

ENUMERATION OF NODAL CURVES ON PROJECTIVE SURFACES

by

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PREFACE

Enumerative geometry is an ancient subject in mathematics. For instance, the problem of counting conics passing through 5 distinct points in the plane can be traced back to as early as the antiquity. More generally, a typical question in enumerative geometry would be "How many geometric structures of a given type satisfy a given collection of geometric requirements?" Visibly, such questions appear simple, but there is one immediate problem: The solution depends on the configuration of the given figures. Even worse, if one were to ask "How many points lie on each of two given lines in \mathbb{R}^2 ?" the answer would generally be 1, sometimes 0 (if the lines are parallel), or ∞ , if the lines coincide. This shows that if we want a well-defined enumerative problem (to which the answer is expected to be finite) we must be careful to choose the geometric requirements sufficiently general for the question to make sense. Also, one avoids the problem of empty intersections by passing to a projective space.

With the development of powerful tools such as Schubert calculus (subject of Hilbert's 15th problem), intersection theory and moduli theory, enumerative geometry is still a flourishing subject in algebraic geometry. In this thesis, we will concern ourselves with the subject of *nodal curves*. More specifically, the basic problem is, given some positive integer δ , the enumeration of δ -nodal curves satisfying geometric requirements which ensure that their number is finite (this is more clearly stated in Chapter 1).

Conventions: By a *variety* we will usually mean an *irreducible, reduced algebraic scheme* over \mathbb{C} . In particular, we will be interested in *surfaces* (varieties of dimension 2), and these will always be assumed to be *projective*. At some points we will consider surfaces which are not irreducible, but this will be clear from the context. Further notations will be introduced at the appropriate moments.

The thesis is structured as follows: Chapter 1 is devoted to a review of some of the most important *general* results concerning the enumeration of nodal curves, along with the main conjectures of Göttsche on this subject. We include an overview of the ideas motivating these conjectures, in particular the theorems of Bryan–Leung, which provide a proof of Göttsche's second conjecture in the case of $K3$ and abelian

surfaces. Finally, we use this conjecture to calculate the fundamental polynomials a_i , which intervene in Theorem 1.1.1, for $1 \leq i \leq 15$, and use these numerical results to provide some new observations concerning the behaviour of these polynomials for large i .

In Chapter 2 we concentrate on the case of \mathbb{P}^2 , which is classically the most studied case. We provide a partial proof of a conjecture concerning the shape of the polynomials $N(\delta, d)$ which enumerate δ -nodal curves of degree d , and look at two recursive procedures for the calculation of these polynomials, one due to Caporaso–Harris, the second established by Ziv Ran.

The case of *rational* nodal curves in \mathbb{P}^2 is, in many ways, completely solved by the recursive formula of Kontsevich. We have included a separate study of this beautiful theory in Chapter 3, since it illustrates several important methods of modern enumerative geometry (moduli spaces, Gromov–Witten invariants, quantum cohomology etc.).

Finally, in Chapter 4 we provide a new proof of the non-numerical part of Kleiman–Piene’s theorem (Theorem 1.1.1). This proof is based on intersection theory on a compactification of a certain configuration space, extensively reviewed in Section 4.2. It is hoped that one could obtain the numerical part through the use of residual intersection theory on the same space, although this is in itself a big project.

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CHAPTER 1

NODAL CURVES ON PROJECTIVE SURFACES

Contents

1.1	Main results and conjectures	7
1.2	The case of $K3$ and abelian surfaces	12
1.3	Göttsche's conjecture and the polynomials a_i	15

In this chapter we review some general theory concerning the enumeration of nodal curves on projective surfaces; we present both established theorems and more general conjectures, due to Göttsche. Conjecture 1.1.3 is particularly important, because it expresses the generating function of the numbers of δ -nodal curves in terms of five functions, three of which are known. The theorems of Bryan–Leung confirm this conjecture in the case of $K3$ and abelian surfaces. We also use this conjecture of Göttsche to calculate polynomials a_i which intervene in the enumeration of nodal curves, and present some new observations concerning the behaviour of these polynomials for large i .

1.1 Main results and conjectures

Consider the complex projective plane \mathbb{P}^2 . The complete linear system of curves of degree d is $|\mathcal{O}_{\mathbb{P}^2}(d)|$, that is, $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)))$, which forms a projective space $\mathbb{P}^{d(d+3)/2}$. The closure of the locus of reduced (but possibly reducible) curves having δ simple nodes as only singularities forms a subvariety $V(\delta, d)$ of this space, the *Severi variety*, the irreducibility of which was first properly established by Joe Harris in [Har]. The degree of this variety, $N(\delta, d)$, corresponds to the number of plane curves having δ nodes and passing through $d(d+3)/2 - \delta$ fixed general points. It is referred to as the *Severi degree* of the corresponding variety. Generally speaking, it can be expressed as a polynomial in d , naturally named the *Severi polynomial*. One might

also be interested in counting irreducible curves only; the variety parametrizing such curves is referred to as the *classical Severi variety*.

More generally, for a smooth, projective, irreducible surface S over \mathbb{C} and a complete N -dimensional linear system of (reduced) curves $|\mathcal{L}| = \mathbb{P}(H^0(S, \mathcal{L}))$ on S , we may consider the linear subsystem of curves having δ nodes as only singularities, trying to find an expression for the number of such curves passing through $N - \delta$ fixed general points on S . We let $N_S(\delta, \mathcal{L})$ denote the number of such curves: it is the degree of the corresponding Severi variety.

Notation: In the following, if \mathcal{L}, \mathcal{K} are line bundles we let $\mathcal{L}\mathcal{K}$ denote the degree of $c_1(\mathcal{L})c_1(\mathcal{K})$.

The theorem below, due to Kleiman and Piene, will be our starting point. The main objective of the last chapter will be to prove a partial generalization using intersection theory on a compactification of a certain configuration space.

THEOREM 1.1.1 ([KP1], Theorem 1.1) Kleiman–Piene. *For $\delta \leq 8$ and $m \geq 3\delta$, if \mathcal{L} can be written as $\mathcal{M}^{\otimes m} \otimes \mathcal{N}$ where \mathcal{M} is very ample and \mathcal{N} is globally generated, then $N_S(\delta, \mathcal{L})$ can be written as a polynomial in the four Chern numbers $\partial = \mathcal{L}^2, k = \mathcal{L}\mathcal{K}_S, s = \mathcal{K}_S^2, x = c_2(S)$, where \mathcal{K}_S is the canonical sheaf on S . More specifically, the expressions are*

$$N_S(\delta, \mathcal{L}) = P_\delta(\partial, k, s, x)/\delta!$$

where we have the formal identity $\sum_{\delta \geq 0} P_\delta t^\delta / \delta! = \exp(\sum_{l \geq 1} a_l t^l / l!)$ in t , and the polynomials a_l are the following, for $l \leq 8$:

$$\begin{aligned} a_1 &= 3\partial + 2k + x \\ a_2 &= -42\partial - 39k - 6s - 7x \\ a_3 &= 1380\partial + 1576k + 376s + 138x \\ a_4 &= -72360\partial - 95670k - 28842s - 3888x \\ a_5 &= 5225472\partial + 7725168k + 2723400s + 84384x \\ a_6 &= -481239360\partial - 778065120k - 308078520s + 7918560x \\ a_7 &= 53917151040\partial + 93895251840k + 40747613760s - 2465471520x \\ a_8 &= -7118400139200\partial - 13206119880240k - 6179605765200s + 516524964480x \end{aligned}$$

The above defines the polynomials P_δ as the *complete exponential Bell polynomials* of the a_l (see Appendix B); we have $P_0 = 1, P_1 = a_1, P_2 = a_1^2 + a_2, P_3 = a_1^3 + 3a_2a_1 + a_3$. Note that in the following, we will often write $N(\delta, \mathcal{L})$ and not mention the surface: the important aspect is that these numbers are expressed as polynomials in variables depending only on the numerical properties of S and \mathcal{L} .

