Risk measures in jump diffusion markets

by

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Introduction

For essentially all investors and financial institutions it is vital to measure levels of risk. History gives enough examples on the consequenses of ignoring or miscalculating the risky part of investments. As time and fluctuations taught investors costly lessons, VaR (Value at Risk) came forth as the preferred risk measure. Practitioners quickly embraced VaR as it has advantages like wide applicability and universality. Convex risk measures could not compete with the simplicity of VaR, and was written off as something describing an ideal fantasy world. This contributed to an increasing gap between practitioners and researchers, as VaR failed to fulfill one or more properties defining a convex risk measure. These properties were set to define what we want a risk measure to tell us. Violating these properties can therefore lead to strange conclusions, e.g. diversification gives higher risk, as in the example in Chapter 3. So when banks, regulators and others with huge impact on a nations financial health uses non-convex measures of risk this is a cause to worry.

Generalizations from the contiuous setting to the non-continuous setting is a very popular subject in todays stochastic analysis. Articles written before the millennium changed is in general restricting themselves to the continuous setting. In more recent studies that involves stochastic processes, the Itô-Lévy setting is a popular generalization from the setting with only the Brownian Motion, since these processes seem to elicit more of the same behavior as real world prices. As the title reveales, this thesis is no exception, since we will work in the Itô-Lévy setting.

The primary target of this thesis is to familiarize the different types of risk measures, with a more in depth view on convex risk measures. The secondary target is to represent, and give applications to, the stochastic games arising from risk minimization. The teriary target is to generalize theory on risk measures in the contiuous setting, to the non-continuous setting. To meet these goals we will refer to a selection of articles by others, give comments and examples, and lastly generalize some of these results.

This thesis is organized as follows. In the first chapter we give a short recap on Lévy measures and Poisson random measures. In the second chapter we look at the families of risk measures arising from the axioms stated for risk measures in general. In the third chapter we give an example that excludes Value at Risk from the class of convex risk measures. In the fourth chapter we give a more detailed look at convex risk measures. In the fifth chapter we minimize the risk of a stochastic variable, dependent of the choice of portifolio. This is given in the context where our random variable is represented as the end point of a value process. This problem will be formulated as an HJBI problem, and the problem is solved by the results represented in [20]. In the sixth chapter we give a short introduction to pricing of contingent claims in incomplete markets, and look at an example of risk indifference pricing based on the work in [18]. In the seventh chapter we look at g-expectations, and their relation to convex risk measures.

This thesis will often give reference to results or statements given by other authors. This will be done by inserting a square parenthesis on each side of a number, for instance [1]. When this is written in bold text behind a definition or proposition, it means that the definition or proposition is taken from the article or book this reference leads to. Sometimes a footnote connected to the reference will give a further explanation of how the referred article or book translates to the given setting. All such refrences are listed on the last page.

Contents

1 Lévy measures and Poisson random measures

As stated in the introduction and in the title of this thesis, we will work in the non-continuous setting. Whilst the Brownian Motion setting is thoroughly treated in many articles and graduate courses, the generalization to the jump setting is lacking the same level of attention. Therefore we start this thesis by a short introduction to the fundamentals of Lévy measures and Poisson random measures.

Let $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space. An $\mathcal{F}_t$-adapted process $\{\eta_t\}_{t\geq 0} \subset \mathbb{R}$ is called a Lévy process if

(i) $\eta_0 = 0$

(ii) η_t is continuous in probability, i.e.

$$\lim_{s\to t} P(\{\omega \in \Omega : |\eta_s - \eta_t| > \epsilon\}) = 0$$

(iii) η_t has stationary increments, i.e. for some suitable t_1, t_2 and s

$$\eta_{t_1+s} - \eta_{t_1} \overset{\text{dist}}{=} \eta_{t_2+s} - \eta_{t_2}$$

(iv) η_t has independent increments, i.e. for $A, B \subset \mathbb{R}$

$$P(\eta_{t_1+s} - \eta_{t_1} \in A \cap \eta_{t_2+s} - \eta_{t_2} \in B) = P(\eta_{t_1+s} - \eta_{t_1} \in A)P(\eta_{t_2+s} - \eta_{t_2} \in B)$$

If X_t, Y_t are stochastic processes, we say that X_t *is a version of* Y_t if

$$P\big(\{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\}\big) = 1 \qquad \text{for all } t \in [0, T]$$

which is a weaker assumption than X_t and Y_t being *indistinguishable*, i.e.

$$P\big(\{\omega \in \Omega : X_t(\omega) = Y_t(\omega), \text{ for all } t \in [0, T]\}\big) = 1.$$

It is well known that a Lévy process has a càdlàg version (right continuous with left limits), which is also a Lévy process. In the remainder any Lévy process will be assumed to be càdlàg. Now define the jump of η_t at time $t > 0$ by

$$\Delta\eta_t = \eta_t - \eta_{t^-}$$

and let

$$\mathbf{B}_0 = \{U \subset \mathbb{R} : U \text{ is a Borel set}, 0 \notin \bar{U}\}.$$

Then, for $U \in \mathbf{B}_0$, define the Poisson random variable

$$N(t,U) = \sum_{0<s\leq t} \chi_U(\Delta\eta_s)$$

called the *Poisson random measure* of η_t. It is shown in [15] remark 1.3, that $N(t,U)$ is finite for all $U \in \mathbf{B}_0$, hence the function $U \to N(t,U)$ is σ-finite on $\mathbf{B}_0$. Then the Lévy measure ν of η_t, defined by

$$\nu(U) = E\left[N(1,U)\right]$$

also becomes a σ-finite measure on $\mathbf{B}_0$. If η_t is a Lévy process it is common to assume

$$E[|\eta_t|] < \infty \qquad \text{for all } t \geq 0$$

which ensures that η_t has the decomposition

$$\eta_t = \alpha t + \sigma B(t) + \int_{\mathbb{R}_0} z\tilde{N}(t,dz); \qquad \mathbb{R}_0 = \mathbb{R}\backslash\{0\}$$

for some constants $\alpha, \sigma \in \mathbb{R}$, and where

$$\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt.$$

This points us in the direction of stochastic integrals on the form[1]

$$X(t) = X(0) + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dB(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds,dz) \qquad (1.1)$$

where $\alpha(t) = \alpha(t,\omega)$, $\sigma(t) = \sigma(t,\omega)$ and $\gamma(t,z) = \gamma(t,z,\omega)$ are adapted processes. We call such processes *Itô-Lévy processes.*

As in the Brownian motion setting, a central result for Itô-Lévy processes is the Itô Formula, which is used several times in this thesis. It states that if $X(t)$ is an Itô-Lévy process on the form (1.1), $f \in C^2(\mathbb{R}^2)$ and we define $Y(t) = f(t,X(t))$, then (in the 1-dimensional case)

[1]In the appendix we simulate $X(t)$ to give a graphic example of the behavior of such processes.

$$
\begin{aligned}
dY(t) &= \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))[\alpha(t,\omega)dt + \beta(t,\omega)dB(t)] \\
&+ \frac{1}{2}\beta^2(t,\omega)\frac{\partial^2 f}{\partial x^2}(t, X(t))dt \\
&+ \int_{\mathbb{R}_0} \Big\{ f(t, X(t^-) + \gamma(t,z)) - f(t, X(t^-)) \\
&\qquad - \frac{\partial f}{\partial x}(t, X(t^-))\gamma(t,z) \Big\} \nu(dz)dt \\
&+ \int_{\mathbb{R}_0} \{ f(t, X(t^-) + \gamma(t,z)) - f(t, X(t^-)) \} \tilde{N}(dt, dz).
\end{aligned}
$$

If α, σ and γ are *time homogeneous*, $X(t)$ is called a *jump diffusion* or a *Lévy diffusion.* Then, for $f \in C_0^2(\mathbb{R}^n)$, the generator $Af(x)$ of $X(t) \in \mathbb{R}^n$ is given by

$$
\begin{aligned}
Af(x) &= \sum_{i=1}^{n} \alpha_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n} (\beta\beta^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \\
&+ \int_{\mathbb{R}_0} \sum_{k=1}^{l} \{ f(x + \gamma^{(k)}(x,z)) - f(x) - \nabla f(x) \cdot \gamma^{(k)}(x,z) \} \nu_k(dz_k).
\end{aligned}
$$

For a more detailed introduction to Itô-Lévy processes see [15] or [1].

2 Introduction to the axiomatic theory of risk measures

We view a financial position as a P-a.e. bounded and $\mathcal{F}_T$-measurable function $X : \Omega \to \mathbb{R}$ where Ω is a fixed set of scenarios. The set of all such functions is denoted by $\mathcal{X}$, where $\mathcal{X} = L^\infty(\Omega, \mathcal{F}_T, P)$. From here we define three families of risk measures to assess the risk of a financial position.

2.1 Three families of risk measures

Definition 2.1 **[3]** *A coherent risk measure is a map $\rho : \mathcal{X} \to \mathbb{R}$ which has the following properties:*

(1) *Subadditivity:* $\rho(X + Y) \leq \rho(X) + \rho(Y)$

- This property gives that the risk of an aggregate position is bounded by the sum of the risk of each position.

(2) *Positive Homogenity: If* $\lambda \geq 0$, *then* $\rho(\lambda X) = \lambda\rho(X)$

- This property suggest linear growth in risk with the size of the position. We deduce that $\rho(0) = 0$.

(3) *Monotonicity: If* $X \leq Y$, *then* $\rho(X) \geq \rho(Y)$

- This property states that the risk decreases if the payoff increases.

(4) *Translation Invariance: If* $m \in \mathbb{R}$, *then* $\rho(Y + m) = \rho(Y) - m$

- This property gives that if we add the amount m to the position in a risk free manner, the capital requirement decreases with the same amount m. We deduce that $\rho(X + \rho(X)) = 0$.

Property (2) suggest a linear growth in risk with the size of the position, which in many situations can be too restrictive. Therefore we may relax some of these properties and define a convex risk measure.

Definition 2.2a **[21]** *A 'strong' convex risk measure is a map $\rho : \mathcal{X} \to \mathbb{R}$ which has the following properties:*

(3) *Monotonicity: If* $X \leq Y$, *then* $\rho(X) \geq \rho(Y)$

(4) *Translation Invariance: If* $m \in \mathbb{R}$, *then* $\rho(Y + m) = \rho(Y) - m$

(5) *Convexity:* $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for $\lambda \in [0,1]$

- This property ensures that the risk of a diversified investemt is bounded by the weighted average of each position's risk[2].

Definition 2.2b **[19]** *A 'weak' convex risk measure is a map* $\rho : \mathcal{X} \to \mathbb{R}$ *which has the following properties:*

(3) *Monotonicity: If* $X \leq Y$*, then* $\rho(X) \geq \rho(Y)$

(5) *Convexity:* $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ *for* $\lambda \in [0,1]$

In light of definitions 2.1, 2.2a and 2.2b we can see that any coherent risk measure is also a convex risk measure, but that the converse is not true. The set of convex risk measures is the most important set of risk measures in this thesis. In the remainder of this thesis, a 'strong' convex risk measure will be refferd to as just a convex risk measure. The third definition on risk measures that will be represented here, is the one on monetary risk measures.

Definition 2.3 **[3]** *A monetary risk measure is a map* $\rho : \mathcal{X} \to \mathbb{R}$ *which has the following properties:*

(3) *Monotonicity: If* $X \leq Y$*, then* $\rho(X) \geq \rho(Y)$

(4) *Translation Invariance: If* $m \in \mathbb{R}$*, then* $\rho(Y+m) = \rho(Y) - m$

2.2 The *acceptance set* of risk measures

We will call a financial position $X \in \mathcal{X}$ *acceptable* if $\rho(X) \leq 0$. If we let $\rho(0) = 0$, we can interpret $\rho(X)$ as the amount that should be added to X in a risk free manner to make it acceptable under the given risk measure ρ. It is then natural to define the set

$$\mathcal{A}_\rho := \{X \in \mathcal{X} \mid \rho(X) \leq 0\} \tag{2.1}$$

[2]For more information on the interpretation of axioms (1)-(5) see e.g. [3].

which we call the *acceptance set* of ρ, i.e. the set of all positions that do not need additional investments to pass as acceptable. Conversely, if we start with a acceptance set $\mathcal{A}_\rho \subset \mathcal{X}$, the risk measure

$$\rho_{\mathcal{A}_\rho}(X) = \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_\rho\}$$

fulfills $\rho_{\mathcal{A}_\rho}(X) = \rho(X)$. This last statement is true because

$$\begin{aligned} \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}_\rho\} &= \inf\{m \in \mathbb{R} \mid \rho(m + X) \leq 0\} \\ &= \inf\{m \in \mathbb{R} \mid \rho(X) \leq m\} \\ &= \rho(X). \end{aligned}$$

We also see that all constant functions are in $\mathcal{X}$, and if ρ fulfills translation invariance, all non-negative constant functions are in $\mathcal{A}_\rho$.

