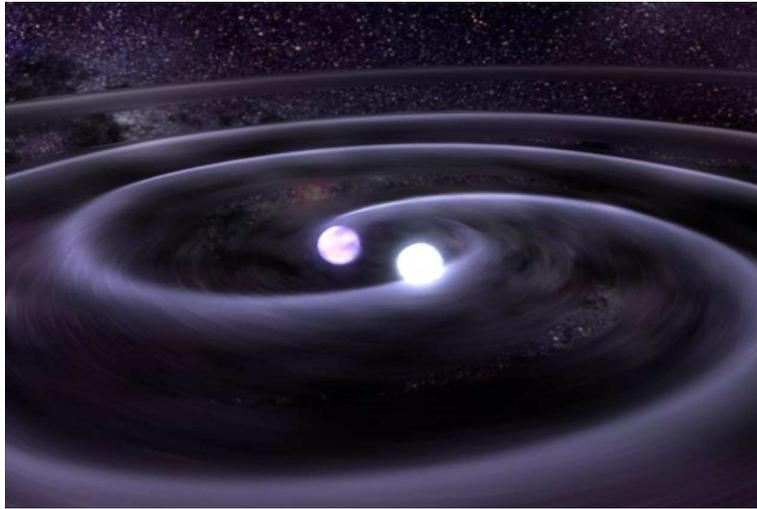


Chameleon fields and gravitational waves



**Master of Science thesis by
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Introduction

For the last 5 to 6 billion years the expansion of the Universe has been accelerating. The simplest way to explain this is that the energy density of the Universe contains a vacuum energy contribution. However, there are a couple of problems with this explanation. The first problem is that the energy density needed to explain the observations is much smaller than the vacuum energy density from fluctuations in quantum field theory. The second problem is the so-called coincidence problem: How can it be that the vacuum energy density and the matter density in the Universe are of the same order of magnitude today? This was the motivation to construct so-called quintessence models. In these models, the acceleration is driven by a scalar field, and through a self-interacting potential its dynamics can be adjusted so that the coincidence problem is avoided. Another problem arise though; such fields will couple to matter, and unless this coupling is unnaturally small, it will cause a long-ranged fifth force which is in conflict with known tests of gravity.

In a paper published in 2004, Justin Khoury and Amanda Weltman [11] introduced the chameleon field. This is a scalar field theory made to be consistent with known constraints on the magnitude and range on a fifth force. The chameleon field has a mass that depends on the density in its surroundings: where the density is high the mass is high, while it is light where the density is low. Thus, the field is very light on cosmological scales, while it has a large mass in the solar system and inside stars.

Chameleon field theory is a kind of scalar-tensor theory for gravitation. General relativity is a pure tensor theory, while the first known scalar-tensor theory was the Brans-Dicke theory. General relativity predicts the existence of gravitational waves. Such waves have never been observed directly, but observations of the Hulse-Taylor pulsar has provided indirect evidence of their existence. Because the scalar field can function as an additional gravitational field, gravitational waves behave different in scalar-tensor theories than in general relativity.

The goal of this thesis is to calculate the gravitational radiation from a quadrupole source in chameleon field theory, and compare the result with the predictions from general relativity. I will also make comparisons with calculations of gravitational waves in Brans-Dicke theory.

Structure

I have divided my thesis into 6 chapters:

- In chapter 1 I give a brief presentation of Einstein's general theory of relativity. I give a short description of its formalism and further derive Einstein's field equations. The chapter is finished with a presentation of the equivalence principle and Mach's principle.
- In chapter 2 I present the RW metric and the Friedmann equations and apply them to simple cosmological models before I go on with the accelerating universe and dark energy. In the last part of the chapter I look at quintessence as a model for dark energy and show how it can be designed to avoid the coincidence problem. Most of the examples and calculations in this chapter are taken from [4] and [9].

Together, the first two chapters establish the theoretical foundations of the thesis and are meant to motivate to the topics of the next two chapters.

- In chapter 3 I present the chameleon field theory and derive the chameleon equation of motion before I present its most important properties and the thin shell effect. The chapter is ended with a discussion of known constraints on chameleon parameters. A large part is based on [11], from which I have taken the examples and figures in this chapter.
- Chapter 4 is the most extensive chapter of the thesis. After a short introduction, the gravitational wave-equation is derived, and further, the formulae for gravitational radiation and period decay of binary systems. Then I present the results from the Hulse-Taylor pulsar. Finally, I make a short review of ongoing GW-detecting experiments. Many important derivations are taken from [2].

Having established the two main topics of the thesis, the focus is turned to gravitational waves in scalar-tensor theories.

- Chapter 5 starts with a presentation of the Brans-Dicke theory and a derivation of its field equations. Then we again consider gravitational waves and derive the gravitational energy loss formula in this theory. This is then applied to a neutron star-white dwarf system. The derivations in this chapter are mostly taken from [30].
- Finally, in chapter 6 I do the main work of this thesis when I study gravitational waves in chameleon field theory and present my results and conclusion.

Notations and conventions

Even though I will make it clear what different symbols means and which conventions I use throughout, I find it convenient to give an overview of the most important ones here.

- I will use Einstein's summation convention, and sum over all repeated indices, e.g.,

$$\sum x^\mu x_\mu = x^\mu x_\mu,$$

where greek indices means summation from 0 to 3, while latin indices means summation from 1 to 3.

- Spatial vectors are denoted by bold fonts, e.g., \mathbf{v} , and the components are denoted v^i .
- Time derivatives are often denoted by a dot, e.g.,

$$\dot{Q} = \frac{\partial Q}{\partial t}, \quad \ddot{Q} = \frac{\partial^2 Q}{\partial t^2}, \quad \text{etc.}$$

- The derivative with respect to given spacetime coordinates are expressed as

$$\frac{\partial}{\partial x^\mu} = \partial_\mu = ,_\mu .$$

- I often use units where $\hbar = c = 1$ so that we have

$$[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1},$$

though I have not been very consistent. But I will make it clear throughout which convention I use.

Chapter 1

General relativity

Einstein's general theory of relativity (throughout referred to as general relativity or GR) represents our most fundamental understanding of space, time and gravitation. It was published by Albert Einstein in 1916 in order to find a geometric theory of gravitation, and is today the current description of gravity in modern physics. The theory is a unification of special relativity and Newton's law of gravity, and describes gravity as a property of the geometry of spacetime.

I will not go through the entire development of Einstein's theory here, but rather present some of its most basic principles and the most important ingredients of GR that may be useful throughout this thesis.

1.1 Important tensors in GR

The general theory of relativity is a so-called *tensor-theory* of gravitation. This means that it is described by tensor equations, which are valid in *all* reference frames. In other words, Einstein developed a theory of gravitation that is independent of choice of reference system.

The perhaps most important tensor in GR is the *metric tensor*, $g_{\mu\nu}$. This is a symmetric covariant tensor (rank 2) that describes the distance between two points, x^μ and $x^\mu + dx^\mu$:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

It also defines the scalar product between two (spatial) vectors \mathbf{A} and \mathbf{B} ;

$$\mathbf{A} \cdot \mathbf{B} = g_{ij} A^i B^j.$$

We can consider the metric tensor as a $n \times n$ matrix (where n is the number of dimensions), and its determinant is denoted by $g = \det(g_{\mu\nu})$. If the determinant is non-zero, there exists an inverse (contravariant) metric tensor,

$g^{\alpha\nu}$, that satisfies

$$g_{\mu\alpha}g^{\alpha\nu} = \delta_{\mu}^{\nu} = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases}.$$

The metric tensor and its inverse can be used to respectively lower and raise indices, e.g.,

$$\begin{aligned} T_{\mu}^{\nu} &= g_{\mu\alpha}T^{\alpha\nu}, \\ T_{\nu}^{\mu} &= g^{\mu\alpha}T_{\alpha\nu}. \end{aligned}$$

The metric of flat spacetime is called the *Minkowski metric* and is given by

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1),$$

where I use the metric signature $(-, +, +, +)$, so that the line element of flat spacetime is given by¹

$$ds^2 = \eta_{\mu\nu}dx^{\mu}dx^{\nu} = -dt^2 + dx^2 + dy^2 + dz^2.$$

Another important tensor in GR is the *Riemann tensor*, which is related to the curvature of spacetime. It is given by

$$R_{\nu\beta\alpha}^{\mu} = \Gamma_{\nu\alpha,\beta}^{\mu} - \Gamma_{\nu\beta,\alpha}^{\mu} + \Gamma_{\sigma\beta}^{\mu}\Gamma_{\nu\alpha}^{\sigma} - \Gamma_{\sigma\alpha}^{\mu}\Gamma_{\nu\beta}^{\sigma}. \quad (1.1)$$

The Γ 's are called *Christoffel symbols*, and are given by

$$\Gamma_{\mu\nu}^{\sigma} = \frac{1}{2}g^{\rho\sigma}(g_{\nu\rho,\mu} + g_{\mu\rho,\nu} - g_{\mu\nu,\rho}). \quad (1.2)$$

We also have the *Ricci tensor* which is obtained by contracting two indices in the Riemann tensor:

$$R_{\mu\nu} = \Gamma_{\mu\nu,\alpha}^{\alpha} - \Gamma_{\mu\alpha,\nu}^{\alpha} + \Gamma_{\beta\alpha}^{\alpha}\Gamma_{\mu\nu}^{\beta} - \Gamma_{\beta\nu}^{\alpha}\Gamma_{\mu\alpha}^{\beta}. \quad (1.3)$$

The *Ricci scalar* is given by

$$R = g^{\mu\nu}R_{\mu\nu}. \quad (1.4)$$

Finally, we have the *Einstein tensor*, given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \quad (1.5)$$

An important property of the Einstein tensor is that it is covariant divergence free, i.e. $G^{\mu\nu}_{;\nu} = 0$. We will soon see why this is important.

The covariant derivative is defined by

$$A^{\mu}_{;\nu} \equiv A^{\mu}_{,\nu} + \Gamma_{\alpha\nu}^{\mu}A^{\alpha}.$$

¹I will stick to this signature throughout the thesis.

One last tensor we need to know is the *energy-momentum tensor*, also known as the stress-energy tensor, $T_{\mu\nu}$. There exists many different energy-momentum tensors for different cases (e.g., mass distributions, electromagnetic fields, etc.). Here, we will mostly deal with the one representing a *perfect fluid*, that is, a fluid with no viscosity or heat conduction. It is generally described by the mass/energy density, $\rho(x)$, the pressure, $p(x)$ and a four-velocity field u^μ . The trace of the energy-momentum tensor in this case is given by

$$T = g^{\mu\nu}T_{\mu\nu} = -\rho + 3p,$$

where we have put $c = 1$. By considering mass- and momentum-conservation in fluid mechanics (see e.g., [2]), one can show that the energy-momentum tensor, like the Einstein tensor, is covariant divergence free, $T^{\mu\nu}_{;\nu} = 0$. We then have two such tensors. The Einstein tensor is related to the geometry of spacetime, while the energy-momentum tensor is related to mass, energy and pressure. Einstein postulated that these two tensors had to be proportional,

$$G_{\mu\nu} = \kappa T_{\mu\nu}.$$

The proportionality constant κ can be found by taking the Newtonian limit of the field equations. This will give us

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (1.6)$$

These are Einstein's field equations, which are the relativistic generalization of Newton's law of gravitation. They tell us that matter-/energy distributions dictates the curvature of spacetime.

1.2 Einstein's field equations

We will now go on and derive Einstein's field equations using the Lagrangian formalism [2]. Given a Lagrangian density \mathcal{L} , the action is defined as the integral of \mathcal{L} over all four spacetime dimensions:

$$S = \int \mathcal{L} d^4x. \quad (1.7)$$

The field equations can then be found by using *Hamilton's principle* on the action integral,

$$\delta S = \int \delta \mathcal{L} d^4x = 0 \quad (1.8)$$

and vary with respect to the metric.

The Lagrangian can be split into two terms, one representing the metric

field and one representing matter/energy fields (in the case where matter and energy is present, that is);

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M.$$

The Lagrangian density for the metric field, \mathcal{L}_G , is given by

$$\mathcal{L}_G = \frac{M_{Pl}^2}{2} \sqrt{-g} R, \quad (1.9)$$

where $\sqrt{-g} = \sqrt{-\det g_{\mu\nu}}$, $M_{Pl} = \frac{1}{\sqrt{8\pi G}}$ is the reduced Planck mass and R is the Ricci scalar. The Ricci scalar, constructed from the Riemann tensor, $R^\alpha_{\beta\mu\nu}$, is the simplest scalar involving curvature. The action integral for the metric field, which is often called the *Einstein-Hilbert action*, is then given by

$$S_G = \frac{M_{Pl}^2}{2} \int R \sqrt{-g} d^4x. \quad (1.10)$$

I am going to vary the action inside an infinitesimal region V and let the variation of the metric and its derivative vanish on the boundary of V . The variation of S_G is

$$\delta S_G = \frac{M_{Pl}^2}{2} \int (\sqrt{-g} \delta R + R \delta(\sqrt{-g})) d^4x = 0. \quad (1.11)$$

The Ricci scalar can be expressed as $R = g^{\mu\nu} R_{\mu\nu}$ and thus the variation δR is given by

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} + \delta R_{\mu\nu} g^{\mu\nu}. \quad (1.12)$$

The Ricci tensor is given by

$$R_{\mu\nu} = \Gamma_{\mu\nu,\lambda}^\lambda - \Gamma_{\mu\lambda,\nu}^\lambda, \quad (1.13)$$

and thus the variation can be written

$$\delta R_{\mu\nu} = \delta \Gamma_{\mu\nu,\lambda}^\lambda - \delta \Gamma_{\mu\lambda,\nu}^\lambda = (\delta \Gamma_{\mu\nu}^\lambda)_{,\lambda} - (\delta \Gamma_{\mu\lambda}^\nu)_{,\nu}, \quad (1.14)$$

since the variation commutes with the partial derivatives. The partial derivatives of the metric vanish in V , and we can then write

$$g^{\mu\nu} \delta R_{\mu\nu} = (g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda - g^{\mu\lambda} \delta \Gamma_{\mu\nu}^\nu)_{,\lambda}. \quad (1.15)$$

We define the term in the parenthesis as a vector \mathbf{A} , i.e.

$$A^\lambda = g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda - g^{\mu\lambda} \delta \Gamma_{\mu\nu}^\nu. \quad (1.16)$$

We can now write eq. (1.15) on the form

$$g^{\mu\nu} \delta R_{\mu\nu} = A^\mu_{,\mu}. \quad (1.17)$$

This is a total divergence, and from the Gauss integral theorem or Stokes' theorem we have that this integral only contributes with a boundary term. Since the metric and its derivatives vanishes on the boundary, we get

$$\int g^{\mu\nu} \sqrt{-g} \delta R_{\mu\nu} d^4x = 0, \quad (1.18)$$

so the first term in eq. (1.11) do not contribute to δS_G .

Let us then consider the other term. The variation of $\sqrt{-g}$ is given by

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g, \quad \delta g = \frac{\partial g}{\partial g^{\mu\nu}}\delta g^{\mu\nu}. \quad (1.19)$$

To find $\frac{\partial g}{\partial g^{\mu\nu}}$ we use the formula

$$g = \sum_{\alpha} g_{\alpha\beta} (-1)^{(\alpha+\beta)} |C_{\alpha\beta}| \quad (1.20)$$

where $|C_{\alpha\beta}|$ is the determinant of the cofactor matrix of the metric $g_{\alpha\beta}$. The index β is arbitrary, so we can choose $\beta = \mu$. We find

$$\frac{\partial g}{\partial g^{\mu\nu}} = \sum_{\alpha} \frac{\partial g_{\alpha\mu}}{\partial g^{\mu\nu}} (-1)^{(\alpha+\mu)} |C_{\alpha\mu}|, \quad (1.21)$$

where

$$\frac{\partial g_{\alpha\mu}}{\partial g^{\mu\nu}} = -g_{\alpha\mu} g_{\mu\nu}. \quad (1.22)$$

We then get

$$\frac{\partial g}{\partial g^{\mu\nu}} = -g_{\mu\nu} \sum_{\alpha} g_{\alpha\mu} (-1)^{(\alpha+\mu)} |C_{\alpha\mu}| = -g_{\mu\nu} g. \quad (1.23)$$

Inserting this into eq. (1.19) gives us

$$\delta\sqrt{-g} = \frac{g}{2\sqrt{-g}} g_{\mu\nu} \delta g^{\mu\nu} = -\frac{\sqrt{-g}}{2} g_{\mu\nu} \delta g^{\mu\nu}. \quad (1.24)$$

The variation integral δS_G is then given by

$$\delta S_G = \frac{M_{Pl}^2}{2} \int \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} d^4x = 0. \quad (1.25)$$

If we also include variation of the matter Lagrangian we find the total variation with respect to the metric $g^{\mu\nu}$:

$$\delta S = \int d^4x \sqrt{-g} \left[\frac{M_{Pl}^2}{2} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} \right] \delta g^{\mu\nu} = 0. \quad (1.26)$$

This requires that the expression inside the bracket parenthesis is equal to zero, which gives:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{1}{M_{Pl}^2} \frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}}. \quad (1.27)$$

The energy-momentum tensor, $T_{\mu\nu}$, can thus be defined as:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}}. \quad (1.28)$$

This will then give us the field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu}, \quad (1.29)$$

where the energy-momentum tensor is given by

$$T_{\mu\nu} = \text{diag}(\rho, p, p, p), \quad (1.30)$$

for a perfect fluid.

1.3 The equivalence principle

The equivalence principle (EP) is perhaps the most fundamental principle upon which general relativity is built. *The weak equivalence principle* (WEP) states the equality between inertial mass (appearing in Newton's 2nd law, $\mathbf{F} = m_I \mathbf{a}$), and gravitational mass (appearing in Newton's law of gravity, $\mathbf{F} = -m_g \nabla \Phi$), i.e.

$$m_I = m_g.$$

This was verified already in the 17th century by Galileo in his famous experiment where he dropped different objects from the leaning tower of Pisa. This equivalence between gravitation and inertia inspired Einstein to go further and postulate that *physical experiments performed in a freely falling frame in a gravity field is equivalent to physical experiments performed in an inertial frame without gravity*.

Confined to the physics of mechanics, the equivalence principle of Einstein is just a re-statement of $m_I = m_g$. But Einstein extended this equivalence to *all* physics, and this generalized statement of the EP is often called the *strong equivalence principle*. It can be shown that the strong EP implies many kinds of general-relativistic effects, like the bending of light rays, gravitational redshift, etc. (see e.g., [3] or any textbook in GR).

1.4 Mach's principle

A fundamental question concerning the theory of relativity is: If all motion is relative, how can we then measure the inertia of a body?

Let's make a thought experiment where we assume that the Universe is only containing two particles, connected by a string. If the particles are rotating about each other, will the string be stretched due to centrifugal forces? According to Newton and his idea of absolute space, it will. However, if there is no absolute space the particles can rotate relatively to, the answer is not obvious. If the particles were in rest and an observer was rotating around them, the string would not appear to stretch. But this situation should, since all motion is relative, be equivalent to the first case where the string is stretched.

This problem led Ernst Mach, an Austrian philosopher, to conclude that the motion of a particle in an empty universe is not defined. Rather, all motion is relative to the cosmic background i.e. to "the great masses of the Universe". If no such cosmic masses existed, there would be no inertial forces. So in an empty universe, the string from our example would not be stretched.

Mach's arguments likely inspired Einstein when he constructed the general theory of relativity, and some Machian effects has also been shown to follow from its equations². However, Mach's principle is not a fundamental assumption of general relativity, and thus, it may be that not all the requirements set by Mach's principle are met. This has often been among the main motivations behind alternative gravitational theories, first and foremost the *Brans-Dicke theory* (which we will study in chapter 5), which has attempted to give a more complete incorporation of Mach's principle. None of these theories have been too successful though.

1.5 Summary

In this chapter we have made a brief presentation of general relativity. The most important things to remember are:

- General relativity is a geometric theory of gravitation, i.e. it describes gravity as a property of spacetime geometry.
- The theory is described by tensor equations, which are valid in all reference frames.
- Einstein's field equations are given by

$$G_{\mu\nu} = 8\pi GT_{\mu\nu},$$

²An important example is the Lense-Thirring effect, see *Phys. Z.*, 19:156 (1918)

and are a relativistic generalization of Newton's law of gravity. They can be derived by use of Hamilton's principle.

- From the weak equivalence principle, which states that *inertial mass is equivalent to gravitational mass*, Einstein went on to postulate the strong equivalence principle, which says that *physical experiments performed in a freely falling frame in a gravity field is equivalent to physical experiments performed in an inertial frame without gravity*. This is the most fundamental principle upon which general relativity is built.
- There are uncertainties whether Mach's principle, which can be stated *all motion is relative to the great masses in the universe*, is fully incorporated in general relativity. This has been the motivation behind many alternative theories of gravitation.

General relativity has many applications in modern astrophysics, and models made from the theory play important roles in research areas like gravitational lensing, black holes and gravitational waves (as we will see in chapter 4). Another important application is to cosmology, which will be the topic of the next chapter.

Chapter 2

Cosmology

2.1 Spacetime curvature and expansion

Our description of spacetime and cosmological models are based on some important assumptions. The most fundamental is the *cosmological principle* which says that the Universe, at the largest scales, is

- *homogeneous*, i.e. has a smooth background density,
- *isotropic*, i.e. looks the same in all directions.

General relativity is the essential framework in which we describe physical cosmology. We remember that the equivalence principle states that we cannot distinguish between physical experiments performed in a uniformly accelerated reference frame and those performed in a uniform gravitational field. In an accelerated frame, light-rays follows a curved path. According to the equivalence principle, this must also be the case in a uniform gravitational field. This leads to the concept of curved spacetime; since we define the trajectories of light-rays to be straight lines, it must mean that spacetime itself is curved. We can thus interpret the gravitational field as spacetime curvature.

During the first years after Einstein developed his theory (1916), he realized that he could use it to construct universe models as solutions of the relativistic field equations. At this time, the expansion of the universe was an unknown phenomenon. However, the solutions Einstein found described dynamic universe models. Thus, he introduced the *cosmological constant* as an extra term into his equations to evade this “problem”. But, from Hubble’s discovery of the redshift-distance relation of galaxies in 1929, it was clear that the universe indeed is expanding. This made Einstein consider the cosmological constant as the biggest blunder of his life, but many years later the cosmological constant would get its renaissance, as we will come back to later.

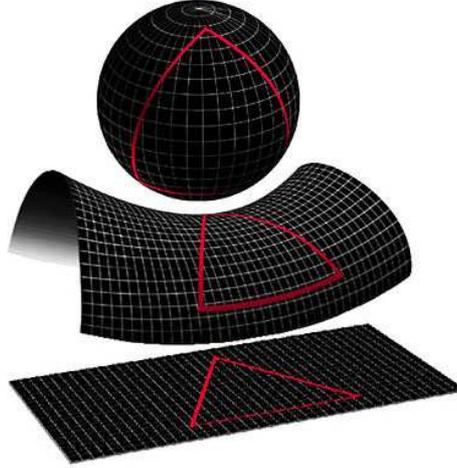


Figure 2.1: Illustration of the different spacetime curvatures; positive (spherical), negative (hyperbolic) and flat (euclidean). Note also the path of light rays in the different cases.

