CREDIT
CONTAGION

by

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Preface

The aim of my study has been to review models for credit contagion finalizing the study to the computation of derivative prices. Credit contagion is a fairly new field to be studied. Kusuoka introduced a way to model dependent defaults in 1998 and Davis and Lo, [4], introduced a model for default contagion in 2001. Credit contagion is an element of credit risk. Credit risk consists of individual risk elements, such as default probability and recovery rates, and it consists of portfolio risk elements like default correlation. There are roughly two approaches to model credit risk; structural modeling and intensity based modeling. The structural models (also called firm value models) goes back to Merton. In these models, default is triggered when the value process (which might be modeled by a standard geometric Brownian motion) of a firm falls below a pre-determined default boundary. In the intensity based models (also known as reduced form models) default is typically described as a jump time of a jump process (for instance a Poisson process).

Default is, as mentioned, an element of credit risk. Modeling credit risk is importaint when it comes to the modeling of derivative pricing, such as the prices of credit default swaps, (CDS), and collateral debt obligations, (CDO), which are basic protection contracts against default of firms in a portfolio. CDS' and CDOs have been largely talked about during the latest financial crisis since, among others, credit rating agencies (which evaluate the default probability of issuers of debt securities) failed to adequately account for large risks when rating these products. Credit rating agencies, like Moody's and Credit Suisse, calculate the default likelihood of firms. To model correlations between the default behavior of firms, Credit Suisse uses the correlations in equity values as a replacement for the correlations in the default probabilities (also known as correlations in credit quality). Moody's uses the 'diversity score' which is based on the binomial expansion technique, where independence between firms is assumed. Moody's idea on how to capture correlations in a binomial distribution is to make a hypothetical portfolio consisting of less firms than the original one, and having the hypothetical firms being independent. Then a default in the 'new' portfolio would correspond to, say, 2 defaults in the original portfolio. Other ways to model default and credit contagion might be to introduce primary and secondary firms, as in the approach of Jarrow and Yu, referred to in [6]. In the model by Jarrow and Yu the defaults of the primary firms are influenced by macoreconomic conditions (i.e. influenced by the gross domestic product, unemployment rate and inflation rate), but not by the credit risk of counterparties. The default of the secondary firm depends on the status of other firms, so it suffices to focus on securities issued by secondary firms. Kusuoka's approach to model default dependence, which is also referred to in [6], is based on a change of probability measure. Kusuoka assumed that the default times were exponentially distributed. The probability measure is then changed so that the parameter of the exponential law belonging to one firm will jump to a pre-determined value as soon as the default of another firm occurs. Yet another approach is the one by

Davis and Lo, [4]. They model default in a portfolio by independent Bernoulli variables where default can occur due to direct default of a company or by contagion. The model suggested by Biagini, Fuschini and Klüppelberg is based on the one by Davis and Lo.

My studies of the modeling of credit contagion and pricing of derivatives are based on the papers of Biagini, Fuschini and Klüppelberg [2] and of Hatchett and Kühn [3]. The paper is organized as follows: First I will give a short introduction to contagion and default, as well as a general calculation of derivative prices. In chapter 2 I will describe the default intensities for the discrete time model as in [3]. In chapter 3 the continuous time model will be described as in [2]. I will also compare some elements of the two models. The pricing of derivatives will be presented in the two chapters where the model description is taking place. Finally, I will present an extention of the continuous time contagion description in chapter 4. This extension consists in expanding the economic relationship between firms from not just being present or absent, but to try to say how much they can influence each other economically if they are in an economic relationship. This means that the extension is trying to describe which type of economic relationship the firms are in - a competitive or cooperative economic relationship.

Contents

Chapter 1

Introduction

Credit contagion arises when a company is in economic distress or if it defaults. The default of a company will have implications for any firm that is economically connected to this given company. The effect of the default, and thus the effect of the credit contagion, depends on which economic relation the defaulting company has with other firms. If they were in a cooperative relationship, the default would have a negative effect on the firms that are connected to the defaulting company. For instance, if a company goes bankrupt, it will have a negative effect on the credit situation of its service provider or on its bank connection. On the other hand, if they were in a competitive economic relation, the default would have a positive effect on the firms that are connected to the defaulting company. For example, if there is a default within a business section, the number of orders might increase for the surviving firms.

One of the main worries when investing in a portfolio consisting of defaultable bonds is to not recieve the promised payment at the date of maturity, and this may occur if a bond defaults. If one bond defaults, there might be the risk of default contagion resulting in several defaults within the portfolio. Hence, the loss will be even larger. This is one of the reasons why one is interested in credit contagion.

1.1 Intensity based default

Default in an intensity based model is specified in terms of a jump process, and the jump occurs at time τ which is typically modeled as a jump time of a jump process. What drives contagion are the default intensities of the firms within the portfolio. The default probability is the probability that the obligor or counterparty will default on its contractual obligations to repay its debt, [5]. Denote the random time of default by $\tau : \Omega \to \mathbb{R}^+$ which is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. τ is assumed to be unbounded and non-negative. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. Further, consider the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, where, for any t, $\mathcal{G}_t$ is some given σ-algebra which contains all the null sets of $\mathcal{F}_t$ and is right-continuous on the given probability space with $\mathcal{F}_t \subset \mathcal{G}_t$.

1.1.1 Definitions regarding default

Default in an intensity based model is regarded as a stopping time with respect to a given filtration, and one has to consider the two different cases of continuous or discrete time. Starting with the definition for a discrete time model.

Definition (Stopping time in discrete time) An $\mathbb{F}$-stopping time on $(\Omega, \mathcal{F}, \mathbb{P})$ is a random variable $\tau : \Omega \to \mathbb{N} \cup \{ \infty \}$ such that $\{ \omega \in \Omega : \tau(\omega) = n \}$ is in $\mathcal{F}_n$ for all n in $\mathbb{N}$.

The definition of the default time for a continuous time model is as follows:

Definition (Stopping time in continuous time) An $\mathbb{F}$-stopping time on $(\Omega, \mathcal{F}, \mathbb{P})$ is a random variable $\tau : \Omega \to [0, \infty]$ such that $\{ \omega \in \Omega : \tau(\omega) \leq t \}$ is in $\mathcal{F}_t$ for all t in [0,T].

Let F be the cumulative distribution function of τ, then $F(t) = \mathbb{P}(\tau \leq t)$ for every t in $\mathbb{R}^+$ is the default probability, and $1 - F(t) = \mathbb{P}(\tau > t)$ is the survival probability. If $\mathbb{P}(\tau \in (0, \infty)) > 0$ the stopping time is non-trivial. The following definition is from [6].

Definition (Hazard and intensity function) An increasing function $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ given by the formula

$$\Gamma(t) := -ln(1 - F(t)) \text{ for all t in } \mathbb{R}^+,$$

is called the *hazard function* of τ. If the cumulative distribution function F is absolutely continuous with respect to the Lebesgue measure - that is, when $F(t) = \int_0^t f(u)du$, for a Lebesgue integrable function $f : \mathbb{R}^+ \to \mathbb{R}^+$, then

$$F(t) = 1 - e^{-\Gamma(t)} = 1 - e^{-\int_0^t \gamma(u)du},$$

where $\gamma(t) = f(t)(1 - F(t))^{-1}$. The function γ is called the *intensity function* (or the *hazard rate*) of the random time τ.

By assuming that $F(t) < 1$, the hazard function Γ_t is well defined for any t in $\mathbb{R}^+$ since the function f is positive and the cumulative distribution function also is positive, the intensity function γ is non-negative.

In order to give some examples of the different types of default intensities, one has to define some processes. Remember that a counting process *N* is defined through an increasing sequence $\{ T_0, T_1, \ldots \}$ of random variables, or random times, in

[0,∞]. The process *N* is called non-explosive if $\lim_n T_n = +\infty$ almost surely. Recall as well that a random variable X with outcomes $\{\ 0, 1, 2, \ldots\}$ is Poisson distributed with parameter λ in $(0, \infty)$, written $X \sim \mathcal{P}o(\lambda)$, if $\mathbb{P}(X = x) = \frac{\lambda^x}{x!}e^{-\lambda}$ where $0! = 1$. The following definitions are from [7].

Definition (Poisson Process) *A Poisson process* is a $\mathbb{G}$-adapted non-explosive counting process *N* with deterministic intensity $\lambda > 0$ such that $\int_0^t \lambda_s ds$ is finite dt-almost everywhere for all t, with the property that, for all t and $s > t$, conditional on $\mathcal{G}_t$, the random variable $(N_s - N_t) \sim \mathcal{P}o(\int_t^s \lambda_u du)$.

The filtration $(\mathcal{G}_t)_{t\geq 0}$ has been fixed in advance for the purpose of the definitions. Alternatively, for $s > t$, one can say, since the increment $(N_s - N_t)$ is independent of the σ-field $\sigma(N_u : u \leq t)$, that $\mathbb{P}\Big((N_s - N_t) = k|\mathcal{G}_t\Big) = \mathbb{P}\Big((N_s - N_t) = k\Big) = \frac{(\lambda(s-t))^k}{k!}e^{-\lambda(s-t)}$.

Definition (Doubly Stochastic Process) Let N be a $\mathbb{G}$-adapted non-explosive counting process with intensity $\lambda > 0$. N is *doubly stochastic*, driven by $\mathbb{F}$, if λ is $\mathbb{F}$-predictable and if, for all t and $s > t$, conditional on the filtration $\mathcal{G}_t \vee \mathcal{F}_s$, $(N_s - N_t) \sim \mathcal{P}o(\int_t^s \lambda(\omega, u))du$. A doubly stochastic process is also called a *Cox process*.

The intuition behind a doubly stochastic counting process is that $\mathcal{F}_t$ contains enough information to uncover the default intensity λ_t, but not information to uncover the jump times of the counting process.

1.1.2 Some examples of default intensities

This thesis is not considering structural modeling, but just to have it mentioned: In the basic Merton model of default, τ happens when the value of the firm at the time of maturity, T, falls below the face value of the bond. Thus, default is only possible at T. And in the model of Black and Cox, τ is modeled as a first passage time in which default happens when the value process of the firm reaches the level of its debt for the first time. In these kinds of models default is economic motivated. In the intensity based models defaults happen when an intensity based process makes a jump. One can mention three types of default intensities: constant-, deterministic- and stochastic default intensities.

In the three following cases, let τ be exponentially distributed with intensity parameter λ and $\tau := inf\{\ t > 0 : N_t = 1\}$, where N_t is a Poisson process. This means that the time of default can be seen as the time of the first jump of N_t.

In a model where the default intensity is constant, the default probability is $\mathbb{P}(\tau \leq t) = 1 - e^{-\lambda t}$. The intensity function is $\gamma(t) = \lambda$ for all t in $\mathbb{R}^+$ and it is constant for all t. A Poisson process with constant intensity $\lambda > 0$ is called a time-homogeneous Poisson process.

If one has a deterministic default intensity, then $N_t \sim \mathcal{P}o(\int_0^t \lambda_u du)$. The default probability would be $\mathbb{P}(\tau \leq t) = 1 - e^{-\int_0^t \lambda_u du}$. The intensity function is $\gamma(t) = \lambda(t)$ and it varies with the time t. A Poisson process with deterministic intensity $\lambda > 0$ is called a time-inhomogeneous Poisson process.

When one has the case of stochastic default intensity, N_t is a doubly stochastic process which is Poisson distributed with parameter $\int_0^t \lambda(\omega, u)du$. The parameter of the exponential distribution is $\lambda(\omega, u)$, and τ is a $\mathbb{G}$-stopping time. Both the intensity and the stopping time are stochastic, and this is why a Cox process is sometimes called a doubly stochastic Poisson process. The general probability of default would be, for $t \leq s$,

$$\mathbb{P}(t < \tau \leq s|\mathcal{G}_t) = \mathbb{E}(1 - e^{-\int_t^s \lambda(\omega,u)du}|\mathcal{G}_t).$$

And for $t = 0$, the default probability becomes

$$\mathbb{P}(\tau \leq s) = \mathbb{E}(1 - e^{-\int_0^s \lambda(\omega,u)du}).$$

The expectations are under $\mathbb{P}$. In these two cases the intensity function is $\gamma(s) = \lambda(\omega, s)$. The two expressions can be evaluated by the same means as in calculating the price of a default free zero coupon bond (a contract paying one unit of currency at the time of maturity), by letting λ_t be the short term interest rate, r_t, and solve the stochastic differential equation by, for instance, the Vasicek or CIR models. As an illustration one can consider the short term interest rate to be the instantaneous spot rate and the bank account to grow at each time instant t at a rate of r_t. One can look, for example, at *the fundamental pricing formula*, found in [12]:

The price of an attainable contingent claim with payoff H_T at time $T > t$ is given by

$$V_t = \mathbb{E}_{\mathbb{Q}}(e^{-\int_t^T r_s ds} H_T|\mathcal{F}_t) \tag{1.1}$$

where the risk neutral measure $\mathbb{Q} \sim \mathbb{P}$ is assumed to exist. Before moving on with the calculations, one can recall the meaning of an attainable claim:
A *contingent claim* is an $\mathcal{F}_T$-measurable random variable F in $L_2(\mathbb{Q})$. The contingent claim F is *attainable* on the given market model if there exists an admissible portfolio Z such that the value process of the portfolio at time T is $V_y^Z(T) = F$ where $y = V^Z(0)$. The portfolio Z is *admissible* if it is self-financing and lower bounded, $V_t^Z \geq -K$, K > 0 for all t P-almost surely. By *self-financing* one means that $dV_t^Z = Z_t \cdot dX_t$ where $X_t(i)$ is the price of security i at time t, and the value process has to be integrable.

Moving on with the calculations of equation (1.1), let the short rate r_t be the the intensity function λ_t and $H_T = 1$ as the face value of the zero coupon bond. By applying the Vasicek model, the dynamics of λ_t is given by the stochastic differential equation

$$d\lambda_t = a(b - \lambda_t)dt + \sigma dB_t \tag{1.2}$$

where a,b and σ are positive constants. (The CIR model is similar to the expression in equation (1.2), but where the term $\sigma dB_t = \sigma\sqrt{\lambda_t}dB_t$). B_t is a standard one-dimensional Brownian motion generating $\mathcal{F}_t$. Note that $\mathbb{F}$ coinsides with the filtration generated by λ_t. By letting $X_t = -(b - \lambda_t)$ one gets that (1.2) can be written as

$$\begin{cases} dX_t = -aX_t dt + \sigma dB_t \\ X_0 = \lambda_0 - b \end{cases}$$

which is an Ohrnstein-Uhlenbeck process, and it is an affine process which means that it is Markovian and that there exists an explicit expression for (1.1). The Ornstein-Uhlenbeck process is solved by applying Itô's formula with integrating factor e^{at}, so

$$d(X_t e^{at}) = ae^{at}X_t dt + e^{at}(-aX_t + \sigma dB_t)$$

and by integrating from s to t and dividing by the integrating factor one gets

$$X_t = X_s e^{-a(t-s)} + e^{-at}\int_s^t \sigma e^{au} dB_u, \qquad s \leq t.$$

By substituting $X_t = -(b - \lambda_t)$, the answer to equation (1.2) is

$$\lambda_t = \lambda_s e^{-a(t-s)} + b(1 - e^{-a(t-s)}) + \sigma \int_s^t e^{-a(t-u)} dB_u. \tag{1.3}$$

The process λ_t is Gaussian. By looking at $\mathbb{E}(\lambda_t|\mathcal{F}_s)$ and $Var(\lambda_t|\mathcal{F}_s)$, one finds that
$\lambda_t \sim \mathcal{N}\left(\lambda_s e^{-a(t-s)} + b(1 - e^{-a(t-s)}), \frac{\sigma^2}{2a}(1 - e^{-2a(t-s)})\right)$. By looking at $\lim_{t\to\infty} \mathbb{E}(\lambda_t) = b$, one can regard b as a long term average intensity.

