Dynamic and Stochastic Survival Models

A Comparison of Europe

by

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Throughout this thesis we have used data sets drawn from published death rates and life tables for national populations by Human mortality database (HMD). The database is maintained by the University of California, Berkeley, USA and Max Planck Institute for Demographic Research, Germany. My sincere thanks to all involved in developing this database.

Finally, I would like to thank my parents, the entire family and all my good friends for their love, endless support and for providing patient encouragement while working on this thesis and throughout my studies.

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1. Introduction

This chapter provides an introduction to the work by briefly introducing the problem and motivation of the thesis and the objectives. The document structure is described in the layout section.

1.1 Problem

In many countries, Europe in particular, statistical evidence shows a decline in human mortality over the 20th century. There is no sign of mortality rates leveling out or improvement rates even slowing down in the near future. The continuing decline in morality have far reaching consequences for pension funds and for the future financing of public health care and the state pension system. Mortality improvements have caused life offices to incur losses on the life annuity business. The problem lies in the fact that pensioners are living much longer than anticipated. As a result life offices are paying out much longer than what was forecasted, and their profit margins are being eroded in the process. The insurance industry is therefore bearing the costs of unexpected higher longevity. Looking forward, possible changes in lifestyle and medical advances are likely to make future improvements to life expectancy very unpredictable as well.

1.2 A possible solution

The growing mortality concern mentioned above has led actuaries to think differently. Actuaries have traditionally been using static and deterministic mortality intensity, which is a function of the age only. Here, we are modelling the mortality intensity as a dynamic and stochastic process. The advantage of introducing stochastic mortality intensity is twofold. Firstly, it gives more realistic life tables, and secondly, it quantifies the risk of the insurance companies and pension funds associated with the underlying mortality intensity.

1.3 Objective

Several of today's dynamic survival models of stochastic type are based on the Perks model. Master thesis will study estimation within a more flexible Gompertz-Makeham framework and compare with results under the other model, namely, Perks. The objective of this thesis is to decide as to which model is most suitable for Europe. Another objective is to study if there is any correlation in the development between different European countries and among genders.

1.4 Layout

The layout of this thesis is as follows. We start with chapter 2 (Data) where we present the data. Chapter 3 (Dynamic and Survival Modelling) presents the fundamental concepts and models which the reader needs to look at in order to follow our work. Chapter 4 (Model parameters) look at the parameters estimated for the two models. In Chapter 5 (Testing the models) we present the results of our work on the historic data and test the two models. Next we have chapter 6 (Forecasting Future Mortality Trends) in which we forecast mortality in the future and compare the results. Chapter 7 (Economic Consequences) focuses on the financial side of improvements in mortality. And finally chapter 8 (Conclusion) where we conclude our findings.

2. Data

This chapter presents the data requirements for this thesis.

The data used in this thesis is taken from the Human mortality database (HMD), which offers free access to original calculations of death rates and life tables for national populations, as well as the raw data used in constructing those tables. It also contains life expectancy and exposure data for a long list of countries. The database is maintained by the University of California, Berkeley, USA and Max Planck Institute for Demographic Research, Germany.

The data we are using consists of gender specific $q(t,x)$ death rates and the corresponding exposure to risk $E(t,x)$ for a range of years t and ages x. More precisely, death rates gives the probability of death occurring in calendar year t among people aged x and exposure to risk gives the total number of years lived during calendar year t by people of age x. Seven main countries of Europe have been used for comparison which are: *England & Wales, France*, *East & West Germany*, *Italy*, *Russia and Spain*. There is no combined data available for Germany, instead divided into east and west; therefore we are considering each of them as individual countries. The availability of the data was different for each country. Some had data starting way back from the nineteenth century, while the others had data starting from the mid fifties. The latest data also varied from country to country. So for those countries which had enough historic data, the starting year for them is considered to be *1950*, while the others have their original starting year from the database. For England & Wales, the data used for estimation is from *1950-2003, 1950-2005* for France, *1956-2004* for East & West Germany, *1950-2004* for Italy, *1959-2006* for Russia and *1950-2005* for Spain. The data is available for ages 0 to 110+, where the last age is an open-ended interval covering age 110 and above.

As far as the accuracy of the data is concerned, apart from Russia and Spain, the rest have no quality issues. The quality of the data for Russia for *1959-1969* is lower than in later years. There is also a growing problem with the quality of population estimates at ages 90+ in the second half of the *1990s*. It results in underestimation of mortality at these ages especially for males. Spain's data from *1950-1960* display age heaping problems and this might affect our results a bit.

3. Dynamic survival modelling

In this chapter we present the fundamental concept of dynamic survival modelling and explain the two models we are testing along with other theoretical elements used throughout the thesis.

3.1 What is dynamic modelling?

Mortality assumptions have played a central role throughout the whole history of life insurance and pension mathematics, whose origins can be traced back to the second half of the 17th century. Despite this long history, it was not until the construction of a long series of mortality observations that trends in mortality clearly emerged, and hence the concept of "dynamic" mortality was achieved, namely at the beginning of the 20th century.

A dynamic model accounts for the element of time where as a static model does not. As time is very important when we forecast mortality, actuaries came up with the idea of dynamic modelling. Mortality in a dynamic context is assumed to be a function of both the age x and the year t. For example, the expected lifetime for a newborn is denoted by E_0 in a non-dynamic context, but in a dynamic context it is represented by $E_0(t)$, a function of the calendar year t (namely the year of birth). Similarly, the general death rate in a given population can be represented by a function $q(t)$, where t denotes the calendar year in which the population is considered. Actuarial calculations for pension plans and in life insurance involve the use of mortality assumptions expressed as $q(t,x)$, which is the underlying probability that an individual aged exactly x at time t will die before time $t+1$. The period t to $t+1$ will also be referred to as year t. The next section will introduce us with the two dynamic survival models which are used in this thesis.

3.2 Survival models

3.2.1 Gompertz-Makeham

Humans have been trying to understand life and death for as long as they existed. In 1825, a British actuary, Benjamin Gompertz presented his version of the survival probability formula, based on the recognition that human mortality displayed exponential patterns for most ages. He found that the probability of dying was high at birth but then declined until sexual maturity. After this it increased at an exponential rate. His result is believed to be the most influential parametric mortality model in the literature. Some years later, in 1860, Makeham noticed that Gompertz's model was not adequate for higher ages and amended it in an effort to correct this deficiency. This amended model is called the Gompertz-Makeham model and the mortality intensity is modeled as follows:

$$\mu(x) = \theta_1 + \theta_2 e^{\theta_3 x} \tag{3.2.1}$$

The link to the death probabilities is

$$q_{tx} = 1 - e^{-\left(\theta_1 t + \frac{\theta_2}{\theta_3}\left(e^{\theta_3 (t+x)} - e^{\theta_3 x}\right)\right)} \tag{3.2.2}$$

Here, t is time in years, x is the age and $\theta_1,$ θ_2 *and* θ_3 are the parameters estimated from the maximum likelihood program specifically designed for Gompertz-Makeham parameter estimation. This model estimates the death probabilities which will be used in all the calculations involving Gompertz-Makeham.

3.2.2 Perks

Following on from Gompertz work, Perks (1932) proposed that instead of using an exponential function, the force of mortality was best described by a logistic function of age. In this thesis we will restrict ourselves to the following model for the mortality curve. This is a special case of what is known as the Perks model.

$$q_{tx} = \frac{e^{(\theta_1 + (x+t)\theta_2)}}{1 + e^{(\theta_1 + (x+t)\theta_2)}} \tag{3.2.3}$$

Like the first model, this one also has *t*, which represents time in year, *x* representing the age, and θ_1 and θ_2 are the parameters estimated using another maximum likelihood program for Perks parameter estimation. Perks model involves two stochastic factors. The first affects mortality at all ages in an equal number, whereas the second has an effect on morality that is proportional to age. The model will provide the death probabilities which are then used throughout the thesis for assessing Perks.

3.3 Random Walk

A commonly used model in finance is the random walk. To make forecast of the future distribution of θ for both models, we will model θ as random walk with drift:

$$\hat{\theta}_{it} = \mu_{it} + \hat{\theta}_{it-1} + \xi_{it}, \tag{3.3.1}$$

where $i=1,2,3$ for Gompertz-Makeham and $i=1,2$ for Perks, *t* is the calendar year e.g.2005...2050, $\hat{\theta}_{it}$ is the new estimated stochastic parameter, μ_{it} is the drift and ξ_{it} are multivariate normally distributed random variables with mean zero and a covariance matrix,

$$\xi_{it} \sim N(\mu, \Sigma) \tag{3.3.2}$$

Here Σ is a 2x2 matrix for Perks and 3x3 for Gompertz-Makeham.

4. Model Parameters

This chapter presents the method used to estimate the parameters of the survival models and briefly examines them.

Parameters for our models are estimated using maximum likelihood estimation programs which were provided by the supervisor, one program for each model. In order to get the desired parameters, the following steps are followed:

1. Get the required data from HMD database for a specific country and gender.
2. Adjust the maximum likelihood program according to the requirements.
3. Run the program and find the estimated parameters.

These routines were followed for each model, each country and for each gender separately. After a lot of hard work and effort, we finally get the estimated parameters in a form which can then be used to estimate morality. To give an example of the type of parameters we get from these maximum likelihood programs, the following figures are plotted:

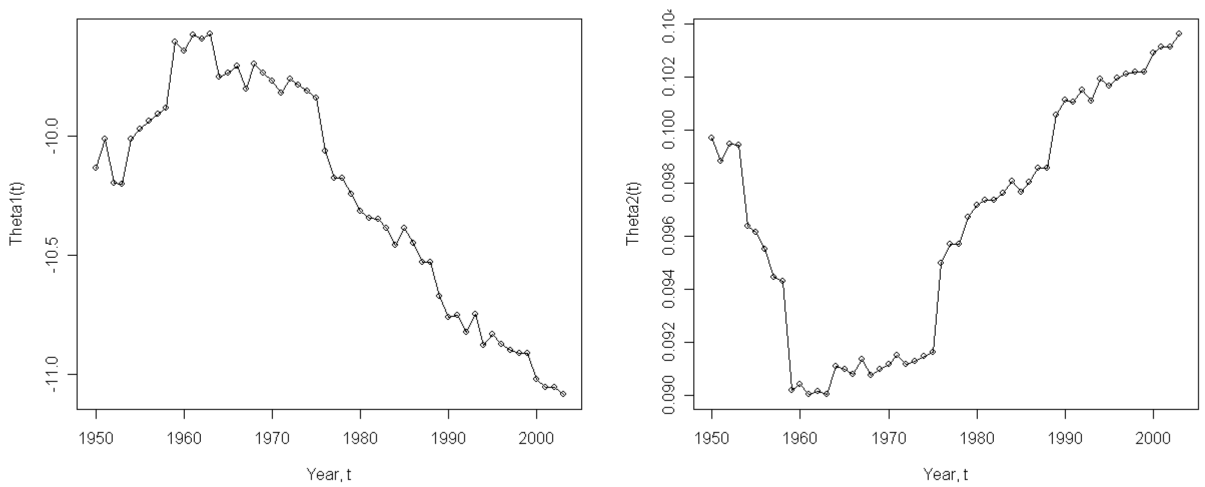


Fig 4.1: Parameters of the Perks model of female mortality in England & Wales for years 1950-2003.

Estimated values for Perks θ_1 and θ_2 for years *1950-2003* for England & Wales female are plotted in the figure 4.1. The downward trend in θ_1 reflects general improvement in mortality over time at all ages. The increasing trend in θ_2 means that the curve is getting slightly steeper over time: that is, mortality improvements have been greater at lower ages.

Similarly, figure 4.2 shows the development of three estimated parameters of Gompertz-Makeham over 54 years. While the first parameter is age-independent, the remaining two are age-dependent. The development of θ_2 and θ_3 is almost the same as we saw for Perks parameters above.

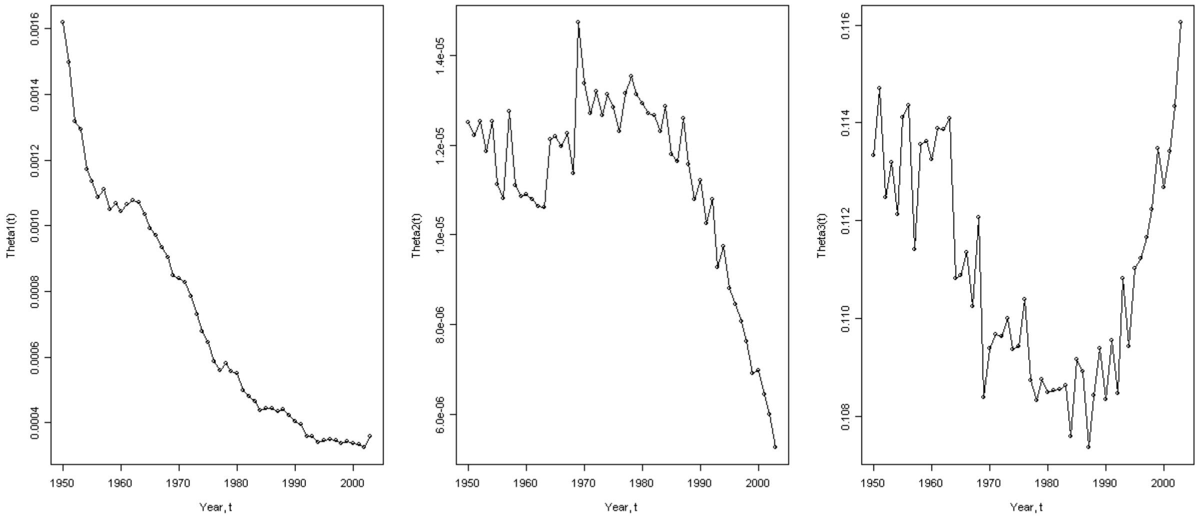


Fig 4.2: Parameters of the Gompertz-Makeham model of female mortality in England & Wales for years 1950-2003.

Note that the Gompertz-Makeham model depends on three parameters while the Perks have two. Another thing to mention here is the range of age. The maximum likelihood program is not able to estimate the parameters for all ages. It stops estimation at a certain point, but this varies from country to country. Russia has the age range 0-111, England & Wales 0-101, France 0-99, East & West Germany 0-98, Italy 0-101 and Spain 0-104.

5. Testing the models

This chapter examines each model separately and compare their results. The calculations of this chapter revolve around historic data. The idea is to test how the two models work and how well each model fits the historic data.

5.1 Aim

We are aiming at a detailed investigation of mortality trends over a particular period for both genders from age 0 to 111. The intention here is to give a broad picture of the main features of mortality and a detailed analysis using the two models. The choice of which countries to include in the comparison of trends, is influenced by the completeness, reliability and uniformity of the data, but, at the same time, by the desire to widen the representation. Period covered by the study varies from country to country, but on average we are looking at fifty-three years of data ranging from 1950-2006 for seven different countries.

The same procedure will be applied for each model separately, after which the two models are compared. Here we have chosen to show the results for East Germany only. The other countries will also be reviewed, but in a more generalized manner.

5.2 Gompertz-Makeham model

5.2.1 Mortality Curves

The first thing we are looking at is the mortality curve. We use the maximum likelihood program to estimate the three parameters of Gompertz-Makeham for age 0-111, from year 1956 to 2004. Using equation 3.2.2 and the parameters for each year one by one, death probabilities are calculated. Figure 5.2.1 illustrates the death probability of female aged 0-111 for year 1956, 1976, 1996 and 2004. The first figure is drawn using the historic data while the second shows the data estimated by Gompertz-Makeham model.

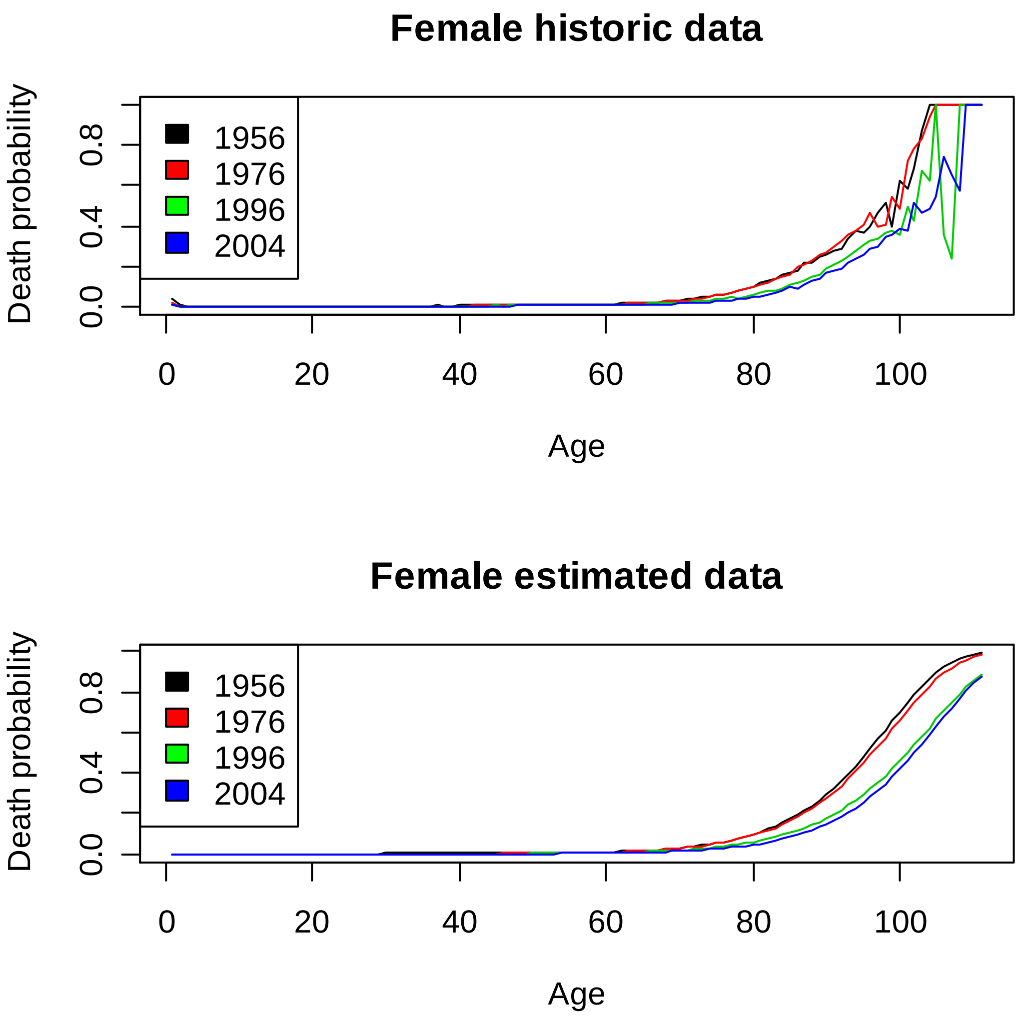


Fig.5.2.1: Historic and estimated mortality plot for female

One can see a decline in mortality over the years. The two figures are presented together in order to give a picture of the goodness or closeness of our estimation compared to historic data. Though the curves are not exactly the same, they are more or less similar in nature. Both plots show a decline in mortality over time. Our estimated plot seems to have a bit higher death probabilities compared to the historic plot at the old ages. Remember that maximum likelihood program for Gompertz-Makeham estimated parameters till age 98 for East Germany, meaning that we can only compare our estimates with the historic data up to age 98. The fluctuating data in the historic plot after age 100 is bad data and is not considered so important. So it does not make much of a difference if we skip data comparison after age 98 as we have bad data after this age and also that not many live up to this age.