2.1.1 The RW line-element and the Friedmann equations

The Robertson-Walker line-element is the line-element describing a curved, expanding spacetime, and is given by

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad (2.1)$$

where

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2.$$

Here, r, θ and ϕ are coordinates in a spherical coordinate system that follows the expansion of the universe (so-called *comoving coordinates*), $a(t)$ is the *scale factor* which represents the expansion, and k is the *curvature parameter* which can take the values $-1, 0$ and $+1$, representing negatively curved, flat, and positively curved space respectively. In other words, the RW line-element represents all homogeneous and isotropic universe models. The dynamics of an expanding universe are determined by Einstein's field equations in Robertson-Walker geometry, with the source term (i.e. the right hand side of the field equations) described by a perfect fluid. That is, the energy-momentum tensor is given by eq. (1.30). This way, the field equations relates the curvature k and the scale factor a to the density ρ and the pressure p of the cosmic "fluid".

This leads to the *Friedmann equations*, which is the basic set of cosmic equations. I will skip the more thorough derivations here, but what I can tell is that we end up with two non-vanishing components of the field equations,

called the first and the second Friedmann equation. The first one (F1) is given by

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G}{3}\rho, \quad (2.2)$$

where $\dot{}$ represents the (cosmic) time derivative. Given the definition of the Hubble parameter, $H \equiv \dot{a}/a$, F1 can be written

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho.$$

The second Friedmann equation (F2) is given by

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \quad (2.3)$$

The term $\rho + 3p$ represents in a sense the effective gravitational mass, so $p < -\frac{1}{3}\rho$ implies repulsive gravitation. This will in turn imply that $\ddot{a} > 0$, i.e. a positive second time derivative of the scale factor, which implies accelerated expansion. We will come back to this later.

To solve the Friedmann equations, we need two additional equations, since we have four unknowns (a, k, ρ and p). The first equation follows from conservation of energy and momentum (i.e. from the vanishing divergence of the energy-momentum tensor). We find

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \quad (2.4)$$

The last equation we need is an *equation of state*, which relates the pressure and the density, given by

$$p = w\rho, \quad (2.5)$$

where w is a constant. For example, a universe dominated by non-relativistic matter (dust) has an equation of state where $w = 0$, since dust has zero pressure.

We can now integrate the differential equation (2.4) and find a relation between the density and the scale factor. This will help us see how the universe evolves when dominated by a given "fluid". We find

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}, \quad (2.6)$$

where $\rho_0 = \rho(t_0)$ and $a_0 = a(t_0)$ represents the present day values of the density and the scale factor respectively. For non-relativistic matter, we get

$$\rho_m = \rho_{m0} \left(\frac{a_0}{a}\right)^3.$$

For radiation (here analogue to a gas of ultra-relativistic particles), $w = 1/3$. This gives $\rho_r \propto a^{-4}$ where the extra factor $1/a$ comes from redshift. The

cosmic redshift (i.e. redshift caused by the expansion of the universe), z , is given by

$$1 + z = \frac{a(t_o)}{a(t_e)}, \quad (2.7)$$

where $a(t_e)$ represents the value of the scale factor at the time of emission from the light source and $a(t_o)$ is the scale factor at the time when it is observed.

2.1.2 The Einstein model

As we have mentioned earlier, Einstein wanted a static solution to his equations. It was assumed that the universe was dominated by non-relativistic matter, which gives $p = 0$. To get a static solution, we can try to put the scale factor constant, i.e. $a = a_0 = \text{const.}$ The second Friedmann equation then states

$$0 = \frac{4\pi G}{3}\rho.$$

But this must imply that $\rho = 0$, i.e. the universe is empty! So Einstein had to try something else; he introduced the *cosmological constant* into the field equations as an extra term. The Friedmann equations are then given by

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3}, \quad (2.8)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_m + \frac{\Lambda}{3}, \quad (2.9)$$

where Λ is the cosmological constant. For $a = a_0 = \text{const.}$, F2 now gives

$$\Lambda = 4\pi G\rho_m.$$

Inserting this into F1 gives

$$k = 4\pi G\rho_m a_0^2,$$

which implies $k > 0$, i.e. a universe with positive curvature.

It is usual to include Λ into the density and the pressure in the Friedmann equations. We rewrite eqs. (2.8) and (2.9) as

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda), \quad (2.10)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_m + \rho_\Lambda + 3p_\Lambda). \quad (2.11)$$

This gives

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}, \quad \frac{\Lambda}{3} = -\frac{4\pi G}{3}(\rho_\Lambda + 3p_\Lambda).$$

Combining these two expressions will give us

$$p_{\Lambda} = -\rho_{\Lambda}, \quad (2.12)$$

that is, an equation of state with $w = -1$, so the cosmological constant has a negative pressure. From the second Friedmann equation, (2.9), we see that if the cosmological constant term dominates, we will get $\ddot{a} > 0$, i.e. accelerating expansion of the universe.

Even though Einstein later dismissed the cosmological constant, it has made a comeback into the equations after it was discovered that the universe actually *is* accelerating. Physically, the cosmological constant represents vacuum energy density. Much more about the cosmological constant and the accelerating universe will soon be discussed.

2.2 Some examples of universe models

In this section, we will solve the Friedmann equations for a couple of simple universe models and see how the scale factor evolves with time. But before we start, we should introduce the term *critical density*. Let's look at F1 with $a = a_0$ and $\rho = \rho_0$. What must the density be as for the universe to be spatially flat today? We can find this out by simply putting $k = 0$ and solve for ρ_0 . We find

$$\rho_0 = \frac{3H_0^2}{8\pi G} \equiv \rho_{c0}, \quad (2.13)$$

where $H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0}$ is the present-day Hubble-parameter, called the *Hubble constant*, and where ρ_{c0} is the present value of the critical density. If $\rho_0 > \rho_{c0}$ the universe has a positive curvature, while the curvature is negative if $\rho < \rho_{c0}$. We also introduce *density parameters*. The density parameter for a given type i of matter/energy is defined by

$$\Omega_{i0} \equiv \frac{\rho_{i0}}{\rho_{c0}}. \quad (2.14)$$

For a spatially flat universe, the sum of the density parameters (representing matter, radiation, vacuum energy, etc.) should then be equal to 1. We may find yet another useful relation. Consider the first Friedmann equation

$$\frac{\dot{a}}{a} = H_0 \left(\frac{a}{a_0}\right)^{-3(1+w)/2}$$

where we have inserted eq. (2.6) for ρ and used (2.13). By integrating this and putting $a = 0$ at $t = 0$ we can find the age of a universe dominated by a given kind of fluid:

$$t_0 = \frac{2}{3(1+w)H_0}. \quad (2.15)$$

As we see, this expression is not valid for fluids with $w = -1$ (we will soon discuss the implications for a universe dominated by the cosmological constant).

2.2.1 Flat, matter-dominated universe

For this model, the Friedmann equation F1 is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m. \quad (2.16)$$

Inserting eq. (2.6) for ρ_m gives

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{m0}\left(\frac{a_0}{a}\right)^3, \quad (2.17)$$

since $w = 0$ for dust. The second Friedmann equation gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_{m0}\left(\frac{a_0}{a}\right)^3, \quad (2.18)$$

which implies that we at any time will have $\ddot{a} < 0$ in this model. Let us now solve the first Friedmann equation to find the time evolution of the scale factor. By using eq. (2.13) we can rewrite it as (assuming $\Omega_{m0} = 1$):

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{a_0}{a}\right)^3.$$

Then we take the square root and integrate:

$$\frac{1}{a_0^{3/2}}\int_{a_0}^a a^{1/2}da = H_0\int_{t_0}^t dt,$$

By doing some simple algebra and using that $a(t = 0) = 0$ we will find

$$a = a_0\left(\frac{t}{t_0}\right)^{2/3}, \quad (2.19)$$

which is called the *Einstein-de Sitter solution*. We see that an Einstein-de Sitter universe is forever expanding, but decelerating. Inserting the present value of the Hubble-constant, we find that the present age of the Einstein-de Sitter universe, $t_0 = \frac{2}{3H_0}$, is about 11 Gyr, while there has been observed objects in the universe with an age of ~ 12 Gyr! So the main problem with this model is that it won't give an old enough universe (except for unnaturally low values of the Hubble-constant). However, the Einstein-de Sitter model is not completely irrelevant, since it actually describes the evolution of the universe in the matter-dominated phase.

2.2.2 Flat universe with cosmological constant

The second example we will look at is a flat universe dominated by the cosmological constant (called the *de Sitter-model*). As we remember, in this case the equation of state gives $w = -1$ and the density is $\rho = \rho_\Lambda = \frac{\Lambda}{8\pi G} = \text{constant}$. The first Friedmann equation simply becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3}, \quad (2.20)$$

while the second Friedmann equation gives

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3}, \quad (2.21)$$

where we have used that $p_\Lambda = -\rho_\Lambda c^2$. As we see from the second equation, we have accelerated expansion. Taking the square root of the first equation, we get

$$\frac{\dot{a}}{a} = H = \sqrt{\frac{\Lambda}{3}} = \text{const.} = H_0.$$

Solving the equation, we find

$$a = Ae^{H_0 t},$$

where A is a constant. We choose A so that $a(t = t_0) = a_0$. This gives

$$a = a_0 e^{H_0(t-t_0)}. \quad (2.22)$$

From this, we see that $a \rightarrow 0$ in the limit $t - t_0 \rightarrow -\infty$, so we don't have any Big Bang in this model, the universe is infinitely old. Notice also that this model describes an empty universe (contains vacuum energy only). But still, this is in some ways an interesting model. Even though it can't be used to describe the entire evolution of the universe, it is relevant since we today have accelerated expansion. It may also be used to describe the inflation phase in the early universe.

Now it is time to discuss the accelerating universe and the observational evidence for this. Then we will take a look at the most popular universe model nowadays, i.e. the model in best accordance with recent cosmological observations, called the Λ CDM-model, which represents a universe filled with cold dark matter and the cosmological constant (vacuum energy), the components we believe make up most of the matter-/energy density in the universe today.

2.3 The accelerating universe and dark energy

2.3.1 Observational evidence from Type Ia Supernovae

Until 1998 the leading cosmological model was that of a flat universe with matter (baryonic and CDM) as its main constituent, implying a decelerating expansion (similar to the Einstein-de Sitter model). From inflation it was predicted that the universe must have a nearly flat geometry. However, observations indicated that the total matter density was significantly lower than the critical density, so there had to be a “missing” energy in order to have a flat universe. Also, as we remember, a flat matter-dominated universe has a lower age than the oldest observed stars. It was suggested that a possible solution to the missing-energy problem was the existence of a nonzero cosmological constant.

So, in 1998, a team of astrophysicists (the High- z Supernova Search Team), lead by Adam G. Riess, released a paper [5] where they summarized the results of their observations of a sample of distant Type Ia Supernovae. The observations had been performed to measure (that is, put constraints on) a range of cosmological parameters, among them the Hubble-parameter, the (eventual) density of the cosmological constant, Ω_Λ , and the *deceleration parameter*, q , which is a quantity that measures the expansion rate of the universe (positive q means a decreasing expansion rate, i.e. deceleration). How can such observations tell us something about the expansion of the universe?

Type Ia Supernovae arise from the explosion of white dwarf stars that has exceeded their Chandrasekhar limit. All Type Ia Supernovae have a similar characteristic light curve (i.e. their luminosity as a function of time after the explosion), thus they can be used as *standard candles* in extragalactic astronomy. It is then relatively easy to find the distances and the redshifts of these supernovae. The (cosmological) redshift tells us about how fast the object is moving away from us because of the expansion of the universe.

And what they found was that the supernovae at large redshifts were fainter than what was expected in a decelerating universe. The observations indicated an eternally expanding and accelerating universe. Thus, the missing energy density should have a negative pressure, like the cosmological constant. This missing energy, causing the acceleration, is generally labelled *dark energy*, with the cosmological constant being one among several candidates (as we will come back to in the next section). We will temporarily stick to the cosmological constant, as did Riess et. al. These results were shortly after confirmed by another team (the Supernova Cosmology Project) lead by Saul Perlmutter [6], which had a larger number of samples. The analysis they performed also indicated (assuming a flat universe) that $\Omega_{m0} \sim 0.3$. The contribution from the cosmological constant should then be $\Omega_{\Lambda0} \sim 0.7$.

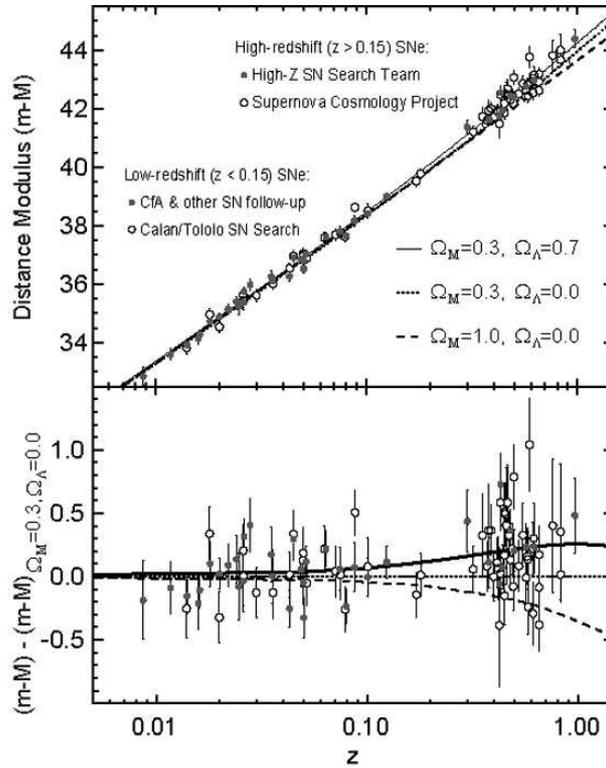


Figure 2.2: Plot of magnitude vs redshift of SNe Ia from the High-z Supernova Search team (Riess et al.) and the Supernova Cosmology Project (Perlmutter et al.) along with the low-redshift samples from the Calan-Tololo survey (1990-1995), see [5], [6], [7].

Further confirmations of the accelerated expansion of the universe has later been made by measurements of CMB anisotropies. The most recent observations, performed by WMAP, indicates that the dark energy makes up 74% of the total matter-/energy density in the universe, while (baryonic+cold dark) matter contributes the remaining 26% [8]. Thus, the leading cosmological model today is that of a flat universe containing matter and a cosmological constant, known as the (flat) Λ CDM model.

2.3.2 The Λ CDM model

There is observational evidence that most of the matter in the universe consists of cold, dark matter (denoted CDM), that is, matter that interacts only through gravitation. However, in our cosmological models all kinds of matter are equivalent to dust, that is, a perfect fluid with zero pressure. In accordance with observations, the flat Λ CDM model is a cosmological

model with $\Omega_{m0} \approx 0.3$ and $\Omega_{\Lambda 0} = 1 - \Omega_{m0} \approx 0.7$. Written in terms of density parameters and the Hubble parameter, the first Friedmann equation for this model gives

$$\frac{H^2(t)}{H_0^2} = \Omega_{m0} \left(\frac{a_0}{a}\right)^3 + (1 - \Omega_{m0}). \quad (2.23)$$

Notice that since $\Omega_{m0} < 1$, the right hand side of the Friedmann equation is always positive, and hence the universe is always expanding in this model (also in accordance with observations). As we have seen, the matter density goes as a^{-3} as the universe expands. We also know that the energy density of the cosmological constant is, by definition, constant. So at some time, for some value of the scale factor a , these densities have been equal. This can be found by putting

$$\Omega_{m0} \left(\frac{a_0}{a_{m\Lambda}}\right)^3 = \Omega_{\Lambda} = 1 - \Omega_{m0},$$

which will give us

$$a_{m\Lambda} = a_0 \left(\frac{\Omega_{m0}}{1 - \Omega_{m0}}\right)^{1/3}. \quad (2.24)$$

So for $a < a_{m\Lambda}$ the universe is matter-dominated, while for $a > a_{m\Lambda}$ the universe is dominated by the cosmological constant. This can also be used to find the present age of the universe in this model, by solving F1 for t and inserting a_0 for a , which will give us

$$t_0 = \frac{2}{3H_0\sqrt{1 - \Omega_{m0}}} \sinh^{-1} \left(\sqrt{\frac{1 - \Omega_{m0}}{\Omega_{m0}}} \right). \quad (2.25)$$

Inserting $\Omega_{m0} = 0.3$ and the present value of the Hubble constant¹, we find $t_0 = 13.5$ Gyr, which is consistent with the age of the oldest observed objects in the universe. We can also calculate when the cosmological constant became dominant from the above equation;

$$t_{m\Lambda} = \frac{2}{3H_0\sqrt{1 - \Omega_{m0}}} \sinh^{-1}(1), \quad (2.26)$$

which gives $t_{m\Lambda} \simeq 9.8$ Gyr. So in this model, the universe has been dominated by the cosmological constant for the last 3.7 billion years.

It is also possible to show that the universe at some point starts to accelerate by considering the second Friedmann equation. We can write it as

$$\frac{\ddot{a}}{a} = -\frac{H_0^2}{2} \left[\Omega_{m0} \left(\frac{a_0}{a}\right)^3 - 2\Omega_{\Lambda 0} \right]. \quad (2.27)$$

¹According to the five-year WMAP data, the current value of the Hubble constant is $H_0 \simeq 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}$, $h \approx 0.7$

We get $\ddot{a} > 0$ when the term inside the bracket parenthesis is negative. It can be shown [4] that the crossover from deceleration to acceleration occurs at a value of the scale factor a_{acc} , given by

$$a_{acc} = \left(\frac{1}{2}\right)^{1/3} a_{m\Lambda}, \quad (2.28)$$

which corresponds to when the universe was about 7.3 Gyr old, about 2.5 billion years before the cosmological constant became dominant. Hence, the universe has been accelerating for the last 6.2 billion years.

By considering the first Friedmann equation in the limit $a \ll a_{m\Lambda}$, it can be shown that in the early universe, $a \propto t^{2/3}$. So in the matter-dominated phase, the Λ CDM model is in accordance with the Einstein-de Sitter model as one might expect. Similarly, in the Λ -dominated phase, when $a \gg a_{m\Lambda}$, we find $a \propto \exp(\sqrt{1 - \Omega_{m0}}H_0t)$, as expected from our discussion of the de Sitter model.

2.4 Quintessence

As mentioned, the cosmological constant is just one candidate among several dark energy candidates and thus the explanation of the accelerated expansion of the universe. It is the simplest form of dark energy, and the Λ CDM model seems to fit observations very well. So why consider other candidates? Well, after all, there are a couple of difficulties concerning the cosmological constant. The first one: We remember that the cosmological constant is corresponding to vacuum energy density. The energy density of dark energy is of order 10^{-47} GeV^4 , which is 14 orders of magnitude smaller than the vacuum energy density that occurs in particle physics. So, if the cosmological constant represents dark energy, why is the energy density so small compared to what is found at particle physics scales?

The second problem is called the *coincidence problem*. We have seen that the vacuum energy density is constant, while the matter density is decreasing as the universe is expanding. How can it be then, that the matter density and the vacuum energy density is of the same order today? Their ratio must be set to a specific, infinitesimal value in the very early universe for the two densities to nearly coincide today, close to 15 billion years later. This was the main motivation to introduce *quintessence models*, that is, models where the dark energy is associated with a quintessence field or a *scalar field*, ϕ . Quintessence is a dynamical field, i.e. it can vary in time and space, in contrast to the cosmological constant. It couples to matter, but must have a light mass for it not to clump and form structures. It is also self-interacting through its own potential, $V(\phi)$. This potential is constructed to fit observations. Another important feature of (some) quintessence models is that they are very insensitive to initial conditions, as we soon will see.

Let's now take a look at how the quintessence field may give an accelerated universe. The density and the pressure of the scalar field are given by

$$\begin{aligned}\rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi), \\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi).\end{aligned}\tag{2.29}$$

To cause accelerated expansion of the universe, the scalar field must have a negative pressure, which require $\frac{1}{2}\dot{\phi}^2 < V(\phi)$. That is, the scalar field must evolve slowly down its potential (the kinetic energy density must be less than the potential energy density).

For the equation of state we find that

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}.\tag{2.30}$$

In the limit $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$ we get $w_\phi \approx -1$, the same as for the cosmological constant. From the second Friedmann equation it is possible to show that, to have positive acceleration, we need $w_\phi < -1/3$, and with the given density and pressure of the scalar field, we generally have that $-1 \leq w_\phi < -1/3$ (though there are also existing hypothetical models of quintessence, known as *phantom energy*², which has an equation of state where $w < -1$). Given the energy density of the scalar field and using the adiabatic equation (2.4), we can find the differential equation

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0,\tag{2.31}$$

where $V'(\phi) = \frac{dV}{d\phi}$. This is an equation of a damped harmonic oscillator. It may represent a particle with a one-dimensional coordinate ϕ moving in a potential $V(\phi)$ with a frictional force $-3H\dot{\phi}$. With time, the field will go toward lower values of $V(\phi)$ and finally come to rest at a field value corresponding to a minimum of the potential. However, many quintessence potentials do not have any minimum for finite values of ϕ , so one should perhaps rather say that the field is converging towards a minimum value of $V(\phi)$.

2.4.1 Tracker fields

To evade the coincidence problem, the quintessence model must have a so-called *tracker behaviour*. In these models, the scalar field rolls down a potential (i.e. the potential term in the energy density will with time dominate over the kinetic term) according to an attractorlike solution to the equation

²See R.R. Caldwell, M. Kamionkowski and N.N. Weinberg, *Phys. Rev. Lett.* **91** (2003)

of motion (2.31). Attractorlike means that a very wide range of initial conditions for ϕ and $\dot{\phi}$ rapidly approach a common evolutionary track, which will make the cosmology insensitive to initial conditions. The ratio of the ϕ -energy density to the background radiation- or matter density changes steadily as ϕ rolls down its potential. This will make the energy density of the scalar field overtake the background density and drive the universe into a phase of accelerated expansion. Such scalar fields are called tracker fields.

Let's look at an illustrative example of a tracker field (we will follow calculations given in [9]). Consider a scalar field with a potential given by

$$V(\phi) = M^{4+n} \phi^{-n}, \quad (2.32)$$

where M is a constant with unit of mass, and where n also is a constant which is positive but otherwise arbitrary. Though tracker fields are insensitive to initial conditions, we have to make an important assumption for the field to get the right behaviour; at early times (i.e. in the radiation-dominated phase) we require $\rho_\phi \ll \rho_r$. In a radiation-dominated universe, we have $a \propto t^{1/2}$, which gives $H = \frac{\dot{a}}{a} = \frac{1}{2t}$. The equation of motion becomes

$$\ddot{\phi} + \frac{3}{2t} \dot{\phi} - nM^{4+n} \phi^{-n-1} = 0. \quad (2.33)$$

This equation has a solution given by

$$\phi = \left(\frac{n(2+n)^2 M^{4+n} t^2}{6+n} \right)^{\frac{1}{2+n}}. \quad (2.34)$$

From this we can find that both $\dot{\phi}^2$ and $V(\phi)$ goes as $t^{-\frac{2n}{2+n}}$ (which means that $\rho_\phi \propto t^{-\frac{2n}{2+n}}$). Since we have that $\rho_r \propto a^{-4} \propto t^{-2}$, ρ_ϕ must be less than ρ_r at early times. This solution, for $t \rightarrow 0$, is known as the tracker solution.