The dynamics for λ_t is under $\mathbb{P}$. In order to use the pricing formula given in equation (1.1), one has to find the dynamics for the intensity λ_t under the measure $\mathbb{Q}$. By Girsanov Theorem (theorem 8.6.6 in [13]) one gets that $\tilde{B_t} = B_t + \int_0^t q ds$, where q is in $\mathbb{R}$ and $q = \frac{a(b-\lambda_t)-\alpha}{\sigma}$. Notice that q depends on t through λ_t, but Vasicek assumed that the instantaneous spot rate (which is now λ_t) under the measure $\mathbb{P}$ evolves as an Ornstein-Uhlenbeck process with constant coefficients. By the given choice of q it is also assumed that the coefficients are constant under $\mathbb{Q}$ as well. The dynamics in equation (1.2) becomes

$$d\lambda_t = a(b - \lambda_t)dt + \sigma dB_t \qquad \text{under } \mathbb{P}$$
$$d\lambda_t = a(b - \frac{\sigma q}{a})dt - a\lambda_t dt + \sigma d\tilde{B}_t \qquad \text{under } \mathbb{Q}.$$

The last equality could be stated as $\alpha_t dt + \sigma d\tilde{B}_t$, but in this case let $(b - \frac{\sigma q}{a}) = \tilde{b}$. Then $\tilde{X}_t = \lambda_t - \tilde{b}$ implies that

$$d\tilde{X}_t = -a\tilde{X}_t dt + \sigma d\tilde{B}_t. \tag{1.4}$$

This is an Ornstein-Uhlenbeck process under the measure $\mathbb{Q}$ as well, so it is Gaussian with continuous paths. For $H_T = 1$, the pricing formula given in equation (1.1) can be written as

$$V_t = \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T \lambda_s ds} | \mathcal{F}_t\Big) = \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T \tilde{X}_s + \tilde{b}\ ds} | \mathcal{F}_t\Big)$$
$$= e^{-\int_t^T \tilde{b}\ ds}\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T \tilde{X}_s ds} | \mathcal{F}_t\Big). \tag{1.5}$$

Since the coefficients in the Ornstein-Uhlenbeck equation are time-independent, one can write

$$\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T \tilde{X}_s ds} | \mathcal{F}_t\Big) = F(T - t, \tilde{X}_t) \tag{1.6}$$

where F is the function defined by $F(\theta, x) = \mathbb{E}_{\mathbb{Q}}(e^{-\int_0^\theta \tilde{X}_s^x ds})$ and $\tilde{X}_s^x$ is the unique solution of equation (1.4) satisfying $\tilde{X}_0^x = x$. In this case $x = \lambda_0 - \tilde{b}$ and $\tilde{X}_s^x$ will just be written $\tilde{X}_s$. Recall that the Laplace transformation of a random variable, for u in $\mathbb{R}$, is $\mathbb{E}(e^{uX}) = \int_{\mathbb{R}} e^{ux}\mathbb{P}_X(dx)$. The expectation is calculated by Laplace transformation of a Gaussian random varibale:

$$\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_0^\theta \tilde{X}_s ds}\Big) = e^{\Big(-\mathbb{E}_{\mathbb{Q}}(\int_0^\theta \tilde{X}_s ds) + \frac{1}{2}Var_{\mathbb{Q}}(\int_0^\theta \tilde{X}_s) ds\Big)}.$$

The expectation becomes, where the first equality is due to Fubini's,

$$\mathbb{E}_{\mathbb{Q}}(\int_0^\theta \tilde{X}_s ds) = \int_0^\theta \mathbb{E}_{\mathbb{Q}}(\lambda_s - \tilde{b})ds$$
$$= \int_0^\theta \mathbb{E}_{\mathbb{Q}}(\lambda_0 e^{-a(s)} + \tilde{b}(1 - e^{-a(s)}) + \sigma \int_0^s e^{-a(s-u)} dB_u - \tilde{b})ds$$
$$= (\frac{\lambda_0 - \tilde{b}}{a})(1 - e^{-a\theta}). \tag{1.7}$$

To calculate $Var_{\mathbb{Q}}(\int_0^\theta \tilde{X}_s ds)$ one starts out with writing the variance as the covariance. Recall that for two random variables X and Y with expectation μ_X and μ_Y, respectively, the covariance is defined as $Cov(X,Y)=\mathbb{E}\Big((X-\mu_X)(Y-\mu_Y)\Big)$. Then

$$Var_{\mathbb{Q}}(\int_0^\theta \tilde{X}_s ds) = \int_0^\theta \int_0^\theta Cov(\tilde{X_t},\tilde{X_u})du\, dt$$
$$= \int_0^\theta \int_0^\theta (\mathbf{1}_{\{\ t>u\}}\ + \mathbf{1}_{\{\ t\leq u\}}\ )Cov(\tilde{X_t},\tilde{X_u})du\, dt. \tag{1.8}$$

Looking at the expression for the covariance without the integrals and putting $\tilde{X}_t = \lambda_t - \tilde{b}$ and by using the expression given in equation (1.3) , for $s=0$, one gets

$$Cov(\tilde{X_t},\tilde{X_u}) =$$
$$\mathbb{E}_{\mathbb{Q}}\Big(\big(\lambda_0 e^{-at}+\tilde{b}(1-e^{-at})+\sigma\int_0^t e^{-a(t-s)}dB_s-\tilde{b}-(\lambda_0 e^{-at}+\tilde{b}(1-e^{-at})-\tilde{b})\big)$$
$$\times\big(\lambda_0 e^{-au}+\tilde{b}(1-e^{-au})+\sigma\int_0^t e^{-a(u-s)}dB_s-\tilde{b}-(\lambda_0 e^{-au}+\tilde{b}(1-e^{-au})-\tilde{b})\big)\Big)$$
$$=\mathbb{E}_{\mathbb{Q}}\Big((\sigma e^{-at}\int_0^t e^{as}dB_s)(\sigma e^{-au}\int_0^u e^{as}dB_s)\Big)$$
$$=\sigma^2 e^{-a(t-u)}\int_0^{t\wedge u} e^{2as}ds$$
$$=\sigma^2 e^{-a(t-u)}\frac{(e^{2a(t\wedge u)}-1)}{2a}.$$

Inserting this answer into equation (1.8), one is left with calculating

$$\int_0^\theta\int_0^\theta \sigma^2 e^{-a(t-u)}\frac{(e^{2a(t\wedge u)}-1)}{2a}dudt$$
$$=\frac{\sigma^2}{2a}\int_0^\theta\Big(\int_0^u e^{-at-au+2at}-e^{-at-au}dt\Big)du$$
$$+\frac{\sigma^2}{2a}\int_0^\theta\Big(\int_0^t e^{-at-au+2at}-e^{-at-au}du\Big)dt$$
$$=\frac{\sigma^2\theta}{a^2}-\frac{\sigma^2}{a^3}(1-e^{-a\theta})-\frac{\sigma^2}{2a^3}(1-e^{-a\theta})^2. \tag{1.9}$$

Finally, by putting together the expressions for the expectation in equation (1.7) and the variance in equation (1.9), and by using equation (1.6), one gets that the pricing formula in equation (1.5) becomes:

$$exp\{\ -\tilde{b}(T-t) - (\frac{\lambda_0 - \tilde{b}}{a})(1-e^{-a(T-t)})\}$$
$$\times exp\{\ \frac{1}{2}\Big(\frac{\sigma^2(T-t)}{a^2} - \frac{\sigma^2}{a^3}(1-e^{-a(T-t)}) - \frac{\sigma^2}{2a^3}(1-e^{-a(T-t)})^2\Big)\}$$

which is more frequently expressed as

$$P(t,T) = e^{-(T-t)R(T-t,\lambda_t)}, \tag{1.10}$$

which is the affine structure mentioned previously, and where $R(T-t,\lambda_t)$ is given by

$$R(\theta,\lambda) = (\tilde{b} - \frac{\sigma^2}{2a^2}) - \frac{1}{a\theta}\Big((\tilde{b} - \frac{\sigma^2}{2a^2} - \lambda)(1-e^{-a\theta}) - \frac{\sigma^2}{4a^2}(1-e^{-a\theta})^2\Big).$$

If the model for λ_t is under the measure $\mathbb{P}$, historical data should be used to estimate the drift and volatility. If the model is under the measure $\mathbb{Q}$, the risk adjusted drift and volatility can only be inferred from existing prices. One drawback of the Vasicek model is if $b = \lambda_t$, then the dynamics become $d\lambda_t = \sigma dB_t$ and one has a random walk. The Brownian component can take positive and negative values, so the intensity λ_t might be negative, and default intensities are supposed to be positive . In the CIR model the default intenstiy will not become negative, but the process is not Gaussian and explicit formulae are more difficult to come by.

Chapter 2

Contagion model in discrete time

The paper of Hatchett and Kühn, [3], describes credit contagion in a discrete time framework. They are using probability theory to express their findings. The results are based on the use of the Law of Large Numbers and the Central Limit Theorem. In order to use the Law of Large Numbers and the Central Limit Theorem the random variables (the firms) has to be independent and identically distributed, and the number of firms, *m*, has to tend to infinity. The firms and their enviroment within the portfolio are assumed to be fairly homogeneous, so the firms are thus assumed to be similar to each other or of the same type. The only possible states for the firms are solvent or defaulted. They describe the default process $Z_t(i)$ by a discrete time Markov chain where the probability of default of firm *i* in a given time step depends on the state of its economic partners at the beginning of that given time period, as well as on the macroeconomic interference. The time period $[0, T] = \{\ 0, 1, \ldots, T\}$ describes a one year range. It is assumed that the defaulting state is absorbing and that there is one single macroeconomic factor which is constant over the time period of one year. Hatchett and Kühn did not consider pricing in their paper, so in section 2.4 there will be given some examples of pricing by using their default intensity.

2.1 The framework

Let the number of firms $m \to \infty$. For i in $\{\ 1, 2, \ldots, m\}$ the default process of the portfolio is denoted by $Z_t(i)$ for $t = 0, \ldots, T$ and is described by a binary indicator variable, meaning

$$Z_t(i) = \begin{cases} 0 & \text{firm } i \text{ is solvent at time } t, \\ 1 & \text{firm } i \text{ has defaulted at time } t \text{ by itself.} \end{cases}$$

The default process is a Markov chain and it evolves accordingly:

$$\begin{cases} Z_{t+1}(i) = Z_t(i) + (1 - Z_t(i))\mathbf{1}_{\{\ W_t(i)<0\}}\ , \\ Z_0(i) = 0 \end{cases} \tag{2.1}$$

The $Z_t(i)$ are functions of the wealth, which is stochastic. It is assumed that a firm defaults when its wealth falls below zero. The wealth process, $W_t(i)$, is the value of the total wealth of firm *i* at time t and it is given by

$$\begin{cases} W_t(i) = \vartheta_i - \sum_{j=1}^m C_{ij} Z_t(j) - \eta_t(i) & t = 1, 2, \ldots, T \\ W_0(i) = \vartheta_i > 0 \end{cases} \tag{2.2}$$

The constant ϑ_i is the initial wealth of firm *i* at time $t = 0$. Note that the initial wealth ϑ_i does not depend on time, so the model does not say anything about how firm i is making or loosing money in each time epoch. By letting the wealth process $W_t(i)$ depend on t and not on $t+1$, i.e. not $W_{t+1}(i)$, means that the possible effect of credit contagion does not happen immediately. This means that the default process can have its first default state at $Z_2(i)$. If the wealth process was depending on $t+1$, i.e. $W_{t+1}(i)$, then any default contagion effect would influence the portfolio default process immediately.

The matrix C_{ij} describes the credit contagion relation of firm *i* and *j*, so for firm $i \neq j$ in $\{\ 1, \ldots, m\}$ one has

$$\begin{cases} C_{ij} > 0 & \text{firm i and j in a cooperative economic relation} \\ C_{ij} = 0 & \text{firm i and j independent} \\ C_{ij} < 0 & \text{firm i and j in a competitive economic relation.} \end{cases}$$

The case $C_{ii} = 0$. If company *j* defaults and $C_{ij} > 0$ it means that the two firms had a cooperative credit relation, and the default of *j* would contribute to a decrease in the wealth of firm *i*. If $C_{ij} < 0$, then *j* and *i* had a competitive relation and $W_t(i)$ would increase due to the default of firm j. The case $C_{ij} = 0$ means that the firms are not in an economic relation at all. Further description of the contagion term C_{ij} will be given in section 2.2.

The fluctuating forces disturbing the wealth process of a company given in equation (2.2) is the random variable $\eta_t(i) \sim \mathcal{N}(0, \sigma_i^2)$, and it is decomposed into a term describing individual fluctuations (for instance, extremely productive employees or defect production equipments) and another term describing the macroeconomic factor, which is one dimensional. In other words,

$$\eta_t(i) = \sigma_i(\sqrt{\rho_i}\eta_0 + \sqrt{1-\rho_i}\xi_t(i)). \qquad (2.3)$$

Here, σ_i is a scaling parameter. The random variables $\xi_t(i) \sim \mathcal{N}(0,1)$ are independent and they describe the individual fluctuations of firm i. The macroeconomic factor is described by $\eta_0 \sim \mathcal{N}(0,1)$, but it is assumed to be constant over the time horizon of one year in this model. The information of η_0 is known at time $t = 0$, so η_0 is treated as a known constant element of the model.

In choosing the correlations of $\eta_t(i)$ and $\eta_t(j)$, Hatchett and Kühn followed the prescription given by BASEL II (which are recomondations on banking laws and regulations) which sets

$$\rho_i \approx 0.12(1 + e^{-50PD_i}), \qquad (2.4)$$

where PD_i is the probability of self default of firm *i* over one year, ignoring credit contagion effects. (More on PD_i is found in section 2.3.)

2.2 The contagion model

The contagion quantities C_{ij} which describe the loss or gain of firm *i* caused by the default of firm *j* are given by

$$C_{ij} = c_{ij}\Big(\frac{C_0}{c} + \frac{C}{\sqrt{c}}x_{ij}\Big), \qquad (2.5)$$

where the random variables $x_{ij} \sim \mathcal{N}(0,1)$ are assumed to be pairwise independent. The c_{ij} are randomly fixed in the sence that the way to assign the value of c_{ij} is according to a random generator. The c_{ij} are fixed numbers that are either 0 or 1, and they describe the absence or presence of a business connection between firm i and j. The c_{ij} has a probability distribution given by

$$\mathbb{P}(c_{ij} \in \{\,0,1\}\,) = \frac{c}{m}\delta_{c_{ij}}(1) + (1 - \frac{c}{m})\delta_{c_{ij}}(0), \quad c_{ij} = c_{ji},$$

where the c_{ij} is in $\{\,0,1\}$ and

$$\delta_{c_{ij}}(1) = \begin{cases} 1 & , c_{ij} = 1 \\ 0 & , c_{ij} = 0. \end{cases}$$

The quantity $(\frac{C_0}{c} + \frac{C}{\sqrt{c}}x_{ij}) \sim \mathcal{N}(\frac{C_o}{c}, \frac{C^2}{c})$ by linear transformation (see appendix A). It gives the size of the contagion strength and the size is not symmetric. This can be understood by the following example, which is from [8]: Let there be one small supplier with one large company taking the majority of its orders. If the larger company defaults then the small supplier may default as well. However, if the small supplier defaults then the larger company is less likely to suffer terminal distress. The number c is the average number of connections that a firm has, and the connections are symmetric, i.e. $c_{ij} = c_{ji}$. They also satisfy transitivity, i.e. $c_{ih}c_{hj} \leq c_{ij}$ or even some bigger loop. The value c is assumed to be large.

