
Coarsening Natural Deduction Proofs I: Finding Perfect Proofs

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Abstract

This paper explores how, given a proof, we can systematically transform it into a proof that contains no irrelevancies and which is as strong as possible. I define a weaker and stronger notion of what counts as a proof with no irrelevancies, calling them perfect proofs and gaunt proofs, respectively. Using classical core logic to study classical validities and core logic to study intuitionistic validities, I show that every core proof or classical core proof can be transformed into a perfect proof. In a sequel paper, I show how proofs in core logic can also be transformed into gaunt proofs and I observe that this property fails for classical core logic.

1 Introduction

When we formalize an argument and provide a proof to demonstrate its validity, oftentimes redundancies, irrelevancies, and unnecessary detours creep into the proof. For instance, we may not find the simplest or most direct proof. Or we may come up with a proof that contains unnecessary detours in the sense, familiar to proof theorists, of having maximal sentence occurrences.

It can also happen, when formalizing an argument, that the original argument itself contains irrelevancies or is unnecessarily weak. For instance, the argument may have contained irrelevant premises or it may have a weaker conclusion than the premises warranted.

Either of these situations is suboptimal from a logical perspective. While there may be practical reasons to be satisfied with a proof that contains irrelevancies or detours—for instance, because it was the easiest to find and it serves your purposes—the simplest and most direct proof has a distinguished standing from a proof-theoretical point of view. Similarly, there may be rhetorical or dialectical reasons to provide a proof for a weaker argument even when a stronger argument is available, but from a logical point of view we generally want to prove the strongest result possible. We can ask, then, whether it is in general possible to avoid these logically suboptimal situations, and if so how.

This leads to the *guiding question* of this paper: given a natural deduction proof, is there a way to systematically transform it into a proof that contains no irrelevancies and in which the argument proved to be valid is as strong as possible? In brief, the answer is yes, though it will depend to some extent on whether we are interested in classical or intuitionistic validity and exactly how we define a proof with no irrelevancies. With this guiding question and its intuitive motivation on the table, I turn now to making it precise, so that it admits of a definitive answer.

1.1 Making the Question Precise

In what follows I will use the convention that Φ, Ψ, \dots are metavariables ranging over sets of formulas, ϕ, ψ, \dots are metavariables ranging over formulas, and A, B, \dots are metavariables ranging over atoms.

Instead of arguments, I will talk of sequents. A sequent, as I will understand it here, is an ordered pair $\Delta : \Phi$ of sets of sentences in a given language. Since I will be working in a natural deduction

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setting, and natural deduction proofs have at most one conclusion, Φ will be at most a singleton. The absurdity symbol \perp is treated merely as a punctuation device in proofs, not a formula itself; hence a natural deduction proof of \perp from Δ is really a proof of the zero-conclusion sequent $\Delta : \emptyset$. We say $\Delta' : \Phi'$ is a subsequent of $\Delta : \Phi$ iff $\Delta' \subseteq \Delta$ and $\Phi' \subseteq \Phi$; if either of these inclusions is proper, then $\Delta' : \Phi'$ is a *proper* subsequent.

I will be concerned with both classical validity and intuitionistic validity. As usual, $\Delta : \Phi$ is valid when every interpretation that makes every member of Δ true also makes at least one member of Φ true. The difference between classical validity and intuitionistic validity is a matter of what an interpretation is. In the classical case an interpretation can be taken to be a truth-assignment using the familiar truth-tables. In the intuitionistic case, interpretations are given in terms of Kripke models (or something equivalent, such as Heyting algebras). I will assume the reader is familiar with Kripke models for intuitionistic logic and their basic features; see [23, Ch. 2] for a standard reference.

With these preliminary remarks in place, we can now offer a precise definition of what counts as a proof with no irrelevancies or detours. In fact, there are two precisifications of this notion that I will study. The first is based on the idea of perfect validity:

DEFINITION 1 (Perfect validity; perfect proofs).

A sequent $\Delta : \Phi$ is *perfectly valid* iff it is valid but has no proper subsequent that is valid. A proof Π is a *perfect proof* iff every subproof, including Π itself, is a proof of a perfectly valid sequent.¹

EXAMPLE 1

The following are perfectly valid, both classically and intuitionistically:

- $A \wedge B : A \wedge B$.
- $A, A \rightarrow B : A \wedge B$.
- $\neg A : A \rightarrow B$.

The following are *not* perfectly valid, either intuitionistically or classically:

- $A \wedge B, C : B$, since this has the valid proper subsequent $A \wedge B : B$.
- $A, B, (A \vee B) \rightarrow C : C$, since this has the valid proper subsequents $A, (A \vee B) \rightarrow C : C$ and $B, (A \vee B) \rightarrow C : C$.
- $A, \neg A : B$, since this has the valid proper subsequent $A, \neg A : \emptyset$.

It is also possible for a sequent to be perfectly valid intuitionistically but *not* classically, for example:

- $P : P \vee \neg P$.

The basic idea is that if any element of the sequent were irrelevant, then it could be omitted without loss. Since no premise or conclusion can be omitted from a perfectly valid sequent without rendering it invalid, no element of the sequent is irrelevant. A perfect proof, then, is a proof of a sequent with no irrelevancies, and moreover it does not contain any detours through subproofs with irrelevancies.

¹As Neil Tennant pointed out to me, the property of being a perfect proof will be decidable in the case of propositional logic, but will be undecidable in general. This follows from the observation in [3] that Church's Theorem precludes there being a recursive enumeration of the perfectly valid sequents of first-order logic.

The second way to precisify the idea of a proof with no irrelevancies or detours is based on the notion of gaunt validity. Introducing this requires an auxiliary definition:

DEFINITION 2 (Coarsening).

A sequent $\Delta' : \Phi'$ is a *coarsening* of $\Delta : \Phi$ iff there is a uniform substitution σ (that is, a mapping from atoms to formulas) such that $\sigma(\Delta') = \Delta$ and $\sigma(\Phi') = \Phi$.

A coarsening $\Delta' : \Phi'$ of $\Delta : \Phi$ is trivial just in case the mapping σ that gives $\sigma\Delta' = \Delta$ and $\sigma\Phi' = \Phi$ is a one-one mapping of atoms to atoms. Thus a trivial coarsening is a mere relettering. Note that if σ maps all the atoms in Δ' to other atoms it will be considered non-trivial provided it is not one-one.

DEFINITION 3 (Gaunt validity; gaunt proofs).

A sequent $\Delta : \Phi$ is *gauntly valid* iff it is perfectly valid and has no non-trivial coarsening that is perfectly valid. A proof Π is a *gaunt proof* iff every subproof, including Π itself, is a proof of a gauntly valid sequent.

EXAMPLE 2

The following are all gauntly valid, both classically and intuitionistically:

- $A : A \vee B$.
- $(A \wedge B) \wedge C : A$.
- $\neg A, A \vee B : B$.

The following are *not* gauntly valid, either classically or intuitionistically:

- $A \wedge B : A \wedge B$, since it has the perfectly valid proper coarsening $A : A$.
- $A, A \rightarrow (C \wedge D) : C \wedge D$, since it has the perfectly proper coarsening $A, A \rightarrow B : B$.
- $A \wedge B : B \wedge A$, since it has perfectly valid proper coarsening $A \wedge C, D \wedge B : B \wedge A$.²

Again, there are also sequents that are intuitionistically gauntly valid but not classically so, for example:

- $\neg\neg\neg A : \neg A$.