What happens when the universe goes into the matter-dominated phase (that is, when ρ_r drops below ρ_m)? The only change in the equation of motion is in the numerical factor in front of the $\dot{\phi}$ -term. So the tracker solution continues to go as $t^{\frac{2}{2+n}}$, and we still have that

$$\rho_\phi \propto t^{-\frac{2n}{2+n}}.$$

The scale factor now grows as $t^{2/3}$. However, we still assume that the background density goes as a^{-4} . This gives

$$\rho_m \propto t^{-8/3}.$$

So both ρ_r and ρ_m decrease faster than ρ_ϕ , and will eventually fall below the scalar field energy density. At what time will this happen? Under matter-domination we can approximate the first Friedmann equation by

$$H^2 \simeq H_0^2 = \frac{4}{9t^2} = \frac{8\pi G}{3} \rho_m,$$

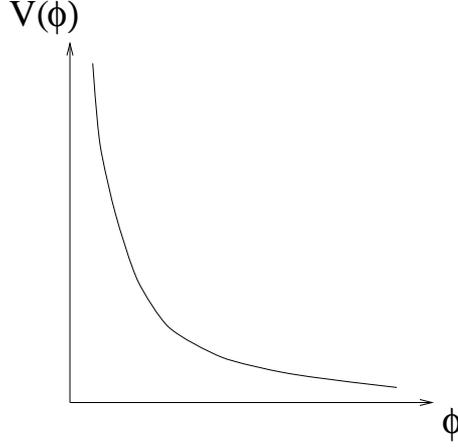


Figure 2.3: Sketch of a runaway potential.

which gives

$$\rho_m = \frac{1}{6\pi G t^2}. \quad (2.35)$$

From eqs. (2.32) and (2.34) we find

$$\rho_\phi \approx M^{\frac{2(4+n)}{(2+n)}} t^{-\frac{2n}{2+n}}, \quad (2.36)$$

which makes the time $t_{m\phi}$, when $\rho_\phi = \rho_m$, to be of order

$$t_{m\phi} \approx M^{-\frac{4+n}{2}} G^{-\frac{2+n}{4}} \sim H_0^{-\frac{2+n}{n}}. \quad (2.37)$$

2.4.2 ϕ -dominated universe

When ρ_m falls below ρ_ϕ , the equation of motion becomes

$$\ddot{\phi} + \sqrt{24\pi G \rho_\phi} \dot{\phi} - nM^{4+n} \phi^{n-1} = 0. \quad (2.38)$$

We may assume that the damping term in this equation will slow the growth of ϕ , so that the kinetic term $\dot{\phi}^2$ will become less than the potential $V(\phi)$ at late times. We may also guess that the term $\ddot{\phi}$ will become negligible compared to the other two terms in the equation of motion. Then we find

$$\sqrt{24\pi G M^{4+n} \phi^{-n}} \dot{\phi} = nM^{4+n} \phi^{n-1},$$

which gives

$$\dot{\phi} = \frac{nM^{\frac{2+n}{2}} \phi^{-\frac{n}{2}-1}}{\sqrt{24\pi G}}. \quad (2.39)$$

The solution of this equation is

$$\phi = M \left(\frac{n(2+n/2)t}{\sqrt{24\pi G}} \right)^{\frac{1}{2+n/2}}. \quad (2.40)$$

Let's now check if our assumptions, which we made when deriving eq. (2.40), hold. From the solution we find that

$$\dot{\phi}^2 \sim t^{-\frac{2+n}{2+n/2}}$$

while

$$V(\phi) \sim t^{-\frac{n}{2+n/2}},$$

so the kinetic term decreases faster with time than the potential. We also find that

$$\ddot{\phi} \sim t^{-\frac{3+n}{2+n/2}}$$

while

$$V'(\phi) \sim t^{-\frac{1+n}{2+n/2}},$$

so the $\ddot{\phi}$ -term will decrease faster with time than the other two terms in the equation of motion, thus it may be neglected at late times. The solution (2.40) is then a valid asymptotic solution of eq. (2.38) for $t \rightarrow \infty$. Not only that; it can also be shown (from numerical calculations) that the tracker solution (2.34) will take this form when $t \rightarrow \infty$. At late times, we then have that the scalar field is driving the expansion of the universe and that

$$\rho_\phi \propto V(\phi) \propto t^{-\frac{n}{2+n/2}}.$$

The first Friedmann equation will then give us that

$$\frac{\dot{a}}{a} \propto t^{-\frac{n}{4+n}},$$

which has a solution

$$\ln a \propto t^{\frac{4}{4+n}}. \quad (2.41)$$

We remember that, for the cosmological constant, we had $\ln a \propto t$. So since n is positive, the expansion of a ϕ -dominated universe is similar to, but less rapid than that of a Λ -dominated universe. Though it can be shown for tracker models with exponential potentials (rather than an inverse power-law potential, which we have worked with here), that $n \rightarrow 0$ as the universe ages, causing $w_\phi \rightarrow -1$ (the latter may also be seen directly from eq. (2.30) since the potential term becomes dominant at late times).

We can then make the conclusion that the coincidence problem can be evaded by using tracker fields. Thus, quintessence is also a possible explanation for the accelerated expansion of the universe and work as an effective cosmological constant at late times.

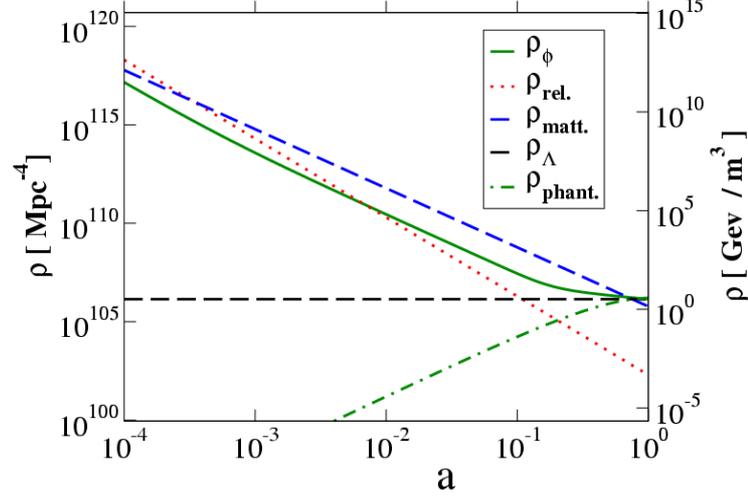


Figure 2.4: Plot of energy density vs scale factor where the tracker behaviour is illustrated. Figure taken from www.thphys.uni-heidelberg.de

2.4.3 Problems with ϕ ?

Quintessence may be the solution of the mystery of dark energy. But we still don't know today what the dark energy is, so what is the problem with quintessence models? First and foremost, a problem I will not be able to deal with in this thesis, is that there is not yet detected any such scalar fields in nature (though their existence is predicted in stringtheory and supergravity).

Secondly, if such a scalar field exists, it couples to matter, and this will lead to violations of the equivalence principle (since the mass then will depend on the scalar field, and thus, on time and space), violations which has not yet been detected in local tests of gravity.

Since w is variable for quintessence (at least over large time scales), we may assume that the scalar field energy must have varied significantly over the last Hubble-time, $t_H = H_0^{-1}$. From this we can calculate the required mass of the scalar field. The mass corresponds to a compton wavelength $\lambda = ct_H = c/H_0$. This gives (when re-inserting c and h):

$$m_\phi = \frac{hc}{\lambda} = \frac{h}{c^2} H_0. \quad (2.42)$$

A reasonable choice of the value of the Hubble parameter is $H_0 \simeq 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ which, in SI-units, corresponds to $H_0 \sim 10^{-18} \text{ s}^{-1}$.

An order-of-magnitude estimate of m_ϕ gives

$$m_\phi c^2 \sim 10^{-34} \text{J}_s \times 10^{-18} \text{s}^{-1} \sim 10^{-52} \text{J} \sim 10^{-33} \text{eV}.$$

The scalar field is essentially massless at both cosmological and solar system scales. The low mass means that it has a cosmological range of interaction. So if the scalar field is coupled to mass, it will cause a so-called fifth force. From laboratory experiments on Earth, this fifth force should have very strict constraints which forces the gravitational coupling strength to be very small or the interaction range very short.

This finally leads us to the motivation to introduce the *chameleon field*, a scalar field with properties that allow us to evade these problems. This will be the topic of the next chapter.

2.5 Summary

Before we move on, I would like to give an overview of the most important things to remember from this chapter:

- The cosmological principle, which states that *the universe is homogeneous and isotropic at large scales* is the most fundamental principle of modern physical cosmology.
- The Friedmann equations are the basic set of cosmic equations. They are derived from Einstein's field equations in Robertson-Walker geometry. From these we can construct curved or flat universe models containing cosmic fluids like matter, radiation or vacuum energy. Universe models dominated by the latter gives forever expanding and accelerating solutions.
- The discovery of the accelerating universe, made in 1998, indicated that the energy density in the universe must be dominated by dark energy, presumably represented by a cosmological constant (=vacuum energy). The Λ CDM model, describing a universe containing vacuum energy and cold dark matter, is in good accordance with observations and is currently the most reliable universe model.
- The tiny energy density of the cosmological constant compared to what is predicted at particle physics scales, in addition to the coincidence problem, has been motivations to consider other candidates for dark energy. Quintessence models, in which the dark energy is represented by a dynamical scalar field, has been constructed in order to evade the coincidence problem.

- By possessing a tracker-behaviour, the energy density of the scalar field behaves in such a way that it overtakes the matter energy density at the “right” time according to observations, and it evolves at cosmological time scales.
- The scalar fields in quintessence models couples to matter. Unless the coupling is tuned to an unnaturally small value, this will lead to a fifth force which has a cosmological range of interaction, which further will lead to an unacceptable large violation of the equivalence principle.

Chapter 3

Chameleon fields

3.1 Introduction

In the last section we saw that the accelerated expansion of the universe may be explained by the existence of a scalar field that couples to matter. But to avoid violence of the equivalence principle, this coupling must be tuned to unnaturally small values.

The chameleon field [11, 12] is a scalar field that has matter couplings of order unity, but still evolves on cosmological time scales today. This is because of the characteristic property that the chameleon field has a mass that depends on the background matter density. In areas with high densities (e.g., on Earth or inside stars) the interaction of the field is short ranged, while at large (i.e. cosmological) scales it is long ranged. In other words: the chameleon field changes its properties to fit its surroundings. Thus, it can be strongly coupled to matter and at the same time remain light over solar system and cosmological scales. On Earth, where the matter density is about 10^{30} times larger than in the cosmological background, the Compton wavelength (i.e. the range) of the field ($\lambda \sim 1/m_\phi$) is sufficiently small to satisfy all existing tests of gravity. In the solar system, the density is several orders of magnitude smaller, and thus the Compton wavelength is much larger (it can even be larger than the solar system itself). This means that the scalar field may have a mass in the solar system that is much lower than previously thought allowed. At cosmological scales the density is even lower and the field even lighter, and the energy density of the field evolves slowly over cosmological time-scales. Here, the mass of the field is of the same order as the present Hubble-parameter, $H_0 \approx 10^{-33}$ eV, as we calculated for quintessence in the last section. Thus, the chameleon field can function as an effective cosmological constant.

Beacuse of its mentioned properties, the chameleon field is essentially invisible to searches for EP-violations and fifth force in laboratories on Earth.

3.2 Chameleon equation of motion

The chameleon equation of motion can be found by varying the chameleon Lagrangian with respect to the field, ϕ [13]. We now have a Lagrangian for the chameleon field, \mathcal{L}_ϕ , given by

$$\mathcal{L}_\phi = -\sqrt{-g} \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi) \right) \quad (3.1)$$

in addition to the gravitational and the matter Lagrangian we had in the derivation of Einstein's field equations. Thus the action governing the dynamics of the chameleon field can be written as

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{Pl}^2}{2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + \int d^4x \mathcal{L}_M(\psi^{(i)}, g_{\mu\nu}^{(i)}), \quad (3.2)$$

where $\psi^{(i)}$ are matter fields coupled to the metrics $g_{\mu\nu}^{(i)}$ in the so-called *Jordan frame*. These are related to the *Einstein frame* metric by a conformal transformation

$$g_{\mu\nu}^{(i)} = \Omega_{(i)}^2 g_{\mu\nu}, \quad (3.3)$$

where $\Omega_{(i)}$ are (model dependent) functions of the scalar field, ϕ .¹ Often, $\Omega_{(i)}$ is given by the exponential function $e^{\beta_i \phi / M_{Pl}}$. This gives

$$g_{\mu\nu}^{(i)} = \exp(2\beta_i \phi / M_{Pl}) g_{\mu\nu}.$$

However, we will temporarily stick to the more general $\Omega_{(i)}$ in this derivation. The Lagrangian representing the metric, $\mathcal{L}_G = \frac{1}{M_{Pl}^2} R$, does not depend on ϕ , so we will only get a contribution from \mathcal{L}_M and \mathcal{L}_ϕ . Variation of the matter Lagrangian gives us

$$\delta \mathcal{L}_M = \frac{\partial \mathcal{L}_M}{\partial \phi} \delta \phi = \sum_i \frac{\partial \mathcal{L}_M}{\partial g_{(i)}^{\mu\nu}} \frac{\partial g_{(i)}^{\mu\nu}}{\partial \phi} \delta \phi.$$

Further, we have

$$g_{(i)}^{\mu\nu} = \frac{1}{g_{\mu\nu}^{(i)}} = \Omega_{(i)}^{-2} g^{\mu\nu} \quad (3.4)$$

which gives

$$\frac{\partial g_{(i)}^{\mu\nu}}{\partial \phi} = -2\Omega_{(i)}^{-3} \partial_\phi \Omega_{(i)} g^{\mu\nu}.$$

¹Note that this implies that the chameleon field is *non-minimal coupled* to matter, in contrast to the "traditional" minimal coupled quintessence which we studied in chapter 2.

Inserted into the expression for $\delta\mathcal{L}_M$, we get

$$\begin{aligned}\delta\mathcal{L}_M &= \sum_i \frac{\partial\mathcal{L}_M}{\partial g^{(i)\mu\nu}} \left(-2\Omega_{(i)}^{-3} \partial_\phi \Omega_{(i)} \right) g^{\mu\nu} \delta\phi \\ &= - \sum_i 2 \frac{\partial_\phi \Omega_{(i)}}{\Omega_{(i)}} \frac{\partial\mathcal{L}_M}{\partial g^{(i)\mu\nu}} g^{(i)\mu\nu} \delta\phi.\end{aligned}\quad (3.5)$$

The variation of the chameleon Lagrangian is

$$\delta\mathcal{L}_\phi = -\sqrt{-g} \left(\frac{1}{2} \delta[\partial_\mu \phi \partial^\mu \phi] + \delta V(\phi) \right).$$

The differential operators δ and ∂_μ commute, so by using the chain rule the full expression for $\delta\mathcal{L}_\phi$ can be written as

$$\delta\mathcal{L}_\phi = \sqrt{-g} (\partial_\mu \partial^\mu \phi - \partial_\phi V(\phi)) \delta\phi - \sqrt{-g} \partial_\mu [\partial^\mu \phi \delta\phi]. \quad (3.6)$$

The total variation with respect to ϕ is then

$$\begin{aligned}\delta S &= \int d^4x \sqrt{-g} \left[\partial_\mu \partial^\mu \phi - \partial_\phi V(\phi) + \sum_i \frac{\partial_\phi \Omega_{(i)}}{\Omega_{(i)}} \left(-\frac{2}{\sqrt{-g}} \frac{\partial\mathcal{L}_M}{\partial g^{(i)\mu\nu}} \right) g^{(i)\mu\nu} \right] \delta\phi \\ &\quad - \int d^4x \sqrt{-g} \partial_\mu [\partial^\mu \phi \delta\phi] = 0.\end{aligned}\quad (3.7)$$

If we assume that the variation $\delta\phi$ is zero at the boundary, we find that the last term vanishes by using Gauss' theorem. The remaining integral is then equal to zero, and we can find the equations of motion:

$$\partial_\mu \partial^\mu \phi = \partial_\phi V(\phi) - \sum_i \Omega_{(i)}^3 \partial_\phi \Omega_{(i)} \left(-\frac{2}{\sqrt{-g_{(i)}}} \frac{\partial\mathcal{L}_M}{\partial g_{(i)}^{\mu\nu}} \right) g_{(i)}^{\mu\nu}, \quad (3.8)$$

where we have used that

$$\frac{1}{\sqrt{-g}} = \frac{\Omega_{(i)}^4}{\sqrt{-g_{(i)}}}. \quad (3.9)$$

Further, the energy-momentum tensor in the Jordan frame is given by

$$T_{\mu\nu}^{(i)} \equiv -\frac{2}{\sqrt{-g_{(i)}}} \frac{\partial\mathcal{L}_M}{\partial g_{(i)}^{\mu\nu}}. \quad (3.10)$$

This gives

$$\partial_\mu \partial^\mu \phi = \partial_\phi V(\phi) - \sum_i \Omega_{(i)}^3 \partial_\phi \Omega_{(i)} T_{\mu\nu}^{(i)} g^{(i)\mu\nu}. \quad (3.11)$$

For simplicity we restrict ourselves to a universal matter coupling (i.e. we assume that all matter fields $\psi^{(i)}$ has the same coupling to the scalar field). We can then write $g_{\mu\nu}^{(i)} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu}$. The equation of motion then becomes

$$\partial_\mu \partial^\mu \phi = \partial_\phi V(\phi) - \Omega^3 \partial_\phi \Omega \tilde{T}_{\mu\nu} \tilde{g}^{\mu\nu}.$$

If we use the notations $\partial_\mu \partial^\mu = -\frac{\partial^2}{\partial t^2} + \nabla^2 = \square$, $\partial_\mu = ,_\mu$ and $\frac{\partial}{\partial \phi} = \partial_\phi = ,_\phi$, we can write it as

$$\square \phi = V_{,\phi}(\phi) - \Omega^3 \Omega_{,\phi} \tilde{g}^{\mu\nu} \tilde{T}_{\mu\nu},$$

Inserting $e^{\beta\phi/M_{Pl}}$ for Ω , we get

$$\square \phi = V_{,\phi}(\phi) - \frac{\beta}{M_{Pl}} e^{4\beta\phi/M_{Pl}} \tilde{g}^{\mu\nu} \tilde{T}_{\mu\nu}. \quad (3.12)$$

For non-relativistic matter we have that $\tilde{g}^{\mu\nu} \tilde{T}_{\mu\nu} = -\tilde{\rho}$, where $\tilde{\rho}$ is the energy density. However, it is more convenient to use an energy density which is conserved in Einstein frame, given by

$$\rho \equiv \tilde{\rho} e^{3\beta\phi/M_{Pl}}. \quad (3.13)$$

The equation of motion for the chameleon field is then given by

$$\square \phi = V_{,\phi}(\phi) + \frac{\beta}{M_{Pl}} \rho e^{\beta\phi/M_{Pl}}. \quad (3.14)$$

We can also find the Einstein equations in the presence of the chameleon field. Then we must vary the total action (3.2) with respect to $g^{\mu\nu}$. From our calculations of the ordinary field equations we know that variation of the first term will give us the Einstein-tensor, while the second term will give us the energy-momentum tensor for the scalar field. To find this, we will use the Lagrangian of the scalar field together with eq. (1.28). We have that

$$T_{\mu\nu}^\phi = -\frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}_\phi}{\partial g^{\mu\nu}}, \quad \mathcal{L}_\phi = -\sqrt{-g} \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi) \right). \quad (3.15)$$

To save some work, I express the Lagrangian as

$$\mathcal{L}_\phi = \sqrt{-g} \mathcal{L}_\phi^0, \quad \mathcal{L}_\phi^0 = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi).$$

This gives

$$T_{\mu\nu}^\phi = -\frac{2}{\sqrt{-g}} \frac{\partial[\sqrt{-g} \mathcal{L}_\phi^0]}{\partial g^{\mu\nu}} = -2 \frac{\mathcal{L}_\phi^0}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial g^{\mu\nu}} - 2 \frac{\partial \mathcal{L}_\phi^0}{\partial g^{\mu\nu}}$$

$$= -\frac{\mathcal{L}_\phi^0}{g} \frac{\partial g}{\partial g^{\mu\nu}} - 2 \frac{\partial \mathcal{L}_\phi^0}{\partial g^{\mu\nu}}. \quad (3.16)$$

Using what we found in eq. (1.23);

$$\frac{\partial g}{\partial g^{\mu\nu}} = -g g_{\mu\nu},$$

we find

$$T_{\mu\nu}^\phi = -2 \frac{\partial \mathcal{L}_\phi^0}{\partial g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}_\phi^0. \quad (3.17)$$

Inserting the expression for \mathcal{L}_ϕ^0 , we finally end up with

$$T_{\mu\nu}^\phi = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi) \right). \quad (3.18)$$

Variation of the matter Lagrangian with respect to $g^{\mu\nu}$ gives

$$\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} = \frac{\partial \mathcal{L}_M}{\partial \tilde{g}^{\mu\nu}} \frac{\partial \tilde{g}^{\mu\nu}}{\partial g^{\mu\nu}}$$

where I still assume the field to have the same coupling to all kinds of matter. From (3.4) we find that

$$\frac{\partial \tilde{g}^{\mu\nu}}{\partial g^{\mu\nu}} = \Omega^{-2},$$

which gives

$$\frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} = \Omega^{-2} \frac{\partial \mathcal{L}_M}{\partial \tilde{g}^{\mu\nu}}.$$

The total variation then becomes

$$\delta S = \int d^4x \sqrt{-g} \left[\frac{M_{Pl}^2}{2} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) + \frac{1}{\sqrt{-g}} \left(\frac{\partial \mathcal{L}_\phi}{\partial g^{\mu\nu}} + \Omega^{-2} \frac{\partial \mathcal{L}_M}{\partial \tilde{g}^{\mu\nu}} \right) \right] \delta g_{\mu\nu} = 0.$$

Using (3.9), we recognize the last term as the energy-momentum tensor in Jordan frame, eq.(3.10). Thus, we get

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{M_{Pl}^2} (T_{\mu\nu}^\phi + \Omega^2 \tilde{T}_{\mu\nu}) \\ &= 8\pi G (T_{\mu\nu}^\phi + T_{\mu\nu}) \end{aligned} \quad (3.19)$$

which are Einstein's field equations in presence of a chameleon field.

3.3 Important properties

Let us again write up the chameleon equation of motion:

$$\square\phi = V_{,\phi}(\phi) + \frac{\beta}{M_{Pl}}\Omega_{,\phi}(\phi)\rho.$$

The density-dependent mass of the chameleon field is generated by the interplay of the two source terms on the right hand side. The first one is a self-interacting term, represented by the potential of the field, $V(\phi)$. This potential need not have a minimum; rather, it is monotonically decreasing and of a runaway form. That is, the potential satisfies

$$\lim_{\phi \rightarrow \infty} V = 0, \quad \lim_{\phi \rightarrow \infty} \frac{V_{,\phi}}{V} = 0, \quad \lim_{\phi \rightarrow \infty} \frac{V_{,\phi\phi}}{V_{,\phi}} = 0 \dots \quad (3.20)$$

as well as

$$\lim_{\phi \rightarrow 0} V = \infty, \quad \lim_{\phi \rightarrow 0} \frac{V_{,\phi}}{V} = \infty, \quad \lim_{\phi \rightarrow 0} \frac{V_{,\phi\phi}}{V_{,\phi}} = \infty \dots \quad (3.21)$$

As mentioned, the second term represents the coupling of the chameleon field to matter, and is described by the model dependent function $\Omega(\frac{\beta\phi}{M_{Pl}})\rho$, where ρ is the background density. Ω is a monotonic increasing function of ϕ (in the examples throughout this section we will substitute Ω with the exponential function), β is the chameleon-to-matter coupling constant and $M_{Pl} = (8\pi G)^{-1/2}$ is the reduced Planck mass. The coupling constants need not be small; as mentioned earlier they can have values of order unity or greater. For simplicity, we will here assume that the chameleon field couples to all species of matter in the same way, that is, β is the same for all kinds of matter.