It it assumed that $Z_t(i), \sum_j C_{ij}Z_t(j)$ and $\xi_t(i)$ (appearing in equation (2.3)) are uncorrelated. The parameters C_0 and C determine the mean and variance of C_{ij}, which means that if $C_0 > 0$ the firms are not independent on average. For future calculations, one needs the distribution of

$$\sum_{j=1}^{m} c_{ij}\Big(\frac{C_0}{c} + \frac{C}{\sqrt{c}}x_{ij}\Big)Z_t(j). \tag{2.6}$$

The number c_{ij} in $\{\ 0, 1\}$ is known in advance. The $\frac{C_0}{c} + \frac{C}{\sqrt{c}}$ are numbers as well, so the only randomness in the expression in (2.6) is from x_{ij}, which are Gaussian with finite moments, and from $Z_t(j)$. Through equation (2.1) one sees that the $Z_t(i)$ are correlated, but Hatchett and Kühn argue that the $Z_t(i)$ are sufficiently weakly correlated for the limit theorems to be applied. The contributions to $\sum_{j=1}^{m} C_{ij}Z_t(j)$ are sufficiently weakly correlated because:
There are two ways that the neighbor firms of i can be correlated through the dynamics. Either through firm i or through some other loop, i.e. one can have C_{hi} and C_{ij} so firm h and j are correlated through firm i or one can have $C_{h\ell}$ and $C_{\ell j}$ so firm h and j are not correlated through firm i. However, as long as $Z_t(i) = 0$, $Z_t(h)$ and $Z_t(j)$ can not influence each other through firm i. But when $Z_t(i) = 1$, the correlation firm i induces on h and j are irrelevant for its own dynamics, i.e. for the dynamics of firm i.

By assumption, both the average connectivity c, which was assumed to be large, and the number of firms $m \to \infty$, and $c/m \to 0$ with $c = \mathcal{O}(log(m))$, which means that c goes more slowly to ∞ than m does. The variables $(\frac{C_0}{c} + \frac{C}{\sqrt{c}}x_{ij})$ are independent and identically distributed (i.i.d.). The default process is a binary indicator variable taking the value 0 or 1. To find the distribution of the sum in (2.6), exploit the fact that $(\frac{C_0}{c} + \frac{C}{\sqrt{c}}x_{ij})$ and $Z_t(j)$ are independent. Hatchett and Kühn treats the sum in (2.6) as a sum of Gaussian variables. This is seen form the way the find the distributin of the sum.

$$\sum_{j=1}^{m} c_{ij}\Big(\frac{C_0}{c} + \frac{C}{\sqrt{c}}x_{ij}\Big)Z_t(j) = \sum_{j=1}^{m} c_{ij}\frac{C_0}{c}Z_t(j) + \sum_{j=1}^{m} c_{ij}\Big(\frac{C}{\sqrt{c}}x_{ij}\Big)Z_t(j).$$

They apply the Law of Large Numbers (appendix A) to the first sum, and using $c = c_m \to \infty$

$$\sum_{j=1}^{m} c_{ij} \frac{C_0}{c} Z_t(j) \quad \xrightarrow{P} \quad C_0 \mathbb{E}(Z_t(j)).$$

This is only valid if the $Z_t(j)$ are i.i.d., and if so $\mathbb{E}(Z_t(j)) = \mathbb{E}(Z_t(1))$. Since $x_{ij} \sim \mathcal{N}(0,1)$, the expectation of the second part in the sum is zero. By using that $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$ for a random variable X, and the Continuity Theorem (appendix A), since $f(u) = u^2$ is a continuous function, then

$$\sum_{j=1}^{m} (c_{ij} \Big(\frac{C}{\sqrt{c}} x_{ij} \Big) Z_t(j))^2 \xrightarrow{P} \quad C^2 Var(c_{ij} Z_t(j))$$

as $c = c_m \to \infty$. Again, the variables have to i.i.d. If so, then $C^2 Var(c_{ij} Z_t(j)) = C^2 (\mathbb{E}(Z_t(1)))^2$. Then they use the Central Limit Theorem to find the asymptotic distribution of the sum in (2.6) and finds that

$$\sum_{j=1}^{m} c_{ij} \Big(\frac{C_0}{c} + \frac{C}{\sqrt{c}} x_{ij} \Big) Z_t(j) \xrightarrow{L} \quad \mathcal{N}(C_0 \mathbb{E}(Z_t(1)), C^2 (\mathbb{E}(Z_t(1)))^2).$$

Alternatively, one could argue that the $Z_t(j)$ are random variables taking the value 0 or 1 where the randomness comes from the underlying stochastic wealth process $W_t(j)$ which is Gaussian, thus the $Z_t(j)$ are Gaussian as well. They are independent of x_{ij}. The expectation of each element in the sum becomes, due to independence,

$$\mathbb{E}(c_{ij} \Big(\frac{C_0}{c} + \frac{C}{\sqrt{c}} x_{ij} \Big) Z_t(j)) = \frac{C_0}{c} \mathbb{E}(c_{ij} Z_t(j)),$$

and the variance

$$Var(c_{ij} \Big(\frac{C_0}{c} + \frac{C}{\sqrt{c}} x_{ij} \Big) Z_t(j)) = \frac{C^2}{c} Var(c_{ij} Z_t(j)).$$

Use linear transformation on the Gaussian sum and obtain that the distribution of the sum in (2.6)

$$\sum_{j=1}^{m} c_{ij}\Big(\frac{C_0}{c}+\frac{C}{\sqrt{c}}x_{ij}\Big)Z_t(j) \sim \mathcal{N}\Big(\frac{C_0}{c}\sum_{j=1}^{m}\mathbb{E}(c_{ij}Z_t(j)), \frac{C^2}{c}\sum_{j=1}^{m}Var(c_{ij}Z_t(j))\Big). \tag{2.7}$$

Let this be the distribution for the sum for future calculations.

2.3 The default intensities

The description of the default model is as seen from a structural point of view since default happens when the wealth process $W_t(i) < 0$. The starting point in an intensity based model is the modeling of the intensity process. The basic idea behind an intensity based model is that there are two states, solvent or defaulted. By letting $Z_t(i)$ be the state at time t of firm i and $\lambda_t(i)$ the transition intensity from solvent to default, then the transition intensity is interpreted as the probability of Z_t going from solvent to default in a short time interval.

Starting out with the very general case first, the variable $\eta_t(i)$ is now assumed to be standard normal. In order to find the default intensity in this model for the very general case, one has to look at

$$\begin{aligned}
\mathbb{P}\Big(Z_{t+1}(i)=1\Big|Z_t(i)=0,\sum_{j=1}^{m}C_{ij}Z_t(j)\Big) &= \mathbb{P}\Big(\mathbf{1}_{\{\ W_t(i)<0\}}\ =1\Big|\sum_{j=1}^{m}C_{ij}Z_t(j)\Big)\\
=\mathbb{P}\Big(W_t(i)<0\Big|\sum_{j=1}^{m}C_{ij}Z_t(j)\Big) &= \mathbb{P}\Big(\eta_t(i)>\vartheta_i-\sum_{j=1}^{m}C_{ij}Z_t(j)\Big|\sum_{j=1}^{m}C_{ij}Z_t(j)\Big)\\
&= \Phi(y-\vartheta_i)\Big|_{y=\sum_{j=1}^{m}C_{ij}Z_t(j)}.
\end{aligned} \tag{2.8}$$

The function $\Phi(.)$ denotes the cumulative distribution function of the standard normal density. This is then the general default intensity of $Z_t(i)$ where default can occure due to both self default or default by contagion, i.e. $\lambda_t(i) = \Phi(y-\vartheta_i)$.

One can interpret the variables ϑ_i and C_{ij} in terms of the default probabilities. Let the default probability without contagion of firm *i* be denoted by p_i, so p_i is the self default probability and $\sum_{j=1}^{m} C_{ij}Z_t(j) = 0$. Then

$$p_i = \mathbb{P}(W_t(i) < 0|\sum_{j=1}^{m}C_{ij}Z_t(j) = 0) = \Phi(-\vartheta_i) \tag{2.9}$$

and the initial wealth of firm *i* can be expressed as $\vartheta_i = -\Phi^{-1}(p_i)$. With p_i as the monthly default probability, then PD_i in equation (2.3) is approximately $12\Phi(-\vartheta_i)$.

On the other hand, the expected default of firm *i* given that only one firm, say, *j* has defaulted would be

$$p_{i|j} = \mathbb{P}(W_t(i) < 0 \big| C_{ij} Z_t(j) = 1) = \Phi(C_{ij} - \vartheta_i).$$

This leads to the following expression of the contagion term:

$$\Phi^{-1}(p_{i|j}) = C_{ij} + \Phi^{-1}(p_i)$$

$$\Longleftrightarrow$$

$$C_{ij} = \Phi^{-1}(p_{i|j}) + \Phi^{-1}(p_i).$$

By moving on to the less general case where one uses the expression for $\eta_t(i)$ as given in equation (2.3), so $\eta_t(i) = \sigma_i(\sqrt{\rho_i}\eta_0 + \sqrt{1-\rho_i}\xi_t(i))$, one can start describing the default intensities in the cases of independent firms and firms exposed to credit contagion.

2.3.1 Independent default intensity

The following default models are for one time epoch, i.e. for *t* in [t,t+1]. The first focus is on the case where the firms are independent, $C_{ij} = 0$ for all *i* and *j*, so firm i is thus not in an economic relation with any other fimrs.

$$\mathbb{P}(Z_{t+1}(i) = 1 | Z_t(i) = 0, \sum_{j=1}^{m} C_{ij} Z_t(j) = 0)$$

$$= \mathbb{P}(\mathbf{1}_{\{\ W_t(i)<0\}}\ = 1 \big| \sum_{j=1}^{m} C_{ij} Z_t(j) = 0)$$

$$= \mathbb{P}(W_t(i) < 0 | \sum_{j=1}^{m} C_{ij} Z_t(j) = 0) = \mathbb{P}(\vartheta_i - \eta_t(i) < 0 | \sum_{j=1}^{m} C_{ij} Z_t(j) = 0)$$

$$= 1 - \mathbb{P}(\sigma_i(\sqrt{\rho_i}\eta_0 + \sqrt{1-\rho_i}\xi_t(i)) < \vartheta_i | \sum_{j=1}^{m} C_{ij} Z_t(j) = 0).$$

By using that $\sigma_i \equiv 1$ and η_0 is constant over a risk horizon of a year, the expression becomes

$$1 - \mathbb{P}(\sqrt{1-\rho_i}\xi_t(i) < \vartheta_i - \sqrt{\rho_i}\eta_0 | \sum_{j=1}^{m} C_{ij} Z_t(j) = 0)$$

$$= \Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}}). \tag{2.10}$$

This is thus the *default probability without contagion effect*.

2.3.2 Contagion default intensity

Moving on to the other case where $C_{ij} \neq 0$ for all *i* and *j*, one gets

$$\mathbb{P}(Z_{t+1}(i) = 1 | Z_t(i) = 0, \sum_{j=1}^{m} C_{ij} Z_t(j) \neq 0) = \mathbb{P}(W_t(i) < 0 | \sum_{j=1}^{m} C_{ij} Z_t(j) \neq 0)$$

$$= \mathbb{P}(\vartheta_i - \sum_{j=1}^{m} C_{ij} Z_t(j) - \sigma_i(\sqrt{\rho_i}\eta_0 + \sqrt{1-\rho_i}\xi_t(i)) < 0 | \sum_{j=1}^{m} C_{ij} Z_t(j) \neq 0)$$

Where again, by using that $\sigma_i \equiv 1$ and η_0 is constant over a risk horizon of a year, the expression becomes

$$\mathbb{P}(-\sum_{j=1}^{m} c_{ij}\Big(\frac{C_0}{c} + \frac{C}{\sqrt{c}} x_{ij}\Big) Z_t(j) - \sqrt{1-\rho_i}\xi_t(i) < -\vartheta_i + \sqrt{\rho_i}\eta_0 | \sum_{j=1}^{m} C_{ij} Z_t(j) \neq 0). \tag{2.11}$$

By linear transformation one gets, since $\sqrt{1-\rho_i}\xi_t(i) \sim \mathcal{N}(0, 1-\rho_i)$, that

$$\sum_{j=1}^{m} c_{ij}\Big(\frac{C_0}{c} + \frac{C}{\sqrt{c}} x_{ij}\Big) Z_t(j) + \sqrt{1-\rho_i}\xi_t(i)$$

is normally distributed with mean $\frac{C_0}{c}\sum_{j=1}^{m} \mathbb{E}(c_{ij} Z_t(j))$ and variance $1 - \rho_i + \frac{C^2}{c}\sum_{j=1}^{m} Var(c_{ij} Z_t(j))$. Returning to equation (2.11), one gets by standardizing

$$\mathbb{P}(-\sum_{j=1}^{m} c_{ij}\Big(\frac{C_0}{c} + \frac{C}{\sqrt{c}} x_{ij}\Big) Z_t(j) - \sqrt{1-\rho_i}\xi_t(i) < -\vartheta_i + \sqrt{\rho_i}\eta_0 | \sum_{j=1}^{m} C_{ij} Z_t(j) \neq 0)$$

$$= \Phi\Big(\frac{a + \sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i+b}}\Big)\Big|_{a=\frac{C_0}{c}\sum_{j=1}^{m}\mathbb{E}(c_{ij}Z_t(j)), b=\frac{C^2}{c}\sum_{j=1}^{m} Var(c_{ij}Z_t(j))} \tag{2.12}$$

which is the *default probability with contagion effect* of firm *i*.