Göttsche has formulated a conjecture generalizing the theorem above (see [Got] for more details):

CONJECTURE 1.1.2 ([Got], Conjecture 2.1) Göttsche's First Conjecture. *For all $\delta \geq 0$ there is a universal polynomial $T_\delta(u, v, z, t)$ having degree δ , such that given δ , a surface S and a very ample line bundle \mathcal{M} , there is an $m_0 > 0$ such that for all $m \geq m_0$ and all very ample line bundles \mathcal{N} , if $\mathcal{L} = \mathcal{M}^{\otimes m} \otimes \mathcal{N}$ then:*

$$N(\delta, \mathcal{L}) = T_\delta(\mathcal{L}^2, \mathcal{L} \mathcal{K}_S, \mathcal{K}_S^2, c_2(S))$$

We introduce the expressions $T_\delta^S(u, v) = T_\delta(u, v, \mathcal{K}_S^2, c_2(S))$ (fixing the surface S), $t_\delta^S(\mathcal{L}) = T_\delta^S(\mathcal{L}^2, \mathcal{L} \mathcal{K}_S)$ (fixing the line bundle \mathcal{L}) and finally the associated generating function $T(S, \mathcal{L})(y) = \sum_{\delta \geq 0} t_\delta^S(\mathcal{L}) y^\delta$.

CONJECTURE 1.1.3 ([Got], Conjecture 2.4) Göttsche's Second Conjecture. *There exist universal (independent of S and \mathcal{L}) power series $B_1, B_2 \in \mathbb{Q}[[q]]$, such that*

$$T(S, \mathcal{L})(DG_2(\tau)) = \sum_{\delta \geq 0} t_\delta^S(\mathcal{L})(DG_2(\tau))^\delta = \frac{(DG_2(\tau)/q)^{\chi(\mathcal{L})} B_1(q)^{\mathcal{K}_S^2} B_2(q)^{\mathcal{L} \mathcal{K}_S}}{(\Delta(\tau) D^2 G_2(\tau)/q^2)^{\chi(\mathcal{O}_S)/2}}$$

where $G_2(\tau)$ is the second Eisenstein series, a quasimodular form (see Appendix B), and Δ is the modular form $\Delta(\tau) = q \prod_{m>0} (1 - q^m)^{24}$. q denotes the expression $e^{2\pi i \tau}$ and D is the differential operator $\frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}$.

Note that the ring of quasimodular forms is closed under differentiation (see [KZ]), so DG_2 and D^2G_2 are also quasimodular. This conjecture implies that $t_\delta^S(\mathcal{L})$ is a polynomial of degree δ in $\mathcal{L}^2, \mathcal{L} \mathcal{K}_S, \mathcal{K}_S^2$ and $c_2(S)$. Indeed, by Noether's formula (see, for instance, [Beau] I.14), we have $\chi(\mathcal{O}_S) = \frac{1}{12}(c_1(S)^2 + c_2(S))$ where $c_1(S)^2 = \mathcal{K}_S^2$, and also $\chi(\mathcal{L}) = \chi(\mathcal{O}_S) + \frac{1}{2}\mathcal{L}(\mathcal{L} - \mathcal{K}_S)$. In fact, the conjecture even gives us that $\delta! t_\delta^S(\mathcal{L})$ must be the δ th Bell polynomial in δ polynomials which are linear in ∂, k, s and x , so we get the non-numerical part of Kleiman–Piene's theorem. Indeed, the conjecture implies that $T(S, \mathcal{L})(y)$ can be written as $A_1(y)^\partial A_2(y)^k A_3(y)^s A_4(y)^x$ for power series $A_i \in \mathbb{Q}[[y]]$. Suppose we have the formal identities

$$\sum_{\delta=0}^{\infty} t_\delta^S(\mathcal{L}) y^\delta = \exp\left(\sum_{l \geq 1} a_l y^l / l!\right) \quad \text{and} \quad A_i(y) = \exp\left(\sum_{l \geq 1} b_l^{(i)} y^l\right) \quad \text{for } 1 \leq i \leq 4,$$

then taking logarithms on both sides of $T(S, \mathcal{L})(y) = A_1(y)^\partial A_2(y)^k A_3(y)^s A_4(y)^x$, we get

$$\sum_{l \geq 1} a_l y^l / l! = \partial \sum_{l \geq 1} b_l^{(1)} y^l + k \sum_{l \geq 1} b_l^{(2)} y^l + s \sum_{l \geq 1} b_l^{(3)} y^l + x \sum_{l \geq 1} b_l^{(4)} y^l,$$

and identifying coefficients, a_l is a linear combination of ∂, k, s, x , whereas $\delta! t_\delta^S(\mathcal{L}) = P_\delta(a_1, \dots, a_\delta)$.

For $\delta \leq 8$, Conjecture 1.1.3 coincides numerically with the results of Kleiman and Piene, giving $B_1(q), B_2(q)$ up to degree 8. Later we will see that in the case of the projective plane (and also, thanks to R. Vakil, for Hirzebruch surfaces), there is

a recursive formula for the number $N(\delta, d)$ of degree d curves with δ nodes, proved by Caporaso and Harris. Using this recursion, Göttsche calculates the coefficients of the $B_i(q)$ up to degree 28. We include the expressions up to degree 8:

$$\begin{aligned} B_1(q) &= 1 - q - 5q^2 + 39q^3 - 345q^4 \\ &\quad + 2961q^5 - 24866q^6 + 207759q^7 - 1737670q^8 + \dots \\ B_2(q) &= 1 + 5q + 2q^2 + 35q^3 - 140q^4 \\ &\quad + 986q^5 - 6643q^6 + 48248q^7 - 362700q^8 + \dots \end{aligned}$$

REMARK 1.1.4 ([Got], Proposition 2.3). We can give some evidence of the conjecture above. More precisely, we will show that if Göttsche's first conjecture holds, then there *does* exist universal power series $A_{i,n} \in \mathbb{Q}[[y]]$, $1 \leq i \leq 4$, such that

$$T(S_n)(y) = A_{1,n}(y)^{\mathcal{L}_n^2} A_{2,n}(y)^{\mathcal{L}_n \mathcal{K}_{S_n}} A_{3,n}(y)^{\mathcal{K}_{S_n}^2} A_{4,n}(y)^{c_2(S_n)}.$$

for reducible surfaces S_n (with a line bundle \mathcal{L}_n) of a particular form introduced below. Göttsche uses a somewhat obscure limit and density argument to show that this implies the existence of universal power series $A_i \in \mathbb{Q}[[y]]$, $1 \leq i \leq 4$, such that

$$T(S, \mathcal{L})(y) = A_1(y)^{\mathcal{L}^2} A_2(y)^{\mathcal{L} \mathcal{K}_S} A_3(y)^{\mathcal{K}_S^2} A_4(y)^{c_2(S)}.$$

for arbitrary S, \mathcal{L} .