Although the two source terms are monotonic functions of ϕ , their combined effect is that of an effective potential which displays a minimum. It is given by

$$V_{eff}(\phi) = V(\phi) + \Omega\left(\frac{\beta\phi}{M_{Pl}}\right)\rho. \quad (3.22)$$

This effective potential depends on the local matter density, ρ , and thus both the field value at the minimum and the mass of small fluctuations depend on ρ as well. The latter will increase as a function of the density. Let's see how this is achieved;

The mass of small fluctuations about the minimum is given by

$$m = \sqrt{V_{,\phi\phi}^{eff}(\phi_{min})}, \quad (3.23)$$

where ϕ_{min} is the corresponding value of the field. If we write out this expression, we get

$$m^2 = V_{,\phi\phi}(\phi_{min}) + \frac{\beta^2}{M_{Pl}^2}\Omega_{,\phi\phi}(\phi_{min})\rho. \quad (3.24)$$

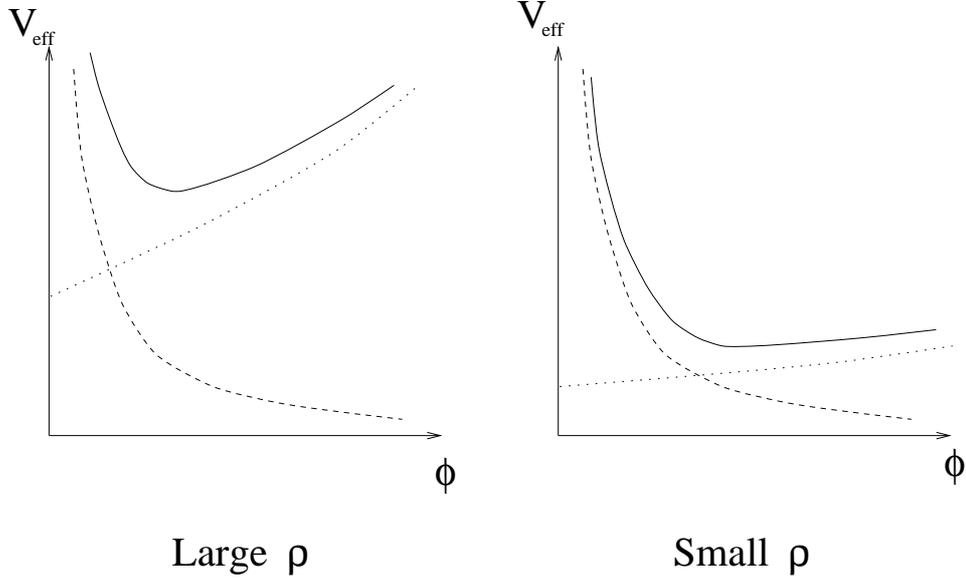


Figure 3.1: Effective chameleon potential for high and low densities [11]. The dashed curve represents the self-coupling potential $V(\phi)$ while the dotted curve represents the coupling to the matter density ρ .

Usually, we have that $|V_{,\phi\phi}| \gg |\Omega_{,\phi\phi}\rho|$, so m is almost entirely determined by the form of $V(\phi)$ and the value of ϕ_{min} (given that V is neither constant, linear nor quadratic in ϕ). Finally, for this to be a chameleon field theory, the mass must increase as the background density increases. To see this, remember that

$$V_{,\phi}^{eff}(\phi_{min}) = V_{,\phi}(\phi_{min}) + \frac{\beta}{M_{Pl}} \Omega_{,\phi}(\phi_{min})\rho = 0. \quad (3.25)$$

We know that $V_{,\phi}$ must be negative and thus the other term is positive. We see that larger values of ρ correspond to smaller values of ϕ_{min} (remember the properties of Ω). Further, if we again consider the expression for the mass, eq. (3.24), and remember the properties of $V(\phi)$ (and especially of its second derivative), we see that lower values of ϕ_{min} implies larger m (equivalent: larger $\rho \Rightarrow$ larger m).

3.4 Thin shell effect

Near-future experiments like STEP (Satellite Test of the Equivalence Principle) [16] and Galileo Galilei (GG) [17] will test the universality of free fall in orbit with high expected accuracies. It is predicted that these experiments may show eventual violations of the EP. However, all existing

constraints from planetary orbits are well satisfied within our model. This is because of something called the *thin shell effect*.

It can be shown that the only contribution to the chameleon-mediated force from a large compact object like the Earth or the Sun comes from a *thin shell* near the surface of the body. We will here show how such a thin shell is developed and how large a body must be to have a thin shell (mostly following calculations given in [11]).

Consider a large, isolated compact body with radius R_c and homogeneous density ρ_c , immersed in a background of homogeneous density ρ_b . The equation of motion in this case is given by

$$\frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} = V_{,\phi} + \frac{\beta}{M_{Pl}} \rho(r) e^{\beta\phi/M_{Pl}}, \quad (3.26)$$

where

$$\rho(r) = \begin{cases} \rho_c, & r < R_c \\ \rho_b, & r > R_c \end{cases}$$

We denote the field values which minimize V_{eff} for $r < R_c$ and $r > R_c$ as ϕ_c and ϕ_b respectively. This gives us

$$V_{,\phi}(\phi_c) + \frac{\beta}{M_{Pl}} \rho_c e^{\beta\phi_c/M_{Pl}} = 0, \quad (3.27)$$

$$V_{,\phi}(\phi_b) + \frac{\beta}{M_{Pl}} \rho_b e^{\beta\phi_b/M_{Pl}} = 0. \quad (3.28)$$

Since the equation of motion is a second order differential equation, we require two boundary conditions. To avoid singularities at the origin, we choose $d\phi/dr = 0$ at $r = 0$. Moreover, far from the body we have $\rho = \rho_b$, so we choose $\phi \rightarrow \phi_b$ as $r \rightarrow \infty$.

We may think of this problem as an analogue to a dynamical problem in classical mechanics, with ϕ representing the position of a particle and r representing a time coordinate. The particle moves along the inverted potential $-V_{eff}$ (see fig. 3.2 a). The second term on the l.h.s. in the equation of motion (proportional to $1/r$) will function as a damping term. Notice that the potential is “time dependent” since the density ρ depends on r , so the potential undergoes a jump when r reaches the radius of the body, R_c .

The particle is initially at rest (since $\frac{d\phi}{dr}(r=0) = 0$). We denote the initial field value as $\phi_i \equiv \phi(r=0)$. At early “times” (i.e. small r) the particle is essentially frozen at $\phi = \phi_i$, since the damping term is dominant. As r grows, the damping term will become negligible and ϕ will start to roll down the potential. It rolls down until it reaches $r = R_c$. At this time, the potential suddenly changes shape as the density changes from ρ_c to ρ_b . However, we can match ϕ and $d\phi/dr$ to be continuous at the jump. Once $r > R_c$, ϕ starts climbing up the “new” potential (see fig. 3.2 b). If we

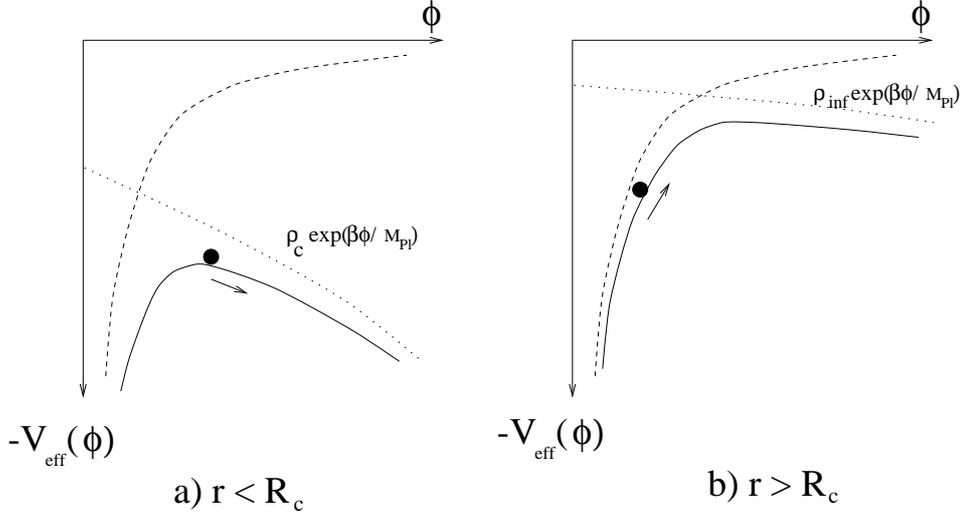


Figure 3.2: Sketch of the inverted potentials [11]. a) represents the potential inside the body, b) represents the potential outside.

choose the initial value ϕ_i carefully, it will reach the far-background value ϕ_b as $r \rightarrow \infty$. Thus our problem is reduced to determining the value ϕ_i . Let us first consider the case $(\phi_i - \phi_c) \ll \phi_c$, that is, $\Delta R_c/R_c \ll 1$, which correspond to the *thin shell regime*. This means that the initial value ϕ_i is very close to ϕ_c which is a local extremum of V_{eff} . Thus the driving term $V_{,\phi}^{eff}$ is initially negligible, and the damping term strongly dominates the dynamics. Until r has grown sufficiently for ϕ to start rolling, the field value remains approximately constant, that is

$$\phi(r) \approx \phi_c, \quad 0 < r < R_{roll}, \quad (3.29)$$

where R_{roll} is the value of r at which ϕ starts rolling. Just after the field has begun to roll, its value is close to ϕ_c , so we have that $M_{Pl}|V_{,\phi}| \ll \beta\rho e^{\beta\phi/M_{Pl}}$. So in the regime $R_{roll} < r < R_c$ we can write the equation of motion as

$$\frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} = \frac{\beta}{M_{Pl}} \rho_c, \quad (3.30)$$

since $\phi/M_{Pl} \ll 1$ for all relevant times [14]. When using the boundary conditions $\phi = \phi_c$ and $d\phi/dr = 0$ at $r = R_{roll}$, we find the solution

$$\phi(r) = \frac{\beta\rho_c}{3M_{Pl}} \left(\frac{r^2}{2} + \frac{R_{roll}^3}{r} \right) - \frac{\beta\rho_c R_{roll}^2}{2M_{Pl}} + \phi_c, \quad R_{roll} < r < R_c. \quad (3.31)$$

The equations (3.29) and (3.31) then describes the full interior solution, that is, for the regime $0 < r < R_c$.

Outside the body ($r > R_c$), we assume that we can linearize the equation of motion about the value of ϕ in the far background. According to the expression

$$m_c^2 = V_{,\phi\phi}^{eff},$$

we write the equation of motion as

$$\frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} = m_b^2(\phi - \phi_b). \quad (3.32)$$

The solution which satisfies $\phi \rightarrow \phi_b$ as $r \rightarrow \infty$ is

$$\phi(r) = -\frac{Ae^{-m_b(r-R_c)}}{r} + \phi_b, \quad r > R_c, \quad (3.33)$$

where A is a constant. The two unknowns R_{roll} and A is found by matching ϕ and $d\phi/dr$ at $r = R_c$ using the solutions (3.31) and (3.33). This will give us the exterior solution

$$\phi(r) \approx -\left(\frac{\beta}{4\pi M_{Pl}}\right) \left(\frac{3\Delta R_c}{R_c}\right) \frac{M_c e^{-m_b(r-R_c)}}{r} + \phi_b, \quad (3.34)$$

where

$$\frac{\Delta R_c}{R_c} = \frac{\phi_b - \phi_c}{6\beta M_{Pl} \Phi_c} \approx \frac{R_c - R_{roll}}{R_c} \ll 1, \quad (3.35)$$

and where Φ_c is the Newtonian potential of the body, given by

$$\Phi_c = \frac{M_c}{8\pi M_{Pl}^2 R_c}.$$

The thin shell condition $\Delta R_c/R_c$ is equivalent to the condition $m_c R_c \gg 1$, i.e. the assumption that the range of the interior field is very short compared to the size of the body. All perturbations in ϕ will die off quickly over a distance $\lambda_c \sim 1/m_c$, so the field inside the body is almost constant, $\phi \approx \phi_c$. All variations in the field thus takes place within the thin shell, which is of thickness $\Delta R \approx 1/m_c$.

The chameleon force on a test particle with mass M and coupling β is given by

$$F_\phi = -\frac{\beta}{M_{Pl}} M \nabla \phi. \quad (3.36)$$

Thus we see that the interior field (which is constant) will give no contribution to the chameleon force on a particle outside the body.

Let us now go back and look at another regime, where $\phi_i \gtrsim \phi_c$. This is called the *thick shell regime*. We assume the field initially to be sufficient displaced from ϕ_c that it immediately starts rolling. Hence there is no place

where the damping term is dominant. The interior solution for ϕ is obtained by taking eq. (3.31) in the limit $R_{roll} \rightarrow 0$ and replacing ϕ_c by ϕ_i . When we match this to the exterior solution, (3.33), we find

$$\phi(r) = \frac{\beta \rho_c r^2}{6M_{Pl}} + \phi_i, \quad 0 < r < R_c \quad (3.37)$$

and

$$\phi(r) \approx - \left(\frac{\beta}{4\pi M_{Pl}} \right) \frac{M_c e^{-m_b(r-R_c)}}{r} + \phi_b, \quad r > R_c. \quad (3.38)$$

If we equate these two solutions at $r = R_c$, we find that

$$\phi_i = \phi_b - 3 \frac{\beta \Phi_c}{M_{Pl}}.$$

Using the definition (3.35) of $\Delta R_c/R_c$, we find that this implies

$$\frac{\Delta R_c}{R_c} > 1. \quad (3.39)$$

Bodies with “thick shells” have sizes that are comparable to the range of the chameleon field, $1/m_c$. Thus they will not develop any thin shell. The field cannot adjust itself to the value ϕ_c which minimizes the effective potential inside the body.

Sufficiently small objects, like the satellites that will be used in the STEP- and GG-experiments will not have any thin shell, thus their entire mass will contribute to the exterior field. According to chameleon field theories, we should then expect that these experiments will detect EP-violations since the satellites don’t have any thin shell that will suppress the fifth force caused by the chameleon field.

3.4.1 Example: Thin shell at the Earth

Let us try to estimate the thickness of the thin shell for the Earth and show that the atmosphere must have a thin shell as well. We model the Earth as a sphere of radius $R_\oplus \simeq 6 \times 10^6$ m with homogeneous density $\rho_\oplus = 10^4$ kg/m³. Furthermore, we let the atmosphere be a 10 km thick layer around the Earth ($R_{atm} = R_\oplus + 10$ km) with homogeneous density 1 kg/m³. We treat the Earth as an isolated body, i.e. we neglect effects from compact objects as the Moon and the Sun. Far away from the Earth we assume that the matter density is approximated by the density of gas and dark matter in our local neighborhood of the galaxy, $\rho_G = 10^{-21}$ kg/m³.

We remember that the exterior solution of the equation of motion for a body with a thin shell is given by

$$\phi(r) \simeq - \left(\frac{\beta}{4\pi M_{Pl}} \right) \left(\frac{3\Delta R_c}{R_c} \right) \frac{M_c e^{-m_b r}}{r} + \phi_b,$$

with the thin shell condition

$$\frac{\Delta R_c}{R_c} = \frac{\phi_b - \phi_c}{6\beta M_{Pl} \Phi_c} \ll 1, \quad (3.40)$$

where Φ_c is the Newtonian potential at the surface of the object. So for the atmosphere we then have the thin shell condition

$$\frac{\Delta R_{atm}}{R_{atm}} = \frac{\phi_G - \phi_{atm}}{6\beta M_{Pl} \Phi_{atm}} \ll 1, \quad (3.41)$$

where $\Phi_{atm} = \rho_{atm} R_{atm}^2 / 6M_{Pl}^2$. For the atmosphere to have a thin shell, clearly the thickness of the shell must be less than the thickness of the atmosphere itself, which is about $10^{-3} R_{atm}$. Hence we require

$$\Delta R_{atm} / R_{atm} \lesssim 10^{-3}.$$

Using that $\rho_{atm} \approx 10^{-4} \rho_{\oplus}$, and thus $\Phi_{atm} \approx 10^{-4} \Phi_{\oplus}$, gives us

$$\frac{\Delta R_{\oplus}}{R_{\oplus}} = \frac{\phi_G - \phi_{atm}}{6\beta M_{Pl} \Phi_{\oplus}} < 10^{-7}. \quad (3.42)$$

Inserting the Earth radius, we find $\Delta R_{\oplus} \lesssim 10^{-1}$ m.

It can also be shown that EP-tests actually require the atmosphere to have a thin shell. The *Eötvös parameter* η , which is a measure of the difference in relative free-fall acceleration for two different bodies, can be approximated by

$$\eta \equiv 2 \frac{|a_1 - a_2|}{a_1 + a_2} \sim 10^{-4} \beta^2 \frac{\Delta R_{\oplus}}{R_{\oplus}}. \quad (3.43)$$

If we contradict (3.42) and put $\Delta R_{\oplus} / R_{\oplus} > 10^{-7}$ (that is, assuming that the atmosphere have no thin shell) and use β of order 1, it will clearly lead to a violation of the current bound, which is at $\eta \sim 10^{-13}$ [15].

3.5 Constraints on chameleon parameters

Let us finally take a brief look at some further constraints on the parameters in the chameleon field theory that are found from fifth force searches and EP-tests. We will use these along with the above results to estimate the interaction range of the chameleon field at different scales.

Since EP-tests and fifth force searches are usually done in vacuum, let us first find an approximate solution for the chameleon field inside a vacuum chamber. We model the chamber as perfectly empty and spherical with radius R_{vac} . Outside the chamber, we assume that $\rho = \rho_{atm}$. It is found that the vacuum solution is analogous to the solution for a compact object

with a thin shell. The chameleon field takes the value $\phi \sim \phi_{vac}$ inside the chamber, and the corresponding mass of the field is

$$m_{vac} = \sqrt{V_{,\phi\phi}(\phi_{vac})} = R_{vac}^{-1}, \quad (3.44)$$

so the Compton wavelength of small fluctuations about the field value ϕ_{vac} is equal to the radius of the vacuum chamber, R_{vac} .

From fifth force searches, one has obtained a bound on ϕ_{vac} at

$$\phi_{vac} \lesssim 10^{-28} M_{Pl}. \quad (3.45)$$

EP-tests using the current bound on the Eötvös parameter, $\eta \sim 10^{-13}$, gives a weaker constraint, $\phi_{vac} \lesssim 10^{-26} M_{Pl}$, that we henceforth will ignore.

Let us now use these results along with the thin shell conditions for the Earth and apply them to a given potential, $V(\phi)$. We choose the inverse power-law potential that we used in the tracker field example in chapter 1;

$$V(\phi) = M^{4+n} \phi^{-n}, \quad (3.46)$$

where we remember that M , which may represent the energy scale of the field, has units of mass and that n is an arbitrary, positive constant. We are now interested in finding constraints on the energy scale, M . We saw that EP-tests required the thin shell condition

$$\frac{\Delta R_{\oplus}}{R_{\oplus}} = \frac{\phi_G - \phi_{atm}}{6\beta M_{Pl} \Phi_{\oplus}} < 10^{-7}. \quad (3.47)$$

By definition, ϕ_G is the value of ϕ which minimizes the effective potential with $\rho = \rho_G$, i.e.

$$V_{,\phi}^{eff}(\phi_G) = V_{,\phi}(\phi_G) + \frac{\beta}{M_{Pl}} \rho_G e^{\beta\phi_G/M_{Pl}} = 0. \quad (3.48)$$

Substituting for $V(\phi)$ and solving for ϕ_G gives us the solution

$$\phi_G = \left(\frac{nM^{4+n}M_{Pl}}{\beta\rho_G} \right)^{\frac{1}{n+1}}. \quad (3.49)$$

Inserting $\rho_G = 10^{-21} \text{ kg/m}^3$ and $\Phi_{\oplus} = 10^{-9}$, we can rewrite the thin shell condition (3.47) as a bound on M . This gives

$$M < \left(\frac{6^{n+1}}{n} \right)^{\frac{1}{n+4}} \beta^{\frac{n+2}{n+4}} \times 10^{\frac{15n-7}{n+4}} \times (1\text{mm})^{-1}. \quad (3.50)$$

We can also make a constraint on M from the results of the fifth force search. We had that $\phi_{vac} \lesssim 10^{-28} M_{Pl}$. If we insert our potential into eq. (3.44) and choose $R_{vac} = 1 \text{ m} = 10^{34} M_{Pl}^{-1}$ we get

$$R_{vac}^{-2} = 10^{-68} M_{Pl}^2 = n(n+1)M^{4+n}\phi_{vac}^{-(n+2)}. \quad (3.51)$$

From this we can find

$$M \lesssim [n(n+1)]^{-1/(4+n)} \times 10^{3n/(4+n)} \times (1\text{mm})^{-1}. \quad (3.52)$$

If we insert n and β of order unity in eqs. (3.50) and (3.52) we find that in both cases, the bound on M is of order $(1\text{mm})^{-1}$, or equivalently, 10^{-3} eV. It is interesting that this is in fact the same energy scale that is associated with the dark energy.

Let us now use these constraints to find bounds on the interaction range of the chameleon field. From eq. (3.24) (still using the exponential function for Ω) we may find the range of interaction for the field in the atmosphere (m_{atm}^{-1}), in the solar system (m_G^{-1}) and on cosmological scales (m_0^{-1}):

$$\begin{aligned} m_{atm}^2 &= V_{,\phi\phi}(\phi_{atm}) + \frac{\beta^2}{M_{Pl}^2} \rho_{atm} e^{\beta\phi_{atm}/M_{Pl}}, \\ m_G^2 &= V_{,\phi\phi}(\phi_G) + \frac{\beta^2}{M_{Pl}^2} \rho_G e^{\beta\phi_G/M_{Pl}}, \\ m_0^2 &= V_{,\phi\phi}(\phi_0) + \frac{\beta^2}{M_{Pl}^2} \rho_0 e^{\beta\phi_0/M_{Pl}}, \end{aligned} \quad (3.53)$$

where $\rho_0 \approx 10^{-26}$ kg/m³ is the current energy density of the universe and ϕ_0 is the corresponding field value at cosmological scales. If we substitute our bounds on M and use $n \lesssim 2$ and β of order unity, we find

$$\begin{aligned} m_{atm}^{-1} &\lesssim 1\text{mm} - 1\text{cm}, \\ m_G^{-1} &\lesssim 10 - 10^4\text{AU}, \\ m_0^{-1} &\lesssim 10^{-1} - 10^3\text{pc}. \end{aligned} \quad (3.54)$$

Here we see some of the earlier discussed features of the chameleon field. It is short ranged in the atmosphere, while in the solar system it is sufficiently long ranged to be essentially free.

A weakness with our potential is, as we see, that it implies an interaction range which is smaller than $H_0^{-1} \sim 10^9$ pc on cosmological scales. Then it follows that m_0 is too large to be rolling on cosmological time scales today. Hence, we should consider other potentials in chameleon theories for them to be viable quintessence models. In ref. [14] it is shown that, given a potential

$$V(\phi) = M^4 \exp\left(\frac{M^n}{\phi^n}\right), \quad (3.55)$$

with $M \approx 10^{-3}$ eV, we can obtain a so-called *attractor solution* for the chameleon field, which is analogous to the tracker solution for quintessence. For a wide range of initial conditions, the field value will follow the minimum of the effective potential which will change as the matter density in

the universe changes.

It is further shown that when the vacuum energy density becomes completely dominant at some time in the future, we get $V \rightarrow M^4$, that is, the universe will be driven towards a pure de Sitter phase. The matter density will then decrease exponentially (see (2.22)), and the dynamics of the chameleon field will to a good approximation be determined solely by $V(\phi)$. Thus, the evolution of the chameleon will in the future converge towards that of normal quintessence.

