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SPECIAL ISSUE ARTICLE

Replies

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I am immensely grateful to the commentators for all of their thoughts and probing questions about *TO*. My replies will focus on one or two central points from each article. Inevitably, I need to be selective.

1 | ABSTRACTION AND GROUNDING: CARRARA, DE FLORIO AND POGGIOLESI

As Carrara, De Florio and Poggiolesi (henceforth *CDP*) correctly observe, my approach to thin objects is connected with grounding. In particular, 'the (equivalence relations holding between) thick objects ground (the identity and distinctness of) thin objects.' The idea is that in cases of permissible abstraction, the equivalence of two specifications grounds the identity of their corresponding abstracts. Using the customary symbol '$<$' for strict and full grounding, a natural first shot at formalising this idea is:[1]

$$\alpha \sim \beta \rightarrow (\alpha \sim \beta < \S\alpha = \S\beta). \qquad (\S <)$$

CDP proceed to present some important observations and challenges.

First, for ($\S<$) to hold, the facts that figure on the two sides of '$<$' need to be distinct. I agree. Indeed, this is what distinguishes my asymmetric approach to abstraction from the more traditional symmetric approach espoused by Frege, the neo-Fregeans and Rayo, where the two sides of an abstraction principle are regarded as merely different ways to 'carve up' one and the same fact.[2]

Second, ($\S<$) raises tricky questions about the individuation of facts. Suppose two distinct specifications are equivalent: $\alpha \sim \beta$. Are $\S\alpha = \S\beta$ and $\S\alpha = \S\alpha$ one and the same fact? The answer

[1] This is a generalisation of the so-called 'Schwartzkopff–Rosen principle', first proposed in Schwartzkopff (2011) and Rosen (2010).
[2] See, e.g., Frege (1953, §64), Hale & Wright (2001, esp. essay 4), and Rayo (2013).

will depend on how abstraction terms contribute to the representation of facts. One option is that the abstraction terms make a 'purely referential' contribution (in the sense of Quine (1960, 177)): an abstraction term first picks out an object, which is then slotted into the fact. On this analysis, the fact that $\S\alpha = \S\beta$ is identical with the fact that $\S\alpha = \S\alpha$. Another option is that an abstraction term contributes both a specification and the object thus specified. Then, '$\S\alpha = \S\beta$' represents a fact a concerning the two specifications α and β, namely, that they determine one and the same abstract. Since the language for which I try to secure an interpretation is an artificial one, I get to choose. I choose the former option, on the grounds of its greater simplicity (cf. *TO*, p. 19).

Third, suppose my choice is granted, such that '$\S\alpha = \S\beta$' represents the fact that a certain object is self-identical. This raises the question of what it would be to ground a fact of self-identity. Understandably, CDP find this question problematic. There is neither room nor need for any grounding here, they think, because self-identity is 'a kind of "logical fact", a universal feature of reality' (p. 12). They conclude that 'although metaphysical grounding might sound as a natural and interesting connection it is not a viable road' (*ibid.*).

A better option, they propose, would be to invoke conceptual grounding, understood as 'a relation amongst truths, which is objective, non-causal and explanatory in nature (as is the case with metaphysical grounding), but which holds in virtue of the concepts these truths contains'. (Sereni and Zanetti make a similar proposal.) While I am agnostic about the prospects of appeals to conceptual grounding, I believe CDP are too quick to give up on metaphysical grounding. (Henceforth, I omit the qualification 'metaphysical'.)

I agree with CDP that it would be obscure to ask of an existing object what *more* might be required to ground its self-identity. However, following much of the abstractionist tradition, I use a negative free logic where $\S\alpha = \S\alpha$ is tantamount to the statement that $\S\alpha$ exists.[3] And I do not find it an obscure idea that $\alpha \sim \alpha$, if true, should ground the existence of $\S\alpha$. More importantly, this idea figures at the heart of an account of abstraction and grounding that Louis deRosset and I have recently worked out. We start with the thought that a sufficiency statement $\varphi \Rightarrow \psi$ records an explanatory argument (or a 'grounding potential', as I put it in *TO*). We proceed to use such arguments to derive information about grounding.[4] This yields an attractive theory of abstraction and grounding, we argue, which solves CDP's problems – as well as some additional problems identified by Donaldson (2017).

2 | MODALLY FRAGILE ABSTRACTION: DONALDSON

No sooner have deRosset and I solved some problems due to Donaldson than he points out another. The new problem concerns specifications that have only contingently the properties relevant to some form of abstraction. A clay sphere might instead have been shaped as a cube. A pen pointing north might have pointed east. How, then, might shape or direction abstraction on these specifications work? This is not an idle question. For as Donaldson observes, 'abstractionism is attractive in large part because it offers us a general theory of types and tokens' (p. 11).

Here is one manifestation of the problem. Let $\cong$ be geometrical congruence. Then we wish to endorse:

[3]See e.g. Hale & Wright (2009a, pp. 463–464) and Linnebo (2018, Appendix 2.B). An alternative would be to appeal to the account of Fine (2016) of how identities are grounding.