The idea behind gaunt validity is to ask not only whether a premise or conclusion is needed for the validity of a sequent, but whether all of the internal logical structure of a premise or conclusion is needed for its validity.

The guiding question of this paper now admits of two precise versions: (1) Can any natural deduction proof be systematically transformed into a perfect proof that is appropriately related to the original proof? (2) Can any natural deduction proof be systematically transformed into a gaunt proof that is appropriately related to the original proof?

²This example also reveals that a gauntly valid sequent can be *longer* than its substitution instances.

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1.2 Outline of the Paper and Main Results

In Section 2, I will present the natural deduction systems for core logic and classical core logic, which will be the main tools I use to study perfect and gaunt validity. To a first approximation, classical core logic can be thought of as subtracting *ex falso quodlibet* from classical logic and adding disjunctive syllogism. This is a way of achieving the minimal revision of classical logic that does not make the so-called Lewis Paradox $A, \neg A : B$ provable. Core logic bears the analogous relation to intuitionistic logic as classical core logic does to classical logic.

In Section 3, I will elaborate the historical background and conceptual motivation for our key notions of perfect validity and gaunt validity. This will include their relation to the concept of *relevance*, their connection to core logic and classical core logic, and the relation to the main results proved in this paper.

Sections 4 and 5 contain the technical contributions of this paper. The main result below is that for every proof Π of the sequent $\Delta : \Phi$ in classical core logic or core logic there is respectively a classical core proof or core proof Π' of a perfectly valid sequent $\Delta' : \Phi'$ such that for some uniform substitution $\sigma, \sigma\Delta'$ and $\sigma\Phi'$ are trivially equivalent to Δ and Φ , respectively (in a sense made precise below).

Finally, in Section 6, I remark on the relation of these results to Tennant's soundness conjecture for core logic.

In the sequel to this paper I will show how, in the case of core logic, by adapting the methods of this paper we can find a proof Π' of a *gauntly* valid coarsening $\Delta' : \Phi'$ that is similarly related to $\Delta : \Phi$. This latter result is unique to core logic, as there are gauntly valid sequents in classical logic which do not have gaunt natural deduction proofs.

2 Classical Core Logic

Classical core logic has the following rules of natural deduction. The vertical dots indicate the possible occurrence of further proofwork, and where these are missing it is understood that the node in question must be a leaf-node in the proof tree. In particular, major premises of eliminations (MPEs) are not allowed to have any proofwork above them. Also, vacuous discharge is not allowed in any of the rules: the premises to be discharged must really have been used.

$$\wedge I \quad \frac{\begin{array}{c} \vdots \quad \vdots \\ \phi \quad \psi \end{array}}{\phi \wedge \psi}$$

$$\wedge E \quad \frac{\begin{array}{c} \frac{\quad}{\phi} \text{ (i)} \quad \frac{\quad}{\psi} \text{ (i)} \\ \dots\dots\dots \\ \phi \wedge \psi \end{array}}{\theta} \text{ (i)}$$

Here either ϕ , ψ , or both may occur as premises in the rightmost subproof.

$$\begin{array}{l} \vee I \\ \frac{\begin{array}{c} \vdots \\ \phi \end{array}}{\phi \vee \psi} \quad \frac{\begin{array}{c} \vdots \\ \psi \end{array}}{\phi \vee \psi} \\ \\ \vee E \\ \frac{\phi \vee \psi \quad \frac{\overline{\phi} \text{ (i)}}{\theta/\perp} \quad \frac{\overline{\psi} \text{ (i)}}{\theta/\perp}}{\theta/\perp} \text{ (i)} \end{array}$$

Here either θ or \perp may occur as the conclusion of each subproof. If one or both subproofs end in θ , then the conclusion of the rule is θ , and if both subproofs end in \perp , then the conclusion of the rule is \perp .

$$\begin{array}{l} \rightarrow I \\ \frac{\frac{\overline{\phi} \text{ (i)}}{\psi} \text{ (i)} \quad \frac{\overline{\phi} \text{ (i)}}{\perp} \text{ (i)}}{\phi \rightarrow \psi} \text{ (i)} \quad \frac{\begin{array}{c} \vdots \\ \psi \end{array}}{\phi \rightarrow \psi} \end{array}$$

$$\begin{array}{l} \rightarrow E \\ \frac{\phi \rightarrow \psi \quad \frac{\overline{\psi} \text{ (i)}}{\phi} \quad \frac{\overline{\theta} \text{ (i)}}{\theta}}{\theta} \text{ (i)} \end{array}$$

$$\begin{array}{l} \neg I \\ \frac{\overline{\phi} \text{ (i)}}{\frac{\perp}{\neg\phi} \text{ (i)}} \end{array}$$

$$\begin{array}{l} \neg E \\ \frac{\neg\phi \quad \begin{array}{c} \vdots \\ \phi \end{array}}{\perp} \end{array}$$

$$\begin{array}{l} CR \\ \frac{\overline{\neg\phi} \text{ (i)}}{\frac{\perp}{\phi} \text{ (i)}} \end{array}$$

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Core logic is the constructive system that results from omitting the rule *CR*.

The notion of a subproof will be essential for what follows:

DEFINITION 4 (Subproof).

If Π is a proof tree and n is a node in Π , the subtree of Π above n is a *subproof* of Π iff n is not a major premise of an elimination (MPE).

This definition is motivated by the view of inference rules as clauses in an inductive definition of proofhood. (This view is typically left implicit in the presentation of a proof system, but see [11] and [13, 52-55] for explicit statements). Then the subproofs of Π are the proofs that occur as parts of the inductive construction of Π .

A proof of a sequent $\Delta : \phi$ is a proof with conclusion ϕ whose undischarged premises are exactly the members of Δ . A proof of $\Delta : \perp$ is a proof with conclusion \perp whose undischarged premises are exactly the members of Δ . Note that, while a sequent is just a formal object which may be valid or invalid, $\Delta \vdash \phi$ is an *assertion* that there is a proof of ϕ whose premises are among Δ .

3 Background

3.1 Perfect Validity

While the notion of perfect validity is of obvious logical interest for any logician, it was first isolated in connection with relevance logic and the problem of analyzing the concept of entailment. The basic idea, as it was originally put by von Wright [25], Geach [6], Smiley [12], and perhaps most clearly enunciated by Geach in [7], was that ϕ entails ψ if it is possible to come to know ψ by a deduction from ϕ whose validity did not turn on ψ being logically true or on ϕ being logically false.³ The definitions of Geach, von Wright, and Smiley only applied to single-premise entailment, but a natural way to generalize this idea to multiple-premise deductions is to add the requirement that in coming to know ψ by deduction from ϕ_1, \dots, ϕ_n one must use *all* the ϕ 's. This leads us to the idea of a sequent $\phi_1, \dots, \phi_n : \psi$ being valid and having no valid proper subsequents, or in other words, being *perfectly valid*. Finally, Geach and von Wright were particularly interested in deduction; and since deduction is arguably a formal matter, it should be closed under substitution. Thus, they thought that entailments should also be closed under substitution. Later on, Burgess [5] would coin a term for sequents that are obtained from perfectly valid sequents by substitution.⁴

DEFINITION 5

A sequent $\Delta : \Gamma$ is *perfectible* iff there is a perfectly valid sequent $\Delta_0 : \Gamma_0$ and a substitution σ such that $\sigma(\Delta_0) = \Delta$ and $\sigma(\Gamma_0) = \Gamma$.