Proof. Note that it is enough to show the result up to order δ_0 in y for all $\delta_0 \geq 1$. Let us first fix some notation. Göttsche's first conjecture essentially claims that for sufficiently ample line bundles \mathcal{L} on the surface S (with respect to the number of nodes δ considered), the locally closed subset $W_\delta^S(\mathcal{L})$ of elements in $|\mathcal{L}|$ consisting of δ -nodal curves has codimension δ and degree $t_\delta^S(\mathcal{L})$. If S is a surface with several connected components, $W_\delta^S(\mathcal{L})$ includes only those $C \in |\mathcal{L}|$ which do not vanish identically on any components of S .

Now let $\delta_0 \geq 1$ be fixed and consider first a surface S of the form $S_1 \sqcup S_2$, together with a line bundle \mathcal{L} . Define $\mathcal{L}_i = \mathcal{L}|_{S_i}$ and assume that they are both sufficiently ample for $W_\delta^{S_i}(\mathcal{L}_i)$ to have codimension δ and degree $t_\delta^{S_i}(\mathcal{L}_i)$ in $|\mathcal{L}_i|$ for all $\delta < \delta_0$. There is an obvious surjective morphism $p : U \rightarrow |\mathcal{L}_1| \times |\mathcal{L}_2|$ defined by sending $C + D$ to (C, D) , where $U \subset |\mathcal{L}|$ is the open set consisting of curves which have a non-vanishing component on both S_1 and S_2 . We have

$$W_\delta^S(\mathcal{L}) = p^{-1} \left(\coprod_{\delta_1 + \delta_2 = \delta} W_{\delta_1}^{S_1}(\mathcal{L}_1) \times W_{\delta_2}^{S_2}(\mathcal{L}_2) \right)$$

so that $\text{codim}(W_\delta^S(\mathcal{L}), |\mathcal{L}|) = \delta$ for all $\delta < \delta_0$. Considering degrees, it also follows that for all $\delta < \delta_0$,

$$t_\delta^S(\mathcal{L}) = \sum_{\delta_1 + \delta_2 = \delta} t_{\delta_1}^{S_1}(\mathcal{L}_1) t_{\delta_2}^{S_2}(\mathcal{L}_2)$$

but then necessarily $T(S, \mathcal{L})(y) \equiv T(S_1, \mathcal{L}_1)(y) \cdot T(S_2, \mathcal{L}_2)(y) \pmod{y^{\delta_0}}$.

Let n be a positive integer chosen so that Göttsche's first conjecture holds for $Z_{j,n} := (\mathbb{P}_2, \mathcal{O}(jn))$, $1 \leq j \leq 3$, and for $Z_{4,n} := (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(n, n))$ for all $\delta < \delta_0$. We will use the notation $Z_{j,n} = (S_{j,n}, \mathcal{L}_{j,n})$. Define, for d_j non-negative integers:

$$S_n := S_n(d_1, \dots, d_4) = \prod_{j=1}^4 \prod_{i=1}^{d_j} S_{j,n}.$$

From what we established above, $T(S_n, \mathcal{L}_n)(y) = \prod_j T(Z_j)(y)^{d_j} \pmod{y^{\delta_0}}$, with \mathcal{L}_n denoting the sheaf on S_n which restricts to the appropriate sheaves on the components. Write $A_{i,n} = \exp B_{i,n}$ for each $1 \leq i \leq 4$ and suppose $\sum_{\delta=0}^{\infty} t_{\delta}^{S_n}(\mathcal{L}_n) y^{\delta} = \exp(\sum_{l \geq 1} a_l(S_n, \mathcal{L}_n) y^l / l!)$. Then we wish to show that there exist power series $B_{i,n} \in \mathbb{Q}[[y]]$ such that

$$\sum_{l \geq 1} a_l(S_n, \mathcal{L}_n) y^l / l! = \mathcal{L}_n^2 B_{1,n}(y) + \mathcal{L}_n \mathcal{K}_{S_n} B_{2,n}(y) + \mathcal{K}_{S_n}^2 B_{3,n}(y) + c_2(S_n) B_{4,n}(y)$$

up to some degree in y which increases with δ_0 . But from the equality $T(S_n, \mathcal{L}_n)(y) = \prod_j T(Z_j)(y)^{d_j} \pmod{y^{\delta_0}}$ we know that, for some number (which increases when δ_0 increases) of values of $l \geq 1$, we have

$$a_l(S_n, \mathcal{L}_n) = \sum_{j=1}^4 d_j a_l(S_{j,n}, \mathcal{L}_{j,n})$$

Note that the Chern numbers, as well, are additive for a disjoint union of surfaces. Let the coefficient of y^l in $B_{i,n}$ be $b_{l,n}^{(i)} / l!$ — we wish to show that these can be chosen so that we have, independently of the values of the d_j ,

$$a_l(S_n, \mathcal{L}_n) = \mathcal{L}_n^2 b_{l,n}^{(1)} + \mathcal{L}_n \mathcal{K}_{S_n} b_{l,n}^{(2)} + \mathcal{K}_{S_n}^2 b_{l,n}^{(3)} + c_2(S_n) b_{l,n}^{(4)}$$

up to some value of l , but this is equivalent to finding rational numbers $b_{l,n}^{(i)}$ such that the following holds independently of the d_j :

$$\sum_{j=1}^4 d_j a_l(S_{j,n}, \mathcal{L}_{j,n}) = \sum_{j=1}^4 d_j \left(\mathcal{L}_{j,n}^2 b_{l,n}^{(1)} + \mathcal{L}_{j,n} \mathcal{K}_{S_{j,n}} b_{l,n}^{(2)} + \mathcal{K}_{S_{j,n}}^2 b_{l,n}^{(3)} + c_2(S_{j,n}) b_{l,n}^{(4)} \right)$$

This is possible simply because the vectors $\bar{\partial}_n, \bar{k}_n, \bar{s}_n$ and \bar{x}_n (defined, for instance, by $\bar{\partial}_n = (\partial_{1,n}, \partial_{2,n}, \partial_{3,n}, \partial_{4,n})$) form a \mathbb{Q} -basis for \mathbb{Q}^4 . Indeed, the matrix having these vectors as row vectors is the invertible matrix

$$\begin{bmatrix} n^2 & 4n^2 & 9n^2 & 2n^2 \\ -3n & -6n & -9n & -4n \\ 9 & 9 & 9 & 8 \\ 3 & 3 & 3 & 4 \end{bmatrix}$$

of determinant $72n^3$. This concludes the proof. \square

1.2 The case of $K3$ and abelian surfaces

By the second conjecture of Göttsche, we express the numbers of curves of arbitrary genus in a suitably ample linear system in terms of three quasimodular forms and two universal power series. For surfaces with *numerically trivial canonical divisor* ($K3$ and abelian surfaces), only the quasimodular forms appear.