Proposition 2.1 *A monetary risk measure $\rho_{\mathcal{A}}$ is convex if and only if $\mathcal{A}$ is convex.*

Proof. Let $\rho_{\mathcal{A}}$ be convex, and $\lambda \in [0, 1]$. Then

$X, Y \in \mathcal{A} \Rightarrow 0 \geq \lambda\rho_{\mathcal{A}}(X) + (1-\lambda)\rho_{\mathcal{A}}(Y) \geq \rho_{\mathcal{A}}(\lambda X + (1-\lambda)Y) \Rightarrow (\lambda X + (1-\lambda)Y) \in \mathcal{A}.$

Conversely, let $\mathcal{A}$ be convex and $X, Y \in \mathcal{X}$. If $\rho_{\mathcal{A}}(X) = m$ and $\rho_{\mathcal{A}}(Y) = n$ then $m + X$ and $n + Y$ are both in $\mathcal{A}$, and by convexity $\lambda(m + X) + (1 - \lambda)(n + Y)$ are also in $\mathcal{A}$. So

$$0 \geq \rho_{\mathcal{A}}(\lambda(m + X) + (1 - \lambda)(n + Y)) = \rho_{\mathcal{A}}(\lambda X + (1 - \lambda)Y) - (\lambda m + (1 - \lambda)n).$$

Hence,

$$\rho_{\mathcal{A}}(\lambda X + (1 - \lambda)Y) \leq \lambda\rho_{\mathcal{A}}(X) + (1 - \lambda)\rho_{\mathcal{A}}(Y)$$

□

3 VaR in light of the axiomatic properties of risk measures

In computing VaR (Value at Risk), we basically compute the quantile corresponding to the distribution of X, given a probability measure P. One definition among many equivalent definitions is the following.

Definition 3.1 *Let $\alpha \in (0,1)$. We define $VaR(X)$ with level α as*

$$VaR_{\alpha}(X) := \inf\{\ m \mid P(X+m<0) \leq \alpha\}.$$

Next we look at an example that rules out VaR as an convex risk measure.

3.1 Example

Let the risk free rate of return be $r \geq 0$. Our investment options is two independent defaultable bonds, both with equal probability of default p, rate of return $\tilde{r}$ which staisfies $r \leq \tilde{r} \leq 1+2r$, and cost ω. For bond i we then have that the discounted net gain is

$$X_i = \begin{cases} -\omega & \text{in case of default} \\ \frac{\omega(\tilde{r}-r)}{1+r} & \text{otherwise.} \end{cases}$$

We then get

$$P\left(X_i - \frac{\omega(\tilde{r}-r)}{1+r} < 0\right) = P(\text{default}) = p.$$

Hence, if $\alpha = 0.1$ and $p = 0.09$

$$VaR_{0.1}(X_i) = -\frac{\omega(\tilde{r}-r)}{1+r} < 0.$$

The interpretation of this is that the position do not need additional risk free investments to pass as acceptable. This is without concern for the size of the potential loss. If we now choose to diversify our investment by investing $\omega/2$ in each of the two bonds, the discounted net gain becomes $Y = (X_1+X_2)/2$, i.e.

$$Y = \begin{cases} -\omega & \text{in case of both default} \\ -\frac{\omega}{2}\left(1-\frac{(\tilde{r}-r)}{1+r}\right) & \text{in case only one default} \\ \frac{\omega(\tilde{r}-r)}{1+r} & \text{otherwise.} \end{cases}$$

Then the probability that Y is negative is the same as the probability that one or two of the bonds default: $P(\text{one or two bonds default}) = p(2-p)$.
Since we have $p(2-p) > \alpha$

$$VaR_{0.1}(Y) = \frac{\omega}{2}\left(1 - \frac{(\tilde{r}-r)}{1+r}\right) > 0.$$

We conclude that

$$VaR_{0.1}(Y) > \frac{1}{2}VaR_{0.1}(X_1) + \frac{1}{2}VaR_{0.1}(X_2).$$

This example contradicts property (5) from the definition of convex risk measures, and hence VaR is not a convex risk measure. VaR gives the minimal loss that can occur within the α-quantile. By other words VaR will compute the best scenario among all bad scenarios, but it does not answer 'how bad is bad'. If risk is measured with respect to VaR, this example shows that one might be inclined to concentrate the investment in one single asset, without regard to the expected loss in case of default. When, as in this example, VaR discourage diverification it clearly works against what we want from a risk measure.

When constructing such an example it is not needed much creativity. One only need to exploit the fact that VaR fails to recognice the severity of 'bad' scenarios. In general, VaR fails to fulfill convexity when dealing with small probabilities of default. Consequently, the worst deployment of VaR is in the case where we deal with small probabilities and big losses.

4 A characterization of a convex risk measure

To assess the risk of a financial position one could simply look at the 'worst case scenario' measure.

$$\rho_{\text{wcs}}(X) = -\inf_{\omega\in\Omega} X(\omega).$$

It is readily seen that ρ_{wcs} is a coherent, and therefore also convex, risk measure.

- $X \geq Y$ gives that $\inf_{\omega\in\Omega} X(\omega) \geq \inf_{\omega\in\Omega} Y(\omega)$ i.e. $\rho_{\text{wcs}}(X) \leq \rho_{\text{wcs}}(Y)$
- $m \in \mathbb{R}$ gives that $\rho_{\text{wcs}}(X+m) = \rho_{\text{wcs}}(X) - m$
- $\rho_{\text{wcs}}(X+Y) \leq \rho_{\text{wcs}}(X) + \rho_{\text{wcs}}(Y)$
- $\lambda \geq 0$ gives that $\rho_{\text{wcs}}(\lambda X) = \lambda\rho_{\text{wcs}}(X)$

Here, the acceptance set $\mathcal{A}_{\rho_{\text{wcs}}}$ is given by all non-negative functions in $\mathcal{X}$. Since this measure does not take into account the probability of such a 'worst case scenario' it is very conservative. It may even not be interesting when dealing with big losses on a set of measure zero. In fact, it is the most conservative of all coherent risk measures on $\mathcal{A}_{\rho_{\text{wcs}}}$. This is seen by letting ρ be any coherent risk measure on $\mathcal{A}_{\rho_{\text{wcs}}}$, then by properties (2), (3) and (4) from Definition 2.1

$$\begin{aligned}
\rho(X) &\overset{(3)}{\leq} \rho(\inf_{\omega\in\Omega} X(\omega)) \\
&\overset{(2)}{=} \inf_{\omega\in\Omega} X(\omega)\rho(1) \\
&\overset{(4)}{=} \inf_{\omega\in\Omega} X(\omega)[\rho(0) - 1] \\
&= \rho_{\text{wcs}}(X).
\end{aligned}$$

If we let $\mathcal{M}$ be the set of all probability measures on $(\Omega, \mathcal{F})$ we see that ρ_{wcs} has the alternative representation

$$\rho_{\text{wcs}}(X) = \sup_{Q\in\mathcal{M}} E_Q[-X].$$

To define a less conservative measure we let $\mathcal{Q} \subseteq \mathcal{M}$ and $\gamma : \mathcal{Q} \to [-\infty, \infty)$, with $\sup_{Q\in\mathcal{Q}} \gamma(Q) < \infty$, where $\gamma(Q)$ will represent some 'lower bound' relative to the probability measure Q. Now let the set of acceptable financial positions be

$$\mathcal{A}' = \{X \in \mathcal{X} \mid E_Q[X] \geq \gamma(Q) \text{ for all } Q \in \mathcal{Q}\}.$$

Observe that if $X, Y \in \mathcal{A}'$ then $\gamma(Q) \leq E_Q[X]$, and $\gamma(Q) \leq E_Q[Y]$ for all Q. Then we have that

$$\begin{aligned}
\gamma(Q) - E_Q[\lambda X + (1-\lambda)Y] &= \gamma(Q) - \lambda E_Q[X] - (1-\lambda)E_Q[Y] \\
&\leq \gamma(Q) - \lambda\gamma(Q) - (1-\lambda)\gamma(Q) \\
&= 0.
\end{aligned}$$

So the acceptance set $\mathcal{A}'$ is convex. By Definition 2.3 and Proposition 2.1 we see that the corresponding risk measure is convex. From the form of $\mathcal{A}'$ we deduce that the corresponding risk measure gets the form

$$\rho_{\mathcal{A}'}(X) = \sup_{Q \in \mathcal{Q}} (\gamma(Q) - E_Q[X]).$$

From this point on ρ and $\rho_{\mathcal{A}'}$ will have the same meaning.

An equivalent form of ρ is

$$\rho(X) = \sup_{Q \in \mathcal{Q}} (E_Q[-X] - \zeta(Q)) \tag{4.1}$$

where $\zeta(Q) = -\gamma(Q)$. We will call ζ a 'penalty' function, and say that ρ is represented by ζ.

In (4.1) we have taken $\mathcal{Q}$ to be some arbitrary subset of $\mathcal{M}$. A more specific characterization of $\mathcal{Q}$ may be given when noted that it is natural to restrict our class of convex risk measures to those who have the following property on $\mathcal{X}$.

$$\rho(X) = \rho(Y) \text{ if } X = Y \text{ } P\text{-a.s.} \tag{4.2}$$

This assumption on ρ is not only useful in the study of $\mathcal{Q}$. It also has the practical interpertation that two financial positions with the same payoff will be regarded as equally risky. If this assumption is violated one could have unexpected results, e.g. an option being more risky than its replicating portfolio.

***Proposition 4.1*,[10]** *Let ρ be a convex risk measure given by (4.1) with property (4.2). Furthermore, let ζ be as in* (4.1). *Then $\zeta(Q) = \infty$ for all Q not absolutely continous with respect to P.*

Proof. Let

$$\rho(X) = \sup_{Q \in \mathcal{Q}} (E_Q[-X] - \zeta(Q))$$

then

$$\zeta(Q) \geq E_Q[-X] - \rho(X)$$

for all Q and X. Hence

$$\begin{aligned} \zeta(X) &\geq \sup_{X \in \mathcal{X}} (E_Q[-X] - \rho(X)) \\ &\geq \sup_{X \in \mathcal{A}_\rho} (E_Q[-X] - \rho(X)) \\ &\geq \sup_{X \in \mathcal{A}_\rho} (E_Q[-X]) . \end{aligned}$$

Here, $\mathcal{A}_\rho$ is as in (2.1). Now let $A \in \mathcal{F}_\mathrm{T}$ be such that $P(A) = 0$. Then if $X \in \mathcal{A}_\rho$ and $X_n = X - nI_A$ then $X = X_n$ P-a.e., so $\rho(X) = \rho(X_n)$. Hence $X_n \in \mathcal{A}_\rho$. Now let $Q(A) > 0$, then

$$\zeta(Q) \geq \sup_{X \in \mathcal{A}_\rho} E_Q[-X] \geq E_Q[-X_n] = E_Q[-X] + nQ(A) \to \infty$$

when $n \to \infty$. □

By Proposition 4.1 we see that there is nothing to gain by including probability measures not absolutely continuous with respect to P in $\mathcal{Q}$.

***Proposition 4.2*, [8],[9]** *Every convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is of the form*

$$\rho(X) = \sup_{Q \in \mathcal{P}} \big(E_Q[-X] - \zeta(Q)\big)$$

for some family $\mathcal{P}$ of probability measures absolutely continuous with respect to P, and some 'penalty' function $\zeta : \mathcal{P} \to (-\infty, \infty]$.

By the form of ρ we see that $\zeta(Q) \geq \inf_{\mu \in \mathcal{P}}\{\zeta(\mu)\} = -\sup_{\mu \in \mathcal{P}}\{-\zeta(\mu)\} = -\rho(0)$ for all $Q \in \mathcal{P}$. And in retrospect, we see that the existence of the alternative representation of ρ_{WCS} follows from Proposition 4.2.

The following is another important result on convex risk measures, which says that choosing a convex risk mesaure is equivalent to choosing a family of measures $\mathcal{P}$, and a penalty function ζ.

***Proposition 4.3*, [8],[9]** *A map $\rho : \mathcal{X} \to \mathbb{R}$ is a convex risk measure if and only if there exists a family $\mathcal{P}$ of measures $Q \ll P$ on $\mathcal{F}_T$ and a convex 'penalty' function $\zeta : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ with $\inf_{Q\in\mathcal{P}} \zeta(Q) = 0$ such that*

$$\rho(X) = \sup_{Q\in\mathcal{P}} \big(E_Q[-X] - \zeta(Q)\big)$$

In Proposition 4.3, we see that the assumtption $\inf_{Q\in\mathcal{P}} \zeta(Q) = 0$ ensures that $\rho(0) = 0$, i.e. the risk $\rho(X)$ can be interpered as the amount needed to make the investment X acceptable. Thanks to Proposition 4.3, the interpretation of the acceptance set of ρ in Chapter 2.2 now holds for all convex risk measures. As Proposition 4.2 and Proposition 4.3 does not single out one spesific risk measure, one can to some degree let the final choice of risk measure allow economic considerations and risk aversion to come into play. For instance, since not all investors has the same level of risk aversion, or even the same regulating framework, the choice of ζ may differ to suit the situation at hand.

Example A well known family of measures absolutely continuous with respect to P is the family arising from Girsanov transformations, i.e. on the form

$$dQ = Z_\theta(T)dP.$$

Example One choice of ζ is the relative entropy[3] of Q with respect to P

$$\zeta(Q) = E\left[\frac{dQ}{dP} log \frac{dQ}{dP}\right] = H(Q;P)$$

where the expectation is taken with respect to P, and $\frac{dQ}{dP}$ denotes the Radon-Nikodym derivative of Q with respect to P.