³The main (and perhaps better-known) alternative to the von Wright, Geach, Smiley approach to entailment is, of course, represented by [1] and [2].

⁴The notion of a perfectly valid sequent was first explicitly isolated in [16]; the notion of a substitution instance of a perfect sequent was also present in that work, though the term 'perfectible' was not used. Verdée, de Bal, and Samonek [24] also define a notion of relevant implication as substitution instances of perfectly valid sequents. Their approach to relevance is similar to that in [3], though my emphasis was on the philosophical analysis of relevance while Verdée, de Bal, and Samonek go further in the technical study of the notion. There is further interest in the fact that, while I offered my account as an alternative to the Anderson-Belpap tradition of relevance, Verdée, de Bal, and Samonek bridge these two approaches by defining a faithful translation from their system **NTR** of classical relevance to the logic **R** (see §4.3 of [24]).

A different, though related, route to perfect validity is offered by the explication of *relevance* in [3]. In that paper, I aimed to identify a robust notion of relevance. The goal was not to develop a system of ‘relevance logic’ as an alternative to classical logic, however, but rather to study the notion of relevance in the context of classical logic. The basic idea was that relevance is a relation between sentences and sequents: a sentence is relevant to a sequent if it contributes to the validity of that sequent. I then argued that this standard of relevance is best exemplified by the perfectly valid sequents, since every sentence in the sequent is essential for the validity of the sequent: remove any sentence from a perfectly valid sequent and you are left with an invalid sequent.⁵

Where von Wright, Geach, Smiley, and I arrived at the notion of perfect validity by conceptual analysis or explication, Tennant approaches it from a more top-down methodology. In the early programmatic work [14], he argued that a theory of entailment must account for ordinary deductive reasoning, and in [15] and [16] he developed a proof system that arguably does so. This is an early version of what would he would come to call classical core logic.⁶ Specifically, Tennant argues that all entailments should have normalizable natural deduction proofs, that the entailment relation should be transitive at least when the premises are consistent, and that entailment should be preserved under uniform substitution. On this basis, he defined entailment to be perfectibility: Δ entails ϕ iff $\Delta : \phi$ is perfectible. While Tennant has refined his system in various ways over the years, the main properties established in these early works continue to hold for the up to date system of classical core logic. The main properties of (classical) core logic are as follows; the results for classical core logic were established in [15] and the results for core logic were established in [17].

THEOREM 1

1. $A, \neg A : B$ is not provable in core logic or classical core logic.
2. If $\Delta : \Phi$ is provable in classical logic (intuitionistic logic, respectively), then there is some subsequent $\Delta' : \Phi'$ provable in classical core logic (core logic).

This theorem arguably substantiates Tennant’s claim that his system suffices to account for ordinary deductive reasoning as it occurs in science and mathematics. It also follows from claim 2 that classical core logic and core logic are complete with respect to the classes of perfectible sequents of classical or intuitionistic logic, respectively. In [16] he further established the converse soundness claim for classical core logic:

THEOREM 2

If $\Delta : \Phi$ is provable in classical core logic, then it is a substitution instance of a classically perfectly valid sequent.

In other words, every sequent provable in classical core logic is a substitution instance of a classically perfectly valid sequent. Theorems 1 and 2 can be informally glossed as together showing that classical core logic exactly captures the relevance-sensitive kernel of classical logic.

The analogue of Theorem 2 for core logic remains open, however, and is Tennant’s soundness conjecture for core logic:

⁵In [3], such sequents were mistakenly called ‘perfect’ rather than ‘perfectly valid’. I also defined another notion of relevance that is weaker than perfect validity, though that notion will not matter to us here.

⁶Before 2012, Tennant called this system CR for classical relevance logic, and the intuitionistic counterpart was called IR. See [22] for a thorough, up to date account of core logic and classical core logic.

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CONJECTURE 1

If $\Delta : \Phi$ is provable in core logic, then it is a substitution instance of an intuitionistically perfectly valid sequent.

This conjecture is significant because it would establish that core logic adequately formalizes the relevance-sensitive kernel of intuitionistic logic in the same way that classical core logic does for classical logic.

3.2 Gaunt Validity

The notion of gaunt validity has less history than perfect validity. It was first isolated in print by Tennant in [20, Ch. 11], though it is prefigured in Quine's *Maxim of Shallow Analysis*: "expose no more logical structure than seems useful for the deduction or other inquiry at hand."⁷ Part of Tennant's motivation derives from the circuitous nature of proofs such as:

$$\frac{\frac{\overline{A} \text{ (1)} \quad \overline{B} \text{ (1)}}{A \wedge B} \quad A \wedge B}{A \wedge B} \text{ (1)}$$

Such a proof is unnecessarily prolix, having gone through an entirely dispensable detour through the atoms A and B , which is somewhat awkward. In addition to the intuitive oddity of such proofs, Tennant is partly motivated by work in automated proof search. An algorithm that returned such circuitous proofs would be indulging in unnecessary work. A proof search algorithm that found gaunt proofs could potentially be more efficient.⁸

Alternatively, we can modify the ideas from [3] to motivate studying gaunt validity. As mentioned above, the core idea in that paper was that relevance meant contributing to the validity of a sequent. That paper was focused on the question of whether a premise is relevant to a sequent, but we can equally ask whether the *internal structure* of a premise is relevant in the sense of contributing to the validity of a sequent. If a complex formula could just as well be an atom without affecting the validity of a sequent, then the internal logical complexity of that formula does not really contribute anything to the validity of the sequent.⁹ For instance, say $\phi, \phi \rightarrow \psi : \psi$ is perfectly valid; then even if ϕ and ψ are highly complex sentences, their internal complexity is not essential to the validity of this sequent. We could coarsen ϕ and ψ to atoms, revealing the bare logically necessary structure of this sequent: $A, A \rightarrow B : B$. Just as perfectly valid sequents epitomize the ideal of a sequent in which every member contributes something to its validity, gauntly valid sequents epitomize the ideal of a sequent in which every iota of logical structure is needed for its validity.

⁷[10, 160]. Tennant has also explored gaunt validity at greater length in the unpublished work [21]. That work does not, however, overlap with the results proved in this paper. Tennant's terminology is slightly different from my own. He reserves the term 'gaunt' to apply to proofs and where I use 'gauntly valid' to describe sequents he uses the expression 'skeletonally valid'. I prefer my usage because it emphasizes the uniformity between the concept that applies to sequents and the concept that applies to proofs, and because 'skeletonally valid' is a bit of a mouthful. Both expressions, however, are meant to evoke the idea that there should be no extra flesh on the bare logically necessary bones of a sequent.

⁸This conjecture is partly born out by the observation in [4] that classically gauntly valid sequents all have polynomially sized proofs, which entails that classical gaunt validity is polynomially decidable. As noted in footnote 2, however, gauntly valid sequents can be much longer than their substitution instances. So an efficient way to find gaunt proofs does not necessarily provide an efficient way to find proofs in general.

⁹This connection between gaunt validity and relevance can be read as implicit in the discussion of [20], but is not explicitly drawn out there.

