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Low-dimensional Cox-Ingersoll-Ross process

Yuliya Mishura[a,b], Andrey Pilipenko[c,d] and Anton Yurchenko-Tytarenko[e]

[a]Department of Probability, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine; [b]Division of Mathematics and Physics, Mälardalen University, Västerås, Sweden; [c]Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, Ukraine; [d]Igor Sikorsky Kyiv Polytechnic Institute, Kyiv, Ukraine; [e]Department of Mathematics, University of Oslo, Oslo, Norway

ABSTRACT

The present paper investigates Cox-Ingersoll-Ross (CIR) processes of dimension less than 1, with a focus on obtaining an equation of a new type including local times for the square root of the CIR process. To derive this equation, we utilize the fact that non-negative diffusion processes can be obtained by the transformation of time and scale of a certain reflected Brownian motion. The equation mentioned above turns out to contain a term characterized by the local time of the corresponding reflected Brownian motion. Additionally, we establish a new connection between low-dimensional CIR processes and reflected Ornstein-Uhlenbeck (ROU) processes, providing a new representation of Skorokhod reflection functions.

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1. Introduction

1.1. Background and motivation

The squared Bessel process

$$X(t) = x_0 + at + 2\int_0^t \sqrt{X(s)}\,\mathrm{d}W(s) \tag{1}$$

as well as its generalization Cox-Ingersoll-Ross (CIR) process

$$X(t) = x_0 + \int_0^t (a - bX(s))\,\mathrm{d}s + \sigma\int_0^t \sqrt{X(s)}\,\mathrm{d}W(s) \tag{2}$$

where $x_0 \geq 0$, $a, \sigma > 0$, $b \in \mathbb{R}$, and their respective square roots are widely used in various fields, in particular physics (see e.g. [8,20] and the overview in [17, Section I]) and finance [11–13,19]. One of the reasons for the popularity of these processes lies in the well-known fact (see e.g. [21, Chapter IV, Example 8.2]) that $a > 0$ in (2) implies that $X(t) \geq 0$ for all $t \geq 0$ with probability 1, which is a natural property for multiple real-life phenomena. Furthermore, if the *Feller condition* $2a \geq \sigma^2$ is satisfied, the paths of X in (2) are strictly positive

CONTACT Anton Yurchenko-Tytarenko antony@math.uio.no Department of Mathematics, University of Oslo, Moltke Moes vei 35, 0851 Oslo, Norway

a.s., which turns out to be very useful in multiple cases. For example, the well-known Heston model [19] utilizes $Y := \sqrt{X}$ as stochastic volatility and, under the Feller condition, Y has the dynamics of the form

$$Y(t) = \sqrt{x_0} + \frac{1}{2}\int_0^t \left(\frac{a - \frac{\sigma^2}{4}}{Y(s)} - bY(s) \right) \mathrm{d}s + \frac{\sigma}{2} W(t), \quad t \geq 0, \tag{3}$$

since it is evident that

$$\int_0^t \frac{1}{Y(s)} \mathrm{d}s < \infty \tag{4}$$

with probability 1 for all $t \geq 0$. This equation can be used for e.g. simulation purposes (see, for example, [1,15,28]); moreover, the measure change procedure associated with the Heston model naturally involves the inverse volatility $1/Y$ which has far more transparent properties when $X > 0$ a.s.

At the same time, empirical considerations indicate that the Feller condition $2a \geq \sigma^2$ can sometimes be too restrictive and models perform better when it is not satisfied. For instance, [23, Section 3.4] reports that the joint SPX-VIX fit of the Heston model turns out to be substantially better when the Feller condition is not demanded from the model parameters. Additionally, [2, Example 10.2.6] indicates that the Heston model with violated Feller condition can reproduce the upward VIX 'smirk'. In other words, there are cases when the process $Y = \sqrt{X}$ under relatively small values of a turns out to be more relevant for reflecting real-life phenomena despite the associated analytical challenges. Nevertheless, the majority of sources in the literature pay more attention to the case when the Feller condition is satisfied. Among notable exceptions, we mention [4,7,9,10] which discussed the SDEs of the type (3) when $\frac{\sigma^2}{4} < a < \frac{\sigma^2}{2}$. It is worth to note more recent papers [18,27] which establish a connection between $Y = \sqrt{X}$ and a reflected Ornstein-Uhlenbeck (ROU) process

$$Y_0(t) = \sqrt{x_0} - \frac{b}{2}\int_0^t Y_0(s)\,\mathrm{d}s + \frac{\sigma}{2} W(t) + L_0(t),$$

where L_0 is the corresponding Skorokhod reflection function, i.e. a continuous non-decreasing process that has points of growth exclusively when $Y_0(t) = 0$ and such that $Y_0(t) \geq 0$. In particular, it is established that $Y_0 = \sqrt{X}$ when $a = \frac{\sigma^2}{4}$ in (2). Additionally, [27, Theorem 2.4] provides a new representation of L_0 in terms of a limit of the CIR processes: with probability 1, for any positive sequence $\{\varepsilon_n,\ n \geq 1\}$ such that $\varepsilon_n \downarrow 0$, $n \to \infty$, and for all $T > 0$

$$\sup_{t \in [0,T]} \left| L_0(t) - \frac{1}{2}\int_0^t \frac{\varepsilon_n}{\sqrt{X_{\varepsilon_n}(s)}} \mathrm{d}s \right| \to 0, \quad n \to \infty, \tag{5}$$

where

$$X_{\varepsilon_n}(t) = x_0 + \int_0^t \left(\frac{\sigma^2}{4} + \varepsilon_n - bX_{\varepsilon_n}(s) \right) \mathrm{d}s + \sigma \int_0^t \sqrt{X_{\varepsilon_n}(s)}\,\mathrm{d}W(s).$$

The representation of L_0 from [27] described above essentially concerns convergence of the CIR square roots as $a \to \frac{\sigma^2}{4}+$ and does not cover what happens when $a \to \frac{\sigma^2}{4}-$. The reason is that analytic challenges associated to the process $Y = \sqrt{X}$ are especially acute when $0 < a < \frac{\sigma^2}{4}$, i.e. when the *dimension* (see e.g. [24]) $k := \frac{4a}{\sigma^2}$ of the process (2) is less than 1. Indeed, the integral in (4) is infinite after the first moment of hitting zero, the representation (3) does not hold and, furthermore, the process $Y = \sqrt{X}$ *is not a semimartingale* (see e.g. Example 1.2 and Appendix 1 in [25] or [16, p. 100]). In this regard, one must mention important contributions [5,6] which shed light on the behaviour of $Y = \sqrt{X}$ when X is the squared Bessel process (1) of dimension $k = a \in (0,1)$. There, it is shown that Y satisfies the equation of the form

$$Y(t) = \sqrt{x_0} + W(t) + L(t), \tag{6}$$

where

$$L(t) := \frac{a-1}{2}\int_0^\infty y^{a-2}(\ell(t,y) - \ell(t,0))\,\mathrm{d}y \tag{7}$$

with ℓ being a jointly continuous in (t,y) normalized local time such that for any bounded measurable function f

$$\int_0^t f(Y(s))\,\mathrm{d}s = \int_0^\infty f(y)y^{a-1}\ell(t,y)\,\mathrm{d}y. \tag{8}$$

