have that $\psi = \sqrt{\frac{\kappa}{6}}a$ and $\varphi = a\phi$:

$$\Rightarrow E = \frac{1}{2}a^2 \left[\frac{d}{dt}(a\phi) \right]^2 - \frac{18}{\kappa^2}a^2 \left[\frac{d}{dt} \left(\sqrt{\frac{\kappa}{6}}a \right) \right]^2 + \lambda a^4 \phi^4 + \frac{36\Lambda}{\kappa^3} \frac{\kappa^2}{36} a^4 \tag{4.37}$$

$$\Rightarrow E = \frac{1}{2}a^2 \phi^2 \left[\frac{da}{dt} \right]^2 - \frac{18}{\kappa^2}a^2 \frac{\kappa}{6} \left[\frac{da}{dt} \right]^2 + \lambda a^4 \phi^4 + \frac{\Lambda}{\kappa} a^4. \tag{4.38}$$

Next, we use that

$$a = e^{\sqrt{\frac{\Lambda}{3}}t}, \quad \frac{da}{dt} = \sqrt{\frac{\Lambda}{3}} e^{\sqrt{\frac{\Lambda}{3}}t}, \quad \text{and} \quad \phi = \sqrt{-\frac{\Lambda}{6\lambda}}: \tag{4.39}$$

$$\Rightarrow E = \frac{1}{2} e^{2\sqrt{\frac{\Lambda}{3}}t} \left(-\frac{\Lambda}{6\lambda} \right) \frac{\Lambda}{3} e^{2\sqrt{\frac{\Lambda}{3}}t} - \frac{18}{\kappa^2} e^{2\sqrt{\frac{\Lambda}{3}}t} \frac{\kappa \Lambda}{6} \frac{\Lambda}{3} e^{2\sqrt{\frac{\Lambda}{3}}t} + \lambda e^{4\sqrt{\frac{\Lambda}{3}}t} \frac{\Lambda^2}{36\lambda^2} + \frac{\Lambda}{\kappa} e^{4\sqrt{\frac{\Lambda}{3}}t} \quad (4.40)$$

$$\Rightarrow E = -\frac{\Lambda^2}{36\lambda} e^{4\sqrt{\frac{\Lambda}{3}}t} - \frac{\Lambda}{\kappa} e^{4\sqrt{\frac{\Lambda}{3}}t} + \frac{\Lambda^2}{36\lambda} e^{4\sqrt{\frac{\Lambda}{3}}t} + \frac{\Lambda}{\kappa} e^{4\sqrt{\frac{\Lambda}{3}}t} \quad (4.41)$$

$$\Rightarrow E = \left[-\frac{\Lambda^2}{36\lambda} - \frac{\Lambda}{\kappa} + \frac{\Lambda^2}{36\lambda} + \frac{\Lambda}{\kappa} \right] e^{4\sqrt{\frac{\Lambda}{3}}t}. \quad (4.42)$$

All terms cancel out, and we finally get that

$$E = 0 \quad (4.43)$$

also in this case.

In this chapter I have applied F&C's method for calculating the total energy of the universe to the de Sitter model, for both positive and zero spatial curvature. In both cases, their method gives the result of zero energy. This is in agreement with the reasoning based on quantum cosmology.

In the next chapter I will look at Einstein's static universe.

Chapter 5

Einstein's Static Universe

In this chapter, I will consider Einstein's static universe. Proposed by Einstein in 1917, this is a model of a universe that is neither expanding nor contracting, has positive spatial curvature, and which contains dust and a cosmological constant. In fact, a static solution with matter is only possible with the presence of a cosmological constant, and this is the reason why Einstein introduced the cosmological constant in the first place. It is worth noting that this model is highly unstable and therefore unphysical: If the (constant) scale factor is increased by just a tiny bit, the mass density will decrease and the cosmological constant will dominate, causing a runaway expansion of the universe. And, if the scale factor is decreased, the mass density will increase, and gravity will dominate, causing the universe to collapse. This being said, this model is still interesting to look at from a theoretical point of view.

Based on our intuition, we might expect the energy of this universe to be zero, since it is balanced perfectly between expansion and contraction.

I will try to find an expression for the scalar field ϕ , enabling us to calculate the energy of this universe.

For this model, we have that the scale factor $a = \text{constant}$, the scalar field $\phi = \text{constant}$, $m = 0$, and the spatial curvature $k = 1$. From this, we find that $\dot{\phi} = \ddot{\phi} = 0$ and $\dot{a} = \ddot{a} = 0$. We also use $c = 1$ as before. First, we calculate the Ricci scalar:

$$R = 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right] = \frac{6}{a^2}. \quad (5.1)$$

Note that a Ricci scalar different from 0 implies that this universe cannot be described using Minkowski coordinates. This model has curved spacetime. Now, we find an expression for the potential $V(\phi)$ using equation (3.60).

Inserting the expression for R gives

$$\begin{aligned} \frac{6}{a^2} - 4\kappa V + \kappa\phi \frac{dV}{d\phi} &= 0 \\ \Rightarrow \kappa\phi \frac{dV}{d\phi} - 4\kappa V &= -\frac{6}{a^2} \\ \Rightarrow \phi \frac{dV}{d\phi} - 4V &= -\frac{6}{a^2\kappa}. \end{aligned} \quad (5.2)$$

Let us try a solution on the form

$$V(\phi) = \lambda\phi^4 + V_0 \quad \Rightarrow \quad \frac{dV}{d\phi} = 4\lambda\phi^3. \quad (5.3)$$

Substituting this into the differential equation (5.2) gives

$$\phi 4\lambda\phi^3 - 4\lambda\phi^4 - 4V_0 = -\frac{6}{a^2\kappa}. \quad (5.4)$$

This is satisfied if we require that

$$V_0 = \frac{3}{2a^2\kappa}, \quad (5.5)$$

so that

$$\underline{V(\phi) = \lambda\phi^4 + \frac{3}{2a^2\kappa}}. \quad (5.6)$$

We wish to find an expression for ϕ . Using equation (3.61), we get

$$\frac{1}{6} \frac{6}{a^2} \phi + 4\lambda\phi^3 = 0 \quad (5.7)$$

$$\frac{1}{a^2} \phi + 4\lambda\phi^3 = 0 \quad (5.8)$$

$$\frac{1}{a^2} + 4\lambda\phi^2 = 0 \quad (5.9)$$

$$4\lambda\phi^2 = -\frac{1}{a^2} \quad (5.10)$$

$$\Rightarrow \phi = \sqrt{-\frac{1}{4\lambda a^2}}. \quad (5.11)$$

We see that if $\lambda < 0$, there is a real solution for ϕ .