3.3 Relation to the present work

The results in Section 4 on classical core logic are closely related to the work of [16] (stated as Theorem 2 above). I will improve on that work in a few regards, however. A small difference is that Tennant omitted the connective \rightarrow from his language, and it was not until [18] that he settled on a particular set of rules for the arrow. I will consider the full language $\{\neg, \wedge, \vee, \rightarrow\}$ in what follows. More significantly, while Tennant established that every provable *sequent* is a substitution instance of perfectly valid sequent, I will consider properties not only of provable sequents, but of proofs as wholes, showing that every *proof* can be coarsened to a perfect proof. Tennant's approach also turned on his use of a multi-conclusion sequent calculus, while I will focus on natural deduction proofs, limiting us to a single conclusion. (Natural deduction and sequent calculi are each of interest in their own right, so this is a mere difference from [16] rather than an improvement per se.) Finally, and this is the most novel aspect of the current paper, I will build on the results about perfect validity and classical core logic to establish entirely new results in §5 about perfect proofs in core logic. The sequel paper will push these results about core logic even further, showing how core proofs can be transformed into *gaunt* core proofs.

The present work is thus a contribution to Tennant's program of studying (classical) core logic. It is also a contribution to the program initiated in [3] of studying perfect validity within the context of classical logic. The provable sequents of classical core logic are a proper subset of the provable sequents of classical logic; but as noted above, whenever $\Delta : \Phi$ is provable in classical logic there is some subsequent $\Delta' : \Phi'$ provable in classical core logic. Hence classical core logic includes every classically perfectly valid sequent, and we can use classical core logic as a tool for studying perfect validity within classical logic.

4 Coarsening Classical Core Proofs

In this section I address the guiding question of this paper as it applies to classical logic, and for this section a proof will refer to a proof in classical core logic unless otherwise stated. The main result will be that a classical core proof of $\Delta : \Phi$ can be transformed into a perfect proof of a sequent $\Delta' : \Phi'$ which is trivially equivalent to $\Delta : \Phi$ in the following sense.

DEFINITION 6 (c-variant).

A formula ϕ_c is a *contraction-variant* of ϕ (or c-variant, for short) iff ϕ can be obtained from ϕ_c by successively replacing occurrences of $\psi \vee \psi$ with ψ or replacing occurrences of $\psi \wedge \psi$ with ψ .

EXAMPLE 3

Each of the following is a c-variant of $A \vee B$:

- $A \vee (B \vee B)$
- $(A \vee B) \vee (A \vee (B \wedge B))$
- $((A \wedge A) \vee B) \wedge (A \vee B)$

Note that being a c-variant is a transitive and anti-symmetric relation. Moreover, if ϕ_c is a c-variant of ϕ then $\neg\phi_c$ is a c-variant of $\neg\phi$ and $\phi_c \circ \psi$ is a c-variant of $\phi \circ \psi$ and $\psi \circ \phi_c$ is a c-variant of $\psi \circ \phi$, where \circ is any of the binary connectives.

The definition of a c-variant is primarily technical rather than philosophical in nature, being motivated by its usefulness in the proof of Theorem 3. The philosophical significance of c-variants,

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however, is that a formula is *trivially equivalent* to its c-variants. Determining whether ϕ and ψ are c-variants is neither conceptually nor computationally demanding. And although applying the notion of synonymy to logical formulas is notoriously vexed, it would not be absurd to offer *being c-variants* as a surrogate for synonymy in propositional logic.¹⁰

I begin with a lemma which will be used in proving the main theorem.

LEMMA 1

If A is an atom and Π is a perfect proof, then the result of uniformly substituting $\neg A$ for A in Π is also a perfect proof.

PROOF. It will suffice to show that the result of uniformly substituting $\neg A$ for A in a perfectly valid sequent $\Delta : \Phi$ is also perfectly valid. Denote the result of this substitution $\Delta[A/\neg A] : \Phi[A/\neg A]$. Note that for any model M there is a model N such that every atom other than A is true in N iff it is true in M , and A is true in M iff it is false in N . Then M will be a countermodel to any subsequent of $\Delta : \Phi$ iff N is a countermodel to the corresponding subsequent of $\Delta[A/\neg A] : \Phi[A/\neg A]$. Thus $\Delta : \Phi$ has valid proper subsequents iff $\Delta[A/\neg A] : \Phi[A/\neg A]$ has valid proper subsequents. \square

THEOREM 3

If Π is a proof of $\Delta : \Phi$, then there is perfect proof Π' of $\Delta' : \Phi'$ and a substitution σ such that $\sigma\Delta'$ consists of c-variants of the members of Δ and $\sigma\Phi'$ is a c-variant of Φ . That is, if $\Phi = \emptyset$, then $\Phi' = \emptyset$ and if $\Phi = \{\phi\}$ then $\Phi' = \{\phi'\}$ and $\sigma\phi'$ is a c-variant of ϕ .

PROOF. Induction on the height of Π , defined as the length of the longest branch in Π . In the basis case we have the trivial proof of ϕ from the premise ϕ . Take as Π' the trivial proof of A from A , with $\sigma : A \mapsto \phi$.

For the induction step, I proceed according to the last rule applied in Π . I will begin with the cases where the last rule is an I-rule.

First, assume the last rule applied in Π is $\wedge I$, so Π is of the form:

$$\frac{\begin{array}{cc} \Delta_1 & \Delta_2 \\ \Pi_1 & \Pi_2 \\ \psi & \theta \end{array}}{\psi \wedge \theta}$$

By i.h. we have perfect proofs:

$$\begin{array}{cc} \Delta'_1 & \Delta'_2 \\ \Pi'_1 & \text{and} \quad \Pi'_2 \\ \psi' & \theta' \end{array}$$

And we have the two substitutions σ_1 and σ_2 such that $\sigma_i\Delta'_i = \Delta_i$ and $\sigma_1\psi'$ is a c-variant of ψ and $\sigma_2\theta'$ is a c-variant of θ . Without loss of generality we can assume that Π'_1 and Π'_2 are atom-disjoint, since we can always reletter if need be. Then putting $\sigma := \sigma_1 \cup \sigma_2$, and taking Π' to be the following

¹⁰A simple argument for this claim uses the following two plausible assumptions: (1) ϕ is synonymous with $\phi \wedge \phi$ and $\phi \vee \phi$, and (2) if ϕ is synonymous with ϕ' , then the result $\psi[\phi/\phi']$ of replacing an occurrence of ϕ with ϕ' in a formula ψ is synonymous with ψ . Given these two premises, any definition of synonymy in propositional logic would be at least as inclusive as *being c-variants*.

proof will provide our result.

$$\frac{\begin{array}{cc} \Delta'_1 & \Delta'_2 \\ \Pi'_1 & \Pi'_2 \\ \psi' & \theta' \end{array}}{\psi' \wedge \theta'}$$

Second, assume the last rule applied in Π is \vee I, so Π is:

$$\frac{\begin{array}{c} \Delta \\ \Pi_1 \\ \psi \end{array}}{\psi \vee \theta}$$

Then let Π'_1 and σ_1 be as guaranteed by i.h., and let A be a new atom not occurring in Π_1 .¹¹ Letting $\sigma := \sigma_1 \cup \{A \mapsto \theta\}$, we have Π' as follows.

$$\frac{\begin{array}{c} \Delta' \\ \Pi'_1 \\ \psi' \end{array}}{\psi' \vee A}$$

Third, assume the last rule applied in Π is \rightarrow I. This can take three forms, giving us three subcases. In the first subcase, we have Π of the form:

$$\frac{\frac{\overline{\psi}^{(1)} \quad \Delta}{\dots \Pi_1 \dots} \quad \theta}{\psi \rightarrow \theta}^{(1)}$$