The morality curve is unclear for the age group 0-60. It is hard to see the trend at early age. But detailed analysis of all the age group is important; therefore the mortality curve is divided into four age groups as shown in the figure below:

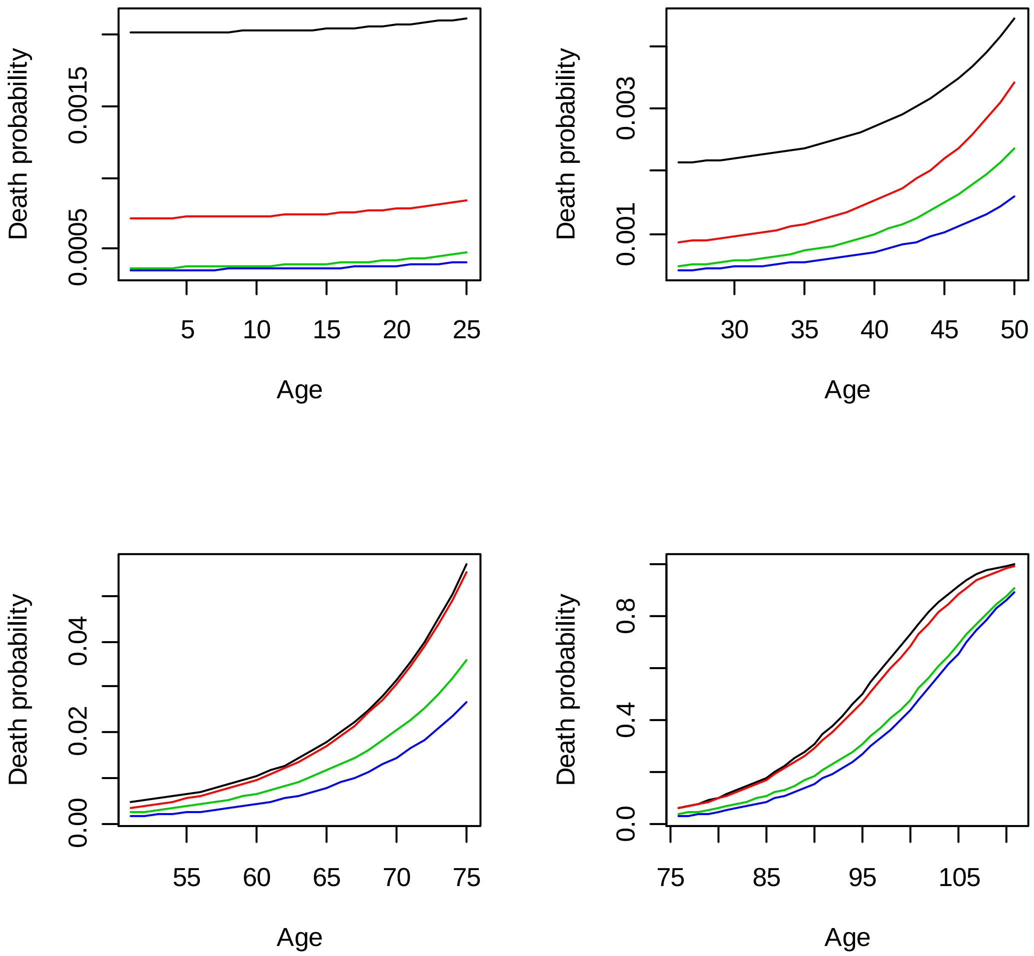


Fig.5.2.2: Gompertz-Makeham female mortality plot divided into four age groups

From figure 5.1.2 it is very clear that there has been mortality decline in 49 years for all age groups. Mortality improvements from 1956 to 1976 can be primarily attributed to improvements in the age-specific death rates for the age group from 0-50, where as mortality decline in the years 1976 to 1996 are mainly due to decline in age-specific death rates for the age group 51-111.

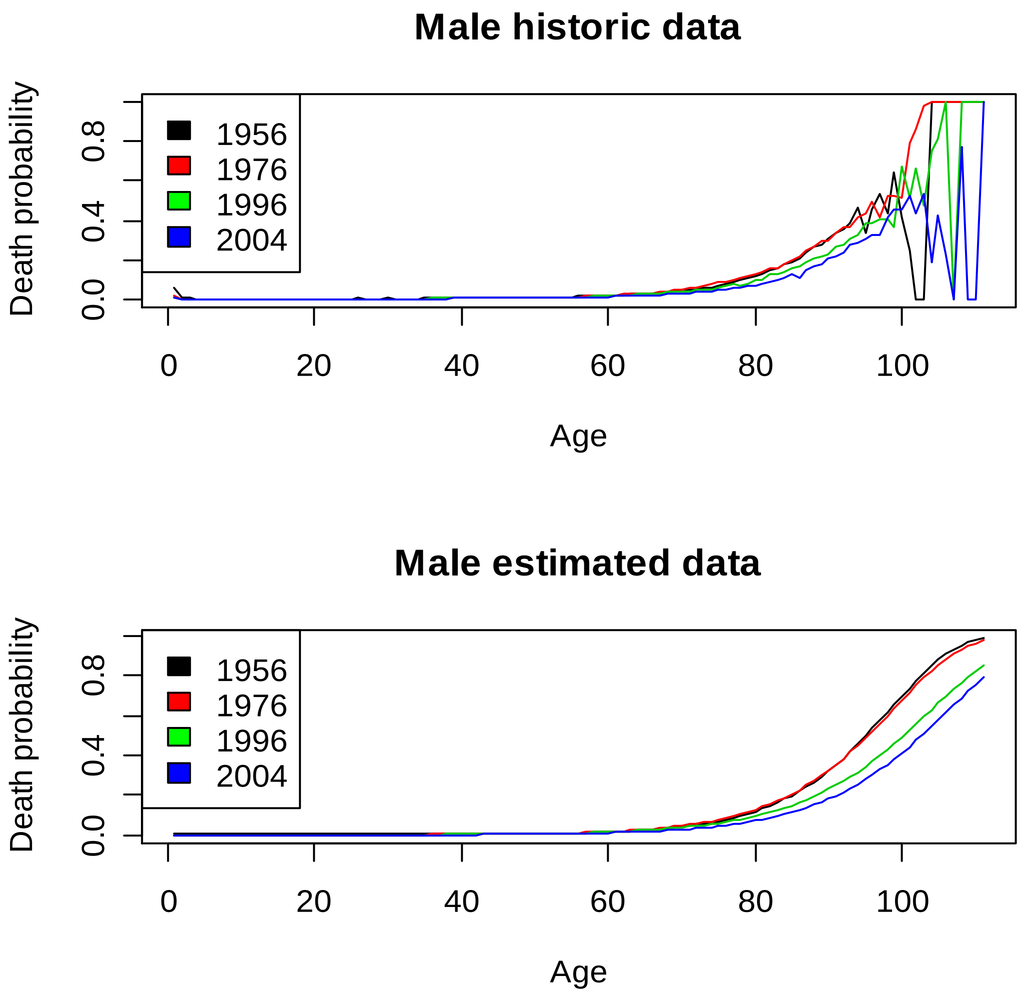


Fig.5.2.3: Historic and estimated mortality plot for male

Mortality rate for male population is shown in figure 5.2.3. The figure has exactly the same type of plots as figure 5.2.1, except that we have changed female to male data. As for the case for female, male mortality rate has also declined over the years. This can be identified by the colors of the curves where black curve is for year 1956 and blue for 2004. One can see an incredible decrease in mortality in 49 years. Both for male and female, the greatest decline has come between years 1976 and 1996. The reason for that might be advances in medicine between these years. Bad data also exists for male in the old age and therefore not taken into consideration. Over estimation at the old age can also be seen here. We will look more into it later.

5.2.2 Male and Female comparison

Next, we are looking at male and female mortality together. It is a well known fact that women live longer than men. With that said it would be interesting to see if that is really the case for East Germany. It might not be the case for every country, but the historic data for East Germany shows that female have a lower mortality rate than men. Comparing male and female mortality using Gompertz-Makeham model, we come up with the following results:

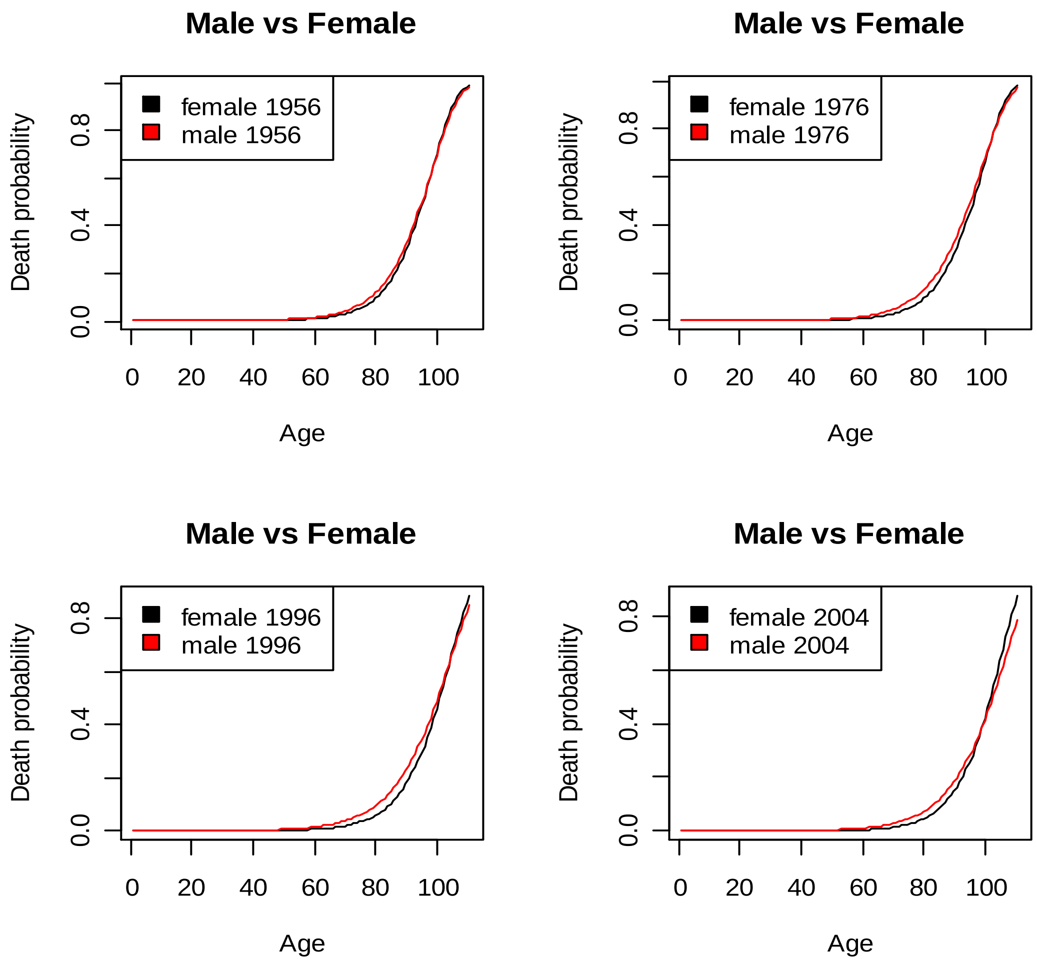


Fig.5.2.4: Male Vs Female mortality plots

Figure 5.2.4 compares the mortality rates of male and female for year 1956, 1976, 1996 and 2004 separately. Red curves indicate male mortality of the given year, while black curves indicate female mortality. One can easily notice that the red curves are over the black ones in all the four plots shown above, indicating that men have a higher mortality rate than women which mean that women live longer than men and thus supporting the fact. Though the curves start to cross each other after the age has passed 90, it is of less importance as not many live above that age. As mentioned earlier, we have bad data at old age and the maximum likelihood program stops estimation after age 98. This can be the reason for over estimation at old age and can also be the cause of male mortality lower than female mortality.

5.2.3 Goodness of fit

Gompertz-Makeham's model seems to be working reasonably well for now, but it is too early to conclude that the model is good. What we are interested in now is to check how good the model fits the historic data. In other words, how well can we estimate mortality using our model. Figures 5.2.5 and 5.2.6 below illustrate Gompertz-Makeham's estimates plotted over the original data. Figure 5.2.5 is a female plot for year 1956, 1976, 1996 and 2004, whereas figure 5.2.6 shows a male plot for the same years. Plots are in log scale. Here, we are plotting log of *q(x)* against age. For both 1956 and 1976 female, one can see an over-estimation for up to age 40, after which the curve fits very well. 1996 female shows an over-estimation by Gompertz-Makeham till age 20 and a slight under-estimation around age 40-50. 2004 female also have similar pattern, over-estimation till age 40 and a slight under estimation for age 40-60.

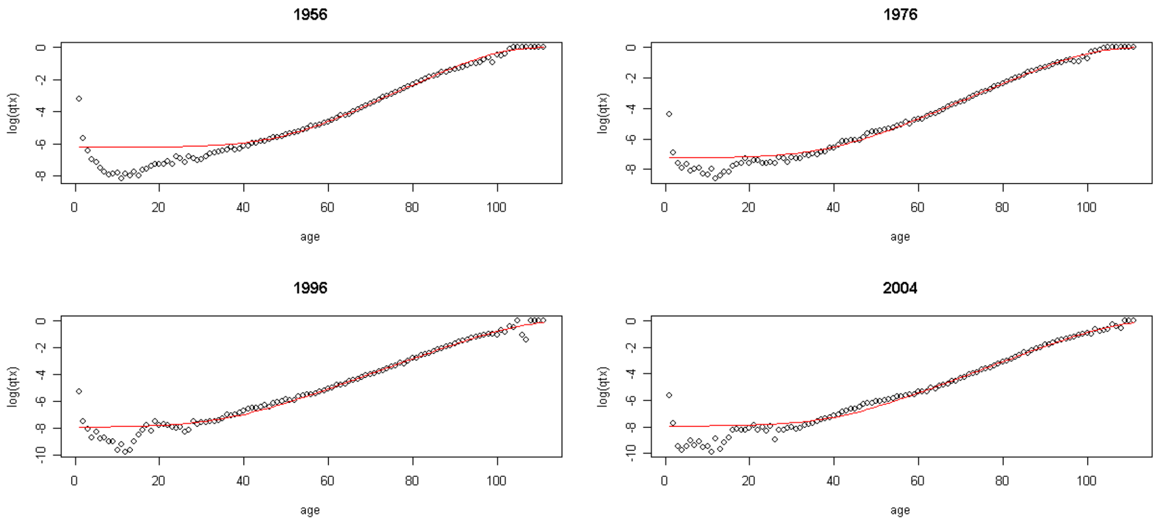


Fig.5.2.5: historic Vs estimated log(qtx) plots for female

Next, we are looking at male plots. 1956 male plot shows Gompertz-Makeham over estimating for young and middle age, but fits well for age 50 and over. Both 1976 and 1996 plots show a good fit over all, although there is an over estimation till age 20. 2004 male also has similar pattern with over estimation at the start age, a slight fluctuation in the middle, and smooth towards the end. Ignoring the early ages, Gompertz-Makeham fits generally very well for the rest of the ages.

In actuarial context, under estimation is not a very good sign. This can lead to many economic problems which we will discuss in the coming chapters. As far as Gompertz-Makeham is concerned, we have seen a slight under estimation on a few occasions and this will be taken into consideration.

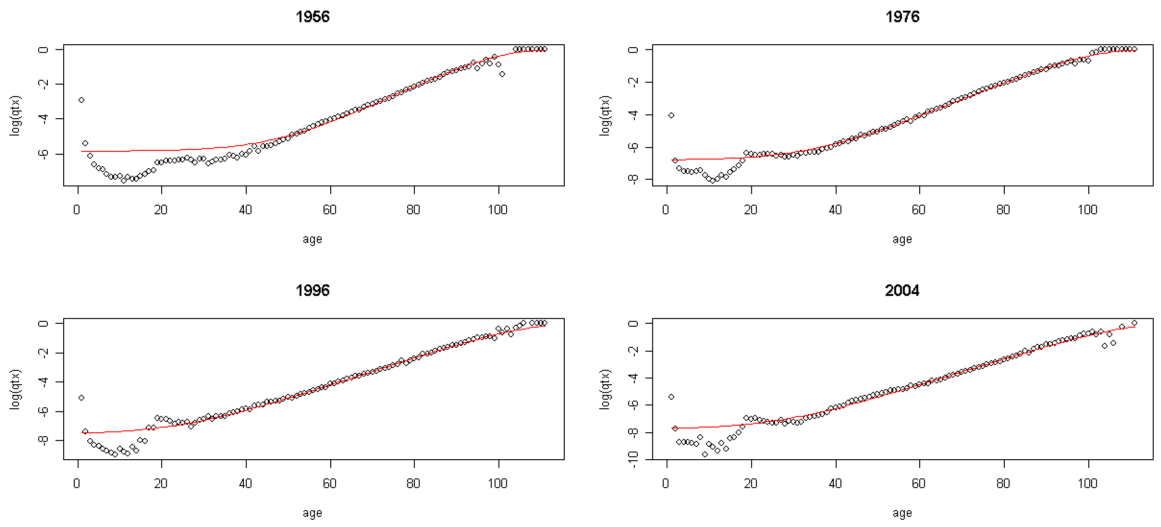


Fig.5.2.6: historic Vs estimated log(qtx) plots for male

5.2.4 Mortality improvement

Decreasing mortality is a problem world wide for insurance companies and pension funds. Next we look at the extent to which mortality is decreasing over the years. Here, we are considering three different population percentiles. Calculation is based on the following:

$$P(X_t^i \le x) = \sum_{x=1}^{111} (1-q_{t0}^i)(1-q_{t1}^i)...(1-q_{tx}^i) = \varepsilon \qquad (5.2.1)$$

where, q_{tx} is the death probability, x is the age and ε = 20%, 50%, 80%, is the percentile. We are calculating that an X year old will be at least x years old at a given time t. Here, i indentifies the age at which this probability equals the percentile.

Figure 5.2.7 is a mere reflection of our calculation. It is indeed an interesting plot. The plot shows three different percentiles of population deaths through years. On the left we have a plot for women and on the right for men. The pattern is the same in both cases. What we see is an increase in death age per year. 20% of the female population died at the age of 63 in 1956, 50% died at 77 and 80% at 85 the same year. On the other hand, 20% of the male population died at the age of 55, 50% at 73, and

80% died at 82 in 1956. By 2004, 20% of the female population died at the age of 75, 50% at 85, and 80% at 92. For male population, 20% died at 66, 50 % at 78, and 80% at 87.

We can now look at the amount of increase in death age over the 49 years. For 20% population, the increase in death age is 12 years for female and 11 for male. 50% population had an increase of 8 years for female and 5 years for male. And finally, 80% of the population's death age increased by 7 years for female and 5 for male.

This analysis has shown us three important things: firstly, there is mortality improvement for both genders. Secondly, men tend to die before women. And finally, the mortality of women is improving a bit more than men's.