1.2. Main results

In our paper, we consider a more general case of the CIR process (2) with $b \in \mathbb{R}$ and $0 < a < \frac{\sigma^2}{4}$ (we call such a process a *low-dimensional* CIR) and study the properties of $Y = \sqrt{X}$. More precisely, we represent Y as a transformation of a reflected Brownian motion $\widetilde{W}$ and use the properties of the local time $L^{\widetilde{W}}$ of the latter to study the local time L^Y of Y. Afterwards, we use the connection between $L^{\widetilde{W}}$ and L^Y to get a representation in the spirit of (6): namely, we prove that Y is a strong solution of the equation

$$Y(t) = \sqrt{x_0} - \frac{b}{2}\int_0^t Y(s)\,\mathrm{d}s + \frac{\sigma}{2}W(t) + L(t), \tag{9}$$

where

$$L(t) = -\frac{1}{2}\left(\frac{\sigma^2}{4} - a\right)\int_0^\infty y^{\frac{4a}{\sigma^2}-2}\left(\ell(t,y) - \ell(t,0)\right)\,\mathrm{d}y \tag{10}$$

with ℓ being an explicitly given normalized transformation of a local time L^Y of the process Y:

$$\ell(t,y) := y^{1-\frac{4a}{\sigma^2}}L^Y(t,y),$$

where $\ell(t,0) := \lim_{y\to 0+}\ell(t,y)$ is defined by continuity.

Finally, we close the gap of [27] mentioned above and obtain a representation of the Skorokhod reflection function for the ROU process in terms of CIR processes of dimension $k = \frac{4a}{\sigma^2} < 1$.

It is worth noting that our approach is simpler than the one in [5,6] and is based on the following machinery: we notice that Itô's formula applied to $\sqrt{X(t)+\varepsilon}$ followed by moving $\varepsilon \downarrow 0$ implies that $Y = \sqrt{X}$ satisfies the equation of the form (9) with L represented as an a.s.-limit

$$L(t) := \lim_{n\to\infty} \frac{1}{2}\int_0^t \left(\frac{a}{\sqrt{X(s)+\varepsilon_n}} - \frac{\sigma^2}{4}\frac{X(s)}{(X(s)+\varepsilon_n)^{\frac{3}{2}}} \right) ds \tag{11}$$

and $\{\varepsilon_n,\ n \geq 0\}$ being some sequence converging to zero. After that, we utilize the fact that the CIR process X is a regular diffusion and hence can be obtained from a reflected Brownian motion $\widetilde{W}$ by a transformation of time and scale (see e.g. [30, Chapter V, Section 7]). We find the explicit shape of this transformation, use it to establish the connection between the local times of $\widetilde{W}$ and Y . Finally, we exploit this link to show that the limit (11) is equal to (10). The technique described above seems to be more transparent than the one employed in [5,6] and additionally allows to get a clear intuition behind the process ℓ in (8).

1.3. Structure of the paper

The paper is organized as follows. In Section 2, we present some preliminary calculations and discuss the representation (9)–(11). Section 3 is devoted to the case $0 < a < \frac{\sigma^2}{4}$ and contains Theorem 3.1 that can be regarded as the main result of the paper. In Section 4, we discuss the results and compare them with the behaviour of the limit in (11) when $a \geq \frac{\sigma^2}{4}$. In Section 5, we establish a new connection between CIR processes of dimension $k = \frac{4a}{\sigma^2} < 1$ and ROU processes and obtain a new representation of Skorokhod reflection function.

2. Preliminary calculations

Let $a, \sigma > 0$, $b \geq 0$, $W = \{W(t),\ t \geq 0\}$ be a Brownian motion, and let us consider the continuous modification of a standard CIR process (2) driven by W. Note that, by [22, Chapter IV, Example 8.2], the paths of X are non-negative with probability 1 provided that $a > 0$ and hence one can define the square-root process $Y = \{Y(t),\ t \geq 0\} := \{\sqrt{X(t)},\ t \geq 0\}$. In order to analyze the dynamics of Y , take $\varepsilon > 0$ and observe that, by Itô's formula,

$$\begin{aligned}\sqrt{X(t)+\varepsilon} &= \sqrt{x_0+\varepsilon} + \frac{1}{2}\int_0^t \left(\frac{a}{\sqrt{X(s)+\varepsilon}} - \frac{\sigma^2}{4}\frac{X(s)}{(X(s)+\varepsilon)^{\frac{3}{2}}} \right) ds \\ &\quad - \frac{1}{2}\int_0^t \frac{bX(s)}{\sqrt{X(s)+\varepsilon}}\, ds + \frac{\sigma}{2}\int_0^t \frac{\sqrt{X(s)}}{\sqrt{X(s)+\varepsilon}}\, dW(s).\end{aligned} \tag{12}$$

Fix an arbitrary $T > 0$ and note that the left-hand side of (12) converges to $Y(t)$ uniformly on $[0, T]$ with probability 1 as $\varepsilon \downarrow 0$. It is also evident that

$$\sup_{t\in[0,T]} \left| \int_0^t \frac{X(s)}{\sqrt{X(s)+\varepsilon}}\,\mathrm{d}s - \int_0^t Y(s)\,\mathrm{d}s \right| \to 0 \quad a.s., \quad \varepsilon \downarrow 0. \tag{13}$$

Next, by [29, Chapter XI], the expectation $\mathbb{E}\int_0^\infty \mathbb{1}_{\{X(s)=0\}}\,\mathrm{d}s = 0$ and hence, by the Burkholder-Davis-Gundy inequality, for any $T > 0$

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T]} \left| \int_0^t \frac{\sqrt{X(s)}}{\sqrt{X(s)+\varepsilon}}\,\mathrm{d}W(s) - W(t) \right|\right)^2 \\
&\le 4\mathbb{E}\int_0^T \left(\frac{\sqrt{X(s)}}{\sqrt{X(s)+\varepsilon}} - 1\right)^2 \mathrm{d}s \\
&= 4\mathbb{E}\int_0^T \left(\frac{\sqrt{X(s)}}{\sqrt{X(s)+\varepsilon}} - 1\right)^2 \mathbb{1}_{\{X(s)>0\}}\,\mathrm{d}s + 4\mathbb{E}\int_0^T \mathbb{1}_{\{X(s)=0\}}\,\mathrm{d}s \\
&= 4\mathbb{E}\int_0^T \left(\frac{\sqrt{X(s)}}{\sqrt{X(s)+\varepsilon}} - 1\right)^2 \mathbb{1}_{\{X(s)>0\}}\,\mathrm{d}s \to 0, \quad \varepsilon \downarrow 0.
\end{aligned}$$