Example To illustrate the ideas of convex risk measures and the relative entropy, we introduce this simple example. Consider a given measure space $(\Omega, \mathcal{F}, P)$, where $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\mathcal{F}$ is the powerset of Ω. We define $\mathcal{P} = \{P, Q\}$, where the distribution of P and Q is given by

[3]See e.g. [4]

	P	Q
ω_1	0.2	0.2
ω_2	0.5	0.6
ω_3	0.3	0.2

The random variable $X(\omega)$ (e.g some financial option) will have payoff

	ω_1	ω_2	ω_3
X	0.2	-0.1	-0.05

The penalty function of ρ is in this example the relative entropy, so $\rho(X)$ will have the form

$$\rho(X) = \max_{Q \in \mathcal{P}} \left\{ E_Q[-X] - \sum_{i=1}^{3} Q(\omega_i) \log\left(\frac{Q(\omega_i)}{P(\omega_i)}\right) \right\}$$

First we compute this expression with resptect to P, then the relative entropy will be 0, and

$$E_P[-X] = -0.2 \times 0.2 + 0.1 \times 0.5 + 0.05 \times 0.3 = 0.025$$

If we use Q we will get

$$\begin{aligned} E_Q[-X] - \sum_{i=1}^{3} Q(\omega_i) \log\left(\frac{Q(\omega_i)}{P(\omega_i)}\right) &= \underbrace{-0.2 \times 0.2 + 0.6 \times 0.1 + 0.2 \times 0.05}_{=0.03} \\ &- (0.2 \times \log 1 + 0.6 \times \log 1.2 + 0.2 \times \log 0.67) \\ &\approx 0.0017 \end{aligned}$$

Here we see that the most 'pessimistic' expectation of X ('optimistic' expectation of $-X$) is achieved with Q. However, if the expectation is computed with respect to Q, we recive a penalty for the distance[4] from Q to P. In this case, this penalty assures that the maximum is attained when P is used. From an intuitional point, ρ (represented by the relative entropy) makes sense as the most pessimistic expectation of X, given that we do not stray too far away from the 'true' probablility measure P.

[4] The relative entropy, also called the 'Kullback-Leibler divergence', measures the distance (in some sense) from one probability distribution to another. It is non-cummutative, i.e. $H(Q;P) \neq H(P;Q)$

5 Risk minimizing portfolios

In this chapter we will look at what happens when our payoff is affected by the choice of portfolio. The $\mathcal{F}_T$-measurable, P-a.e bounded random variable $X : \Omega \times \Pi \to \mathbb{R}$ will represent this payoff, where Π is a given set of allowed portifolios. In this setting it will be of interest to find the portifolio $\pi \in \Pi$ that minimizes the risk, where the risk measure is as in Proposition 4.2. This leads to a min-max problem on the form

$$\inf_{\pi\in\Pi} \sup_{Q\in\mathcal{P}} \left(E_Q[-X^{(\pi)}] - \zeta(Q)\right) \tag{5.1}$$

This problem can be reformulated by introducing the concave function $U(X) := -\rho(X)$, where ρ is a given convex risk measure as in Definition 2.2a. This function $U(X)$ is an example of a *monetary utility function*, which is defined in the following way.

Definition 5.1 *A monetary utility function is a map $U : \mathcal{X} \to \mathbb{R}$ with the following properties:*

(3)' *Monotonicity: If $X \leq Y$, then $U(X) \leq U(Y)$*

(4)' *Translation Invariance: If $m \in \mathbb{R}$, then $U(Y+m) = U(Y) + m$*

(5)' *Concavity: $U(\lambda X + (1-\lambda)Y) \geq \lambda U(X) + (1-\lambda)U(Y)$* for $\lambda \in [0,1]$

We can now reformulate (5.1) in the following manner.

Problem 5.1 Find

$$\Phi(x) := \sup_{\pi\in\Pi} \inf_{Q\in\mathcal{P}} \left(E_Q^x[X^\pi] + \zeta(Q)\right)$$

and find π^* and Q^* such that

$$\Phi(x) = E_{Q^*}^x[X^{\pi^*}] + \zeta(Q^*).$$

In view of the original problem in (5.1) we see that $-\Phi(x)$ will give the solution we set out to find. If $\zeta(Q) = E[f(Q)]$ for some suitable f, this reformulation of (5.1) has the advantage of being easy to translate into the existing theroy on HJBI equations. More on this will follow in the sections below.

5.1 The market model

In our jump diffution market there will be two investment possibilities:

- A non-risky asset with price $S_0(t)$ given by

$$dS_0(t) = r(t)S_0(t)dt; \qquad S_0(0) = 1 \tag{5.2}$$

- A risky asset with price $S_1(t)$ given by

$$dS_1(t) = S_1(t^-)[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)]; \qquad S_1(0) > 0 \tag{5.3}$$

where r, α, β and γ are $\mathcal{F}_t$-adapted processes satisfying

$$E\Big[\int_0^T \{|r(s)| + |\alpha(s)| + \frac{1}{2}\beta^2(s) + \int_{\mathbb{R}_0} |\log(1+\gamma(s,z)) - \gamma(s,z)|\nu(dz)ds\}\Big] < \infty.$$

The solution for $S_0(t)$ is straight forward

$$\begin{gathered} dS_0(t) = r(t)S_0(t)dt \\ \Downarrow \\ S_0(t) = e^{\int_0^t r(s)ds} \end{gathered}$$

By Itô's formula we find the solution for $S_1(t)$ by taking the derivative of $\log S_1(t)$

$$\begin{aligned} d\log S_1(t) = &\frac{1}{S_1(t)} S_1(t)[\alpha(t)dt + \beta(t)dB(t)] - \frac{1}{2S_1^2(t)} S_1^2(t)\beta^2(t)dt \\ &+ \int_{\mathbb{R}} \left[\log(S_1(t^-) + S_1(t^-)\gamma(t,z)) - \log S_1(t^-) - \frac{1}{S_1(t^-)} S_1(t^-)\gamma(t,z)\right] \nu(dz)dt \\ &+ \int_{\mathbb{R}} \left[\log(S_1(t^-) + S_1(t^-)\gamma(t,z)) - \log S_1(t^-)\right] \tilde{N}(dt,dz) \\ &= \alpha(t)dt + \beta(t)dB(t) - \frac{1}{2}\beta^2(t)dt + \int_{\mathbb{R}} [\log(1+\gamma(t,z)) - \gamma(t,z)]\,\nu(dz)dt \\ &+ \int_{\mathbb{R}} \log(1+\gamma(t,z))\tilde{N}(dt,dz) \end{aligned}$$

$$\begin{aligned} \log S_1(t) = \log S_1(0) &+ \int_0^t \left[\alpha(s) - \frac{1}{2}\beta^2(s) + \int_{\mathbb{R}} [\log(1+\gamma(s,z)) - \gamma(s,z)]\,\nu(dz)\right] dt \\ &+ \int_0^t \beta(s)dB(s) + \int_0^t \int_{\mathbb{R}} \log(1+\gamma(s,z))\tilde{N}(ds,dz) \end{aligned}$$

$$S_1(t) = S_1(0) \exp \left\{ \int_0^t \left[\alpha(s) - \frac{1}{2}\beta^2(s) + \int_{\mathbb{R}} [\log(1+\gamma(s,z)) - \gamma(s,z)]\, \nu(dz) \right] dt \right.$$
$$\left. + \int_0^t \beta(s) dB(s) + \int_0^t \int_{\mathbb{R}} \log(1+\gamma(s,z)) \tilde{N}(ds,dz) \right\}$$

We see that this solution is well defined for $\gamma(t,z) > -1$ for a.a. t,z, P-a.s. Here, the random variable $X^{(\pi)} = V^\pi(T)$ will be a linear combination of the risky asset and the non-risky asset. The function $\pi : [0,T] \times \Omega \to \mathbb{R}$ will represent the proportion of wealth invested in the risky asset. Let η_0 and η_1 be number of units held in asset 0 and 1 respectively. The wealth process then becomes

$$\begin{aligned}
dV^\pi(t) &= \eta_0(t) dS_0(t) + \eta_1(t) dS_1(t) \\
&= \eta_0(t) r(t) S_0(t) dt \\
&+ \eta_1(t^-) S_1(t^-) \Big[\alpha(t) dt + \beta(t) dB(t) + \int_{\mathbb{R}} \gamma(t,z) \tilde{N}(dt,dz) \Big] \\
&= (1-\pi(t)) V^\pi(t) r(t) dt \\
&+ \pi(t^-) V^\pi(t^-) \Big[\alpha(t) dt + \beta(t) dB(t) + \int_{\mathbb{R}} \gamma(t,z) \tilde{N}(dt,dz) \Big] \\
&= V^\pi(t^-) \Big[((1-\pi(t)) r(t) + \pi(t)\alpha(t)) dt \\
&+ \pi(t)\beta(t) dB(t) + \pi(t^-) \int_{\mathbb{R}} \gamma(t,z) \tilde{N}(t,z) \Big]
\end{aligned} \tag{4.4}$$

By the same method used to find $S_1(t)$ we get

$$V^\pi(t) = V^\pi(0) \exp \left\{ \int_0^t \Big[(1-\pi(s)) r(s) + \pi(s)\alpha(s) - \frac{1}{2}\pi^2(s)\beta^2(s) \right.$$
$$+ \int_{\mathbb{R}} [\log(1+\pi(s)\gamma(s,z)) - \pi(s)\gamma(s,z)]\, \nu(dz) \Big] dt$$
$$\left. + \int_0^t \pi(s)\beta(s) dB(s) + \int_0^t \int_{\mathbb{R}} \log(1+\pi(s)\gamma(s,z)) \tilde{N}(ds,dz) \right\}$$

where we assume $\pi(t)\gamma(t,z) > -1$ for a.a. t,z, P-a.s.

5.2 An HJBI equation for zero-sum differential games for jump diffusions

Consider the controlled jump diffusion $Y(t) = Y^u(t)$, where

$$\begin{aligned} dY(t) &= b(Y(t), u_0(t))dt + \sigma(Y(t), u_0(t))dB(t) \\ &+ \int_{\mathbb{R}^k} \gamma_1(Y(t^-), u_1(t^-, z), z)\tilde{N}(dt, dz); \qquad Y(0) = y \in \mathbb{R}^k \end{aligned} \tag{5.5}$$

Here, $u(t,z) = (u_0(t), u_1(t,z))$, where $u_0(t), u_1(t,z) \in K \subseteq \mathbb{R}^p$ represent the control (e.g. π and Q as in Problem 4.1), and it is assumed to be càdlàg and adapted to the filtration $\mathcal{F}_t$ generated by $B(\cdot)$ and $\tilde{N}(\cdot,\cdot)$. Furthermore, $b : \mathbb{R}^k \times K \to \mathbb{R}^k$, $\sigma : \mathbb{R}^k \times K \to \mathbb{R}^{k\times k}$ and $\gamma_1 : \mathbb{R}^k \times K \times \mathbb{R}^k \to \mathbb{R}^{k\times k}$ are given, $B(t)$ is a k-dimensional Brownian motion, and $\tilde{N}(\cdot,\cdot)$ are k independent compensated Poisson random measures.

Now let $\mathcal{S}$ be a open subset of $\mathbb{R}^k$ where $Y(t)$ is 'solvent', and define the stopping time

$$\tau_{\mathcal{S}} = \inf\{t > 0; Y(t) \notin \mathcal{S}\}.$$

Let $\mathcal{K}$ be the set of admissible controls satisfying integrability conditions, and such that (5.5) has a unique strong solution. For $u \in \mathcal{K}$ define the performance functional

$$J^u(y) = E^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y(t), u_0(t))dt + g(Y(\tau_{\mathcal{S}}))\right]$$

where $f : \mathbb{R}^k \times K \to \mathbb{R}$, and $g : \mathbb{R}^k \to \mathbb{R}$ are given.

If we let $u_0(t) = (\theta_0(t), \pi_0(t))$, and $u_1(t,z) = (\theta_1(t,z), \pi_1(t,z))$ we can view $\theta(t,z) = (\theta_0(t), \theta_1(t,z)) \in K_1$ as the control for player 1, and $\pi(t,z) = (\pi_0(t), \pi_1(t,z)) \in K_2$ as the control for player 2, for suitable sets K_1 and K_2. Denote the set of such controls by Θ and Π, respectively. We then formulate the following problem.

Problem 5.2 Find $\Phi(y)$ and $(\theta^*, \pi^*) \in \Theta \times \Pi$ such that

$$\Phi(y) = \sup_{\pi\in\Pi}\left(\inf_{\theta\in\Theta} J^{\theta,\pi}(y)\right) = J^{\theta^*,\pi^*}(y)$$

where (θ^*, π^*) is called an optimal control if it exist.

We will not look at the general family of adapted controls, but focus on Markov controls. This may not be a big loss of generality, since in many cases a Markov control gives the same performance as the more general family of adapted controls[5]. This means that our control will only depend on the current state of the system, and not e.g. the starting point.