2.3.3 The default number

Hatchett and Kühn introduce the fraction of defaulted firms and use the Law of Large Numbers in their result. This means that the $Z_t(j)$ must be i.i.d. for the results hold. They present the fraction of defaulted firms and it is $m_t = \frac{1}{m}\sum_{j=1}^{m} Z_t(j)$. Assume that $\mathbb{E}(Z_t(j)) = \xi$ and that $Var(Z_t(j)) < \infty$. By the Law of Large Numbers, as $m \to \infty$,

$$m_t = \frac{1}{m}\sum_{j=1}^{m} Z_t(j) \xrightarrow{P} \mathbb{E}(Z_t(j)).$$

The dynamics of the fraction of defaulted firms, m_t, can be found by looking at equation (2.1). Then one gets that

$$m_{t+1} = \frac{1}{m}\sum_{j=1}^{m} Z_{t+1}(j) = \frac{1}{m}\sum_{j=1}^{m}\Big(Z_t(j) + (1 - Z_t(j))\mathbf{1}_{\{\ W_t(j)<0\}}\Big)$$

$$= m_t + \frac{1}{m}\sum_{j=1}^{m}(1 - Z_t(j))\mathbf{1}_{\{\ \vartheta_j - \sum_{i=1}^{m} C_{ji}Z_t(i) - \sigma_j(\sqrt{\rho_j}\eta_0 + \sqrt{1-\rho_j}\xi_t(j))<0\}}\ . \quad (2.13)$$

Let $\sigma_j \equiv 1$ and exploit that $Z_t(j), \xi_t(j)$ and $\sum_{i=1}^{m} C_{ji}Z_t(i)$ are uncorrelated. By applying the Law of Large Numbers, as $m \to \infty$, one gets that

$$\frac{1}{m}\sum_{j=1}^{m}(1 - Z_t(j))\mathbf{1}_{\{\ \vartheta_j - \sum_{i=1}^{m} C_{ji}Z_t(i) - (\sqrt{\rho_j}\eta_0 + \sqrt{1-\rho_j}\xi_t(j))<0\}}$$

$$\xrightarrow{P} \quad \mathbb{E}(1 - Z_t(j))\mathbb{E}(\mathbf{1}_{\{\ \vartheta_j - \sum_{i=1}^{m} C_{ji}Z_t(i) - (\sqrt{\rho_j}\eta_0 + \sqrt{1-\rho_j}\xi_t(j))<0\}} \mid \sum_{i=1}^{m} C_{ji}Z_t(i))$$

$$= \mathbb{E}(1 - Z_t(j))\mathbb{P}\Big(\vartheta_j - \sum_{i=1}^{m} C_{ji}Z_t(i) - \sqrt{\rho_j}\eta_0 - \sqrt{1-\rho_j}\xi_t(j) < 0 \big| \sum_{i=1}^{m} C_{ji}Z_t(i)\Big)$$

$$= \mathbb{E}(1 - Z_t(j))\Phi\Big(\frac{a + \sqrt{\rho_j}\eta_0 - \vartheta_j}{\sqrt{1-\rho_j+b}}\Big)\Big|_{a=\frac{C_0}{c}\sum_{i=1}^{m}\mathbb{E}(c_{ij}Z_t(i)),\, b=\frac{C^2}{c}\sum_{i=1}^{m} Var(c_{ji}Z_t(i))}.$$

If the $Z_t(j)$ are not identically distributed, but independent, one could still find out how the expected number of defaults evolves. Generally, by using equation (2.8) one gets

$$\mathbb{E}(m_{t+1}) = \mathbb{E}(m_t) + \frac{1}{m}\sum_{j=1}^{m}\mathbb{E}(1 - Z_t(j))\Phi(y - \vartheta_j)\big|_{\{\ y=\sum_i C_{ji}Z_t(i)\}}\ .$$

To find the distribution of m_t, notice that m_t is a monotone increasing function of the macroeconomic factor $\eta_0 \sim \mathcal{N}(0,1)$, so $m_t = m_t(\eta_0)$ and one gets that the cumulative density function, cdf, of the fraction of defaulted firms is

$$\mathbb{P}(m_t(\eta_0) < m) = \Phi(m_t^{-1}m) \tag{2.14}$$

The density function is found by differentiating the cdf, i.e. $(\Phi(m_t^{-1}m))' = \phi(\frac{m}{m_t})(\frac{m}{m_t})'$. Hatchett and Kühn found that the evolution of the fraction of defaulted firms when considering credit contagion hardly differed from the case when there were only independent firms in the portfolio. However, when they looked at the probability density function of the fraction of defaulted firms they found that the tail of the distribution was fatter when one included credit contagion. i.e. when $(C_0, C) = (1,1)$ the tail was fatter then when $(C_0, C) = (0,0)$.

2.4 Pricing

Hatchett and Kühn did not consider pricing in their paper, but they did present the default probabilities which are to be interpreted as the default intensities. The default intensity in the general case was found to be

$$\mathbb{P}(Z_{t+1}(i) = 1 | Z_t(i) = 0, \sum_{j=1}^{m} C_{ij} Z_t(j)) = \mathbb{P}(W_t(i) < 0 \big| \sum_{j=1}^{m} C_{ij} Z_t(j))$$

$$= \mathbb{P}(\eta_t(i) > \vartheta_i - \sum_{j=1}^{m} C_{ij} Z_t(j) | \sum_{j=1}^{m} C_{ij} Z_t(j))$$

$$= 1 - \mathbb{P}(\eta_t(i) \le \vartheta_i - \sum_{j=1}^{m} C_{ij} Y_t(j) | \sum_{j=1}^{m} C_{ij} Z_t(j)) = \Phi(y - \vartheta_i)\Big|_{\{\ y = \sum_{j=1}^{m} C_{ij} Z_t(j)\}} .$$

On the underlying stochastic basis $(\Omega, \mathcal{G}, \mathbb{Q}, \mathbb{F})$ with $\mathbb{F} = (\mathcal{F}_t)_{t \ge 0}, \mathcal{F}_t \subset \mathcal{G}$, the risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$ is assumed given. The σ-algebra $\mathcal{F}_t$ contains the market information up to time t. The macroeconomic factor η_0 is $\mathcal{F}_0$-measurable and it is not a trivial σ-algebra. One can define a random time, or the default time, by

$$\tau_i = inf\{\ t > 0 : Z_t(i) = 1\}\ .$$

For i in $\{\ 1, \ldots, m\}$, $t = 0, 1, \ldots, T$ one has $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau_i \le u : u \le t)$, and $\mathcal{G}_t \supseteq \mathcal{F}_t \supseteq \mathcal{F}_0 \supseteq \sigma(\eta_0)$. Recall that the information of η_0 is know at $t = 0$, so it is treated as a known constant that does not change during the time period of this model. Note that τ_i are $\mathbb{G}$-stopping times.

Define the counting process (Poisson process) by

$$N_t(i) = \begin{cases} 1 & \text{if } \tau_i \le t \\ 0 & \text{if } \tau_i > t \end{cases}$$

which means that the event of a stopping time occurs in time as a Poisson process with intensity $\lambda_i > 0$. The survival probability of the Poisson process is

$$\mathbb{P}(\tau_i > t) = \mathbb{P}(\text{no jumps until t}) = \mathbb{P}(N_t(i) = 0) = e^{-\lambda_i t} \sim \mathcal{E}xp(\lambda_i).$$

Thus, by substituting λ_i for the intensity presented by Hatchett and Kühn, one gets that $\tau_i \sim \mathcal{E}xp(\Phi(\sum_j C_{ij} Z_t(j) - \vartheta_i))$.

Assume that the stochastic interest rate r_t is bounded and continuous. The time t *price of a defaultable zero coupon bond* on firm i with maturity T is given by the following formula:

$$V_t(i) = \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s ds}\mathbf{1}_{\{\ \tau_i > T\}}\ |\mathcal{G}_t\Big). \tag{2.15}$$

Equation (2.15) is, by the tower property of conditional expectations, equal to

$$\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{E}_{\mathbb{Q}}(e^{-\int_t^T r_s ds}\mathbf{1}_{\{\ \tau_i > T\}}\ |\mathcal{G}_T)\ \ |\mathcal{G}_t\Big) = \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s ds}\mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{\{\ \tau_i > T\}}\ |\mathcal{G}_T)\ \ |\mathcal{G}_t\Big)$$

$$= \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T (r_s + \lambda_s(i))ds}|\mathcal{G}_t\Big).$$

And if r_s and $\lambda_s(i)$ are independent, then the expression becomes

$$V_t(i) = \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s ds}|\mathcal{G}_t\Big)\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T \lambda_s(i) ds}|\mathcal{G}_t\Big).$$

Moving on and looking at the price of a claim when the default intensities are specified as in sections 2.3.1 and 2.3.2, and starting with the case in which the default probability is without contagion effect. Recall that the default intensity of firm i not being exposed to default contagion is $\Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}})$ as given in equation (2.10).

Let r_t and the default intensities be independent. By substituting $\lambda_s(i)$ with $\Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}})$ one obtains that the price of a defaultable zero coupon bond at time t in $\{\ 0, 1, \ldots, T\}$ issued by firm i, where the issuer is not in an economic relation with other firms, would amount to calculate

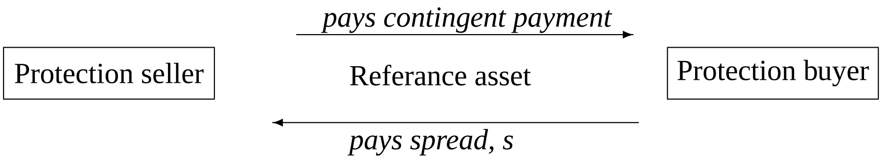

Figure 2.1: Credit Default Swap

$$V_t(i) = \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T (r_s + \Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}}))ds} | \mathcal{G}_t\Big). \tag{2.16}$$

$$= \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s ds} | \mathcal{F}_t\Big)\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T \Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}})ds} | \eta_0\Big).$$

$$= e^{-\Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}})(T-t)}\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s ds} | \mathcal{F}_t\Big)$$

where the last equality follows from the assumption that η_0 is known and constant. The default intensity $\Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}})$ is constant for the given firm i. Since the short rate is stochastic the price in equation (2.16) would be, if one uses the Vasicek model as described in section 1.1.2 ,

$$V_t(i) = e^{-\Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}})(T-t)} P(t,T), \tag{2.17}$$

where $P(t,T)$ is given in equation (1.10).

Another element of credit risk is the recovery rate, R, which says how much of the face value of the bond that can be recovered if the obligor defaults. If one includes a stochastic and independent recovery rate R_i which is paid at maturity T, then the payoff for a defaultable zero coupon bond is $\Big(e^{-\int_t^T r_s ds}\mathbf{1}_{\{\tau_i > T\}} + R_i e^{-\int_t^T r_s ds}\mathbf{1}_{\{\tau_i \leq T\}}\Big)$, and the price given in equation (2.15) will then be

$$V_t(i) = \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s ds}\mathbf{1}_{\{\tau_i > T\}} + R_i e^{-\int_t^T r_s ds}\mathbf{1}_{\{\tau_i \leq T\}} | \mathcal{G}_t\Big)$$

$$= \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s ds}(\mathbf{1}_{\{\tau_i > T\}} + R_i(1 - \mathbf{1}_{\{\tau_i > T\}})) | \mathcal{G}_t\Big)$$

$$= \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s ds}(R_i + (1-R_i)\mathbf{1}_{\{\tau_i > T\}}) | \mathcal{G}_t\Big)$$

$$= \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s ds} R_i | \mathcal{G}_t\Big) + \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s ds}(1-R_i)\mathbf{1}_{\{\tau_i > T\}}) | \mathcal{G}_t\Big)$$

$$= \mathbb{E}_{\mathbb{Q}}\Big(R_i e^{-\int_t^T r_s ds} | \mathcal{G}_t\Big) + \mathbb{E}_{\mathbb{Q}}\Big((1-R_i)e^{-\int_t^T (r_s + \lambda_s(i))ds} | \mathcal{G}_t\Big)$$

and with the default intensities inserted, the price will be

$$\mathbb{E}_{\mathbb{Q}}\Big(R_i e^{-\int_t^T r_s ds} | \mathcal{G}_t\Big) + e^{-\Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}})(T-t)}\mathbb{E}_{\mathbb{Q}}\Big((1-R_i)e^{-\int_t^T r_s ds} | \mathcal{G}_t\Big). \tag{2.18}$$

A credit default swap, CDS, is a protection contract against default of a firm where the protection buyer pays periodic payments to the protection seller who in turn pays a one-off contingent payment if default occurs before maturity of the bond (referance asset) issued by company i. The periodic payments to the protection seller continues until either default or maturiy. If $\tau_i \leq T$, assume that the protection seller has to pay the protection buyer $(1 - R_i)$, called loss given default and is the loss in percentage in case of default. The protection buyer has to pay a fixed amount s, called spread, set at the time of evaluation such that the contract is fair. The payment dates for the spread are $0 = t_0 < t_1 < \ldots < t_n = T$ which are assumed to be equally spaced and no accured interest rate is paid. (If the accured interes rate is paid then an extra term is added to the value of the claim of the seller). Then the value at time $t \leq t_j$ for j in $\{\, 0, 1, \ldots, n\}$ of the claim regarding firm i of the protection buyer is

$$V_t^{Buyer}(i) = \mathbb{E}_{\mathbb{Q}}\Big(s \sum_{j=0}^{n} e^{-\int_t^{t_j} r_s ds} \mathbf{1}_{\{\, \tau_i > t_j\}} \,|\mathcal{G}_t\Big),$$

and by equation (2.15) the above price becomes

$$\begin{aligned} V_t^{Buyer}(i) &= \mathbb{E}_{\mathbb{Q}}\Big(s \sum_{j=0}^{n} e^{-\int_t^{t_j} (r_s + \Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}}))ds} |\mathcal{G}_t\Big) \\ &= s \sum_{j=0}^{n} \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^{t_j} (r_s + \Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}}))ds} |\mathcal{G}_t\Big) \\ &= s \sum_{j=0}^{n} e^{-\Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}})(t_j - t)} \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^{t_j} r_s ds} |\mathcal{F}_t\Big) \end{aligned} \tag{2.19}$$

where the last equality follows from the default intensity of firm i being constant. If the Vasicek model is used, then the expression in equation (2.19) could be written as

$$V_t^{Buyer}(i) = s \sum_{j=0}^{n} e^{-\Phi(\frac{\sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1-\rho_i}})(t_j - t)} P(t, t_j). \tag{2.20}$$

For the protection seller one gets that the value of the claim is, for R_i constant,

$$\begin{aligned} V_t^{Seller}(i) &= \mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^{\tau_i} r_s ds}(1 - R_i)\mathbf{1}_{\{\, t < \tau_i \leq T\}} \,|\mathcal{G}_t\Big) \\ &= (1 - R_i)\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^{\tau_i} r_s ds} \mathbf{1}_{\{\, t < \tau_i \leq T\}} \,|\mathcal{G}_t\Big). \end{aligned} \tag{2.21}$$

Since $e^{-\int_t^{\tau_i} r_s ds}$ is bounded and continuous, and the default intensity is continuous, corollary 5.1.3 in [6] gives that equation (2.21) becomes

$$(1-R_i)\Phi(\frac{\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i}})\mathbf{1}_{\{\ \tau_i>t\}}\ e^{\int_0^t \Phi(\frac{\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i}})ds}$$
$$\times\mathbb{E}_{\mathbb{Q}}\Big(\int_t^T e^{-\int_t^u r_s ds}e^{-\int_0^u \Phi(\frac{\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i}})ds}du|\mathcal{F}_t\Big)$$
$$=(1-R_i)\Phi(\frac{\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i}})\mathbf{1}_{\{\ \tau_i>t\}}\ e^{\Phi(\frac{\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i}})t}$$
$$\times\int_t^T e^{-\Phi(\frac{\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i}})u}\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^u r_s ds}|\mathcal{F}_t\Big)du. \tag{2.22}$$

If one was using the Vasicek model to calculate the expectation, then equation (2.22) would be

$$V_t^{Seller}(i)=(1-R_i)\Phi(\frac{\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i}})\mathbf{1}_{\{\ \tau_i>t\}}\ e^{\Phi(\frac{\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i}})t}$$
$$\times\int_t^T e^{-\Phi(\frac{\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i}})u}P(t,u)du. \tag{2.23}$$

The contract values are set to be zero at initiation of the contract, so in order to find the fair spread, s, one equates the two expressions, i.e. $V_t^{Buyer}(i) = V_t^{Seller}(i)$ and solves for s.