[4]First, we use the explanatory arguments to derive relations of *weak* grounding. (Unlike strict grounding, which is concerned with the explanation of one fact in terms of some strictly more fundamental facts, weak grounding tolerates explanations that 'move horizontally' in the explanatory hierarchy of truths.) Second, unidirectional weak grounding (that is, cases where there is no weak grounding in the reverse direction) is strict grounding. In this way, we show that (§<) is not generally valid but needs to be reformulated with weak grounding ($\leq$) in place of strict ($<$). See deRosset & Linnebo (2022) for details.

$$a \cong b \rightarrow (a \cong b < \text{shape}(a) = \text{shape}(b)).$$

Let a be our clay sphere. It follows that a's self-congruence grounds the self-identity (and thus existence) of its shape, say s. But a ground is generally taken to necessitate the grounded, which in this case yields:

$$\Box(a \cong a \rightarrow E(s)). \tag{1}$$

But this is problematic. For a might have been a cube, say, rather than a sphere. Why, then, should a's existence (and thus self-congruence) necessitate the existence of a shape that a might not even have?

In Section 6.3.3 of *TO*, I discuss whether it is permissible to abstract on a relation $\sim$ that is only contingently a partial equivalence. I defend a negative answer on the grounds that the language in which we talk about the resulting abstracts would be too modally fragile: while $\alpha \sim \beta$ is in fact a basis for asserting $\S\alpha = \S\beta$, it might easily not have been so. Donaldson's problem is related but different. My problem is that what *is* a partial equivalence might fail to be so *at other possible worlds*. His problem is, loosely speaking, that what is a partial equivalence might fail to be so *across possible worlds*. In our clay example: a as it is fails to be congruent with a as it might have been.

Two possible responses suggest themselves. Extending the strict policy of *TO*, I could prohibit the offending forms of abstraction on the grounds of excessive modal fragility. A promising alternative, though, would be to permit the abstraction but acknowledge its modal fragility. One way to do so would be to let the abstraction work, not on ordinary objects such as our piece of clay, but on objects *qua* shaped in a particular way. It is entirely plausible that a-qua-thus-shaped should necessitate the existence of s. On this proposal, the specifications would be precisely as modally fragile as the abstraction that they underwrite.

By contrast, it would be a bad idea to respond by denying (1). For I wish to use principles of this form to explain why cardinal numbers are necessary objects, thus meeting a request by Donaldson. Letting ee be the empty plurality, we obtain $\Box(ee \approx ee \rightarrow E(0))$.[5] Since the antecedent holds of (logical) necessity, it follows that $\Box E(0)$. This, in turn, ensures that the singleton-plurality consisting of 0 only is self-equinumerous – indeed necessarily so. Since this self-equinumerosity necessitates the existence of 1, we easily derive $\Box E(1)$. Continuing in this way, we can prove that every natural number exists necessarily. An analogous argument shows that every pure set exists necessarily.

A second manifestation of Donaldson's new problem remains to be addressed. While a (necessitated) abstraction principle enables comparisons of the shapes of objects *within* any one possible world, its ability to underwrite comparisons *across* possible worlds is unclear. But such comparisons seem possible. To borrow an example from Donaldson, we can meaningfully say that a projected church would have had exactly the same shape regardless of which of two competing proposals, differing only with respect to their choice of materials, had prevailed. Moreover, I agree with him it would be unwise for abstractionists to secure such cross-world comparisons by embracing Lewisian modal realism, according to which actual and merely possible specifications exist on a par and so are straightforwardly comparable.

To illustrate my alternative proposal, consider cardinal numbers. Suppose we followed Frege and the neo-Fregeans in abstracting cardinal numbers from (Fregean) concepts. This would result in modally fragile abstraction, with the associated problems discussed above. Let F be true of all and only my children. Although cardinality abstraction on F in fact yields the number 2, it might easily have given a different number. It is better, therefore, to proceed as in *TO* and do cardinality abstraction on pluralities, which have their numbers essentially. For

[5]Concerning the empty plurality, see the *Précis*, fn. 3.

example, let *cc* be my children. Then necessarily, if *cc* exist, they are two. To enable cross-world comparisons of cardinal numbers, all that remains is to find pluralities that exist across all possible worlds. Pluralities of pure sets or of cardinal numbers are natural options. These pluralities can serve as rigid 'measuring rods' that enable the desired comparisons. For example, while *cc* are equinumerous with the plurality of 0 and 1, it might easily have been the case that there are some objects who are all and only my children and who are not equinumerous with said measuring rod.

In the case of shapes, our rigid measuring rods might be regions of space, as opposed to physical objects located at these regions. For a region plausibly has its shape essentially and can – *modulo* familiar (and serious) Leibnizian worries – be compared across possible worlds.

3 | THE FLEXIBLE CONCEPTION AND THE QUESTION OF REALISM: SERENI AND ZANETTI

Sereni and Zanetti (henceforth, *S & Z*) explain how two versions of 'lightweight' platonism – namely, Rayo's trivialism and my account of thin objects – are naturally seen as forms of Aristotelianism. On both views, abstract mathematical objects exist but are somehow derivative from non-mathematical facts. I thank them for this astute observation.