Before stating Proposition 5.1 we need some notes of nomenclature. The generator of a process $Y(t) \in \mathbb{R}^n$, with $Y(0) = y = (s, y_1, \cdots, y_n)$ will be denoted $A^{\theta,\pi}\varphi(y)$, and let $\mathcal{T} = \{\tau : \tau \text{ is a } \mathcal{F}_t\text{-stopping time}, \tau \leq \tau_{\mathcal{S}}\}$.

***Proposition 5.1*,[20]** *Suppose there exist a function $\varphi \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$ and a Markov control $(\hat{\theta}(y), \hat{\pi}(y)) \in \Theta \times \Pi$ such that*

(i) $A^{\theta,\hat{\pi}(y)}\varphi(y) + f(y, \theta, \hat{\pi}(y)) \geq 0$ *for all* $\theta \in K_1, y \in \mathcal{S}$

(ii) $A^{\hat{\theta}(y),\pi}\varphi(y) + f(y, \hat{\theta}(y), \pi) \leq 0$ *for all* $\pi \in K_2, y \in \mathcal{S}$

(iii) $A^{\hat{\theta}(y),\hat{\pi}(y)}\varphi(y) + f(y, \hat{\theta}(y), \hat{\pi}(y)) = 0$ *for all* $y \in \mathcal{S}$

(iv) $Y^{\theta,\pi}(\tau_{\mathcal{S}}) \in \partial\mathcal{S}$ *a.s. on* $\{\tau_{\mathcal{S}} < \infty\}$ *and*
$\lim_{t\to\tau_{\mathcal{S}}^-} \varphi(Y^{\theta,\pi}(t)) = g(Y^{\theta,\pi}(\tau_{\mathcal{S}}))\chi_{\{\tau_{\mathcal{S}}<\infty\}}$
a.s. for all $(\theta, \pi) \in \Theta \times \Pi, y \in \mathcal{S}$

(v) *The family of* $\{\varphi(Y^{\theta,\pi}(\tau))\}_{\tau\in\mathcal{T}}$ *is uniformly integrable, for all* $y \in \mathcal{S}, (\theta, \pi) \in \Theta \times \Pi$.

Then

$$\begin{aligned}\varphi(y) = \Phi(y) &= \sup_{\pi\in\Pi}\left(\inf_{\theta\in\Theta} J^{\theta,\pi}(y)\right) = \inf_{\theta\in\Theta}\left(\sup_{\pi\in\Pi} J^{\theta,\pi}(y)\right) \\ &= \sup_{\pi\in\Pi} J^{\hat{\theta},\pi}(y) = \inf_{\theta\in\Theta} J^{\theta,\hat{\pi}}(y) = J^{\hat{\theta},\hat{\pi}}(y)\end{aligned}$$

and

$(\hat{\theta}, \hat{\pi})$ *is an optimal (Markov) control.*

For proof of Proposition 5.1 see [20].

5.3 Examples

One way to start the process of constructing an example regarding risk minimizing portfolios, is to look at more or less trivial situations. However, this may not give the desired results, as the following example will show.

[5] As proven for the case without jumps in [17].

Trivial case with constant coefficients

Let us study the problem (5.1) in the context where $\Pi = \mathbb{R}$, $\Theta = \{0\}$, and $\zeta = 0$. Then (5.1) gets the form

$$\inf_{\pi \in \Pi} E[-V^{(\pi)}(T)].$$

Let π be the number of assets held in the risky asset $S(t)$, defined as in (5.3) by

$$dS(t) = S(t^-)[\alpha dt + \beta dB(t) + \int_{\mathbb{R}_0} \gamma \tilde{N}(dt, dz)]$$

for some given constants α, β and γ. If we let the risk free asset $S_0(t) = 1$, i.e. $r(t) = 0$, our value process $V^{(\pi)}(t)$ from (4.4) becomes

$$dV^{(\pi)}(t) = \pi(t^-)S(t^-)[\alpha dt + \beta dB(t) + \int_{\mathbb{R}_0} \gamma \tilde{N}(dt, dz)].$$

Thanks to Theorem 5.3.5 (Martingale representation 2) in [1], our problem can be written as

$$\inf_{\pi \in \Pi} E[-(x + \int_0^T \pi(t)S(t)\alpha dt)] = -x - \alpha \sup_{\pi \in \Pi} E[\int_0^T \pi(t)S(t)dt].$$

which is degenerate, in the sense that

$$\sup_{\pi \in \Pi} E[\int_0^T \pi(t)S(t)dt] = \infty.$$

This example shows that even close to trivial cases may not have a solution.

An example with constant coefficients and no Brownian motion

In this example we will look at the following version of Problem 5.1.

$$\sup_{\pi \in \Pi} \inf_{\theta \in \Theta} E_{Q_\theta}[V^{(\pi)}(T)]$$

where $\Pi = \mathbb{R}$ and $\Theta = \{\theta : E[Z_\theta(T)] = 1\}$, with $Z_\theta(t)$ given by

$$dZ_\theta(t) = -Z_\theta(t^-) \int_{\mathbb{R}_0} \theta(t, z)\tilde{N}(dt, dz)$$

and

$$dQ_\theta(\omega) = Z_\theta(T)dP(\omega), \text{ on } \mathcal{F}_T.$$

In this case we consider the process

$$dY(t) = \begin{bmatrix} dY_0(t) \\ dY_1(t) \\ dY_2(t) \end{bmatrix} = \begin{bmatrix} dt \\ dZ_\theta(t) \\ dV^{(\pi)}(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \pi(t)S(t^-)\alpha \end{bmatrix} dt + \int_{\mathbb{R}_0} \begin{bmatrix} 0 \\ -\theta(t,z)Z_\theta(t^-) \\ \pi(t)S(t^-)\gamma \end{bmatrix} \tilde{N}(dt,dz)$$

with $Y(0) = (s, y_1, y_2)$. Then, the generator of $Y(t)$ has the form

$$\begin{aligned} A^{\pi,\theta}\varphi(y) = \varphi_s(y) + \pi y_2 \alpha \varphi_2(y) + \int_{\mathbb{R}_0} \big[&\varphi(s, y_1 - \theta y_1, y_2 + \pi y_2 \gamma) - \varphi(y) \\ &+ \theta_1 y_1 \varphi_1(y) - \pi y_2 \gamma \varphi_2(y) \big] \nu(dz) \end{aligned}$$

In light of (iv) from Proposition 5.1, we guess

$$\varphi(y) = h(s)y_1y_2$$

for some $h : [0,T] \to \mathbb{R}$ with $h(T) = 1$. This gives that the generator can be written as

$$A^{\pi,\theta}\varphi(y) = h'(s)y_1y_2 + \pi\alpha h(s)y_1y_2 - \int_{\mathbb{R}_0} [h(s)\theta y_1 \pi y_2 \gamma]\nu(dz).$$

If we now maximize $A^{\pi,\hat{\theta}}\varphi(y)$ over all $\pi \in \Pi$ we get the following first order condition for $\hat{\pi}$.

$$\begin{aligned} \alpha h(s)y_1y_2 - \int_{\mathbb{R}_0} [h(s)\hat{\theta} y_1 y_2 \gamma]\nu(dz) = 0, \text{ i.e.} \\ \alpha = \int_{\mathbb{R}_0} [\hat{\theta}\gamma]\nu(dz). \end{aligned}$$

By this, we see that the market will choose a $\hat{\theta}$ that corresponds to Q_θ being a 'risk-free' measure, as in Girsanov Theorem II in [15]. Also, we note that $\hat{\theta}$ will be independent of t. The first order condition obtained when minimizing $A^{\hat{\pi},\theta}\varphi(y)$ over all $\theta \in \Theta$ is

$$\int_{\mathbb{R}_0} [h(s)y_1\hat{\pi}y_2\gamma]\nu(dz) = 0$$

which leads to $\hat{\pi} = 0$, as the optimal choice for the agent. By condition (iii) in Proposition 5.1, the differential equation for the deterministic function $h(s)$ becomes

$$h'(s)y_1y_2 = 0,$$

So

$$h(s) = 1 \text{ for } t \in [0, T].$$

We conclude that the solution to the problem in this example is

$$\Phi(x) = y_1 y_2.$$

An example with constant coefficients

As a deployment of Proposition 5.1 we will study Problem 5.1 in the case where α, β and γ are constants, and $r(t) = 0$.

Let $\mathcal{P}$ from Proposition 4.2 be the set of all probability measures on the form

$$dQ_\theta(\omega) = Z_\theta(T)dP(\omega) \text{ on } \mathcal{F}_T$$

where

$$Z_\theta(t) = \exp\Big\{ -\int_0^t \theta_0(s)B(s) - \frac{1}{2}\int_0^t \theta_0^2(s)ds + \int_0^t \int_{\mathbb{R}_0} \log(1-\theta_1(s,z))\tilde{N}(ds,dz) + \int_0^t \int_{\mathbb{R}_0} [\log(1-\theta_1(s,z)) + \theta_1(s,z)]\nu(dz)ds \Big\}.$$

By Itô‘s formula we get

$$dZ_\theta(t) = -Z_\theta(t^-)\Big[\theta_0(t)dB(t) + \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{N}(dt,dz)\Big].$$

Next, we let

$$\Theta = \Big\{\theta : E[Z_\theta(T)] = 1\Big\}.$$

In this case, our process will be given by

$$dY(t) = \begin{bmatrix} dY_0(t) \\ dY_1(t) \\ dY_2(t) \end{bmatrix} = \begin{bmatrix} dt \\ dZ_\theta(t) \\ dV(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \pi(t)V(t^-)\alpha \end{bmatrix} dt + \begin{bmatrix} 0 \\ -\theta_0(t)Z_\theta(t^-) \\ \pi(t)V(t^-)\beta \end{bmatrix} dB(t) + \int_{\mathbb{R}_0} \begin{bmatrix} 0 \\ -\theta_1(t)Z_\theta(t^-) \\ \pi(t)V(t^-)\gamma \end{bmatrix} \tilde{N}(dt,dz)$$

with $Y(0) = y = (s, y_1, y_2)$. The penalty function will be assumed to have the form

$$\zeta(Q_\theta) = E\left[\left(\frac{dQ_\theta}{dP}\right)^2\right] = E[Z_\theta(T)^2].$$

We see that Problem 5.1 can be written in the HJBI context as

$$\Phi(y) = \sup_{\pi\in\Pi}\Big\{\inf_{\theta\in\Theta}\{E^y\left[Y_1(T-s)Y_2(T-s) + Y_2(T-s)^2\right]\}\Big\}.$$

Here, γ_1 from (5.5) becomes

$$\gamma_1(Y(t^-), u_1(t^-, z), z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -y_1\theta_1(t,z) & 0 \\ 0 & \pi(t)y_2\gamma & 0 \end{bmatrix}$$

If we denote the i'th column of γ_1 by $\gamma_1^{(i)}$, the generator of $Y(\cdot)$ becomes

$$\begin{aligned} A^{\theta,\pi}\varphi(y) &= \varphi_s(y) + \pi y_2\alpha\varphi_2(y) + \frac{1}{2}\theta_0^2 y_1^2\varphi_{22}(y) - \theta_0 y_1\pi y_2\beta\varphi_{12}(y) + \frac{1}{2}\pi^2 y_2^2\beta^2\varphi_{22}(y) \\ &\quad + \int_{\mathbb{R}}\left[\varphi(y+\gamma_1^{(2)}) - \varphi(y) - \nabla\varphi(y)\cdot\gamma_1^{(2)}\right] \\ &= \varphi_s(y) + \pi y_2\alpha\varphi_2(y) + \frac{1}{2}\theta_0^2 y_1^2\varphi_{22}(y) - \theta_0 y_1\pi y_2\beta\varphi_{12}(y) + \frac{1}{2}\pi^2 y_2^2\beta^2\varphi_{22}(y) \\ &\quad + \int_{\mathbb{R}}\Big[\varphi(s, y_1 - y_1\theta_1, y_2 + y_2\pi\gamma) - \varphi(s, y_1, y_2) + \theta_1 y_1\varphi_1(y) - \pi y_2\gamma\varphi_2(y)\Big]\nu(dz). \end{aligned}$$

Now that we have the form of $A^{\theta,\pi}\varphi(y)$ we see from Proposition 5.1 (iii) and (iv) that the equation to be solved is

$$\begin{cases} \sup_{\pi\in\Pi}\left(\inf_{\theta\in\Theta} A^{\theta,\pi}\varphi(y)\right) = 0 \\ \varphi(T, y_1, y_2) = y_1 y_2 + y_2^2 \end{cases} \tag{5.6}$$

From the form of (5.6), one possible choice of φ is

$$\varphi(y) = h(s)y_1y_2 + y_1^2$$

for some $h : [0,T] \to \mathbb{R}$ with $h(T) = 1$. Then

$$\begin{aligned}
A^{\theta,\pi}\varphi(y) &= h'(s)y_1y_2 + \pi y_2\alpha h(s)y_1 + \theta_0^2y_1^2 - \theta_0y_1\pi y_2\beta h(s) \\
&+ \int_{\mathbb{R}_0} \Big[h(s)(y_1 - y_1\theta_1)(y_2 + y_2\pi\gamma) + (y_1 - y_1\theta_1)^2 \\
&- h(s)y_1y_2 - y_1^2 + \theta_1y_1(h(s)y_2 + 2y_1) - \pi y_2\gamma h(s)y_1\Big]\nu(dz) \\
&= h'(s)y_1y_2 + \pi y_2\alpha h(s)y_1 + \theta_0^2y_1^2 - \theta_0y_1\pi y_2\beta h(s) \\
&+ \int_{\mathbb{R}_0} \Big[y_1^2\theta_1^2 - h(s)y_1\theta_1y_2\pi\gamma\Big]\nu(dz).
\end{aligned}$$