3.6 Summary

It is time to give an overview of what we have learned in this chapter:

- The chameleon field is a non-minimal coupled scalar field constructed to evade EP-violations, that is, to satisfy known local tests of gravity at the Earth.
- The effective potential makes the mass of the field dependent on the local matter density.
- Large compact objects develop thin shells, which suppress the chameleon force, i.e. only the thin shell gives a contribution to the force. The thin shell condition is given by (3.35). At the Earth, the thin shell condition gives

$$\Delta R_{\oplus}/R_{\oplus} < 10^{-7}.$$

- Sufficiently small objects do not develop thin shells. Thus, the chameleon theory predicts future space-borne tests of gravity to show EP-violations.
- Given an inverse power-law potential, $V(\phi) = M^{4+n}\phi^{-n}$, we can, by considering local tests of gravity, show the mass scale M to be of the same order as dark energy, i.e. $M \sim 10^{-3}$ eV.

If there are existing chameleon fields, this will lead to a slightly different gravitational theory than that of general relativity (because of the coupling to matter). Many future experiments, especially in space, will test the viability of general relativity and eventually rule out the chameleon field and other alternative gravitational theories. In this thesis, we wish to put the chameleon theory onto another testing ground, namely; how will the presence of a chameleon field affect *gravitational waves*?

Chapter 4

Gravitational waves

4.1 Introduction

Our knowledge of the Universe is almost entirely based upon observations of electromagnetic radiation detected by telescopes and satellite-borne detectors. As new technology has developed in the modern ages, we are now able to observe parts of the electromagnetic spectrum, like x-ray or gamma radiation, that earlier has been impossible because of the disturbing influence of the Earth's atmosphere. This has made us able to observe exotic phenomena like gamma ray bursts and processes around black holes. Hopefully, in a not-too-far future, a whole new area of observations of the Universe will become available to us, which is not based upon electromagnetic radiation, but upon gravitational waves.

Gravitational waves may simply be considered as ripples in spacetime, propagating with the speed of light, moving outward from a massive object or some other mass distribution. Though they never have been observed directly, there are indirect evidence of their existence from observations of the *Hulse-Taylor binary*, a system of two inspiraling neutron stars (which we will consider later in this chapter).

Gravitational waves do not exist in Newtonian theory, they are a fully relativistic phenomenon. This can be seen if we consider the expression for the Newtonian gravitational potential $\Phi(\mathbf{r}, t)$ in a point at a distance \mathbf{r} from a mass distribution with density $\rho(\mathbf{r}', t)$:

$$\Phi(\mathbf{r}, t) = -G \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'. \quad (4.1)$$

We see that the time t occurs at both sides of the equation. This means that a change in the source will instantaneously find place in \mathbf{r} and thus, no wave phenomena are possible. Besides, since \mathbf{r} is at an arbitrary distance from the source, this is in conflict with special relativity; no information can travel faster than the speed of light! Thus, we require a gravitational

theory where this constraint is fulfilled. This naturally leads us to general relativity.

4.2 Linearization of Einstein's field equations

In this section we will see how the equation for gravitational waves can be derived from Einstein's field equations.

To derive the wave-equation, we first need to make a linear approximation of the field equations. In this approximation we assume that the gravitational field is weak. That is, if we have a limited extended mass distribution, the gravitational field is weak at distances much larger than the Schwarzschild radius of the mass.

I will here follow the outline given in [2], section 9.1.

We start by looking at small perturbations, $h_{\mu\nu}$, in Minkowski spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$

where $\eta_{\mu\nu}$ is the Minkowski metric. If we make a coordinate transform of the metric components, we get

$$g_{\rho'\sigma'} = \frac{\partial x^\mu}{\partial x^{\rho'}} \frac{\partial x^\nu}{\partial x^{\sigma'}} g_{\mu\nu}. \quad (4.2)$$

Then we look at an infinitesimal coordinate transform at a point P:

$$x^{\mu'}(P) = x^\mu(P) + \xi^\mu(P), \quad |\xi^\mu| \ll |x^\mu|$$

and we get

$$g_{\rho'\sigma'}|_{x^\mu} = \frac{\partial x^\mu}{\partial x^{\rho'}} \frac{\partial x^\nu}{\partial x^{\sigma'}} g_{\mu\nu}|_{(x^{\mu'} - \xi^\mu)}. \quad (4.3)$$

We do all calculations only to first order in $h_{\mu\nu}$ and ξ^μ and their derivatives. Since

$$\frac{\partial x^\mu}{\partial x^{\rho'}} = \delta_\rho^\mu - \xi_{,\rho}^\mu$$

and

$$g_{\mu\nu}|_{(x^{\mu'} - \xi^\mu)} = \eta_{\mu\nu} + h_{\mu\nu},$$

we get, to first order

$$g_{\rho'\sigma'} = (\delta_\rho^\mu - \xi_{,\rho}^\mu)(\delta_\sigma^\nu - \xi_{,\sigma}^\nu) \approx \eta_{\rho\sigma} + h_{\rho\sigma} - \xi_{\sigma,\rho} - \xi_{\rho,\sigma}. \quad (4.4)$$

Since we also have

$$g_{\rho'\sigma'} = \eta_{\rho\sigma} + h_{\rho'\sigma'},$$

we get

$$h_{\rho'\sigma'} = h_{\rho\sigma} - \xi_{\sigma,\rho} - \xi_{\rho,\sigma}. \quad (4.5)$$

This kind of transformation is called a *gauge transformation*. But, as we can see, the components of the metric tensor are not gauge invariant in this approximation.

The Riemann curvature tensor is given by

$$R_{\nu\alpha\beta}^{\mu} = \Gamma_{\nu\beta,\alpha}^{\mu} - \Gamma_{\nu\alpha,\beta}^{\mu} + \Gamma_{\nu\beta}^{\rho}\Gamma_{\rho\alpha}^{\mu} - \Gamma_{\nu\alpha}^{\rho}\Gamma_{\rho\beta}^{\mu}. \quad (4.6)$$

To first order in $h_{\mu\nu}$ we may neglect products of Christoffel symbols, and the Riemann tensor is thus reduced to

$$R_{\alpha\mu\beta\nu} = \Gamma_{\alpha\mu\nu,\beta} - \Gamma_{\alpha\mu\beta,\nu}, \quad (4.7)$$

where

$$\Gamma_{\alpha\mu\nu} = \frac{1}{2}(h_{\mu\alpha,\nu} + h_{\nu\alpha,\mu} - h_{\mu\nu,\alpha}). \quad (4.8)$$

If we insert this into the Riemann tensor, we get

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}(h_{\nu\alpha,\mu\beta} + h_{\mu\beta,\alpha\nu} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu}). \quad (4.9)$$

Thus, the Ricci tensor to first order is

$$R_{\mu\nu} = \eta^{\alpha\beta}R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\nu,\alpha\mu}^{\alpha} + h_{\mu,\alpha\nu}^{\alpha} - h_{,\mu\nu} - \square h_{\mu\nu}) \quad (4.10)$$

where $\square = \partial_{\mu}\partial^{\mu} = -\frac{\partial^2}{\partial t^2} + \nabla^2$ is the d'Alembert wave operator in Minkowski spacetime. Contracting once more with $\eta^{\mu\nu}$, we obtain the Ricci scalar:

$$R \equiv \eta^{\mu\nu}R_{\mu\nu} = h_{,\mu\nu}^{\mu\nu} - \square h, \quad h \equiv h_{\mu}^{\mu}. \quad (4.11)$$

Finally, we obtain the Einstein tensor:

$$G_{\mu\nu} = \frac{1}{2} \left[h_{\nu,\alpha\mu}^{\alpha} + h_{\mu,\alpha\nu}^{\alpha} - h_{,\mu\nu} - \square h_{\mu\nu} - \eta_{\mu\nu} \left(h_{,\alpha\beta}^{\alpha\beta} - \square h \right) \right]. \quad (4.12)$$

The linearised field equations are thus given by

$$h_{\nu,\alpha\mu}^{\alpha} + h_{\mu,\alpha\nu}^{\alpha} - h_{,\mu\nu} - \square h_{\mu\nu} - \eta_{\mu\nu} \left(h_{,\alpha\beta}^{\alpha\beta} - \square h \right) = 16\pi GT_{\mu\nu}. \quad (4.13)$$

If we introduce

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \quad (4.14)$$

the field equations are simplified to

$$\bar{h}_{\nu,\alpha\mu}^{\alpha} + \bar{h}_{\mu,\alpha\nu}^{\alpha} - \square \bar{h}_{\mu\nu} - \eta_{\mu\nu} \bar{h}_{,\alpha\beta}^{\alpha\beta} = 16\pi GT_{\mu\nu}. \quad (4.15)$$

We can simplify the field equations further by performing a gauge transformation of the metric:

$$\bar{h}'_{\alpha\beta} = \bar{h}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta}\xi_{,\mu}^{\mu}.$$

The transformed divergence of $\bar{h}_{\alpha,\beta}$ then becomes

$$\bar{h}'^{\beta}_{\alpha,\beta} = \bar{h}^{\beta}_{\alpha,\beta} - \square \xi_{\alpha}. \quad (4.16)$$

If we choose gauge conditions such that $\square \xi_{\alpha} = \bar{h}'^{\beta}_{\alpha,\beta}$, we obtain

$$\bar{h}^{\beta}_{\alpha,\beta} = 0. \quad (4.17)$$

(where I have dropped the prime on the \bar{h}). This is called a Lorenz gauge. In this gauge, the field equations are reduced to

$$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \quad (4.18)$$

This is the wave equation for gravitational waves. The waves are represented by the metric perturbations, $\bar{h}_{\mu\nu}$, while the energy-momentum tensor represents the source. Re-inserting c , the solution of the wave equation can be written in terms of retarded potentials as

$$\bar{h}_{\mu\nu}(\mathbf{r}, t) = \frac{4G}{c^4} \int \frac{T_{\mu\nu}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'. \quad (4.19)$$

Note the similarity and the difference with the Newtonian gravitational potential (4.1). We see that general relativity provides a physical solution since it satisfies the causality principle (i.e. the gravitational interaction can not propagate faster than c), in contrast with Newton's law of gravitation.

4.3 Gravitational waves in vacuum

Gravitational waves are represented by plane wave solutions of the linearized field equations. In empty space (vacuum), the field equations are reduced to

$$\square \bar{h}_{\mu\nu} = 0. \quad (4.20)$$

A possible solution is then given by

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \cos(k_{\alpha} x^{\alpha}), \quad (4.21)$$

where $A_{\mu\nu}$ is a constant symmetric tensor of rank 2 representing the amplitude, and k_{α} is the wave 4-vector in the direction of propagation. Inserting the plane wave solution into the vacuum field equation, we find that

$$k_{\alpha} k^{\alpha} = 0. \quad (4.22)$$

This means that k_{α} is a so-called null vector, which implies that the gravitational waves are propagating with the speed of light in vacuum. The components of the wave vector can be written

$$k^{\mu} = (\omega, k^1, k^2, k^3),$$

where

$$\omega = -k_{\mu} U^{\mu}$$

is the frequency measured by an observer with four-velocity U^{μ} .

4.3.1 Polarization

We will now go on to show that, for a given wave vector k_α , there are only two possible polarizations of gravitational waves. By using the Lorenz gauge (4.17) we can find that

$$k^\alpha A_{\alpha\beta} = 0, \quad (4.23)$$

which means that the wave is transverse (i.e. orthogonal) to $A_{\alpha\beta}$. To specify the gauge further, we can choose ξ_μ (see 4.16 and 4.17) so that we can require

$$U^\alpha A_{\alpha\beta} = A_\alpha^\alpha = 0.$$

Then, we have only two free components of $A_{\alpha\beta}$, which represent two polarizations for a plane gravitational wave. The gauge we have chosen is called the *transverse traceless gauge* (referred to as the TT gauge). In a co-moving frame, i.e. a reference frame where $U^\mu = (1, 0, 0, 0)$, the TT-gauge gives us

$$h_{\mu 0}^{TT} = 0, \quad h_{jk,k}^{TT} = 0, \quad h_{ii}^{TT} = 0.$$

The first equation tells us that only the spatial components of $h_{\mu\nu}$ are non-zero, the second tells us that the spatial components are divergence-free, while the last one says they are trace-free. In this gauge, we have no distinction between $\bar{h}_{\mu\nu}$ and $h_{\mu\nu}$ since $h = h^\mu_\mu = 0$. The components of the metric perturbations can thus be written as

$$h_{\mu\nu}^{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx} & h_{xy} & 0 \\ 0 & h_{xy} & -h_{yy} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.24)$$

for a wave travelling in the z-direction.

The two possible polarizations of gravitational waves are called + polarization and \times polarization. + polarization correspond to the case where $h_{xx} \neq 0$ or $h_{yy} \neq 0$ and $h_{xy} = 0$, while \times polarization correspond to $h_{xx} = h_{yy} = 0$ and $h_{xy} \neq 0$. Thus, the components can also be written as

$$h_{\mu\nu}^{TT} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (4.25)$$

4.3.2 Physical effects

The physical effect of gravitational waves can be illustrated by how the waves will affect a ring (made up of some material) perpendicular to the

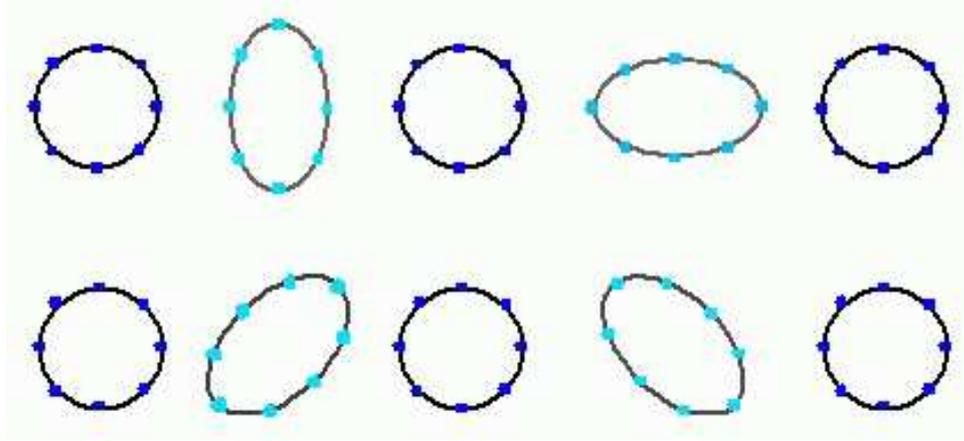


Figure 4.1: Illustration of how a passing gravitational wave with + (top) or \times (bottom) polarization affects a ring of particles.

direction of propagation (see figure). The wave can be observed as a periodical deformation of the ring. Let the ring have radius R . When hit by a gravitational wave, it will get a relative elongation given by

$$h(\mathbf{x}, t) = \frac{2\Delta R}{R}, \quad (4.26)$$

where $h(\mathbf{x}, t)$ is the wave amplitude at time t and position \mathbf{x} .

However, gravitational waves have a rather tiny amplitude when reaching Earth, even from phenomena like Supernovae or merging of black holes. We will discuss the difficulties of detecting gravitational waves later in this chapter.

4.4 Gravitational radiation from a quadrupole source

According to the gravitational wave equation (4.18), a source of gravitational radiation can be represented by the energy-momentum tensor, $T_{\mu\nu}$. We let the gravitational waves be represented by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h,$$

where $h_{\mu\nu}$ is a solution of the gravitational wave equation, $\eta_{\mu\nu}$ is the metric of Minkowski spacetime, and $h = h^\mu{}_\mu$. Let's consider a matter distribution localised near the origin O , in which the particles are moving slowly compared to the speed of light. We will use a far field approximation, i.e. we calculate the field at a distance r that is large compared to the extension of the matter distribution. Then we can write the solution (4.19) as

$$\bar{h}_{\mu\nu}(t, \mathbf{r}) = \frac{4G}{r} \int T_{\mu\nu}(t - r, \mathbf{r}') dV', \quad (4.27)$$

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where we have put $c = 1$. Here \mathbf{r} represents the spatial coordinates of the field point at which $\bar{h}_{\mu\nu}$ is determined, while \mathbf{r}' represents the spatial coordinates of the source.

In this zone far from the source, the radiation looks like plane waves, in which case the radiative part of $\bar{h}_{\mu\nu}$ is determined by the spatial part, \bar{h}_{ij} , only. Thus, we need only to consider $\int T^{ij} dV$. We remember that, from conservation of momentum and energy, follows $T^{\mu\nu}_{;\nu} = 0$. This gives us the component equations

$$T^0{}_{,0} + T^{0k}{}_{,k} = 0, \quad (4.28)$$

$$T^{i0}{}_{,0} + T^{ik}{}_{,k} = 0. \quad (4.29)$$

Furthermore, we will also use the integral identity

$$\int (T^{ik} x^j)_{,k} dV = \int T^{ik}{}_{,k} x^j dV + \int T^{ij} dV, \quad (4.30)$$

where we have used that $x^j{}_{,k} = \delta_k^j$. The integrals are taken over a region of space enclosing the source, so that $T^{\mu\nu} = 0$ on the boundary of the region. If we then transform the integral on the left hand side by using Gauss' integral theorem, we see that it vanishes, and we are left with

$$\int T^{ij} dV = - \int T^{ik}{}_{,k} x^j dV = \int T^{i0}{}_{,0} x^j dV = \frac{d}{dt} \int T^{i0} x^j dV. \quad (4.31)$$

Using the symmetry in the indices i and j , we can write

$$\int T^{ij} dV = \frac{1}{2} \frac{d}{dt} \int (T^{i0} x^j + T^{j0} x^i) dV. \quad (4.32)$$

Furthermore, we have

$$\int (T^{0k} x^i x^j)_{,k} dV = \int T^{0k}{}_{,k} x^i x^j dV + \int (T^{i0} x^j + T^{j0} x^i) dV. \quad (4.33)$$

Again, we see that the left hand side vanishes when we use Gauss' integral theorem. By using equation (4.28) we get

$$\int (T^{i0} x^j + T^{j0} x^i) dV = \int T^{00}{}_{,0} x^i x^j dV = \frac{d}{dt} \int T^{00} x^i x^j dV. \quad (4.34)$$

Since the source particles are non-relativistic, we have that $T^{00} \approx \rho$. Using eq. (4.32) and (4.34) we then get

$$\int T^{ij} dV = \frac{1}{2} \frac{d^2}{dt^2} \int \rho x^i x^j dV. \quad (4.35)$$

The integral term on the right hand side is defined as the *quadrupole moment* of the source:

$$q^{ij} = \int \rho x^i x^j dV. \quad (4.36)$$

Equation (4.27) can then be written as

$$\bar{h}_{ij}(t, \mathbf{r}) = \frac{2G}{r} \frac{d^2}{dt^2} \left[\int \rho x_i x_j dV \right]_{t'=t-r} = \frac{2G}{r} \ddot{q}_{ij}, \quad (4.37)$$

where the indice $t' = t - r$ indicates that the integral is evaluated at the retarded time. This tells us that the gravitational radiation produced by a non-relativistic, isolated object is proportional to the second time derivative of the quadrupole moment of the source at the emission time.

Note: Remember that we assumed the distance from the source to the field point to be large compared to the size of the source. If we had not made this assumption, (4.37) would have contained extra terms, indicating radiation from higher order multipole moments than the quadrupole [18].

Hence, now that the assumption is made, we can ignore higher order multipoles.

Now, let's try to find an expression for the total power radiated gravitationally by our non-relativistic source. We start by expanding the Newtonian potential ϕ in powers of r :

$$\phi = - \left(\frac{M}{r} + \frac{d_j n^j}{r^2} + \frac{3Q_{ij} n^i n^j}{2r^3} + \dots \right), \quad n^i = \frac{x^i}{r}, \quad (4.38)$$

where d_j is the dipole moment of the source, defined by

$$d_j \equiv \int \rho x_j dV, \quad (4.39)$$

and Q_{ij} is the trace-free part of the quadrupole moment, given by

$$Q_{ij} \equiv \int \rho \left(x_i x_j - \frac{1}{3} \delta_{ij} r^2 \right) dV = q_{ij} - \frac{1}{3} \delta_{ij} q^k_k. \quad (4.40)$$

In the transverse traceless gauge, we can introduce an effective energy-momentum tensor for gravitational waves:

$$T_{\mu\nu}^{GW} = \frac{1}{32\pi} \langle h_{ik,\mu} h_{ik,\nu} \rangle, \quad (4.41)$$

where $\langle \rangle$ denotes average over wavelengths.

The total power crossing a sphere of radius r at a time t is given by

$$P(t, r) = \int T_{0r}^{GW} r^2 d\Omega. \quad (4.42)$$

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Using eq. (4.37) in the transverse traceless gauge, we find

$$\begin{aligned} T_{0r}^{\text{GW}} &= \frac{1}{32\pi} \langle \dot{h}_{ik} \dot{h}_{ik} \rangle \\ &= \frac{1}{8\pi r^2} \left\langle \ddot{Q}_{jk} \ddot{Q}_{jk} - 2n_i \ddot{Q}_{ij} \ddot{Q}_{jk} n_k + \frac{1}{2} \left(n_j \ddot{Q}_{jk} n_k \right)^2 \right\rangle, \end{aligned} \quad (4.43)$$

which is the energy flux. The n 's are the components of the unit vector \mathbf{n} representing the direction of the waves. The total radiated power (i.e. the energy-loss) can then be found by averaging the flux over all directions and multiplying it with 4π . For this calculation we need the expressions

$$\begin{aligned} \int d\Omega &= 4\pi, \quad \int n_i n_j d\Omega = \frac{4\pi}{3} \delta_{ij}, \\ \int n_i n_j n_k n_l d\Omega &= \frac{4\pi}{15} (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}). \end{aligned} \quad (4.44)$$

Now we can insert eq. (4.43) into (4.42) and use (4.44), and thus find the expression for the energy-loss from a slowly moving source:

$$L_{\text{GW}} = P(t, r) = \frac{G}{5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle. \quad (4.45)$$

4.4.1 Example: Binary systems

Let's consider two stars with masses m_1 and m_2 in a circular orbit of radius r about each other [19, 20]. We treat the stars as point masses. Let r_1 and r_2 be their respective distances from their common center of mass. Then we have

$$m_1 r_1 = m_2 r_2 = \mu r, \quad (4.46)$$

where μ is the reduced mass of the system, given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

We choose the z -axis to be the axis of rotation and Ωt to be the azimuthal angle from the x -axis to the line joining the masses, where Ω is the orbital angular velocity. The xx -component of the quadrupole moment is then given by

$$Q_{xx} = (m_1 r_1^2 + m_2 r_2^2) \cos^2 \Omega t,$$

where we have used that $m_i r_i^2/3 = \text{constant}$, $i = 1, 2$, for the two masses. We can rewrite this equation as

$$Q_{xx} = \frac{1}{2} \mu r^2 \cos 2\Omega t + \text{constant}. \quad (4.47)$$

Similarly, we get the other components:

$$Q_{yy} = -\frac{1}{2}\mu r^2 \cos 2\Omega t, \quad (4.48)$$

$$Q_{xy} = Q_{yx} = \frac{1}{2}\mu r^2 \sin 2\Omega t. \quad (4.49)$$

Then we can take the 3rd time derivatives of these components and calculate the energy-loss from eq. (4.45). This will give us

$$L_{GW} = \frac{G}{5}(2\Omega)^6 \left(\frac{1}{2}\mu r^2\right)^2 \langle 2 \sin^2 2\Omega t + 2 \cos^2 2\Omega t \rangle. \quad (4.50)$$

If we use the identity $\sin^2 2\Omega t + \cos^2 2\Omega t = 1$ and Kepler's 3rd law

$$\Omega^2 = \frac{GM}{r^3}, \quad (4.51)$$

where $M = m_1 + m_2$, we will find

$$L_{GW} \equiv \left(-\frac{dE}{dt}\right)_{GW} = \frac{32}{5} \frac{G^4 M^3 \mu^2}{r^5}. \quad (4.52)$$

The total energy of the system is

$$\begin{aligned} E &= \frac{1}{2} (m_1 r_1^2 + m_2 r_2^2) \Omega^2 - \frac{Gm_1 m_2}{r} \\ &= -\frac{1}{2} \frac{G\mu M}{r}. \end{aligned} \quad (4.53)$$

The loss of energy leads to a decrease in the orbital separation r and hence a decrease in the orbital period P . By using Kepler's 3rd law and the above expression for energy, we find

$$\begin{aligned} \frac{1}{P} \frac{dP}{dt} &= \frac{3}{2} \frac{1}{r} \frac{dr}{dt} \\ &= -\frac{3}{2} \frac{1}{E} \frac{dE}{dt} \\ &= -\frac{96}{5} \frac{G^3 M^2 \mu}{r^4}. \end{aligned} \quad (4.54)$$

4.4.2 The Hulse-Taylor binary pulsar

A radio survey for pulsars¹ in our galaxy made by Russell Hulse and Joseph Taylor in 1974, at the Arecibo Radio Telescope in Puerto Rico, discovered the unusual system PSR 1913+16 (later known as the Hulse-Taylor binary pulsar). This is a system that consists of two neutron stars with masses about $1.4 M_{\odot}$. Observations of this system allowed Hulse and Taylor to test general relativity to great precision including the verification for the existence of gravitational waves as predicted by GR.