In fact, this result has been proved by Bryan and Leung (see [BL1] and [BL2] for proofs and details). More specifically, if S is an algebraic $K3$ surface and C is a smooth curve on S representing a primitive homology class, we define, for any g, δ satisfying $C^2 = 2g + 2\delta - 2$, an invariant $N_g(\delta)$ counting the number of curves of geometric genus g and δ nodes, in the linear system $|C|$. This number is well-defined for generic (S, C) , and for fixed g , we consider the generating function

$$\Gamma_g(q) = \sum_{\delta=0}^{\infty} N_g(\delta) q^{g+\delta-1}$$

Note the meaning of this: if S and C are chosen there are only finitely many pairs (g, δ) satisfying our requirement ($C^2 = 2g + 2\delta - 2$), but given g and δ we can always find surfaces S and curves C such that we have $C^2 = 2g + 2\delta - 2$, and the number $N_g(\delta)$ is then independent of which surface and which curve we have chosen. In the algebraic case, we can state the $K3$ theorem of Bryan–Leung as follows:

THEOREM 1.2.1 ([BL2], Theorem 1.1) Bryan–Leung. *Let S be a $K3$ surface and C be a smooth irreducible curve on S representing a primitive homology class. For any g, δ satisfying $C^2 = 2g + 2\delta - 2$, let $N_g(\delta)$ be the number of curves of geometric genus g and δ nodes in the linear system $|C|$. Then, for any g , the generating function $\Gamma_g(q)$ defined above is given by*

$$\Gamma_g(q) = \frac{(DG_2)^g}{\Delta}(\tau)$$

This gives for instance

$$\begin{aligned} \Gamma_0(q) &= q^{-1} + 24 + 324q + 3200q^2 + \dots \\ \Gamma_1(q) &= 1 + 30q + 480q^2 + 5460q^3 + \dots \\ \Gamma_2(q) &= q + 36q^2 + 672q^3 + 8728q^4 + \dots \\ \Gamma_3(q) &= q^2 + 42q^3 + 900q^4 + 13220q^5 + \dots \end{aligned}$$

The proof uses methods from symplectic geometry and is based on the consideration of moduli spaces of stable maps. The proof of the following corollary concerning rational curves is, however, within the scope of this thesis. It was first given implicitly by Yau and Zaslow, and we include it partially here for illustrative purposes:

COROLLARY 1.2.2. *Let S be a $K3$ surface and C be a smooth curve on S representing a primitive homology class such that $C^2 = 2\delta - 2$. Let $N_0(\delta)$ be the number of rational curves having δ nodes in the linear system $|C|$. We have*

$$\Gamma_0(q) = \sum_{\delta=0}^{\infty} N_0(\delta)q^{\delta-1} = \Delta^{-1}(q).$$

Proof. Since $C^2 = 2\delta - 2$, the adjunction formula $2g - 2 = C(C + \mathcal{K}_S) = C^2 = 2\delta - 2$ implies that the genus of C is δ , and by the Riemann–Roch theorem (plus a moving lemma) one can show that $|C| \cong \mathbb{P}^\delta$. So imposing δ general nodes, we expect to obtain a finite number $N_0(\delta)$ of rational curves in $|C|$ with δ nodes. Now consider the universal jacobian $\pi : \tilde{J} \rightarrow |C|$ for the system. If all the curves in $|C|$ have at most nodal singularities, it is possible to show that whenever $C' \in |C|$, the Euler characteristic $\chi(\pi^{-1}(C'))$ is always 0 unless C' is a rational curve with δ nodes, in which case $\chi(\pi^{-1}(C')) = 1$, so it follows that $N_0(\delta) = \chi(\tilde{J})$. If the members of $|C|$ are reduced and irreducible, one can show that \tilde{J} is birational to the Hilbert scheme \mathcal{H}_δ of δ points in S . By a result of Göttsche, the Euler characteristics $\chi(\mathcal{H}_\delta)$ satisfy

$$\sum_{\delta=0}^{\infty} \chi(\mathcal{H}_\delta)q^\delta = \prod_{m=1}^{\infty} (1 - q^m)^{-24} = q\Delta^{-1}(q)$$

Since compact, birationally equivalent, projective Calabi–Yau manifolds (of which \tilde{J} and \mathcal{H}_δ are examples) have the same Betti numbers, $N_0(\delta) = \chi(\mathcal{H}_\delta)$, so $\Gamma_0(q) = \sum_{\delta=0}^{\infty} N_0(\delta)q^{\delta-1} = \Delta^{-1}(q)$, which completes the proof. \square

On the other hand, if S is an (algebraic) abelian surface and C is a smooth curve representing a primitive homology class on S , with g, δ satisfying $C^2 = 2g + 2\delta - 2$, there is a g -dimensional space of curves of genus g in the class of C . So to define an enumerative problem one must impose g conditions on these curves – one can either count curves passing through g generic points – this number is denoted by $N_g(\delta, C)$ – or one can count the curves in the fixed linear system $|C|$ passing through $g - 2$ generic points – this number is denoted by $N_g^{FLS}(\delta, C)$.

THEOREM 1.2.3 ([BL1], Theorem 1.1) Bryan–Leung. *The numbers $N_g(\delta, C)$ and $N_g^{FLS}(\delta, C)$ defined above are given by the following generating functions:*

$$\begin{aligned} \sum_{\delta=0}^{\infty} N_g(\delta, C)q^{\delta+g-1} &= g(DG_2)^{g-1}(\tau) \\ \sum_{\delta=0}^{\infty} N_g^{FLS}(\delta, C)q^{\delta+g-1} &= (DG_2)^{g-2}D^2G_2(\tau) \\ &= (g-1)^{-1}D((DG_2)^{g-1})(\tau) \end{aligned}$$

PROPOSITION 1.2.4 ([Got], Remark 2.6). *The theorems of Bryan–Leung imply the second conjecture of Göttsche in the case of $K3$ and abelian surfaces.*