If we now maximize $A^{\hat{\theta},\pi}\varphi(y)$ over all π, we get the following first order condition for the minimum point $\hat{\pi}$.

$$y_2\alpha h(s)y_1 - \hat{\theta}_0y_1y_2\beta h(s) - \int_{\mathbb{R}_0} h(s)y_1\hat{\theta}_1y_2\gamma\nu(dz) = 0$$

or equivalently

$$\alpha\pi y_2 - \hat{\theta}_0\pi y_2\beta - \int_{\mathbb{R}_0} [\hat{\theta}_1\pi y_2\gamma]\nu(dz) = 0$$

which gives that the optimal choice for the market is to choose θ such that Q_θ is an equivalent martingale measure (or *risk-free* measure) for $X(t)$. Since α, β and γ are constants, we note that $\hat{\theta}$ is deterministic. To find the optimal choice for the agent, we note that $A^{\theta,\pi}\varphi(y)$ is linear in π, and can be written on the form

$$\begin{aligned}
A^{\theta,\pi}\varphi(y) &= \pi\left(y_2\alpha h(s)y_1 - \theta_0y_1y_2\beta h(s) - \int_{\mathbb{R}_0}[h(s)y_1y_2\theta_1\gamma]\nu(dz)\right) \\
&+ h'(s)y_1y_2 + \theta_0^2y_1^2 + \int_{\mathbb{R}_0}[y_1^2\theta_1^2]\nu(dz)
\end{aligned}$$

which gives that the optimal choice for the agent is

$$\hat{\pi} = \begin{cases} \infty, & \text{if positive coefficient} \\ -\infty, & \text{if negative coefficient} \\ \text{remiss}, & \text{if the coefficient is 0} \end{cases}$$

For instance, we can let $\hat{\pi}(\hat{\theta}) = 0$. Then we have our candidate for the optimal control $(\hat{\theta}, \hat{\pi})$, and the solution of

$$\begin{cases} A^{\hat{\theta},\hat{\pi}}\varphi(y) = 0 \\ \varphi(T, y_1, y_2) = y_1y_2 + y_2^2 \end{cases}$$

will be our candidate for $\Phi(y)$ in Problem 5.2. It remains to determine $h(s)$. By (iii) in Proposition 5.1, and that the market will choose an equivalent martingale measure, we have

$$A^{\hat{\theta},\hat{\pi}}\varphi(y) = h'(s)y_1y_2 + \hat{\theta}_0^2 y_1^2 + \int_{\mathbb{R}_0} \hat{\theta}_1^2 y_1^2 \nu(dz) = 0.$$

This equation has a unknown solution, if any, since $h(s)$ is assumed to be a deterministic function of s alone.

6 Pricing of contingent claims

6.1 Pricing in complete and incomplete markets

In a complete market we have one and only one equivalent martingale measure. If we find ourselves in the complete market setting we can also always find an $x \in \mathbb{R}$ and a self-financing portifolio π such that for any contingent claim F with maturity $T > 0$

$$V_x^{(\pi)}(T) = F \qquad P\text{-a.s.}$$

and the 'fair' price is given by

$$p(F) = e^{-\int_0^T r(t)dt} E_Q [F]$$

for some risk free rate $r : [0, T] \to \mathbb{R}^+$ (see e.g. [6]).

In general, markets driven by jump processes such as in this thesis are not complete. This leads to problems when trying to give a 'fair' price to a contingent claim for two reasons. Firstly there is infinitely many equivalent martingale measures, and secondly there may not exist a replicating portifolio. When using the replicating portifolio argument the seller will not settle for less than

$$p_{up} = \inf \left\{ x \mid \exists \text{ self-financing } \pi \text{ such that } V_x^{(\pi)}(T) \geq F \right\}$$

and the buyer will not pay more than

$$p_{low} = \sup \left\{ x \mid \exists \text{ self-financing } \pi \text{ such that } V_x^{(\pi)}(T) \leq F \right\}.$$

In a complete market we will have $p_{up} = p_{low}$, but in our setting there may be a big gap between the acceptable prices for the two partisipants. Other prices in incomplete markets are arising from utility indifference arguments and risk indifference arguments, where the latter will be dicussed here.

Our market, equipped with a fixed terminal time $T > 0$, will consist of two investment possibilities.

(i) A risk free investment, with discounted unit price $S_0(t) = 1$, $t \in [0, T]$.

(ii) A risky investment, with discounted unit price $S(t)$, given by

$$\begin{cases} dS(t) &= S(t^-) \left[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz) \right]; \quad t \in [0,T] \\ S(0) &= s > 0 \end{cases}$$

A portfolio will in this chapter represent the number of units held in asset (ii). Then the discounted wealth process $V(t) = V^{(\pi)}(t)$ becomes

$$\begin{cases} dV(t) &= \pi(t)dS(t) = \pi(t^-)S(t^-)\big[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,z)\tilde{N}(dt,dz)\big] \\ V(0) &= x > 0 \end{cases}$$

6.2 Risk indifference pricing

A seller can either agree to sell a contingent claim $F \in \mathcal{X}$, or not. If the seller has the amount x and agree to sell the contingent claim F for some price p, the minimal risk involved is

$$\Phi_F(x+p) = \inf_{\pi \in \Pi} \rho\big(V^{(\pi)}_{x+p}(T) - F\big) \tag{6.1}$$

whereas the minimal risk if no sale is made is

$$\Phi_0(x) = \inf_{\pi \in \Pi} \rho\big(V^{(\pi)}_{x}(T)\big) \tag{6.2}$$

where Π is some subset of all admissible portifolios. We recall that a self-financing portfolio π is called *admissible* if

$$\int_0^T \Big\{|\alpha(t)||\pi(t)|S(t) + \beta^2(t)\pi^2(t)S^2(t) + \pi^2(t)S^2(t)\int_{\mathbb{R}_0} \gamma^2(t,z)\nu(dz)\Big\}dt < \infty$$

and

$$V^{(\pi)}(t) \geq 0 \quad \text{for } t \in [0,T] \text{ a.s.}$$

***Definition 6.1*,[18]** *The seller's risk indifference price* p^{seller} *of the claim* $F \in \mathcal{X}$ *is the solution* p *of the equation*

$$\Phi_F(x+p) = \Phi_0(x). \tag{6.3}$$

So one way to solve this problem would be to solve the stochastic games (6.1) and (6.2). If we let

$$dY(t) = \begin{bmatrix} dY_0(t) \\ dY_1(t) \\ dY_2(t) \end{bmatrix} = \begin{bmatrix} dt \\ dK_\theta(t) \\ dS(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ S(t^-)\alpha(t) \end{bmatrix} dt$$
$$+ \begin{bmatrix} 0 \\ K_\theta(t^-)\theta_0(t) \\ S(t^-)\beta(t) \end{bmatrix} dB(t) + \int_{\mathbb{R}_0} \begin{bmatrix} 0 \\ K_\theta(t^-)\theta_1(t,z) \\ S(t^-)\gamma(t,z) \end{bmatrix} \tilde{N}(dt,dz)$$

it is shown in [18] that we can view the the sellers risk indifference price p^{seller} of a contingent claim $F = g(S(T))$, as

$$p^{seller}(F) = \sup_{Q_\theta \in \mathcal{L}} \{E_{Q_\theta}[F] - \zeta(Q_\theta)\} - \sup_{Q_\theta \in \mathcal{L}} \{ - \zeta(Q_\theta)\} \tag{6.4}$$

and the buyers risk indifference price p^{buyer} becomes

$$p^{buyer}(F) = \inf_{Q_\theta \in \mathcal{L}} \{E_{Q_\theta}[F] + \zeta(Q_\theta)\} - \inf_{Q_\theta \in \mathcal{L}} \{\zeta(Q_\theta)\}. \tag{6.5}$$

Here $\mathcal{L}$ denotes the set of equivalent martingale measures for $Y(t)$.

6.3 Prelude to example

As Proposition 5.1 assumes $\varphi \in C^2(\mathcal{S})$, we see that for $A = \{K - y_2 \geq 0\}$

$$\varphi(s, y_1, y_2, y_3) = h(s)y_1(y_3 - (K - y_2)\chi_A) + y_1 \log y_1$$

will not meet this assumption. However, if we let

$$f_{(n)}(y_2) = C_0^n + C_1^n y_2 + C_2^n y_2^2 + C_3^n y_2^3 + C_4^n y_2^4 + C_5^n y_2^5$$

and determine (for each n) the constants $C_0^n, \cdots, C_5^n$ by the equations

$$\begin{aligned} f_{(n)}(K + \frac{1}{n}) &= 0, & f_{(n)}(K - \frac{1}{n}) &= \frac{1}{n} \\ f'_{(n)}(K + \frac{1}{n}) &= 0, & f'_{(n)}(K - \frac{1}{n}) &= -1 \\ f''_{(n)}(K + \frac{1}{n}) &= 0, & f''_{(n)}(K - \frac{1}{n}) &= 0 \end{aligned}$$

we can define, for $y = (s, y_1, y_2, y_3)$

$$\varphi_{(n)}(y) = \begin{cases} \varphi(y), & \text{if } y_2 \notin [K - \frac{1}{n}, K + \frac{1}{n}] \\ h(s)y_1(y_3 - f_{(n)}(y_2)) + y_1 \log y_1, & \text{if } y_2 \in [K - \frac{1}{n}, K + \frac{1}{n}] \end{cases}$$

to approximate $\varphi(y)$. Since $\varphi_{(n)}(y) \in C^2(\mathcal{S})$, our problem can be contained to an arbitrary small interval around K. Also, the singularity is not severe, so this problem is omitted in the following example.

6.4 Example: Sellers risk indifference price

In this example we will solve (6.3) in the case where $F = (K - S(t))\chi_{\{K-S(t)\geq 0\}}$, and

$$\zeta(Q) = E\left[\frac{dQ}{dP}\log\frac{dQ}{dP}\right]$$

called the *relative entropy*, as in Chapter 4. Further we assume that α, β and γ are constants, $\Pi = \mathbb{R}$, and that

$$\Theta = \{\theta : E[Z_\theta(T)] = 1\}$$

where

$$dZ_\theta(t) = -Z_\theta(t^-)\Big[\theta_0(t) + \int_{\mathbb{R}_0}\theta_1(t,z)\tilde{N}(dt,dz)\Big].$$

First we try to solve the stochastic differential game (6.1). From Chapter 5, we can study the problem

$$\Psi_F(x+p) = \sup_{\pi\in\Pi}\inf_{\theta\in\Theta} E\Big[Z_\theta(T)\Big(V^{(\pi)}(T) - (K - S(T))\chi_A\Big) + Z_\theta(T)\log Z_\theta(T)\Big]$$

where $A = \{K - S(T) \geq 0\}$. The process needed to solve this problem is given by

$$dY(t) = \begin{bmatrix} dY_0(t)\\ dY_1(t)\\ dY_2(t)\\ dY_3(t)\end{bmatrix} = \begin{bmatrix} dt\\ dZ_\theta(t)\\ dS(t)\\ dV(t)\end{bmatrix}$$
$$= \begin{bmatrix} 1\\ 0\\ \alpha S(t^-)\\ \alpha\pi S(t^-)\end{bmatrix} dt + \begin{bmatrix} 0\\ -\theta_0(t)Z_\theta(t^-)\\ \beta S(t^-)\\ \beta\pi S(t^-)\end{bmatrix} dB(t) + \int_{\mathbb{R}_0}\begin{bmatrix} 0\\ -\theta_1(t,z)Z_\theta(t^-)\\ \gamma S(t^-)\\ \gamma\pi S(t^-)\end{bmatrix}\tilde{N}(dt,dz)$$

with $Y(0) = (s, y_1, y_2, y_3)$.