This implies that for each $T > 0$

$$\sup_{t\in[0,T]} \left| \int_0^t \frac{\sqrt{X(s)}}{\sqrt{X(s)+\varepsilon_n}}\,\mathrm{d}W(s) - W(t) \right| \xrightarrow{\mathbb{P}} 0, \quad \varepsilon \downarrow 0, \tag{14}$$

where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability. In particular, (12) as well as convergences (13) and (14) imply that the left-hand side of

$$\begin{aligned}
&\sqrt{X(t)+\varepsilon} - \sqrt{x_0+\varepsilon} + \frac{1}{2}\int_0^t \frac{bX(s)}{\sqrt{X(s)+\varepsilon}}\,\mathrm{d}s - \frac{\sigma}{2}\int_0^t \frac{\sqrt{X(s)}}{\sqrt{X(s)+\varepsilon}}\,\mathrm{d}W(s) \\
&= \frac{1}{2}\int_0^t \left(\frac{a}{\sqrt{X(s)+\varepsilon}} - \frac{\sigma^2}{4}\frac{X(s)}{(X(s)+\varepsilon)^{\frac{3}{2}}}\right)\mathrm{d}s
\end{aligned}$$

converges uniformly on compacts in probability as $\varepsilon \downarrow 0$. Therefore, there exists a ucp-limit

$$L(t) := \lim_{\varepsilon\downarrow 0} \frac{1}{2}\int_0^t \left(\frac{a}{\sqrt{X(s)+\varepsilon}} - \frac{\sigma^2}{4}\frac{X(s)}{(X(s)+\varepsilon)^{\frac{3}{2}}}\right)\mathrm{d}s \tag{15}$$

and the process $Y = \sqrt{X}$ satisfies the SDE of the form

$$Y(t) = Y(0) - \frac{b}{2}\int_0^t Y(s)\,\mathrm{d}s + \frac{\sigma}{2}W(t) + L(t),$$

where $Y(0) = \sqrt{x_0}$.

Remark 2.1: Ucp convergence in (15) implies that for an arbitrary $T > 0$ there exists a sequence $\{\varepsilon_n,\ n \geq 1\}$ (depending on T) such that, for any $t \in [0, T]$,

$$\left| L(t) - \frac{1}{2}\int_0^t \left(\frac{a}{\sqrt{X(s) + \varepsilon_n}} - \frac{\sigma^2}{4}\frac{X(s)}{(X(s)+\varepsilon_n)^{\frac{3}{2}}} \right) \mathrm{d}s \right| \to 0 \tag{16}$$

with probability 1 as $n \to \infty$. Later on, we will see that the a.s. convergence (16) holds for an *arbitrary* sequence $\{\varepsilon_n,\ n \geq 1\}$ such that $\varepsilon_n \downarrow 0$ as $n \to \infty$. Moreover, it will be shown that the set of full probability where (16) holds can be chosen independently of a particular sequence $\{\varepsilon_n,\ n \geq 1\}$.

Remark 2.2: Note that the process L defined by (15) is continuous a.s. since

$$L(t) = Y(t) - Y(0) + \frac{b}{2}\int_0^t Y(s)\,\mathrm{d}s - \frac{\sigma}{2}W(t).$$

3. Stochastic representation of L when $0 < a < \frac{\sigma^2}{4}$

Our strategy for the analysis of L will be as follows. Since the CIR process X in (2) is a non-negative regular diffusion, it can be represented (see e.g. [30, Chapter V, Section 7]) in the form

$$X(t) = S^{-1}\left(\widetilde{W}(\tau_t)\right)$$

for certain change of time τ and change of scale S of a reflected Brownian motion $\widetilde{W} = \{\widetilde{W}(t),\ t \geq 0\}$. Then, we re-write the integral in the limit (15) in terms of the local time $L^{\widetilde{W}} = L^{\widetilde{W}}(t, x)$, $t \geq 0$, $x \geq 0$, of $\widetilde{W}$ and exploit Hölder continuity of the latter to find an explicit representation of L in terms of $L^{\widetilde{W}}$.

3.1. CIR process as the transformation of a reflected Brownian motion

In order to implement our approach, we first need to represent the CIR process as a transformation of a reflected Brownian motion. For a given set of parameters a, b, σ of the SDE (2), define a *scale function* S: $[0, \infty) \to [0, \infty)$ by

$$S(x) := \int_0^x y^{-\frac{2a}{\sigma^2}} e^{\frac{2by}{\sigma^2}}\,\mathrm{d}y \tag{17}$$

and observe that, since S is strictly increasing and $S(\infty) = \infty$, there exists its inverse S^{-1}. Define also a *speed measure*

$$m\,(\mathrm{d}x) = \rho(x)\,\mathrm{d}x,$$

where

$$\rho(x) := \frac{1}{\sigma^2} x^{\frac{4a}{\sigma^2}-1} e^{-\frac{4b}{\sigma^2}x} \mathbb{1}_{x>0}. \tag{18}$$

Proposition 3.1: *Let X be the unique strong solution to the CIR Equation (2) with $0 < a < \frac{\sigma^2}{4}$. Then there exists a reflected Brownian motion $\widetilde{W}$ starting at $S(x_0)$ such that*

$$X(t) := S^{-1}\left(\widetilde{W}(\tau_t)\right), \tag{19}$$

where

$$\tau_t := \varphi_t^{-1} \tag{20}$$

with

$$\varphi_t := \int_0^t \rho\left(S^{-1}\left(\widetilde{W}(s)\right)\right) \mathrm{d}s. \tag{21}$$

Before moving to the proof of Proposition 3.1, let us make some remarks regarding its formulation.

Remark 3.1: (1) The process φ in (21) is well-defined. Indeed, let $L^{\widetilde{W}} = \{L^{\widetilde{W}}(t,x),\ t \geq 0,\ x \geq 0\}$ be the local time of $\widetilde{W}$, i.e. for any bounded measurable f,

$$\int_0^t f(\widetilde{W}(s))\,\mathrm{d}s = \int_0^\infty f(x) L^{\widetilde{W}}(t,x)\,\mathrm{d}x \quad a.s.$$

Then, with probability 1,

$$\begin{aligned}
\varphi_t &= \int_0^t \rho\left(S^{-1}\left(\widetilde{W}(s)\right)\right) \mathrm{d}s \\
&= \int_0^\infty \rho\left(S^{-1}\left(y\right)\right) L^{\widetilde{W}}(t,y)\,\mathrm{d}y \\
&= \int_0^\infty \rho\left(x\right) S'(x) L^{\widetilde{W}}(t,S(x))\,\mathrm{d}x \\
&= \frac{1}{\sigma^2}\int_0^\infty x^{\frac{2a}{\sigma^2}-1} e^{-\frac{2b}{\sigma^2}x} L^{\widetilde{W}}(t,S(x))\,\mathrm{d}x \\
&< \infty
\end{aligned}$$

because $\frac{2a}{\sigma^2} - 1 > -1$ and $L^{\widetilde{W}}(t,S(x)) = 0$ for $x > S^{-1}(\max_{s\in[0,T]} \widetilde{W}(s))$.

(2) Since φ is strictly increasing and $\varphi_\infty = \infty$ with probability 1, its inverse τ in (20) is well-defined a.s.

Remark 3.2: The transformation (19) is invertible. Indeed, it is straightforward to check that, with probability 1,

$$\tau_t = \int_0^t \frac{1}{\rho\left(X(s)\right)}\,\mathrm{d}s,$$

therefore,

$$\widetilde{W}(t) = S\left(X\left(\varphi_t\right)\right),$$

where $\varphi = \tau^{-1}$ can be expressed as the inverse of the mapping $t \mapsto \int_0^t \frac{1}{\rho(X(s))}\,\mathrm{d}s$. For more details on transformations of this type, we refer the reader to [22, Chapter IV, §7].

Proof: We will split the proof into two steps. First, we will follow [30, Chapter V, Section 7, § 48] to prove that, for *some given reflected Brownian motion* $\widetilde{W}$, the process $X(t) := S^{-1}(\widetilde{W}(\tau_t))$ is the weak solution to the SDE (2). Then we will utilize the invertibility of transformation (19) outlined in Remark 3.2 to establish the existence of a reflected Brownian motion $\widetilde{W}$ together with the required representation for the given CIR process X.