By i.h. we have a substitution σ and a perfect proof:

$$\frac{\psi' \quad \Delta'}{\dots \Pi'_1 \dots} \quad \theta'$$

Using the same σ , we form the perfect proof Π' by an application of \rightarrow I:

$$\frac{\frac{\overline{\psi'}^{(1)} \quad \Delta'}{\dots \Pi'_1 \dots} \quad \theta'}{\psi' \rightarrow \theta'}^{(1)}$$

Note, however, that there may be multiple occurrences of ψ in Π_1 that are discharged in Π by the rule \rightarrow I. And in the process of coarsening Π_1 , those occurrences might be coarsened to distinct

¹¹The trick of using a new atom (here and elsewhere in this work) comes from [21].

instances of $\wedge E$ to the bottom of Π'_1 as necessary to obtain:

$$\frac{\frac{\frac{\overline{\psi'_1}^{(1)} \quad \dots \quad \overline{\psi'_n}^{(n-1)} \quad \Delta'_1}{\Pi'_1} \quad \frac{\chi'}{\wedge E \text{ rules}}^{(n-1)}}{\psi'_1 \wedge \dots \wedge \psi'_n}^{(1)} \quad \frac{\chi'}{\wedge E \text{ rules}}^{(2)}}{(\psi'_1 \wedge \dots \wedge \psi'_n) \wedge A} \quad \chi'}{\chi'}^{(1)}$$

Second, assume the last rule applied is $\vee E$, so Π has the form:

$$\frac{\frac{\overline{\psi}^{(1)} \quad \Delta_1}{\Pi_1} \quad \frac{\overline{\theta}^{(1)} \quad \Delta_2}{\Pi_1}}{\psi \vee \theta} \quad \frac{\chi/\perp}{\chi/\perp} \quad \frac{\chi/\perp}{\chi/\perp}^{(1)}$$

By i.h. we have the two perfect proofs, with their corresponding substitutions σ_1 and σ_2 :

$$\frac{\overline{\psi'} \quad \Delta'_1}{\Pi'_1} \quad \text{and} \quad \frac{\overline{\theta'} \quad \Delta'_2}{\Pi'_2} \quad \frac{\chi'_1/\perp}{\chi'_1/\perp} \quad \frac{\chi'_2/\perp}{\chi'_2/\perp}$$

And we can assume that Π'_1 and Π'_2 are atom-disjoint. As expected we set $\sigma := \sigma_1 \cup \sigma_2$. In either of these proofs Π'_1 and Π'_2 , if ψ or θ have been coarsened to distinct formulas ψ'_i or θ'_i , then again use the trick of adding $\wedge E$ to the bottom of Π'_1 or Π'_2 as necessary. If one or both of these proofs has \perp as the conclusion, then we can obtain our desired proof Π' by the obvious application of $\vee E$ with Π'_1 and Π'_2 as subproofs. On the other hand, if neither Π'_1 nor Π'_2 have \perp as a conclusion, then we take the following proof for Π' .

$$\frac{\frac{\overline{\psi'}^{(1)} \quad \Delta'_1}{\Pi'_1} \quad \frac{\overline{\theta'}^{(1)} \quad \Delta'_2}{\Pi'_2}}{\psi' \vee \theta'} \quad \frac{\chi'_1}{\chi'_1 \vee \chi'_2} \quad \frac{\chi'_2}{\chi'_1 \vee \chi'_2}}{\chi'_1 \vee \chi'_2}^{(1)}$$

Now $\sigma(\chi'_1 \vee \chi'_2) = \sigma_1(\chi'_1) \vee \sigma_2(\chi'_2) = \chi \vee \chi$, which is a c-variant of χ , as required.

Third, assume the last rule applied is $\rightarrow E$, so Π has the following form:

$$\frac{\Delta_1 \quad \frac{\overline{\theta}^{(1)} \quad \Delta_2}{\Pi_2}}{\psi \rightarrow \theta} \quad \frac{\psi}{\psi} \quad \frac{\chi}{\chi}^{(1)}$$

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By i.h. we have the two perfect proofs, and again we can ensure that they are atom-disjoint:

$$\begin{array}{c} \Delta'_1 \\ \Pi'_1 \\ \psi' \end{array} \quad \text{and} \quad \begin{array}{c} \theta' \dots \Delta'_2 \\ \dots \dots \dots \\ \Pi'_2 \\ \chi' \end{array}$$

If θ has been coarsened to distinct formulas θ'_i , add steps of $\wedge E$ to the end of Π'_2 . Now we apply the obvious instance of $\rightarrow -E$ to obtain a proof of $\Delta', \psi' \rightarrow \theta' : \chi'$. Note that $\sigma\psi'$ may be a c-variant of ψ , and that $\sigma\theta'$ may also be a c-variant of θ .

Fourth, assume the last rule applied is $\neg E$, so Π has the form:

$$\frac{\begin{array}{c} \Delta \\ \Pi_1 \\ \neg\psi \quad \psi \end{array}}{\perp}$$

By i.h. we have the perfect proof:

$$\begin{array}{c} \Delta' \\ \Pi'_1 \\ \psi' \end{array}$$

Note again that $\sigma\psi'$ might be a c-variant of ψ . Since $\Delta' : \psi'$ is perfectly valid, it follows that $\Delta', \neg\psi' : \emptyset$ is also perfectly valid. This is because if there were a valid proper subsequent of the form $\Delta'_1, \neg\psi' : \emptyset$ or $\Delta'_1 : \emptyset$ (with $\neg\psi' \notin \Delta'_1$), then we would have the respective valid sequents $\Delta'_1 : \psi$ or $\Delta'_1 : \emptyset$, contradicting the perfect validity of $\Delta' : \psi$. This allows us to form the desired perfect proof Π :

$$\frac{\begin{array}{c} \Delta' \\ \Pi'_1 \\ \neg\psi' \quad \psi' \end{array}}{\perp}$$

This completes the cases with E-rules. The final case for our induction is when the last rule applied is CR , so that Π is of the form:

$$\frac{\frac{\overline{\neg\psi} \quad (1) \quad \Delta}{\dots \dots \dots} \quad \Pi_1}{\frac{\perp}{\psi} \quad (i)}$$

By the i.h. we have a substitution σ and perfect proof of the following form:

$$\frac{\begin{array}{c} \theta \dots \Delta' \\ \dots \dots \dots \\ \Pi'_1 \\ \perp \end{array}}$$

In this case the i.h. guarantees that $\sigma\theta = \neg\psi$, but it is not immediate that θ is of the form $\neg(\psi')$. Since we know that $\neg\psi$ is a substitution instance of θ , however, we know that either θ is of the form $\neg(\psi')$ or θ is an atom A . And in the latter case Lemma 1 guarantees that we can substitute $\neg A$ for A in Π'_1 and then apply CR to the result to obtain our desired perfect proof Π' of $\Delta' : \psi'$.