They proceed to formulate a good challenge. As discussed, it is natural to explicate this Aristotelianism in terms of grounding (by which I continue to mean *metaphysical* grounding). Is this appeal to grounding compatible with the flexible conception of reality that both Rayo and I embrace?[6] '[M]ost proponents of grounding', S & Z claim, 'seem to think of relations of determination as metaphysically rigid' (p. 10). If correct, this poses an obvious danger: 'if abstraction principles are interpreted as claims of grounding, then even if the relevant principle is coherent, the world might fail to respond to our stipulation' (*ibid*.).

I am not in a position to assess S & Z's claim about 'most proponents of grounding'. Let me instead explain why I believe the flexible conception offers all the metaphysical realism that we need, both in general and in order to make sense of grounding. The flexible conception, as I define it, is first and foremost a metasemantic thesis to the effect that there is no uniquely right way to apply the apparatus of first-order logic to reality. *We* choose which objects to pick out. But, I claim, the objects that we pick out are real and typically exist independently of us.

Let me first make some remarks on realism.[7] As I understand it, realism about some domain has to do with reality providing an objective answer to every meaningful question about the domain. The flexible conception does not restrict reality's ability to provide answers to meaningful questions. Rather, it is a thesis to the effect that more work than what one might have expected is required to formulate a meaningful question, which we can then put to reality. Consider the question 'How many objects are there in my office?' As it stands, this is not a fully meaningful question: we additionally need to know what kind of objects the questioner has in mind. On the flexible conception, it takes more work than naively expected to formulate a meaningful question. Yet once we have formulated a meaningful question, reality provides an objective answer. In short, we must not conflate reality's inability to answer questions that are not even fully meaningful with some form of anti-realism.

Next, there is the concern that on the flexible conception the objects we pick out will somehow be counterfactually dependent on being thus picked out. That is, had we (or someone else) not picked out the objects, they would not have existed. I do not see why any such thing should follow. Consider my account of reference to numbers or physical bodies. In both cases, we obtain an account of what it takes to ground the existence of the relevant referents: the existence

[6] I gloss over some nuances and terminological differences between Rayo's view and mine.
[7] This paragraph and the next derive from Linnebo (2023b).

of 2, for example, is grounded in the existence of any pair, while the existence of a stick is grounded in the existence of parcels of matter appropriately arranged. In neither case, then, do the objects depend on us for their existence. Numbers and physical bodies are therefore fundamentally different from objects whose existence *is* counterfactually dependent on us and our activities. Contracts and marriages provide examples. By withholding your assent or vow, you could have prevented a contract or a marriage from coming into existence.

There is a good question, though, of whether our way of picking out objects might in some sense 'rub off' on the objects. S & Z propose what seems to me a promising way to sharpen this loose idea.

> Fine's hylomorphism appears to be a way of emphasising the role of conceptual elements in the constitution of objects. To a certain extent, it may be argued that principles of embodiment are provided by concepts, and thus that objects are constituted also in terms of the concepts that 'we bring to bear'. (p. 12)

Their main concern is that this may give rise to 'multiple and different metaphysical hierarchies […] based on multiple coherent applications of conceptual relations […] to "the most basic forms of matter" ' (*ibid.*). Thankfully, this concern can be addressed. The ways of picking out objects licensed by my account are tightly controlled by the reducibility (or 'predicativity') requirement discussed in the *Précis*, Section 5. With this requirement in place, one can prove that all the objects we can pick out cohere in a single ontological hierarchy (cf. *TO*, ch. 9).

4 | ARITHMETIC AND THE METAPHYSICS OF NUMBERS: PANTSAR

Pantsar first discusses my approach to arithmetic before he concludes with some remarks about the metaphysics of numbers.

There are two different abstractionist conceptions of the natural numbers. The Fregean tradition has almost universally regarded the numbers as *cardinals*, in the sense that they are obtained by abstraction on equinumerosity, as discussed above. In ch. 10 of *TO*, I develop an alternative abstractionist conception, which regards the natural numbers as *ordinals*, obtained by abstraction on numerals under the equivalence of occupying matching positions in their respective numeral sequences. While I regard both conceptions as legitimate, I argue that the ordinal conception better matches our actual arithmetical thought and language.[8]

As Pantsar notes, this latter claim is in large part empirical. This means that 'we should study the trajectory of cognitive development of arithmetical cognition. In some parts, our best understanding of this trajectory is potentially in conflict with Linnebo's account of the epistemology of arithmetic' (p. 4). I accept this observation and enthusiastically welcome work, such as Pantsar's own, that can help us integrate the philosophical and psychological aspects of the problem. I also agree with much of what he writes, for example when he emphasises the importance of the Object Tracking System for our most basic arithmetical capacities and its conceptual priority to any explicit understanding of equinumerosity or its ordinal equivalent.

In my defense, though, I did signal some distance from empirical claims, for example when I wrote that 'it is generally good methodology to begin by articulating and exploring various "pure" analyses of some phenomenon, even while admitting that the phenomenon might turn out to be too messy to be fully captured by any single pure analysis' (p. 178). In retrospect,

[8]This claim has spurred a fair amount of discussion; see, e.g., Snyder et al. (2018), Assadian & Buijsman (2019), and Buijsman (2021).