The energy equation reads:

$$E = \frac{1}{2}\varphi'^2 - \frac{18}{\kappa^2}\psi'^2 + \frac{3m^2}{\kappa}\psi^2\varphi^2 + \lambda\varphi^4 + \frac{36}{\kappa^2}V_0\psi^4 + \frac{1}{2}k\varphi^2 - \frac{18}{\kappa^2}k\psi^2. \quad (5.12)$$

Setting $m = 0$, $k = 1$ and $V_0 = \frac{3}{2a^2\kappa}$, this becomes

$$E = \frac{1}{2}\varphi'^2 - \frac{18}{\kappa^2}\psi'^2 + \lambda\varphi^4 + \frac{36}{\kappa^2} \frac{3}{2a^2\kappa}\psi^4 + \frac{1}{2}\varphi^2 - \frac{18}{\kappa^2}\psi^2 \quad (5.13)$$

$$\Rightarrow E = \frac{1}{2}\varphi'^2 - \frac{18}{\kappa^2}\psi'^2 + \lambda\varphi^4 + \frac{54}{\kappa^3 a^2}\psi^4 + \frac{1}{2}\varphi^2 - \frac{18}{\kappa^2}\psi^2. \quad (5.14)$$

Inserting $\varphi' = \frac{d\varphi}{d\eta}$ and $\psi' = \frac{d\psi}{d\eta}$ we get

$$E = \frac{1}{2} \left[\frac{d\varphi}{d\eta} \right]^2 - \frac{18}{\kappa^2} \left[\frac{d\psi}{d\eta} \right]^2 + \lambda\varphi^4 + \frac{54}{\kappa^3 a^2}\psi^4 + \frac{1}{2}\varphi^2 - \frac{18}{\kappa^2}\psi^2. \quad (5.15)$$

Next, we use that $\frac{d}{d\eta} = a\frac{d}{dt}$ and get

$$E = \frac{1}{2} \left[a \frac{d\varphi}{dt} \right]^2 - \frac{18}{\kappa^2} \left[a \frac{d\psi}{dt} \right]^2 + \lambda\varphi^4 + \frac{54}{\kappa^3 a^2} \psi^4 + \frac{1}{2} \varphi^2 - \frac{18}{\kappa^2} \psi^2. \quad (5.16)$$

Using $\psi = \sqrt{\frac{\kappa}{6}}a$ and $\varphi = a\phi$ this becomes

$$E = \frac{1}{2} a^2 \left[\frac{d}{dt} (a\phi) \right]^2 - \frac{18}{\kappa^2} a^2 \left[\frac{d}{dt} \left(\sqrt{\frac{\kappa}{6}}a \right) \right]^2 + \lambda a^4 \phi^4 + \frac{54}{\kappa^3 a^2} \frac{\kappa^2}{36} a^4 + \frac{1}{2} a^2 \phi^2 - \frac{18}{\kappa^2} \frac{\kappa}{6} a^2. \quad (5.17)$$

Since both a and ϕ are constants, this is reduced to

$$E = \lambda a^4 \phi^4 + \frac{3a^2}{2\kappa} + \frac{1}{2} a^2 \phi^2 - \frac{3}{\kappa} a^2. \quad (5.18)$$

Inserting $\phi = \sqrt{-\frac{1}{4\lambda a^2}}$ gives

$$E = \lambda a^4 \frac{1}{16\lambda^2 a^4} + \frac{3a^2}{2\kappa} - \frac{1}{2} a^2 \frac{1}{4\lambda a^2} - \frac{3}{\kappa} a^2 \quad (5.19)$$

$$\Rightarrow E = \frac{1}{16\lambda} + \frac{3a^2}{2\kappa} - \frac{1}{8\lambda} - \frac{3a^2}{\kappa} \quad (5.20)$$

$$\Rightarrow E = -\frac{1}{16\lambda} - \frac{3a^2}{2\kappa}. \quad (5.21)$$

Here, we also get that the energy is constant, but it is zero only for one particular choice of the integration constant λ . If we choose

$$\lambda = -\frac{\pi G}{3a^2} \quad (5.22)$$

the energy is zero.

One might immediately think that choosing this particular value of λ is the same thing as choosing a zero level for the energy that makes the energy equal to zero. F&C claim that the zero level is arbitrary. However, there are fundamental problems related to just putting the zero level where we want. The vacuum energy has a clear physical meaning; without it Einstein's static universe would collapse. Remember that this was the very reason why Einstein introduced the cosmological constant, to allow for a static solution of Einstein's field equations. My calculations show that F&C's approach gives the result of constant energy for this universe model, but this constant can take on any value, depending on the constant of integration λ .

In the following chapter I will consider the Milne universe.

Chapter 6

The Milne Model

In this chapter I will apply the results of chapter 3 to the Milne model. The Milne model is a model of an empty universe where the scale factor $a = t$ and the spatial curvature $k = -1$. I will consider this model because it is the only model that actually can be described using Minkowski coordinates. In other words, the metric describing this model will be the Minkowski metric if we use Minkowski coordinates. According to the logic of Faraoni and Cooperstock's article, the energy of this model should turn out to be zero, since they argue that flat spacetime implies the absence of energy. I will test if their method for calculating the energy gives a result that is in agreement with this.

First, I will show that the Milne model really can be described using Minkowski coordinates. Then, I will calculate the energy of this universe model, to see if Cooperstock and Faraoni's results make sense.

6.1 Change of coordinates to Minkowski coordinates

Expressed in RW-coordinates, the Milne model has the line element

$$ds^2 = dt^2 - t^2 \left[\frac{dr^2}{1+r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]. \quad (6.1)$$

We can define new coordinates

$$r = \frac{R}{\sqrt{T^2 - R^2}} \quad \text{and} \quad t = \sqrt{T^2 - R^2} \quad (6.2)$$

so that

$$R = tr \quad \text{and} \quad T = t\sqrt{1+r^2} \quad (6.3)$$

where we have used $c = 1$.

Using this set of coordinates, the Milne universe will appear as Minkowski space. Imagine looking at this universe from a position on the 'outside'. You will see a spatially flat space where testparticles move away from the origin with speeds between 0 and c . On the other hand, if we use coordinates including the cosmic time t , we would be describing the universe

as it would appear for an observer sitting on one of the testparticles. That observer would observe an expanding (FRW) universe with negative spatial curvature.

6.2 Equations of motion in Minkowski coordinates

The task at hand is to find equations of motion in the 'new' Minkowski coordinates R and T , for the Milne model. We begin again with equations (3.60) and (3.61):

$$R - 4\kappa V + \kappa\phi \frac{dV}{d\phi} = 0 \quad (6.4)$$

and

$$\ddot{\phi} + 3H\dot{\phi} + \frac{1}{6}R\phi + \frac{dV}{d\phi} = 0. \quad (6.5)$$

What does the Ricci scalar look like in this specific case? (Not to be confused with the radial coordinate!) We already know the answer to this, since we know that Minkowski spacetime is flat; that is $R = 0$. We check that this is really true: We have

$$R = 6 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right]. \quad (6.6)$$

Since the Ricci scalar is just that, a scalar, it makes no difference what coordinates we use to calculate its value. We choose to use the cosmic time coordinate t . From $a = t$ it follows that $\dot{a} = 1$ and $\ddot{a} = 0$. Remembering that $k = -1$ for the Milne model, we get

$$R = 6 \left[0 + \frac{1}{t^2} - \frac{1}{t^2} \right] \quad (6.7)$$

and

$$R = 0 \quad (6.8)$$

as expected. This means that the Milne model has flat spacetime.

Equation (3.60)

Using $R = 0$, equation (3.60) reduces to

$$-4\kappa V + \kappa\phi \frac{dV}{d\phi} = 0 \quad (6.9)$$

$$\Rightarrow \phi \frac{dV}{d\phi} - 4V = 0. \quad (6.10)$$

This is a differential equation, and it is easy to solve. The result is

$$\underline{V(\phi) = \lambda\phi^4}, \quad (6.11)$$

where λ is a constant.