Proof. Let us first see that the conjecture of Göttsche can be slightly reformulated. Given a surface S and a sheaf \mathcal{L} on S , we introduce the following expression for all $l, m, r \in \mathbb{Z}$:

$$n_r^S(l, m) = T_{l+\chi(\mathcal{O}_S)-1-r}^S(2l+m, m).$$

When the sheaf \mathcal{L} is sufficiently ample with respect to $\delta := \chi(\mathcal{L}) - 1 - r$ the number

$$\begin{aligned} n_r^S\left(\frac{1}{2}(\mathcal{L}^2 - \mathcal{L}\mathcal{K}_S), \mathcal{L}\mathcal{K}_S\right) &= T_{\frac{1}{2}(\mathcal{L}^2 - \mathcal{L}\mathcal{K}_S) + \chi(\mathcal{O}_S) - 1 - r}^S(\mathcal{L}^2, \mathcal{L}\mathcal{K}_S) \\ &= T_{\chi(\mathcal{L}) - 1 - r}^S(\mathcal{L}^2, \mathcal{L}\mathcal{K}_S) = T_\delta^S(\mathcal{L}^2, \mathcal{L}\mathcal{K}_S) \end{aligned}$$

is nothing but the number of δ -nodal curves in a general sublinear system of $|\mathcal{L}|$ of codimension r , with $\delta + r = \dim |\mathcal{L}|$. In other words, r represents the number of constraints (points to pass through) which we must impose on our curves to get an enumerative problem. In this case we have the following formulation of Göttsche's conjecture:

$$\sum_{l \in \mathbb{Z}} n_r^S(l, m) q^l = B_1(q)^{\chi_S^2} B_2(q)^m (DG_2(\tau))^r \frac{D^2 G_2(\tau)}{(\Delta(\tau) D^2 G_2(\tau))^{\chi(\mathcal{O}_S)/2}}.$$

Note that when r and m are fixed not all the $n_r^S(l, m)$ have any enumerative meaning. For instance, in the case of $S = \mathbb{P}^2$, the number m must be a negative multiple of 3, say $-3d$ for some $d \geq 1$, and then the value of l must be $d(d+3)/2$ for us to have an enumerative problem.

Returning to the subject, if S is a surface with numerically trivial canonical divisor, as in the case of the theorems of Bryan and Leung, $n_r^S(\mathcal{L}^2/2, 0)$ is, for \mathcal{L} sufficiently ample, the number of curves with $\delta = \chi(\mathcal{L}) - r - 1$ nodes in a sublinear system of $|\mathcal{L}|$ of codimension r . By the preceding formulation of Göttsche's conjecture we should have:

$$\sum_{l \in \mathbb{Z}} n_r^S(l, 0) q^l = (DG_2(\tau))^r / \Delta(\tau)$$

if S is a $K3$ surface, as $\chi(\mathcal{O}_S) = 2$ (indeed, $\chi(\mathcal{O}_S) = 1 - q + p_g = 1 - 0 + 1 = 2$), and

$$\sum_{l \in \mathbb{Z}} n_r^S(l, 0) q^l = (DG_2(\tau))^r D^2 G_2(\tau)$$

if S is an abelian surface, as $\chi(\mathcal{O}_S) = 0$ (here $p_g = 1$ and $q = 2$). But this is what the theorems of Bryan and Leung state, so we have a proof of the conjecture of Göttsche in these cases. \square

1.3 Göttsche's conjecture and the polynomials a_i

In the theorem of Kleiman–Piene, the node polynomials $N(\delta, \mathcal{L})$ appear, up to a multiplicative factor $r!$, as the Bell polynomials of polynomials a_i in the four basic Chern numbers ∂, k, s, x . The first eight a_i are constructed as a natural part of the proof of this theorem, but at the moment it is not clear whether there is any pattern in these polynomials. However, assuming the general validity of Göttsche's conjecture, we are able to calculate (at least theoretically) the first 28 a_i . It is hoped that this could lead to a somewhat better understanding of them.

Unfortunately, Göttsche's conjecture involves a power series, $DG_2(\tau)$, instead of simply being of the form $\sum t_\delta^S(\mathcal{L})y^\delta$. To account for this, we put $y = DG_2(\tau)$, which is a power series in q , and use the inversion theorem of Lagrange to express q as a power series in y . More specifically, we have

$$y = DG_2(\tau) = f(q) = \sum_{n=1}^{\infty} n\sigma(n)q^n, \text{ where } \sigma(n) = \sum_{d|n} d.$$

Since $f(0) = 0$ while $f'(0) = 1 \neq 0$, we can invert the series in a neighborhood of 0, and the inverted series has the form

$$q = g(y) = \sum_{n=1}^{\infty} \frac{d^{n-1}}{dq^{n-1}} \left(\frac{q}{f(q)} \right)_{|q=0}^n \frac{y^n}{n!}.$$

While this is hard to work out manually, Maple gives the following expression for $g(y)$ up to order 15:

$$\begin{aligned} g(y) &= y - 6y^2 + 60y^3 - 748y^4 + 10482y^5 - 157740y^6 \\ &+ 2489960y^7 - 40674000y^8 + 681756159y^9 - 11659122666y^{10} \\ &+ 202627975572y^{11} - 3568373043012y^{12} + 63537740326630y^{13} \\ &- 1141968772084740y^{14} + 20690126107206360y^{15} + O(y^{16}) \end{aligned}$$

The commands given were:

```
with(numtheory);
Order:=16;
f:=q->sum(n * sigma(n) * q^n, n = 1..infinity);
solve(series(f(x), x) = y, x);
```

In order to calculate the polynomials a_i for $i \geq 9$, we first obtain the polynomial P_i expressed as a function of ∂, k, s and x . If we know a_j for $j \leq i - 1$ and P_i , the fact that P_i is the i th Bell polynomial in a_1, \dots, a_i suffices to extract the expression of a_i as a polynomial in ∂, k, s and x . But the conjecture of Göttsche now has the form

$$\sum_{\delta=0}^{\infty} t_\delta^S(\mathcal{L})y^\delta = \frac{(y/g(y))^{\frac{\partial-k}{2}} B_1(g(y))^s B_2(g(y))^k}{\left(\Delta(g(y)) D^2 G_2(g(y)) / g(y)^2 \right)^{\frac{s+x}{24}}},$$

where $\partial = \mathcal{L}^2, k = \mathcal{L}\mathcal{K}_S, s = \mathcal{K}_S^2, x = c_2(S)$ (here we consider Δ and D^2G_2 as functions of $q = g(y)$). We are interested in extracting the expressions of the $t_\delta^S(\mathcal{L})$ as polynomials in ∂, k, s, x for $\delta \leq 15$ (for higher values of δ we seem to lack the necessary computer power). The right hand side above involves a certain number of power series in y , but since we are only interested in the $t_\delta^S(\mathcal{L})$ for $\delta \leq 15$, we only need to know their expressions up to order 16. Also note that since the operator D corresponds to $q \frac{d}{dq}$ we have

$$D^2G_2(q) = \sum_{n=1}^{\infty} n^2 \sigma(n) q^n.$$

Using only order 16 expressions, we define in Maple a function $n(y)$ corresponding to the right hand side of Göttsche's conjecture and then extract the polynomials $t_\delta^S(\mathcal{L})$, which are the coefficients in this generating function. Next we proceed, as indicated above, to recursively extract the a_i , collected in the table on the next page.