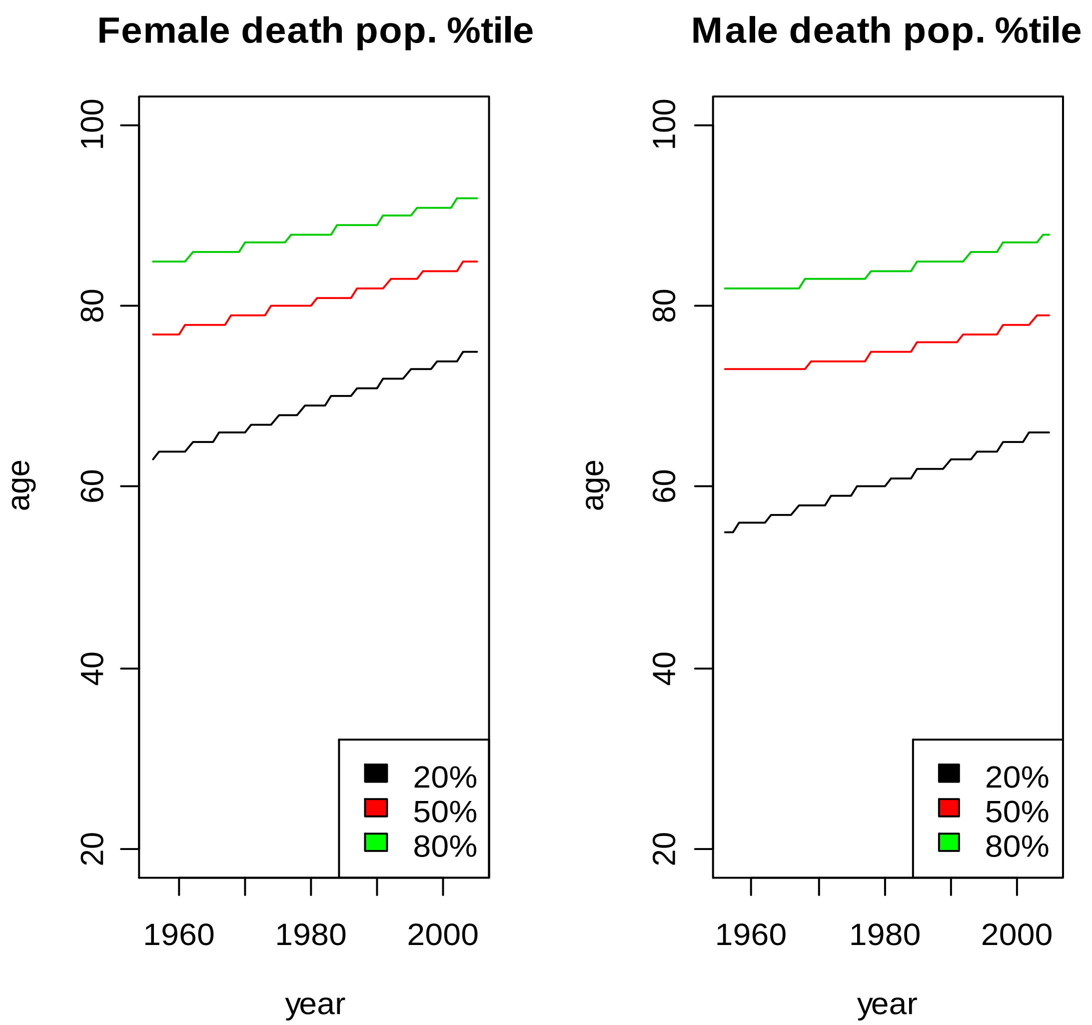


Fig.

Fig.5.2.7: Male and female death population percentile

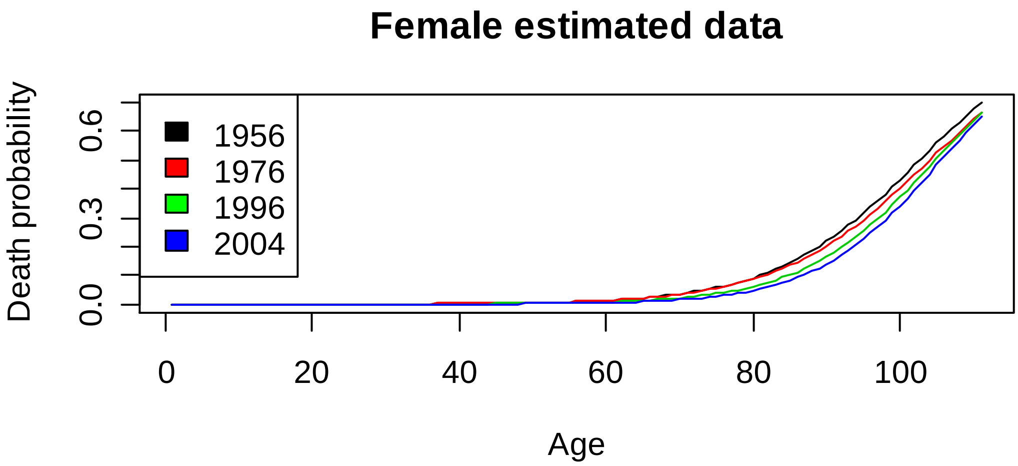
5.3 Perks

The next model we are looking at is the Perks model. Validation process is the same as used for Gompertz-Makeham.

5.3.1 Mortality Curves

We have already seen how the historic mortality looks like for East Germany male and female and how well Gompertz-Makeham model estimates. Now we are checking if Perks estimation is any closer to historic data. Figure 5.3.1 shows two plots, one for female and one for male. The mortality curves are produced by first estimating the two parameters for Perks using the maximum likelihood estimate program, then using these parameters in equation 3.2.3 to get the death probabilities. The plots produced by Perks look also good. The declining mortality trend over the years as seen by Gompert-Makeham for both the genders can also be seen here. One interesting thing to notice is the death probability. While the historic and Gompertz-Makeham's estimation showed a death probability ranging from 0-1, Perks give us probability range from 0-0.7. There is a huge difference in the mortality range among the two models. Therefore, it would be interesting to see the results of the two models together in order to compare them. The next section will cover the comparison part. For now, we will continue to look at Perks estimation only.

While the Gompertz-Makeham's mortality curves looked very similar to historic data, Perks plots are a bit different in nature. Though we have seen earlier that mortality is decreasing for every year, it is not in case of Perks male plot. Here we see 1976 mortality greater than for 1956. In general, we see slightly different mortality curves produced by Perks, than one we got from Gompertz-Makeham.



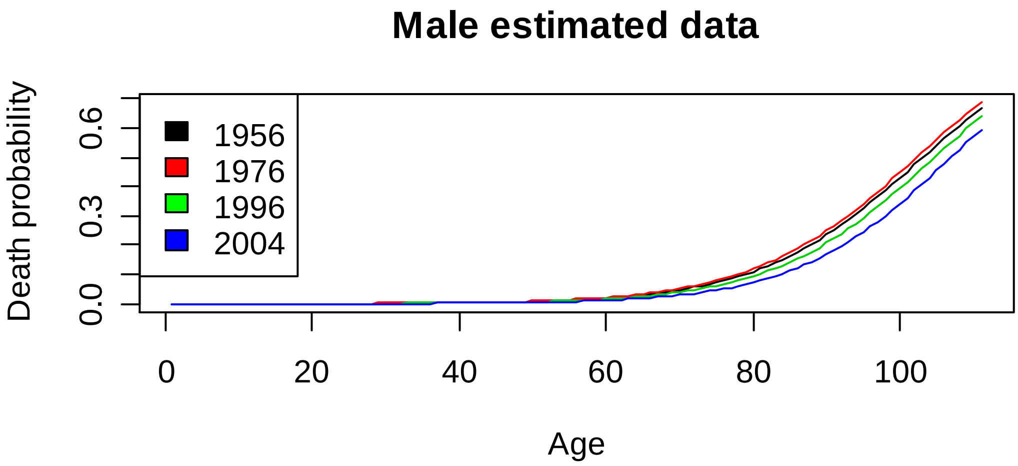


Fig.5.3.1: Perks estimated mortality plot for female and male

5.3.2 Male and Female comparison

The next part is to compare the mortality rates of male and female respectively. Figure 5.3.2 illustrates this. Again, we get similar pattern to the one we found using Gompertz-Makeham. Male mortality is above female mortality throughout the curve, except for the last few ages which we can be neglected.

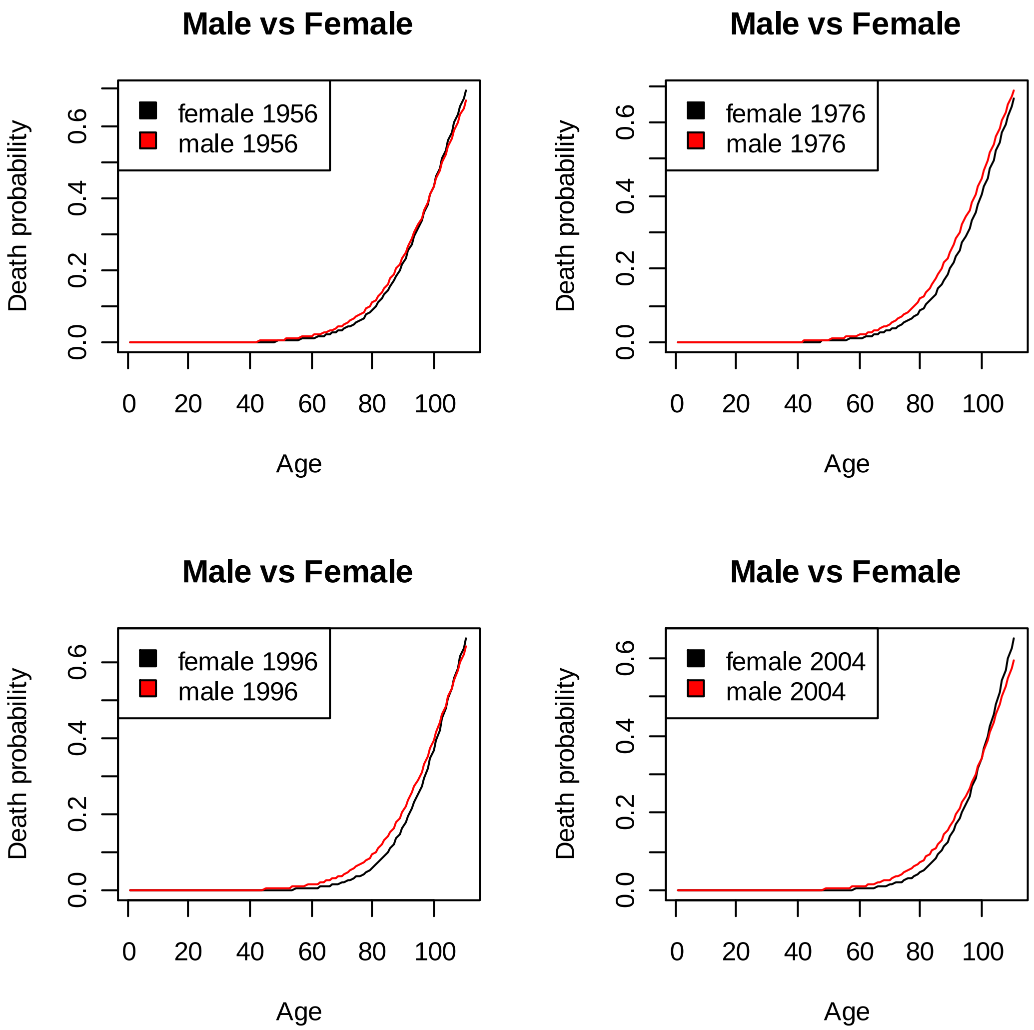


Fig.5.3.2: Male Vs Female mortality plots

At a first glance Perks estimates do not look very promising. As far as goodness of fit of Perks model is concerned, the results above tell little about that therefore it is too early to say anything about the model.

5.3.3 Goodness of fit

The next thing of interest is the goodness of fit. We need to see how well Perks model estimates the mortality. As we did for Gompertz-Makeham, we are plotting historic data and estimated data together, where the dotted plot is historic data and red line represent the estimates. Figure 5.3.3 illustrates the case for female population for year 1956, 1976, 1996 and 2004. Plots for year 1956 and 1976 are pretty similar. We see an under estimation for the early age group, followed by a slight over estimation in the middle age, and under estimation again for the later age group. Plots for 1996 and 2004 reveal an under estimation till about age 50-60, followed by a good fit. We have just looked at four cases. It is hard to see a pattern by looking at just these plots. But we do get a slight idea that Perks does not fit very well.

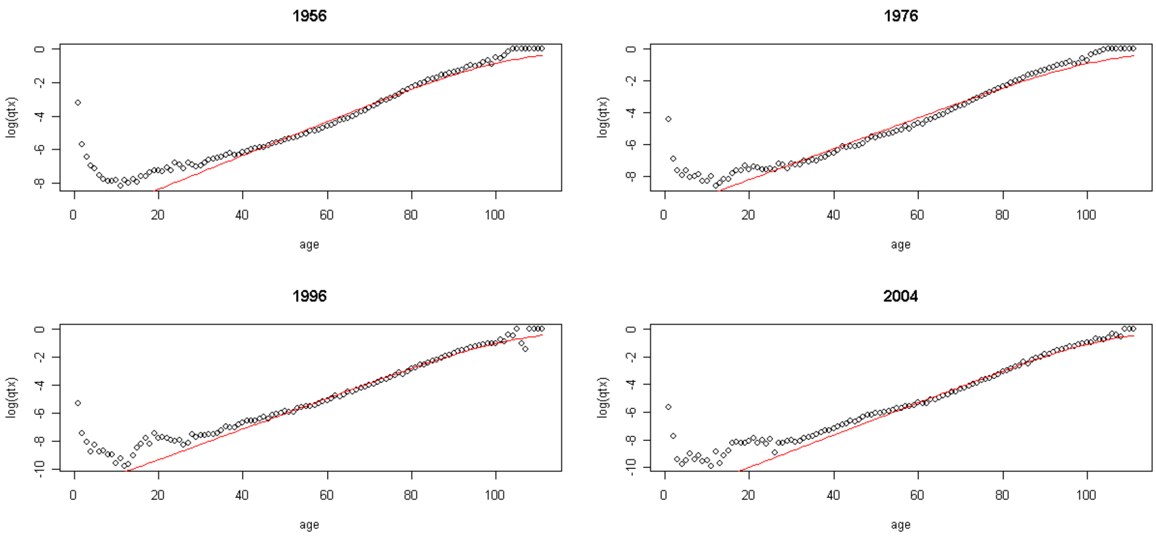


Fig.5.3.3: historic Vs estimated log(qtx) plots for female

5.3.4 Mortality improvement

We want to see how Perks model estimates death ages and what do the estimates say about increase or decrease in mortality. Figure 5.3.4 reflects our calculation. The plot shows three percentiles of population died through years. The pattern is the same in both cases. What we see is an increase in death age per year. 20% of the female population died at the age of 65 in 1956, 50% died at 77 and 80% at 85 the same year. On the other hand, 20% of the male population died at the age of 60, 50% at 73, and 80% died at 82 in 1956. By 2004, 20% of the female population died at the age of 75, 50% at 84, and 80% at 92. For male population, 20% died at 66, 50 % at 78, and 80% at 87.

We can now look at the amount of increase in death age over the 49 years. For 20% population, the increase in age is 9 years for female and 6 for male. 50% population had an increase of 7 years for female and 5 years for male. And finally, 80% of the population's death age increased by 7 years for female and 5 for male.

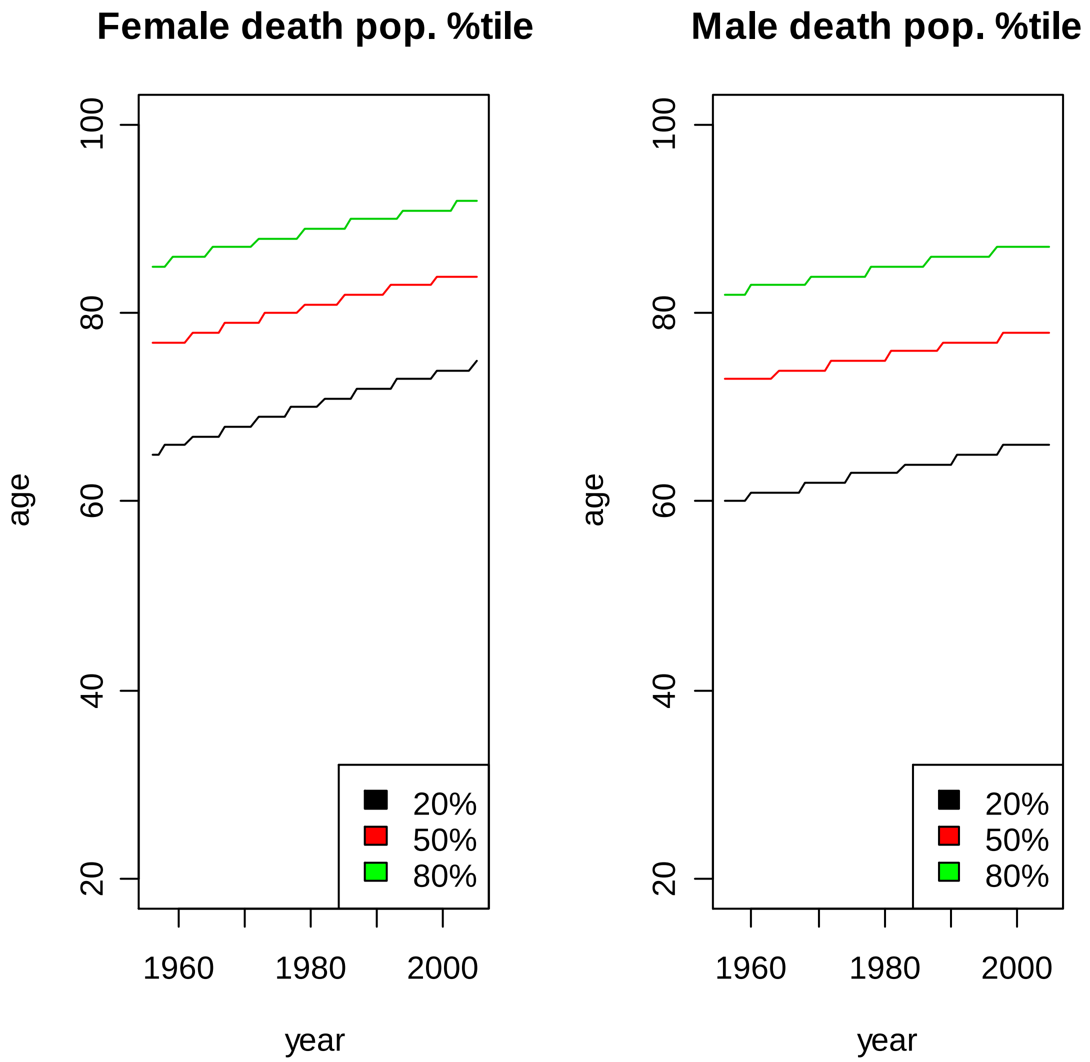


Fig.5.3.4: historic Vs estimated log(tQx) plots for male

5.4 Comparing the two models

After working on each model separately, we have got a bit of idea of the way the two models work and estimate. In this section our aim is to compare the results of the two models together. There are two things we are interested in:

1. Goodness of fit
2. Estimated change in death age overthe years.

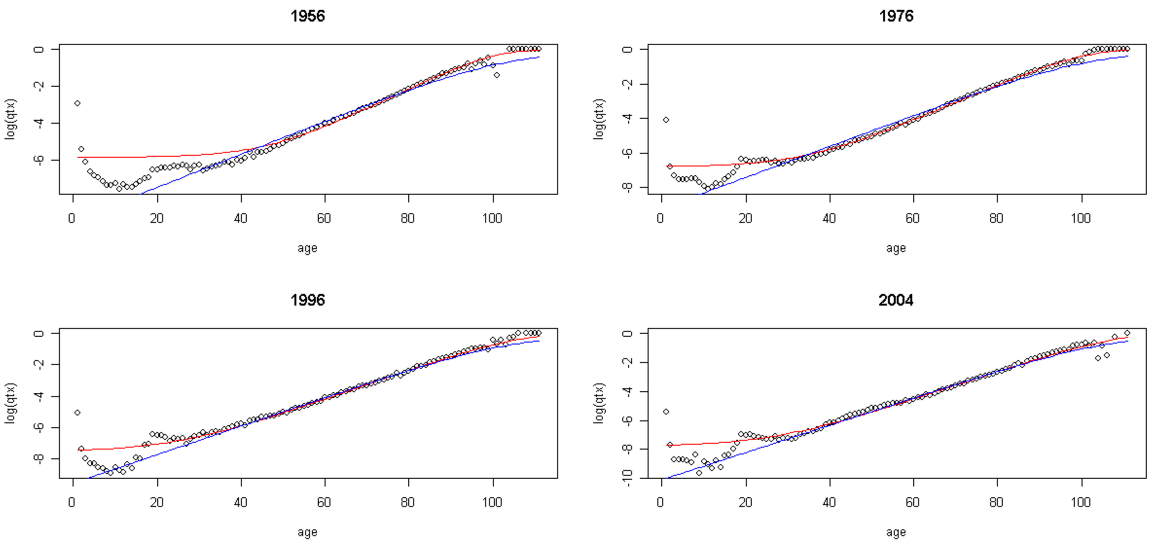


Fig.5.4.1: historic Vs estimated log(qtx) plots for male for both G-M and Perks

We have seen how the two models fit the historic data. In order to see the similarities or differences in their fit, we have plotted them together in figure 5.4.1. Historic data is presented with dots, red line shows Gompertz-makeham estimation, while the blue line indicates Perks. The four plots are for the male population for year 1956, 1976, 1996 and 2004. One notices that Gompertz-Makeham starts with an over estimation, comes in line and then estimates well all the way. Perks on the other hand starts with an under estimation, estimates well in the middle and loses track again right at the end.