Equation (3.61)

Equation (3.61) boils down to

$$\ddot{\phi} + 3\frac{1}{t}\dot{\phi} + \frac{dV}{d\phi} = 0 \quad (6.12)$$

Understanding that this might get ugly, we find $\ddot{\phi}$ first.

The differential operator $\frac{d}{dt}$:

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial R}{\partial t} \frac{\partial}{\partial R} + \frac{\partial T}{\partial t} \frac{\partial}{\partial T} \\ &= r \frac{\partial}{\partial R} + \sqrt{1+r^2} \frac{\partial}{\partial T} \\ &= \frac{R}{\sqrt{T^2 - R^2}} \frac{\partial}{\partial R} + \sqrt{1 + \frac{R^2}{T^2 - R^2}} \frac{\partial}{\partial T} \\ &= \frac{R}{\sqrt{T^2 - R^2}} \frac{\partial}{\partial R} + \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial}{\partial T} \end{aligned} \quad (6.13)$$

Now, we can find $\dot{\phi}$ and $\ddot{\phi}$:

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{R}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial R} + \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial T} \quad (6.14)$$

$$\begin{aligned} \ddot{\phi} &= \frac{d}{dt} \left[\frac{d\phi}{dt} \right] \\ &= \frac{R}{\sqrt{T^2 - R^2}} \frac{\partial}{\partial R} \left[\frac{R}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial R} + \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial T} \right] \\ &\quad + \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial}{\partial T} \left[\frac{R}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial R} + \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial T} \right] \end{aligned} \quad (6.15)$$

$$\begin{aligned} \Rightarrow \ddot{\phi} &= \frac{d}{dt} \left[\frac{d\phi}{dt} \right] \\ &= \frac{R}{\sqrt{T^2 - R^2}} \left[\frac{\partial}{\partial R} \left(\frac{R}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial R} \right) + \frac{\partial}{\partial R} \left(\frac{T}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial T} \right) \right] \\ &\quad + \frac{T}{\sqrt{T^2 - R^2}} \left[\frac{\partial}{\partial T} \left(\frac{R}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial R} \right) + \frac{\partial}{\partial T} \left(\frac{T}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial T} \right) \right] \end{aligned} \quad (6.16)$$

We find the four derivatives one at a time:

$$\begin{aligned}
\frac{\partial}{\partial R} \left[\frac{R}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial R} \right] &= \frac{R}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial R^2} \\
&\quad + \frac{\partial \phi}{\partial R} \left[\frac{1}{\sqrt{T^2 - R^2}} + R \left(-\frac{1}{2} \right) (T^2 - R^2)^{-\frac{3}{2}} (-2R) \right] \\
&= \frac{R}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial R^2} \\
&\quad + \frac{1}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial R} + R^2 (T^2 - R^2)^{-\frac{3}{2}} \frac{\partial \phi}{\partial R} \\
&= \frac{R}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial R^2} + \frac{T^2}{(c^2 T^2 - R^2)^{\frac{3}{2}}} \frac{\partial \phi}{\partial R} \tag{6.17}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial T} \left[\frac{T}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial T} \right] &= \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial R \partial T} \\
&\quad + \frac{\partial \phi}{\partial T} \left[T \left(-\frac{1}{2} \right) (T^2 - R^2)^{-\frac{3}{2}} (-2R) \right] \\
&= \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial R \partial T} + \frac{RT}{(T^2 - R^2)^{\frac{3}{2}}} \frac{\partial \phi}{\partial T} \tag{6.18}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial T} \left[\frac{R}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial R} \right] &= \frac{R}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial T \partial R} \\
&\quad + \frac{\partial \phi}{\partial R} \left[R \left(-\frac{1}{2} \right) (T^2 - R^2)^{-\frac{3}{2}} (2T) \right] \\
&= \frac{R}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial T \partial R} - \frac{RT}{(T^2 - R^2)^{\frac{3}{2}}} \frac{\partial \phi}{\partial R} \tag{6.19}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial T} \left[\frac{T}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial T} \right] &= \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial T^2} \\
&\quad + \frac{\partial \phi}{\partial T} \left[\frac{1}{\sqrt{T^2 - R^2}} + T \left(-\frac{1}{2} \right) (T^2 - R^2)^{-\frac{3}{2}} (2T) \right] \\
&= \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial T^2} + \frac{1}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial T} \\
&\quad - \frac{T^2}{(T^2 - R^2)^{\frac{3}{2}}} \frac{\partial \phi}{\partial T} \\
&= \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial T^2} - \frac{R^2}{(T^2 - R^2)^{\frac{3}{2}}} \frac{\partial \phi}{\partial T} \tag{6.20}
\end{aligned}$$

Our expression for $\ddot{\phi}$ now becomes

$$\begin{aligned}
\ddot{\phi} = \frac{R}{\sqrt{T^2 - R^2}} & \left[\frac{R}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial R^2} + \frac{T^2}{(T^2 - R^2)^{\frac{3}{2}}} \frac{\partial \phi}{\partial R} \right. \\
& \left. + \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial R \partial T} + \frac{RT}{(T^2 - R^2)^{\frac{3}{2}}} \frac{\partial \phi}{\partial T} \right] \\
+ \frac{T}{\sqrt{T^2 - R^2}} & \left[\frac{R}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial T \partial R} - \frac{RT}{(T^2 - R^2)^{\frac{3}{2}}} \frac{\partial \phi}{\partial R} \right. \\
& \left. + \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial^2 \phi}{\partial T^2} - \frac{R^2}{(T^2 - R^2)^{\frac{3}{2}}} \frac{\partial \phi}{\partial T} \right] \tag{6.21}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \ddot{\phi} = \frac{R^2}{T^2 - R^2} \frac{\partial^2 \phi}{\partial R^2} + \frac{RT^2}{(T^2 - R^2)^2} \frac{\partial \phi}{\partial R} + \frac{RT}{T^2 - R^2} \frac{\partial^2 \phi}{\partial R \partial T} \\
+ \frac{R^2 T}{(T^2 - R^2)^2} \frac{\partial \phi}{\partial T} + \frac{TR}{T^2 - R^2} \frac{\partial^2 \phi}{\partial T \partial R} - \frac{RT^2}{(T^2 - R^2)^2} \frac{\partial \phi}{\partial R} \\
+ \frac{T^2}{T^2 - R^2} \frac{\partial^2 \phi}{\partial T^2} - \frac{TR^2}{(T^2 - R^2)^2} \frac{\partial \phi}{\partial T} \tag{6.22}
\end{aligned}$$

We finally get

$$\ddot{\phi} = \frac{R^2}{T^2 - R^2} \frac{\partial^2 \phi}{\partial R^2} + \frac{T^2}{T^2 - R^2} \frac{\partial^2 \phi}{\partial T^2} + \frac{2RT}{T^2 - R^2} \frac{\partial^2 \phi}{\partial R \partial T}. \tag{6.23}$$

The second term of equation (6.12) is

$$3 \frac{1}{t} \dot{\phi} = 3 \frac{1}{\sqrt{T^2 - R^2}} \frac{R}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial R} + 3 \frac{1}{\sqrt{T^2 - R^2}} \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial T}. \tag{6.24}$$