If we guess

$$\varphi(y) = h(s)y_1(y_3 - (K - y_2)\chi_A) + y_1\log y_1,$$

we get that the partial derivatives of φ is

$$\begin{aligned}
&\varphi_s(y) = h'(s)y_1(y_3 - (K - y_2)\chi_A); && \varphi_3(y) = h(s)y_1\\
&\varphi_1(y) = h(s)(y_3 - (K - y_2)\chi_A) + \log y_1 + 1; && \varphi_{33}(y) = 0\\
&\varphi_{11}(y) = \tfrac{1}{y_1}; && \varphi_{12}(y) = h(s)\chi_A\\
&\varphi_2(y) = h(s)y_1\chi_A; && \varphi_{13}(y) = h(s)\\
&\varphi_{22}(y) = 0; && \varphi_{23}(y) = 0
\end{aligned}$$

and the generator becomes

$$
\begin{aligned}
A^{\theta,\pi}\varphi(y) &= h'(s)y_1(y_3-(K-y_2)\chi_A)+y_2\alpha h(s)y_1\chi_A+\pi y_2\alpha h(s)y_1\\
&+\frac{1}{2}\theta_0^2 y_1-\theta_0 y_1 y_2\beta h(s)\chi_A-\theta_0 y_1\pi y_2\beta h(s)\\
&+\int_{\mathbb{R}_0}\Big\{h(s)(y_1-y_1\theta_1)(y_3+\pi y_2\gamma-(K-y_2-y_2\gamma)\chi_A)\\
&+(y_1-y_1\theta_1)\log(y_1-y_1\theta_1)-h(s)y_1(y_3-(K-y_2)\chi_A)\\
&-y_1\log y_1+y_1\theta_1[h(s)(y_3-(K-y_2)\chi_A)+\log y_1+1]\\
&-y_2\gamma h(s)y_1\chi_A-\pi y_2\gamma h(s)y_1\Big\}\nu(dz)\\
&=h'(s)y_1(y_3-(K-y_2)\chi_A)+y_2\alpha h(s)y_1\chi_A+\pi y_2\alpha h(s)y_1\\
&+\frac{1}{2}\theta_0^2 y_1-\theta_0 y_1 y_2\beta h(s)\chi_A-\theta_0 y_1\pi y_2\beta h(s)\\
&+\int_{\mathbb{R}_0}\Big\{\theta_1 y_1-\theta_1 y_1 h(s)y_2\gamma\chi_A+y_1\log(1-\theta_1)\\
&-\theta_1 y_1\log(1-\theta_1)-y_1\theta_1\pi y_2\gamma h(s)\Big\}\nu(dz).
\end{aligned}
$$

To maximize $A^{\hat{\theta},\pi}\varphi(y)$ for all π we obtain the first order condition

$$
\begin{aligned}
\alpha y_1 y_2 h(s)-\hat{\theta}_0\beta y_1 y_2 h(s)-&\int_{\mathbb{R}_0}[\hat{\theta}_1\gamma y_1 y_2 h(s)]\nu(dz)=0,\ \text{i.e.}\\
\alpha y_2-\hat{\theta}_0\beta y_2-&\int_{\mathbb{R}_0}[\hat{\theta}_1\gamma y_2]\nu(dz)=0,\ \text{and}\\
\alpha y_2\pi-\hat{\theta}_0\beta y_2\pi-&\int_{\mathbb{R}_0}[\hat{\theta}_1\gamma y_2\pi]\nu(dz)=0
\end{aligned}
$$

We see that the market will choose a 'risk-free' measure, as in the last example in Chapter 5. Also $A^{\theta,\pi}\varphi(y)$ is linear in π, so the optimal choice for the agent is again dependent on the coefficient for π, i.e.

$$
\hat{\pi}=\begin{cases}\infty, & \text{if positive coefficient}\\ -\infty, & \text{if negative coefficient}\\ \text{remiss}, & \text{if the coefficient is 0}\end{cases}
$$

To determine $h(s)$ we use (iii) from Proposition 5.1, and write

$$\begin{aligned}0 = A^{\hat{\theta},\hat{\pi}}\varphi(y) &= h'(s)y_1(y_3 - (K - y_2)\chi_A) + \frac{1}{2}\theta_0^2 y_1 \\ &+ \int_{\mathbb{R}_0} \Big\{\theta_1 y_1 + y_1 \log(1-\theta_1) - \theta_1 y_1 \log(1-\theta_1)\Big\}\nu(dz) \\ &= h'(s)(y_3 - (K - y_2)\chi_A) + \frac{1}{2}\theta_0^2 \\ &+ \int_{\mathbb{R}_0} \Big\{\theta_1 + \log(1-\theta_1) - \theta_1 \log(1-\theta_1)\Big\}\nu(dz).\end{aligned}$$

So

$$h'(s) = \frac{-1}{y_3 - (K - y_2)\chi_A}\Big[\frac{1}{2}\theta_0^2 + \int_{\mathbb{R}_0} \{\theta_1 + \log(1-\theta_1) - \theta_1 \log(1-\theta_1)\}\nu(dz)\Big].$$

Again, this equation has no known solution. The case where $F = 0$ is somewhat similar to the above and the last example of Chapter 5. The stochastic differential game (6.2) can be formulated as

$$\Psi(x) = \sup_{\pi\in\Pi} \inf_{\theta\in\Theta} E\Big[Z_\theta(T)V^{(\pi)}(T) + Z_\theta(T)\log Z_\theta(T)\Big].$$

The process used will be

$$\begin{aligned}dY(t) = \begin{bmatrix} dY_0(t) \\ dY_1(t) \\ dY_2(t)\end{bmatrix} &= \begin{bmatrix} dt \\ dZ_\theta(t) \\ dV(t)\end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \\ \alpha\pi S(t^-)\end{bmatrix} dt + \begin{bmatrix} 0 \\ -\theta_0(t)Z_\theta(t^-) \\ \beta\pi S(t^-)\end{bmatrix} dB(t) + \int_{\mathbb{R}_0} \begin{bmatrix} 0 \\ -\theta_1(t,z)Z_\theta(t^-) \\ \gamma\pi S(t^-)\end{bmatrix} \tilde{N}(dt,dz)\end{aligned}$$

Here, we guess $\varphi(y) = h(s)y_1y_2 + y_1 \log y_1$. Then the corresponding generator gets the form

$$\begin{aligned}A^{\theta,\pi}\varphi(y) &= h'(s)y_1y_2 + \pi y_1 y_2 \alpha h(s) + \frac{1}{2}\theta_0^2 y_1 - \theta_0 y_1 y_2 \pi\beta h(s) \\ &+ \int_{\mathbb{R}_0} \{\theta_1 y_1 - h(s)y_1y_2\theta_1\pi\gamma + y_1\log(1-\theta_1) - y_1\theta_1\log(1-\theta_1)\}\nu(dz)\end{aligned}$$

As before, the maximization of $A^{\hat{\theta},\pi}$ for all π, gives the first order condition

$$\alpha y_2\pi - \hat{\theta}_0\beta y_2\pi - \int_{\mathbb{R}_0} [\hat{\theta}_1\gamma y_2\pi]\nu(dz) = 0,$$

and since $A^{\theta,\pi}$ is linear in π we get

$$\hat{\pi} = \begin{cases} \infty, & \text{if positive coefficient} \\ -\infty, & \text{if negative coefficient} \\ \text{remiss}, & \text{if the coefficient is 0} \end{cases}$$

To find $h(s)$ we use as before

$$0 = A^{\hat{\theta},\hat{\pi}}\varphi(y) = h'(s)y_1y_2 + \frac{1}{2}\theta_0^2 y_1 + \int_{\mathbb{R}_0} \Big\{\theta_1 y_1 + y_1 \log(1-\theta_1) - \theta_1 y_1 \log(1-\theta_1)\Big\}\nu(dz)$$

So

$$h'(s) = \frac{-1}{y_2}\Big[\frac{1}{2}\theta_0^2 + \int_{\mathbb{R}_0} \{\theta_1 + \log(1-\theta_1) - \theta_1 \log(1-\theta_1)\}\nu(dz)\Big].$$

Similar problem as before comes to light here, as $h(s)$ is a deterministic function of s.

7 Convex risk measures and g-expectation

In this section we will look at another representation of convex risk measures. This will be in terms of g-expectaion and backward stochastic differential equations (BSDEs). Let $F \in \mathcal{X}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be a convex function. Also, let $X(t)$, $Y(t)$ and $K(t,z)$ be square-integrable, $\mathcal{F}_t$-predictable processes, such that the BSDE

$$\begin{cases} dX(t) &= -g(X(t), Y(t), K(t,\cdot))dt + dM(t) \\ X(T) &= F \end{cases} \tag{7.1}$$

where

$$dM(t) = Y(t)dB(t) + \int_{\mathbb{R}_0} K(t,z)\tilde{N}(dt,dz)$$

has a unique solution. By [1], $M(t)$ is a martingale, and hence

$$\begin{aligned} X(t) + \int_0^t g(X(s), Y(s), K(s,\cdot))ds &= E\Big[X(0) + \int_0^T Y(s)dB(s) \\ &\quad + \int_0^T \int_{\mathbb{R}_0} K(s,z)\tilde{N}(ds,dz)\Big|\mathcal{F}_t\Big] \\ &= E\left[X(T) + \int_0^T g(X(s), Y(s), K(s,\cdot))ds\middle|\mathcal{F}_t\right]. \end{aligned}$$

Since $\int_0^t g(X(s), Y(s), K(s,\cdot))ds$ is $\mathcal{F}_t$-measurable we can rearrange the terms and get

$$X(t) = E\left[F + \int_t^T g(X(s), Y(s), K(s,\cdot))ds\middle|\mathcal{F}_t\right], \quad 0 \le t \le T. \tag{7.2}$$

From this last argument we deduce that $M(t)$ can be written on the form

$$M(t) = E\left[F + \int_0^T g(X(s), Y(s), K(s,\cdot))ds\middle|\mathcal{F}_t\right], \quad 0 \le t \le T.$$

In some arguments in this chapter we will use that

$$
\begin{aligned}
X(t) &= E\left[F + \int_t^T g(X(s), Y(s), K(s,\cdot))ds \Big| \mathcal{F}_t\right] \\
&= F + \int_t^T g(X(s), Y(s), K(s,\cdot))ds + M(t) \\
&- F - \int_0^T g(X(s), Y(s), K(s,\cdot))ds \\
&= F + \int_t^T g(X(s), Y(s), K(s,\cdot))ds + M(t) - M(T). \qquad (7.3)
\end{aligned}
$$

So (7.1), (7.2) and (7.3) are all representations of the same BSDE.

7.1 A static convex risk measure induced by g-expectation

***Definition 7.1*, [21]** *The risk $\rho_g(F)$ (associated to the convex function g) of a financial position $F \in \mathcal{X}$ is defined by*

$$\rho_g(F) := \mathcal{E}_g[-F] := X_g^{-F}(0) \in \mathbb{R},$$

where $X_g^{-F}(0)$ is the value at $t = 0$ of the solution $X(t)$ of the BSDE (7.1) with terminal value $-F$.

Inspired by Theorem 2.3 in [16], we now state the following fundamental proposition for this chapter.

Proposition 7.1 *Let*

$$dX_i(t) = -g_i(X_i(t), Y_i(t), K_i(t,\cdot))dt + Y_i(t)dB(t) + \int_{\mathbb{R}_0} K_i(t,z)\tilde{N}(dt,dz), \quad i = 1, 2.$$

and

$$\begin{cases} X_1(T) &= F \\ X_2(T) &= G \end{cases}$$

where $X_i(t)$, $Y_i(t)$ and $K_i(t,z)$ is as in (6.1). If $F \geq G$, $g_1 \geq g_2$ P-a.s., then

$$X_1(t) \geq X_2(t) \;\; P\text{-a.s. for all } t \in [0,T].$$

Proof. Since

$$
\begin{aligned}
X_1(t) - X_2(t) &= F - G + \int_t^T g_1(X_1(s), Y_1(s), K_1(s,\cdot)) - g_2(X_2(s), Y_2(s), K_2(s,\cdot))ds \\
&+ (M_1(t) - M_2(t)) - (M_1(T) - M_2(T)) \\
&= E\Big[F - G + \int_t^T g_1(X_1(s), Y_1(s), K_1(s,\cdot)) \\
&- g_2(X_2(s), Y_2(s), K_2(s,\cdot))ds \Big| \mathcal{F}_t\Big] \\
&\geq 0,
\end{aligned}
$$

the result follows. □

Note that $g_1 \leq g_2$ imply $\rho_{g_1}(F) \leq \rho_{g_2}(F)$ by Proposition 7.1. So a plausible interpertation of g is that it represents some measure of risk aversion. From the form (7.2), the solution $X(t)$ of (7.1) has the intuitive interpretation of being the conditional expectation of the payoff, added to a measure g of risk aversion. Another interpretation is that for $\rho(F) \leq 0$, $-\rho(F)$ represents the 'certainty equivalent', i.e. the price that makes an agent indifferent in regards to keeping or selling the risky asset F. In the remainder, g will be assumed to be independent of $X(t)$[6]. This assumption is in compliance with the interpretation of g being a measure of risk aversion, since it is only affected by the scaling of the Brownian Motion and the scaling of the Poisson Random Measure, and not by the state of $X(t)$.

Proposition 7.2 **(i)** *ρ_g is a convex risk measure as in Definition 2.2a.* **(ii)** *If $g(y,k)$ is sublinear in (y,k), then ρ_g is a coherent risk measure.*

Proof. **(i)** The monotonicity property is a direct consequense of Proposition 7.1. To show translation invariance we use that g is independent of $X(t)$, and define $X'(t) = X(t) + a$, for some $a \in \mathbb{R}$. Then we have

$$
\begin{aligned}
\rho_g(F + a) &= \mathcal{E}_g\left[-(F + a)\right] \\
&= E\left[-(F + a) + \int_0^T g(Y(s), K(s,\cdot))ds\right] \\
&= E\left[-F + \int_0^T g(Y(s), K(s,\cdot))ds\right] - a \\
&= \mathcal{E}_g\left[-F\right] - a \\
&= \rho_g(F) - a.
\end{aligned}
$$

[6]As assumed in e.g. [14].