Step 1. Let $\widetilde{Z} = \{\widetilde{Z}(t),\ t \geq 0\}$ be a standard Brownian motion starting at $\widetilde{Z}(0) = S(x_0)$. Consider a reflected Brownian motion

$$\widetilde{W}(t) := |\widetilde{Z}(t)| = S(x_0) + Z(t) + L^{\widetilde{Z}}(t),$$

where $Z(t) := \int_0^t \operatorname{sign}\widetilde{Z}(s)\, \mathrm{d}\widetilde{Z}(s)$ is a Brownian motion and $L^{\widetilde{Z}}(t)$ is the local time of $L^{\widetilde{Z}}$ at zero. Put $V(t) := S^{-1}(\widetilde{W}(t))$ and observe that, by the extension of Itô's formula in [30, Lemma IV.45.9],

$$\begin{aligned}
&V(t) - V(0)\\
&\quad = \int_0^t \left(S^{-1}\left(\widetilde{W}(u)\right)\right)^{\frac{2a}{\sigma^2}} e^{-\frac{2bS^{-1}(\widetilde{W}(u))}{\sigma^2}} d\widetilde{W}(u)\\
&\qquad + \int_0^t \frac{1}{\sigma^2}\left(S^{-1}\left(\widetilde{W}(u)\right)\right)^{\frac{4a}{\sigma^2}-1} e^{-\frac{4b}{\sigma^2}S^{-1}(\widetilde{W}(u))}\left(a - bS^{-1}\left(\widetilde{W}(u)\right)\right) du\\
&\quad = \int_0^t (V(u))^{\frac{2a}{\sigma^2}} e^{-\frac{2b}{\sigma^2}V(u)} dZ(u)\\
&\qquad + \int_0^t \frac{1}{\sigma^2}(V(u))^{\frac{4a}{\sigma^2}-1} e^{-\frac{4b}{\sigma^2}V(u)}(a - bV(u))\, du.
\end{aligned}$$

Hence, by Itô's formula, for any infinitely differentiable function with compact support h,

$$C_t(h) := h(V(t)) - h(V(0)) - \int_0^t \frac{1}{\sigma^2}(V(s))^{\frac{4a}{\sigma^2}-1} e^{-\frac{4b}{\sigma^2}V(s)}\mathcal{A}h(V(s))\, \mathrm{d}s \tag{22}$$

is a local martingale, where

$$\mathcal{A}h(x) := (a - bx)h'(x) + \frac{\sigma^2 x}{2}h''(x)$$

is the generator of (2). Recall that

$$X(t) = V(\tau_t)$$

and observe that, by (22), for any infinitely differentiable function with compact support h, simple change of variables yields that

$$\begin{aligned}
C_{\tau_t}(h) &= h(V(\tau_t)) - h(V(0)) - \int_0^{\tau_t} \frac{1}{\sigma^2}(V(s))^{\frac{4a}{\sigma^2}-1} e^{-\frac{4b}{\sigma^2}V(s)}\mathcal{A}h(V(s))\, \mathrm{d}s\\
&= h(X(t)) - h(x_0) - \int_0^t \mathcal{A}h(X(s))\, \mathrm{d}s.
\end{aligned}$$

Since $C_{\tau_t}(h)$, $t \geq 0$, is a local martingale by the optional stopping theorem,

$$h(X(t)) - h(x_0) - \int_0^t \mathcal{A}h(X(s))\,\mathrm{d}s$$

is also a local martingale and therefore, by [30, V.19–V.20], X is the weak solution to (2).

Step 2. Let now X be the unique strong solution to (2). By Remark 3.2 and Step 1, the process

$$\widetilde{W}(t) := S(X(\varphi_t)),$$

where φ is defined as the inverse of the mapping $t \mapsto \int_0^t \frac{1}{\rho(X(s))}\,\mathrm{d}s$, is a reflected Brownian motion for which X admits the representation (19). ■

3.2. Characterization of L in terms of $L^{\widetilde{W}}$

Having the representation (19) at our disposal, we are now ready to characterize the process L from (15) in terms of the local time $L^{\widetilde{W}}$ of the corresponding reflected Brownian motion.

Let X be the unique strong solution to the SDE (2) with $0 < a < \frac{\sigma^2}{4}$ and $\widetilde{W}$ be the reflected Brownian motion such that

$$X(t) := S^{-1}\left(\widetilde{W}(\tau_t)\right).$$

Denote $L^{\widetilde{W}} = \{L^{\widetilde{W}}(t,x),\ t \geq 0,\ x \geq 0\}$ the jointly continuous modification of the local time of $\widetilde{W}$ so that for any bounded measurable f,

$$\int_0^t f(\widetilde{W}(s))\,\mathrm{d}s = \int_0^\infty f(x) L^{\widetilde{W}}(t,x)\,\mathrm{d}x, \quad t \geq 0,$$

with probability 1.

First of all, let us express the local time $L^Y = L^Y(t,y)$ of the process $Y = \sqrt{X}$ in terms of $L^{\widetilde{W}}$.

Proposition 3.2: *Let*

$$Y(t) := \sqrt{X(t)} = \sqrt{S^{-1}\left(\widetilde{W}(\tau_t)\right)}$$

be the square root of the CIR process X. Then, for any bounded measurable f,

$$\int_0^t f(Y(s))\,\mathrm{d}s = \int_0^\infty f(y) L^Y(t,y)\,\mathrm{d}y,$$

where, with probability 1,

$$L^Y(t,y) = \frac{2}{\sigma^2} y^{\frac{4a}{\sigma^2}-1} e^{-\frac{2b}{\sigma^2}y^2} L^{\widetilde{W}}(\varphi_t, S(y^2)) \tag{23}$$

with φ being defined by (21).

Proof: For any bounded measurable f, we can write

$$\begin{aligned}\int_0^t f(Y(s))\,\mathrm{d}s &= \int_0^t f\left(\sqrt{S^{-1}\left(\widetilde{W}(\tau_u)\right)}\right) du \\ &= \int_0^{\varphi_t} f\left(\sqrt{S^{-1}\left(\widetilde{W}(z)\right)}\right) \rho\left(S^{-1}\left(\widetilde{W}(z)\right)\right) dz \\ &= \int_0^{\infty} f\left(\sqrt{S^{-1}(x)}\right) \rho\left(S^{-1}(x)\right) L^{\widetilde{W}}\left(\varphi_t, x\right)\,\mathrm{d}x \\ &= \int_0^{\infty} f(y)\, \rho\left(y^2\right) L^{\widetilde{W}}\left(\varphi_t, S(y^2)\right) 2yS'(y^2)\,\mathrm{d}y \\ &=: \int_0^{\infty} f(y) L^Y(t,y)\,\mathrm{d}y.\end{aligned}$$

The final result is obtained by recalling that

$$\begin{aligned}L^Y(t,y) &= \rho\left(y^2\right) L^{\widetilde{W}}\left(\varphi_t, S(y^2)\right) 2yS'(y^2) \\ &= \frac{2}{\sigma^2} y^{\frac{4a}{\sigma^2}-1} e^{-\frac{2b}{\sigma^2}y^2} L^{\widetilde{W}}(\varphi_t, S(y^2)).\end{aligned}$$

■

Define a normalized local time of the process Y as follows. Set

$$\ell(t,y) := y^{1-\frac{4a}{\sigma^2}} L^Y(t,y), \quad y > 0, \tag{24}$$

and

$$\ell(t,0) := \lim_{y\to 0+} \ell(t,y).$$

Note that $\ell(t,y)$ is continuous in (t,y) because

$$\ell(t,y) = \frac{2}{\sigma^2} e^{-\frac{2b}{\sigma^2}y^2} L^{\widetilde{W}}(\varphi_t, S(y^2)).$$

However, we want to stress that $\ell(t,y)$ is a function of the local time L^Y of the process Y without mentioning the auxiliary Brownian motion $\widetilde{W}$.