If the issuer of the defaultable claim is in an economic relation with other firms, then the default intensity is $\Phi(\frac{a_t+\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i+b_t}})$, $a_t = \frac{C_0}{c}\sum_{j=1}^m \mathbb{E}(c_{ij}Z_t(j))$ and $b_t = \frac{C^2}{c}\sum_{j=1}^m Var(c_{ij}Z_t(j))$ as given in equation (2.12). The price of a defaultable zero coupon bond at time t, where firm i is in an economic relation with other firms and the short rate is independent of the default intensity, is

$$V_t(i)=\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T(r_s+\Phi\big(\frac{a_s+\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i+b_s}}\big))ds}|\mathcal{G}_t\Big) \tag{2.24}$$
$$=\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T r_s ds}|\mathcal{F}_t\Big)\mathbb{E}_{\mathbb{Q}}\Big(e^{-\int_t^T\Phi\big(\frac{a_s+\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i+b_s}}\big)ds}|\mathcal{G}_t\Big).$$

When the firms are in an economic relation with each other, the default intensities are deterministic since $a_t = \frac{C_0}{c}\sum_{j=1}^m \mathbb{E}(c_{ij}Z_t(j))$ and $b_t = \frac{C^2}{c}\sum_{j=1}^m Var(c_{ij}Z_t(j))$ are numbers that change in time. So with deterministic default intensities that includes the contagion effect, the price in equation (2.17) would be

$$V_i(i) = e^{-\int_t^T \Phi\left(\frac{a_s+\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i+b_s}}\right)ds} P(t,T).$$

If the stochastic recovery rate is included, one would get that equation (2.18) would be

$$\mathbb{E}_{\mathbb{Q}}\Big(R_i e^{-\int_t^T r_s ds}|\mathcal{G}_t\Big) + e^{-\int_t^T \Phi(\frac{a_s+\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i+b_s}})ds}\mathbb{E}_{\mathbb{Q}}\Big((1-R_i)e^{-\int_t^T r_s ds}|\mathcal{G}_t\Big).$$

When the default intensities include the contagion term, the price of a CDS would be, for equation (2.20)

$$V_t^{Buyer}(i) = s\sum_{j=0}^{n} e^{-\int_t^{t_j} \Phi(\frac{a_s+\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i+b_s}})ds} P(t,t_j).$$

And for the protection seller equation (2.23) would be

$$V_t^{Seller}(i) = (1-R_i)\mathbf{1}_{\{\ \tau_i>t\}}\ e^{\int_0^t \Phi(\frac{a_s+\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i+b_s}})ds}$$
$$\times \int_t^T e^{-\int_0^u \Phi(\frac{a_s+\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i+b_s}})ds}\Phi(\frac{a_u+\sqrt{\rho_i}\eta_0-\vartheta_i}{\sqrt{1-\rho_i+b_u}})P(t,u)du.$$

2.5 Remarks

The portfolio default process is a binary indicator variable taking the values 0 and 1, so it can be understood as Bernoulli random variables

$$Z_t(i) = \begin{cases} 1 & \text{with probability } \ p_i \\ 0 & \text{with probability } \ 1-p_i, \end{cases}$$

The self default probability $p_i = \Phi(-\vartheta_i)$, is as found in equation (2.9). This means that the $Z_t(i)$ are not identically distributed. They might be independent by the argument given by Hatchett and Kühn in section 2.2. Since the $Z_t(i)$ are not i.i.d., the Law of Large Numbers and the Central Limit Theorem can not be applied to find the asymptotic distribution of the sum in expression (2.6), and the dynamics of the fraction of defaulted firms can not converge in probability.

In general, there is no clear description of default caused by bad performance of firm i itself and default caused by contagion from firm j in this model. In the model presented by Hatchett and Kühn, a firm defaults if its wealth falls below zero, and the wealth is the difference of the assets and liabilities of firm i. This means that firm i defaults if it has more liabilities than it has assets. If one wants to introduce a model that distinguishes between self default and default by contagion one could proceed as follows:

Let the wealth of firm i be denoted $V_t(i)$ and let

$$\begin{cases} V_t(i) = \vartheta_i - \eta_t(i) & t = 1, \ldots, T \\ V_0(i) = \vartheta_i > 0 \end{cases}$$

where ϑ_i and $\eta_t(i)$ are given as in [3]. Then introduce a self default process $Y_t(i)$ given by

$$Y_t(i) = \mathbf{1}_{\{ V_t(i) < \ell_i \}} = \begin{cases} 1 & \text{self default of firm i} \\ 0 & \text{firm i solvent} \end{cases}$$

where $\ell_i < 0$ is the admissible level of liabilities. Let the portfolio default process be, for $t = 0, \ldots, T$

$$\begin{cases} Z_{t+1}(i) = Z_t(i) + (1 - Z_t(i))\Big(Y_{t+1}(i) - (1 + Y_{t+1}(i))\mathbf{1}_{\{ W_{t+1}(i) < 0\}}\Big) \\ Z_0(i) = 0. \end{cases}$$

The wealth process is depending on $t+1$ now, which means that the model captures default immediately. The portfolio default process is still 1 if firm i has defaulted, and 0 otherwise. The $Y_t(i)$ are independent, but not identically distributed, but they are driven by some Gaussian noice. The probability of default in the portfolio for firms that are not in an economic relation with other firms is

$$\begin{aligned} &\mathbb{P}(Z_{t+1}(i) = 1 \big| Z_t(i) = 0) \\ = \mathbb{P}\Big(&Y_{t+1}(i) + (1 - Y_{t+1})\mathbf{1}_{\{ W_{t+1}(i) < 0\}} = 1 \big| Y_t(i) = 0, \mathbf{1}_{\{ W_{t+1}(i) < 0\}} = 0\Big) \\ &= \Phi\Big(\frac{\ell_i - \vartheta_i + \sqrt{\rho_i}\eta_0}{\sqrt{1-\rho_i}}\Big). \end{aligned}$$

The default probability for firm i being in an economic relation with other firms is the same as in the model suggested by Hatchett and Kühn, i.e.

$$
\begin{aligned}
&\mathbb{P}(Z_{t+1}(i) = 1 \big| Z_t(i) = 0) \\
&= \mathbb{P}\Big(Y_{t+1}(i) + (1 - Y_{t+1})\mathbf{1}_{\{\ W_{t+1}(i) < 0\}}\ = 1 \big| Y_t(i) = 0, \mathbf{1}_{\{\ W_{t+1}(i) < 0\}}\ = 1\Big) \\
&= \Phi\Big(\frac{a + \sqrt{\rho_i}\eta_0 - \vartheta_i}{\sqrt{1 - \rho_i + b}}\Big)\Big|_{a = \frac{C_0}{c}\sum_{j=1}^m \mathbb{E}(c_{ij}Y_{t+1}(j)), b = \frac{C^2}{c}\sum_{j=1}^m Var(c_{ij}Y_{t+1}(j))}.
\end{aligned}
$$

Chapter 3

Contagion model in continuous time

Biagini, Fuschini and Klüppelberg, [2], present a contagion model in continuous time which is based on [4]. They chose to let the default intensities depend on a long range dependent process describing the macroeconomic factor because macroeconomic factors tend show a long range dependence effect. As in the discrete time model, a default in the portfolio can be caused by either self default or default by contagion. Opposite to [3], their paper is considering the pricing of defaultable derivatives, where the derivatives depend on the macroeconomic process and are exposed to default contagion. Biagini, Fuschini and Klüppelberg are able to give explicit pricing formulas for derivatives. It is assumed that the primary assets on the market (a primary asset in banking might be the bank's reserves or loans) are not driven by a long range dependent process. Both self default and default by contagion happen instantaneously and the defaulting state is absorbing.

3.1 The default model

There are only two states for the firms in the portfolio: defaulted or solvent. Let a portfolio consist of *m* firms, where each firm is indexed by *i* in $\{1, 2, \ldots, m\}$. The *portfolio default process* is taking the values $\{0, 1\}^m$ and is described by

$$Z_t = (Z_t(1), \ldots, Z_t(m)), \qquad t \geq 0,$$

where each random component $Z_t(i)$ describes if firm *i* has defaulted or not by time *t*, meaning that

$$Z_t(i) = \begin{cases} 0 & \text{firm } i \text{ is solvent at time } t, \\ 1 & \text{firm } i \text{ has defaulted at time } t\,. \end{cases}$$

Since one is interested in credit contagion, one has to distinguish between default in the portfolio which is caused by the firm itself or by contagion from the default of other firms. The *self default indicator process*, which is a random vector taking the values $\{0,1\}^m$, is described by

$$Y_t = (Y_t(1), \dots, Y_t(m)), \qquad t \geq 0,$$

where each $Y_t(j)$ is given by

$$Y_t(j) = \begin{cases} 0 & \text{firm } j \text{ is solvent at time } t, \\ 1 & \text{firm } j \text{ has defaulted at time } t \text{ by itself.} \end{cases}$$

Denote by $\tau_j = \tau_j(\omega)$ the default time of firm *j* for j in $\{1, 2, \dots, m\}$. Then $Y_t(j) = \mathbf{1}_{\{\tau_j \leq t\}}$, $t \geq 0$. The random variable $Y_t(j)$ generates the natural filtration denoted $\mathcal{F}_t^{Y(j)} := \sigma(Y_u(j) : u \leq t)$. The self default processes are assumed to be independent.

In [2], the suggested modeling of credit contagion in continuous time is through a *contagion matrix indicator process*. The matrix C_t is in $\mathbb{R}^{m \times m}$ and its coefficients indicate if there is contagion between the firms or not. This means that if firm *i* defaults, then $C_t(i,j)$ will determine if there was any infection from firm *i* to firm *j* at time *t*. For any time $t \geq 0$, the coefficients in the matrix are described by

$$C_t(i,j) = \begin{cases} 0 & \text{no infection of default,} \\ 1 & \text{if default of firm } i \text{ causes firm } j \text{ to default at time } t. \end{cases}$$

This way of describing contagion differs from the one in the discrete time model. In the discrete time model, the entries in the contagion matrix can be both positive and negative, but in the continuous time model the entries are either 0 or 1. In the discrete time model the contagion is more of an average contagion between the firms that may be in a competitive or cooperative economic relationship, whereas in the continuous time model the default contagion is not divided into 'good' and 'bad' contagion. The contagion is a pure default contagion from firm *i* to firm *j*, and no other firms can get firm *j* back in business. In the discrete time model there is no clear description of the self default process as it is in the continuous time model.

The contagion matrix process generates the filtration $\mathcal{F}_t^{C_{ij}} = \sigma(C_u(i,j) : u \leq t)$ for every i, j in $\{1, \dots, m\}$, $i \neq j$. One can express the portfolio default indicator process of firm *j* as

$$Z_t(j) = Y_t(j) + (1 - Y_t(j))\Big(1 - \prod_{i \neq j}(1 - C_{t \wedge \tau_i}(i,j) Y_t(i))\Big), \qquad t \geq 0. \quad (3.1)$$

Since firm *j* obviously is influencing itself, $C_t(j,j) \equiv 1$, the portfolio default process can be written in a shorter form;

$$Z_t(j) = 1 - \prod_{i=1}^{m}(1 - C_{t\wedge\tau_i}(i,j)Y_t(i)), \qquad t \geq 0. \tag{3.2}$$

Defaults in the portfolio are caused by fluctuations in the macroeconomic factor, and defaults happen at τ_i, and the stopping time τ_i has an intensity λ^i which is driven by an underlying stochastic process $\Psi = (\Psi_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ representing the evolution of the macroeconomy. Ψ generates the filtration $\mathcal{F}_t^{\Psi} = \sigma(\Psi_u : u \leq t)$. Ψ will be described in details later in section 3.4.

3.2 The probability space and assumptions

The system is described by the process $(\Psi_t, Y_t, C_t)_{t\geq 0}$ on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}_t := F_t^{\Psi} \vee \mathcal{F}_t^{Y} \vee \mathcal{F}_t^{C}$. The larger filtration $\mathcal{G}_t := \mathcal{F}_\infty^{\Psi} \vee \mathcal{F}_t^{Y} \vee \mathcal{F}_t^{C}$ contains information about the whole path of $(\Psi_t)_{t\geq 0}$. All filtrations are assumed to be right-continuous and $\mathbb{P}$-augmented. It is aslo assumed that the investors have knowledge about $(\mathcal{F}_t)_{t\geq 0}$, that the investors know the contagion structure and if a firm has defaulted or not. Further assumptions are

1. Ψ is not affected by Y and Z, meaning that for every bounded $\mathcal{F}_\infty^{\Psi}$-measurable random variable η, $\mathbb{E}(\eta|\mathcal{F}_t) = \mathbb{E}(\eta|\mathcal{F}_t^{\Psi}), t \geq 0$.

2. The processes $(Y_t(i))_{t\geq 0}$ and $(C_t(i,j))_{t\geq 0}$ are conditionally orthogonal of the filtration $(\mathcal{G}_t)_{t\geq 0}$, meaning that for every $\{ i_1, \ldots, i_k\} \subseteq \{ 1, \ldots, m\}$ and for every choice of $(\alpha_1, \beta_1), \ldots, (\alpha_l, \beta_l)$ in $\{ (i,j) \in \{ 1, \ldots, m\}^2 | i \neq j\}$ one gets that for all $t_j \geq t, j = 1, \ldots, k$ and $s_n \geq t, n = 1, \ldots, l$

$$\mathbb{E}\Big(\prod_{j=1}^{k}\prod_{n=1}^{l} f(Y_{t_j}(i_j))g(C_{s_n}(\alpha_n, \beta_n))|\mathcal{G}_t\Big)$$

$$= \prod_{j=1}^{k}\mathbb{E}\Big(f(Y_{t_j}(i_j))|\mathcal{F}_\infty^{\Psi}\vee\mathcal{F}_t^{Y(i_j)}\Big)\prod_{n=1}^{l}\mathbb{E}\Big(g(C_{s_n}(\alpha_n,\beta_n))|\mathcal{F}_\infty^{\Psi}\vee\mathcal{F}^{C(\alpha_n,\beta_n)}\Big)$$

   for *f, g* : $\{0,1\} \to \mathbb{R}$ and *i, j* in $\{ 1, \ldots, m\}$, $i \neq j$.