It is worth noting the following new observation: if we put $a_i = a_i^{(\partial)}\partial + a_i^{(k)}k + a_i^{(s)}s + a_i^{(x)}x$, then each component of a_i is divisible by $(i-1)!$ and that if we let $b_i = a_i/(i-1)! = b_i^{(\partial)}\partial + b_i^{(k)}k + b_i^{(s)}s + b_i^{(x)}x$, then it would seem like we have

$$\lim_{i \rightarrow \infty} \frac{b_i^{(\partial)}}{b_{i-1}^{(\partial)}} \approx -20$$

and similarly for the other components, as indicated in the table below.

i	$\frac{b_i^{(s)}}{b_{i-1}^{(s)}}$	$\frac{b_i^{(k)}}{b_{i-1}^{(k)}}$	$\frac{b_i^{(\partial)}}{b_{i-1}^{(\partial)}}$	$\frac{b_i^{(x)}}{b_{i-1}^{(x)}}$
2	—	-19,5	-14	-7
3	-31,33	-20,21	-16,43	-9,86
4	-25,57	-20,23	-17,48	-9,39
5	-23,61	-20,19	-18,05	-5,43
6	-22,62	-20,14	-18,42	18,77
7	-22,04	-20,11	-18,67	-51,89
8	-21,67	-20,09	-18,86	-29,93
9	-21,4	-20,08	-19,01	-25,54
10	-21,21	-20,07	-19,12	-23,71
11	-21,06	-20,06	-19,21	-22,73
12	-20,95	-20,06	-19,29	-22,13
13	-20,85	-20,06	-19,36	-21,73
14	-20,78	-20,06	-19,41	-21,45
15	-20,72	-20,06	-19,46	-21,24

$a_1 =$	$2k + 3\partial + x$
$a_2 =$	$-6s - 39k - 42\partial - 7x$
$a_3 =$	$376s + 1576k + 1380\partial + 138x$
$a_4 =$	$-28842s - 95670k - 72360\partial - 3888x$
$a_5 =$	$2723400s + 7725168k + 5225472\partial + 84384x$
$a_6 =$	$-308078520s - 778065120k - 481239360\partial + 7918560x$
$a_7 =$	$40747613760s + 93895251840k + 53917151040\partial - 2465471520x$
$a_8 =$	$-6179605765200s - 13206119880240k - 7118400139200\partial + 516524964480x$
$a_9 =$	$1057994510106240s + 2121324101971200k + 1082298739737600\partial - 105531591674880x$
$a_{10} =$	$-201938068481143680s - 383178257123397120k - 186244876934645760\partial + 22522077486397440x$
$a_{11} =$	$42529950621208512000s + 76882882686451430400k + 35785074342095769600\partial - 5120189378609356800x$
$a_{12} =$	$-9799242960045675628800s - 16965814444711292160000k - 7593954156671416934400\partial + 1246637955659688345600x$
$a_{13} =$	$2452287375661994231961600s + 4083791314361072077209600k + 1764002599954269954048000\partial - 325131495890223904358400x$
$a_{14} =$	$-662444750461765046378803200s - 1064857909823340069685248000k - 445196702136181894778880000\partial + 90666752530924449021542400x$
$a_{15} =$	$192137539658526071385289113600s + 299017798634897453079185817600k + 121304301227469541054089216000\partial - 26963216698297962471175987200x$

CHAPTER 2

THE CASE OF \mathbb{P}^2

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In this chapter we consider the particular case of \mathbb{P}^2 . This is the case which classically has been studied the most profoundly. After some observations of how the results in Chapter 1 specialize in the case of \mathbb{P}^2 , we provide a partial proof of a conjecture concerning the shape of the node polynomials $N(\delta, d)$ enumerating δ -nodal curves of degree d (see Proposition 2.1.4). We also look at two other approaches to the problem, both of which are recursive in nature and which generalize the study of rational nodal curves, described in Chapter 3.

2.1 The Severi degree for projective plane curves

We return to the case of \mathbb{P}^2 and the Severi degree $N(\delta, d)$ for plane curves. We have $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(d)$, and we write $t_\delta(d)$ for $t_\delta^{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(d))$ as introduced above. Since $\mathcal{K}_S = \mathcal{O}_{\mathbb{P}^2}(-3)$ we get

$$\mathcal{L}^2 = d^2; \quad \mathcal{L}\mathcal{K}_S = -3d; \quad \mathcal{K}_S^2 = 9; \quad \chi(\mathcal{O}_S) = 1; \quad \chi(\mathcal{L}) = \binom{d+2}{2}$$

This means that since $t_\delta^S(\mathcal{L})$ was a polynomial in ∂, k, s and x , $N(\delta, d)$ will be, for a sufficiently ample linear system (i.e. with certain restrictions on the relation between δ and d), a polynomial in d only. In fact, Göttsche conjectures an upper bound for the validity of the polynomial expression of these nodal numbers:

CONJECTURE 2.1.1 ([Got], Conjecture 4.1) Göttsche's Third Conjecture. *For all $\delta \leq 2d - 2$ we have $N(\delta, d) = t_\delta(d)$, where the polynomial $t_\delta(d)$ appears as a*

coefficient in Göttsche's generating function.

In the following we will assume the validity of the three conjectures of Göttsche mentioned above, and establish a series of results concerning the enumeration of nodal curves in \mathbb{P}^2 that would follow if these conjectures were proved. We will start by making some remarks concerning the universal polynomials $P_i, i \geq 0$. Recall that these are defined as the complete exponential Bell polynomials of the a_i , which are polynomials of degree 1 in ∂, k, s, x defined for $i \geq 1$. It will, however, be of practical interest to define a polynomial $a_0 = 1$. We have the formal identity in t :

$$\sum_{r=0}^{\infty} \frac{P_r t^r}{r!} = \exp \left(\sum_{q=1}^{\infty} \frac{a_q t^q}{q!} \right).$$

LEMMA 2.1.2. For all $r \geq 0, P_r$ is a polynomial of degree r in the $a_i, 0 \leq i \leq r$.

Proof. We have

$$\exp \left(\sum_{q=1}^{\infty} \frac{a_q t^q}{q!} \right) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{q=1}^{\infty} \frac{a_q t^q}{q!} \right)^m = 1 + \sum_{m=1}^{\infty} \left(\sum_{\sum_{i=1}^m i q_i = m} \prod_{j=1}^m \frac{\binom{a_j}{j!}^{q_j}}{q_j!} \right) t^m.$$

By identification we have $P_0 = 1$ and

$$\forall r \geq 0, P_r / r! = \sum_{\sum_{i=1}^r i q_i = r} \prod_{j=1}^r \frac{\binom{a_j}{j!}^{q_j}}{q_j!} = \frac{a_1^r}{r!} + \frac{a_1^{r-2}}{(r-2)!} \cdot \frac{a_2}{2} + \dots$$

We clearly see that P_r expressed as a polynomial in the a_i is constructed from only the first r polynomials a_i . In addition, the leading term is a_1^r , which concludes the proof. (In fact, we get even more; assigning to each polynomial a_i a weight i , we see that P_r is a weighted homogenous polynomial in a_1, \dots, a_r). \square

Note that it follows from this that P_r is a polynomial of degree r in ∂, k, s, x , since Conjecture 1.1.3, as stated in Chapter 1, implies that the a_i are linear polynomials in these variables.