I admit I may have gone a bit overboard in my enthusiasm for the ordinal conception and downplaying of cardinal aspects of our arithmetical competence. But I was fed up with the cardinal conception's hegemony within abstractionism. And I still regard it as important to develop an alternative, "pure" ordinal conception – even though this purity is bound to be compromised when we seek a better, empirically informed match with our actual abilities.

Deeper disagreements emerge when we turn to the metaphysics of numbers. In particular, Pantsar disagrees with my claim that numbers are counterfactually independent of us. His argument begins with an innocuous observation: 'Had there been no intelligent agents, the cognitive trajectory outlined in [his paper] would never have taken place' (p. 12). But then things take a problematic turn. '[W]hat reason is there to believe in the existence of numbers independently of the existence of number concepts?' (p. 13). Whereas I take the existence of numbers to make no substantial demand on the world, Pantsar writes: 'My account makes one such demand, namely, that there are agents that possess number concepts' (p. 14). Why make this demand? Pantsar appears to be conflating representations with things represented. It is certainly true that *relations of representation* found in our thought and language are constituted, at least in part, by facts about us. But this does not imply that *the objects represented* are thus constituted. In short, whereas I ascribe to thin objects an Aristotelian dependence profile, where they depend on the existence of appropriate specifications, Pantsar ascribes to them a Kantian dependence profile, where they depend on being represented by appropriate agents (cf. the *Précis*, Section 8).

I can be more specific. Consider the fact (written '$\mathrm{Ref}(t, \S\alpha)$') that a singular term t refers to the abstract $\S\alpha$. On my account, this fact is partially grounded in the fact (written '$\mathrm{Assoc}(t, \alpha, \sim)$') that t is associated, in the way detailed in *TO*, with the specification α and the unity relation $\sim$. Additionally, we need the fact that α is in the field of the partial equivalence: $\alpha \sim \alpha$. Thus, we have the following relation of grounding (figuring two grounds on the left):

$$\mathrm{Assoc}(t, \alpha, \sim), \alpha \sim \alpha < \mathrm{Ref}(t, \S\alpha). \tag{Ref <}$$

The first of these grounds says something about us: it is *we* who associate t with α and $\sim$. After all, it is our use of the language that endows it with meaning. By contrast, the existence of the referent has a ground entirely independent of us:[9]

$$\alpha \sim \alpha \leq E(\S\alpha). \tag{E ≤}$$

On my account, then, facts about us make an essential contribution to the constitution of relations of *reference* but typically play no role in grounding the existence of the *referents*.

5 | THE EPISTEMOLOGY OF ABSTRACTION: PLEBANI, SAN MAURO AND VENTURI

Plebani, San Mauro and Venturi (henceforth *PSV*) investigate the epistemology of abstraction, making the important observation that 'the doctrine of thin objects might make it easy to prove that mathematical objects a and b exist but hard to know whether they are the same object or not' (pp. 1–2).

Let me explain this observation and place it in context. As discussed in the *Précis*, Section 5, the ground for asserting an identity '$\S\alpha = \S\beta$' is the equivalence of the associated specifications, namely $\alpha \sim \beta$. In many of the cases discussed in *TO*, this equivalence is comfortably within our

[9]Recall that $E(\S\alpha)$ abbreviates $\S\alpha = \S\alpha$. So this is an instance of (§<), only revised with weak ground rather than strict, for the reasons mentioned in footnote 4.

epistemic reach; examples include the parallelism of two lines and the 'co-typicality' of two tokens. Not all cases are that straightforward, however. In some cases, the specifications α and β may be infinite pluralities of objects, which are not in fact within our epistemic reach, but figure only in a highly idealised account of how reference can be constituted. In other cases, it takes some work to determine whether the specifications are equivalent. A case in point are physical bodies, which are specified by parcels of matter (cf. the *Précis*, Section 4). Whether two parcels of matter stand in the appropriate unity relation is not intrinsic to these two parcels but turns on the physical environment to which they belong. This explains the epistemic gulf that separates physical bodies from abstract objects (cf. *TO*, Sections 11.3 and 11.4). To decide whether the bodies specified by two parcels of matter are identical, it is not sufficient to examine these two specifications. By contrast, to decide whether the cardinal numbers specified by two pluralities are identical, it *is* sufficient to examine these two pluralities; for whether or not they are equinumerous is intrinsic to the two pluralities in question.[10]

PSV identify a new and different type of case, namely, where the specifications in question are finite and the objects that they specify are abstract, yet where there is no effective procedure to decide whether two specifications are equivalent. Here is a nice example. '[C]onsider the case where D is the set of the Turing machines' programs (conceived here as concrete tokens, say, sequences of marks on a surface) and the equivalence relation is the one that holds between two programs just in case for any (code of a) numerical input they return the same output' (p. 4). As PSV observe, the resulting unity relation is not decidable, which means it can be hard to tell whether two mathematical objects thus specified are identical or not.