3. The self default process $(Y_t(i))_{t\geq 0}$ is a doubly stochastic process with respect to the filtration $(\mathcal{F}_\infty^{\Psi} \vee \mathcal{F}_t^{Y})_{t\geq 0}$. The stochastic intensity of $(Y_t(i))_{t\geq 0}$ is denoted $\lambda^i(t, \Psi_t)$ for $\lambda^i : \mathbb{R}^2 \to \mathbb{R}^+$. This means that

$$\mathbb{E}(1 - Y_s(i)|\mathcal{G}_t) = (1 - Y_t(i))e^{-\int_t^s \lambda^i(u,\Psi_u)du}, \qquad s \geq t.$$

4. The contagion processes $(C_t(i,j))_{t\geq 0}$ for $i \neq j$ are $\mathcal{F}_\infty^\Psi$-conditionally time-inhomogeneous Markov chains, i.e. for every function $f : \{ 0,1\} \to \mathbb{R}$,

$$\mathbb{E}\Big(f(C_s(i,j))|\mathcal{F}_\infty^\Psi \vee \mathcal{F}_t^{C(i,j)}\Big) = \mathbb{E}\Big(f(C_s(i,j))|\mathcal{F}_\infty^\Psi \vee \sigma(C_t(i,j))\Big), s \geq t.$$

5. For all *i,j* in $\{ 1,\ldots,m\}$, $i \neq j$, and states *h,k* in $\{ 0,1\}$, the conditional transition probabilities are denoted by

$$p_{ts}^{ij}(k,h) = \mathbb{P}\Big(C_s(i,j) = h|\mathcal{F}_\infty^\Psi \vee \sigma(C_t(i,j) = k)\Big)$$

and the process $(p_{ts}^{ij}(k,h))_{s\in\mathbb{R}^+}$ is assumed to be continuous for everty *t* in $\mathbb{R}^+$, *i,j* in $\{ 1,\ldots,m\}$ and *k,h* in $\{ 0,1\}$.

3.3 Contagion classes

By assuming that the matrix C is *time-independent* and *deterministic* one can divide the *m* firms in the portfolio into fixed *contagion classes*. Firms belonging to the same contagion class need to satisfy the following:

1. Reflexivity: $C(i,i) = 1$ for all *i* in $\{ 1,\ldots,m\}$.
2. Symmetry: $C(i,j) = C(j,i)$ for all *i,j* in $\{ 1,\ldots,m\}$.
3. Transitivity: $C(i,h)C(h,j) \leq C(i,j)$ for all *i,j,h* in $\{ 1,\ldots,m\}$.

The contagion classes are disjoint and denoted by

$$[i] := \{ j \in \{ 1,\ldots,m\} |C(i,j) = 1\}$$

where it is assumed that the portfolio consists of $k \leq m$ contagion classes $[i_1],\ldots,[i_k]$. The contagion classes might represent local markets. There can not be two different contagion classes defaulting at the same time, otherwise the two classes would actually be the same. Since the matrix C now is assumed to be time-independent and deterministic the portfolio default process in equation (3.1) becomes

$$Z_t(j) = Y_t(j) + (1 - Y_t(j))\Big(1 - \prod_{i\neq j}(1 - C(i,j)Y_t(i))\Big).$$

One sees that $Z_t(j) = 1$ if $Y_t(j) = 1$ and $Z_t(j) = 1 - \prod_{i\neq j}(1 - C(i,j)Y_t(i))$ if $Y_t(j) = 0$. The contagion part $\Big(1-\prod_{i\neq j}(1-C(i,j)Y_t(i))\Big) = 1$ and gives default in the portfolio in position j if there exists some i such that $C(i,j)Y_t(i) = 1$. This means that there must be at least one i in $[j]$ in order to have a default in the portfolio in position j. On the other hand, if $\Big(1 - \prod_{i\neq j}(1 - C(i,j)Y_t(i))\Big) = 0$ then $C(i,j)Y_t(i) = 0$ for all i. In order for $C(i,j)Y_t(i) = 0$, either $Y_t(i) = 0$ or

$Y_t(i) = 1$ and $C(i,j) = C(j,i) = 0$, by the symmetry assumption. This means that $Y_t(i) = 0$ for all j in $[i]$, and one gets that

$$Z_t(i) = 0 \qquad \Longleftrightarrow \qquad Z_t(j) = 0 \quad \text{for all } j \text{ in [i]},$$

which means that either all firms in the same contagion class have defaulted or that all firms are solvent. According to Biagini, Fuschini and Klüppelberg this form of classification of the firms makes their modeling different from usual credit risk contagion modeling since usual modeling would increase the default hazard of all the other firms in the same class when a default in that class occured. All firms belonging to the same contagion class $[i]$ have a default intensity given by

$$\lambda_t^{[i]} = \sum_{j\in[i]} \lambda^j(t, \Psi_t).$$

Since the contagion matrix is assumed to be time-independent and deterministic, the default intensities of the portfolio default process, $(Z_t)_{t\geq 0}$, are as the default intensities of the self default indicator process $(Y_t(j))_{t\geq 0}$. The different contagion classes are independent.

In the discrete time model all the firms and their enviroments were assumed to be fairly homogeneous, but they could not be put in the type of contagion classes described above since the contagion matrix is not symmetric. In the discrete time model the contagion matrix is deterministic and the c_{ij} which describes if there exists a connection between the firms i and j is symmetric and transient.

3.3.1 The default number

Like in the discrete time model, one can find the average number of defaulted firms within the portfolio. In the continuous time model, the default number process is linked to the contagion classes. All the firms in the portfolio are split into $l \leq m$ homogeneous groups, denoted by $G_1, \ldots, G_l$, where each group contains all the firms that have the same default intensity. The groups might represent firms with identical credit rating (the probability of the issuer being able to pay its debt) or firms belonging to the same industry. For h in $\{1, \ldots, l\}$, G_h can be written as the disjoint union of contagion classes, i.e.

$$G_h = \bigcup_{k=1}^{s_h} [j_k^h]$$

where s_h is the number of contagion classes that group G_h consists of. Then the weighted average number of defaults within group G_h is given by

$$m_t(h) := \frac{1}{s_h}\Big(\sum_{i\in[j_1^h]} \frac{Z_t(i)}{n_1^h} + \ldots + \sum_{i\in[j_{s_h}^h]} \frac{Z_t(i)}{n_{s_h}^h}\Big) \tag{3.3}$$

where n_i^h is the cardinality of the contagion class $[j_i^h]$ for i in $\{ 1, \ldots, s_h\}$ and $m_t := (m_t(1), \ldots, m_t(l))$. Since the contagion classes are conditionally independent of the filtration $\mathcal{G}_t$, by assumption in section 3.2, the summands of the process m_t are conditionally independent as well.

Recall that there can not be simultaneous defaults of contagion classes and all firms within a contagion class default at the same time. By following the reasoning of the proof of Lemma 3.4 in [11] one gets that, for *h* in $\{ 1, \ldots, l\}$, the counting process $(m_t)_{t\geq 0}$ will jump from a state u in $\mathbb{R}^l$ with $u = (u_1, \ldots, u_l) = \Big((\frac{v_1}{s_1}, \ldots, \frac{v_l}{s_l}) : v_h \in \{ 0, \ldots, s_h\} \Big)$ to a state of the form $u + \frac{e_h}{s_h}$ if and only if the next defaulting firm belongs to group G_h. The e_h is the *h*-th element in the standard basis of $\mathbb{R}^l$. The u_h increases only in steps of $\frac{1}{s_h}$. The transition intensity of $m_t(h)$ from the state *u* into the state $u + \frac{e_h}{s_h}$ is given by

$$\lambda_t^{m_t(h)}(u, u + \frac{e_h}{s_h}) = s_h(1 - u_h)\lambda^{G_h}(t, \Psi_t)$$

where $\lambda^{G_h}(t, \Psi_t)$ is the default intensity of every firm belonging to group G_h, i.e. the default intensiyt of G_h, and u_h is the proportion of firms that have defaulted in group G_h at time *t*.

The following example is ment to illustrate the contagion matrix and group structure.

Example: Let there be two groups, $G_i, i = 1, 2$, with 3 firms in one group and 4 firms in the other group. Let the contagion matrix be deterministic. Then the contagion matrix is as follows:

$$C = \begin{pmatrix} C_{3\times 3} & C_{3\times 4} \\ C_{4\times 3} & C_{4\times 4} \end{pmatrix}$$

Let $\mathbf{I}_d$ be the identity matrix in $\mathbb{R}^d$, $\mathbf{0}_{d\times k}$ be the zero matrix and let $\mathbf{1}_{d\times k}$ be the matrix with only entries 1. Then one can consider the two following contagion cases:

$$C_1 = \begin{pmatrix} \mathbf{1}_{3\times 3} & \mathbf{0}_{3\times 4} \\ \mathbf{0}_{4\times 3} & \mathbf{1}_{4\times 4} \end{pmatrix} \qquad C_2 = \begin{pmatrix} \mathbf{1}_{3\times 3} & \mathbf{0}_{3\times 4} \\ \mathbf{1}_{4\times 3} & \mathbf{I}_{4\times 4} \end{pmatrix}$$

The interpretation of the case in C_1 is that there is default contagion between the 3 firms in group 1, as well as it is contagion between the 4 firms in group 2. The zero matrices tells that there is no default contagin from G_1 and over to G_2 and no contagion the other way around either. The matrix C_2 models contagion within G_1, no default contagion from G_1 to G_2, default contagin from group 2 over to group 1 and no contagion within group 2. If the matrix $C = \mathbf{1}_{12\times 12}$ there is default contagion between all the 12 firms, and the case $C = \mathbf{I}_{12}$ means that there is no default contagion between the firms. To explain the contagion effect in the case of C_2 a little bit more in detail, assume that the firms in group G_1 are called a_1, a_2 and a_3, and the firms in G_2 are called b_1, b_2, b_3 and b_4. Then $\mathbf{1}_{3\times 3}$ means that $C_t(a_i, a_j) = 1$ for all $i, j = 1, 2, 3$. $\mathbf{I}_{4\times 4}$ means that $C_t(b_i, b_i) = 1$ for all $i = 1, 2, 3, 4$. The case where $\mathbf{0}_{3\times 4}$ tells that $C_t(a_i, b_j) = 0$ for all i, j, and $\mathbf{1}_{4\times 3}$ means that $C_t(b_i, a_j) = 1$ for all i, j as well as it has to include self default, $C_t(b_i, b_i) = 1$.

3.4 The macroeconomic process

The macroeconomic process Ψ is chosen to be modeled as a one dimensional fractional Brownian motion, *fBm*, with Hurst index $H > \frac{1}{2}$, (See appendix B). Since the process is one dimensional it might be seen to represent a weighted mean of a vector of macroeconomic variables. The *fBm* was chosen to represent the macroeconomic factors (such as supply and demand, unemploymentrate and inflation) since these factors often show a long range dependence. Since *fBm* is a long range time dependent process, it is not Markovian. The macroeconomic variable is given by

$$\Psi_t^H := \psi\Big(\int_0^t g(s)dB_s^H\Big), \qquad t \in [0, T], \tag{3.4}$$

where ψ is an invertible continuous function, and g is a deterministic function in $H^\mu([0, T])$ (see appendix B for more) such that $\frac{1}{g(s)}$ is defined for all *s* in [0,T]. Since g is a deterministic function the integral in equation (3.4) can be understood in a pathwise Riemann-Stieltjes sense by using the formula for integration by parts. (See page 124 in [1].)

Biagini, Fuschini and Klüppelberg restricted themself to the case where, for all i in $\{ 1, \ldots, m\}$ the default intensities of the self default processes $(Y_t(i))_{t\geq 0}$ are stochastic and of the form

$$\lambda^i(t, \Psi_t^H) = \beta^i(t)\int_0^t g(s)dB_s^H + \gamma^i(t), \qquad t \in [0, T] \tag{3.5}$$

where β^i and γ^i are continuous functions.

The modeling choice of both [2] and [3] when it comes to the macroeconomic process is thus a zero mean Gaussian process. The disturbing element to the wealth process in equation (2.3) is decomposed into one term handling the macroeconomic factor and another term describing the individual fluctuations disturbing the wealth of a firm, whereas in the continuous time model only the macroeconomic factor is described explicitly. One big difference in the two models studied is regarding the macroeconomic factor. In the discrete case, the macroeconomic factor is constant over the time horizon of one year, but in the continuous model it is a stochastic process representing a mean of macroeconomic variables. The macroeconomic process in the continuous time model is a fractional Brownina motion, so it is not a Markovian process. If the macroeconomic variable in the discrete time model was not constant, it would be Gaussian and thus a (standard) Brownian motion as well as it would be Markovian.

3.5 The price of credit derivatives

The prices of derivatives are influenced by the contagion matrix C and by the macroeconomic factor Ψ. Before presenting the pricing formulas, there are some assumptions that need to be stated:

1. The information that the investor has at time *t* is given by $\mathcal{F}_t$, i.e. the investor knows the processes Ψ, the self default process Y and the contagion matrix C up to time *t*.

2. The default free interest rate is deterministic, and it is set equal to 0.

3. The risk neutral pricing measure $\mathbb{Q}$ exists and is known such that the price at time *t* of any $\mathcal{F}_T$-measurable claim L_T in $L_1(\Omega, \mathbb{Q})$ is given by $\mathbb{E}_{\mathbb{Q}}(L_T|\mathcal{F}_t) = L_t$ for $0 \leq t \leq T$.

It is not assumed that the pricing measure necessarily is unique. Without the specific expression of the macroeconomic process Ψ as given in equation (3.4), Biagini, Fuschini and Klüppelberg formulated the following pricing formula which is given without any restrictions on the matrix C, i.e. the matrix is stochastic and depends on time:

Theorem 1. *Let $f : \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be a bounded measurable function. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1 \ldots, \beta_m)$ and $z = (z_1, \ldots, z_m)$ be in $\{ 0, 1\}^{\ m}$ and $h^{(i)}, k^{(i)}$ be in $\{ 0, 1\}^{\ m-1}$ for i = $1, \ldots, m$. Set $h_{ii} = k_{ii} := 1$ for i = $1, \ldots, m$, $h_{ij} := [h^{(i)}]_j$ and $k_{ij} := [k^{(i)}]_j$ for $i \neq j$. Then for t in [0,T]*

$$\mathbb{E}_{\mathbb{Q}}(f(\Psi_T, Z_T)|\mathcal{F}_t) = \sum_{\alpha,\beta,z \in \{ 0,1\}^{\ m}} (-1)^{\sum_{j=1}^m \alpha_j z_j} \prod_{j=1}^m z_j^{1-\alpha_j}$$

$$\times \prod_{i=1}^m \Big((Y_t(i) a_t(i))^{1-\beta_i} (1 - Y_t(i))^{\beta_i} \Big) \ \mathbb{E}_{\mathbb{Q}} \Big(f(\Psi_T, z) \prod_{i=1}^m b_{t,T}(i)^{\beta_i} | \mathcal{F}_t^{\Psi} \Big), \quad (3.6)$$

with

$$a_t(i) = \sum_{h^{(i)} \in \{ 0,1\}^{\ m-1}} \mathbf{1}_{\{ \tilde{h_i}(\alpha,h)=0\}} \ \mathbf{1}_{\{ C^{(i)}_{\tau_i} = h^{(i)}\}}$$

$$b_{t,T}(i) = \sum_{h^{(i)},k^{(i)} \in \{ 0,1\}^{\ m-1}} \mathbf{1}_{\{ C^{(i)}_t = k^{(i)}\}}$$

$$\times \ \Big(\int_T^\infty \lambda^i(u, \Psi_u) e^{-\int_t^u \lambda^i(s,\Psi_s)ds} p_{t,u}(k^{(i)}, h^{(i)}) du$$

$$+ \mathbf{1}_{\{ \tilde{h_i}(\alpha,h)=0\}} \int_t^T \lambda^i(u, \Psi_u) e^{-\int_t^u \lambda^i(s,\Psi_s)ds} p_{t,u}(k^{(i)}, h^{(i)}) du \Big)$$

where

$$\tilde{h_i}(\alpha, h) := \begin{cases} 0 & \text{if} \quad \sum_{j=1}^m \alpha_j h_{ij} = 0, \\ 1 & \text{otherwise} . \end{cases} \quad (3.7)$$

and $p_{t,\tau_i}(k^{(i)}, h^{(i)}) := \prod_{j=1}^m p^{ij}_{t\tau_i}([k^{(i)}]_j, [h^{(i)}]_j)$ denotes the joint transition probabilities of the random vector $C^{(i)}_{\tau_i}$ from time t to time τ_i.