Inserting all this into (6.12), we get

$$\begin{aligned}
& \frac{R^2}{T^2 - R^2} \frac{\partial^2 \phi}{\partial R^2} + \frac{T^2}{T^2 - R^2} \frac{\partial^2 \phi}{\partial T^2} + \frac{2RT}{T^2 - R^2} \frac{\partial^2 \phi}{\partial R \partial T} \\
+ 3 \frac{1}{\sqrt{T^2 - R^2}} \frac{R}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial R} + 3 \frac{1}{\sqrt{T^2 - R^2}} \frac{T}{\sqrt{T^2 - R^2}} \frac{\partial \phi}{\partial T} + \frac{dV}{d\phi} = 0 \tag{6.25}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{R^2}{T^2 - R^2} \frac{\partial^2 \phi}{\partial R^2} + \frac{T^2}{T^2 - R^2} \frac{\partial^2 \phi}{\partial T^2} + \frac{2RT}{T^2 - R^2} \frac{\partial^2 \phi}{\partial R \partial T} \\
+ \frac{3R}{T^2 - R^2} \frac{\partial \phi}{\partial R} + \frac{3T}{T^2 - R^2} \frac{\partial \phi}{\partial T} + \frac{dV}{d\phi} = 0 \tag{6.26}
\end{aligned}$$

Next, we multiply through by $T^2 - R^2$:

$$\Rightarrow R^2 \frac{\partial^2 \phi}{\partial R^2} + T^2 \frac{\partial^2 \phi}{\partial T^2} + 2RT \frac{\partial^2 \phi}{\partial R \partial T} + 3R \frac{\partial \phi}{\partial R} + 3T \frac{\partial \phi}{\partial T} + (T^2 - R^2) \frac{dV}{d\phi} = 0 \quad (6.27)$$

From equation (6.11) we find that

$$\frac{dV}{d\phi} = 4\lambda\phi^3, \quad (6.28)$$

and we finally arrive at

$$\underline{R^2 \frac{\partial^2 \phi}{\partial R^2} + T^2 \frac{\partial^2 \phi}{\partial T^2} + 2RT \frac{\partial^2 \phi}{\partial R \partial T} + 3R \frac{\partial \phi}{\partial R} + 3T \frac{\partial \phi}{\partial T} + (T^2 - R^2) 4\lambda\phi^3 = 0.} \quad (6.29)$$

Now, we wish to find a solution to the differential equation (6.29). This is a second order, partial, non-linear differential equation, and finding a solution is not trivial. Instead, we first solve equation (6.12) in its original form, and then make the coordinate change when we have found a solution expressed by t . The equation to be solved is

$$\ddot{\phi} + 3\frac{1}{t}\dot{\phi} + \frac{dV}{d\phi} = 0. \quad (6.30)$$

Using

$$\frac{dV}{d\phi} = 4\lambda\phi^3, \quad (6.31)$$

we get

$$\ddot{\phi} + 3\frac{1}{t}\dot{\phi} + 4\lambda\phi^3 = 0. \quad (6.32)$$

We now assume a solution on the form

$$\phi(t) = Ct^p \quad (6.33)$$

where C and p are constants.

It then follows that

$$\dot{\phi} = Cpt^{p-1}, \quad (6.34)$$

$$\ddot{\phi} = Cp(p-1)t^{p-2} \quad (6.35)$$

and

$$\phi^3 = C^3 t^{3p}. \quad (6.36)$$

Inserting these expressions into the differential equation (6.30) gives the result

$$Cp(p-1)t^{p-2} + 3\frac{1}{t}Cpt^{p-1} + 4\lambda C^3t^{3p} = 0 \quad (6.37)$$

$$Cp(p-1)t^{p-2} + 3Cpt^{p-2} + 4\lambda C^3t^{3p} = 0. \quad (6.38)$$

We can determine the value of p :

$$p-2 = 3p \quad \Rightarrow \quad 2p = -2 \quad \Rightarrow \quad \underline{\underline{p = -1}} \quad (6.39)$$

...and C :

$$Cp(p-1) + 3Cp + 4\lambda C^3 = 0 \quad \Rightarrow \quad 2-3 + 4\lambda C^2 = 0 \quad (6.40)$$

$$\Rightarrow C^2 = \frac{1}{4\lambda} \quad \Rightarrow \quad \underline{\underline{C = \frac{1}{2\sqrt{\lambda}}}} \quad (6.41)$$

It is easy to check that

$$\phi(t) = \frac{1}{2\sqrt{\lambda}} \frac{1}{t} \quad (6.42)$$

really is a solution to (6.30). Using the variables T and R , this scalar field becomes

$$\phi(R, T) = \frac{1}{2\sqrt{\lambda}} \frac{1}{\sqrt{T^2 - R^2}}. \quad (6.43)$$

It is straightforward to check that this is a solution of the not-so-nice equation (6.29).

We start out by calculating the differentials:

$$\frac{\partial \phi}{\partial R} = \frac{1}{2\sqrt{\lambda}} \left(-\frac{1}{2}\right) (T^2 - R^2)^{-\frac{3}{2}} (-2R) = \frac{1}{2\sqrt{\lambda}} R (T^2 - R^2)^{-\frac{3}{2}} \quad (6.44)$$

$$\frac{\partial \phi}{\partial T} = \frac{1}{2\sqrt{\lambda}} \left(-\frac{1}{2}\right) (T^2 - R^2)^{-\frac{3}{2}} 2T = -\frac{1}{2\sqrt{\lambda}} T (T^2 - R^2)^{-\frac{3}{2}} \quad (6.45)$$

$$\begin{aligned} \frac{\partial^2 \phi}{\partial T^2} &= -\frac{1}{2\sqrt{\lambda}} \left[(T^2 - R^2)^{-\frac{3}{2}} + T \left(-\frac{3}{2}\right) (T^2 - R^2)^{-\frac{5}{2}} 2T \right] \\ &= -\frac{1}{2\sqrt{\lambda}} (T^2 - R^2)^{-\frac{3}{2}} + \frac{1}{2\sqrt{\lambda}} 3T^2 (T^2 - R^2)^{-\frac{5}{2}} \end{aligned} \quad (6.46)$$

$$\begin{aligned} \frac{\partial^2 \phi}{\partial R^2} &= \frac{1}{2\sqrt{\lambda}} \left[(T^2 - R^2)^{-\frac{3}{2}} + R \left(-\frac{3}{2}\right) (T^2 - R^2)^{-\frac{5}{2}} (-2R) \right] \\ &= \frac{1}{2\sqrt{\lambda}} (T^2 - R^2)^{-\frac{3}{2}} + \frac{1}{2\sqrt{\lambda}} 3R^2 (T^2 - R^2)^{-\frac{5}{2}} \end{aligned} \quad (6.47)$$

$$\frac{\partial^2 \phi}{\partial T \partial R} = \frac{1}{2\sqrt{\lambda}} R \left(-\frac{3}{2}\right) (T^2 - R^2)^{-\frac{5}{2}} 2T = -\frac{1}{2\sqrt{\lambda}} 3RT (T^2 - R^2)^{-\frac{5}{2}} \quad (6.48)$$

The last term of equation (6.29) becomes

$$(T^2 - R^2) 4\lambda\phi^3 = (T^2 - R^2) 4\lambda \frac{1}{8\lambda\sqrt{\lambda}} (T^2 - R^2)^{-\frac{3}{2}} = \frac{1}{2\sqrt{\lambda}} (T^2 - R^2)^{-\frac{1}{2}}. \quad (6.49)$$