To show convexity, let

$$dX_i(t) = -g(Y_i(t), K_i(t,\cdot))dt + Y_i(t)dB(t) + \int_{\mathbb{R}_0} K_i(t,z)\tilde{N}(dt,dz), \quad i = 1,2,3.$$

$$\begin{cases} X_1(T) &= -F \\ X_2(T) &= -G \\ X_3(T) &= -\left(\lambda F + (1-\lambda)G\right) \end{cases}$$

where $X_i(t)$, $Y_i(t)$ and $K_i(t,z)$ satisfy the same assumptions as $X(t)$, $Y(t)$ and $K(t,z)$ in (7.1) respectively. Then define

$$\begin{aligned} dW(t) &= d\big(\lambda X_1(t) + (1-\lambda)X_2(t)\big) \\ &= -\Big(\lambda g(Y_1(t), K_1(t,\cdot)) + (1-\lambda)g(Y_2(t), K_2(t,\cdot))\Big)dt \\ &+ \Big(\lambda Y_1(t) + (1-\lambda)Y_2(t)\Big)dB(t) \\ &+ \int_{\mathbb{R}} \Big(\lambda K_1(t,z) + (1-\lambda)K_2(t,z)\Big)\tilde{N}(dt,dz). \end{aligned}$$

Now let $Y_3(t) = \lambda Y_1(t) + (1-\lambda)Y_2(t)$, and $K_3(t,z) = \lambda K_1(t,z) + (1-\lambda)K_2(t,z)$. Since g is convex in (y,k) we have

$$g\big(Y_3(t), K_3(t,z)\big) \leq \lambda g\big(Y_1(t), K_1(t,z)\big) + (1-\lambda)g\big(Y_2(t), K_2(t,z)\big).$$

Then, again by Proposition 7.1, $X_3(t) \leq W(t)$ P-a.s. for all $t \in [0,T]$. For $t = 0$ we get

$$\rho_g(\lambda F + (1-\lambda)G) = X_3(0) \leq W(0) = \lambda\rho(F) + (1-\lambda)\rho_g(G) \qquad P\text{-a.s.}$$

(ii) Now let g be sublinear, and $X_1(t)$ and $X_2(t)$ be as in **(i)**. Also let

$$\begin{cases} dX_3(t) &= -g\big(Y_3(t), K_3(t,\cdot)\big)dt + Y_3(t)dB(t) + \int_{\mathbb{R}_0} K_3(t,z)\tilde{N}(dt,dz) \\ X_3(T) &= -(F+G) \end{cases}$$

with $Y_3(t) = Y_1(t) + Y_2(t)$ and $K_3(t,z) = K_1(t,z) + K_2(t,z)$. Define $W'(t) = X_1(t) + X_2(t)$. Then

$$dW'(t) = -\Big(g\big(Y_1(t), K_1(t,\cdot)\big) + g\big(Y_2(t), K_2(t,\cdot)\big)\Big)dt + \Big(Y_1(t) + Y_2(t)\Big)dB(t)$$
$$+ \int_{\mathbb{R}_0} \Big(K_1(t,z) + K_2(t,z)\Big)\tilde{N}(dt,dz).$$

Since g is sublinear we have that

$$g(Y_3(t), K_3(t,z)) \le g(Y_1(t), K_1(t,z)) + g(Y_2(t), K_2(t,z)).$$

By Proposition 7.1 we then have that $X_3(t) \le W'(t)$ P-a.s. for all $t \in [0,T]$. For $t = 0$ we get

$$\rho_g(F+G) = X_3(0) \le W'(0) = \rho_g(F) + \rho_g(G) \qquad P\text{-a.s.}$$

To show positive homogenity we let $\lambda \ge 0$, and then by the positive homogenity of g

$$\begin{aligned}\rho_g(\lambda F) &= E\left[-\lambda F + \int_0^T g(\lambda Y(s), \lambda K(s,\cdot))ds\right] \\ &= \lambda E\left[-F + \int_0^T g(Y(s), K(s,\cdot))ds\right] \\ &= \lambda \rho_g(F).\end{aligned}$$

Monotonicity is established in the same way as in **(i)**. Since g is sublinear it is also convex, and by **(i)** ρ_g is translation invariant. Hence ρ_g is a coherent risk measure. □

From Proposition 4.3, we know that since ρ_g is a convex risk measure, there exists a family $\mathcal{P}$ of measures absolutely continuous with respect to P, and a convex penalty function $\zeta : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ such that

$$\rho_g(F) = \sup_{Q \in \mathcal{P}} \big(E_Q[-F] - \zeta(Q)\big).$$

To find $\mathcal{P}$ and ζ in this case, we recall from the beginning of this chapter that

$$\rho_g(F) = E\Big[-F + \int_0^T g(Y(t), K(t,\cdot))ds\Big] = E[-F] - E\Big[\int_0^T -g(Y(t), K(t,\cdot))dt\Big].$$

So the convex risk measure induced by g-expectation is represented by $\mathcal{P} = \{P\}$, and

$$\zeta(Q) = E_Q\Big[\int_0^T -g(Y(t), K(t,\cdot))dt\Big],$$

where ζ trivially becomes a constant function on $\mathcal{P}$.

In the case without jumps, it is assumed that g-expectations fulfill the *strict monotonicity* property[7], which is explained in more detail below. We will generalize this result to include the case with jumps. The following result will be instrumental in this regard.

Proposition 7.3 *Let*

$$dX_i(t) = -g_i(Y_i(t), K_i(t,\cdot))dt + Y_i(t)dB(t) + \int_{\mathbb{R}_0} K_i(t,z)\tilde{N}(dt,dz), \quad i = 1,2.$$

and

$$\begin{cases} X_1(T) & = F \\ X_2(T) & = G \end{cases}$$

where $X_i(t)$, $Y_i(t)$ and $K_i(t,z)$ is as in (7.1). If $F \geq G$, $g_1 \geq g_2$ P-a.s. and $X_1(0) = X_2(0)$, then

$$X_1(t) = X_2(t) \;\; P\text{-a.s. for all } t \in [0,T].$$

Proof. Let

$$\hat{X}(t) = E\left[\hat{F} + \int_t^T \hat{g}(\hat{Y}(s), \hat{K}(s,\cdot)ds \Big| \mathcal{F}_t\right]$$

where $\hat{F} \geq 0$ and $\hat{g}(y,k) \geq 0$. If $\hat{X}(0) = 0$, then $\hat{F} + \int_0^T \hat{g}(\hat{Y}(s), \hat{K}(s,\cdot)ds = 0$ P-a.s. Since $\hat{F} \geq 0$ and $\hat{g} \geq 0$, we have by Corallary 4.10 in [5] that $\hat{F} = 0$ P-a.s. and $\hat{g} = 0$ $dt \times P$-a.s. This gives that $\hat{X}(t) = 0$, $t \in [0,T]$. Define

$$W(t) = X_1(t) - X_2(t)$$

where $F - G \geq 0$, and $g_1 - g_2 \geq 0$. Now let $W(0) = 0$, then

$$W(t) = X_1(t) - X_2(t) = 0 \text{ for } t \in [0,T].$$

□

[7] See e.g. [11] for an explanation in the continuous setting.

Proposition 7.4 *ρ_g has the strict monotonicity property, i.e.*

(I) *If $F \geq G$ P-a.s. then $\rho_g(F) \leq \rho_g(G)$*

(II) *If $F \geq G$ P-a.s. then $\rho_g(F) = \rho_g(G) \iff F = G$ P-a.s.*

Proof. (I) This is a special case of Proposition 7.1.
(II) Sufficiency: This is a special case of Proposition 7.3. Necessity: Since $F = G$ P-a.s., $F \leq G$ and $F \geq G$. By (I) $\rho_g(F) = \rho_g(G)$ □

The following uniqueness result is an example of results deriving from strict monotonicity.

***Proposition 7.5,* [11]**[8] *Let $F, G \in \mathcal{X}$.* If *$\rho_g(F\chi_A) = \rho_g(G\chi_A)$ for all $A \in \mathcal{F}_T$, then $F = G$.*

Proof. Let $A = \{F \leq G\} \in \mathcal{F}_T$. Then $F\chi_A \leq G\chi_A$. Since $\rho_g(F\chi_A) = \rho_g(G\chi_A)$, we have by strict monotonicity $F\chi_A = G\chi_A$. Similarly, let $B = \{G \leq F\} \in \mathcal{F}_T$. Then $G\chi_B \leq F\chi_B$. Again, since $\rho_g(F\chi_B) = \rho_g(G\chi_B)$, and by strict monotonicity, $F\chi_B = G\chi_B$. Hence $F = G$. □

In the case without jumps, there are made some assumptions on g (see e.g. [13]). A generalization to the setting with jumps will be

(A) $\exists a, b > 0$ such that $\forall\ y_1, y_2, k_1, k_2 \in \mathbb{R}$

$$|g(y_1, k_1) - g(y_2, k_2)| \leq a|y_1 - y_2| + b|k_1 - k_2|$$

(B) $g(y, k) \in L^2(\Omega, \mathcal{F}_T, P)\ \forall\ y, z \in \mathbb{R}$

(C) $g(0, \cdot) \equiv 0$

***Proposition 7.6,* [7]**[9] *Let g be a given function which fulfills (A), (B) and (C). Then for every $F \in \mathcal{X}$ and $\epsilon \in (0, 1]$, there exist a constant C_ϵ such that*

$$|\rho_g(F)| \leq C_\epsilon \|F\|_{1+\epsilon}$$

where

$$\|F\|_p = \left(\int_\Omega |F|^p dP(\omega) \right)^{\frac{1}{p}} \quad \text{for } p \geq 1.$$

[8] In [11], strict monotonicity is assumed in the continuous setting.
[9] In [7], a shorter proof is given in the continuous setting.

Proof. Let

$$\begin{cases} dX(t) &= -g(Y(t), K(t,\cdot))dt + Y(t)dB(t) + \int_{\mathbb{R}_0} K(t,z)\tilde{N}(dt,dz) \\ X(T) &= -F \end{cases}$$

and

$$\begin{cases} u(s) &= -\frac{g(Y(s),K(s,\cdot))}{Y(s)} \\ \theta(s,z) &= 0 \end{cases} \qquad \text{(where we use the convention } \tfrac{0}{0} = 0\text{)}$$

Then define the measure Q on $\mathcal{F}_T$ by

$$dQ(\omega) = Z(T)dP(\omega)$$

where

$$Z(t) = \exp\left\{-\int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u^2(s)ds\right\}.$$

Since

$$-\int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u^2(s)ds \sim N\Big(-\frac{1}{2}\int_0^t u^2(s)ds, \int_0^t u^2(s)ds\Big)$$

and

$$x \sim N(\mu,\sigma^2) \text{ implies } E[e^x] = e^{\mu+\frac{\sigma^2}{2}}$$

we see that

$$E[Z(T)] = 1.$$

By the Girsanov Theorem III in [15] we get that

$$dB_Q(t) = u(t)dt + dB(t)$$

is a Brownian Motion with respect to Q, and

$$\tilde{N}_Q(dt,dz) = \theta(t,z)\nu(dz)dt + \tilde{N}(dt,dz) = \tilde{N}(dt,dz)$$

is the Q-compensated Poisson random measure of $N(dt,dz)$. Consequently

$$\begin{aligned}
X(t) &= X(0) - \int_0^t g(Y(s), K(s,\cdot))dt + \int_0^t Y(s)dB(s) + \int_0^t \int_{\mathbb{R}_0} K(s,z)\tilde{N}(ds,dz) \\
&= X(0) + \int_0^t Y(s)dB_Q(s) + \int_0^t \int_{\mathbb{R}_0} K(s,z)\tilde{N}_Q(ds,dz) \\
&= E_Q\left[X(0) + \int_0^T Y(s)dB_Q(s) + \int_0^T \int_{\mathbb{R}_0} K(s,z)\tilde{N}_Q(ds,dz)|\mathcal{F}_t\right] \\
&= E_Q\left[-F|\mathcal{F}_t\right].
\end{aligned}$$

Since g is Lipschitz in (y,k) we have that $Z(T) \in L^p(\Omega, \mathcal{F}_T, P)$ for all $p \in [1,\infty)$. So, by Hölder

$$\begin{aligned}
|\rho_g(F)| &= |E_Q[-F]| \\
&= |E[Z(T)(-F)]| \\
&\le E[|Z(T)(-F)|] \\
&\le \|Z(T)\|_{\frac{1+\epsilon}{\epsilon}} \|-F\|_{1+\epsilon} \\
&= \|Z(T)\|_{\frac{1+\epsilon}{\epsilon}} \|F\|_{1+\epsilon}
\end{aligned}$$

□

7.2 A dynamic convex risk measure induced by conditional g-expectation

In Definition 7.1 ρ_g is an example of a static risk measure, i.e. not dependent on t. A natural question in this setting would be how to view the risk of a $\mathcal{F}_T$-measurable random variable at a general time $t \in [0,T]$. This introduces the notion of dynamic risk measures, defined in the continuous case in [14] as the following. For $t \in [0,T]$, ρ_g^t is a dynamic risk measure if

- $\rho_g^t : \mathcal{X} \to L^2(\Omega, \mathcal{F}_t, P), \quad \forall t \in [0,T]$.
- ρ_g^0 is a static risk measure.
- $\rho_g^T(F) = -F \quad P$-a.s. $\forall\ F \in \mathcal{X}$.