For proof, see [2].

If one assumes a time-independent contagion matrix which might be random, meaning that

$$C_t(i, j) = C_\omega(i, j), \quad t \geq 0, \quad (3.8)$$

where the $C_\omega(i, j)$ are given by i.i.d. random variables which are independent of the processes Y and Ψ. Then the filtration
$\mathcal{F}_t = \mathcal{F}_t^\Psi \vee \mathcal{F}_t^Y \vee \sigma(C)$ for *t* > 0. Again, without specifying the macroeconomic process the general pricing formula becomes as in the following theorem:

Theorem 2. *If the contagion matrix is of the form (3.8), the pricing formula (3.6) for 0 < t* $\leq T$ *is given by*

$$\mathbb{E}_{\mathbb{Q}}(f(\Psi_T, Z_T)|\mathcal{F}_t) = \sum_{\alpha,z\in\{0,1\}^m} \sum_{h\in\{0,1\}^{m(m-1)}} (-1)^{\sum_{i=1}^m \alpha_i z_i}$$

$$\times \prod_{i=1}^m z_i^{1-\alpha_i}(1-Y_t(i))^{\tilde{h_i}(\alpha,h)}$$

$$\times \ \mathbf{1}_{\{C=h\}} \ \ \mathbb{E}_{\mathbb{Q}}\Big(f(\Psi_T, z)e^{-\int_t^T \sum_{i=1}^m \tilde{h_i}(\alpha,h)\lambda^i(u,\Psi_u)du}\Big|\mathcal{F}_t^{\Psi}\Big) \qquad (3.9)$$

and for t = 0 the pricing formula becomes

$$\mathbb{E}_{\mathbb{Q}}(f(\Psi_T, Z_T)) = \sum_{\alpha,z\in\{0,1\}^m} \sum_{h\in\{0,1\}^{m(m-1)}} (-1)^{\sum_{i=1}^m \alpha_i z_i}$$

$$\times \prod_{i=1}^m z_i^{1-\alpha_i}\mathbb{Q}(C=h) \ \mathbb{E}_{\mathbb{Q}}\Big(f(\Psi_T, z)e^{-\int_0^T \sum_{i=1}^m \tilde{h_i}(\alpha,h)\lambda^i(u,\Psi_u)du}\Big) \qquad (3.10)$$

where $\tilde{h_i}(\alpha, h)$ *is as in (3.7) with* $h_{ii} := 1$ *for i = 1,* $\ldots, m$ *and* $h_{ij} := [h]_{ij}$ *for i* $\neq$ *j.*

For proof, see [2].

If the contagion matrix is deterministic, i.e. for every *i,j* in $\{1, \ldots, m\}$ and all $t \geq 0$,

$$C_t(i,j)(\omega) = C_t(i,j) \quad \text{for all} \quad \omega \in \Omega,$$

one has that the filtration $\mathcal{F}_t^C = \{\emptyset, \Omega\}$ for every *t* in [0,T]. In this situation, the pricing formula (3.6) becomes:

Corollary 1. *Assuming that the contagion matrix is deterministic, the pricing formula (3.6) simplifies to*

$$\mathbb{E}_{\mathbb{Q}}(f(\Psi_T, Z_T)|\mathcal{F}_t) = \sum_{\alpha,z\in\{0,1\}^m} (-1)^{\sum_{i=1}^m \alpha_i z_i} \prod_{i=1}^m z_i^{1-\alpha_i}(1-Y_t(i))^{\tilde{h_i}(\alpha)}$$

$$\times \ \mathbb{E}_{\mathbb{Q}}\Big(f(\Psi_T, z)e^{-\int_t^T \sum_{i=1}^m \tilde{h_i}(\alpha)\lambda^i(u,\Psi_u)du}|\mathcal{F}_t^{\Psi}\Big) \qquad (3.11)$$

where

$$\tilde{h_i}(\alpha) := \begin{cases} 0 & \textit{if} \quad \sum_{j=1}^m \alpha_j C_T(i,j) = 0, \\ 1 & \textit{otherwise}\ . \end{cases}$$

The following is an example of the use of formula (3.11).

Example: To find the price of a defaultable bond of a firm in group G_i for $i = 1, 2$, let there be one bond of one firm in G_i at one time. Recall that firms in the same group have the same default intensity. The short rate was assumed deterministic and it was set to be 0. The bond has payoff $f(\Psi_T, z) = (1 - Z_T^{[i]})$, and one has to calculate

$$V_0^{[i]} = \mathbb{E}_{\mathbb{Q}}\Big((1 - Z_T^{[i]})e^{-\int_0^T \sum_{i=1}^m \tilde{h_i}(\alpha)\lambda^i(u,\Psi_u)du}\Big).$$

Since the macroeconomic process has not been specified yet, one can specify the default intensity $\lambda^i(u, \Psi_u) = \lambda^i(u, B_u) = \lambda_u^i$ by, for instance the Vasicek model as was done in section 1.1.2. The number of firms, m, is equal to the total number of firms in the two groups.

If one is to consider the pricing of a CDS in this setting, one would get that the value of the pricing formula at time $t = 0$ for the protection buyer of a defaultable bond with payoff $(1 - Z_T^{[i]})$ would be, if the spread is paid continuously until default,

$$\text{Buyer} \quad V_0^{[i]} = s\int_0^T \mathbb{E}_{\mathbb{Q}}\Big((1 - Z_T^{[i]})e^{-\int_0^u \sum_{i=1}^m \tilde{h_i}(\alpha)\lambda^i(s,\Psi_s)ds}\Big)du,$$

and for the protection seller one gets

$$\text{Seller} \quad V_0^{[i]} = \int_0^T \mathbb{E}_{\mathbb{Q}}\Big((1 - Z_T^{[i]} - R^{[i]})e^{-\int_0^u \sum_{i=1}^m \tilde{h_i}(\alpha)\lambda^i(s,\Psi_s)ds} \times \sum_{i=1}^m \tilde{h_i}(\alpha)\lambda^i(u, \Psi_u)\Big)du.$$

And to find the fair spread, one equates the expression for the buyer and seller and solves for the spread s.

By specifying the macroeconomic process as done in section 3.4, the pricing formula for a long range dependent macroeconomic state variable process is given in the following theorem.

Theorem 3. *Assume that the contagion matrix C is deterministic and that for all i in* $\{1,\ldots,m\}$ *the intensities of the self default processes* $Y_i = (Y_t(i))_{t\geq 0}$ *are of the form*

$$\lambda^i(t,\Psi_t^H) := \beta^i(t)\int_0^t g(s)dB_s^H + \gamma^i(t) \quad t\geq 0,$$

where β^i *and* γ^i *are continuous functions. g is in* $H^\mu([0,T]) \subset L_2^H([0,T])$ *with* μ *> 1-H and such that* $\frac{1}{g(s)}$ *is well defined for all s in [0,T]. Let* $f(\cdot,z)$ *and* $\psi(\cdot)$ *be deterministic continuous functions and denote for all z in* $\{0,1\}^m$

$$f^\psi(x,z) := f(\psi(x),z), \qquad x\in\mathbb{R}$$

and

$$f_\alpha^\psi(x,z) := e^{-\alpha x}f^\psi(x,z), \qquad \alpha,x\in\mathbb{R},$$

where $f^\psi = f\circ\psi$*. Assume that there exists some a in* $\mathbb{R}$ *such that* $f_a^\psi(\cdot,z)$ *and its Fourier transform* $\hat{f}_a^\psi(\cdot,z)$ *belong to* $L_1(\mathbb{R})$ *for all z in* $\{0,1\}^m$*. Finally, let* ψ *be invertible and set*

$$\Psi_t^H := \psi(\int_0^t g(s)dB_s^H).$$

Then the price (3.11) at time t in [0,T] is given by the following formula

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}}(f(\Psi_T,Z_T)|\mathcal{F}_t) = \\ &\sum_{\alpha,z\in\{0,1\}^m}(-1)^{\sum_{i=1}^m\alpha_iz_i}\prod_{i=1}^m\Big(z_i^{1-\alpha_i}(1-Y_t(i))^{\tilde{h_i}(\alpha)}\Big)e^{-\int_t^T\sum_{i=1}^m\tilde{h_i}(\alpha)\gamma^i(u)du} \\ &\times\ e^{\int_0^t\sum_{i=1}^m\tilde{h_i}(\alpha)\beta^i(u)\int_0^u g(s)dB_s^H du}\frac{1}{2\pi}\int_{\mathbb{R}}e^{\frac{1}{2}\int_t^T\int_0^t\eta(s,\xi)\eta(u,\xi)|u-s|^{2H-2}dsdu} \\ &\times\ e^{\int_0^t\eta(s,\xi)dB_s^H} \qquad (3.12) \\ &\times\ e^{\int_0^t\left(I_{t^-}^{-(H-\frac{1}{2})}\left(I_{T^-}^{-(H-\frac{1}{2})}\left((\eta(s,\xi)\mathbf{1}_{[t,T]}(s))^{H-\frac{1}{2}}\right)\right)\right)^{H-\frac{1}{2}}dB_s^H}\hat{f_\alpha^\psi}(\xi,z)d\xi \end{aligned}$$

where

$$\tilde{h_i}(\alpha) := \begin{cases} 0 & \textit{if} \ \sum_{j=1}^m\alpha_jC_T(i,j)=0, \\ 1 & \textit{otherwise}, \end{cases}$$

and

$$\eta(s,\xi) := g(s)\Big(a+i\xi-\int_s^T\sum_{i=1}^m\tilde{h_i}(\alpha)\beta^i(u)du\Big), \quad s\in[0,T]$$

and for $\alpha = H-\frac{1}{2}$ *in (0,*$\frac{1}{2}$*)*

$$(I_{t^-}^{-\alpha}\eta)(s) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{ds}\Big(\int_s^t\eta(r)(r-s)^{\alpha-1}dr\Big)$$

For a proof, see [2], and for more on the Fourier transform in this case, see appendix A.

Chapter 4

An extension of the contagion model in continuous time

This chapter is aiming at an extension of the description of the contagion matrix introduced in [2]. Instead of having a zero-one model indicating whether there is default contagion or not, one could try to describe the possible default contagion of firm i as a positive or negative contagion relative to firm j as was done in [3]. The self default of firm j is the indicator variable $Y_t(j) = \mathbf{1}_{\{\tau_j \leq t\}}$, $t \geq 0$. The $Y_t(j)$ are assumed to be independent of each other.

The probability space and assumptions are the same as in the continuous time model which are stated in sections 3.2 and 3.5.

4.1 The contagion model

The default model of firm j in $\{1, \ldots, m\}$ is the same used by [2] and is given by

$$D_j(t) = Y_t(j) + (1 - Y_t(j))\Big(1 - \prod_{i \neq j}(1 - C_{t\wedge\tau_i}(i,j)Y_t(i))\Big), \qquad t \geq 0, \quad (4.1)$$

and since $C_t(i,i) \equiv 1$ the process $D_t(j)$ can be written in a shorter form:

$$D_t(j) = 1 - \prod_{i=1}^{m}(1 - C_{t\wedge\tau_i}(i,j)Y_t(i)), \qquad t \geq 0. \quad (4.2)$$

Let $x \geq 1$ be in $\mathbb{N}$. The contagion matrix is now of the form

$$C_t(i,j) = \begin{cases} 1 - \frac{1}{x} & \text{firm } i \text{ has a negative contagion effect on firm } j, \\ 0 & \text{firm } i \text{ has no contagion effect on firm } j, \\ 1 - x & \text{firm } i \text{ has a positive contagion effect on firm } j\text{ .} \end{cases}$$

Figure 4.1: Possible values of $D_t(j)$.

which means that if $x = 1$ then $C_t(i, j) = 0$ for all $i \neq j$ in $\{ 1, \ldots, m\}$ so there does not exist any economic relations between the different firms in the portfolio and they are thus all independent of each other. The expression 'negative contagion effect' is to be understood as firm i and j were in a cooperative economic relation and the default of firm i was not good for firm j, whereas 'positive contagion effect' means that firm i and j were in a competitive business relation and the default of firm j was good for firm i.

By looking at the contagion term $\prod_{i=1}^{m}(1 - C_{t\wedge\tau_i}(i, j)Y_t(i))$ when $Y_t(i) = 1$ one gets that, for $i \neq j$,

$$\prod_{i=1}^{m}(1 - C_t(i, j)) = \begin{cases} \prod_{i=1}^{m} \frac{1}{x_{ij}} = y & \in (0, 1] \\ 1 & \\ \prod_{i=1}^{m} x_{ij} = z & \in [1, \infty]. \end{cases}$$

Then the default model expressed in equation (4.2) gives the following interpretation for all i, j in $\{ 1, \ldots, m\}$, including the case when $i = j$:

$$D_t(j) = 1 - \prod_{i=1}^{m}(1 - C_{t\wedge\tau_i}(i, j)Y_t(i))$$

$$= \begin{cases} 1 & & \text{self-default of firm j} \\ 1 - y > 0 & \in [0, 1) & \text{default by contagion for firm j} \\ 0 & & \text{firm j is solvent} \\ 1 - z < 0 & \in [-\infty, 0] & \text{firm j is solvent after contagion} \end{cases}$$

where the last equality follows from firm i having defaulted. So this means that $D_t(j)$ is taking values in $[-\infty, 1]$. If $D_t(j) = (1 - y) > 0$ it means that firm j has defaulted by contagion from the default of firm i, i.e. there has been a *negative contagion* which was bad for firm j. If $D_t(j) = 1$ firm j has defaulted by itself. If $D_t(j) = 0$ there was no default effect in the portfolio and firm j is solvent, and if $D_t(j) = 1 - z < 0$ then the firms were in a competitive economic relation and firm j is better off than it was previously, i.e. there has been a *positive contagion*. Thus, the closer $D_t(j)$ is to $-\infty$, the better it is for firm j as illustrated in figure 4.1. All in all, if $D_t(j) > 0$ there is a default in the portfolio at postition j and if $D_t(j) \leq 0$ firm j in the portfolio is solvent. One could then describe the process $Z_t(j)$ given in the continuous time model in chapter 3 by the indicator function $Z_t(j) = \mathbf{1}_{\{ D_t(j)>0\}}$.