After observing the plots of both Gompertz-Makeham and Perks, we can conclude that Gompertz-Makeham gives an over estimation for early ages which in our case is from age 0 to age 40 but estimates very well the death probability for later ages. This gives us a range of good estimation of about 70 years. Perks on the other hand under estimates for the early ages (0-40) and for the later ages (80-111), but works well for the middle age group. This gives Perks a range of about 40 years of good estimation. From what we have seen, it is clear that Gompertz-Makeham is working better.

Another thing we need to look at is the difference in the mortality improvements estimated by each model. As seen earlier, Gompertz-Makeham estimates a 12 year improvement in death age for 20% of the female population in 49 years, compared to 9 of Perks. For men we see Gompertz-Makeham showing 11 years of improvement, whereas Perks showed only 6. For the other two percentiles, the results are closer. But overall, Perks estimate less morality improvement than Gompertz-Makeham. This can also be verified by looking at the goodness of fit plot for each model where we see Perks under estimating on many occasions.

By looking at East Germany, we can easily say that Gompertz-Makeham fits better to the historic data and therefore is a more trust worthy model. Whether it works better for the other European countries as well is yet to be found in the next section.

5.5 A look at other European countries

This section deals with the remaining six countries. Though we have just shown the results for East Germany, the calculations done on the remaining countries also show mortality decline over the years. Figure 5.5.1 visualizes the mortality trends for each country. Age is considered from 0-90. The plots are for female mortality. Male mortality also follows similar trend.

Apart from Russia, all the other countries experience the same declining mortality trend. Russia has had a very strange mortality trend over the past 50 years or so. This can be true due to war and all, but there is also a possibility of bad data. Looking closely at Russia one can see almost no change in mortality trend over the years. In fact, it is fluctuating up and down over the years but still with not much effect. It is hard to believe that on some occasions the mortality is increasing instead of decreasing like seen elsewhere in Europe. It would be interesting to see how our models deal with Russia's data when we introduce forecasting the future in the next chapter.

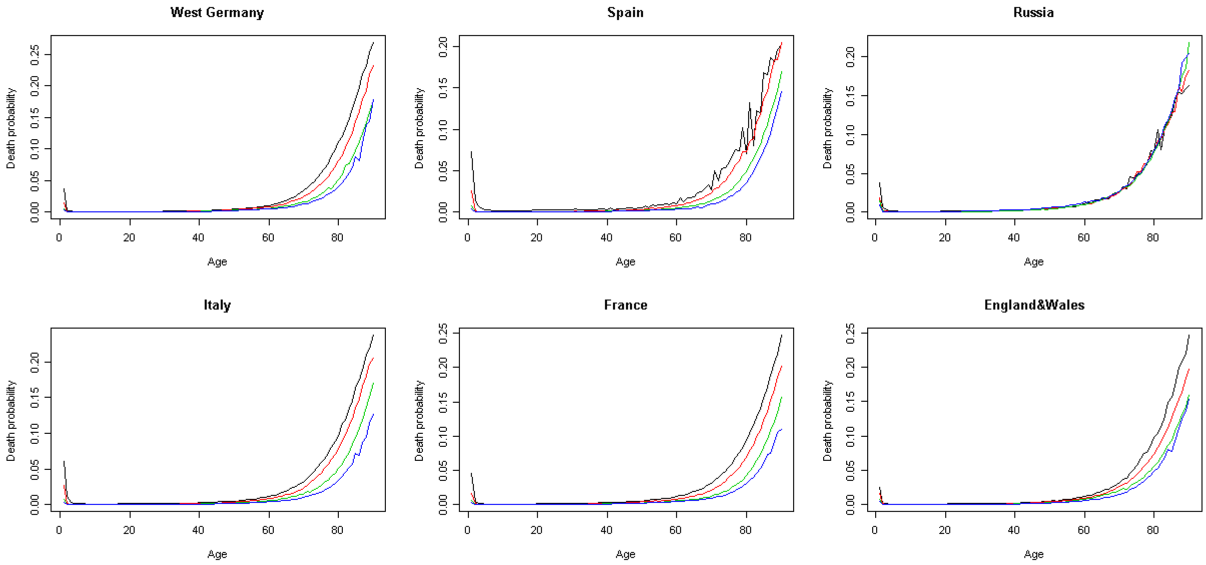
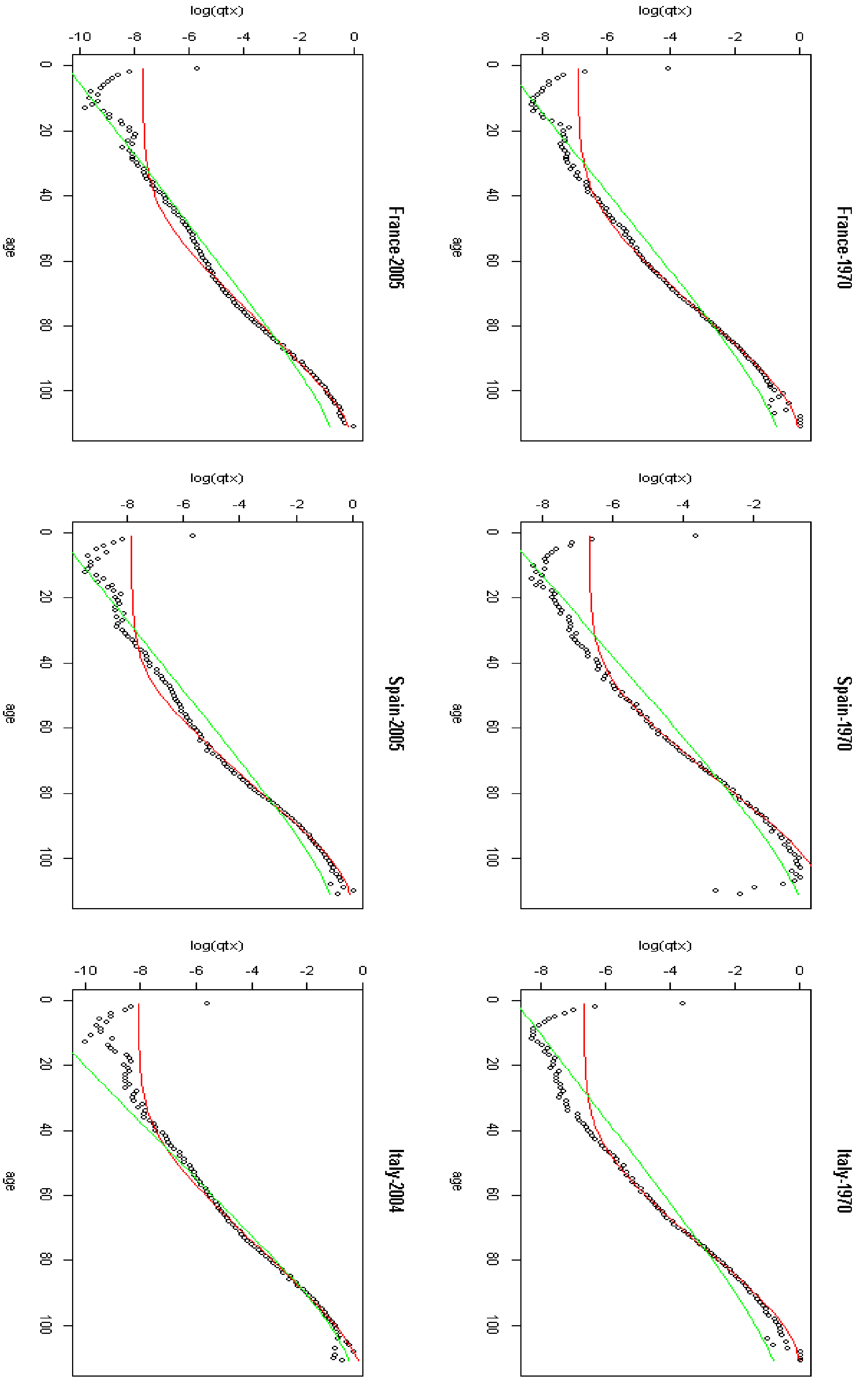
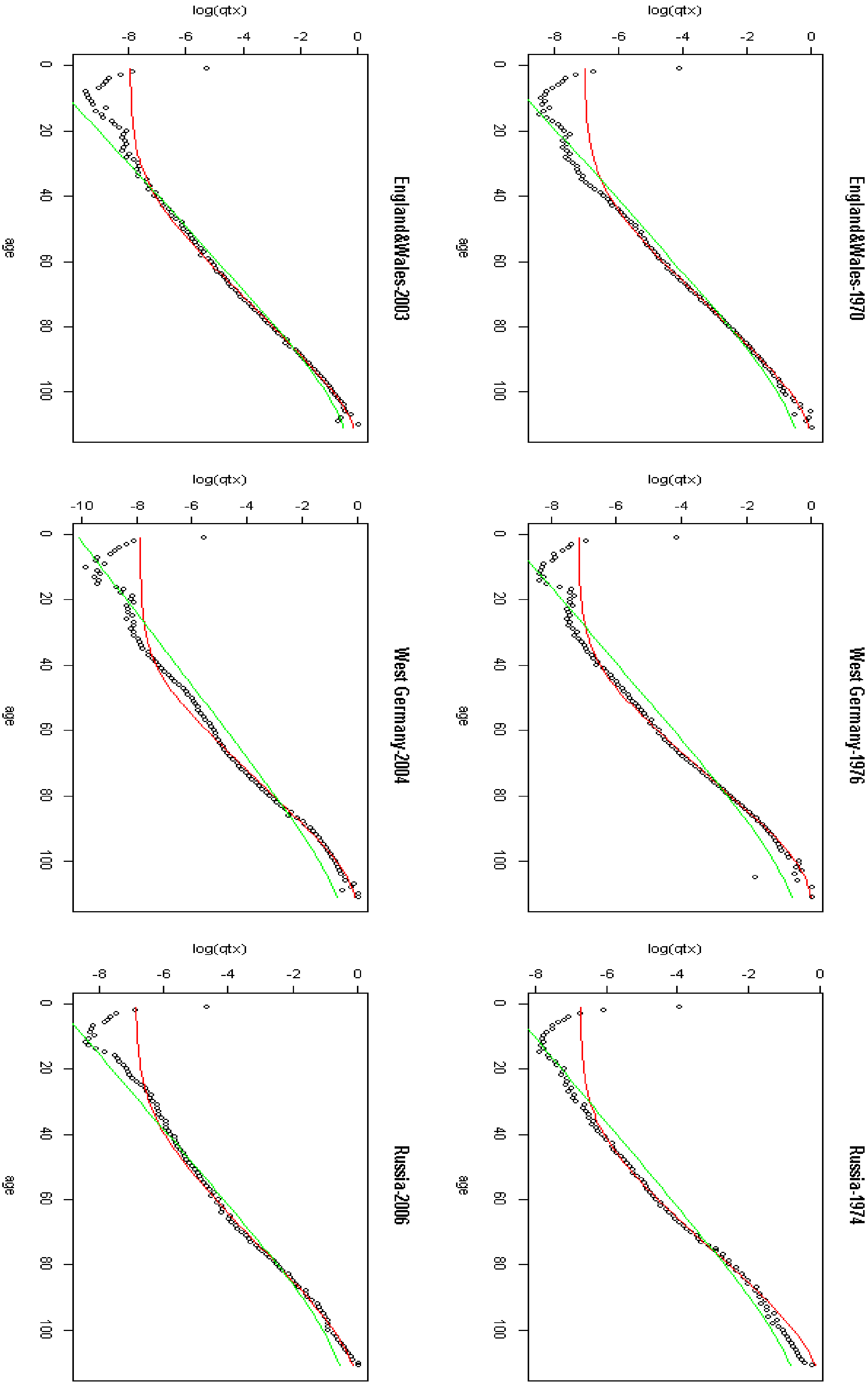


Fig.5.5.1: Historic female mortality trends for Europe

5.5.1 Goodness of fit

Plots of goodness of fit for female of each country are shown in the figures below. Both the models are plotted over the historic data. We found this to be the best way to visualize how each model fits to the historic data of each country. This gives us the general idea about the way each model works. The plots are of the same nature as we have seen earlier. We have chosen to draw the plots of latest available data and picked one random year from the middle. There is no specific reason for it. It does not make any difference which year to choose as plot for every year is unique. We are showing data for two years for each country in order to get a general idea of how these models fit.


France-1970
France-2005
Spain-1970
Spain-2005
Italy-1970
Italy-2004
log(qtx)
age


England&Wales-1970
West Germany-1976
Russia-1974
England&Wales-2003
West Germany-2004
Russia-2006
log(qtx)
age

Starting with Gompertz-Makaham first (red curve), we see a general pattern which this model follows. As seen earlier, Gompertz-Makeham over estimates at early ages. This is the case for almost every year and for every country we have tested. The age-group for which the model over estimates is seen to be round about 0-40. It may be less or more on few occasions as in case of Spain-1970 or Italy-1970, where it over estimate for the range 0-50. In general, we see a good, smooth fitting of the estimates for the rest of ages. But there are a few instances where the model under estimate for the middle ages from 40-60, like for West Germany-2004, France-2005 or Spain-2005. Overall, we can conclude that Gompertz-Makaham fits well except from the start ages ranging from 0-40.

Next we look at the green curves which represent Perks estimates. Perks model also follows a general pattern. It is not a flexible model as Gompertz-Makeham is; therefore we see very bad fit to the historic data. It is hard to see anything about the age groups for which the model fits. But one thing is prominent in almost every plot we have studied. There is a clear under estimation for the early age, over estimation for the middle age, and under estimation again towards the old age. We see a straight line going through the historic data, with very little correct estimation. The curve bends a little for old age. The best fit is seen for Italy-2004.

After looking at all the seven countries, there is no doubt remaining in our minds that Gomeprtz-Makeham fit way better than Perks. The reason for that seems to be the extra parameter Gompertz-Makeham contains. This makes the model more flexible and help capture the bends in mortality curves.

5.6 Correlation

The focus of this section will be on correlation. We are interested in looking at the mortality relation between different European countries and among male and female, if any.

Correlation indicates the strength and direction of a linear relationship between two random variables. The correlation coefficient $\rho_{X,Y}$ between two random variables X and Y with expected values μ_X and μ_Y and standard deviations σ_X and σ_Y is defined as:

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{E((X-\mu_X)(Y-\mu_Y))}{\sigma_X \sigma_Y} \tag{5.6.1}$$

We wanted to check if mortality trend in one country or in one gender had any effect on the trend of another country or gender. And if so, what kind of effect? Will there be a mortality decline in France if we see mortality decline in Spain? If male population experience an improvement in mortality, will the female population of the same country also experience it? These are the types of question which can be answered with the help of correlation and that was our aim. But unfortunately, due to limited time we were not able to analyze correlation. It would have been interesting to see the linear relationships between European countries and genders. It can be a useful study for insurance companies which operate throughout Europe.

6. Forecasting future mortality trends

This chapter focuses on forecasting mortality in the future. Survival probabilities and life expectancies for the two models are compared.

Time series forecasting is the use of a model to describe the likely outcome of the future events based on known past events. In this chapter we are looking at mortality in the future. Our aim is to describe future age patterns of mortality on the basis of the mortality trends we have experienced in the previous chapter. Using the random walk mentioned in chapter 3, we estimate the paramters for our models in order to forecast the future mortality trends. From there death probabilties are calculated. The idea is to forecast and compare the future mortality of different European countries using the two mortality models and compare the results.

6.1 Forecasting mortality

We have seen mortality rates have fallen dramatically at all ages for all the countries observed, with Russia an exception. Improvement rates have been significantly different at different ages. Since rates of improvement have varied over time and have been different at different ages, there will be considerable uncertainty in forecasting what rates of improvement will be in the future. It would be intersting to see how much moratlity will decrease over the coming years.

6.1.1 Parameter distribution

The results are based on 1000 independent simulations. The parameters we get are stochastic in nature, meaning that every time we simulate we get a new value for every parameter. In order to make the parameters more consistent, 1000 simulations have been conducted for every parameter and their mean value is used for further calculations. An example of the 1000 simulations per parameter is as follows:

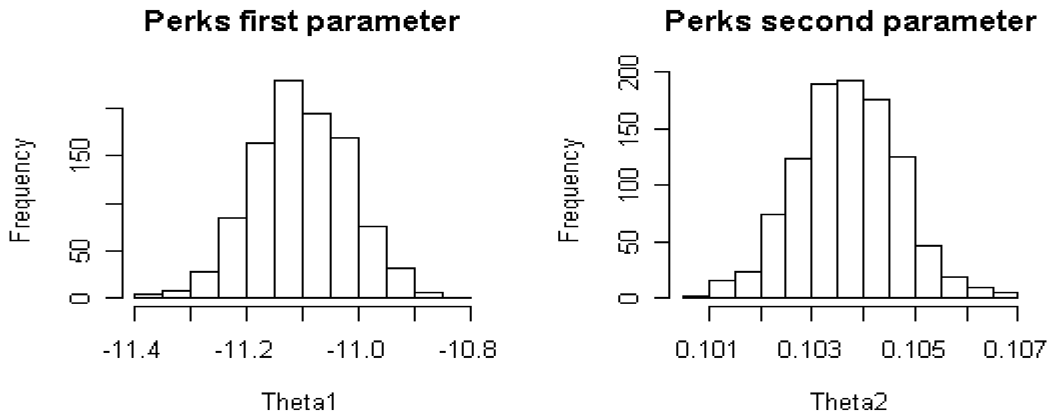


Fig.6.1.1: Perks parameter distributions

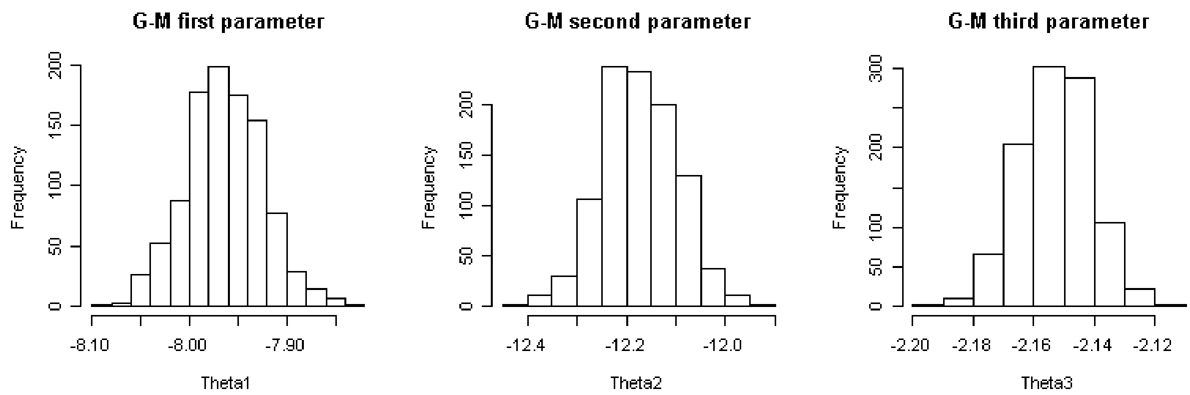


Fig.6.1.2: Gompertz-Makeham parameter distributions

Starting with Gompertz-Makeham model first, in order to make forecasts of the future distribution of $\theta_{it} = (\theta_{1t}, \theta_{2t}, \theta_{3t})$, we will model θ_{it} as a three-dimensional random walk with drift as mentioned in *equation 3.3.1*. This will give us the desired parameters for our model. This procedure is followed 1000 times for every parameter of every year and their mean values are used in order to make our results more consistent. Using Gompertz-Makeham's survival model, we can then calculate the death probabilities in future. We are forecasting morality up to 2050.