Substituting all this into the differential equation gives:

$$\begin{aligned} & \frac{1}{2\sqrt{\lambda}} R^2 (T^2 - R^2)^{-\frac{3}{2}} + \frac{1}{2\sqrt{\lambda}} 3R^4 (T^2 - R^2)^{-\frac{5}{2}} \\ & - \frac{1}{2\sqrt{\lambda}} T^2 (T^2 - R^2)^{-\frac{3}{2}} + \frac{1}{2\sqrt{\lambda}} 3T^4 (T^2 - R^2)^{-\frac{5}{2}} \\ & - \frac{1}{2\sqrt{\lambda}} 6R^2 T^2 (T^2 - R^2)^{-\frac{5}{2}} + \frac{1}{2\sqrt{\lambda}} 3R^2 (T^2 - R^2)^{-\frac{3}{2}} \\ & - \frac{1}{2\sqrt{\lambda}} 3T^2 (T^2 - R^2)^{-\frac{3}{2}} + \frac{1}{2\sqrt{\lambda}} (T^2 - R^2)^{-\frac{1}{2}} = 0 \end{aligned} \quad (6.50)$$

Multiplying each term by $2\sqrt{\lambda} (T^2 - R^2)^{\frac{5}{2}}$ gives

$$\begin{aligned} & R^2 (T^2 - R^2) + 3R^4 - T^2 (T^2 - R^2) + 3T^4 - 6R^2 T^2 \\ & + 3R^2 (T^2 - R^2) - 3T^2 (T^2 - R^2) + (T^2 - R^2) \\ & = R^2 T^2 - R^4 + 3R^4 - T^4 + R^2 T^2 + 3T^4 - 6R^2 T^2 \\ & + 3R^2 T^2 - 3R^4 - 3T^4 + 3R^2 T^2 + T^4 - 2R^2 T^2 + R^4 = 0 \end{aligned} \quad (6.51)$$

All the terms on the left hand side cancel out, and we see that

$$\phi(R, T) = \frac{1}{2\sqrt{\lambda}} \frac{1}{\sqrt{T^2 - R^2}} \quad (6.52)$$

satisfies equation (6.29).

6.3 The energy of the Milne universe

We will now return to the Robertson-Walker interpretation of the Milne model. Now that we have an expression for the scalar field $\phi(t)$, we can find the actual value of the energy of the Milne universe. We start out with the energy equation (3.59):

$$E = \frac{1}{2}\varphi'^2 - \frac{18}{\kappa^2}\psi'^2 + \frac{3m^2}{\kappa}\psi^2\varphi^2 + \lambda\varphi^4 + \frac{36}{\kappa^2}V_0\psi^4 + \frac{1}{2}k\varphi^2 - \frac{18}{\kappa^2}k\psi^2 \quad (6.53)$$

In this case, we have $k = -1$, $V_0 = 0$ and $m = 0$. We get

$$E = \frac{1}{2}\varphi'^2 - \frac{18}{\kappa^2}\psi'^2 + \lambda\varphi^4 - \frac{1}{2}\varphi^2 + \frac{18}{\kappa^2}\psi^2 \quad (6.54)$$

$$\Rightarrow E = \frac{1}{2} \left[\frac{d\varphi}{d\eta} \right]^2 - \frac{18}{\kappa^2} \left[\frac{d\psi}{d\eta} \right]^2 + \lambda\varphi^4 - \frac{1}{2}\varphi^2 + \frac{18}{\kappa^2}\psi^2. \quad (6.55)$$

Using $\frac{d}{d\eta} = a \frac{d}{dt}$ leads to

$$E = \frac{1}{2} \left[a \frac{d\varphi}{dt} \right]^2 - \frac{18}{\kappa^2} \left[a \frac{d\psi}{dt} \right]^2 + \lambda\varphi^4 - \frac{1}{2}\varphi^2 + \frac{18}{\kappa^2}\psi^2. \quad (6.56)$$

Substituting $\psi = \sqrt{\frac{\kappa}{6}}a$ and $\varphi = a\phi$ gives

$$E = \frac{1}{2}a^2 \left[\frac{d}{dt} (a\phi) \right]^2 - \frac{18}{\kappa^2}a^2 \left[\frac{d}{dt} \left(\sqrt{\frac{\kappa}{6}}a \right) \right]^2 + \lambda a^4 \phi^4 - \frac{1}{2}a^2 \phi^2 + \frac{18}{\kappa^2} \frac{\kappa}{6} a^2. \quad (6.57)$$

For the Milne model, the scale factor $a = t$, and the scalar field $\phi(t) = \frac{1}{2\sqrt{\lambda}} \frac{1}{t}$:

$$\begin{aligned} E &= \frac{1}{2}t^2 \left[\frac{d}{dt} \left(t \frac{1}{2\sqrt{\lambda}} \frac{1}{t} \right) \right]^2 - \frac{18}{\kappa^2}t^2 \frac{\kappa}{6} \left[\frac{d}{dt} (t) \right]^2 + \lambda t^4 \left(\frac{1}{2\sqrt{\lambda}} \right)^4 \frac{1}{t^4} - \frac{1}{2}t^2 \left(\frac{1}{2\sqrt{\lambda}} \right)^2 \frac{1}{t^2} + \frac{3}{\kappa}t^2 \\ &= 0 - \frac{3}{\kappa}t^2 + \lambda \frac{1}{16\lambda^2} - \frac{1}{2} \frac{1}{4\lambda} + \frac{3}{\kappa}t^2 \\ &= \frac{1}{16\lambda} - \frac{1}{8\lambda} = \frac{1}{16\lambda} - \frac{2}{16\lambda} \\ &= -\frac{1}{16\lambda} \end{aligned} \quad (6.58)$$

This is under no circumstances zero.

This is an interesting result, since it demonstrates that Faraoni and Cooperstock's argumentation is self contradictory. They argue that the energy of the universe is 0, based on the fact that Minkowski space has zero energy. But here I have shown that Minkowski space, according to their method, has non-zero energy.

Chapter 7

Summary and Conclusions

In Newtonian cosmology, conservation of energy is not a problem. But our universe is not Newtonian, we need the general theory of relativity in order to make a satisfactory description of the universe. In the first chapter, I look at the dynamics and total energy of a Newtonian universe, and show that a cosmological constant and accelerated expansion is completely consistent with energy conservation. Then I explain why the Newtonian analysis is not sufficient in describing our universe, and I also look at the problems that arise when we are dealing with curved spacetimes. There is nothing revolutionary or 'new' about the contents of this introduction to my work, but a similar presentation of the Newtonian case has not been made before.

I have used the method of Faraoni and Cooperstock [1] to find an expression for the total energy of the universe. While they in their calculations assume that $k = 0$, I've followed the same procedure in a more general way, allowing for any spatial curvature. Faraoni and Cooperstock's equations are found to represent a special case of the equations I've derived, by setting $k = 0$. I've found that for any k , the energy of the universe should be constant. This is in agreement with Faraoni and Cooperstock's results. Next, they argue that this constant value should be zero. They use two different arguments to support this claim. First, they state that 'open or critically open FRW cosmologies have Minkowski space as their asymptotic state'. This might at first sound reasonable, because in an expanding universe filled with matter the density will approach zero with increasing time. In the infinite future, one could say that the density will reach zero, and then the universe will be empty. In an empty universe spacetime is flat (Minkowski). They show that Minkowski space represents zero energy by setting up the Schwarzschild metric:

$$ds^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (7.1)$$

Here, the mass parameter m gives the total energy including the contribution from gravity. We see that if m is set to 0, the metric becomes the Minkowski metric that represents flat spacetime. Furthermore, they argue that since the energy of this universe is 0 at one point, it must be zero at all times, due to conservation of energy.