4.2 Pricing formula

By considering the case where the contagion matrix C is deterministic, one gets the following pricing formula:

Theorem 1. *Let $f : \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}$ be a bounded measurable function. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be in $\{ 0,1\}^m$, d_j in $[-\infty, 1]$ for each j in $\{ 1, \ldots, m\}$. For t in [0,T]*

$$\mathbb{E}_{\mathbb{Q}}(f(\Psi_T, D_T)|\mathcal{F}_t) = \sum_{d\in[-\infty,1]^m} \sum_{\alpha\in\{ 0,1\}^m} (-1)^{\sum_j \alpha_j}$$
$$\times \prod_{j=1}^{m}\prod_{i=1}^{m}\Big(1 - C_T(i,j) + C_T(i,j)(1 - Y_t(i))\Big)^{\alpha_j}$$
$$\times \ \mathbb{E}_{\mathbb{Q}}\Big(f(\Psi_T, d)(e^{-\int_t^T \lambda^i(u,\Psi_u)du})^{\alpha_j}|\mathcal{F}_t^{\Psi}\Big).$$

Proof. By the law of total probability (see appendix A) it follows that

$$\mathbb{E}_{\mathbb{Q}}\Big(f(\Psi_T, D_T)|\mathcal{F}_t\Big) = \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{E}_{\mathbb{Q}}(f(\Psi_T, D_T)|\mathcal{G}_t)|\mathcal{F}_t\Big)$$

$$= \mathbb{E}_{\mathbb{Q}}\Big(\sum_{d\in[-\infty,1]^m} f(\Psi_T, d) \ \ \mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{\{ \vec{D_T}=\vec{d}\}} |\mathcal{G}_t) \ \ |\mathcal{F}_t\Big). \tag{4.3}$$

Equation (4.2) becomes $D_T(j) = 1 - \prod_{i=1}^{m}(1 - C_T(i,j)Y_T(i))$. Starting by focusing on the inner expectation $\mathbb{E}(\mathbf{1}_{\{ \vec{D_T}=\vec{d}\}} |\mathcal{G}_t)$ and looking at

$$\mathbf{1}_{\{ D_T(j)=d_j\}} = \begin{cases} 1 & D_j = d_j, \\ 0 & \text{otherwise,} \end{cases}$$

for d_j in $[-\infty, 1]$ for each j in $\{ 1, \ldots, m\}$, and putting

$$\mathbf{1}_{\{ D_T(j)=d_j\}} = 1 - a_T(j).$$

Then

$$\mathbf{1}_{\{ \vec{D_T}=\vec{d}\}} = \prod_{j=1}^{m}(1 - a_T(j)) \tag{4.4}$$

where $a_T(j) = \prod_{i=1}^{m}(1 - C_T(i,j)Y_T(i))$.

By applying the following identity:

$$\prod_{j=1}^{m}(A_j + B_j) = \sum_{\alpha\in\{ 0,1\}^m} \prod_{j=1}^{m}(A_j^{1-\alpha_j} B_j^{\alpha_j}),$$

where $\alpha_j \in \{0,1\}$ for $j = 1,\ldots,m$. Setting $0^0 := 1$ the formula also holds if there exists some j in $\{1,\ldots,m\}$ such that $A_j = 0$ or $B_j = 0$. Applying this formula to equation (4.4) with

$$A_j = 1 \quad \text{and} \quad B_j = (-a_T(j))$$

one gets that

$$\prod_{j=1}^{m}(1-a_T(j)) = \sum_{\alpha\in\{0,1\}^m}\prod_{j=1}^{m}(-a_T(j))^{\alpha_j},$$

and the inner conditional expectation in equation (4.3) becomes

$$\mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{\{\vec{D_T}=\vec{d}\}}|\mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(\sum_{\alpha\in\{0,1\}^m}(-1)^{\sum_{j=1}^m\alpha_j}\prod_{j=1}^{m}(a_T(j))^{\alpha_j}|\mathcal{G}_t)$$

$$= \sum_{\alpha\in\{0,1\}^m}(-1)^{\sum_{j=1}^m\alpha_j}\mathbb{E}_{\mathbb{Q}}\Big(\prod_{j=1}^{m}\prod_{i=1}^{m}\Big((1-C_T(i,j)Y_T(i))\Big)^{\alpha_j}|\mathcal{G}_t\Big).$$

Since T $\geq$ t, assumption 2 in section 3.2 holds and

$$= \sum_{\alpha\in\{0,1\}^m}(-1)^{\sum_{j=1}^m\alpha_j}\prod_{j=1}^{m}\prod_{i=1}^{m}\mathbb{E}_{\mathbb{Q}}(\Big((1-C_T(i,j)Y_T(i))\Big)^{\alpha_j}|\mathcal{G}_t). \qquad (4.5)$$

Since

$$\mathbb{E}_{\mathbb{Q}}(\Big((1-C_T(i,j)Y_T(i))\Big)^{\alpha_j}|\mathcal{G}_t)$$

$$= \mathbb{E}_{\mathbb{Q}}(\Big((1-C_T(i,j)+C_T(i,j)-C_T(i,j)Y_T(i))\Big)^{\alpha_j}|\mathcal{G}_t)$$

$$= \Big(1^{\alpha_j} - C_T(i,j)^{\alpha_j} + C_T(i,j)^{\alpha_j}\mathbb{E}_{\mathbb{Q}}((1-Y_T(i))^{\alpha_j}|\mathcal{G}_t)\Big)$$

is the same as $\mathbb{E}_{\mathbb{Q}}(\Big((1-C_T(i,j)Y_T(i))\Big)^{\alpha_j}|\mathcal{G}_t)$ for $\alpha_j = 0,1$, and since the expression is either 1 for $\alpha_j = 0$ or $\Big(1 - C_T(i,j) + C_T(i,j)\mathbb{E}_{\mathbb{Q}}((1-Y_T(i))|\mathcal{G}_t)\Big)$ for $\alpha_j = 1$, then the α_j can come out. The product of a measurable fuction is still measurable, so by assumption 3 in section 3.2 one gets that equation (4.5) is equal to the following:

$$\sum_{\alpha\in\{0,1\}^m}(-1)^{\sum_{j=1}^m\alpha_j}\prod_{j=1}^{m}\prod_{i=1}^{m}\Big(1-C_T(i,j)+C_T(i,j)(1-Y_t(i))e^{-\int_t^T\lambda^i(u,\Psi_u)du}\Big)^{\alpha_j}.$$

Returning to the aim, which is equation (4.3), one gets that

$$\mathbb{E}_{\mathbb{Q}}(f(\Psi_T,D_T)|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}\Big(\sum_{d\in[-\infty,1]^m}f(\Psi_T,d)\ \mathbb{E}_{\mathbb{Q}}(\mathbf{1}_{\{\vec{D_T}=\vec{d}\}}\ |\mathcal{G}_t)\ |\mathcal{F}_t\Big)$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbb{Q}}\Big(\sum_{d\in[-\infty,1]^m} \sum_{\alpha\in\{\ 0,1\}\ ^m} (-1)^{\sum_{j=1}^m \alpha_j} \prod_{j=1}^m \prod_{i=1}^m f(\Psi_T, d) \\
&\quad \times \ \Big[1 - C_T(i,j) + C_T(i,j)(1 - Y_t(i)) e^{-\int_t^T \lambda^i(u,\Psi_u)du}\Big]^{\alpha_j} \ \ |\mathcal{F}_t\Big)
\end{aligned}
$$

$$
\begin{aligned}
= \sum_{d\in[-\infty,1]^m} \sum_{\alpha\in\{\ 0,1\}\ ^m} (-1)^{\sum_{j=1}^m \alpha_j} \prod_{j=1}^m \prod_{i=1}^m \Big(1 - C_T(i,j) + C_T(i,j)(1 - Y_t(i))\Big)^{\alpha_j} \\
\times \ \mathbb{E}_{\mathbb{Q}}\Big(f(\Psi_T, d)(e^{-\int_t^T \lambda^i(u,\Psi_u)du})^{\alpha_j} \ \ |\mathcal{F}_t^{\Psi}\Big),
\end{aligned}
$$

where the last equatlity follows from assumption 1 in section 3.2.

□

Appendix A

Elements on analysis and probability theory

Most of the contents in this section is from [9].

Weak Law of Large Numbers
Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) with mean $E(X_i) = \xi$ and variance $\sigma_i < \infty$. Then the average $\overline{X} = \frac{(X_1 + \ldots + X_n)}{n}$ satisfies

$$\overline{X} \xrightarrow{P} \xi \quad \text{as } n \to \infty.$$

The meaning of 'weak' is that the convergence is only for the *n'th* element in the sequence, versus the strong Law of Large Numberes where the convergance is for the whole sequence.

Linear transformation I
If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$ for a and b constants.

Linear transformation II
If X and Y are independent random variables with distributions $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$, then the sum $X + Y \sim \mathcal{N}(\mu_X + \mu Y, \sigma_X^2 + \sigma_Y^2)$.

The Continuity Theorem
If X_n is a sequence of random variables such that $X_n \xrightarrow{P} X$ and if the function *f* is continuous at X, then

$$f(X_n) \xrightarrow{P} f(X) \quad \text{as } n \to \infty.$$

Central Limit Theorem
Let $X_1, \dots, X_n$ be i.i.d. with mean $E(X_i) = \xi$ and variance $\sigma_i < \infty$. Then

$$\frac{\sqrt{n}(\overline{X} - \xi)}{\sigma} \quad \xrightarrow{L} \quad \mathcal{N}(0,1) \qquad \text{as } n \to \infty.$$

The next definition is the definition of expectation of a random variable.

Expectation
A random variable $X : \Omega \to \mathbb{R}^n$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that X is $\mathcal{F}$-measurable has *expectation* $\mathbb{E}(X) = \int_\Omega X(\omega) d\mathbb{P}(\omega)$ if $\int_\Omega |X(\omega)| d\mathbb{P}(\omega) < \infty$. And for some A in $\mathcal{F}$,

$$\mathbb{E}(\mathbf{1}_A) = \int_\Omega \mathbf{1}_A d\mathbb{P}(\omega) = \int_A d\mathbb{P}(\omega) = \mathbb{P}(A).$$

The law of total probability is as follows:

The Law of Total Probability
Let $B_1, \dots B_m$ be such that $\cup_{i=1}^m B_i = \Omega$ and $B_i \cap B_j = \emptyset$ for $i \neq j$ with $\mathbb{P}(B_i) > 0$ for all i. Then, for any event A in Ω,

$$\mathbb{P}(A) = \mathbb{P}(A \cap \Omega) = \sum_{i=1}^m \mathbb{P}(A \cap B_i) = \sum_{i=1}^m \mathbb{E}(\mathbf{1}_{A \cap B_i}) = \sum_{i=1}^m \mathbb{E}(\mathbf{1}_A \mathbf{1}_{B_i}).$$

Some more on the Fourier transform which occurs in equation (3.12).
Recall that the Fourier transformation of a function $f(x)$ is the characteristic function of $f(x)$, i.e. if $\int_\mathbb{R} |f(x)| dx < \infty$, then $\phi_X(u) = \mathbb{E}(e^{iuX}) = \int_\mathbb{R} e^{iux} f(x) dx$. The formula $f(x) = \frac{1}{2\pi} \int_\mathbb{R} e^{-iux} \phi(u) du$ determines the density $f(x)$ of a random variable X.

Fourier transform
For *a* and *x* in $\mathbb{R}$ and for $f \circ \psi := f^\psi$, define the function $f_a^\psi := e^{-ax} f^\psi(x)$ and its Fourier transform by $\widehat{f_a^\psi}(\xi) := \int_\mathbb{R} e^{-i\xi x} f_a^\psi(x) dx$ for ξ in $\mathbb{R}$. Assume that *f* and ψ are such that

$$A := \{\ a \in \mathbb{R} | f_a^\psi(\cdot) \in L_1(\mathbb{R}) \ \ and \ \ \widehat{f_a^\psi}(\cdot) \in L_1(\mathbb{R})\} \ \neq \emptyset.$$

Then the following inversion formula holds:

$$f_a^\psi(x) = \frac{1}{2\pi} \int_\mathbb{R} e^{i\xi x} \widehat{f_a^\psi}(\xi) d\xi, \qquad x \in \mathbb{R}.$$

Appendix B

Elements on fractional Brownian motion

Most of the topic regarding the fractional Brownian motion is from [1].

Definition (Fractional Brownian Motion)
Let H be a constant belonging to (0,1). A *fractional Brownian motion, fBm,* $(B_t^H)_{t\geq 0}$ of Hurst index H is a continuous centered Gaussian process with covariance function

$$\mathbb{E}(B_t^H, B_s^H) := \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \qquad t, s \in \mathbb{R}^+.$$

A fBm has the following properties:

- $B_0^H = 0$ and $\mathbb{E}(B_t^H) = 0$ for all $t \geq 0$.
- $B_{t+s}^H - B_t^H$ has the same distribution as B_t^H for $s, t \geq 0$, i.e. the increments of B^H are homogeneous.
- B^H is a Gaussian process and $\mathbb{E}\Big((B_t^H)^2\Big) = t^{2H}, t \geq 0$ for all H in (0,1).
- B^H has continuous trajectories.

The covariance between $(B_{t+h}^H - B_t^H)$ and $(B_{s+h}^H - B_s^H)$ with $s + h \leq t$ and $t - s = nh$ is

$$\rho_H(n) = Cov(B_{t+h}^H - B_t^H, B_{s+h}^H - B_s^H) = \mathbb{E}\Big((B_{t+h}^H - B_t^H)(B_{s+h}^H - B_s^H)\Big)$$

$$= -|t-s|^{2H} + \frac{1}{2}|t+h-s|^{2H} + \frac{1}{2}|t-s-h|^{2H}$$

$$= \frac{|h|^{2H}}{2}\Big((n+1)^{2H} + (n-1)^{2H} - 2|n|^{2H}\Big) \ \sim H(2H-1)n^{2H-2} \ \ , n \to \infty$$

and since

$$\lim_{n\to\infty} \frac{\rho_H(n)}{H(2H-1)n^{2H-2}} = 1$$

then B_t^H exhibits *long-range dependence* for $H > \frac{1}{2}$. Two increments of the form $(B_{t+h}^H - B_t^H)$ and $(B_{t+2h}^H - B_{t+h}^H)$ are positively correlated for $H > \frac{1}{2}$, negatively correlated for $H < \frac{1}{2}$ and for $H = \frac{1}{2}$ the fBm is a standard Brownian motion which has independent increments.

For a comparison to the standard Brownian motion the following definition and remark are from [10].

Definition (Standard, one-dimensional Brownian Motion). *A one-dimensional Brownian motion* is a continuous, adapted process $B = \{ B_t, \mathcal{F}_t; 0 \le t < \infty\}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the properties that $B_0 = 0$ a.s. and for $0 \le s < t$, the increment $(B_t - B_s)$ is independent of $\mathcal{F}_s$ and is normally distributed with mean zero and variance *(t-s)*.

Remark: A one-dimensional Brownian motion is a zero mean Gaussian process with covariance

$$cov(B_t, B_s) = s \wedge t; \qquad s, t \ge 0.$$

Some more details regarding the space $H^\mu([0,T])$:
If the deterministic function g is in the Schwartz space $\mathcal{S}([0,T])$, then the norm of g is

$$||g||_H^2 := \int_0^T \int_0^T g(s)g(t)H(2H-1)|s-t|^{2H-1} dsdt < \infty.$$

If $\mathcal{S}([0,T])$ is equipped with the inner product

$$< f, g >:= \int_0^T \int_0^T f(s)g(t)H(2H-1)|s-t|^{2H-1} dsdt < \infty \quad f, g \in \mathcal{S}([0,T]),$$

then the completion of $\mathcal{S}([0,T])$ is the separable Hilbert space $L_2^H([0,T])$. The space of Hölder continuous functions $H^\mu([0,T]) \subset L_2^H([0,T])$, $\mu > 1-H$. In [2], $g \in H^\mu([0,T])$.

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