Some of the most important observed data are summarized in [21]:

$$\begin{aligned}
 \text{pulsar mass } M_p &= 1.4408 \pm 0.0003 M_{\odot}, \\
 \text{companion mass } M_c &= 1.3873 \pm 0.0003 M_{\odot}, \\
 \text{eccentricity } e &= 0.6171338 \pm 0.000004, \\
 \text{binary orbit period } P &= 0.322997462727 d, \\
 \text{orbit decay rate } \dot{P} &= (-2.4211 \pm 0.0014) \times 10^{-12} \text{s/s}.
 \end{aligned} \tag{4.55}$$

We will now show that we can calculate the decrease of orbit period due to gravitational radiation from these numbers. The calculations will be pretty similar to those we performed in the last example. To make things a bit simpler, let us first make the approximation $M_p \approx M_c = m$. The components of the quadrupole moment of the system are then given by

$$\begin{aligned}
 Q_{xx} &= mR^2 \cos 2\Omega t, \\
 Q_{yy} &= -mR^2 \cos 2\Omega t, \\
 Q_{xy} &= Q_{yx} = mR^2 \sin 2\Omega t,
 \end{aligned} \tag{4.56}$$

where R is the distance from a star to the center of mass of the system. We can express the energy loss as (when c is re-inserted)

$$-\frac{dE}{dt} = \frac{G}{5c^5} (2\Omega)^6 \langle \ddot{Q}_{xx}^2 + \ddot{Q}_{yy}^2 + 2\ddot{Q}_{xy}^2 \rangle = \frac{128}{5} \frac{G}{c^5} \Omega^6 m^2 R^4. \tag{4.57}$$

Then we will use the Keplerian relation

$$\frac{dP}{P} = -\frac{3}{2} \frac{dE}{E}$$

to find the orbit decay. The total energy of a binary pair with equal masses m separated by a distance $2R$ is given by

$$E = mv^2 - \frac{Gm^2}{2R} \tag{4.58}$$

¹Generally, a pulsar is a rapidly rotating, magnetized neutron star. This generates a circulating plasma that serves as a source of beamed radio waves detectable on Earth as periodic pulses.

where the velocity is determined by the Newtonian equation of motion

$$m \frac{v^2}{R} = \frac{Gm^2}{(2R^2)} \rightarrow v^2 = \frac{Gm}{4R}.$$

The total energy comes out to be

$$E = -\frac{Gm^2}{4R}. \quad (4.59)$$

We want to express the energy in terms of the orbit period. This can be done by replacing R using the expression for v^2 and the relation $v = \Omega R = \frac{2\pi R}{P}$:

$$R = \frac{Gm}{4} \left(\frac{P}{2\pi R} \right)^2 \rightarrow R^3 = \frac{Gm}{16\pi^2} P^2.$$

Inserting this into the energy expression gives

$$E = -m^{5/3} \left(\frac{\pi G}{2} \right)^{2/3} P^{-2/3}. \quad (4.60)$$

Then we can express the period decrease to the energy loss by

$$\frac{dP}{dt} = \dot{P} = -\frac{3P}{2E} \frac{dE}{dt}. \quad (4.61)$$

By inserting for dE/dt , E and R we find

$$\dot{P} = -\frac{48\pi}{5c^5} \left(\frac{4\pi Gm}{P} \right)^{5/3}. \quad (4.62)$$

To find the correct expression for the Hulse-Taylor binary, we must take into account that the orbit is elliptical with eccentricity as given in the found data above. This is done by multiplying by a function (see [20])

$$f(e) = \frac{1 + (73/24)e^2 + (37/96)e^4}{(1 - e^2)^{7/2}}. \quad (4.63)$$

The two stars also have different masses. It can be shown [19] that this is taken into account by making the replacement

$$(2m)^{5/3} \rightarrow 4M_p M_c (M_p + M_c)^{-1/3}.$$

Thus the exact GR prediction of the orbit decrease is given by

$$\begin{aligned} \dot{P}_{GR} &= -\frac{192\pi M_p M_c}{5c^5 (M_p + M_c)^{1/3}} \left(\frac{2\pi G}{P} \right)^{5/3} f(e) \\ &= -(2.40247 \pm 0.00002) \times 10^{-12} \text{ s/s} \end{aligned} \quad (4.64)$$

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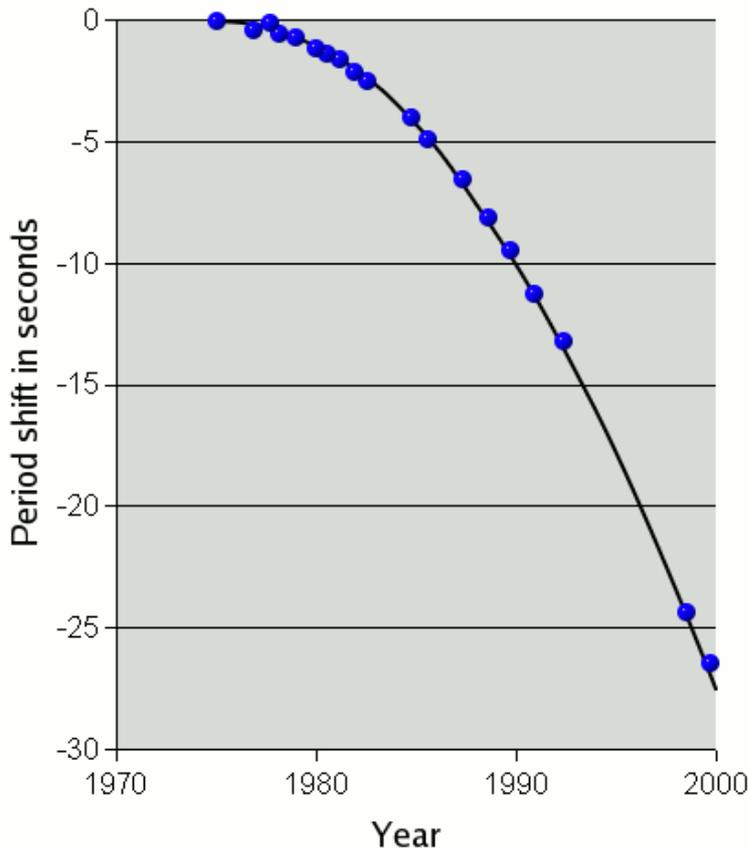


Figure 4.2: Illustration of the decrease in period of the Hulse-Taylor binary. The solid line represents the prediction from GR, while the dots are the observed values [21].

when the data in (4.55) are inserted. The observed value of \dot{P} requires a small correction, \dot{P}_{Gal} because of the relative acceleration between the solar system and the binary pulsar system, projected onto the line of sight [21]. The currently best available value of this correction is $\dot{P}_{Gal} = -(0.0128 \pm 0.0050) \times 10^{-12}$, which gives

$$\dot{P}_{corrected} = \dot{P}_{obs} - \dot{P}_{Gal} = -(2.4056 \pm 0.0051) \times 10^{-12} \text{ s/s}, \quad (4.65)$$

which is in excellent agreement with the theoretical prediction. This result gives a strong confirmation of the existence of gravitational radiation as predicted by GR.

This result is pretty incredible, taken into consideration that we have made a lot of simplifying assumptions, like the point mass approximation, and used simple classical relations like Kepler's laws.

But first and foremost, this result was a victory for general relativity, and Hulse and Taylor were awarded the Nobel Prize in Physics in 1993 for their observations.

4.5 Other sources?

Are quadrupoles the *only* sources for gravitational waves? Though we have argued that we can ignore higher order multipoles in our calculations above, we have not yet considered the dipole. We know from electromagnetism that dipole radiation is produced from sources with low internal velocities. The radiated effect is given by

$$L_{em} = \frac{dE}{dt} = \frac{2}{3c^3} \ddot{\mathbf{d}}_j \ddot{\mathbf{d}}_j, \quad (4.66)$$

where \mathbf{d} is the dipole moment evaluated at the retarded time $t - r/c$. If we express it as $d_j = ex_j$, where e is the elementary charge, we recognize eq. (4.66) as the Larmor formula.

Now, let's consider gravitational radiation. On dimensional grounds we might expect that the radiated effect from a dipole source would be given by

$$L_{GW}^d \propto \frac{G}{c^3} \ddot{\mathbf{d}}_j \ddot{\mathbf{d}}_j,$$

where the gravitational dipole moment is

$$d_j = \sum_A m_A x_j^A$$

and we have let $e^2 \rightarrow Gm^2$ in eq. (4.66). However, the second time derivative of the dipole moment will give us

$$\ddot{\mathbf{d}}_j = \sum_A m_A \ddot{x}_j^A = \sum_A \dot{p}_j^A, \quad (4.67)$$

where \mathbf{p}^A is the momentum of particle A . But since the total momentum of the system is conserved, $\dot{p}_j^A = 0$, we get $\ddot{\mathbf{d}}_j = 0$. So there is no dipole radiation in general relativity.

In electromagnetism there are magnetic dipoles as well, but we can show that the "mass magnetic" dipole moment is also zero in general relativity. The magnetic dipole moment will here be analogous to angular momentum:

$$\boldsymbol{\mu} \equiv \frac{1}{c} \sum_A \mathbf{x}^A \times (m_A \dot{\mathbf{x}}^A) = \frac{1}{c} \sum_A \mathbf{j}^A, \quad (4.68)$$

where \mathbf{j}^A is the angular momentum of particle A . Conservation of angular momentum gives $\dot{\boldsymbol{\mu}} = 0$. We can then conclude that in general relativity, gravitational radiation can be produced by quadrupole sources only.

4.6 Detection of gravitational waves

The observations of the Hulse-Taylor binary pulsar was a breakthrough since it was the first observation of the effects of gravitational waves. But, as mentioned earlier, this was just an *indirect* evidence of their existence. Astrophysicists have yet to make a direct observation or detection of gravitational waves.

I will finish this chapter with a short review of the most important ongoing and future projects concerning detection of gravitational waves.

As we have seen earlier in this chapter, an object will be slightly stretched or squeezed when hit by a gravitational wave. The main problem when trying to detect a gravitational wave this way, is that this stretching or squeezing is extremely small. The amplitude of a gravitational wave falls off as $1/r$ where r is the distance from the source. Even extreme phenomena like collisions of black holes will only result in rather tiny gravitational effects on Earth. A typical amplitude when reaching Earth is $h \sim 10^{-20}$. This means, according to (4.26), that an object with a length of 1 meter will be stretched or squeezed 10^{-20} meter when hit by the wave, that is, 10^{-5} times the size of a proton. So the main challenge is to make the detection equipment sufficiently sensitive.

4.6.1 LIGO

The largest ongoing GW-detection project is LIGO (Laser Interferometer Gravitational-wave Observatory), which started in 2002. It is operating from two observatories in unison; one in Livingston, Louisiana and one in Hanford, Washington. LIGO's concept for GW detection is not deformation of objects, but *laser interferometry*. The detectors (one at each observatory) consists of two 4 km long vacuum-tubes forming an L and containing laser interferometers. The concept behind interferometry is to find phase-shifts, in this application caused by incoming gravitational waves (see [22] for a more detailed description). The detector is most sensitive for waves with frequencies of order $10^2 - 10^3$ Hz, a spectrum which has coalescing black holes or Supernova core collapses as typical sources [23]. However, LIGO (or other ground based detectors) is not sensitive to frequencies below ~ 10 Hz because of seismic vibrations (disturbances in the surface of the Earth, traffic, etc.). It is desirable to detect waves with frequencies lower than 1 Hz and down to mHz scale, since these correspond to emission from binary systems, massive black holes merging at the center of galaxies and, possibly, even a cosmological GW background, i.e. relics from the Big Bang. To make such observations possible, space borne detectors are required.

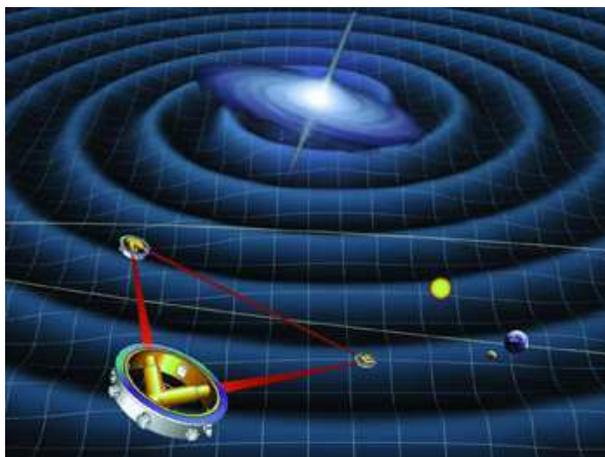


Figure 4.3: Illustration of the LISA detector [23].

4.6.2 LISA

LISA (Laser Interferometer Space Antenna) is a future project that has as its main goal to detect gravitational waves in the frequency band $0.03 \text{ mHz} - 0.1 \text{ Hz}$, and will thus be able to measure signals from the sources mentioned above and complement the signal spectrum from LIGO. Like LIGO, it will use laser interferometry, but at a much larger scale. LISA will consist of three identical spacecraft in orbit around the Sun, each separated by a distance of 5 million kilometers, forming an equilateral triangle. The center of this triangle will trace an Earth-like orbit in the ecliptic plane, about 20 degrees behind the Earth. Each spacecraft will contain two free-falling test masses. An incoming wave will cause a change in the distance between the test-mass pairs. This change will then be detected by interferometry between the laser beams travelling along the three arms. See more details at the LISA website, [23]. The large “armlength” of the LISA detector will give a higher sensitivity than any Earth-based detector. Also, since LISA will be space-borne, the problem of seismic vibrations is avoided. However, there are other sources, like pulsars or binary systems in our own galaxy that will make a foreground noise which may disturb the signals from interesting sources further away. Temperature changes or even quantum fluctuations in the detection device itself may also constrain the sensitivity.

Unfortunately, there is still some time until LISA will be launched. The project has suffered several delays already, and the earliest possible launch will take place around 2018-2020.

4.7 Summary

So let's finish off this chapter by giving a summary of the most important points.

- Gravitational waves are relativistic phenomena and do not exist in classical Newtonian theory.
- By considering small perturbations in a flat (Minkowski) background, i.e. put $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we can linearize and find the reduced field equations,

$$\square \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu},$$

which is a wave equation for gravitational waves. The solution can be expressed by retarded potentials, given by (4.19), which ensures that the gravitational waves cannot propagate faster than the speed of light.

- The energy-loss formula for gravitational waves from quadrupole sources is found by making a far-field approximation, i.e. by considering a non-relativistic source from a field point far from the source. We find that the energy-loss depends on the third time derivative of the quadrupole moment of the source. It is given by

$$L_{GW} = \left(-\frac{dE}{dt} \right)_{GW} = \frac{G}{5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle.$$

- For a system of two point masses (with total mass M and reduced mass μ) in a circular orbit with radius r around their common center of mass, the energy-loss is given by

$$\left(-\frac{dE}{dt} \right)_{GW} = \frac{32 G^4 M^3 \mu^2}{5 r^5},$$

which leads to a period decay given by

$$\frac{dP}{dt} = -\frac{96 G^3 M^2 \mu}{5 r^4} P.$$

- Dipole gravitational radiation does not exist according to GR, which follows from a consideration of momentum conservation.
- Observations of the binary pulsar PSR1913+16, made by R. Hulse and J. Taylor, has provided indirect evidence for the existence of gravitational waves. The observed period decay is in exact agreement with what is predicted by GR. Hopefully, ongoing or future projects will provide the first direct detections of gravitational waves.

Chapter 5

Brans-Dicke theory

5.1 An alternative gravitational theory

In the early 1960's Carl Brans and Robert H. Dicke developed a theory that they called a generalization of general relativity [24]. Their motivation to develop an alternative gravitational theory was that there were difficulties in incorporating Mach's principle into general relativity. In other words, they wanted to generalize general relativity into a theory that was compatible with Mach's principle. The theory is based on a scalar field in Riemannian geometry that will imply a varying gravitational "constant", i.e. the gravitational constant should be a function of the varying scalar field. Then, the gravitational interaction is mediated by both the metric tensor field of general relativity *and* by the scalar field. This is what we call a *scalar-tensor theory* of gravity.

See [24, 25] for more about the background of the theory and its incorporation of Mach's principle.

5.2 Equation of motion

Let's look at how the equations of motion will differ from those of general relativity. We have earlier outlined the equations of motion in standard GR. The Einstein-Hilbert action can be written as

$$\delta \int \left(\frac{R}{16\pi G} + \mathcal{L}_M \right) \sqrt{-g} d^4x = 0, \quad (5.1)$$

where R is the scalar curvature and \mathcal{L}_M is the matter Lagrangian, including all non-gravitational fields. In order to "generalize" this equation, a Lagrangian density of a scalar field ϕ is introduced and added inside the parenthesis. By letting the gravitational constant vary as $G \sim \phi^{-1}$, the gravitational part of the action is equal to ϕR . So the Brans-Dicke scalar field is

directly coupled to the gravitational field. The variation of the Brans-Dicke action is thus given by

$$\frac{1}{16\pi} \delta \int \left(\phi R - \frac{\omega_{\text{BD}}}{\phi} \phi^{;\mu} \phi_{;\mu} + 16\pi \mathcal{L}_M \right) \sqrt{-g} d^4x = 0, \quad (5.2)$$

where ω_{BD} is the dimensionless *Brans-Dicke coupling constant* (though it can be a function of the scalar field in generalized scalar-tensor theories). The matter Lagrangian is the same as in the standard case, so the equations of motion of matter are the same as in general relativity.

By varying with respect to ϕ , we obtain the equation of motion for the scalar field:

$$\left(\frac{2\omega_{\text{BD}}}{\phi} \right) \square \phi - \left(\frac{\omega_{\text{BD}}}{\phi^2} \right) \phi^{;\mu} \phi_{;\mu} + R = 0, \quad (5.3)$$

where \square here is the generally covariant d'Alembertian. $\square \phi$ is thus defined as the covariant divergence of $\partial^\mu \phi$:

$$\square \phi \equiv \phi^{;\mu}_{;\mu} = \frac{1}{\sqrt{-g}} [\sqrt{-g} \phi^{;\mu}]_{;\mu}. \quad (5.4)$$

From the scalar equation of motion it is evident that the Lagrangian densities of the gravitational fields (ϕR) and of the scalar field ϕ serves as the source terms for the generation of ϕ -waves. However, it can be shown that this equation can be transformed to make the source term appear as the contracted energy-momentum tensor of matter. So the matter distribution in space can be a source for ϕ , in accordance with Mach's principle.

The main difference between general relativity and Brans-Dicke theory lies in the gravitational field equations rather than in the equations of motion. Therefore, as in general relativity, the energy-momentum tensor of matter must be covariantly divergence free;

$$T^{\mu\nu}_{;\nu} = 0,$$

where we still have that

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial}{\partial g_{\mu\nu}} (\sqrt{-g} \mathcal{L}_M).$$

The field equations for the metric field are obtained in the usual way, by varying the components of the metric tensor and its first derivatives. This will give us

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= \left(\frac{8\pi}{\phi} \right) T_{\mu\nu} + \left(\frac{\omega_{\text{BD}}}{\phi^2} \right) (\phi_{;\mu} \phi_{;\nu} - \frac{1}{2} g_{\mu\nu} \phi_{;\alpha} \phi^{;\alpha}) \\ &\quad + \frac{1}{\phi} (\phi_{;\mu\nu} - g_{\mu\nu} \square \phi). \end{aligned} \quad (5.5)$$

The left hand side is identical to the Einstein-tensor. The first term on the right hand side is also familiar, since it is equal to the source term in general relativity, except that the gravitational constant G is replaced by ϕ^{-1} , which serves as a gravitational coupling parameter. The second term we recognize as the energy-momentum tensor of the scalar field, also coupled with the gravitational coupling ϕ^{-1} . The rather foreign third term results from the presence of second derivatives of the metric tensor in the Ricci scalar R . These second derivatives are eliminated by integration by parts to give a divergence and the extra terms. So in cases where the first term dominates the right hand side, the equations differs from Einstein's field equations by the presence of a variable gravitational constant only.

If we contract the Brans-Dicke equation of motion, we get

$$-R = \frac{8\pi T}{\phi} - \frac{\omega_{\text{BD}}}{\phi^2} \phi_{,\alpha} \phi^{,\alpha} - \frac{3}{\phi} \square \phi, \quad (5.6)$$

where $T = g^{\mu\nu} T_{\mu\nu}$. This can be combined with the scalar wave-equation (5.3) to give

$$\square \phi = \frac{8\pi T}{3 + 2\omega_{\text{BD}}}. \quad (5.7)$$

From this equation we see, as mentioned earlier, that the scalar field has its source given by the trace of the energy-momentum tensor. We also see that the coupling to matter vanishes in the limit $\omega_{\text{BD}} \rightarrow \infty$, where the Brans-Dicke theory approaches standard general relativity. From solar system experiments, a bound on the coupling parameter at $\omega_{\text{BD}} > 3600$ [28] has been obtained. However, the latest found constraint is $\omega_{\text{BD}} > 40\,000$ [26].

We can also compare the BD equations of motion to those of chameleon field theory. The main difference is that the chameleon field is coupled conformally to the matter fields, while the Brans-Dicke scalar field is coupled to the gravitational field. We also see that we here don't have any self-interacting potential term. So the prototype Brans-Dicke model we are working with here cannot be a model for dark energy (after all, in the early 1960s nobody had any idea of this). However, there are existing more recent Brans-Dicke models where a potential is included [27].

5.3 Gravitational radiation in Brans-Dicke theory

Brans-Dicke theory introduces three important effects for systems involving gravitational radiation. The first one is a modification of the effective masses of bodies. These modifications are parametrized by *sensitivites*, which are roughly a measure of the gravitational binding energy per unit mass. So the motion of bodies depends on their internal structure rather than tidal

interactions, violating the equivalence principle. The sensitivity of a body a is defined by

$$s_a \equiv -\frac{\partial(\ln m_a)}{\partial\phi} = \frac{\Omega_a}{m_a}, \quad (5.8)$$

where

$$\Omega_a = -\frac{1}{2} \int_{V_a} \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3x d^3x' \quad (5.9)$$

is the self-gravitational binding energy of body a .