The second argument is this: Since they have shown that the energy of the universe is

constant, they simply say that they can 'choose' this constant to be zero. They support this by saying that the zero level is arbitrary.

However, there are problems with both these lines of reasoning: First, they state that 'open or critically open FRW cosmologies have Minkowski space as their asymptotic state'. This might at first sound reasonable, because in an expanding universe filled with matter the density will approach zero with increasing time. But the density will never become exactly zero! If we look at the FRW metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] \quad (7.2)$$

and let $t \rightarrow \infty$, it won't approach the Minkowski metric

$$ds^2 = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (7.3)$$

unless the universe model in question can be described using Minkowski coordinates in the first place. Actually, there is only one such model, namely the Milne model, which I discuss in chapter 6. For the Milne model, the scale factor $a \propto t$, while for instance the Einstein-de Sitter model has $a \propto t^{2/3}$.

For the de Sitter universe, my calculations show that Cooperstock and Faraoni's method gives $E = 0$. This is in agreement with their claims. Other methods for calculating the energy, for example the pseudotensor method, have also given the result of zero energy.

For Einstein's static universe, I get that the energy is constant. This is in agreement with Cooperstock and Faraoni's first claim. This model has positive spatial curvature, so the second step of F&C's argument does not apply to it. Nevertheless, other ways of reasoning suggest that the energy of the static universe should be zero, so if F&C's method holds, it should give the result of zero energy. I have shown that their method gives zero energy only for one particular choice of the constant of integration λ .

The Milne model has flat spacetime, and can be described in Minkowski coordinates. According to F&C's logic, their method for calculating the energy of this model should give the result of zero energy. But my calculations show that this is not the case. I get that the energy is constant, but it can never be zero.

There seems to be problems with both steps of Faraoni and Cooperstock's argument. Their method for calculating the energy of the universe fails in two ways: First, it fails in the sense that it produces different results than other methods do. And maybe even more disturbing, their method produces results that are in conflict with their own claims. According to their method, Minkowski space does not have zero energy.

F&C give two independent reasons why the constant energy should be zero. There is something wrong with both ways of reasoning. When it comes to their claim that open and critically open FRW universes have zero energy, I'd say that their way of reasoning is not good. Their argument breaks down when they state that FRW cosmologies have Minkowski space as their asymptotic state. Even if it is true that the density of such a universe approaches zero as it expands, the total amount of matter will not change. The universe will at all times be described by the FRW metric, and this metric will not approach the Minkowski metric when t goes to infinity. F&C also state that the zero level is arbitrary, and that we can choose the

value of the constant energy to be zero. It would be nice if this were true, but unfortunately, it does not seem to work that way. This becomes clear if we consider a model of the universe which contains vacuum energy only. Vacuum energy has a clear physical meaning, and we can not make it go away by simply defining that the energy is zero.

I conclude that F&C's method for calculating the energy of the universe is not a success. The fact that energy conservation as we know it is not defined in curved spacetimes is mind bending. Maybe we need to start thinking about energy and energy conservation in a new and different way when applied to the universe as a whole. I am sure there is a perfectly logical solution to this problem, I think we are just missing some pieces of the puzzle.

Appendix A

The Isotropic Form of the Robertson-Walker Line Element

Here, I will show how to get from the ordinary form of the Robertson-Walker line element to the isotropic form.

The homogeneous isotropic universe can be described by the familiar Robertson-Walker line element

$$ds^2 = dt^2 - R^2(t) \left[\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2 \theta d\phi^2 \right] \quad (\text{A.1})$$

where $R(t)$ is the scale factor and k is spatial curvature. We introduce a new radial coordinate defined by

$$\bar{r} \equiv \frac{r}{1 + \frac{kr^2}{4}}. \quad (\text{A.2})$$

Differentiating both sides, we get

$$d\bar{r} = \frac{1 + \frac{kr^2}{4} - r\frac{kr}{2}}{\left(1 + \frac{kr^2}{4}\right)^2} dr = \frac{1 - \frac{kr^2}{4}}{\left(1 + \frac{kr^2}{4}\right)^2} dr. \quad (\text{A.3})$$

From this, we have that

$$\begin{aligned} 1 - k\bar{r}^2 &= 1 - \frac{kr^2}{\left(1 + \frac{kr^2}{4}\right)^2} \\ &= \frac{\left(1 + \frac{kr^2}{4}\right)^2 - kr^2}{\left(1 + \frac{kr^2}{4}\right)^2} \end{aligned} \quad (\text{A.4})$$

and

$$\begin{aligned}
\frac{d\bar{r}^2}{1 - k\bar{r}^2} &= \frac{\left(1 - \frac{kr^2}{4}\right)^2}{\left(1 + \frac{kr^2}{4}\right)^4} dr^2 \cdot \frac{\left(1 + \frac{kr^2}{4}\right)^2}{\left(1 + \frac{kr^2}{4}\right)^2 - kr^2} \\
&= \frac{\left(1 - \frac{kr^2}{4}\right)^2}{\left(1 + \frac{kr^2}{4}\right)^2} \cdot \frac{dr^2}{\left(1 + \frac{kr^2}{4}\right)^2 - kr^2} \\
&= \frac{dr^2}{\left(1 + \frac{kr^2}{4}\right)^2} \cdot \frac{\left(1 - 2\frac{kr^2}{4} + \frac{k^2r^4}{16}\right)}{1 + 2\frac{kr^2}{4} + \frac{k^2r^4}{16} - kr^2} \\
&= \frac{dr^2}{\left(1 + \frac{kr^2}{4}\right)^2} \cdot \frac{1 - \frac{1}{2}kr^2 + \frac{1}{16}k^2r^4}{1 - \frac{1}{2}kr^2 + \frac{1}{16}k^2r^4} \\
&= \frac{dr^2}{\left(1 + \frac{kr^2}{4}\right)^2}. \tag{A.5}
\end{aligned}$$

Substituting this into the line element gives

$$ds^2 = dt^2 - R^2(t) \left[\frac{dr^2}{\left(1 + \frac{kr^2}{4}\right)^2} + \frac{r^2}{\left(1 + \frac{kr^2}{4}\right)^2} d\theta^2 + \frac{r^2}{\left(1 + \frac{kr^2}{4}\right)^2} \sin^2 \theta d\phi^2 \right] \tag{A.6}$$

and finally

$$ds^2 = dt^2 - \frac{R^2}{\left(1 + \frac{kr^2}{4}\right)^2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2). \tag{A.7}$$

This is the isotropic form of the Robertson-Walker line element. Expressed in Cartesian coordinates, this becomes

$$ds^2 = dt^2 - \frac{R^2}{\left(1 + \frac{kr^2}{4}\right)^2} (dx^2 + dy^2 + dz^2). \tag{A.8}$$

Appendix B

Detailed Calculation of the Energy Integral

Here, I will show in detail how we arrive at $E = 0$ in section 1.7.