Theorem 3.1: *Let X be the CIR process satisfying (2) and $\widetilde{W}$ be the reflected Brownian motion such that $X(t) = S^{-1}(\widetilde{W}(\tau_t))$, $t \geq 0$. Then, with probability 1, the process $Y = \sqrt{X}$ satisfies the SDE of the form*

$$Y(t) = \sqrt{x_0} - \frac{b}{2}\int_0^t Y(s)\,\mathrm{d}s + \frac{\sigma}{2}W(t) + L(t), \tag{25}$$

where

$$L(t) = -\frac{1}{2}\left(\frac{\sigma^2}{4} - a\right)\int_0^{\infty} y^{\frac{4a}{\sigma^2}-2}\left(\ell(t,y) - \ell(t,0)\right)\,\mathrm{d}y. \tag{26}$$

Moreover,

$$L(t) := \lim_{\varepsilon\downarrow 0}\frac{1}{2}\int_0^t \left(\frac{a}{\sqrt{X(s)+\varepsilon}} - \frac{\sigma^2}{4}\frac{X(s)}{(X(s)+\varepsilon)^{\frac{3}{2}}}\right) \mathrm{d}s$$
$$= -\lim_{\varepsilon\downarrow 0}\frac{1}{2}\int_0^t \left(\frac{\frac{\sigma^2}{4}-a}{\sqrt{X(s)+\varepsilon}} - \frac{\sigma^2}{4}\frac{\varepsilon}{(X(s)+\varepsilon)^{\frac{3}{2}}}\right) \mathrm{d}s.$$

Remark 3.3: Since the SDE (2) has a strong solution, Theorem 3.1 immediately yields that the SDE (25)–(26) also has a strong solution.

Remark 3.4: Despite the fact that $\frac{4a}{\sigma^2} - 2 \in (-2,-1)$, the integral

$$\int_0^\infty y^{\frac{4a}{\sigma^2}-2}\left|\ell(t,y)-\ell(t,0)\right| \mathrm{d}y$$

is finite with probability 1. Indeed, denote $k := \frac{4a}{\sigma^2} \in (0,1)$ and observe that, by (24) and properties of local time $L^{\widetilde{W}}$,

$$\sup_{y\geq 1}|\ell(t,y)-\ell(t,0)| < \infty \tag{27}$$

with probability 1 for any $t \geq 0$. Moreover, since $L^{\widetilde{W}}(t,\cdot)$ is Hölder continuous of order up to $\frac{1}{2}$ a.s. (see e.g. calculations in [30, Section IV.44]), for any $\delta \in (0,\frac{1}{2})$ and any fixed $t > 0$ there exists a random variable $C > 0$ such that, with probability 1,

$$|\ell(t,y)-\ell(t,0)| \leq C\cdot (S(y^2))^{\frac{1}{2}-\delta}. \tag{28}$$

Hence, on the one hand,

$$\int_1^\infty \left|\ell(t,y)-\ell(t,0)\right| y^{k-2}\,\mathrm{d}y < \infty \text{ a.s.}$$

by (27). On the other hand, take $\delta \in (0,\frac{k}{2(2-k)})$ and observe that (28) implies

$$\int_0^1 \left|\ell(t,y)-\ell(t,0)\right| y^{k-2}\,\mathrm{d}y \leq C\int_0^1 \left(S(y^2)\right)^{\frac{1}{2}-\delta} y^{k-2}\,\mathrm{d}y$$
$$= C\int_0^1 \left(\int_0^{y^2} z^{-\frac{k}{2}}e^{\beta z}dz\right)^{\frac{1}{2}-\delta} y^{k-2}\,\mathrm{d}y$$
$$\leq C\int_0^1 \left(\int_0^{y^2} z^{-\frac{k}{2}}dz\right)^{\frac{1}{2}-\delta} y^{k-2}\,\mathrm{d}y$$
$$\leq C\int_0^1 y^{-1+\frac{k}{2}-\delta'}\,\mathrm{d}y < \infty \text{ a.s.},$$

where $\delta' := (2-k)\delta \in (0,\frac{k}{2})$ and C is a (random) constant that varies from line to line.

Now we are ready to proceed to the proof of Theorem 3.1.

Proof: In Section 2, we obtained the representation (25) with L being a *ucp-limit* of the form

$$L(t) = \lim_{\varepsilon\downarrow 0} \frac{1}{2}\int_0^t \left(\frac{a}{\sqrt{X(s)+\varepsilon}} - \frac{\sigma^2}{4}\frac{X(s)}{(X(s)+\varepsilon)^{\frac{3}{2}}} \right) \mathrm{d}s$$
$$= -\lim_{\varepsilon\downarrow 0} \frac{1}{2}\int_0^t \left(\frac{\frac{\sigma^2}{4}-a}{\sqrt{X(s)+\varepsilon}} - \frac{\sigma^2}{4}\frac{\varepsilon}{(X(s)+\varepsilon)^{\frac{3}{2}}} \right) \mathrm{d}s.$$

Hence, one is left to prove that this limit exists in the sense of a.s. convergence and check that the last equality in (26) holds.

Let $k := \frac{4a}{\sigma^2} \in (0,1)$ denote the dimension of the CIR process, i.e. we have to study the a.s.-limit of the form

$$L(t) := -\lim_{\varepsilon\downarrow 0} \frac{1}{2}\int_0^t \left(\frac{\frac{\sigma^2}{4}-a}{\sqrt{X(s)+\varepsilon}} - \frac{\sigma^2}{4}\frac{\varepsilon}{(X(s)+\varepsilon)^{\frac{3}{2}}} \right) \mathrm{d}s$$
$$= -\frac{\sigma^2}{8}\lim_{\varepsilon\downarrow 0} \int_0^t \left(\frac{1-k}{\sqrt{X(s)+\varepsilon}} - \frac{\varepsilon}{(X(s)+\varepsilon)^{\frac{3}{2}}} \right) \mathrm{d}s.$$