Since $S = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(d)$ we have $\partial = \mathcal{L}^2 = d^2, k = \mathcal{L}\mathcal{K} = -3d, s = \mathcal{K}^2 = 9$ and $x = c_2(\mathbb{P}^2) = 3$ (obtained by Noether's formula), it follows that $N(\delta, d) = P_\delta(d)/\delta!$ is a polynomial in d of degree 2δ whose leading term is $\frac{3^\delta}{\delta!} d^{2\delta}$. Knowing the a_i for $i \leq 8$, it is an easy matter to establish the following (note that these polynomials include the number of reducible curves):

$$\begin{aligned} N(1, d) &= 3d^2 - 6d + 3 \\ N(2, d) &= \frac{9}{2}d^4 - 18d^3 + 6d^2 + \frac{81}{2}d - 33 \\ N(3, d) &= \frac{9}{2}d^6 - 27d^5 + \frac{9}{2}d^4 + \frac{423}{2}d^3 - 229d^2 - \frac{829}{2}d + 525 \end{aligned}$$

$$\begin{aligned}
N(4, d) &= \frac{27}{8}d^8 - 27d^7 + \frac{1809}{4}d^5 - 642d^4 - 2529d^3 + \frac{37881}{8}d^2 + \frac{18057}{4}d - 8865 \\
N(5, d) &= \frac{81}{40}d^{10} - \frac{81}{4}d^9 - \frac{27}{8}d^8 + \frac{2349}{4}d^7 - 1044d^6 - \frac{127071}{20}d^5 + \frac{128859}{8}d^4 \\
&\quad + \frac{59097}{2}d^3 - \frac{3528381}{40}d^2 - \frac{946929}{20}d + 153513 \\
N(6, d) &= \frac{81}{80}d^{12} - \frac{243}{20}d^{11} - \frac{81}{20}d^{10} + \frac{8667}{16}d^9 - \frac{9297}{8}d^8 - \frac{47727}{5}d^7 + \frac{2458629}{80}d^6 \\
&\quad + \frac{3243249}{40}d^5 - \frac{6577679}{20}d^4 - \frac{25387481}{80}d^3 + \frac{6352577}{4}d^2 + \frac{8290623}{20}d - 2699706
\end{aligned}$$

These expressions correspond with the ones obtained by other methods (for instance, the degeneration of \mathbb{P}^2 used by Ran and Choi — see Section 2.3). Direct calculation quickly becomes complicated for large values of δ , but assuming the validity of Göttsche's second conjecture, it is theoretically possible to obtain $N(\delta, d)$ for any value of δ .

From the observation of the $N(\delta, d)$ for low values of δ , Di Francesco and Itzykson originally conjectured (Remark b following Proposition 2 in [DI]), before the works of Göttsche, that $N(\delta, d)$ is a polynomial in d of degree 2δ of the form

$$N(\delta, d) = \frac{3^\delta}{\delta!} \left(q_0(\delta)d^{2\delta} + q_1(\delta)d^{2\delta-1} + \dots \right).$$

Here the q_μ are polynomials in δ of degree μ , more precisely:

$$\begin{aligned}
q_0(\delta) &= 1 \\
q_1(\delta) &= -2\delta \\
q_2(\delta) &= -\frac{1}{3}\delta(\delta - 4) \\
q_3(\delta) &= \frac{1}{6}\delta(\delta - 1)(20\delta - 13) \\
q_4(\delta) &= -\frac{1}{54}\delta(\delta - 1)(69\delta^2 - 85\delta + 92) \\
q_5(\delta) &= -\frac{1}{270}\delta(\delta - 1)(\delta - 2)(702\delta^2 - 629\delta - 286) \\
q_6(\delta) &= \frac{1}{3240}\delta(\delta - 1)(\delta - 2)(6028\delta^3 - 15476\delta^2 + 11701\delta + 4425)
\end{aligned}$$

Of course, these polynomials q_μ are known for all $0 \leq \mu \leq 16$, since the theorem of Kleiman–Piene gives us the expressions of the $N(\delta, d)$ for $\delta \leq 8$. A more general conjecture would be the following:

CONJECTURE 2.1.3. *There exist universal polynomials q_μ in δ for $\mu \geq 0$, such that for $\delta \leq 2d - 2$, $N(\delta, d)$ is a polynomial in d of the form*

$$N(\delta, d) = \frac{3^\delta}{\delta!} \sum_{\mu=0}^{\delta} q_\mu(\delta)d^{2\delta-\mu}.$$

Although there is no complete proof of this conjecture at the moment, we can, assuming the validity of Göttsche's conjectures, prove the following new result:

PROPOSITION 2.1.4. *There exist universal polynomials q_μ (of degree μ) in δ for $0 \leq \mu \leq 6$, such that for $\delta \leq 2d - 2$, $N(\delta, d)$ is a polynomial in d of the form*

$$N(\delta, d) = \frac{3^\delta}{\delta!} \left(q_0(\delta)d^{2\delta} + q_1(\delta)d^{2\delta-1} + \dots + q_6(\delta)d^{2\delta-6} + \dots \right)$$

where the remaining terms are considered unknown. These polynomials are the ones listed above.

Proof. The definition of the complete exponential Bell polynomials P_i given above is equivalent to a recursive one (see Appendix B), given below:

$$\begin{aligned} P_0 &= 1 \\ \forall r \geq 0, P_{r+1}(a_1, \dots, a_{r+1}) &= \sum_{k=0}^r \binom{r}{k} P_{r-k}(a_1, \dots, a_{r-k}) a_{k+1} \end{aligned}$$

For $i \geq 1$, a_i is a quadratic polynomial in d , that is: $a_i(d) = \alpha_i d^2 + \beta_i d + \gamma_i \in \mathbb{Z}[d]$. Express $P_r(d)$ as a polynomial in d . We wish to show that $P_r(d) = 3^r \sum_{\mu=0}^{2r} q_\mu(r) d^{2r-\mu}$ where q_μ is a polynomial of degree μ for $\mu \leq 6$. This is obviously true for $r \leq 8$. Now let $r \geq 8$ and assume it is true for all numbers $\leq r$. We then have, using the recursive formula above:

$$\begin{aligned} P_{r+1}(d) &= \sum_{k=0}^r \binom{r}{k} \left(3^{r-k} \sum_{\mu=0}^{2(r-k)} q_\mu(r-k) d^{2(r-k)-\mu} (\alpha_{k+1} d^2 + \beta_{k+1} d + \gamma_{k+1}) \right) \\ &= \sum_{\mu=0}^{2r} \sum_{k=0}^{\lfloor r-\mu/2 \rfloor} \binom{r}{k} 3^{r-k} q_\mu(r-k) \left(\alpha_{k+1} d^{2(r+1)-(2k+\mu)} \right. \\ &\quad \left. + \beta_{k+1} d^{2(r+1)-(2k+\mu+1)} + \gamma_{k+1} d^{2(r+1)-(2k+\mu+2)} \right) \\ &= \sum_{j=0}^{2(r+1)} d^{2(r+1)-j} \left(\sum_{2k+\mu=j} \binom{r}{k} 3^{r-k} q_\mu(r-k) \alpha_{k+1} \right. \\ &\quad \left. + \sum_{2k+\mu+1=j} \binom{r}{k} 3^{r-k} q_\mu(r-k) \beta_{k+1} + \sum_{2k+\mu+2=j} \binom{r}{k} 3^{r-k} q_\mu(r-k) \gamma_{k+1} \right) \end{aligned}$$