We start out with

$$E = 4\pi \int_0^\infty (-g)^{\frac{1}{2}} (T_0^0 + t_0^0) r^2 dr \quad (\text{B.1})$$

$$= \frac{3R}{2} \int_0^\infty \frac{r^2 dr}{\left(1 + \frac{r^2}{4}\right)^2} - \frac{R}{2} \int_0^\infty \frac{r^4 dr}{\left(1 + \frac{r^2}{4}\right)^3}. \quad (\text{B.2})$$

Let us define

$$I_i \equiv \int_0^\infty \frac{r^2 dr}{\left(1 + \frac{r^2}{4}\right)^2} \quad (\text{B.3})$$

and

$$I_{ii} \equiv \int_0^\infty \frac{r^4 dr}{\left(1 + \frac{r^2}{4}\right)^3} \quad (\text{B.4})$$

so that we can write

$$E = \left(\frac{3R}{2} \cdot I_i\right) - \left(\frac{R}{2} \cdot I_{ii}\right). \quad (\text{B.5})$$

We calculate the first integral first:

$$I_i = \int_0^\infty \frac{r^2}{\left(1 + \frac{r^2}{4}\right)^2} dr = \int_0^\infty \frac{r^2}{\left(\frac{1}{4}(4 + r^2)\right)^2} dr = 16 \int_0^\infty \frac{r^2}{(4 + r^2)^2} dr \quad (\text{B.6})$$

We proceed using the method of partial fractions. We wish to rewrite the integrand as two fractions, on the form

$$\frac{r^2}{(r^2 + 4)^2} = \frac{Ar + B}{r^2 + 4} + \frac{Cr + D}{(r^2 + 4)^2}. \quad (\text{B.7})$$

Multiplying both sides by $(r^2 + 4)^2$ gives

$$\begin{aligned} r^2 &= (Ar + B)(r^2 + 4) + Cr + D \\ &= Ar^3 + 4Ar + Br^2 + 4B + Cr + D \\ &= Ar^3 + Br^2 + (4A + C)r + (4B + D). \end{aligned} \quad (\text{B.8})$$

For this to be satisfied, we must choose $A = 0$, $C = 0$, $B = 1$ and $D = -4$, so that

$$\frac{r^2}{(r^2 + 4)^2} = \frac{1}{r^2 + 4} - \frac{4}{(r^2 + 4)^2} \quad (\text{B.9})$$

and

$$I_i = 16 \int_0^\infty \frac{r^2}{(r^2 + 4)^2} dr = 16 \left(\int_0^\infty \frac{1}{r^2 + 4} dr - \int_0^\infty \frac{4}{(r^2 + 4)^2} dr \right). \quad (\text{B.10})$$

Let us define

$$I_A \equiv \int_0^\infty \frac{1}{r^2 + 4} dr \quad (\text{B.11})$$

and

$$I_B \equiv \int_0^\infty \frac{4}{(r^2 + 4)^2} dr \quad (\text{B.12})$$

so that

$$I_i = 16(I_A - I_B). \quad (\text{B.13})$$

Now, the first integral can be written

$$\int \frac{1}{r^2 + 4} dr = \frac{1}{4} \int \frac{1}{1 + \frac{r^2}{4}} dr = \frac{1}{4} \int \frac{2}{1 + u^2} du \quad (\text{B.14})$$

where we have made the substitution

$$u^2 = \frac{r^2}{4} \quad \Rightarrow r^2 = 4u^2 \quad \Rightarrow r = 2u \quad \Rightarrow dr = 2 du. \quad (\text{B.15})$$

Carrying out the integration, we get

$$\frac{1}{2} \int \frac{1}{1 + u^2} du = \frac{1}{2} [\arctan u] = \frac{1}{2} \left[\arctan \left(\frac{r}{2} \right) \right], \quad (\text{B.16})$$

so that

$$I_A = \int_0^\infty \frac{1}{r^2 + 4} dr = \frac{1}{2} \lim_{b \rightarrow \infty} \left[\arctan \left(\frac{r}{2} \right) \right]_0^b = \frac{1}{2} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{4}. \quad (\text{B.17})$$

The second integral also needs some work:

$$I_B = \int_0^\infty \frac{4}{(r^2 + 4)^2} dr = 4 \int_0^\infty \frac{1}{(4(\frac{1}{4}r^2 + 1))^2} dr = \frac{1}{4} \int_0^\infty \frac{1}{(\frac{1}{4}r^2 + 1)^2} dr \quad (\text{B.18})$$

Making the same substitution as for I_A , we find

$$\frac{1}{4} \int \frac{1}{(\frac{1}{4}r^2 + 1)^2} dr = \frac{1}{4} \int \frac{2}{(1 + u^2)^2} du = \frac{1}{2} \int \frac{1}{(1 + u^2)^2} du. \quad (\text{B.19})$$

We will now make use of a recursion formula: If

$$I_m = \int \frac{du}{(1 + u^2)^m} \quad (\text{B.20})$$

then

$$I_m = \frac{1}{2(m-1)} \frac{u}{(1 + u^2)^{m-1}} + \frac{2m-3}{2(m-1)} I_{m-1}. \quad (\text{B.21})$$

In our case, $m = 2$. We need to find I_1 first:

$$I_1 = \int \frac{du}{1 + u^2} = \arctan u. \quad (\text{B.22})$$

Now

$$\begin{aligned} I_2 &= \int \frac{du}{(1 + u^2)^2} = \frac{1}{2(2-1)} \frac{u}{(1 + u^2)} + \frac{1}{2} \arctan u = \frac{1}{2} \frac{u}{(1 + u^2)} + \frac{1}{2} \arctan u \\ &= \frac{1}{2} \frac{\frac{r}{2}}{\left(1 + \left(\frac{r}{2}\right)^2\right)} + \frac{1}{2} \arctan \left(\frac{r}{2}\right) = \frac{1}{4} \frac{r}{\left(1 + \frac{r^2}{4}\right)} + \frac{1}{2} \arctan \left(\frac{r}{2}\right), \end{aligned} \quad (\text{B.23})$$

so we finally get

$$I_B = \frac{1}{2} \lim_{b \rightarrow \infty} [I_2]_{r=0}^{r=b} = \frac{1}{2} \lim_{b \rightarrow \infty} \left[\frac{1}{4} \frac{r}{\left(1 + \frac{r^2}{4}\right)} + \frac{1}{2} \arctan \left(\frac{r}{2}\right) \right]_0^b = \frac{1}{2} \left[\left(0 + \frac{\pi}{4}\right) - 0 \right] = \frac{\pi}{8}. \quad (\text{B.24})$$

The first integral of equation (B.2) is then

$$I_i = \int_0^\infty \frac{r^2 dr}{\left(1 + \frac{r^2}{4}\right)^2} = 16(I_A - I_B) = 16 \left(\frac{\pi}{4} - \frac{\pi}{8} \right) = 4\pi - 2\pi = \underline{\underline{2\pi}}. \quad (\text{B.25})$$

Let us determine the second integral of equation (B.2) next. First of all, we rewrite it:

$$I_{ii} = \int_0^\infty \frac{r^4}{\left(1 + \frac{r^2}{4}\right)^3} dr = \int_0^\infty \frac{r^4}{\left(\frac{1}{4}(4 + r^2)\right)^3} dr = 4^3 \int_0^\infty \frac{r^4}{(4 + r^2)^3} dr. \quad (\text{B.26})$$