Observe that

$$\int_0^t \frac{1-k}{\sqrt{X(s)+\varepsilon}}\mathrm{d}s = \int_0^t \frac{1-k}{\sqrt{Y^2(s)+\varepsilon}}\mathrm{d}s = \int_0^\infty \frac{1-k}{\sqrt{y^2+\varepsilon}}L^Y(t,y)\,\mathrm{d}y$$
$$= \int_0^\infty \frac{1-k}{\sqrt{y^2+\varepsilon}}y^{k-1}\ell(t,y)\,\mathrm{d}y$$
$$= \int_0^\infty \frac{1-k}{\sqrt{y^2+\varepsilon}}y^{k-1}(\ell(t,y)-\ell(t,0))\,\mathrm{d}y$$
$$+ \ell(t,0)\int_0^\infty \frac{1-k}{\sqrt{y^2+\varepsilon}}y^{k-1}\,\mathrm{d}y$$

and, similarly,

$$\int_0^t \frac{\varepsilon}{(X(s)+\varepsilon)^{\frac{3}{2}}}\mathrm{d}s = \int_0^t \frac{\varepsilon}{(Y^2(s)+\varepsilon)^{\frac{3}{2}}}\mathrm{d}s = \int_0^\infty \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}}L^Y(t,y)\,\mathrm{d}y$$
$$= \int_0^\infty \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}}y^{k-1}\ell(t,y)\,\mathrm{d}y$$
$$= \int_0^\infty \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}}y^{k-1}(\ell(t,y)-\ell(t,0))\,\mathrm{d}y$$
$$+ \ell(t,0)\int_0^\infty \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}}y^{k-1}\,\mathrm{d}y.$$

Let us study separately the asymptotics of

$$I_1(\varepsilon) := \int_0^\infty \frac{1-k}{\sqrt{y^2+\varepsilon}} y^{k-1}(\ell(t,y)-\ell(t,0))\,\mathrm{d}y,$$
$$I_2(\varepsilon) := \int_0^\infty \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}} y^{k-1}(\ell(t,y)-\ell(t,0))\,\mathrm{d}y,$$
$$I_3(\varepsilon) := \ell(t,0)\left(\int_0^\infty \frac{1-k}{\sqrt{y^2+\varepsilon}} y^{k-1}\,\mathrm{d}y - \int_0^\infty \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}} y^{k-1}\,\mathrm{d}y\right)$$

as $\varepsilon \downarrow 0$. First, observe that for any $y \geq 0$

$$\frac{1-k}{\sqrt{y^2+\varepsilon}} y^{k-1}|\ell(t,y)-\ell(t,0)| \leq y^{k-2}|\ell(t,y)-\ell(t,0)|$$

and note that by Remark 3.4,

$$\int_0^\infty y^{k-2}\left|\ell(t,y)-\ell(t,0)\right|\,\mathrm{d}y < \infty \quad \text{a.s.}$$

Thus, by the dominated convergence theorem, with probability 1,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} I_1(\varepsilon) &= \lim_{\varepsilon\downarrow 0}\int_0^\infty \frac{1-k}{\sqrt{y^2+\varepsilon}} y^{k-1}\left(\ell(t,y)-\ell(t,0)\right)\,\mathrm{d}y \\ &= (1-k)\int_0^\infty y^{k-2}\left(\ell(t,y)-\ell(t,0)\right)\,\mathrm{d}y.\end{aligned}$$

Next, observe that, with probability 1,

$$\begin{aligned}&\left|\int_1^\infty \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}} y^{k-1}\left(\ell(t,y)-\ell(t,0)\right)\,\mathrm{d}y\right| \\ &\quad\leq \varepsilon\int_1^\infty \left|\ell(t,y)-\ell(t,0)\right| y^{k-4}\,\mathrm{d}y \\ &\quad\leq \varepsilon C\int_1^\infty y^{k-4}\,\mathrm{d}y \to 0, \quad \varepsilon\downarrow 0. \end{aligned}\tag{29}$$

On the other hand, take an arbitrary $\delta \in (0, \frac{k}{2(2-k)})$, denote $\delta' := (2-k)\delta$ and observe that (28) yields

$$\begin{aligned}&\int_0^1 \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}} y^{k-1}\left|\ell(t,y)-\ell(t,0)\right|\,\mathrm{d}y \\ &\quad\leq C\int_0^1 \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}} y^{k-1}(S(y^2))^{\frac{1}{2}-\delta}\,\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= C\int_0^1 \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}} y^{k-1}\left(\int_0^{y^2} z^{-\frac{k}{2}} e^{\beta z} dz\right)^{\frac{1}{2}-\delta} \mathrm{d}y \\
&\le C\int_0^1 \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}} y^{k-1}\left(\int_0^{y^2} z^{-\frac{k}{2}} dz\right)^{\frac{1}{2}-\delta} \mathrm{d}y \\
&\le C\int_0^1 \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}} y^{\frac{k}{2}-\delta'} \mathrm{d}y \\
&= C\varepsilon^{\frac{k}{4}-\frac{\delta'}{2}} \int_0^1 \frac{\varepsilon^{-\frac{1}{2}}}{\left((y/\sqrt{\varepsilon})^2+1\right)^{\frac{3}{2}}} \left(\frac{y}{\sqrt{\varepsilon}}\right)^{\frac{k}{2}-\delta'} \mathrm{d}y, \qquad (30)
\end{aligned}
$$

where $\beta := \frac{2b}{\sigma^2}$. Hence, by substituting $z = y/\sqrt{\varepsilon}$ in (30), we can write

$$
\begin{aligned}
&\int_0^1 \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}} y^{k-1}\left|\ell(t,y)-\ell(t,0)\right| \mathrm{d}y \\
&\quad \le C\varepsilon^{\frac{k}{4}-\frac{\delta'}{2}} \int_0^\infty \frac{1}{\left(z^2+1\right)^{\frac{3}{2}}} z^{\frac{k}{2}-\delta'} dz \\
&\quad \to 0 \qquad (31)
\end{aligned}
$$

with probability 1 as $\varepsilon \downarrow 0$. Summarizing (29) and (31), we obtain that, with probability 1,

$$
\lim_{\varepsilon\downarrow 0} I_2(\varepsilon) = 0.
$$

Finally, integration by parts yields

$$
\int_0^\infty \frac{y^{k-1}}{\sqrt{y^2+\varepsilon}} \mathrm{d}y = \frac{1}{k}\int_0^\infty \frac{y^{k+1}}{(y^2+\varepsilon)^{\frac{3}{2}}} \mathrm{d}y
$$

and the right-hand side of the last equation is equal to

$$
\begin{aligned}
&\frac{1}{k}\int_0^\infty \frac{y^{k-1}(y^2+\varepsilon-\varepsilon)}{(y^2+\varepsilon)^{\frac{3}{2}}} \mathrm{d}y \\
&\quad = \frac{1}{k}\int_0^\infty \frac{y^{k-1}}{\sqrt{y^2+\varepsilon}} \mathrm{d}y - \frac{1}{k}\int_0^\infty \frac{\varepsilon y^{k-1}}{(y^2+\varepsilon)^{\frac{3}{2}}} \mathrm{d}y.
\end{aligned}
$$

Therefore

$$
\int_0^\infty \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}} y^{k-1} \mathrm{d}y = \int_0^\infty \frac{1-k}{\sqrt{y^2+\varepsilon}} y^{k-1} \mathrm{d}y
$$

and

$$
I_3(\varepsilon) = \ell(t,0)\left(\int_0^\infty \frac{1-k}{\sqrt{y^2+\varepsilon}} y^{k-1} \mathrm{d}y - \int_0^\infty \frac{\varepsilon}{(y^2+\varepsilon)^{\frac{3}{2}}} y^{k-1} \mathrm{d}y\right) = 0.
$$