What are we to make of this? The example shows that abstraction does not on its own guarantee an easy epistemology. This need not be a problem, though. As PSV observe (quoting, Linnebo (2017, p. 128)), 'abstract objects introduced via an abstraction principle "don't pose any additional epistemological problem" ' (p. 8). The equivalence $\alpha \sim \beta$ can be more or less epistemically tractable. But the identity $\S\alpha = \S\beta$ poses no *additional* epistemological problem. Moreover, we can diagnose the source of the problem. The example involves a unity relation where the equivalence of two specifications is not intrinsic to the specifications but rather involves quantification over a countable infinity of numerical inputs. This gives the example an interesting hybrid character. As in the case of physical bodies, the unity relation is extrinsic. But it is still mathematical in character, being concerned with the functions computed by Turing machines.

6 | PHYSICAL BODIES VERSUS ABSTRACTA: PEARCE

Gareth Pearce sees two arguments in *TO* for the thesis of Reference by Abstraction ('RBA', cf. the *Précis*, Section 5): one deductive and another abductive. Finding both arguments wanting, he instead defends the nominalist view that there are no abstract objects.

The former argument seeks a deductive path from the flexible conception of reality to RBA. I doubt there is any viable such path. As explained in the *Précis*, Section 3, the flexible conception was introduced only in order to locate my view in a broader philosophical landscape. The task of being 'entirely clear and specific' (p. 4) falls, not to the flexible conception, but to the later parts of the book. Nor can a rough first approximation of a view be expected to entail a precise statement of it.

[10]Something analogous holds for atomic properties of the objects in question. Whether or not an atomic property holds of some cardinal numbers, for example, can be 'read off' directly from their specifications, whereas this is not the case for physical bodies (cf. *TO*, Section 11.3). I believe these observations go a long way towards explaining the apparent *a priori* character of elementary arithmetic.

The abductive argument seeks a path from my example of the constitution of reference to physical bodies to the full thesis of RBA: 'the best explanation of why the principle holds in the case of physical bodies is that RBA holds generally' (p. 6). Again, this was not my intention. My strategy was to begin with an easily grasped example, before proceeding to the general thesis.

Pearce's discussion does, however, raise the interesting question of whether philosophers of a nominalist pursuasion can accept my account of reference to physical bodies but resist its extension to the realm of the abstract. To answer the question, let us revisit the three steps of my defense of RBA, summarised in the *Précis*, Section 5. *First*: it is permissible to start speaking as if there are Fs. This is defensible because our language has reductive assertibility conditions, formulated solely in terms of antecedently accepted objects. This reducibility requirement applies to physical bodies and Fregean abstracta alike. *Second*: if available, a semantic interpretation that takes the apparent reference to Fs at face value would be preferable. Again, the reasons I adduce play out in similar ways for bodies and Fregean abstracta. *Third*: such an interpretation *is* available. My defense of this claim turns on permitting the disputed form of reference in the metalanguage in which the interpretation is formulated. Once again, there is no relevant difference between the two cases.

My conclusion is that, while the specific example of physical bodies does not lend much abductive support to the general thesis of RBA, the best defense of the former generalises naturally to a defense of the latter.

7 | ABSTRACTION AND MEREOLOGY: LANDO

Lando raises the interesting question of whether my broadly Fregean form of abstraction might shed light on mereology. Let $\Sigma(xx)$ be the mereological sum of the objects xx. We would like to formulate an equivalence relation $\sim$ on pluralities that specifies the conditions under which sums are identical:

$$\Sigma(xx) = \Sigma(yy) \leftrightarrow xx \sim yy. \qquad (\mathrm{CI}-\Sigma)$$

Lando discusses various options but finds that 'that there are [...] issues with all of them' (p. 2).

Here is an example. Suppose we defined '$xx{\sim_a}yy$' as 'every mereological atom is part of one of xx just in case it is part of one of yy'. The resulting criterion of identity is quite attractive — but performs miserably if there are objects that do not decompose into atoms. For example, any two pieces of atomless gunk would be identified. I agree with Lando, not only on this option, but also on the others he considers. My only complaint is that he gives up the search for a suitable criterion too quickly.

Here is my preferred criterion. Let '$x \circ y$' express that x overlaps y, that is, that x and y have a shared part: $\exists z(z \le x \wedge z \le y)$. Further, let '$x \circ yy$' express that x overlaps one of yy, that is, $\exists y(y \prec yy \wedge x \circ y)$. Then, I propose we define the following equivalence:

$$xx \sim_o yy \leftrightarrow_{\mathrm{def}} \forall u(u \circ xx \leftrightarrow u \circ yy). \qquad (2)$$

I contend that the criterion of identity based on this equivalence – let's call it (CI-Σ_o) – performs as well as can be expected.[11] In particular, we can prove:

Proposition 1. *Let $\le$ be a partial order on a domain D that satisfies the mereological principle of Strong Supplementation, namely $x \le y \to \exists z(z \le x \wedge \neg z \circ y)$. Then there is a*

[11]In fact, if the mereology is atomistic, this criterion is equivalent to the aforementioned one based on atoms.

partial order $\leq^+$ extending $\leq$ on a domain D^+ extending D, on which a summing operation Σ is defined, such that:

(i) (CI-Σ_o) holds for all pluralities of members of D^+;
(ii) when xx consist of x only, then $\Sigma(xx) = x$;
(iii) for every object y of D^+ there are xx of D such that $\Sigma(xx) = y$;
(iv) $\leq^+$ and D^+ satisfy the axioms of Classical Extensional Mereology.