We rewrite the integrand using the method of partial fractions:

$$\frac{r^4}{(4 + r^2)^3} = \frac{Ar + B}{r^2 + 4} + \frac{Cr + D}{(r^2 + 4)^2} + \frac{Er + F}{(r^2 + 4)^3} \quad (\text{B.27})$$

After straightforward but tedious calculations, we find that this is satisfied if we choose $A = 0$, $B = 1$, $C = 0$, $D = -8$, $E = 0$ and $F = 16$. Thus, we have

$$\frac{r^4}{(4 + r^2)^3} = \frac{1}{r^2 + 4} - \frac{8}{(r^2 + 4)^2} + \frac{16}{(r^2 + 4)^3} \quad (\text{B.28})$$

so that our integral becomes

$$I_{ii} = 4^3 \int_0^\infty \frac{r^4}{(4 + r^2)^3} dr = 4^3 \left(\int_0^\infty \frac{dr}{r^2 + 4} - 8 \int_0^\infty \frac{dr}{(r^2 + 4)^2} + 16 \int_0^\infty \frac{dr}{(r^2 + 4)^3} \right). \quad (\text{B.29})$$

Oh, joy, three new integrals! Let us define

$$I_C \equiv \int_0^\infty \frac{dr}{r^2 + 4} \quad (\text{B.30})$$

$$I_D \equiv \int_0^\infty \frac{dr}{(r^2 + 4)^2} \quad (\text{B.31})$$

and

$$I_E \equiv \int_0^\infty \frac{dr}{(r^2 + 4)^3} \quad (\text{B.32})$$

so that

$$I_{ii} = 4^3 (I_C - 8I_D + 16I_E). \quad (\text{B.33})$$

Carrying out the integrations one at a time, we get (observe that $I_C = I_A$):

$$I_C \equiv \int_0^\infty \frac{dr}{r^2 + 4} = \frac{\pi}{4} \quad (\text{B.34})$$

and

$$I_D = \int_0^\infty \frac{dr}{(r^2 + 4)^2} = \int_0^\infty \frac{dr}{\left(4\left(\frac{1}{4}r^2 + 1\right)\right)^2} = \frac{1}{16} \int_0^\infty \frac{dr}{\left(\frac{1}{4}r^2 + 1\right)^2}. \quad (\text{B.35})$$

Using our favorite substitution once again, namely $u^2 = \frac{1}{4}r^2$, $dr = 2 du$, and the previous result I_2 , we find

$$\begin{aligned} \frac{1}{16} \int \frac{dr}{\left(\frac{1}{4}r^2 + 1\right)^2} &= \frac{1}{16} \int \frac{2}{(u^2 + 1)^2} du = \frac{1}{8} \int \frac{1}{(u^2 + 1)^2} du = \frac{1}{8} \left[\frac{1}{2} \frac{u}{(1 + u^2)} + \frac{1}{2} \arctan u \right] \\ &= \frac{1}{16} \frac{u}{(1 + u^2)} + \frac{1}{16} \arctan u = \frac{1}{16} \frac{\frac{r}{2}}{\left(1 + \frac{r^2}{4}\right)} + \frac{1}{16} \arctan \left(\frac{r}{2}\right) \\ &= \frac{1}{32} \frac{r}{\left(1 + \frac{r^2}{4}\right)} + \frac{1}{16} \arctan \left(\frac{r}{2}\right). \end{aligned} \quad (\text{B.36})$$

We get

$$I_D = \lim_{b \rightarrow \infty} \left[\frac{1}{32} \frac{r}{\left(1 + \frac{r^2}{4}\right)} + \frac{1}{16} \arctan \left(\frac{r}{2}\right) \right]_0^b = \left(0 + \frac{1}{16} \cdot \frac{\pi}{2}\right) - (0 + 0) = \frac{\pi}{32}. \quad (\text{B.37})$$

The last one needs a little more work:

$$\begin{aligned} I_E &= \int_0^\infty \frac{dr}{(r^2 + 4)^3} = \int_0^\infty \frac{dr}{\left(4\left(\frac{1}{4}r^2 + 1\right)\right)^3} = \frac{1}{4^3} \int_0^\infty \frac{dr}{\left(\frac{1}{4}r^2 + 1\right)^3} \\ &= \frac{1}{4^3} \int_0^\infty \frac{2}{(u^2 + 1)^3} du = \frac{2}{4^3} \int_0^\infty \frac{1}{(u^2 + 1)^3} du. \end{aligned} \quad (\text{B.38})$$

Here, we have also used the same substitution. Using the same recursion formula as before, we find that

$$\begin{aligned} I_1 &= \arctan u \\ I_2 &= \frac{1}{2} \frac{u}{(1 + u^2)} + \frac{1}{2} \arctan u \\ I_3 &= \frac{1}{4} \frac{u}{(1 + u^2)^2} + \frac{3}{4} \left[\frac{1}{2} \frac{u}{(1 + u^2)} + \frac{1}{2} \arctan u \right] \\ &= \frac{1}{4} \frac{u}{(1 + u^2)^2} + \frac{3}{8} \frac{u}{(1 + u^2)} + \frac{3}{8} \arctan u. \end{aligned} \quad (\text{B.39})$$

The third integral then becomes

$$\begin{aligned} I_E &= \left[\frac{2}{4^3} I_3 \right]_0^\infty = \frac{2}{4^3} \left[\frac{1}{4} \frac{u}{(1 + u^2)^2} + \frac{3}{8} \frac{u}{(1 + u^2)} + \frac{3}{8} \arctan u \right]_0^\infty \\ &= \frac{2}{4^4} \left[\frac{r}{(1 + r^2)^2} + \frac{3}{2} \frac{r}{(1 + r^2)} + \frac{3}{2} \arctan r \right]_0^\infty = \left(\frac{2}{4^4} \cdot \frac{3}{2} \cdot \frac{\pi}{2} \right) - 0 = \frac{3\pi}{2 \cdot 4^4}. \end{aligned} \quad (\text{B.40})$$

Finally, the second integral of equation (B.2) becomes

$$\begin{aligned} \int_0^\infty \frac{r^4}{\left(1 + \frac{r^2}{4}\right)^3} dx &= 4^3 [I_C - 8I_D + 16I_E] = 4^3 \left[\frac{\pi}{4} - \frac{8\pi}{32} + 16 \frac{3\pi}{2 \cdot 4^4} \right] \\ &= \frac{4^5 \cdot 3\pi}{2 \cdot 4^4} = \frac{12\pi}{2} = \underline{\underline{6\pi}}. \end{aligned} \quad (\text{B.41})$$

To sum up, I have just shown that

$$\int_0^\infty \frac{r^2 dr}{\left(1 + \frac{r^2}{4}\right)^2} = 2\pi \quad (\text{B.42})$$

and

$$\int_0^\infty \frac{r^4 dr}{\left(1 + \frac{r^2}{4}\right)^3} = 6\pi, \quad (\text{B.43})$$

so that

$$\frac{3R}{2} \int_0^\infty \frac{r^2 dr}{\left(1 + \frac{r^2}{4}\right)^2} - \frac{R}{2} \int_0^\infty \frac{r^4 dr}{\left(1 + \frac{r^2}{4}\right)^3} = \left(\frac{3R}{2} \cdot 2\pi \right) - \left(\frac{R}{2} \cdot 6\pi \right) = 0 \quad (\text{B.44})$$

and

$$E = 0. \quad (\text{B.45})$$

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