Summarizing all of the above and recalling that $k = \frac{4a}{\sigma^2}$, we finally obtain that with probability 1

$$\begin{aligned} L(t) &= -\frac{\sigma^2}{8} \lim_{\varepsilon \downarrow 0} (I_1(\varepsilon) - I_2(\varepsilon) + I_3(\varepsilon)) \\ &= -\frac{\sigma^2}{8} (1-k) \int_0^\infty y^{k-2} \big(\ell(t,y) - \ell(t,0)\big) \, dy \\ &= -\frac{1}{2} \left(\frac{\sigma^2}{4} - a\right) \int_0^\infty y^{\frac{4a}{\sigma^2}-2} \big(\ell(t,y) - \ell(t,0)\big) \, dy \end{aligned}$$

which ends the proof. ∎

Remark 3.5: Theorem 3.1 implies that the limit

$$L(t) = \lim_{n\to\infty} \frac{1}{2} \int_0^t \left(\frac{a}{\sqrt{X(s) + \varepsilon_n}} - \frac{\sigma^2}{4} \frac{X(s)}{(X(s)+\varepsilon_n)^{\frac{3}{2}}} \right) ds \tag{32}$$

exists a.s. for any sequence $\{\varepsilon_n,\ n \geq 1\}$ such that $\varepsilon_n \downarrow 0$ and does not depend on the particular choice of the sequence. Moreover, the proof of Theorem 3.1 yields that the existence of the limit (32) is ensured for all ω such that $L^{\widetilde{W}}(\omega; t, \cdot)$ is Hölder continuous. In other words, the set of full probability where (32) holds can be chosen independently of a particular sequence $\{\varepsilon_n,\ n \geq 1\}$, as anticipated in Remark 2.1.

4. Discussion of the results

It is evident that the nature of the limit in (15) heavily depends on the relation between parameters a and σ. Therefore, in order to put our findings from Section 3 into context, let us provide some relevant results from [27] on the behaviour of Y when $a \geq \frac{\sigma^2}{4}$.

4.1. Square root of the CIR process when $a \geq \frac{\sigma^2}{4}$

4.1.0.1. Case I: $a > \frac{\sigma^2}{4}$. Observe that, if

$$\int_0^t \frac{1}{\sqrt{X(s)}} ds = \int_0^t \frac{1}{Y(s)} ds < \infty \quad a.s., \tag{33}$$

then the limit (15) is equal to

$$L(t) = \frac{1}{2}\left(a - \frac{\sigma^2}{4}\right) \int_0^t \frac{1}{Y(s)} ds$$

by monotone convergence. This is clearly the case for $a \geq \frac{\sigma^2}{2}$: indeed $a \geq \frac{\sigma^2}{2}$ implies that X (and hence Y) has strictly positive paths a.s. (see e.g. [3] or [14]) and therefore (33) holds for all $t \geq 0$. It turns out (see e.g. [27, Theorem 2.1(a)]) that (33) also holds if $\frac{\sigma^2}{4} < a < \frac{\sigma^2}{2}$, i.e. one can prove the following result.

Theorem 4.1 ([27, Theorem 2.1(a)]): *Let $a > \frac{\sigma^2}{4}$. Then, for any $t \geq 0$,*

$$\int_0^t \frac{1}{Y(s)}\,\mathrm{d}s < \infty \quad a.s. \tag{34}$$

and Y a.s. satisfies the SDE of the form

$$Y(t) = \sqrt{x_0} + \frac{1}{2}\left(a - \frac{\sigma^2}{4}\right)\int_0^t \frac{1}{Y(s)}\,\mathrm{d}s - \frac{b}{2}\int_0^t Y(s)\,\mathrm{d}s + \frac{\sigma}{2}W(t). \tag{35}$$

Remark 4.1: Using the same arguments as in [9, Theorem 3.2], it is possible to prove that for $a > \frac{\sigma^2}{4}$ the process $Y = \sqrt{X}$ is the unique *non-negative* strong solution to the SDE (35). However, if $\frac{\sigma^2}{4} < a < \frac{\sigma^2}{2}$, (35) has other strong solutions; moreover, the uniqueness in law does not hold for (35). For a more detailed discussion of this phenomenon, we refer the reader to [7] whereas a comprehensive overview of SDEs of the type (35) can be found in [10].

4.1.0.2. Case II: $a = \frac{\sigma^2}{4}$. The case $a = \frac{\sigma^2}{4}$ turns out to be different from the one described above: in this regime, X can hit zero (see e.g. [3] or [14]) and, as noted in e.g. [27, Theorem 2.1(b)], (34) does not hold for all

$$t > \inf\{s \geq 0 \mid Y(s) = 0\}.$$

However, the limit L from (15) has a simple interpretation in terms of Skorokhod reflections (see e.g. the seminal works [31,32]) as summarized in the following theorem.

Theorem 4.2 ([27, Theorem 2.1(b)]): *Let $a = \frac{\sigma^2}{4}$ and denote $\tau := \inf\{s \geq 0 \mid X(s) = 0\}$.*

(1) *For all $\gamma > 0$,*

$$\int_0^{\tau+\gamma} \frac{1}{Y(s)}\,\mathrm{d}s = \infty \quad a.s.$$

(2) *The processes $Y := \sqrt{X}$ and L defined by (15) is the (unique) solution to Skorokhod problem*

$$Y(t) = \sqrt{x_0} - \frac{b}{2}\int_0^t Y(s)\,\mathrm{d}s + \frac{\sigma}{2}W(t) + L(t), \tag{36}$$

with L being the corresponding Skorokhod reflection function, i.e. a continuous non-decreasing process starting at 0 with points of growth occurring only at zeros of Y and such that $Y(t) \geq 0$.

Remark 4.2: Item 2) of Theorem 4.2 states that, when $a = \frac{\sigma^2}{4}$, the square root process $Y = \sqrt{X}$ coincides with a *reflected Ornstein-Uhlenbeck (ROU) process*. More details on the latter can be found in e.g. [33].

4.2. Comparison to the low-dimensional case

As we have seen in Section 3, the case $0 < a < \frac{\sigma^2}{4}$ is arguably the most challenging one and leads to the most involved value of the limit (15). First of all, note that (33) does not hold due to Theorem 4.2 together with the comparison theorem for solutions of SDEs (see e.g. [21]). Next, the limit L in (15) cannot be non-decreasing in t as it happens when $a \geq \frac{\sigma^2}{4}$. Indeed, consider $\tau \geq 0$ such that $X(\tau) > 0$. Then, by a.s. continuity of X, there exists a neighbourhood $\tau_- < \tau < \tau_+$ such that X is bounded away from zero on (τ_-, τ_+). Denote now $-\delta := a - \frac{\sigma^2}{4}$, $\delta > 0$. Then, with probability 1, for all $\tau_- < t_1 < t_2 < \tau_+$

$$
\begin{aligned}
L(t_2) - L(t_1) &= \lim_{n\to\infty} \frac{1}{2}\int_{t_1}^{t_2} \left(\frac{a}{\sqrt{X(s)+\varepsilon_n}} - \frac{\sigma^2}{4}\frac{X(s)}{(X(s)+\varepsilon_n)^{\frac{3}{2}}} \right) \mathrm{d}s \\
&= \lim_{n\to\infty} \frac{1}{2}\int_{t_1}^{t_2} \left(\frac{\sigma^2}{4}\frac{\varepsilon_n}{(X(s)+\varepsilon_n)^{\frac{3}{2}}} - \frac{\delta}{\sqrt{X(s)+\varepsilon_n}} \right) \mathrm{d}s \\
&= -\frac{1}{2}\int_{t_1}^{t_2} \frac{\delta}{\sqrt{X(s)}}\,\mathrm{d}s \\
&< 0.
\end{aligned}
$$

On the other hand, L is not strictly decreasing on the entire $[0, T]$: if it is strictly decreasing (and, since $L(0) = 0$, non-positive), then $Y \leq U$, where U is the standard Ornstein-Uhlenbeck process defined by

$$
U(t) = Y(0) - \frac{b}{2}\int_0^t U(s)\,\mathrm{d}s + \frac{\sigma}{2}W(t).
$$

However, it is not possible since Y cannot take negative values.