Moreover, $\leq^+$ and D^+ are unique up to isomorphism.

The moral of this proposition is that, starting with some basic mereological assumptions, we can apply sum abstraction to obtain a Classical Extensional Mereology all of whose 'new' objects are sums of 'old' objects.

I think this moral also answers Lando's question of 'Why bother looking for abstraction principles?' (p. 13). One important reason is that we obtain a systematic and well-motivated defense of the existence of arbitrary sums.

8 | EXTENSIONAL VS. INTENSIONAL ABSTRACTION: EKLUND

Eklund asks a perceptive question: 'what is the big deal [with thin objects]?' (p. 2). 'Even before numbers and other mathematical objects […] have entered the stage, Linnebo already has in his metaphysics whatever the predicates employed stand for, in particular given that he, like neo-Fregeans before him, employs a second-order framework with quantification into predicate position' (*ibid.*). The worry, then, is that my defense of thin objects relies on a prior commitment to 'the existence of some thin entities, the predicables' (*ibid.*).

An unexciting answer would be that *TO* is almost exclusively concerned with what we may call *extensional abstraction*, that is, roughly, abstraction on either single objects or pluralities thereof.[12] Unlike Frege and the neo-Fregeans, for example, I base cardinality abstraction on pluralities of objects, not on Fregean concepts (cf. Section 2 of these Replies). Granted the widely held assumption that plural logic introduces no ontological commitments beyond those incurred by its first-order variables, my account avoids any commitment to thin entities and instead progresses directly to thin objects.

I want to give Eklund a more interesting answer as well. Although *TO* focuses on extensional abstraction, I am also friendly to what we may call *intensional abstraction*, understood as abstraction on Fregean concepts or the like. So even though *TO* offers only some brief remarks on intensional abstraction (Appendix 3.A.2), let me attempt a more direct answer to Eklund's challenge. Yes, I regard Fregean concepts as thin entities. (I am not attracted to a 'neutralist' conception of higher-order quantification, on which such quantification avoids any new ontological commitments, for roughly the reasons Eklund lays out.) By characterising a Fregean concept as 'thin', I mean that all it takes to define such a concept is to specify a function from objects to truth-values. (This is much like the view of Hale & Wright (2009b), expressed in the passage quoted by Eklund.) This thinness is then 'inherited' by any abstracta whose specifications are Fregean concepts.

I do not see why this view would trivialise the thinness of the resulting abstract objects. Many philosophers appear to think that first-order (or ontological) commitments are a bigger deal than higher-order (or ideological) commitments. I disagree. I find the two kinds of commitment broadly on a par. Thus, I am happy to make Eklund's words my own: 'there is a kind of

[12]To be precise, we must add the requirement that the unity relation be *intrinsic* to the specifications that it relates; cf. the *Précis*, Section 6.

parity between claims that entail the existence of predicables and claims that entail the existence of objects that are not predicables' (p. 6).

I hasten to add, though, that my view does not render the definition of a Fregean concept a trivial matter. True, all that it takes is to specify a function from objects to truth-values. But doing so can be surprisingly hard! The problem stems from the dynamic character of my account, where domains successively expand, as described in the *Précis*, Section 6. These expansions may disrupt an attempted definition.[13] Suppose we try to use a condition $\varphi(x)$ to define a concept. Suppose an object a satisfies the condition when evaluated relative to one of the expanding domains. But a may fail to satisfy the condition relative to a later, expanded domain. My proposed solution is that the definition of Fregean concepts be subjected to restrictions of a broadly predicativist character.

9 | ABSTRACTION AND ABSOLUTE GENERALITY: STUDD

Since Studd's is a very rich paper, I will briefly comment on some of the questions it raises, before focusing on what I take to be its most weighty challenge.

First: Like Donaldson, Studd requests a better defense of the necessary existence of the objects of pure mathematics. Fair enough: I outline such a defense in Section 2 of these Replies.

Second: Studd claims that 'Linnebo's restriction to predicative abstraction requires him to eschew familiar one-sorted abstraction principles, such as HP' (p. 5). I note that my use of two-sorted abstraction is limited and only a means to an end. I first use a two-sorted analogue of the familiar abstraction principle, before merging the sorts to return to the ordinary one-sorted setting – with a domain that may have expanded (*TO*, p. 58). In *TO*, I develop the resulting dynamic account of expanding *one-sorted* domains by means of modal logic (see the *Précis*, Section 6). I factorise each familiar abstraction principle into separate criteria of existence and identity, which are then transposed to the modal setting. The resulting principles are admittedly a bit less familiar, but there is nothing two-sorted here. Regardless, if readers prefer to retain the familiar abstraction principles rather than the factorise-and-transpose manoeuvre of *TO*, I can cater to that too, namely, by adopting an alternative development of dynamic abstraction based on a 'critical' plural logic rather than modality (cf. the *Précis*, fn. 11).