It is also possible, however, that there are multiple instances of the premise $\neg\psi$ which have been coarsened to distinct formulas ψ'_1, \dots, ψ'_n . In that case, we append a series of $\vee I$ rules and n instances of CR as follows.

$$\begin{array}{c}
 \frac{\overline{\neg\psi'_1} \quad (1) \quad \overline{\neg\psi'_2} \quad (2) \quad \dots \quad \overline{\neg\psi'_n} \quad (n)}{\dots\dots\dots} \\
 \Pi'_1 \\
 \frac{\perp}{\psi'_1} \quad (1) \\
 \hline
 \psi'_1 \vee \dots \vee \psi'_n \\
 \hline
 \frac{\neg(\psi'_1 \vee \dots \vee \psi'_n)}{\perp} \\
 \frac{\perp}{\psi'_2} \quad (2) \\
 \vdots \\
 \frac{\perp}{\psi'_n} \quad (n) \\
 \hline
 \psi'_1 \vee \dots \vee \psi'_n \\
 \hline
 \frac{\neg(\psi'_1 \vee \dots \vee \psi'_n)}{\perp}
 \end{array}$$

Now we can apply an instance of CR discharging $\neg(\psi'_1 \vee \dots \vee \psi'_n)$, giving us a proof of $\psi'_1 \vee \dots \vee \psi'_n$, which is a c-variant of our original formula ψ . \square

REMARK 1

At first glance it may seem like we could simply omit the case of CR and the proof would carry over to core logic. This is too quick, however. In the case of $\neg E$, we inferred from the perfect validity of $\Delta' : \psi'$ to the perfect validity of $\Delta', \neg\psi' : \emptyset$. Classically, this is reasonable, since if there were a proper subset $\Delta'_0 \subset \Delta'$ such that $\Delta'_0, \neg\psi' : \perp$ were valid, then by classical reductio, we would have $\Delta'_0 : \psi'$, contradicting the perfect validity of $\Delta' : \psi'$. But this is a strictly classical move and it can fail intuitionistically. For instance, $A : A \vee \neg A$ is perfectly valid intuitionistically, but $A, \neg(A \vee \neg A) : \emptyset$ is not, since it has the intuitionistically valid subsequent $\neg(A \vee \neg A) : \emptyset$. As we will see in the next section, Theorem 3 does carry over to core logic, but it will take more work.

EXAMPLE 4

Let us look at three simple examples of classical core proofs and the perfect proofs that result from applying the coarsening procedure of Theorem 3.

1. Our first example will be the following proof of $\theta \vee \psi, \psi \rightarrow \theta, \neg\theta : \theta$

$$\frac{\theta \vee \psi \quad \frac{\frac{\psi \rightarrow \theta \quad \frac{\frac{\neg\theta \quad \overline{\theta}}{\perp} \quad (1)}{\psi} \quad (2)}{\perp} \quad (2)}{\theta} \quad (2)}{\theta}$$

The process of coarsening this proof will leave the structure of the tree as is, merely replacing the formulas with atoms. Note that the occurrence of θ in the main disjunction gets coarsened to a

different atom than the occurrence in $\psi \rightarrow \theta$:

$$\frac{A \vee B \quad \frac{\frac{\frac{}{A} (2) \quad B \rightarrow C \quad \frac{\frac{}{B} (2) \quad \frac{\frac{}{\neg C} \quad \frac{}{C} (1)}{\perp} (1)}{\perp} (2)}{A} (2)}}{A \vee B} (2)}}{A \vee B} (2)$$

2. Our second example will be somewhat more complicated. Consider the following proof of the sequent $\emptyset : (\phi \vee \psi) \rightarrow (\psi \vee \phi)$.

$$\frac{\frac{\frac{}{\phi \vee \psi} (2) \quad \frac{\frac{}{\phi} (1)}{\psi \vee \phi} (1)}{\psi \vee \phi} (1) \quad \frac{\frac{}{\psi} (1)}{\psi \vee \phi} (1)}{\psi \vee \phi} (1)}{(\phi \vee \psi) \rightarrow (\psi \vee \phi)} (2)$$

The coarsening procedure applied to this proof will result in the following proof of the sequent $\emptyset : (A \vee B) \rightarrow [(C \vee A) \vee (B \vee D)]$.

$$\frac{\frac{\frac{}{A \vee B} (2) \quad \frac{\frac{\frac{}{A} (1)}{C \vee A}}{(C \vee A) \vee (B \vee D)} (1)}{(C \vee A) \vee (B \vee D)} (1) \quad \frac{\frac{\frac{}{B} (1)}{B \vee D}}{(C \vee A) \vee (B \vee D)} (1)}{(C \vee A) \vee (B \vee D)} (1)}{(A \vee B) \rightarrow [(C \vee A) \vee (B \vee D)]} (2)$$

In outline, here is how we reach this coarsened proof. We begin at the tops of each of the two main subproofs. We coarsen ϕ to A and ψ to B . Then when we move down to the instance of $\vee I$ that immediately follows A , we replace the newly introduced disjunct ψ with a new atom C ; and similarly at the instance of $\vee I$ following B we replace the newly introduced disjunct ϕ with a new atom D . Then at the subsequent occurrence of $\vee E$, the two main subproofs have different conclusions. So, as prescribed by the coarsening procedure for $\vee E$ in the proof of Theorem 3, we insert an instance of $\vee I$ so that the two subproofs have the same conclusion, namely $(C \vee A) \vee (B \vee D)$. Finally, we apply the terminal application of $\rightarrow I$.

Now the substitution that maps $A \mapsto \phi$, $D \mapsto \phi$, $B \mapsto \psi$, and $C \mapsto \psi$ will give the substitution instance $(\phi \vee \psi) \rightarrow [(\psi \vee \phi) \vee (\psi \vee \phi)]$, which is a c-variant of $(\phi \vee \psi) \rightarrow (\psi \vee \phi)$.

(3) Our third example will involve *CR*. Suppose we are given the proof:

$$\frac{\frac{\frac{}{\neg(A \vee \neg A)} (2) \quad \frac{\frac{}{A} (1)}{A \vee \neg A}}{\perp} (1)}{\neg(A \vee \neg A)} (2) \quad \frac{\frac{}{\neg A} (1)}{A \vee \neg A}}{\perp} (1)}{A \vee \neg A} (2)$$

In the coarsening process, the two MPEs $\neg(A \vee \neg A)$ will get coarsened to distinct formulas, so we have to use the trick of inserting $\vee I$ rules and we will end up with a proper coarsening of the original

not in general preserve perfect validity, so the deductive system D will not prove *only* perfectly valid sequents. However, in this deductive system, every perfectly valid sequent will have *some* perfect proof.

5 Perfect Core Proofs

I now want to address the guiding question of this paper to intuitionistic logic. Applying the ideas from above to core logic, I will show how every core proof be transformed into a perfect proof. For this section and the next, unless otherwise stated, a proof will mean a core proof, and validity will mean intuitionistic validity (which is in turn understood in terms of Kripke models).

The aim is to adapt Theorem 3 to show how core proofs can be coarsened into perfect proofs. I begin with a lemma that addresses the difficulties noted in Remark 1.

LEMMA 2

Suppose that Π is a perfect core proof of $\Delta : \phi$ such that

1. Every leaf-node that is not an MPE is an atom;
2. In every application of $\vee I$, the newly introduced disjunct is a formula consisting of atoms not occurring in the subproof above it, and further the new disjunct is not a tautology;
3. In every application of $\rightarrow I$ in which the antecedent is refuted, the consequent of the newly introduced arrow is an atom not occurring in the subproof above it; and in an application of $\rightarrow I$ in which the consequent is derived without the antecedent, the antecedent of the newly introduced arrow is an atom not occurring above it;
4. In any rule that has two main subproofs, those two subproofs are atom-disjoint, except in an application of $\vee E$ where neither subproof has conclusion \perp . In this case, the two subproofs are atom-disjoint aside from their conclusions. Furthermore, in this case, both conclusions are the disjunction $\phi_l \vee \phi_r$, introduced by $\vee I$, where ϕ_l is atom-disjoint from the right subproof and ϕ_r is atom-disjoint from the left subproof; and
5. In any application of $\wedge E$, if one of the conjuncts of the major premise is not used as a premise in the main subproof, then that conjunct is an atom not occurring in the subproof.