5. Connection to Skorokhod reflections

Finally, let us present the connection of low-dimensional CIR processes with Skorokhod problems. For $\delta > -\frac{\sigma^2}{4}$, consider a family of CIR processes $\{X_\delta\}$ with $a = a(\delta) = \frac{\sigma^2}{4} + \delta$ and defined by

$$
X_\delta(t) = x_0 + \int_0^t \left(\frac{\sigma^2}{4} + \delta - bX_\delta(s) \right) \mathrm{d}s + \sigma \int_0^t \sqrt{X_\delta(s)}\,\mathrm{d}W(s). \tag{37}
$$

As described above in Sections 3 and 4, the process $Y_\delta := \sqrt{X_\delta}$ satisfies the SDE of the form

$$
Y_\delta(t) = \sqrt{x_0} - \frac{b}{2}\int_0^t Y_\delta(s)\,\mathrm{d}s + \frac{\sigma}{2}W(t) + L_\delta(t),
$$

where the term L_δ depends on the parameter δ as follows:

- if $-\frac{\sigma^2}{4} < \delta < 0$,

$$L_\delta(t) = -\frac{1}{2}\left(\frac{\sigma^2}{4} - a\right)\int_0^\infty y^{\frac{4a}{\sigma^2}-2}\left(\ell(t,y) - \ell(t,0)\right)\,\mathrm{d}y$$
$$= \frac{\delta}{2}\int_0^\infty y^{\frac{4\delta}{\sigma^2}-1}\left(\ell(t,y) - \ell(t,0)\right)\,\mathrm{d}y,$$

  where $\ell(t,y) = y^{-\frac{4\delta}{\sigma^2}} L^{Y_\delta}(t,y)$ is the normalized local time of Y_δ, see (24);
- if $\delta > 0$,

$$L_\delta(t) = \frac{1}{2}\int_0^t \frac{\delta}{Y_\delta(s)}\,\mathrm{d}s$$

  and the integral is well-defined and finite with probability 1;
- if $\delta = 0$, L_0 is the Skorokhod reflection function, i.e. a continuous non-decreasing process with points of growth occurring only at zeros of Y_0 and such that $Y_0 \geq 0$, which is a symmetric local time of Y_0 at 0; in particular, Y_0 is a reflected Ornstein-Uhlenbeck process.

The dynamics of Y_δ with $\delta \geq 0$ described above allowed [27] to obtain the following alternative representation to the Skorokhod reflection function L_0.

Theorem 5.1 ([27, Theorem 2.4]): *Let $\{\delta_n,\ n \geq 1\}$ be an arbitrary positive sequence such that $\delta_n \downarrow 0$, $n \to \infty$. Then, with probability 1, for any $T > 0$*

$$\sup_{t\in[0,T]} |Y_{\delta_n}(t) - Y_0(t)| \to 0$$

and

$$\sup_{t\in[0,T]} \left|L_0(t) - L_{\delta_n}(t)\right| = \sup_{t\in[0,T]} \left|L_0(t) - \frac{1}{2}\int_0^t \frac{\delta_n}{\sqrt{X_{\delta_n}(s)}}\,\mathrm{d}s\right| \to 0$$

as $n \to \infty$.

Theorem 5.1 essentially concerns the case $\delta \to 0+$ but does not discuss what happens when $\delta \to 0-$, so we finalize the Section by filling this gap.

Theorem 5.2: *Let $\{\delta_n,\ n \geq 1\}$ be an arbitrary positive sequence such that $\delta_n \downarrow 0$, $n \to \infty$. Then, with probability 1, for any $T > 0$*

$$\sup_{t\in[0,T]} |Y_{-\delta_n}(t) - Y_0(t)| \to 0 \tag{38}$$

and

$$\sup_{t\in[0,T]} \left|L_{-\delta_n}(t) - L_0(t)\right| \to 0 \tag{39}$$

as $n \to \infty$.

Proof: By [21, Theorem 1.1], for any $t \geq 0$, $X_{-\delta_n}(t) \leq X_{-\delta_{n+1}}(t) \leq X_0(t)$ a.s. Moreover, by [26, Theorem 4.1],

$$\sup_{t\in[0,T]} \mathbb{E}\left[\left|X_{-\delta_n}(t) - X_0(t)\right|\right] \to 0, \quad n \to \infty,$$

and hence, for all $t \geq 0$,

$$Y_{-\delta_n}(t) \xrightarrow{\mathbb{P}} Y_0(t), \quad n \to \infty,$$

and

$$\int_0^t Y_{-\delta_n}(s)\,\mathrm{d}s \xrightarrow{\mathbb{P}} \int_0^t Y_0(s)\,\mathrm{d}s, \quad n \to \infty.$$

Therefore, since monotone convergence in probability implies almost sure convergence, for any $t \geq 0$

$$Y_{-\delta_n}(t) \to Y_0(t)$$

and

$$\int_0^t Y_{-\delta_n}(s)\,\mathrm{d}s \to \int_0^t Y_0(s)\,\mathrm{d}s$$

a.s. as $n \to \infty$ and hence, with probability 1,

$$\begin{aligned} L_{-\delta_n}(t) &= Y_{-\delta_n}(t) - \sqrt{x_0} + \frac{b}{2}\int_0^t Y_{-\delta_n}(s)\,\mathrm{d}s - \frac{\sigma}{2}W(t) \\ &\to Y_0(t) - \sqrt{x_0} + \frac{b}{2}\int_0^t Y_0(s)\,\mathrm{d}s - \frac{\sigma}{2}W(t) \\ &= L_0(t), \quad n \to \infty. \end{aligned}$$

It remains to note that Y_0 as well as each $Y_{-\delta_n}$ have a.s. continuous paths and $\{Y_{-\delta_n}(t),\ n \geq 1\}$ is non-decreasing a.s. w.r.t. n, which immediately yields (38) by Dini's theorem. Similarly, L_0 as well as all $L_{-\delta_n}$ are continuous with probability 1 and

$$\begin{aligned} L_{-\delta_n}(t) &= Y_{-\delta_n}(t) - \sqrt{x_0} + \frac{b}{2}\int_0^t Y_{-\delta_n}(s)\,\mathrm{d}s - \frac{\sigma}{2}W(t) \\ &\leq Y_{-\delta_{n+1}}(t) - \sqrt{x_0} + \frac{b}{2}\int_0^t Y_{-\delta_{n+1}}(s)\,\mathrm{d}s - \frac{\sigma}{2}W(t) \\ &= L_{-\delta_{n+1}}(t), \end{aligned}$$

which implies (39). ■

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