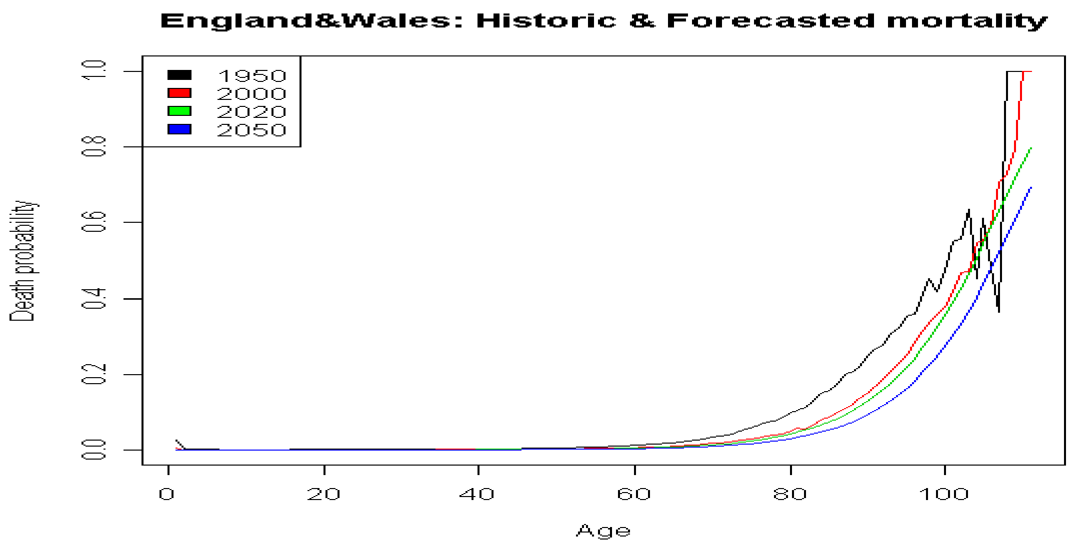


Fig 6.1.3: historic and forecasted female mortality

The figure above shows mortality rates for female of England & Wales. The black and read lines show mortality in year in 1950 and 2000, while the other two lines are forecasted using Gompertz-Makeham model. Clearly, the mortality is decreasing over the years. This was the expected result. The question is how far in the future can we estimate moratlity using these models? We are considering till 2050. Further than that can be a problem. The reason for that is that our parameter estimation is purely based on past experience. In the last fifty years or so, there has been many medical advances which led to this major mortality decline. Decrease in infant mortality and decreasing deaths at old ages has been the driving force in recent mortality improvements. But humans are mortal and will die at one stage. If we forcast using just the the data from the past, we will see that at one stage the forecasted mortality rate in the future will be almost zero for all ages and that is ofcourse not true. So there are limits to these models. We can not forecast for the next hundred years or so as that will give us very unrealistic moratity rates.

We have seen how Gompertz-Makeham model forecasts morality. Now we are looking at Perks. The procedure is the same as before. In order to make forecasts of the future distribution of $\theta_{it} = (\theta_{1t}, \theta_{2t})$, this time we will model θ_{it} as a two-dimensional random walk with drift as mentioned in *equation 3.3.1*. This will give us the desired parameters for our model. And putting these parameters in Perks model, death probabilities are calculated. Again we are forecasting up to 2050 because of the same reasons. Figure 6.1.4 shows Perks forecasted mortality. Along with it we have Gompertz-Makeham's forecasted mortality from the previous plot. Both have the same forecasted years. This is an easy way to compare the results of the two models together. Its clear that Perks also shows a further decrease in mortality over the next years. Up to about the age of 85, both show the same amount of decrease in mortality rate for 2020 and 2050, are which Perks (dotted line) decrease further for the old ages. The results are as expected. We knew from before that Perks under estimate for old ages and that is exactly what we see here.

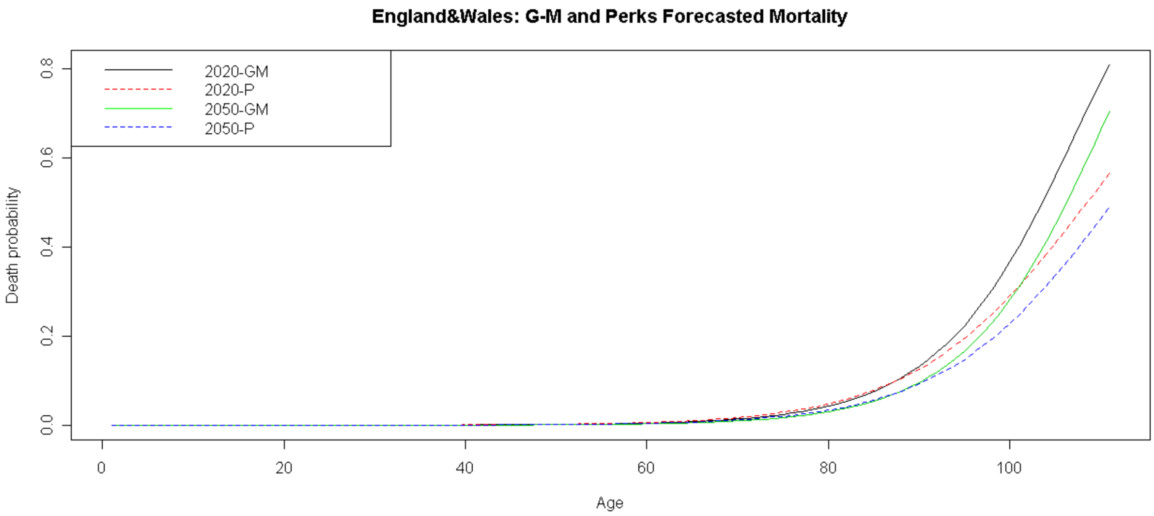


Fig.6.1.4: G-M and Perks forecasted female mortality

6.2 Survival probability

In this section our focus is on forecasting the survival probability. Let X be the length of life of an individual. Then the survival probability is as follows:

$$tPx = P(X > x + t \mid X > x) \qquad (6.2.1)$$

This is the likelihood that a person of age *x* lives at least *t* years longer. As we have calculated q_x previously, we can get *tPx* by using the equation below:

$$ {}_t p_x = {}_{t-1} p_x (1 - q_{x+t-1}) \qquad (6.2.2) $$

Decline in mortality means an automatic improvement in survival probability. We expect survival rate to increase over the coming years. To see how the survival rates differ for the two models and how much they improve over the years, the following is plotted:

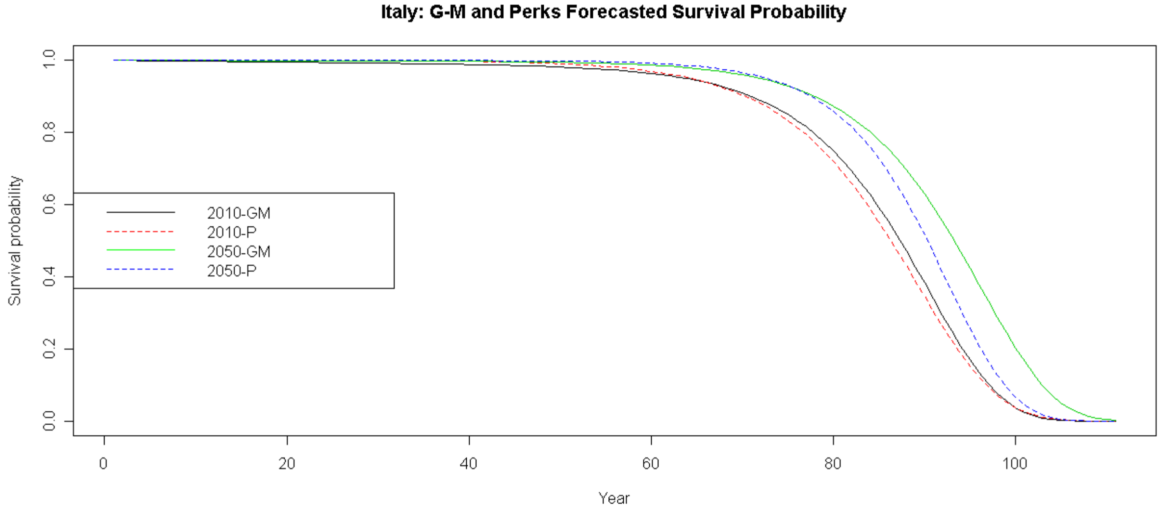


Fig 6.2.1: *G-M and Perks forecasted female survival rate*

Figure 6.2.1 shows the forecasted survival probabilities of new born Italian female for Gompertz-Makeham and Perks for year 2010 and 2050. By new born we mean x = 0. Full curves are Gompertz-Makeham estimates while the dotted shows Perks estimation. Both models estimate an improvement in mortality from 2010 to 2050. If we look at the survival plots for 2010 first, we observe that both models have very close estimates. Up to about the age of 65, Perks has a slightly higher estimation of the survival probability compared to Gompertz-Makeham's, but declines after that. This pattern can also be seen for 2050, where Perks is slightly above Gompertz-Makeham till the age of 75, but declines rapidly after that. As we observed earlier, Perks tend to under estimate at older ages. This is exactly what we are seeing here.

Another thing to notice is the change in survival rate over the years by the two models. Gompertz-Makeham has a greater change.

6.3 Life Expectancy

Life expectancy is the average number of years of life remaining at a given age. It is also the measure of change when describing the effect of improvements in death rates over time and when comparing 'the state of mortality' in different countries. We are interested in seeing the change in life expectancy in the furture for different countries and genders. Using the *tPx* mentioned above (6.2.2), life expectancy can be calculated as follows:

$$E(X) = \int_0^\infty {}_t p_x dt \qquad (6.3.1)$$

This is the general formula for finding the life expectancy of a person aged *x*. In our case x = 0, giving life expectancy at birth.

6.3.1 Results

	2020				2035				2050			
	GM		*Perks*		*GM*		*Perks*		*GM*		*Perks*	
Country	**M**	**F**	**M**	**F**	**M**	**F**	**M**	**F**	**M**	**F**	**M**	**F**
East Germany	76.8	84.4	77.9	84.6	77.6	86.7	79.6	86.4	78.2	88.9	81.3	88.1
West Germany	79.0	84.7	79.1	83.1	81.1	87.1	81.6	85.7	83.0	89.3	84.1	88.3
Russia	54.5	73.3	60.6	74.2	45.9	72.9	60.1	74.3	36.5	72.3	59.5	74.2
England&Wales	79.1	83.2	79.2	82.7	81.2	85.1	81.2	84.4	83.1	86.9	83.3	86.3
France	79.0	86.5	79.2	85.4	81.0	88.9	81.7	87.4	83.0	91.1	84.1	89.1
Spain	79.1	85.7	78.1	84.4	80.9	87.3	79.6	86.2	82.6	88.4	80.8	87.7
Italy	80.3	86.8	80.1	86.5	82.0	89.2	81.7	88.2	83.6	91.5	83.1	89.6

Table 6.3.1: Forecasted life expectancies of new born male and female

Table 6.3.1 contains the forecasted life expectancies of seven countries for both male and female in year *2020, 2035* and *2050* for both Gompertz-Makeham and Perks. The table will be used to compare the countries, gender and the two models. The estimates are for new born and therefore age x = 0 and year t = 1,2,3…111. We can look at East Germany first. The expected life for a new born male in *2020* in East Germany is estimated to be *76.8* years using Gompertz-Makeham model and *77.9* using Perks. In *2005*, male life expectancy was reported to be *75.7*. This means that according to Gompertz-Makeham, life expectancy will increase by about a year in the first fifteen years, while Perks shows an increase of about two years. Though a small margin, this one year difference between these two models can have a substantial impact economically which we will discuss in the next chapter. Moving forward to the next fifteen years we see another one year increase in life expectancy by Gompertz-Makeham and about one and a half year increase in life expectancy for Perks. By *2050*, the expected life for East Germany male will increase to *78.2* using Gompertz-Makeham, and *81.3* using Perks. The difference between the estimates for these two models is increasing the more we forecast in the future. Though both are increasing at a constant rate, it seems that either Perks is estimating very high or Gompertz-Makeham is estimating very low. In the coming forty-five years, Gompertz-Makeham has estimated an increase in life expectancy of 2.5 years while Perks estimates an increase of 5.6 years. This is an enormous difference between the two estimates and can lead to many economic problems if we do not choose the right model.

Let us see if female estimation gives any better results. Female life expectancy at birth in East Germany in 2005 was *81.8* years. By *2020*, this will increase to *84.4* as estimated by Gompertz-Makeham, compared to *84.6* of Perks. *2035* shows an increase to *86.7* and *86.4* years respectively. Finally, for *2050* we get *88.9* and *88.1*. Overall, we have got much closer estimates this time. In forty-five years, female life expectancy is forecasted to increase by 7.1 years by Gompertz-Makeham and an increase of 6.3 years is forecasted by Perks. This time Gompertz-Makeham estimated a higher life expectancy than Perks, but the overall difference for the female estimates is minimal.

A visualization of this result can be seen in figure 6.3.1. The figure is divided into two parts. The left part shows a historic development in the life expectancy from *1956* to *2004* while the right side shows the forecasted estimates of Gompertz-Makeham (gray) and Perks (white). The top plot represents female of East Germany, where as the bottom one is for male. Looking at female plot first we observe that both the models have close estimates with Perks estimating slightly lower than Gompertz-Makeham. The future life expectancy is constantly increasing in both cases. In case of male, we observe Perks model estimating a constant increase whereas Gompertz-Makeham start to flat out and the difference between the two estimates is increasing for every forecasted year.

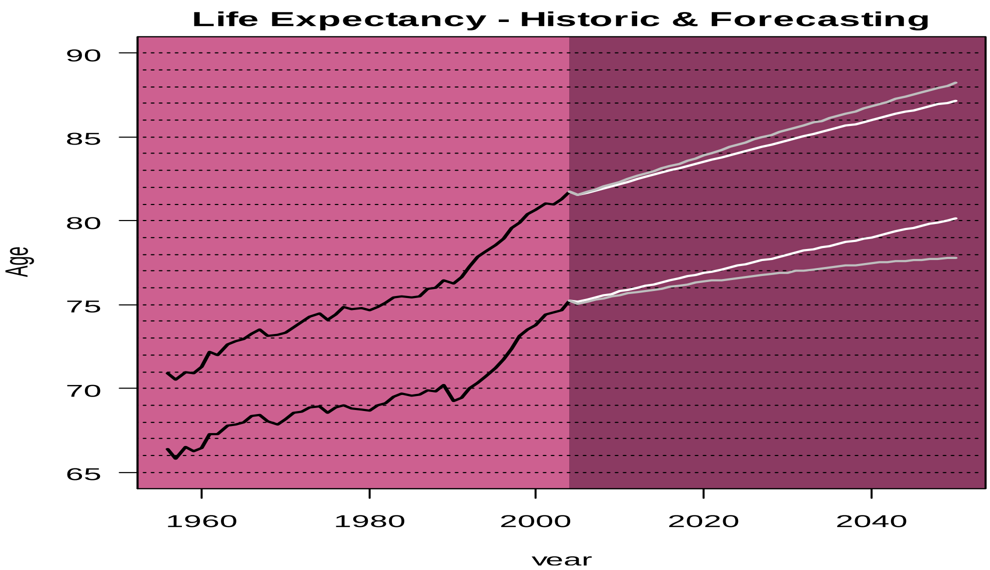


Fig 6.3.1: *G-M and Perks historic & forecasted life expectancy at birth for East Germany*

From 1956 to 2005 East Germany has experienced an increase in life expectancy at birth of about 10-11 years for both sexes. By the coming 45 years Gompertz-Makeham forecasts an increase of 2.5 years for male and 7.1 years for female, where as Perks forecasts an increase of 5.6 for male and 6.3 years for female. We will be looking at the other countries in order to come to a conclusion as it is hard to see any pattern for now.

Though one country, Germany is divided into East and West. For us it would be interesting to see how much the results differ for the two sides. Male life expectancy at birth in West Germany in 2020 is forecasted to be *79* years by Gompertz-Makeham model and *79.1* using Perks. In *2005*, life expectancy at birth for male in West Germany was *76.8*. This means that according to both Gompertz-Makeham and Perks, life expectancy will increase by two years in the first fifteen years. Moving forward to the next fifteen years we see another two year increase in life expectancy by Gompertz-Makeham and about two and a half year increase in life expectancy for Perks. By *2050,* the expected life for East Germany male will increase to *83* according to Gompertz-Makeham, and *84.3* using Perks. For East Germany, the models had very different forecasted life expectancies for male, but in case of West Germany we have very close results.

Moving on to the female estimates, life expectancy of a new born female in West Germany in 2005 was *82* years. By *2020,* this will increase to *84.7* as estimated by Gompertz-Makeham, compared to *83.1* of Perks. *2035* shows an increase to *87.1* and *85.7* years respectively. Finally, for *2050* we get *89.3* and *88.3*. Over the course of next 45 years, female life expectancy will increase by 7.3 years according to Gompertz-Makeham and by 6.3 years according to Perks. Again, we see close estimates for female.

6.3.2 A quick look at the other countries

In order to see the kind of pattern these two models follow, it does not hold with one example only. We need to look at the other countries as well in order to generalize the pattern for the two models. Though the table above contains all the data we need, it is often easier to visualize the results to get a better picture. There is no need to go in detail for every country to get to a conclusion. Here, we will briefly look at each country's estimates using the forecasted life expectancy plots available in the Appendix A.

Starting with England & Wales first, we see a smooth estimation for both the models. The estimates seem to be very close, especially in case of male. There is almost no difference between the two model’s forecasted life expectancies at birth for male. Both Gompertz-Makeham and Perks show a constant improvement in the life expectancies over the coming years for each gender. There is no sign of life expectancy at birth flatting out or improvement slowing down in the near future for England & Wales.