We see that this definition is in compliance with our definition of a risk measure, induced by g-expectation in the jump diffusion setting.

Definition 7.2 *The dynamic risk $\rho_g^t(F)$ (associated to the convex function g) of a financial position $F \in \mathcal{X}$ is defined by*

$$\rho_g^t(F) := \mathcal{E}_g[-F \mid \mathcal{F}_t] := X_g^{-F}(t) \in L^2(\Omega, \mathcal{F}_t, P) \tag{7.4}$$

where $X_g^{-F}(t)$ is the value at time $t \in [0,T]$ of the solution $X(t)$ of the BSDE (7.1) with terminal value $-F$.

Proposition 7.7 **(i)** *ρ_g^t is a convex risk measure as in Definition 2.2a.* **(ii)** *If $g(y,k)$ is sublinear in (y,k), then ρ_g^t is a coherent risk measure.*

The proof of Proposition 7.7 is similar to, and to a large degree contained in, the proof of Proposition 7.2.

7.3 Further generalizations to the Itô-Lévy setting

In the continuous setting, conditional g-exptectations are known to have some of the same properties as conditional expectations. This chapter will generalize some of these properties to the setting with jumps. In [12], conditional g-expectation of F is defined as the unique (P-a.s.) random variable $\eta \in L^2(\Omega, \mathcal{F}_t, P)$ that fulfills

$$\mathcal{E}_g[\chi_A F] = \mathcal{E}_g[\chi_A \eta] \quad \text{for all } A \in \mathcal{F}_t \tag{7.5}$$

and η is denoted by $\mathcal{E}_g[F|\mathcal{F}_t]$. The uniqueness is shown by letting $\eta_1, \eta_2 \in L^2(\Omega, \mathcal{F}_t, P)$ fulfill (7.5). Then

$$\mathcal{E}_g[\chi_A \eta_1] = \mathcal{E}_g[\chi_A F] = \mathcal{E}_g[\chi_A \eta_2] \quad \text{for all } A \in \mathcal{F}_t.$$

In particular

$$\mathcal{E}_g[\chi_{\{\eta_1 \le \eta_2\}} \eta_1] = \mathcal{E}_g[\chi_{\{\eta_1 \le \eta_2\}} \eta_2].$$

Since

$$\chi_{\{\eta_1 \le \eta_2\}} \eta_1 \le \chi_{\{\eta_1 \le \eta_2\}} \eta_2,$$

we have by strict monotonicity that $\chi_{\{\eta_1 \le \eta_2\}} \eta_1 = \chi_{\{\eta_1 \le \eta_2\}} \eta_2$. Similarly we obtain $\chi_{\{\eta_1 \ge \eta_2\}} \eta_1 = \chi_{\{\eta_1 \ge \eta_2\}} \eta_2$, hence

$$\eta_1 = \eta_2 \quad \text{a.s.}$$

Existence comes from the fact that $X_g^F(t) \in L^2(\Omega, \mathcal{F}_t, P)$, i.e. $\mathcal{F}_t$-measurable, and that

$$\mathcal{E}_g[\chi_A X_g^F(t)] = \mathcal{E}_g[\chi_A F].$$

So far we see that the same holds true in the non-continuous case. It is further shown that

(i) If F is $\mathcal{F}_t$-measurable, then $\mathcal{E}_g[F|\mathcal{F}_t] = F$.

(ii) For each t and r, $\mathcal{E}_g[\mathcal{E}_g[F|\mathcal{F}_t]|\mathcal{F}_r] = \mathcal{E}_g[F|\mathcal{F}_{t\vee r}]$.

(iii) If $F \leq G$, then $\mathcal{E}_g[F|\mathcal{F}_t] \leq \mathcal{E}_g[G|\mathcal{F}_t]$.

(iv) For each $A \in \mathcal{F}_t$, $\mathcal{E}_g[\chi_A F|\mathcal{F}_t] = \chi_A \mathcal{E}_g[F|\mathcal{F}_t]$.

Statement (i) follows immediately from (7.5).

Statement (ii) is a consequence of (i) when $r > t$. When $r \leq t$, let $A \in \mathcal{F}_r \subseteq \mathcal{F}_t$, hence

$$\mathcal{E}_g[\chi_A \mathcal{E}_g[\mathcal{E}_g[F|\mathcal{F}_t]|\mathcal{F}_r]] = \mathcal{E}_g[\chi_A F] = \mathcal{E}_g[\chi_A \mathcal{E}_g[F|\mathcal{F}_r]].$$

Consequently, by uniqueness

$$\mathcal{E}_g[\mathcal{E}_g[F|\mathcal{F}_t]|\mathcal{F}_r] = \mathcal{E}_g[F|\mathcal{F}_r].$$

Statement (iii) is shown by letting $\eta_1 = \mathcal{E}_g[F|\mathcal{F}_t]$, $\eta_2 = \mathcal{E}_g[G|\mathcal{F}_t]$. Since, by Proposition 7.1, for each $A \in \mathcal{F}_t$, $\mathcal{E}_g[\chi_A F] \leq \mathcal{E}_g[\chi_A G]$ it follows from (7.5) that $\mathcal{E}_g[\chi_A \eta_1] \leq \mathcal{E}_g[\chi_A \eta_2]$. In particular

$$\mathcal{E}_g[\chi_{\{\eta_1 \geq \eta_2\}} \eta_1] \leq \mathcal{E}_g[\chi_{\{\eta_1 \geq \eta_2\}} \eta_2],$$

yet clearly

$$\chi_{\{\eta_1 \geq \eta_2\}} \eta_1 \geq \chi_{\{\eta_1 \geq \eta_2\}} \eta_2.$$

By the strict monotonicity of $\mathcal{E}_g[\cdot]$ we deduce that

$$\mathcal{E}_g[\chi_{\{\eta_1 \geq \eta_2\}} \eta_1] = \mathcal{E}_g[\chi_{\{\eta_1 \geq \eta_2\}} \eta_2],$$

and

$$\chi_{\{\eta_1 \geq \eta_2\}}\eta_1 = \chi_{\{\eta_1 \geq \eta_2\}}\eta_2,$$

so $\eta_1 \leq \eta_2$.

Statement (iv) will follow from letting $A, B \in \mathcal{F}_t$. Then

$$\mathcal{E}_g[\chi_A \mathcal{E}_g[\chi_B F|\mathcal{F}_t]] = \mathcal{E}_g[\chi_A \chi_B F] = \mathcal{E}_g[\chi_A\{\chi_B \mathcal{E}_g[F|\mathcal{F}_t]\}],$$

so by uniqueness

$$\mathcal{E}_g[\chi_B F|\mathcal{F}_t] = \chi_B \mathcal{E}_g[F|\mathcal{F}_t].$$

So thanks to Proposition 7.1 and the strict monotonicity property (Proposition 7.4) in the non-continuous case, statements (i)-(iv) holds in our Itô-Lévy setting.

8 Closing comments

The primary target of this thesis has been to give a synopsis of risk measures. This has been done by introducing the axiomatic properties of risk measures, and describing the different representations of (in particular) convex risk measures. This has been done through chapters 2, 4 and 7. In these chapters, the most interesting result is in my opinion Proposition 7.2 and the comments given in context to this. To my knowledge, an explicit proof of Proposition 7.2 has not been given previously.

The secondary target of this thesis has been to describe and find solutions of stochastic games regarding risk minimization. After having described these problems in chapters 5 and 6, we found that solutions are not easy to find, given that we do not reproduce examples of others. A solution was only found in a case with constant coefficients and no Brownian Motion. However, the reason why these solutions are hard to find has been thoroughly shown by examples.

The tertiary target of this thesis has been to generalize statements given in the continuous setting, to the non-continuous setting. In this regard, Chapter 7 is of most interest.

When we discuss risk indifference pricing, it is my opinion that the case where our financial position is some common option, e.g. a put or call, is the most interesting since it is a major improvement from the Black & Scholes way of pricing. It is to my knowledge no solution to be found in this case, even when some conditions for Proposition 5.1 are relaxed. If such a solution were to present itself, I would be very happy to learn about it, since the search of such a solution has taken a lot of my time during the work with this thesis.

Throughout this thesis I have tried to mention the intuitional thinking behind the theory presented. The intention behind this was to make it easier to follow the red thread between the capters. My priority in this regard has been comments and examples. On this thought, I finish my thesis with a simulation of a simple Itô-Lévy process.

To build on this thesis, one might continue to generalize results given for g-extectations to the non-continuous setting. Another way to build further is to continue the search for a solution to the problems in chapters 5 and 6. The maximum principle has not been applied here, so a deployment of this would be my first priority. Nevertheless, this thesis has been a contribution to the actualization of convex risk measures, and a further mapping of how the Itô-Lévy setting translates to risk minimization problems.

9 Appendix

As stated in the first chapter, we will look at a graphic example of how the processes in this thesis behave. This is given as an intuitional supplementary to the first chapter. In short terms, we will look at the case where the drift and volatility are constants, and the jumpsizes are normally distributed.

If we use the same notation as in the MatLab code below, the process without jumps, called $X(t)$, is on the form

$$X(t) = at + sB(t).$$

Here, a (drift) and s (volatility) are constants, in this case 0.03 and 1 respectively, and $B(t)$ is a Brownian Motion. Next we define a pure jump process $Z(t)$, given by

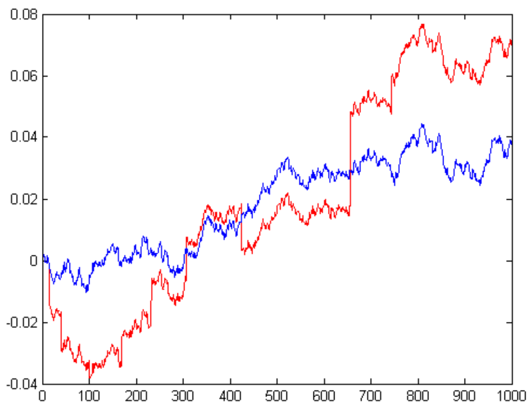
$$Z(t) = \sum_{i=1}^{N(t)} J_i$$

where $N(t) \sim Poisson(\lambda t)$ for some $\lambda \geq 0$, and $J_i \sim N(0, \sigma)$ for $i = 1, ..., N(t)$. Lastly, we make a process with jumps, called $Y(t)$, which is given the form

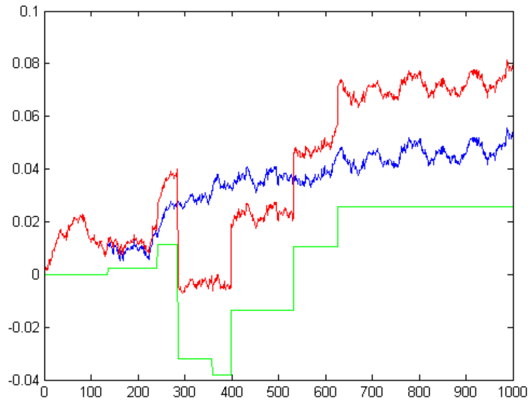
$$Y(t) = X(t) + Z(t).$$

9.1 Simulation

In the first graph we have simulated a process without jumps and made a plot of this process, then added jumps and included the 'new' process with jumps to the plot. In this plot the blue and red line represents $X(t)$ and $Y(t)$, respectively.



In the next graph we include the pure jump process $Z(t)$, represented by the green line.



The code used to obtain this graph is partially a result of the theory expained in [2], and mainly a result of general knowledge of such processes.

9.2 MatLab code

```
%------------ Our setting ------------

% $$$$$$  Ito and Ito-Levy process, starting values.
T = 1; % Maturization.
n = 1000; % Number of timesteps.
t = linspace(0,T,n); % Discretization of [0,T] in n pieces.
E = 7; % Expected number of jumps between 0 and T.
si = 0.02; % Standard deviation of the jump sizes.

% $$$$$$  Coefficients.
a = 0.03; % Drift.
s = 1; % Volatility.


%------------ Continuous process ----------------

% $$$$$$  Draw n normally distributed variables, representing BM.
F = normrnd(0,s,1,n)*(T/n);

% $$$$$$  Cummulative sum of F.
FC = cumsum(F);
X = a*t + FC; % The process without jumps.


%------------ Non-continuous process ------------

% $$$$$$  Draw a Piosson random variable representing number of jumps.
N = poissrnd(E,1);

% $$$$$$  Draw N waiting times.
U = unifrnd(0,1,1,N);

% $$$$$$  Sort the uniform variables, representing waiting times.
US = sort(U);

% $$$$$$  Draw N jumpsizes.
J = normrnd(0,si,1,N);

% $$$$$$  Implementing jumps.
c = 0*linspace(1,n,n); % Jump vector starting at (0,...,0).
j = 1; % Count.
```

```
for i = 1:n
    while N > j
          if US(j) < (i/n)
             c(i) = c(i) + J(j);
          end
          j = j + 1;
    end
    j = 1;
end

Y = X + c; % The process with jumps.
Z = c; % Only the jumps.

%----------- Plot -----------

plot(X,'b')
hold on
plot(Y,'r')
plot(Z,'g')
hold off
```

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