Third: While I argue that each round of dynamic abstraction is epistemically 'for free', Studd observes that this falls short of justifying infinite iterations of such expansions. I agree. Quoting Studd quoting *TO*, my view is that 'our justification for theories that result from infinitely iterated abstraction is "more indirect and less conclusive" (p. 202)' (p. 15). This seems to me unsurprising. There is more robust evidence for elementary mathematics than for infinitary.

Fourth: Might we postpone the need for infinite iterations by allowing abstract objects to exist solely on the basis of the *possible* existence of appropriate specifications? I appeal to such specificationless abstracts some places in *TO* (pp. 45, 188 and 190). While I admit that *TO* offers no worked-out account of such abstraction, I believe an account can be developed – say, by abstracting on actually existing concepts that uniquely characterise the relevant merely possible specifications. For now, I rest content with *TO*'s primary route into the infinite, namely via the long iterations.

Studd's most weighty challenge concerns the possibility of absolutely general quantification. My view here is intermediate between generality relativism and traditional generality absolutism.[14] I call a domain *extensional* if it can be given as a plurality and *merely intensional* if not. On the one hand, I argue, with the relativists, that every *extensional* domain can be surpassed by a larger such domain. On the other hand, I argue, with the absolutists, that there is an absolute domain. But I break with traditional absolutism by insisting that this absolute

[13]See Linnebo & Shapiro (2023b) for details.

[14]See the *Précis*, Section 7. For a broader discussion, see Linnebo (2023a), which expands on many of the views and arguments that follow.

domain is *merely intensional*, that is, without a corresponding plurality. Why this intermediate view? For one thing, the extendability argument that I use to surpass any given extensional domain makes essential use of the assumption of extensionality and thus gets no grip on a merely intensional domain (see the *Précis*, Section 7). For another, there are intensionally specified domains, such as that of every self-identical object, that we can make no sense of surpassing. Given any plurality, we can make good sense of expanding so as to find objects outside and beyond that plurality; but given the concept of being self-identical, we can make no sense of escaping its reach. For wherever we might expand, this concept will follow along.

Studd's challenge takes the form of a new and different extendability argument, which threatens not only extensional domains but also some merely intensional ones. Although the details are complex, the central ideas are simple enough. As observed in the *Précis*, Section 7, the 'modalized quantifiers' $\Box\forall$ and $\Diamond\exists$ behave logically just like genuine quantifiers as far as first-order logic is concerned. We can use this behaviour, Studd argues, to interpret a two-sorted language whose 'extended' sort has variables ranging over all and only the possible values of the 'initial' sort. So far, so good. The key move comes when Studd proposes to add the full resources of plural logic to the extended sort, including a universal plurality comprising all of its objects. The effect of this move would be to introduce an extended stage e, beyond all of the initial stages, at which we 'actualise the infinities that were previously merely potential' (p. 22): for an infinity that is merely potential with respect to the initial stages would then be actual (that is, given as a plurality) with respect to the extended stage e. It is now only a small further step to use the newly obtained universal plurality to define a set, which would lie beyond the potential domain with which we began.

Two types of response are possible. One is to cede no ground and resist any extendability argument that targets the absolute but merely intensional domain. This is the official response in *TO* (and, for that matter, in Florio & Linnebo (2021)). Studd is certainly right that some intensional domains can be completed. As Cantor taught us, Aristotle's potentially infinite domain of natural numbers provides an example. But this domain is the minimal closure under the generating operation of successor, which enables us to make sense of going beyond the domain – thus completing it. The absolute domain, by contrast, is not a minimal closure under any generating operations.

Even if this response suffices to fend off Studd's challenge, the battle is not over. For one thing, one can reasonable demand a more substantive account of how quantification over a merely intensional domain is supposed to work, especially when this domain cannot even be specified as a minimal closure. For another, ever more refined extendability arguments have been developed – not only by Studd but also by Kit Fine and in work of my own.[15] So, the first response looks increasingly perilous.

Another response is to retreat to a robust fallback, which was adumbrated in Appendix 3.A.4 of *TO* and has since been properly developed in Linnebo (2022). The central idea is illustrated by a passage where the great, philosophically minded mathematician Hermann Weyl discusses whether there is a natural number that has some decidable property P.

> Only the finding *that has actually occurred* of a determinate number with the property P can give a justification for the answer 'Yes,' and – since I cannot run a test through all numbers – only the insight, that it lies in the *essence* of number to have the property not-P, can give a justification for the answer 'No'; even for God no other ground for decision is available (Weyl, 1921, p. 97; emphasis in original)[16].