Moving on to France, one can yet again see close and smooth estimates for each model with constant improvements in life expectancies. The only striking thing here is the female curve for Perks. There is an initial decrease in life expectancy the first year from 84 years to 83 years, followed by constant improvement. Italy also follows the same pattern which we have seen till now, where both the models have close estimates and there is a general improvement in life expectancy at birth over the coming years for each gender. Though for female we see a slightly different result from each model. Where Gompertz-Makeham is improving at the same rate as in the past, Perks improvement rate slows down after 2015 and continues to be slow.

Finally we are looking at Spain. While Gompertz-Makeham is following the same trend as seen in all the other countries we have studied, Perks like seen before has the tendency to jump up or down the first year of future forecasting, and that is what we see here again. This time both the gender estimates decline in the first year, then continue to improve at a constant rate in the future.

6.3.3 Generalizing the results

We will not look at each country one by one; instead try to generalize the results. This is done with a help of the table below:

	2020		2035		2050	
Country	**M**	**F**	**M**	**F**	**M**	**F**
East Germany	-1.1	-0.2	-2	0.3	-3.1	0.8
West Germany	-0.1	1.6	-0.5	1.4	-1.1	1
England&Wales	-0.1	0.5	0	0.7	-0.2	0.6
France	-0.2	1.1	-0.7	1.5	-1.1	2
Spain	1	1.3	1.3	1.1	1.8	0.7
Italy	0.2	0.3	0.3	1	0.5	1.9
Average	**-0.1**	**0.8**	**-0.3**	**1.0**	**-0.5**	**1.1**

Table 6.3.2: Difference in the forecasted life expectancies of new born male and female for the two models

Table 6.3.2 contains the differences in the forecasted life expectancies estimated by the two models for both male and female for the six countries in year 2020, 2035 and 2050. Note that Russia is not included in the table. This is because as we mentioned earlier, Russia's data contains bad data which gave strange results. We will discuss Russia separately as it is interesting to see how each model react to Russia's data. For now we concentrate on these six countries. Negative sign shows that Perks has a higher estimate and positive shows that Gompertz-Makeham is estimating higher. Overall, we see that the differences are very little. The row Average contains the average of the differences and that is what we will be looking at.

Starting with the first average value of male 2020 we see the value -0.1, meaning that overall there is 0.1 year difference in the forecasted life expectancy at birth of a male. Here, negative sign indicates that Perks's estimates higher than Gompertz-Makeham. Female for the same year shows 0.8. As the value is positive, it means that in general Gompertz-Makeham is estimating life expectancy at birth of a female 0.8 years more than what Perks has estimated. Moving to 2035 we see -0.3, indicating that Perks is estimating 0.3 years more than Gompertz-Makeham and so on. One thing that strikes is that for every female estimation Gompertz-makeham in general is estimating higher life expectancies at birth than Perks, and for the male it is the opposite. The reason for that is not clear, but is something that strikes at a first glance of the table.

The results of table 6.3.2 gives us the idea that although there are differences in the estimates of each model for each country, in general both the models give very close forecasted life expectancies at birth. Next we want to see how much life expectancy at birth is forecasted to increase over the next 45 years. Remember that we start at 2005 and not 2008; therefore we have 45 years between 2005 and 2050.

Table 6.3.3 shows the forecasted increase in life expectancies at birth from 2005 to 2050 for both genders and for both the models. The table contains the values for each country separately. We can easily compare the countries, genders or models together by looking at the table. East Germany male gives the most unbalanced result. We see a difference of 3.1 years between the estimates of the two models. Here, negative sign suggests that Perks is estimating higher than Gompertz-Makeham. Otherwise, the values are close. To generalize the result in order to come to a conclusion, we are going to look at the average values. Starting with Gompertz-Makeham male, we see that on average an increase in life expectancy at birth of about 5.3 years is forecasted over the 45 years. Perks forecast an increase of 5.8 years. So the difference between these two models on average is of 0.5 years. For female, Gompertz-Makeham forecasts an increase of 6.7 years compared to 5.5 of Perks, thus making the difference to be 1.2 years. Like we saw earlier, Perks seems to estimate a bit higher for male while Gompertz-Makeham estimates higher for female in general.

Country	GMmale	Pmale	Diffmale	GMfem	Pfem	DiffFem
East Germany	2.5	5.6	-3.1	7.1	6.3	0.8
West Germany	6.2	7.3	-1.1	7.3	6.3	1
England&Wales	5.8	6	-0.2	5.5	4.9	0.6
France	6.2	7.3	-1.1	7.3	5.3	2
Spain	5.7	3.9	1.8	4.9	4.2	0.7
Italy	5.4	4.9	0.5	7.8	5.9	1.9
Average	**5.3**	**5.8**	**-0.5**	**6.7**	**5.5**	**1.2**

Table 6.3.3: Forecasted increase in life expectancies of new born male and female

The results so far have been convincing. We have seen a further improvement in future life expectancy. Looking at Europe in general, over the course of the last 45-50 years, life expectancy at birth has increased at a constant rate in the past and is forecasted to continue to rise in the future.

6.3.4 Russia

As seen earlier, Russia has a very unique mortality trend. It is very different from the other countries we have studied. This led us to look at Russia separately. Russia is a very important part of Europe, and therefore it had to be included in the thesis. At that point we had no idea how the data looks and how the results going to look like.

The aim here is to see how each model reacts to the difficult data. Till now we have seen a very fluctuating mortality trend for Russia. Strangely, mortality seems to increase over the years. This can not be true for the future as things are way different now. We expect life expectancy at birth to improve over the coming years. Let us see how the models react to Russia's data. Figure 6.3.2 shows the forecasted life expectancy at birth for Russia for both male and female. The lighter shade shows the development in life expectancy at birth from 1959 to 2006, while the darker portion is the forecasted life expectancy till 2050. Top curve is for female while the bottom curve shows male data and estimation. Perks forecasted life expectancy is the white line while gray line signifies Gompertz-Makeham.

Starting with the female plot first, we see an improvement in life expectancy in the early 1960s, fluctuation till the 80s, a sudden decline in the early 90s followed up by another snake like trend. Male plot shows a similar pattern. With this kind of unbalanced trend as a base for our future forecasting, it is hard to get realistic life expectancies for the future. The key to forecasting in the future is the value of drift factor μ. But with so much uncertainty in the past, it is hard to get a good solid drift value. So the results we see in the figure are not very surprising. This uncertainty has given us the chance to look at each model in an extreme case.

It is not hard to see which model handled the extreme data best. While for female, both models forecast very close life expectancies, male plot gives a very different picture. Perks (white) shows a decline in the future life expectancies at birth, but the rate of decline is low. On the other hand we have Gompertz-Makeham which shows a very high rate of decline in life expectancies in the future. While Perks almost flats out, Gompertz-Makeham is not able to forecast the future life expectancies correctly, especially for male.

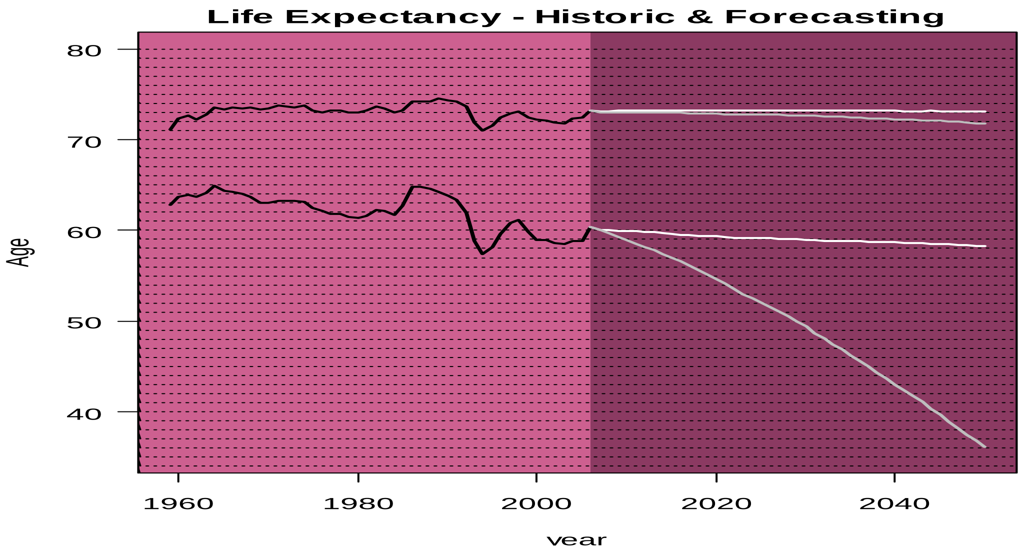


Fig 6.3.2: *G-M and Perks historic & forecasted life expectancy for Russia*

7. Economic Consequences

This chapter focuses on the economic consequences faced by pension funds and insurance companies due to mortality decline. We will look at how much of an effect it makes economically and compare the effects of the two models.

7.1 Longevity: A concern

Longevity is a huge concern for insurance companies and pension planners. When people tend to live longer, a lot more is paid to them after their retirement. If we look at the life expectancy of female in East Germany, we estimated earlier an increase of six to seven years in life expectancy over the next forty-five years. This means that for every single female, the pension fund or the insurance company has to pay an extra payment for six-seven years more than what we are paying now. By now we mean 2005 as this was the latest data year we had for East Germany. Let us look at a scenario. Assume that females in general get a pension of $1000 per year in East Germany. Life expectancy of a female in East Germany is 81.8 years at the moment. By 2050 it is estimated to be around 88.5, meaning that pension planers needs to pay an extra $6700 per female than what they are paying now. Assuming further that the portfolio contains 100 female, the net extra amount that will be paid by 2050 equals $670000, which is an enormous amount.

This is a very serious matter and the insurance companies and pension planers need to work on how to handle these growing mortality concerns. For many years, the governments and insurance companies have had problem like these with female. They tend to live longer than men which results in a lot of extra costs. But now we are having similar problems with men. Though it might sound good that people tend to live longer now than ever before and will continue to improve their survival probability over the years, it is a concern for those in the retirement business. Pension rules are changed every now and then in order to adjust to the growing survival rate.

Actuaries from years have been trying to find new survival models in order to estimate the closest and realistic mortality rate. Here, we have estimated future mortality using the mostly used survival models. And the results have been convincing so far. In the coming section we will look at how much the single premium will increase over the years in order to manage longevity.

7.2 Single Premium

The net single premium is the amount of money that would have to be collected at the time a policy is issued to assure that there will be enough money to pay the death/pension benefit of the policy, assuming that interest is earned at the expected rate and that claims occur at the expected rate. It has the form

$$\sum_{k=\max(l_r - l_0, 0)}^{111} v^t tPx, \tag{7.2.1}$$

where l_0 is the starting age, l_r is the retirement age, v^t is the discount factor $1/(1+r)^t$. We have assumed that there is a constant interest rate of 4% for all the countries with retirement age 67.

Figure 7.2.1 shows the estimated single premium for East Germany female for year *2005*, *2025* and *2050*. The premiums are estimated using the equation 7.2.1. One can see a clear constant increase in premium over the years for all ages. With a mortality decline, an increase in premium is needed to cover up for the extra costs of living longer. Black, green and cyan colored lines show net single premium estimated using Perks model, while the red, blue and violet plots represent the estimates of Gompertz-Makeham. Both models show an increase in premium over the years. Gompertz-Makeham shows a slightly higher premium than Perks. But the difference between the two estimates is minimal.

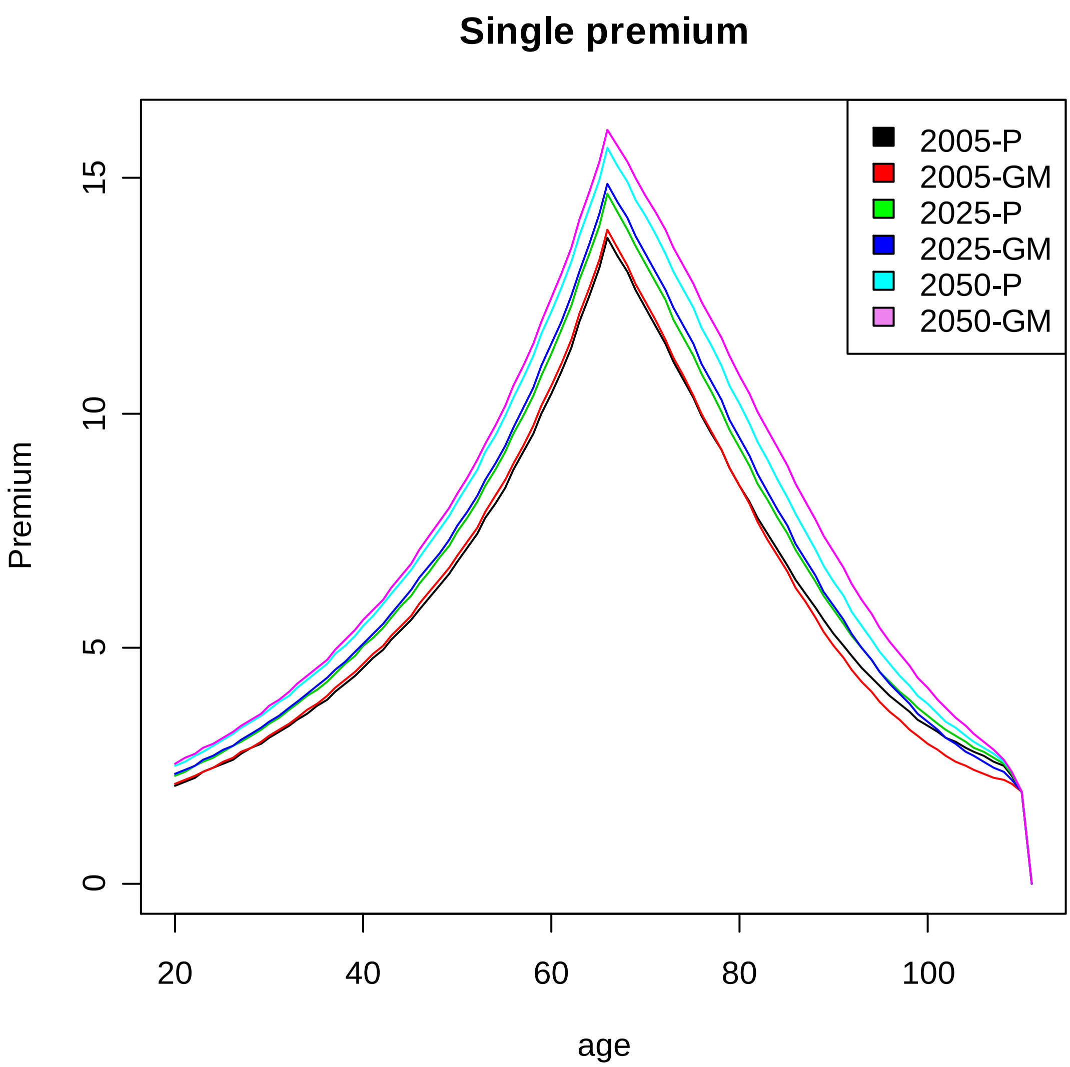


Fig 7.2.1: Forecasted single premiums for East Germany female

7.3 How the single premium works

Premium that we get is not an amount, rather a scaling factor. We can show use of it using an example. We are interested in the total reserve we need to have in order to fulfill person x's pension obligations. Assuming x is 40 years old and has a benefit of $1000, and assuming an administrative cost of 3%, we can find her total reserve:

Total reserve = Single premium * benefit * 1.03 (administrative cost) (7.3.1)

A forty year old female has a premium of about 4 in 2005, which gives us a total reserve of $4120. This means that pension fund need to set aside $4120 for person x in order to fulfill her pension obligations after her retirement. By 2050, premium increases to 5 which then increases total reserve to $5150. This means that over the course of 45 years, the reserve has to be increased by $1030 which gives increase of 25%. Note that we have not taken inflation and other factors into consideration.

7.4 Comparing premiums of the two models

Tables 7.1 and 7.2 show the estimated single premium values for each country for year 2010, 2030 and 2050, and for age 30, 60 and 90. Table 7.1 contains the estimates of Gompertz-Makeham and 7.2 have Perks single premiums. The models have pretty close estimates and both follow a general pattern. Premiums have increased over the years for every age, for every country. The only exception is of Russia like we have seen earlier. In case of Russia, the premiums are decreasing every year. This is because survival probabilities for Russia are decreasing every year. It is very strange, but that is what both the models reveal. So actuaries in Russia must be very careful when forecasting the future mortality and premiums because the estimates we get does not seem to be realistic. It is very tough to forecast future mortality for Russia and other methods need to be considered.

Having seen the tables and plot 7.2.1, one can see that the premiums increase smoothly till a certain age and then follow a downward trend. The top point is met at the age of 67. This is when a person retires. Though retirement age is different from country to country, we have made it a constant for every country we tested in other to compare the results. The top point is at age 67 because after this the member of a pension policy changes state from active to retired. After age 67 the premium value starts to decrease because survival probabilities at old age are lower than at early age, thus automatically reducing the premiums.

Gompertz-Makeham

	2010			2030			2050		
Country \ Age	**30**	**60**	**90**	**30**	**60**	**90**	**30**	**60**	**90**
East Germany	3.0	10.1	4.3	3.3	10.9	5.1	3.5	11.7	6.0
West Germany	3.2	10.9	5.2	3.6	11.8	6.0	3.8	12.6	6.9
Russia	2.3	8.4	4.4	2.2	8.1	4.3	2.1	7.8	4.3
England&Wales	3.1	10.5	5.2	3.4	11.2	5.8	3.6	11.9	6.5
France	3.4	11.5	5.9	3.7	12.4	6.7	4.0	13.1	7.6
Spain	3.3	11.3	5.3	3.5	11.9	5.7	3.6	12.4	5.9
Italy	3.4	11.5	5.8	3.7	12.3	6.7	4.0	13.2	7.8

Table 7.1: Gompertz-Makeham single premiums for each country

Perks

	2010			2030			2050		
Country \ Age	**30**	**60**	**90**	**30**	**60**	**90**	**30**	**60**	**90**
East Germany	3.0	9.9	4.5	3.2	10.7	5.0	3.5	11.5	5.5
West Germany	2.8	9.7	5.5	3.1	10.7	6.7	3.4	11.6	7.9
Russia	2.0	7.5	4.0	2.0	7.2	3.3	2.0	7.0	2.8
England&Wales	2.8	9.5	4.7	3.0	10.2	5.2	3.3	10.9	5.9
France	3.0	10.3	6.9	3.3	11.1	6.2	3.5	11.7	6.5
Spain	2.9	10.0	5.6	3.2	10.7	5.6	3.4	11.3	5.8
Italy	3.2	10.5	5.0	3.4	11.3	5.2	3.6	11.9	5.4

Table 7.2: Perks single premiums for each country

In order to generalize the behavior of the two models, we will consider the results of all the countries together. By thorough investigation we found out that on average Gompertz-Makeham estimates 0.3 premium points higher than Perks for age 30, 1.0 premium point higher for age 60 and 0.4 premium point higher for age 90. This means that overall Gompertz-Makeham estimates higher premiums than Perks for every age and for every year. And the difference in premiums increases more with age moving towards 67 and then starts to decrease again. This can also be seen from figure 7.2.1.