For neutron stars, $s \approx 0.1 - 0.2$ and for black holes $s \equiv 0.5$ [28].

The second effect is that the quadrupole gravitational radiation is modified. For Brans-Dicke theory, the modification of the quadrupole radiation from a two-body system is predicted to be of order $\mathcal{O}(1/\omega_{\text{BD}})$. In general relativity, the quadrupole formula is given by

$$\left(-\frac{dE}{dt}\right)_{\text{quadrupole}} = \frac{32}{5} \frac{G^3 \mu^2 M^2}{r^4} v^2, \quad (5.10)$$

where $v^2 = \frac{GM}{r}$ and $c = 1$.

Finally, the third effect is dipole gravitational radiation. If the bodies are different (in mass and size), the center of gravitational binding energy don't need to coincide with the fixed center of inertial mass. The result of this is that the varying dipole moment is a source of scalar radiation.

5.3.1 Multipole generation of gravitational waves

Let's look at how we calculate the energy-loss formula in Brans-Dicke theory. As we remember, slow motion, weak field sources, has the quadrupole as the dominant multipole contribution to gravitational radiation. The gravitational waveform in the radiation zone was given by

$$\bar{h}_{ij} = \frac{2G}{R} \ddot{q}_{ij}, \quad (5.11)$$

which resulted in an energy flux far from the source given by

$$L_{\text{GW}} = \frac{G}{5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle, \quad (5.12)$$

where Q_{ij} is the trace-free part of the quadrupole moment. For a binary system where we assume that the distance between the bodies is large compared to their size, the energy-loss from quadrupole gravitational radiation is given by (4.52). The resulting decrease in orbital period can be found from Kepler's 3rd law:

$$\frac{1}{P} \frac{dP}{dt} = -\frac{3}{2} \frac{1}{E} \frac{dE}{dt}. \quad (5.13)$$

Nearly every alternative metric theory of gravity predicts the presence of other multipole contributions in addition to quadrupole radiation, first and foremost dipole radiation. For binary systems, this has two important effects on the energy-loss formula; a modification of the numerical coefficients plus generation of an additional term produced by dipole moments. This term depends on the self-gravitational binding energy of the stars in the binary. The resulting energy-loss formula may be written in a form that contains dimensionless parameters whose values depends upon the theory we study. These parameters (often called PM-parameters [20, 30]) are named κ and κ_D . The first one corresponds to the (modified) quadrupole contribution to the gravitational radiation, while the latter parameter refers to the dipole contribution. Schematically, the dipole energy-loss formula can be written

$$\left(-\frac{dE}{dt}\right)_{dipole} = \frac{G}{3}\kappa_D\langle\ddot{d}_j\ddot{d}_j\rangle, \quad (5.14)$$

where d_j are the dipole-moment components of the self-gravitational binding energy Ω_a of the bodies, now given by

$$d_j = \sum_a \Omega_a x_j^a. \quad (5.15)$$

For a binary system, the resulting total energy-loss formula becomes

$$-\left(\frac{dE}{dt}\right)_{GW} = \left\langle \frac{8}{15} \frac{\mu^2 M^2}{r^4} \left(\kappa v^2 + \frac{5}{8} \kappa_D \mathcal{S}^2 \right) \right\rangle, \quad (5.16)$$

where $\mathcal{S} = s_1 - s_2$ is the difference in the sensitivities between the two bodies. We have also put the gravitational constant equal to unity. If we compare this equation to the quadrupole formula, (5.10), we see that in general relativity, $\kappa = 12$, while $\kappa_D = 0$, as expected since we don't get any dipole contribution here.

So let us now calculate these parameters for Brans-Dicke theory. Throughout this calculation, I will use geometrized units, i.e. I will keep the gravitational constant $G = 1$. I have followed calculations given in [30].

Like in general relativity, we start out by linearizing the field equations, which are given by

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \left(\frac{8\pi}{\phi}\right)T_{\mu\nu} + \left(\frac{\omega_{BD}}{\phi^2}\right)(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}) \\ &+ \frac{1}{\phi}(\phi_{;\mu\nu} - g_{\mu\nu}\square\phi), \end{aligned} \quad (5.17)$$

and

$$\square\phi = \frac{8\pi T}{3 + 2\omega_{BD}}. \quad (5.18)$$

We look at small perturbations in the metric and the scalar field:

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}, & \phi &= \phi_0 + \varphi, \\ \theta^{\mu\nu} &= h^{\mu\nu} - \frac{1}{2}h\eta^{\mu\nu} - (\varphi/\phi_0)\eta^{\mu\nu}, \end{aligned} \quad (5.19)$$

where $h_{\mu\nu}$ are, as we remember, small metric perturbations in Minkowskian spacetime and where φ are scalar perturbations, $\varphi \ll \phi_0$. Defining

$$\phi_0 = \frac{4 + 2\omega_{\text{BD}}}{(3 + 2\omega_{\text{BD}})G}$$

gives us that GR is reproduced in the limit $\omega_{\text{BD}} \rightarrow \infty$ which implies $\phi_0 \rightarrow 1/G$. Further, we choose a gauge in which

$$\theta_{,\nu}^{\mu\nu} = 0. \quad (5.20)$$

Then, we can write the reduced field equations on the form

$$\square\theta^{\mu\nu} = -16\pi\tau^{\mu\nu}, \quad \square\varphi = -16\pi S, \quad (5.21)$$

where

$$\tau^{\mu\nu} = \phi_0^{-1}(T^{\mu\nu} + t^{\mu\nu}), \quad \tau_{,\nu}^{\mu\nu} = 0,$$

and where

$$S = -\frac{1}{6 + 4\omega_{\text{BD}}}T \left(1 - \frac{1}{2}\theta - \varphi/\phi_0\right) - \frac{1}{16\pi}(\varphi_{,\mu\nu}\theta^{\mu\nu} + \phi_0^{-1}\varphi_{,\mu}\varphi^{,\mu}) \quad (5.22)$$

is the source of scalar perturbations. T is still the contracted energy-momentum tensor given by $T = g_{\mu\nu}T^{\mu\nu}$. The quantity $t^{\mu\nu}$ is a function of quadratic and higher order terms in $\theta^{\mu\nu}$ and φ .

The reduced field equations can be solved in the far zone (where the distance from the source to the field point, R , is much larger than size of the source, r) like we did in GR, and we find the retarded integral expressions

$$\theta^{\mu\nu} = \frac{4}{R} \sum_{m=0}^{\infty} (1/m!) \left(\frac{\partial}{\partial t}\right)^m \int \tau^{\mu\nu}(t - R, \mathbf{r}') (\hat{\mathbf{n}} \cdot \mathbf{r}')^m d^3x', \quad (5.23)$$

$$\varphi = \frac{4}{R} \sum_{m=0}^{\infty} (1/m!) \left(\frac{\partial}{\partial t}\right)^m \int S(t - R, \mathbf{r}') (\hat{\mathbf{n}} \cdot \mathbf{r}')^m d^3x', \quad (5.24)$$

where $\hat{\mathbf{n}} \equiv \mathbf{r}/|\mathbf{r}|$. Because of the conservation law, $\tau_{,\nu}^{\mu\nu} = 0$, the spatial components θ^{ij} can be re-expressed as second time-derivatives of quadrupole moments, as in general relativity:

$$\theta^{ij} \simeq \frac{4}{R} \int \tau^{ij} d^3x' = \frac{2}{R} \frac{d^2}{dt^2} \int \tau^{00} x'^i x'^j d^3x'. \quad (5.25)$$

We insert

$$\tau^{00} = \frac{1}{2}(1 + \gamma)\rho \quad (5.26)$$

using the lowest quadrupole order, and get

$$\theta^{ij} = (1 + \gamma)R^{-1} \frac{d^2}{dt^2} \sum_a m_a x_a^i x_a^j, \quad (5.27)$$

where γ is a so-called *PPN-parameter* (PPN = parametrized post-Newtonian) given by

$$\gamma = \frac{1 + \omega_{\text{BD}}}{2 + \omega_{\text{BD}}}.$$

This parameter comes into our equations from use of the post-Newtonian metric outlined and given in [30]. We see then that θ^{ij} do not contribute any dipole terms. Specialized to a two-body system, eq. (5.27) yields

$$\theta^{ij} = \frac{2\mu(1 + \gamma)}{R} \left(v^i v^j - \frac{\mathcal{G}_{12} M x^i x^j}{r^3} \right), \quad (5.28)$$

where (still in geometrized units)

$$\mathcal{G}_{12} = 1 - \xi(s_1 + s_2 - 2s_1 s_2), \quad \xi \equiv (2 + \omega_{\text{BD}})^{-1}. \quad (5.29)$$

The source of scalar perturbations, S , does lead to dipole terms. To the lowest post-Newtonian order, S can be written as

$$\begin{aligned} S &\simeq -\frac{1}{6 + 4\omega_{\text{BD}}} T \left(1 - \frac{1}{2}\theta - \varphi/\phi_0 \right) \\ &= \frac{\rho}{6 + 4\omega_{\text{BD}}} \left[1 + \Pi - \frac{3p}{\rho} + \frac{1 + 2\omega_{\text{BD}}}{2 + \omega_{\text{BD}}} \Phi \right] \end{aligned} \quad (5.30)$$

where Π is the *specific energy density*, i.e. the ratio of energy density to rest mass density, and Φ is the Newtonian potential produced by a rest mass density ρ ,

$$\Phi(\mathbf{x}, t) = \int \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}'.$$

Substituted into the integral expression (5.24) we find, for a binary orbit (more detailed outline given in [30])

$$\varphi = -(1 - \gamma)\phi_0 \frac{\mu}{R} \left[v^2 - (\mathbf{n} \cdot \mathbf{v})^2 + \frac{M}{r} + \frac{M}{r^3} (\mathbf{n} \cdot \mathbf{x})^2 + 2\mathcal{S}(\mathbf{n} \cdot \mathbf{v}) \right], \quad (5.31)$$

where s_a is defined by eq. (5.8). Substituting the final expressions for θ^{ij} and φ into the energy-flux formula

$$\frac{dE}{dt} = - \left(\frac{R^2}{32\pi} \right) \phi_0 \int [\theta_{TT,0}^{kl} \theta_{TT,0}^{kl} + (4\omega_{\text{BD}} + 6)\phi_0^{-2} \varphi_{,0} \varphi_{,0}] d\Omega \quad (5.32)$$

where the subscript TT denotes TT-gauge, we obtain the energy-loss formula, eq. (5.16), with

$$\begin{aligned}\kappa &= 12 - \frac{5}{2 + \omega_{\text{BD}}}, \\ \kappa_D &= \frac{2}{2 + \omega_{\text{BD}}}.\end{aligned}\tag{5.33}$$

Here we see the mentioned corrections of order $O(\omega_{\text{BD}}^{-1})$ of the quadrupole contribution.

We can now find the rate of change in orbital period by looking at the average over one orbit and using the Keplerian relations

$$\begin{aligned}E &= -\frac{1}{2} \frac{\mathcal{G}\mu M}{a}, \\ J &= \mu[\mathcal{G}Ma(1 - e^2)]^{1/2}, \\ P/2\pi &= \left(\frac{a^3}{\mathcal{G}M}\right)^{1/2} = \left(\frac{M}{m_1^3 m_2^3}\right) J^3 \mathcal{G}^{-2} (1 - e^2)^{-3/2},\end{aligned}\tag{5.34}$$

where we have replaced the orbital radius r with the semi-major axis a (i.e. we generalize the orbit to be elliptical) and where J is the orbital angular momentum and e the eccentricity of the orbit. The resulting rate of change in orbital period ($\dot{P}/P = -\frac{3}{2}\dot{E}/E$) becomes

$$\frac{\dot{P}}{P} = -\frac{96}{5} \frac{\mu M^2}{a^4} F(e) - \frac{\mathcal{G}^2 \mu M}{a^3} \kappa_D \mathcal{S}^2 G(e),\tag{5.35}$$

where

$$\begin{aligned}F(e) &= \frac{1}{12} (1 - e^2)^{-7/2} \left(1 + \frac{7}{2}e^2 + \frac{1}{2}e^4\right), \\ G(e) &= (1 - e^2)^{-5/2} \left(1 + \frac{1}{2}e^2\right)\end{aligned}\tag{5.36}$$

are geometric corrections.

5.3.2 Example: Period decay in neutron star-white dwarf system

Let us apply these formulas to a two-body system and find how much the dipole contributes to the energy-loss and the period decay. However, it is not favourable to use the Hulse-Taylor pulsar as an example here since the dipole effect will be suppressed in a system where the masses of the two bodies are nearly equal. Rather, we will consider the known pulsar-white dwarf binary system PSR J1141-6545 [31]. Some important data of the system:

$$\begin{aligned}
\text{pulsar mass} \quad M_{\text{ns}} &= 1.27 \pm 0.01 M_{\odot}, \\
\text{companion mass} \quad M_{\text{wd}} &= 1.02 \pm 0.01 M_{\odot}, \\
\text{semimajor axis} \quad a &\simeq 1\,310\,000 \text{ km}, \\
\text{eccentricity} \quad e &= 0.171884, \\
\text{binary orbit period} \quad P &= 0.19765 d.
\end{aligned} \tag{5.37}$$

The energy-loss formula in Brans-Dicke theory for a binary system in an elliptical orbit is given by

$$\frac{dE}{dt} = -\frac{8}{15} \frac{G^3}{c^3} \frac{\mu^2 M^2}{a^4} \left(\frac{GM}{c^2 a} \kappa F(e) + \frac{5}{8} \kappa_D \mathcal{S}^2 F(e) \right), \tag{5.38}$$

where κ and κ_D are given by (5.33) and where we have re-inserted c and let $\mathcal{G} \simeq G$. From this we found the period decay,

$$\frac{\dot{P}}{P} = -\frac{96}{5} \frac{G^3}{c^5} \frac{\mu M^2}{a^4} \kappa F(e) - \frac{G^2}{c^3} \frac{\mu M}{a^3} \kappa_D \mathcal{S}^2 G(e). \tag{5.39}$$

As mentioned, the latest constraint on the Brans-Dicke coupling parameter reads $\omega_{\text{BD}} \geq 40\,000$, which is the value we will use in these calculations. Sensitivities of neutron stars are given in [28] for a range of masses. In this example we get $s_{\text{ns}} \simeq 0.26$, choosing the soft equation of state. We assume further that the sensitivity of the white dwarf is sufficiently small to let $\mathcal{S} \approx s_{\text{ns}}$. Then we have the numbers we need and put them into our equations. The results are

$$\begin{aligned}
\frac{dE}{dt} &\approx -1.8 \times 10^{24} \text{ J/s}, \\
\frac{dP}{dt} &= -4.1 \times 10^{-13} \text{ s/s}.
\end{aligned}$$

Even more interesting would it be to look at the dipole contribution to these numbers compared to the quadrupole contribution. The latter will be similar to what is found in standard GR, since ω_{BD} is relatively large. The results are summarized in table 4.1.

So given the current bound on ω_{BD} , the dipole contribution is of order 10 times smaller than the quadrupole, but the dipole effect should still be measurable in a sufficiently asymmetric system. If we had used $\omega_{\text{BD}} \sim 10^3$, the dipole contribution would have been at the same order of magnitude as the quadrupole.

Parameters	Total	Quadr. contrib.	Dipole contrib.	$\frac{Dipole}{Quadr.}$
$-dE/dt$ (J/s)	1.8×10^{24}	1.7×10^{24}	1.2×10^{23}	0.07
dP/dt (s/s)	-4.1×10^{-13}	-3.9×10^{-13}	-2.3×10^{-14}	0.06

Table 5.1: Energy loss and period decay for the binary system PSR J1141-6545 with $\omega_{BD} = 40\,000$.

5.4 Summary

It's time again to give a summary of the chapter, and I will also give a short discussion on the outlooks of the Brans-Dicke theory.

- Brans-Dicke theory was the first scalar-tensor theory. It was made in order to be a gravitational theory which fully incorporated Mach's principle. To achieve this, an effective gravitational constant dependent on a scalar field ϕ is introduced.
- The scalar field cause a modification of the effective mass of bodies, a modification which is parametrized by a so-called sensitivity, defined by eq. (5.8).
- Brans-Dicke theory, like all scalar-tensor theories, predicts dipole gravitational radiation. The dipole contribution arises from the reduced scalar wave equation, and is given by

$$\left(-\frac{dE}{dt}\right)_{dipole} = \frac{G}{3} \frac{M^2 \mu^2}{r^4} \kappa_D \mathcal{S}^2$$

for a two-body system with total mass M and reduced mass μ in a circular orbit with radius r .

- Neutron star-white dwarf systems are better testing grounds for Brans-Dicke theory than the binary pulsar, since the former will give a larger dipole contribution due to a larger difference in sensitivity.

As new tests of gravity in the solar system have raised the lower limit of the Brans-Dicke coupling constant during the last decades, the difference between Brans-Dicke theory and GR has become harder to measure. Thus the BD-theory may seem to have become somewhat superfluous, and it represents a minor viewpoint in physics today. Concerning Mach's principle, there are different interpretations of what the principle really means and whether or not it is incorporated in GR (Brans and Dicke obviously meant it was *not*).

Chapter 6

Gravitational waves in chameleon field theory

As stated in the introduction of the thesis, we can interpret the chameleon field theory as a scalar-tensor theory, since the chameleon is a scalar field which couples (non-minimally) to matter. As we have discussed, the chameleon field physically plays a different role than the scalar field of Brans-Dicke theory, since it is supposed to be a model for dark energy.

In the last chapter we saw that scalar-tensor theories predict dipole gravitational radiation. I will start this final chapter by showing that this is the case by using a very simple classical approximation.

6.1 Newtonian approximation

Let us consider a scalar field, ϕ , that is coupled to matter and varying as a function of space and time. Let the effective mass of a particle be given by

$$m = m_0 e^{-\alpha\phi}, \quad \phi = \phi(\mathbf{r}), \quad (6.1)$$

where α is some coupling constant and where \mathbf{r} is time dependent. We start by taking a new look at the equations for dipole radiation that we considered in section 4.5. We had that the gravitational dipole moment was given by

$$d_j = \sum_A m_A x_j^A.$$

Since the mass no longer is constant in time (because of ϕ), the time derivative of the dipole moment will get some extra terms:

$$\dot{d}_j = \sum_A (\dot{m}_A x_j^A + m_A \dot{x}_j^A), \quad (6.2)$$

where we recognize the last term as the momentum of the A th particle, and where we have that

$$\dot{m}_A = -\alpha \dot{\phi} m_0^A e^{-\alpha\phi}. \quad (6.3)$$

Conservation of momentum will now give us

$$\sum_A \dot{p}_j^A = \sum_A (\dot{m}_A \dot{x}_j^A + m_A \ddot{x}_j^A) = 0. \quad (6.4)$$

Thus, the second time derivative of the dipole-moment is given by

$$\ddot{d}_j = \sum_A (\ddot{m}_A x_j^A + \dot{m}_A \dot{x}_j^A). \quad (6.5)$$

We see that since \ddot{d}_j is nonzero, we get a dipole contribution to the gravitational radiation.

However, we can easily see that the “mass magnetic” dipole moment, which is given by

$$\mu = \frac{1}{c} \sum_A \mathbf{x}^A \times (m_A \dot{\mathbf{x}}^A) = \frac{1}{c} \sum_A \mathbf{j}^A,$$

will still be zero, as in the standard case.

We will later in this chapter argue that the quadrupole part of the energy-loss formula will be equal to what we found in general relativity.

In analogy to the Larmor formula, we assume that the energy-loss from dipole gravitational radiation is given by

$$\left(-\frac{dE}{dt} \right)_{dipole} = \frac{2}{3} \frac{G}{c^3} \langle \ddot{d}_j \ddot{d}_j \rangle. \quad (6.6)$$

We will now try to find an approximate expression for the dipole energy-loss from a binary system (independent of what we found in the last chapter). Let us again consider the bodies as point masses in a circular orbit with radius r and rotating with an angular frequency Ω . The gravitational dipole moment goes as

$$\mathbf{d} \sim m\mathbf{r},$$

where m is some mass scale, it can be e.g., the reduced mass or the total mass of the system. However, we will here just try to find a dimensional approximation. Assuming the orbit to be in the xy -plane, the components goes as

$$d_x \sim mr \cos \Omega t,$$

$$d_y \sim mr \sin \Omega t.$$

We assume the change in orbital radius dr/dt to be negligible compared to the orbital velocity, so we keep r constant when taking the time derivative

of the dipole moment. Thus, the masses of the bodies also will be constant. We find

$$\begin{aligned}\ddot{d}_x &\sim -mr\Omega^2 \cos \Omega t, \\ \ddot{d}_y &\sim -mr\Omega^2 \sin \Omega t.\end{aligned}$$

Inserting this into (6.6), taking the time average and using

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2}$$

we get

$$\left(-\frac{dE}{dt}\right)_{dipole} \sim \frac{2}{3} \frac{G^3}{c^3} \frac{m^2 M^2}{r^4}. \quad (6.7)$$

From this crude approximation we see that we get something pretty similar to the dipole formula in Brans-Dicke theory, except for the factor $\kappa_D \mathcal{S}^2$. If we take these factors out of consideration for now and compare the dipole formula to the quadrupole formula (4.52), we see that the latter is suppressed by a factor $\frac{GM}{rc^2} = \frac{v_{orb}^2}{c^2}$ in comparison to the dipole formula. So if we have no suppression-factors, we should actually expect the dipole contribution to dominate over the quadrupole.

6.2 Derivation of energy-loss formula

In this section I was supposed to linearize Einstein's field equations in the presence of a chameleon field and further go on to solve it and find the energy-loss formula for gravitational waves in chameleon theory. Unfortunately, this proved to be too hard, and I had to find an easier approach (as we will see in the next section).

However, I will here show how far I came doing it the "hard way", and leave the rest as a problem for you readers to solve!

6.2.1 Linearization

As always, to find the behaviour of gravitational waves, the first step is to linearize the field equations. In chapter 3, we derived the field equations in the presence of a chameleon field and found

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G \left[T_{\mu\nu} + \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu} \left(\frac{1}{2}\phi_{,\mu}\phi^{,\mu} + V(\phi) \right) \right],$$

where I have written out the full expression for $T_{\mu\nu}^{(\phi)}$.

We also need the chameleon equation of motion,

$$\square\phi = V_{,\phi}(\phi) + \frac{\beta}{M_{pl}}\rho,$$

where we have used that $\beta\phi/M_{Pl} \ll 1$.

For simplicity, we assume that perturbations in ϕ are negligible in the weak field limit and thus we have that $\varphi = \delta\phi = 0$. Thus, our only perturbations are in the metric, i.e. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

By multiplying with $\eta^{\alpha\mu}$ on both sides of the Einstein equation, the Einstein-tensor and the ordinary energy-momentum tensor will get one of their indices raised. Thus, we find

$$\begin{aligned} G^\alpha_\nu &= 8\pi G \left[T^\alpha_\nu + \phi^{,\alpha}\phi_{,\nu} - \delta^\alpha_\nu \left(\frac{1}{2}\phi_{,\mu}\phi^{,\mu} + V(\phi) \right) \right] \\ &= 8\pi G (T^\alpha_\nu + T^\alpha(\phi)_{\nu}). \end{aligned} \quad (6.8)$$

The derivation of the reduced field equations should then be no different from the standard case outlined in chapter 4. We end up with

$$\square \bar{h}^\mu_\nu = -16\pi G (T^\mu_\nu + T^\mu(\phi)_\nu). \quad (6.9)$$

So our reduced field equations are on the same form as in general relativity, but with an additional source term for the scalar field. Our next step is to try to find a solution of the chameleon equation of motion to insert for ϕ in the expression for $T^\mu(\phi)_\nu$ in the reduced field equation. To (hopefully) make it easier to solve this equation, I tried concretize things a bit. I chose a potential given by

$$V(\phi) = \phi^{-4} \quad (6.10)$$

and rewrote

$$\square \rightarrow -\frac{\partial^2}{\partial t^2} + \nabla^2.$$

This gives

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) \phi = 4\phi^{-5} + \frac{\beta}{M_{Pl}}\rho. \quad (6.11)$$

But, as it turned out, I was not able to solve this equation. I also saw that, if I had found a solution (which would probably look very nasty) I would still have a very long way to go to find the energy-loss formula. So, I ended up using an easier approach, by looking back at the first section of this chapter...