Weyl's remark that 'even for God no other ground for decision is available' suggests that his point is not only epistemic (about our justification) but also metaphysical (about what might

[15]See Fine, (2023), Crosilla & Linnebo (2023), and Linnebo & Shapiro (2023a).
[16](Weyl, 1921, p. 97; emphasis in original). I use a slightly adapted translation from (Parsons, 2015).

ground or explain a truth). An existential generalisation is made true by the availability of a witness. A universal generalisation, by contrast, cannot always be made true by its instances – since these may not all be available. We are, after all, concerned with generality across a merely intensional domain (corresponding to what we expressed above using the modalised quantifiers, $\Diamond\exists$ and $\Box\forall$). But a universal generalisation need not be made true by its instances, Weyl suggests; it can also be made true by its 'lying in the essence' of the relevant concepts that the generalisation holds. It might, for example, lie in the essence of number that any number have the property not-P, which would explain $\forall x \neg Px$ and $\neg \exists x\, Px$.

This interpretation of the quantifiers achieves something remarkable. Both existential and universal generalisation can be accounted for at a particular stage of the process of expanding our domains, solely on the basis of material that is available at that stage; or, as we might also put it, both types of generalisation can be made true by material available at the stage. This also ensures that both types of generalisation are stable as the process of expanding domains unfolds, not just ephemeral. An existential generalisation is accounted for, or made true, by the availability of a witness at the relevant stage. And this witness will not go away as the generative process unfolds. A universal generalisation is accounted for, or made true, by being underwritten by essences of entities available at the relevant stage. Just as in the case of the witnesses, these essences will not go away as the generative process unfolds. Both types of generalisation, then, admit of local truthmakers, available at a single stage of the process, and are therefore preserved however the process might unfold.

Here is a more mathematical way to make the point. As is well known, on traditional Tarskian semantics, Σ_1-generalisations are preserved upwards under expanding domains, while Π_1-generalisations are not.[17] For whereas a witness to an existential generalisation will always be preserved as the domain expands, new counterexamples to a locally true universal generalisation may be admitted. Things change on the interpretation of the quantifiers suggested by Weyl. Since a quantified statement is made true by the existence of a truthmaker, the truth of any generalisation, even ones of complexity Π_1 and beyond, is analysed as a Σ_1 statement about truthmakers. This analysis ensures that any true generalisation will remain true no matter how the domain might expand.

There is a price to be paid, though. On the most natural semantics, this interpretation of the quantifiers calls for a so-called 'semi-intuitionitic' logic: while every extensional domain obeys the laws of classical logic, quantification over the absolute domain of everything whatsoever obeys only the weaker laws of intuitionistic logic.[18]

To sum up: there are refined extendability arguments that are capable of surpassing a variety of merely intensional domains. So mere intensionality does not, on its own, guarantee unsurpassability. By contrast, non-instance-based generality provides a robust way to develop the idea of an absolute but merely intensional domain. This conception of generality provides an ultimate guarantee against extendability arguments by ensuring that every truth has a local truthmaker. The local truthmaker of a universal generalisation will still be around, no matter what objects we proceed to introduce, and will thus ensure that the universal generalisation remains true.

10 | CONCLUDING REMARKS

I wish to end by summarising what I regard as the most significant developments concerning the views laid out in *TO*: some major clarifications, some new extensions and alternatives, and one outright change.

[17]A Σ_1 (or Π_1)-generalisation consists of a single existential (or universal) quantifier followed by a quantifier-free formula.

[18]This semi-intuitionistic logic blocks any attempt to employ Studd's extendability argument. For this argument assumes that quantification over an allegedly absolute domain behaves logically just like quantification over a plurality, thus enabling us to introduce an extended stage where the allegedly absolute domain forms a plurality (and thus is surpassable).

First: The relation between my notion of sufficiency and grounding has been clarified. As noted in Section 2 of the *Précis*, I now regard sufficiency as a special form of grounding potential; see also Sections 1–4 of these Replies. This paves the way for an alternative perspective on the project of *TO*, namely, in terms of grounding.[19]

Second: By thus admitting grounding into the official statement of the project, its broadly Aristotelian character, which was implicitly there all along, properly comes to the fore. See Section 8 of the *Précis* and Sections 3 and 4 of these Replies.

Third: In *TO*, I chose to develop the idea of dynamic abstraction using the resources of modal logic. An alternative has since emerged. We can eliminate all use of modality and instead use a 'critical' plural logic, which is more restrictive than traditional plural logic.[20] This alternative promises to be more user-friendly. See the *Précis*, Section 6, esp. fn. 9, and Section 9 of these Replies.

Fourth: Where TO focuses almost entirely on *extensional* abstraction (that is, roughly, abstraction on pluralities), there has recently been considerable progress on *intensional* abstraction (that is, abstraction on Fregean concepts or the like).[21] See Sections 8 and 9 of these Replies.

Fifth: With a better understanding of intensional abstraction and merely intensional domains, a threat emerges to *TO*'s defense of absolute generality. To counter this threat, I now wish to avail myself of what was previously only a possible fallback, namely, the idea of non-instance-based generality. See Section 9 of these Replies.[22]

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[19]See deRosset & Linnebo (2022) for details.

[20]Critical plural logic is developed in Florio & Linnebo (2020) and Florio & Linnebo (2021, ch. 12).

[21]See, e.g., Brauer et al. (2022), Linnebo & Shapiro (2023b), Linnebo & Shapiro (2023a), and Roberts (2022).

[22]More details can be found in Linnebo (2022) and Linnebo (2023a).

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