The question is which model should we go for? We can look at an example. Consider a female aged 30, 60 and 90. Using 7.3.1 we will calculate the reserves for both the models and compare the difference. Let benefit = $1000, administrative cost = 3%, single premium for Perks = 1 for each age, single premium for Gompertz-Makeham = 1.3, 2 and 1.4. Total reserve for Perks equals $1030 and $1339 for Gompertz-Makeham for age 30. For age 60 we get $1030 and $2060, and for age 90 we get $1030 and $1442. It was a very simple example with very small values, but there is a striking difference in total reserve for the two models. In real world example, a slight difference in the premium can have such a huge impact when we consider big companies and big portfolios. We believe that it is good to have more in the reserve than to have too little. Therefore Gompertz-Makeham seems to be a better alternative.

7.5 Management of mortaltiy risk

This section gives a brief insight on how to manage and cop with mortality risk. What measures pension funds and other insurance institutes can take to reduce or eliminate the economic affect of mortality risk in the future. One of the key problems facing annuity providers is mortality risk, the risk of underestimating mortality improvements. We have seen two models which give different single premiums and different future mortality trends. Thus the question of choosing the right survival model to explain current mortality trends and forecast future life expectancy is of utmost importance for risk management and valuation of insurance portfolios.

Mortality risk can be significant for financial institutions such as life assurers and pension plans. It might not be the largest risk they face, but it is often significant and one that cannot be ignored. We have assembled a few points under which we feel can be used for managing mortality risk:

- They can enter into a variety of forms of full or partial reinsurance, in order to hedge downside mortality risk.
- Assurers can diversify their mortality risk across product ranges, regions and socioeconomic groups.
- Pension plans can arrange a full or partial buyout of their liabilities by a specialist insurer. Small pension plans in the UK are exposed to considerable non-systematic mortality risk and often, therefore, purchase annuities from a life office for employees at the time of their retirement, thereby removing the tail mortality risk.
- Survivor bonds can be used in helping to hedge mortality risk
- Increase the premium on the products in order to cop with future mortality risk. The more reserve, the better.

Unfortunately, due to lack of time, not much emphasis was put into this section. There are many ways to tackle and manage the mortality risk. The one mentioned above was just a brief idea which we felt need to be introduced as we are forecasting future mortality trend.

8. Conclusion

In this chapter we conclude our work with a summary of the thesis. Known weaknesses of our work are presented. Future work ideas are also discussed in the end.

8.1 Summary and conclusion

In this thesis two dynamic and stochastic survival models are discussed. The objective of this thesis was to test the two models and decide which one is most suitable for Europe. Each model is first run independently on past data for all the countries considered. The mortality trends from the past are simulated and compared to the historic data in order to test the goodness of fit. The tested models are then used to forecast mortality trends in the future. This involves the forecasting of future survival rates and life expectancies at birth. Finally, single premiums are estimated for each model. The results of the two models have been compared along the way to see which model suits best for Europe.

At the minimum, a good model should be consistent with historical patterns of mortality. If that is not the case, much greater doubt must be placed on the validity of any forecasts produced by the model. Our testing of the models on past data revealed that apart from the early age, Gompertz-Makeham fits well and is consistent with the historical patterns of mortality. Perks on the other hand is not able to capture the historical mortality trend that well. It is clear that Perks model under estimates for the young and old age group and over estimates for the middle age. The most important age group in actuarial context is the middle age group, one that needs to be most accurately calculated, but that is exactly where Perks fails. By over estimating the death rate, Perks automatically under estimates the survival rate for the middle age group. Results indicate that Perks mortality model may not be effective in capturing the patterns of decline in mortality in European countries accurately. Therefore, the use of Perks models for predicting future mortality trends is likely to be unwarranted, potentially leading to serious financial, economic and demographic miscalculations.

Having also looked at the future mortality trends forecasted by the two models, we can say that Perks model seem to be unpredictable. It has the tendency to act strangely in certain occasions. This is best observed in the life expectancy plots. While Gompertz-Makeham follows a general pattern for each country, Perks estimates may vary from country to country.

Gompertz-Makeham in general gives higher premiums, thereby calculating better reserves, and thus reducing the future morality risk. Perks on the other hand gives lower premiums and show less improvements in mortality in the future. And knowing its behavior from the past mortality trends, it can not be fully trusted. Having said that, some points are worth mentioning. Both models do have close estimates and both reveal the same picture of the future mortality trends. In case of Russia, Perks seemed to handle the data way better than Gompertz-Makeham.

Our analysis might suggest that Gompertz-Makeham model is satisfactory, but further forensic investigation might reveal some pitfalls that need corrective work. We have a case of over estimation for the early age group which needs to be considered. There are still considerable challenges ahead. The existing model needs further refinement in order to manage mortality risk in the most effective way.

In conclusion, the results from this thesis suggest that Gompertz-Makeham is a more reliable survival model. The extra parameter in Gompertz-Makeham model makes it a more flexible and reliable than Perks. Having survived for 150 years, one can only say that the model is good.

8.2 Known weaknesses

One thing that might have helped in claiming better results is the use of drift μ effectively. In our analysis, we have used a constant drift for every year to calculate the future morality trends. Had we changed the drift for every year, we might have got better estimates. This is thought to be one of the weaknesses in our estimation.

8.3 Future work

Here we outline the ideas and plans for the future work. We look at the things that were planned but could not be completed due to limited time, therefore can be considered as future work.

Firslty, we wanted to check if mortality trend in one country or in one gender had any effect on the trend of another country or gender. Future work should analyze correlation as it would be interesting to see the linear relationships between European countries and genders which can be a useful study for insurance companies which operate throughout Europe.

Secondly, we touched slightly on the topic: management of mortality risk. It is a broad field and one which is very important for both the insurance companies and pension planners. Therefore as a future work one can look at swaps, survivor bonds etc by using our work.

Finally, future work should consist of solidification of the work presented in this thesis. One element to look at is the effect of changing the drift for every year.

Appendix A

Tables showing drifts for all the countries considered.

Gompertz-Makeham

	Female			Male		
Country	μ_1	μ_2	μ_3	μ_1	μ_1	μ_3
East Germany	-3.45E-05	-4.54E-08	-4.88E-05	-5.12E-05	3.67E-07	-3.26E-04
West Germany	-2.69E-05	-9.04E-08	7.71E-05	-4.32E-05	-1.19E-07	-9.23E-05
Russia	-3.02E-05	3.42E-07	-1.99E-04	-7.11E-05	1.58E-05	-6.44E-04
England&Wales	-2.38E-05	-1.37E-07	5.13E-05	-2.82E-05	-2.99E-07	1.26E-05
France	-4.36E-05	-1.20E-07	1.12E-04	-6.10E-05	4.09E-08	-1.87E-04
Italy	-5.40E-05	-5.89E-08	1.18E-06	-6.67E-05	1.59E-08	-1.57E-04
Spain	-7.17E-05	-1.46E-07	3.11E-04	-9.03E-05	-2.05E-08	-1.30E-04

Table1: μ_i, the mean of the difference between Gompertz-makeham parameters, for male and female mortality for ages 0-111 in 7 countries.

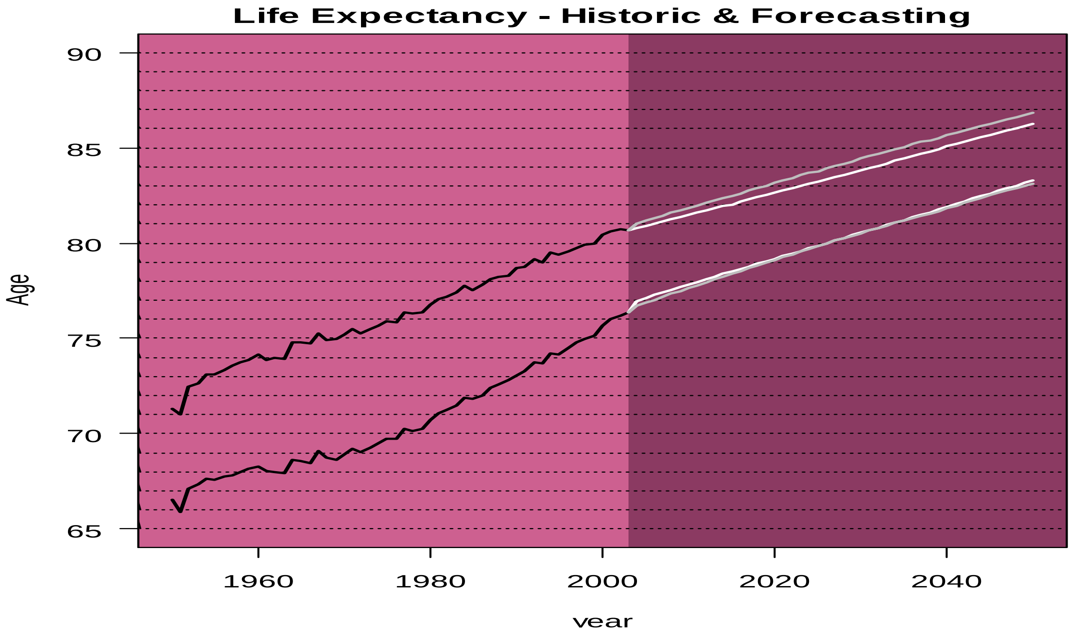
Perks

	Female		Male	
Country	μ_1	μ_2	μ_1	μ_2
East Germany	-4.02E-02	3.26E-04	-1.73E-02	9.66E-05
West Germany	-2.24E-03	-1.76E-04	-4.14E-03	-1.49E-04
Russia	-3.60E-02	5.71E-04	-1.45E-02	3.70E-04
England&Wales	-1.80E-02	7.46E-05	-1.45E-02	1.59E-05
France	-4.78E-02	4.54E-04	-1.80E-02	5.86E-05
Italy	-7.26E-02	7.37E-04	-4.48E-02	4.84E-04
Spain	-5.72E-02	5.99E-04	-5.72E-02	5.99E-04

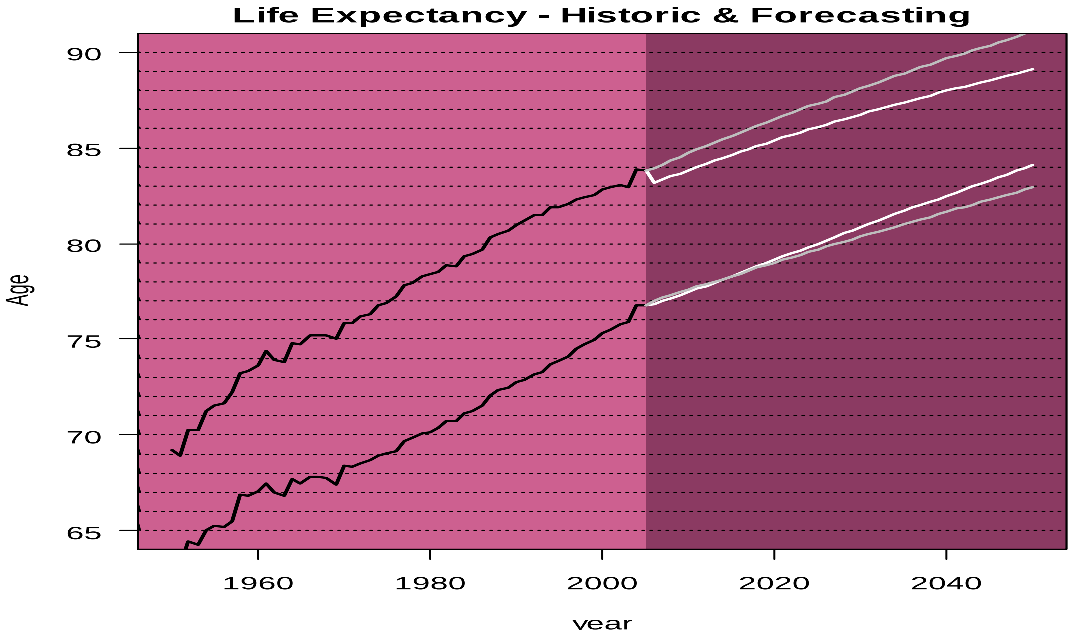
Table1: μ_i, the mean of the difference between Perks parameters, for male and female mortality for ages 0-111 in 7 countries.

Life Expectancy plots:

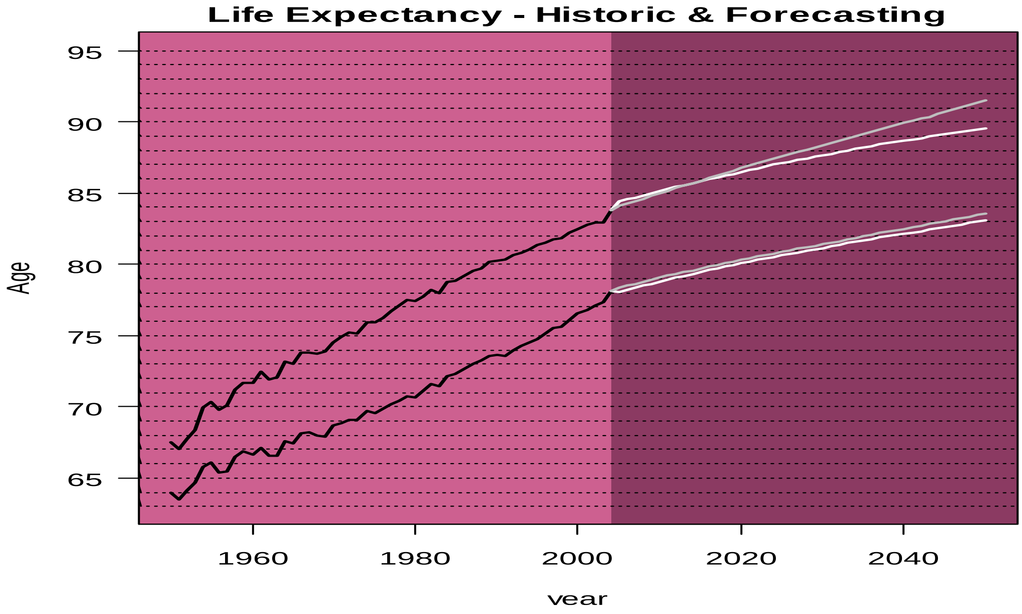
England&Wales:



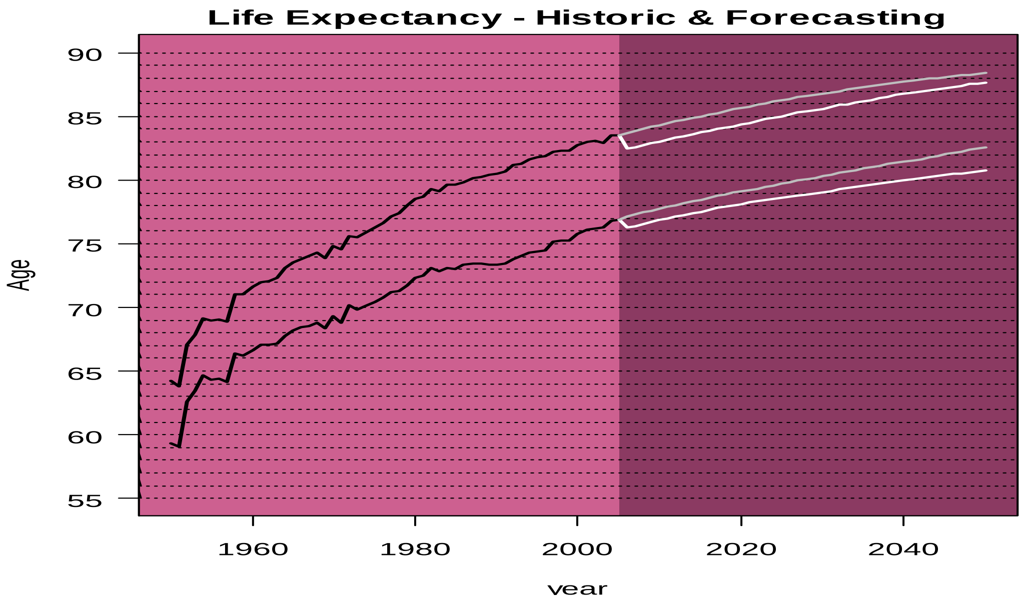
France:



Italy:


Life Expectancy - Historic & Forecasting
Age
95
90
85
80
75
70
65
1960
1980
2000
2020
2040
year

Spain:


Life Expectancy - Historic & Forecasting
Age
90
85
80
75
70
65
60
55
1960
1980
2000
2020
2040
year