6.3 Derivation of energy loss formula - simplified model

6.3.1 Quadrupole gravitational radiation

So let us go back to our Newtonian model where we assumed that the mass of a particle or a body was explicitly dependent of the scalar field ϕ :

$$m(\phi) = m_0 e^{-\alpha\phi}.$$

From this, we will try to find an expression for the energy loss formula for gravitational waves in chameleon theory. Which deviations from the standard GR formula should we expect? As we saw in the first section of this chapter, dipole gravitational radiation is a direct consequence of the mass being dependent of a time-varying scalar field. But what about the quadrupole contribution; will this be affected? The general formula for quadrupole gravitational radiation is given by

$$\left(-\frac{dE}{dt}\right)_{quadrupole} = \frac{G}{5c^5} \langle \ddot{Q}_{ij} \ddot{Q}_{ij} \rangle,$$

where we have re-inserted c . For a binary system, the components are given by (3.105)-(3.107). We see that, as long as we assume that the factor $m_i r_i^2 = \text{const.}$ (like we did in the derivation in chapter 4) there is no reason that the quadrupole expression should be any different from the GR case, and thus we assume that the quadrupole radiation for a binary system is still given by

$$\left(-\frac{dE}{dt}\right)_{quadrupole} = \frac{32}{5} \frac{G^4}{c^5} \frac{M^3 \mu^2}{r^5}, \quad (6.12)$$

assuming a circular orbit with radius r . We can then assume that the only deviation from GR we will get is a dipole contribution, for which we derived an expression, eq. (6.7), in the first section. Of course, this is a rather crude expression, especially for two reasons: 1) We did not say anything about the properties of our scalar field or its matter coupling α , and, 2) according to the dipole-expression in Brans-Dicke theory, it should contain a sensitivity-factor, representing a gravitational "charge", such that the dipole-contribution is exactly zero if the masses in the binary system are equal.

6.3.2 Gravitational charge and EP-violation

Common for all scalar-tensor theories of gravity is that the (time varying) scalar field works as an additional gravitational field (in addition to the metric field) and leads to a gravitational "charge". In Brans-Dicke theory we saw that this charge caused a dipole contribution to the gravitational radiation. I want to stress that the dipole contribution comes from the

(reduced) *scalar wave equation*, while the quadrupole contribution comes from the ordinary reduced field equations (in other words; the “quadrupole waves” are perturbations in the metric, while the “dipole waves” are scalar perturbations). So, in order to have any dipole gravitational radiation, there must exist a specific gravitational charge for each object. So, in our model, we let the gravitational charge of a body be represented by the product of its sensitivity and its mass. We then write the (scalar) dipole moment as

$$d_{(s)}^i = \sum_a m_a s_a r_a^i, \quad (6.13)$$

so according to the electric dipole moment, we have just replaced the electric charge q_a with the gravitational charge $m_a s_a$ (see [32]). As we remember, the sensitivity of body a can be written as

$$s_a = -\frac{d \ln m_a(\phi)}{d\phi},$$

which will give us $s_a = \alpha$ when m_a is given by (6.1), so the sensitivity in this case is equal to the coupling strength of the scalar field to matter.

Since we assume that the mass is dependent on the scalar field, we clearly have a violation of the equivalence principle, since the mass then also depends on location. We have also seen that dipole gravitational radiation is a result of a field-dependent mass. So the dipole radiation should then be a measure of EP-violations hence of deviations from GR. This reasoning may give us an idea of how large dipole contribution we should expect in chameleon field theory. Because of the thin shell effect, we have seen that the chameleon force, and thus, EP-violations can be largely suppressed for compact objects. Thus, we may anticipate a rather small dipole contribution to the gravitational radiation in chameleon theory, and thus expect a total energy loss pretty close to that found in GR. So let us work out our model further and then try to find an expression for the energy loss and period decay of a binary system.

6.3.3 The effective coupling constant

First, let's consider the coupling constant, α . Generally the coupling constant represents the strength of the coupling of the scalar field to matter. In chameleon theory it is given by β , which we have assumed is constant and of order 1. But, since we are considering massive, compact objects, it is convenient to replace β with the *effective coupling* (see [11]), β_{eff} , given by

$$\beta_{eff} = 3\beta \left(\frac{\Delta R_a}{R_a} \right), \quad (6.14)$$

where ΔR_a is the thickness of the thin shell of body a . This is reasonable because these objects have thin shells which effectively suppress the

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chameleon force. We see that for bodies with no thin shell, β_{eff} will approach the ordinary coupling constant β . From chapter 3, we remember that the thin shell condition was given by

$$\frac{\Delta R_a}{R_a} = \frac{\phi_b - \phi_a}{6\beta M_{Pl} \Phi_a} \ll 1, \quad (6.15)$$

where ϕ_b is the chameleon field value in the far background, ϕ_a is the interior value (inside a compact body), and Φ_a is the gravitational potential of the body.

This means that each body will have different effective chameleon-to-matter couplings, and as we will see, this is required in our model in order to have a non-zero dipole contribution to the energy loss formula. If this had not been the case, each body would have equal sensitivities in our model, and there would be no difference in the gravitational charge, and thus no dipole contribution (however, keep in mind that we still assume the ordinary coupling β to be the same for all matter species).

6.3.4 Dipole gravitational radiation

Our next step is to derive the expression for the energy loss in form of dipole gravitational radiation from a binary system. The general energy-loss expression is already known to us;

$$\left(-\frac{dE}{dt} \right)_{dipole} = \frac{2G}{3c^3} \langle \ddot{d}_{(s)}^i \ddot{d}_{(s)}^i \rangle. \quad (6.16)$$

The total dipole moment of a binary system with masses m_1 and m_2 , sensitivities s_1 and s_2 with distances r_1 and r_2 to their common center of mass, is given by

$$\mathbf{d} = m_1 s_1 \mathbf{r}_1 + m_2 s_2 \mathbf{r}_2. \quad (6.17)$$

Choosing the common center of mass as origin, the components of the dipole moment are given by

$$\begin{aligned} d_x &= (m_1 s_1 r_1 - m_2 s_2 r_2) \cos \Omega t, \\ d_y &= (m_1 s_1 r_1 - m_2 s_2 r_2) \sin \Omega t. \end{aligned} \quad (6.18)$$

Using the relation (4.46), we can write this in terms of the reduced mass μ :

$$\begin{aligned} d_x &= \mu r (s_1 - s_2) \cos \Omega t, \\ d_y &= \mu r (s_1 - s_2) \sin \Omega t, \end{aligned} \quad (6.19)$$

where $r = r_1 + r_2$ is the distance between the two bodies. Ω is still the angular velocity of the system, given by

$$\Omega = \left(\frac{GM}{r^3} \right)^{1/2},$$

and $M = m_1 + m_2$ the total mass.

Then we calculate the second time derivatives of the dipole components. Still assuming (for simplicity) the factor μr to be constant under derivation, we get¹

$$\begin{aligned}\ddot{d}_x &= -\mu r (s_1 - s_2) \Omega^2 \cos \Omega t, \\ \ddot{d}_y &= -\mu r (s_1 - s_2) \Omega^2 \sin \Omega t.\end{aligned}\quad (6.20)$$

Taking the time averages, we find

$$\langle (\ddot{d}_x \ddot{d}_x + \ddot{d}_y \ddot{d}_y + 2\ddot{d}_x \ddot{d}_y) \rangle = \mu^2 r^2 \Omega^4 (s_1 - s_2)^2. \quad (6.21)$$

Inserting this into eq. (6.16) and rewriting Ω gives us

$$\left(-\frac{dE}{dt} \right)_{dipole} = \frac{2}{3} \frac{G^3}{c^3} \frac{\mu^2 M^2}{r^4} (s_1 - s_2)^2. \quad (6.22)$$

Finally, we insert for the sensitivities, and we get the dipole energy-loss formula in chameleon theory (at least in our model):

$$\left(-\frac{dE}{dt} \right)_{dipole} = 2\beta^2 \frac{G^3}{c^3} \frac{\mu^2 M^2}{r^4} \left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} \right)^2. \quad (6.23)$$

Combining this with the quadrupole formula, eq. (6.12), we find the total energy loss formula for a binary system in a circular orbit:

$$\left(-\frac{dE}{dt} \right)_{GW} = 2 \frac{G^3}{c^3} \frac{\mu^2 M^2}{r^4} \left[\frac{16}{5} \frac{v^2}{c^2} + \beta^2 \left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} \right)^2 \right], \quad (6.24)$$

where $v^2 = \frac{GM}{r}$ is the orbital velocity.

Then we can go on to find the period decay of the system by using the Keplerian relations

$$\frac{\dot{P}}{P} = -\frac{3}{2} \frac{\dot{E}}{E}, \quad P = 2\pi \left(\frac{r^3}{GM} \right)^{1/2} \quad (6.25)$$

and remembering that the total energy of the system is given by

$$E = -\frac{1}{2} \frac{G\mu M}{r}.$$

This will give us

$$\frac{\dot{P}}{P} = -\frac{96}{5} \frac{G^3}{c^5} \frac{\mu M^2}{r^4} - 6 \frac{G^2}{c^3} \frac{\mu M}{r^3} \beta^2 \left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} \right)^2, \quad (6.26)$$

¹This should be a valid approximation since the orbital velocity is much larger than the radial velocity. And, assuming a homogeneous background, we should get no change in the field value of the chameleon field, and thus the effective mass remains constant.

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where the first and second term represents the quadrupole and the dipole contribution respectively. Note that this equation is valid for circular orbits only. Though, when we are working with systems like the binary pulsar or e.g., a neutron star-white dwarf system, the orbits are often elliptical and with high eccentricities (like the Hulse-Taylor binary). Thus we need to multiply geometric corrections like we did in the GR- and in the Brans-Dicke cases, and we have to replace the circular orbit r with the semi-major axis a ;

$$\begin{aligned} \frac{\dot{P}}{P} &= -\frac{96}{5} \frac{G^3}{c^5} \frac{\mu M^2}{a^4} f(e) - 6 \frac{G^2}{c^3} \frac{\mu M}{a^3} \beta^2 \left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} \right)^2 g(e) \\ &= -6 \frac{G^2}{c^3} \frac{\mu M}{a^3} \left[\frac{16}{5} \frac{G}{c^2} \frac{M}{a} f(e) + \beta^2 \left(\frac{\Delta R_1}{R_1} - \frac{\Delta R_2}{R_2} \right)^2 g(e) \right]. \end{aligned} \quad (6.27)$$

In the quadrupole term it should be reasonable to assume that the eccentricity-factor $f(e)$ is the same as the Peters-Mathews result [20] which we used in the Hulse-Taylor example,

$$f(e) = \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1 - e^2)^{7/2}}. \quad (6.28)$$

We also let $g(e)$ be the same as what is used in the Brans-Dicke case;

$$g(e) = \frac{1 + \frac{1}{2}e^2}{(1 - e^2)^{5/2}}. \quad (6.29)$$

6.3.5 Results

Finally, let us apply these expressions on the binary systems we have considered earlier. Like in Brans-Dicke theory we anticipate to get larger dipole contributions from more asymmetric systems. So we re-visit the binary system PSR J1141-6545, consisting of a pulsar (neutron star) and a companion white dwarf, which we studied in chapter 5. In Brans-Dicke theory we found that the dipole contribution to the energy loss and the period decay was of order 10 smaller than the quadrupole contribution when using the most current constraint on the BD coupling parameter.

In chameleon theory we will as earlier operate with the coupling constant β equal to unity. Then we need to find the relative thickness of the thin shell of a neutron star and of a white dwarf to find the effective coupling and thus, how much the dipole radiation is suppressed. The thin shell expression is given by (6.15). The scalar field values are given by the field value in the minimum of the effective potential. Choosing the power-law potential

$$V(\phi) = M^{4+n} \phi^{-n}$$

we find

$$\phi_c = \left(\frac{nM^{4+n}M_{Pl}}{\beta\rho_c} \right)^{\frac{1}{n+1}}, \quad (6.30)$$

with $M \sim 1 \text{ mm}^{-1}$ corresponding to the energy scale associated with dark energy. However, we will see that it may not be necessary to calculate these field values to find our thin shell constraints. In chapter 3 we considered the thin shell condition for the Earth and found the constraint

$$\frac{\Delta R_\oplus}{R_\oplus} < 10^{-7}.$$

From (6.15) we see that $\Delta R_c/R_c \propto \Phi_c^{-1}$. We have that the dimensionless gravitational potential for a (spherical symmetric) body with mass M_c and radius R_c is given by

$$\Phi_c = \frac{GM_c}{R_c c^2}, \quad (6.31)$$

which for the Earth gives $\Phi_\oplus \simeq 10^{-9}$. So by calculating the gravitational potential for a typical white dwarf and for a neutron star, and compare this with that of the Earth, we can find constraints on the thin shell thicknesses. The neutron star in our system has a mass of $M_{\text{ns}} = 1.27 M_\odot$, while a typical radius R_{ns} of a neutron star is of order 10 km. This will then give us

$$\Phi_{\text{ns}} = \frac{6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \times 1.27 M_\odot}{10^4 \text{ m} \times (3.0 \times 10^8 \text{ ms}^{-1})^2} \simeq 0.2,$$

which is of order 10^8 larger than the gravitational potential on Earth. According to (6.15) we should then assume a relative thickness of the thin shell of

$$\Delta R_{\text{ns}}/R_{\text{ns}} < 10^{-15}.$$

For the white dwarf we have $M_{\text{wd}} = 1.02 M_\odot$ which should give a radius of $R_{\text{wd}} \simeq 5000 \text{ km}$. The gravitational potential then becomes

$$\Phi_{\text{wd}} \simeq 3 \times 10^{-4}.$$

This is of order 10^5 larger than Φ_\oplus which in turn should give

$$\Delta R_{\text{wd}}/R_{\text{wd}} < 10^{-12}.$$

But what about the chameleon field values? Neutron stars and white dwarfs are very dense objects and since (according to (6.30)) the background field value should be much larger than the interior field value, we can put $\phi_b - \phi_c \approx \phi_b$ (the density inside a neutron star is of order $10^{35} - 10^{40}$ larger than in the background, which for $n = 4$ will give a background field value ϕ_b of order 10^7 to 10^8 larger than the field value ϕ_c inside the neutron star). Anyway, the main point is that this will not give us any *lighter* constraints

on the thin shell thicknesses. So if we assume the background density to be $\rho_b \simeq \rho_G = 10^{-21} \text{ kg/m}^3$, i.e. the same as in our galactic neighbourhood (see chapter 3), we can keep our approximations of the thin shell constraints for the neutron star and the white dwarf that we found above.

Then, at last, we can go on to calculate the energy loss and the period decay of our system. If we choose the lightest constraints that we found, we see that $\Delta R_{\text{wd}}/R_{\text{wd}} \approx 10^{-12}$ is of order 10^3 larger than $\Delta R_{\text{ns}}/R_{\text{ns}}$, and thus we choose to neglect the latter term in our calculations. The energy loss and the period decay is then given by

$$\left(-\frac{dE}{dt}\right)_{\text{GW}} = 2\frac{G^3}{c^3}\frac{\mu^2 M^2}{a^4} \left[\frac{16}{5}\frac{G}{c^2}\frac{M}{a}f(e) + \beta^2 \left(\frac{\Delta R_{\text{wd}}}{R_{\text{wd}}}\right)^2 g(e) \right], \quad (6.32)$$

$$\frac{\dot{P}}{P} = -6\frac{G^2}{c^3}\frac{\mu M}{a^3} \left[\frac{16}{5}\frac{G}{c^2}\frac{M}{a}f(e) + \beta^2 \left(\frac{\Delta R_{\text{wd}}}{R_{\text{wd}}}\right)^2 g(e) \right]. \quad (6.33)$$

The orbital eccentricity of our system is $e = 0.17$, which gives, when inserted in (6.28) and (6.29), $f(e) \simeq 1.2$ and $g(e) \simeq 1.1$. Then we insert the rest of the numbers we need from the table (5.37) and compare the quadrupole and the dipole contribution, that is, the two terms in the bracket parentheses in the energy loss- and the period decay expressions. We find that the quadrupole contribution is

$$\frac{16}{5}\frac{G}{c^2}\frac{M}{a}f(e) \sim 10^{-5},$$

while the dipole contribution becomes

$$\beta^2 \left(\frac{\Delta R_{\text{wd}}}{R_{\text{wd}}}\right)^2 g(e) \sim 10^{-24},$$

that is, a factor 10^{19} smaller than the quadrupole contribution. Thus, we can conclude that the period decay from this system will be essentially the same in chameleon theory as in general relativity. The period decay due to dipole radiation will be of order 10^{-32} s/s , which is of course negligible compared to the quadrupole contribution of $\sim 10^{-13} \text{ s/s}$.

For the Hulse-Taylor binary pulsar the effect will be even less (at least about 6 orders of magnitude) since it consists of two neutron stars with almost equal masses. We have in any case mentioned earlier that this system is not an ideal testing ground for alternative gravitational theories.

6.4 Discussion of method and results

Our final result, which tells us that the dipole contribution is more or less negligible, should not be very surprising from a qualitative point of view.

Knowing that the chameleon field is a scalar field “designed” to avoid EP-violations (at least for large compact objects) and that dipole gravitational radiation is a consequence of EP-violation, one should expect results similar to what we have in GR. But how about the quantitative part; how reliable is our simplified model? Difficult to say exactly, since we after all don’t have the more exact calculations from solutions of the reduced field equations. But let us take a quick review of our model and the assumptions we have made throughout.

- We started out by assuming that the mass of a particle or a body was explicitly dependent of the scalar field; $m(\phi) = m_0 e^{-\alpha\phi}$.
- Then we let the coupling constant α be equal to the effective coupling in chameleon field theory,

$$\alpha = \beta_{eff} = 3\beta \left(\frac{\Delta R_c}{R_c} \right),$$

which is reasonable as long as we consider large, compact objects.

- Further, we introduced a sensitivity, which came out to be equal to the effective coupling. It should be mentioned that the sensitivity has various definitions, and in Brans-Dicke theory it is defined as self-gravitational binding energy per rest mass unit (see chapter 4). It is often expressed as the derivative of the mass with respect to the (effective) gravitational constant [28]. Thus the sensitivity in Brans-Dicke theory may not be exactly physically equivalent to the one we use in our model.
- We went on and assumed the gravitational charge to be equal to the mass times the sensitivity, and defined the scalar dipole moment by (6.13). And here is a critical point which was not mentioned above; We argued that the quadrupole gravitational radiation would be no different from that in GR. But why did we not take the sensitivity factor into account in the quadrupole moment? To this I have two arguments. First: Remember that the quadrupole energy-loss formula is concerning *metric* perturbations, and what I assume is simply that the gravitational charge only concerns *scalar* perturbations. The gravitational charge is, as mentioned, an effect caused by the scalar field, and thus I assume that it only affects the scalar dipole- and possibly, other multipole moments.
The second argument is that this is (approximately) what is done in Brans-Dicke theory!
- In the rest of the calculations I have followed standard methods from the examples given in chapters 4 and 5.

But although, remember that I have not found my results within a relativistic framework, i.e. I haven't solved the reduced field equations and found a more exact expression for the gravitational radiation. I have just taken a short cut, used classical equations and made some assumptions based on knowledge and reasonable guesses.

So, I don't think we should lean too heavily on these results until one has provided the more exact solutions of the reduced field equations.

Chapter 7

Summary and conclusion

7.1 Summary

Starting out with a presentation of the major theoretical foundation stone of this thesis, general relativity, I went on to discuss physical cosmology and topics like the accelerated expansion of the universe and candidates for dark energy. The main purpose of the cosmology-chapter was to explain how components like a cosmological constant or quintessence will cause acceleration, and further, to motivate the chameleon field theory.

But as it turned out, the main focus of the thesis came to be on general relativity versus alternative gravitational theories (i.e. scalar-tensor theories) and their description of gravitational waves. More precisely, we have focused on gravitational waves from binary systems as a testing ground for scalar-tensor theories. We first reviewed results from general relativity after presenting the concept of gravitational waves and how they appear from Einstein's field equations. Then we reviewed the Brans-Dicke theory and found that its scalar field caused an extra, non-negligible, dipole contribution to the gravitational radiation. Finally we studied gravitational waves in chameleon field theory, where we ended up using a simplified (classical) model to make an estimate on how the field affects gravitational radiation. From simple reasoning, we argued that we also here would get a dipole contribution. After a discussion of chameleon field-to-matter couplings and EP-violations, we predicted that the dipole effect would be rather small. And the results, summarized in a table below, shows that this is also the case. The dipole radiation, representing deviations from the GR results, gives only a negligible contribution to the energy loss and the period decay of a pulsar-white dwarf system.

Quantity	Chameleon	Brans-Dicke	General relativity
$(-dE/dt)_{quadr}$ (J/s)	2.0×10^{24}	1.8×10^5	2.0×10^{24}
$(-dE/dt)_{dipole}$ (J/s)	2.0×10^5	1.2×10^{23}	0
$(dP/dt)_{quadr}$ (s/s)	-3.9×10^{-13}	-3.9×10^{-13}	-3.9×10^{-13}
$(dP/dt)_{dipole}$ (s/s)	-4.3×10^{-32}	-2.3×10^{-14}	0
$\mathcal{O}(\text{Dipole/Quadrupole})$	10^{-19}	10^{-1}	0

Table 7.1: Overview of quadrupole and dipole contribution to the energy loss and period decay for the binary system PSR J1141-6545 in chameleon theory, Brans-Dicke theory and general relativity.

7.2 Conclusion and outlooks

On this background, we can at last draw a conclusion:

- The extremely small effect the chameleon field cause on gravitational radiation from binary systems indicates that such systems are not able to put any constraints on chameleon fields, that is, binary systems are bad testing grounds for chameleon theories.

According to our model, the dipole effect caused by the chameleon field is strongly suppressed by the thin shell. This, and possibly, the way the chameleon field is coupled to matter may cause the field to play a more passive role in gravitational radiation than the Brans-Dicke scalar field, which is directly coupled to the Ricci scalar.

So given that dipole gravitational radiation is the only existing trace of the chameleon when studying gravitational waves, we should rather consider other testing grounds. This can in a sense be looked upon as unfortunate, since gravitational waves cannot give us any further info or constraints on chameleon theories. On the other hand, there are already many known future tests of gravity that will give us more knowledge, among others the mentioned experiments STEP and GG.

To state my own opinion, I think that of course it would have been a bit funny if it had turned out that chameleon fields play a more significant role in gravitational radiation. But, as I stated in the last chapter, I'm not very surprised that the results turned out as they did. I would have been way more sceptical if I had found the dipole contribution to be very large, not to say dominant, on the background of what I have learned about the properties of chameleon fields and, in particular, the thin shell effect.

Anyway, to get the final answer, two discoveries have to be made first; chameleon fields and gravitational waves.



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