Then $\Delta, \neg\phi : \emptyset$ is perfectly valid.

PROOF. Induction on height of Π . In the basis case, Π is the trivial proof of $\phi : \phi$. Moreover, by assumption, ϕ must be an atom A , and clearly $A, \neg A : \emptyset$ is perfectly valid.

For the induction step, first consider cases where the last rule applied in Π is an E-rule. It cannot be an application of $\neg E$, since we have a proof of ϕ , not \perp .

Next consider the case of $\vee E$. Suppose this application of $\vee E$ is the uniform conclusion version; the other versions can be treated similarly. Then Π is of the form:

$$\frac{\frac{\frac{\overline{\psi}^{(1)}}{\dots} \quad \Delta_1}{\Pi_1} \quad \frac{\overline{\theta}^{(1)}}{\dots} \quad \Delta_2}{\Pi_2}}{\frac{\psi \vee \theta}{\frac{\phi_l}{\phi_l \vee \phi_r} \quad \frac{\phi_r}{\phi_l \vee \phi_r}^{(1)}}}{\phi_l \vee \phi_r}}$$

By i.h. $\Delta_1, \psi, \neg(\phi_l \vee \phi_r) : \emptyset$ and $\Delta_2, \theta, \neg(\phi_l \vee \phi_r) : \emptyset$ are both perfectly valid.

It will follow that $\Delta_1, \Delta_2, \psi \vee \theta, \neg(\phi_l \vee \phi_r) : \emptyset$ is perfectly valid, too. The basic idea is that, since Δ_1, ψ, ϕ_l and Δ_2, θ, ϕ_r are atom-disjoint, for any proper subsequent of $\Delta_1, \Delta_2, \psi \vee \theta, \neg(\phi_l \vee \phi_r)$

ϕ_r) : \emptyset we can create a countermodel by piecing together countermodels for the corresponding proper subsequents of $\Delta_1, \psi, \neg\phi_l$: \emptyset and $\Delta_2, \theta, \neg\phi_r$: \emptyset . In more detail, suppose we have a proper subsequent

$$\Delta'_1, \Delta'_2, [\psi \vee \theta], [\neg(\phi_l \vee \phi_r)] : \emptyset,$$

where the brackets indicate that $\psi \vee \theta$ and $\neg(\phi_l \vee \phi_r)$ may or may not occur in this sequent. Now we want to show that this sequent has a countermodel, that is, a Kripke model M where $\Delta'_1, \Delta'_2, \psi \vee \theta, \neg(\phi_l \vee \phi_r)$ are all true at the root node. Suppose $\Delta'_1 \subset \Delta_1$. Then we know there is a model M_1 where $\Delta'_1, \psi, \neg(\phi_l \vee \phi_r)$ are true at the root node, because $\Delta_1, \psi, \neg(\phi_l \vee \phi_r)$: \emptyset is perfectly valid. Similarly, we can find a model M_2 where $\Delta'_2, \neg(\phi_l \vee \phi_r)$ are true at the root node. By the assumption that Δ_1, ψ, ϕ_l and Δ_2, θ, ϕ_r are atom-disjoint, we can merge these two models into a single model M , and the result is our desired countermodel in which $\Delta'_1, \Delta'_2, \psi \vee \theta, \neg(\phi_l \vee \phi_r)$ are all true at the root node. The argument is similar for the possibilities where $\Delta'_2 \subset \Delta_2$ or where $\psi \vee \theta$ or $\neg(\phi_l \vee \phi_r)$ do not occur in the subsequent in question.

The other cases of E-rules follow similarly by appeal to the induction hypothesis and the condition on main subproofs being atom-disjoint.

Now assume the last rule applied in Π is an I-rule. First, suppose it is \wedge I, so Π has the form:

$$\frac{\begin{array}{cc} \Delta_1 & \Delta_2 \\ \Pi_1 & \Pi_2 \\ \psi & \theta \end{array}}{\psi \wedge \theta}$$

By i.h. $\Delta_1, \neg\psi$: \emptyset and $\Delta_2, \neg\theta$: \emptyset are perfectly valid. By assumption, Δ_1, ψ and Δ_2, θ are atom-disjoint. Similar to the case of \vee E above, these two conditions entail that $\Delta_1, \Delta_2, \neg\psi \vee \neg\theta$: \emptyset is perfectly valid. Now suppose for reductio that some subsequent $\Delta'_1, \Delta'_2, \neg(\psi \wedge \theta)$: \emptyset were perfectly valid. Since $\neg\psi \vee \neg\theta \vdash \neg(\psi \wedge \theta)$, it would follow that $\Delta'_1, \Delta'_2, \neg\psi \vee \neg\theta$: \emptyset is valid, contradicting the perfect validity of $\Delta_1, \Delta_2, \neg\psi \vee \neg\theta$: \emptyset .

Next, assume the last rule applied in Π is \vee I. Then by i.h. we have a perfectly valid sequent Δ : ψ , and a formula θ with entirely new atoms, so Δ : $\psi \vee \theta$ is perfectly valid. Now if some subsequent of $\Delta, \neg(\psi \vee \theta)$: \emptyset were valid, then we could find some subsequent of $\Delta, \neg\psi, \neg\theta$: \emptyset that were valid. Since θ shares no atoms with $\Delta, \neg\psi$ and is not a tautology, it could be omitted from this valid sequent without loss. So if there were a perfectly valid proper subsequent of $\Delta, \neg\psi, \neg\theta$: \emptyset , then $\neg\theta$ would not be in this sequent. Then we would in fact have a valid proper subsequent of $\Delta, \neg\psi$: \emptyset , which is impossible.

Next, assume the last rule applied in Π is \rightarrow I. Begin by assuming that the first form of \rightarrow I is used, so ψ is used in a derivation of θ and discharged. Since Π is perfect, Δ : $\psi \rightarrow \theta$ is perfectly valid, and by i.h. $\Delta, \psi, \neg\theta$: \emptyset is perfectly valid. Now $\psi \wedge \neg\theta \vdash \neg(\psi \rightarrow \theta)$, so if a proper subsequent $\Delta', \neg(\psi \rightarrow \theta)$: \emptyset were valid then $\Delta', \psi, \neg\theta$: \emptyset would be valid too, but that is impossible.

Consider the second form of \rightarrow I, where $\psi \rightarrow \theta$ is inferred from a refutation of ψ . Since Π is perfect, Δ, ψ : \emptyset is perfectly valid. Thus any proper subset of Δ, ψ is satisfiable. By assumption, θ consists of atoms not occurring elsewhere in Π , and so any subset of $\Delta, \psi, \neg\theta$ is satisfiable whenever the corresponding subset of Δ, ψ is. Thus every proper subset of $\Delta, \psi \wedge \neg\theta$ is satisfiable, and hence $\Delta, \neg(\psi \rightarrow \theta)$: \emptyset is perfectly valid. The third form of \rightarrow I is treated similarly.

The final case concerns when the last rule applied in Π is \neg I. So Π is a perfect proof of Δ : $\neg\phi$. It follows that $\Delta, \neg\neg\phi$: \emptyset is also perfectly valid. \square

THEOREM 5

If Π is a core proof of $\Delta : \Phi$, then there is perfect core proof Π' of $\Delta' : \Phi'$ and a substitution σ such that $\sigma \Delta'$ consists of c-variants of members of Δ and $\sigma \Phi'$ is a c-variant of Φ .