Appendix B

CODE:
This is the code for East Germany female. Mainly the code is for Gomepertz-Makeham, but Perks modifications are added. As both the models follow similar routines, the whole of Perks code is not shown. Just the model description and a few other details included. For male and for all the other countries we have similar code with just a few modifications and therefore not included in the Appendix.

```
##FEMALE##
#historic plot
q=read.table("EG1956f.txt")
w=read.table("EG1976f.txt")
e=read.table("EG1996f.txt")
r=read.table("EG2004f.txt")


bind=cbind(q$V1,w$V1,e$V1,r$V1)
par(mfrow=c(2,1))
matplot(bind[,1:4],type="l",ylab="Death
probability",xlab="Age",main="Female historic data",lty=1)
cols = c("black","red","green","blue")
legend("topleft",c("1956","1976","1996","2004"),fill=cols)

#Getting the estimates
fem=read.table("gompEGfemale.txt")


#Finding death probability
Nq.fem=matrix(NA,49,111)
k=1
for(t in 1:49)
{
      for(x in 0:110)
      {
            Nq.fem[t,x+1]=1-exp((-fem[t,2]*k)-
((fem[t,3]/fem[t,4])*(exp(fem[t,4]*(k+x))-exp(fem[t,4]*x))))

      }
}

#Plotting the mortality curve
R=cbind(Nq.fem[1,],Nq.fem[21,],Nq.fem[41,],Nq.fem[49,])
matplot(R[,1:4],type ="l",ylab="Death probability",xlab="Age",main="Female
estimated data",lty=1)
cols = c("black","red","green","blue")
legend("topleft",c("1956","1976","1996","2004"),fill=cols)

#Dividing into four age groups
R=cbind(Nq.fem[1,],Nq.fem[21,],Nq.fem[41,],Nq.fem[49,],q,w,e,r)
par(mfrow=c(2,2))
age1=seq(5,25,by=1)
age2=seq(26,50,by=1)
age3=seq(51,75,by=1)
age4=seq(76,111,by=1)
matplot(age1,R[5:25,],type ="l", xlab="Age", ylab="Death
probability",lty=1)
matplot(age2,R[26:50,],type ="l", xlab="Age", ylab="Death
probability",lty=1)
```

```
matplot(age3,R[51:75,],type ="l", xlab="Age", ylab="Death
probability",lty=1)
matplot(age4,R[76:111,],type ="l", xlab="Age", ylab="Death
probability",lty=1)
cols = c("black","red","green","blue")
legend("topleft",c("1956","1976","1996","2004"),fill=cols)


#MALE + FEMALE#
T=cbind(Nq.fem[1,],Nq.male[1,])
par(mfrow=c(2,2))
matplot(T[,1:2],type ="l",ylab="Death probability",xlab="Age",main="Male
vs Female",lty=1)
cols = c("black","red")
legend("topleft",c("female 1956","male 1956"),fill=cols)

T=cbind(Nq.fem[21,],Nq.male[21,])
matplot(T[,1:2],type ="l",ylab="Death probability",xlab="Age",main="Male
vs Female",lty=1)
cols = c("black","red")
legend("topleft",c("female 1976","male 1976"),fill=cols)

T=cbind(Nq.fem[41,],Nq.male[41,])
matplot(T[,1:2],type ="l",ylab="Death probability",xlab="Age",main="Male
vs Female",lty=1)
cols = c("black","red")
legend("topleft",c("female 1996","male 1996"),fill=cols)

T=cbind(Nq.fem[49,],Nq.male[49,])
matplot(T[,1:2],type ="l",ylab="Death probability",xlab="Age",main="Male
vs Female",lty=1)
cols = c("black","red")
legend("topleft",c("female 2004","male 2004"),fill=cols)


#Observed vs Estimated plots
#Female
par(mfrow=c(2,2))
plot(log(q$V1),xlab="age",ylab="log(qtx)",main="1956")
points(log(Nq.fem[1,]),type="l",col="red")
plot(log(w$V1),xlab="age",ylab="log(qtx)",main="1976")
points(log(Nq.fem[21,]),type="l",col="red")
plot(log(e$V1),xlab="age",ylab="log(qtx)",main="1996")
points(log(Nq.fem[41,]),type="l",col="red")
plot(log(r$V1),xlab="age",ylab="log(qtx)",main="2004")
points(log(Nq.fem[49,]),type="l",col="red")

#Male + Perks estimates combined
par(mfrow=c(2,2))
plot(log(a$V1),xlab="age",ylab="log(qtx)",main="1956")
points(log(Nq.male[1,]),type="l",col="red")
#points(log(Pq.male[1,]),type="l",col="blue")
plot(log(s$V1),xlab="age",ylab="log(qtx)",main="1976")
points(log(Nq.male[21,]),type="l",col="red")
#points(log(Pq.male[21,]),type="l",col="blue")
plot(log(d$V1),xlab="age",ylab="log(qtx)",main="1996")
points(log(Nq.male[41,]),type="l",col="red")
#points(log(Pq.male[41,]),type="l",col="blue")
plot(log(f$V1),xlab="age",ylab="log(qtx)",main="2004")
points(log(Nq.male[49,]),type="l",col="red")
#points(log(Pq.male[49,]),type="l",col="blue")


#Correlation
malepar=male$V2,male$V3,male$V4
fempar=fem$V2,fem$V3,fem$V4
```

```
correlationEG=cor(malepar,fempar)


#Percentiles
#Female
y=49
p.fem1=array(0,y)
p.fem2=array(0,y)
p.fem3=array(0,y)

a=111
pkx=function(x)
{
      P=1-q.fem[x,1]
      for(i in 1:a)
      {
            if(P>0.5){
                  P=P*(1-q.fem[x,1+i])
            }
            else{
                  break
            }
      }
      return(i)
}

for(year in 1:y)
{
      p.fem1[year]= pkx(year)

}

#Plot
year=seq(1956,2005,by=1)
d=p.fem1
e=p.fem2
f=p.fem3

fbind=cbind(d,e,f)
par(mfrow=c(1,2))
matplot(year,fbind[,1:3],ylim=c(20,100),ylab="age",type
="l",lty=1,main="Female death pop. %tile")
cols = c("black","red","green")
legend("bottomright",c("20%","50%","80%"),fill=cols)


#percentage decrease in death probability
a=q.fem[,21]
b=q.fem[,41]
c=q.fem[,61]
d=q.fem[,81]
pa=array(NA,49)
pb=array(NA,49)
pc=array(NA,49)
pd=array(NA,49)

pa[1]=0
for(i in 1:48)
{
      pa[i+1]=((a[i+1]-a[1])/a[1])*100
}
pb[1]=0
for(i in 1:48)
{
      pb[i+1]=((b[i+1]-b[1])/b[1])*100
}
```

```
pc[1]=0
for(i in 1:48)
{
      pc[i+1]=((c[i+1]-c[1])/c[1])*100
}
pd[1]=0
for(i in 1:48)
{
      pd[i+1]=((d[i+1]-d[1])/d[1])*100
}

year=array(seq(1956,2004,by=1),c(49,1))
percentage=cbind(pa,pb,pc,pd)
matplot(year,percentage[,1:4],ylim=c(-1,1),ylab="% of mu1956",type
="l",lty=1)
cols = c("black","red","green","blue")
legend("topright",c("20","40","60","80"),fill=cols)


#Forecasting future mortality trends
#Female
#Estimation
fem=read.table("gompEGfemale.txt")

#Finding mu
Fmu1=mean(diff(log(fem[,2])))
Fmu2=mean(diff(log(fem[,3])))
Fmu3=mean(diff(log(fem[,4])))

#Finding A1, A2 and A3
y=46
Fa1=array(NA,y+1)
Fa1[1]=log(fem[49,2])
Fa2=array(NA,y+1)
Fa2[1]=log(fem[49,3])
Fa3=array(NA,y+1)
Fa3[1]=log(fem[49,4])

a1=diff(log(fem$V2))
b1=diff(log(fem$V3))
c1=diff(log(fem$V4))
d1=cbind(a1,b1,c1)
covF=cov(d1)
mean=c(0,0,0)

Ef=array(NA,c(1000,3))

for(i in 1:y)
{
      for(a in 1:1000)
      {
            Ef[a,]=rmvnorm(1,mean,covF)
      }
      Ef1=mean(Ef[,1])
      Ef2=mean(Ef[,2])
      Ef3=mean(Ef[,3])
      Fa1[i+1]=Fmu1+Fa1[i]+Ef1
      Fa2[i+1]=Fmu2+Fa2[i]+Ef2
      Fa3[i+1]=Fmu3+Fa3[i]+Ef3
}

Fa=cbind(Fa1,Fa2,Fa3)
expFa=exp(Fa)
```

```
#For Perks
mPFa1=array(NA,c(y+1,1000))
mPFa1[1,]=-Pfem[54,2]
mPFa2=array(NA,c(y+1,1000))
mPFa2[1,]=Pfem[54,3]

Pa1=diff(-Pfem$V2)
Pb1=diff(Pfem$V3)
Pc1=cbind(Pa1,Pb1)
PcovF=cov(Pc1)
Pmean=c(0,0)
for(s in 1:1000)
{
      for(i in 1:y)
      {
            Ef=rmvnorm(1,Pmean,PcovF)
            mPFa1[i+1,s]=PFmu1+mPFa1[i,s]+Ef[1]
            mPFa2[i+1,s]=PFmu2+mPFa2[i,s]+Ef[2]
      }
}

meanPFa1=array(NA,48)
meanPFa2=array(NA,48)

#Ploting the parameter distribution
par(mfrow=c(2,2))
hist(mPFa1[2,],main="Perks first parameter",xlab="Theta1")
hist(mPFa2[2,],main="Perks second parameter",xlab="Theta2")


for(m in 1:48)
{
      meanPFa1[m]=mean(mPFa1[m,])
      meanPFa2[m]=mean(mPFa2[m,])
}


#Finding death probability
q.fem=matrix(NA,y,111)
k=1
for(t in 1:y)
{
      for(x in 0:110)
      {
            q.fem[t,x+1]=1-exp((-expFa[t,1]*k)-
((expFa[t,2]/expFa[t,3])*(exp(expFa[t,3]*(k+x))-exp(expFa[t,3]*x))))

      }
}

test=matrix(NA,t,111)
t=50
qkx=function(k,x,sex)
{

}

for(age in 0:110)
{
      for(i in 1:t)
      {
            test[i,age+1]=1-exp(-(fem[1,2]+fem[1,3]*exp(fem[1,4]*x)))

      }
}
```

```
#Transfering qkx to tQx array
tQx=array(1,c(92,92))

for(i in 1:92)
{
      k=18+i
      for(j in 1:92)
      {
            if(j+k>111)
                  break
            else
                  tQx[i,j]=qx[1,j+k]
      }
}

#Survival Probability
tPx=array(0,c(92,92))
tPx[,1]=1

for(x in 1:92)
{
      for(t in 2:92)
      {
            tPx[x,t]=tPx[x,t-1]*(1-tQx[x,t-1])
      }
}

#Single Premium
EP2=array(0,92)
r=0.04

for(x in 1:91)
{
      lo=x+19
      lr=67
      t=max(lr-lo,1)
      l=91
      P=tPx[x,t]
      Premie=P/((1+r)^t)
      for(i in t:l)
      {
            Premie=Premie+((tPx[x,i+1])/((1+r)^(i+1)))
      }
      EP2[x]=Premie
}

}

meanEP=array(NA,92)
for(i in 1:92)
{
      meanEP[i]=mean(EP[,i])
}

varEP=array(NA,92)
for(i in 1:92)
{
      varEP[i]=var(EP[,i])
}

#matplot
age=matrix(seq(20,111,by=1),92,10)
Tage=t(age)
```

```
#EPbind=cbind[]
par(lwd=1)
matplot(Tage,EP,ylab="Premium",type ="l",lty=1)
par(lwd=2)
age2=array(seq(20,111,by=1),c(92,1))
lines(age2,meanEP)

#per year
age=matrix(seq(20,111,by=1),92,1)
EPpy=cbind(EP1,EP2,EP3,EP4,EP5)
matplot(age,EPpy,xlab="age",ylab="Premium",type ="l",main="One-time
premium",lty=1)
cols = c("black","red","green","blue","cyan")
legend("topright",c("2005","2015","2025","2035","2050"),fill=cols)


#Expected life for 20-111
ExpLife4=array(0,92)
for(x in 1:92)
{
      ExpLife4[x]=sum(tPx[x,])
}

age=array(seq(20,111,by=1),c(1,92))
plot(age,ExpLife3,type="l",ylab="Life Expectancy",col="red")

#Life expectancy at birth
#female
tPa=array(0,c(46,111))
tPa[,1]=1

for(a in 1:46)
{
      for(t in 2:111)
      {
            tPa[a,t]=tPa[a,t-1]*(1-q.fem[a,t-1])
      }
}

ExpLifeGMf=array(0,46)
for(x in 1:46)
{
      ExpLifeGMf[x]=sum(tPa[x,])
}



#Forecasted life expectancy plots
require(gplots)

set.seed(120)

# compute the limits of the graph
ylim <- c(65,90)

# prepare the space where to plot
opar <- par(mar=c(4,4,2,2),las=1)

year=array(seq(1956,2050,by=1),c(95,1))

plot(year,EX1,ylim=ylim,type="n",ylab="Age",main="Life Expectancy -
Historic & Forecasting")
usr <- par("usr")

# split the figure in two parts
#   - the part used to fit the model
```

```
rect(usr[1],usr[3],2004,usr[4],border=NA,col="hotpink3")

#   - the part used to make the forecast
rect(2004,usr[3],usr[2],usr[4],border=NA,col="hotpink4")

abline(h=(65:90), col ="black" , lty =3)

lines(1956:2004,EX1[1:49],lwd=2 )
lines(1956:2004,EX2[1:49],lwd=2)
lines(2004:2050,EX1[49:95],lwd=2,col ="white")
lines(2004:2050,EX3[49:95],lwd=2,col ="gray")
lines(2004:2050,EX2[49:95],lwd=2,col ="white")
lines(2004:2050,EX4[49:95],lwd=2,col ="gray")

box()


ELf=read.table("ELfem.txt")
ELm=read.table("ELmale.txt")

ELfem=array(0,0,c(49,1))
ELmale=array(0,0,c(49,1))
for(i in 1:49)
{
      ELfem[i]=ELf[i,]
      ELmale[i]=ELm[i,]
}

EX1=array(0,c(95,1))
for(i in 1:49)
{
      EX1[i]=ELfem[i]
}
for(i in 50:95)
{
      EX1[i]=ExpLifePf[i-49]
}

EX2=array(0,c(95,1))
for(i in 1:49)
{
      EX2[i]=ELmale[i]
}
for(i in 50:95)
{
      EX2[i]=ExpLifePm[i-49]
}

EX3=array(0,c(95,1))
for(i in 1:49)
{
      EX3[i]=ELfem[i]
}
for(i in 50:95)
{
      EX3[i]=ExpLifeGMf[i-49]
}

EX4=array(0,c(95,1))
for(i in 1:49)
{
      EX4[i]=ELmale[i]
}
for(i in 50:95)
{
      EX4[i]=ExpLifeGMm[i-49]
```

```
}


#Goodness of fit for the rest of the countries
par(mfcol=c(2,3))
#FRANCE
plot(log(Fw$V1),xlab="age",ylab="log(qtx)",main="France-1970")
points(log(Fq.fem[21,]),type="l",col="red")
points(log(PFq.fem[21,]),type="l",col="green")
plot(log(Fr$V1),xlab="age",ylab="log(qtx)",main="France-2005")
points(log(Fq.fem[56,]),type="l",col="red")
points(log(PFq.fem[56,]),type="l",col="green")

#SPAIN
plot(log(Sw$V1),xlab="age",ylab="log(qtx)",main="Spain-1970")
points(log(Sq.fem[21,]),type="l",col="red")
points(log(PSq.fem[21,]),type="l",col="green")
plot(log(Sr$V1),xlab="age",ylab="log(qtx)",main="Spain-2005")
points(log(Sq.fem[56,]),type="l",col="red")
points(log(PSq.fem[56,]),type="l",col="green")

#ITALY
plot(log(Iw$V1),xlab="age",ylab="log(qtx)",main="Italy-1970")
points(log(Iq.fem[21,]),type="l",col="red")
points(log(PIq.fem[21,]),type="l",col="green")
plot(log(Ir$V1),xlab="age",ylab="log(qtx)",main="Italy-2004")
points(log(Iq.fem[55,]),type="l",col="red")
points(log(PIq.fem[55,]),type="l",col="green")

par(mfcol=c(2,3))
#ENGLAND&WALES
plot(log(EWw$V1),xlab="age",ylab="log(qtx)",main="England&Wales-1970")
points(log(EWq.fem[21,]),type="l",col="red")
points(log(PEWq.fem[21,]),type="l",col="green")
plot(log(EWr$V1),xlab="age",ylab="log(qtx)",main="England&Wales-2003")
points(log(EWq.fem[54,]),type="l",col="red")
points(log(PEWq.fem[54,]),type="l",col="green")

#WEST GERMANY
plot(log(w$V1),xlab="age",ylab="log(qtx)",main="West Germany-1976")
points(log(q.fem[21,]),type="l",col="red")
points(log(Pq.fem[21,]),type="l",col="green")
plot(log(r$V1),xlab="age",ylab="log(qtx)",main="West Germany-2004")
points(log(q.fem[49,]),type="l",col="red")
points(log(Pq.fem[49,]),type="l",col="green")

#RUSSIA
plot(log(Rw$V1),xlab="age",ylab="log(qtx)",main="Russia-1974")
points(log(Rq.fem[16,]),type="l",col="red")
points(log(PRq.fem[16,]),type="l",col="green")
plot(log(Rr$V1),xlab="age",ylab="log(qtx)",main="Russia-2006")
points(log(Rq.fem[48,]),type="l",col="red")
points(log(PRq.fem[48,]),type="l",col="green")

#Mortality curves for Europe
par(mfrow=c(2,3))
matplot(bind2[1:90,],type="l",ylab="Death
probability",xlab="Age",lty=1,main="West Germany")
matplot(bind3[1:90,],type="l",ylab="Death
probability",xlab="Age",lty=1,main="Spain")
matplot(bind4[1:90,],type="l",ylab="Death
probability",xlab="Age",lty=1,main="Russia")
matplot(bind5[1:90,],type="l",ylab="Death
probability",xlab="Age",lty=1,main="Italy")
matplot(bind6[1:90,],type="l",ylab="Death
probability",xlab="Age",lty=1,main="France")
```

```
matplot(bind7[1:90,],type="l",ylab="Death
probability",xlab="Age",lty=1,main="England&Wales")


#PERKS
#Estimation
Pfem=read.table("perksEGfemale.txt")
par(mfrow=c(2,1))
year=array(seq(1956,2004,by=1),c(1,49))
plot(year,-Pfem[,2],xlab="Year, t",ylab="A-1(t)",type="o")
plot(year,Pfem[,3],xlab="Year, t",ylab="A-2(t)",type="o")

#Finding death probability
a=111
Pq.fem=matrix(NA,49,a)

qkx=function(k,x,sex)
{
      exp(-sex[k,2]+((x+1)*sex[k,3]))/(1+exp(-sex[k,2]+((x+1)*sex[k,3])))
}

for(i in 1:49)
{
      for(age in 0:(a-1))
      {
            Pq.fem[i,age+1]=qkx(i,age,Pfem)

      }
}

R=cbind(Pq.fem[1,],Pq.fem[21,],Pq.fem[41,],Pq.fem[49,])
#par(mfrow=c(2,1))
matplot(R[,1:4],type ="l", xlab="Age", ylab="Death
probability",main="Female estimated data",lty=1)
cols = c("black","red","green","blue")
legend("topleft",c("1956","1976","1996","2004"),fill=cols)

#Forecasting
#Female
#Estimation
Pfem=read.table("perksEGfemale.txt")

#Finding mu
PFmu1=mean(diff(-Pfem[,2]))
PFmu2=mean(diff(Pfem[,3]))


#Finding A1 and A2
y=46
PFa1=array(NA,y+1)
PFa1[1]=-Pfem[49,2]
PFa2=array(NA,y+1)
PFa2[1]=Pfem[49,3]

Pa1=diff(-Pfem$V2)
Pb1=diff(Pfem$V3)
Pc1=cbind(Pa1,Pb1)
PcovF=cov(Pc1)
Pmean=c(0,0)

Ef=array(NA,c(1000,2))
for(i in 1:y)
{
      for(a in 1:1000)
      {
            Ef[a,]=rmvnorm(1,Pmean,PcovF)
```

```
        }
        Ef1=mean(Ef[,1])
        Ef2=mean(Ef[,2])
        PFa1[i+1]=PFmu1+PFa1[i]+Ef1
        PFa2[i+1]=PFmu2+PFa2[i]+Ef2
}

PFa=cbind(PFa1,PFa2)


#Finding death probability
a=111
Pq.fem=matrix(NA,y,a)

qkx=function(k,x,sex)
{
        exp(sex[k,1]+((x+1)*sex[k,2]))/(1+exp(sex[k,1]+((x+1)*sex[k,2])))
}

for(i in 1:y)
{
        for(age in 0:(a-1))
        {
                Pq.fem[i,age+1]=qkx(i,age,PFa)

        }
}
```

References

1) MacDonald A.S, A.J.G Cairns, P.L Gwilt and K.A Miller (1998), *An International comparison of recent trends in population mortality*, British Actuarial Journal, 4: 3–141

2) Ruttermann M (1999), Mortality trends worldwide, *Risk Insights*, General & Cologne RE, 3 (4): 18–20

3) Benjamin B and J H Pollard (1993), *The analysis of mortality and other actuarial statistics*, The Institute of Actuaries, Oxford.

4) Ermanno Pitacco, *Survival models in actuarial mathematics: From Halley to longevity risk.*

5) Mikkel Dahl, Martin Melchior and Thomas Møller, *On systematic mortality risk and risk-minimization with survivor swaps*, Scandinavian Actuarial Journal, 2008, 2-3, 114-146

6) Hans U.Gerber, *Life Insurance Mathematics*: Third edition

7) Ermanno Pitacco, Survival *models in a dynamic context: a survey*. Insurance: Mathematics & Economics volume 35, issue 2.

8) Knut Haakon Brox, *Forelesninger I Livforsikringsmatematikk*, August 1999.

9) Andrew J.G.Cairns. David Blake and Kevin Dowd, *Modelling and management of mortality risk: a review*, Scandinavian Actuarial Journal, 2008, 2-3, 79-113

10) John A. Rice, *Mathematical Statistics and Data Analysis*, Second Edition.

11) A Cairns, D Blake, K Dowd, *A Two-Factor model for Stochastic Mortality with Parameter Uncertainty*. Center for Risk & Insurance Studies.

12) Søren Fiig Jarner, Esben Masotti Kryger and Chresten Dengsøe, *The evolution of death rates and life expectancy in Denmark*, Scandinavian Actuarial Journal, 2008, 2-3, 147-173