PROOF. In virtue of Theorem 3 and Remark 1, it suffices to show that the coarsening procedure applied to applications of $\neg E$ preserve perfection. And this is established in Lemma 2: proofs obtained by the coarsening procedures for $\forall I$, $\rightarrow I$, $\vee E$, and $\wedge E$ from Theorem 3 all meet the conditions on Π that Lemma 2 relies on, so that the coarsening procedure for $\neg E$ then preserves perfection. \square

This theorem does not show that every core-provable sequent is perfectible, so it does not resolve Tennant's Soundness Conjecture 1. It does, however, show that every core-provable sequent is trivially equivalent to a perfectible sequent.

6 The Soundness of Core Logic

Given Tennant's definition of entailment as perfectibility, showing that every sequent provable in core logic is perfectible amounts to showing that core logic is sound for this definition of entailment. This was the Soundness Conjecture 1. Milne [8] offered a counterexample to this conjecture, noting that $A, B, A \vee B : A \wedge B$ is provable in core logic, but is not perfectible. Milne acknowledges, however, that this counterexample depends essentially on treating sequents as having at most one conclusion. If we allow multiple-conclusion sequents, then $A, B, A \vee B : A \wedge B$ is a substitution instance of the intuitionistically perfectly valid sequent $C, D, A \vee B : A \wedge D, C \wedge B$.

It is true that in Gentzen's original formulation of the sequent calculi, the intuitionistic system LJ was obtained from the classical system LK by restricting the conclusion set to having at most one member. But this is not an essential restriction, and there exist multiple-conclusion sequent calculi for intuitionistic logic.¹³ Similar to the classical case, multiple conclusions are given a disjunctive reading: $\Delta : \phi_1, \dots, \phi_n$ is valid just in case $\Delta : \bigvee_i \phi_i$ is valid. Recognizing that Gentzen's restriction to single-conclusion sequents was a technical trick rather than an essential feature of intuitionistic logic, the soundness question for core logic could be posed in terms of multiple-conclusion sequents. Milne even suggested that Tennant's soundness conjecture might hold when posed in these terms.¹⁴

In the natural deduction setting that I have been working in, we do not have multiple conclusions. But we can still capture the same insight by rephrasing the soundness conjecture to state: if there is a core proof of $\Delta : \phi$, then there is an intuitionistically perfectly valid sequent $\Delta_0 : \phi_1, \dots, \phi_n$ and a substitution σ such that $\sigma \Delta_0 = \Delta$ and $\sigma \phi_1 = \dots = \sigma \phi_n = \phi$. While this claim does not follow from what I have proved here, there is a sense in which Theorem 5 is intimately related to this soundness claim.

If a multiple-conclusion sequent $\Delta : \Phi$ is perfectly valid, and has a single-conclusion substitution instance $\Delta' : \phi$ under the substitution σ , this is because σ maps every member of Φ to the same formula ϕ . This effect turns on treating sequents as pairs of *sets* of formulas rather than pairs of *multisets* of formulas. If sequents are pairs of multisets, then to get from $\sigma \Delta : \sigma \Phi$ to $\Delta' : \phi$ would require finitely many applications of the contraction rule. Analogously, if $\Delta_0 : \phi_1, \dots, \phi_n$ is perfectly valid, and we have a natural deduction proof of $\Delta_0 : \phi_1 \vee \dots \vee \phi_n$, then applying the substitution σ will give us a natural deduction proof of $\Delta : \phi \vee \dots \vee \phi$. 'Contracting' this terminal self-disjunction

¹³See, for instance, [9, §5.3].

¹⁴[19] suggested a different reply to Milne, which I will not consider here.

would be the equivalent in the natural deduction setting of applying the structural rule of contraction in the multiset sequent setting.

Theorem 5 gives us a similar way to obtain a core-provable sequent from a perfectible sequent by contracting self-disjunctions and self-conjunctions. That theorem requires a slightly more liberal understanding of what counts as the natural deduction analogue of the structural rule of contraction in the sequent calculus. The structural rule of contraction allows us to eliminate duplicate occurrences of premises or conclusions from a sequent. The straightforward analogy would be to regard $\Delta, \psi : \phi$ and $\Delta, \psi \wedge \psi : \phi$ or $\Delta, \psi : \phi \vee \phi$ as equivalent from the standpoint of natural deduction.¹⁵ In other words, I am suggesting that if every core provable sequent could be obtained from a perfectible sequent by contracting outermost self-conjunctions in the premise set and contracting outermost self-disjunctions in the conclusion, that could naturally be seen as providing a positive solution to the soundness problem for core logic.

What Theorem 5 shows, on the other hand, is that every core provable sequent can be obtained from a perfectible sequent by contracting not-necessarily-outermost self-conjunctions and self-disjunctions in both the premises and the conclusion. The relation to the soundness question is clear and suggestive. But we do not have a direct answer to the soundness question per se.

7 Conclusion

This paper began with the guiding question of whether, given a natural deduction proof, there is a way to systematically transform it into a proof that contains no irrelevancies and which in which the argument proved to be valid is as strong as possible. I offered two different definitions of when a proof contains no irrelevancies: the first is when we have a perfect proof and the second is when we have a gaunt proof. I dealt with perfect proofs in this paper and the sequel paper deals with gaunt proofs.

In order to transform an arbitrary core or classical core proof into a perfect proof, I developed the idea of working top-down through a proof to gradually *coarsen* it into a perfect proof. This method of coarsening is based on four essential ideas:

1. Leaf nodes that are not MPEs can be coarsened to atoms, and this coarsening is then propagated down the proof tree.
2. When an inference such as $\vee I$ introduces a new formula that does appear previously in the proof, that (sub)formula can be coarsened to an atom.
3. When a rule such as $\rightarrow E$ or $\wedge I$ has parallel subproofs, those subproofs can be relettered to be atom-disjoint.
4. When we reach an E-rule R that discharges multiple instances of a formula ϕ , if those instances have been coarsened to distinct formulas ϕ'_1, \dots, ϕ'_n , then we interpolate n instances of $\wedge E$, so that in the coarsened proof we have rule R discharging $\phi'_1 \wedge \dots \wedge \phi'_n$; or, if R is an instance of CR , we interpolate n $\vee I$'s to discharge $\neg(\phi'_1 \vee \dots \vee \phi'_n)$.

¹⁵As a referee points out, contraction on the left in a sequent calculus is usually associated with the discharge of multiple occurrences of a premise in natural deduction. Interestingly, something similar is true of contracting self-disjunctions and self-conjunctions in the way I propose in the main text. In the coarsening procedures of Theorem 3, premises only get replaced by their c-variants when they are MPEs that discharge multiple occurrences of premises. So when we contract self-disjunctions and self-conjunctions to get from $\sigma \Delta'$ to Δ , we are doing so with MPEs that discharge multiple premises. On the other hand, there is not a good analogy in natural deduction of contraction on the right in a sequent calculus. (Though [11] shows how contractions on the right get elegantly implemented using directed proof nets.)

By implementing these four ideas, we saw that a core or classical core proof of $\Delta : \Phi$ can always be coarsened to a perfect proof of $\Delta' : \Phi'$, where $\Delta' : \Phi'$ is coarsening of a c-variant of $\Delta : \Phi$. In the case of core proofs, as I explained, this result is closely related to Tennant's Soundness Conjecture 1 for core logic; while it does not solve the conjecture, perhaps it is not unreasonable to describe it as a partial solution.

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