We may write this as:

$$P_{r+1}(d) = \sum_{j=0}^{2(r+1)} d^{2(r+1)-j} \left(\sum_{k=0}^{\lfloor j/2 \rfloor} \binom{r}{k} 3^{r-k} q_{j-2k}(r-k) \alpha_{k+1} \right)$$

$$\begin{aligned}
& + \sum_{k=0}^{\lfloor (j-1)/2 \rfloor} \binom{r}{k} 3^{r-k} q_{j-2k-1} (r-k) \beta_{k+1} \\
& + \sum_{k=0}^{\lfloor j/2-1 \rfloor} \binom{r}{k} 3^{r-k} q_{j-2k-2} (r-k) \gamma_{k+1}
\end{aligned}$$

Of course, where the index sets are empty the sums are taken to be 0, and it is understood that we introduce zero polynomials q_k for $k < 0$, but these are minor obstacles. More importantly, if we had done the same procedure for P_s for some $s \leq r$, we would have ended up with a similar expression, only replacing $r+1$ with s . This allows us to conclude that the functions q_μ satisfy the following: For all $1 \leq s \leq r$ and all $0 \leq j \leq 2s$ we must have (since $P_s(d) = \sum_{\mu=0}^{2s} 3^s q_\mu(s) d^{2s-\mu}$) the following equality:

$$3^s q_j(s) = \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{s-1}{k} 3^{s-1-k} q_{j-2k} (s-1-k) \alpha_{k+1} + \sum \dots + \sum \dots$$

or, dividing both sides by 3^s :

$$q_j(s) = \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{s-1}{k} \frac{1}{3^{k+1}} q_{j-2k} (s-1-k) \alpha_{k+1} + \sum \dots + \sum \dots$$

our aim being, of course, to show that for $j \leq 6$ we have

$$q_j(r+1) = \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{r}{k} \frac{1}{3^{k+1}} q_{j-2k} (r-k) \alpha_{k+1} + \sum \dots + \sum \dots$$

What we know is that the polynomial in z

$$\Omega_j(z) = q_j(z) - \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{z-1}{k} \frac{1}{3^{k+1}} q_{j-2k} (z-1-k) \alpha_{k+1} - \sum \dots - \sum \dots$$

has a certain number of zeros (since $\binom{z-1}{k} = \frac{(z-1)(z-2)\dots(z-k)}{k!}$ this is indeed a polynomial). On the other hand, since $\alpha_1 = 3$ this polynomial has degree $\leq j-1$. We have a zero s for $\Omega_j(z)$ for each $\max(1, \lceil j/2 \rceil) \leq s \leq r$, that is, we have $r - \max(1, \lceil j/2 \rceil) + 1$ zeros for $\Omega_j(z)$, $0 \leq j \leq 2r$. Since $r \geq 8$ and $j \leq 6$ we have $\lceil j/2 \rceil \leq 3$, so $r - \max(1, \lceil j/2 \rceil) + 1 \geq r - 3 + 1 = r - 2 \geq 6$. This means that $\Omega_j(z)$ has a number of zeros greater than its degree, which is $\leq j-1 \leq 5$, so it is the zero polynomial, and we conclude that we indeed must have

$$q_j(r+1) = \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{r}{k} \frac{1}{3^{k+1}} q_{j-2k} (r-k) \alpha_{k+1} + \sum \dots + \sum \dots$$

for $j \leq 6$. This concludes the proof. \square

REMARK 2.1.5. Note that for all μ , the polynomials q_μ in δ are of the following form

$$q_\mu(\delta) = \frac{1}{\mu!3^{\lfloor \mu/2 \rfloor}} \frac{\delta!}{(\delta - \lfloor \mu/2 \rfloor)!} Q_\mu(\delta)$$

where $Q_\mu(\delta)$ is a polynomial with integer coefficients and degree $\lfloor \mu/2 \rfloor$, such that the only common factors of its terms are powers of 2 and 3. This can be related to Göttsche's comment on p. 530 in [Got], where his $p_\mu(\delta)$ is $3^\delta q_\mu(\delta)$. However, he defines $\lfloor \cdot \rfloor$ to be the integer part, so his denominator should contain the factor $(\delta - \lfloor \mu/2 \rfloor)!$ instead of $(\delta - \lceil \mu/2 \rceil)!$

2.2 Recursive formulas for $N(\delta, d)$

REMARK 2.2.1. For a plane irreducible curve of degree d having δ ordinary nodes (these being the only singularities), the genus g is given by the genus formula:

$$g = \frac{(d-1)(d-2)}{2} - \delta$$

This implies that $\frac{d(d+3)}{2} - \delta = 3d - 1 + g$, so instead of considering curves of a given number of nodes, we might as well consider curves of a certain genus g : let $N_g(d)$ denote the number of such curves. For $g = 0$, the theory of *quantum cohomology* (see Chapter 3) has yielded the celebrated recursive formula of Kontsevich, giving the number of rational curves of degree d and passing through $3d - 1$ general points by:

$$\begin{aligned} N_0(1) &= 1 \text{ and for all } d \geq 2 \\ N_0(d) &= \sum_{d_A+d_B=d} N_0(d_A)N_0(d_B) \left[d_A^2 d_B^2 \binom{3d-4}{3d_A-2} - d_A^3 d_B \binom{3d-4}{3d_A-1} \right] \end{aligned}$$

Note that if we introduce the quantity $n_d = \frac{N_0(d)}{(3d-1)!}$ the formula reads

$$n_d = \sum_{d_A+d_B=d} n_{d_A} n_{d_B} \frac{d_A d_B \left[(3d_A - 2)(3d_B - 2)(d + 2) + 8(d - 1) \right]}{6(3d - 1)(3d - 2)(3d - 3)}$$

and we have a perfect symmetry in d_A and d_B . This will be developed in Chapter 3.

The following table shows the first values given by the recursive formula:

d	$N_0(d)$	$3d - 1$
1	1	2
2	1	5
3	12	8
4	620	11
5	87304	14
6	26312976	17
7	14616808192	20
8	13525751027392	23
9	19385778269260800	26

In [DI], Proposition 3, Di Francesco and Itzykson show that, asymptotically, we have

$$N_0(d) = (3d - 1)! a^d d^{-7/2} b (1 + O(d^{-1})), a \approx 0.138 \text{ and } b \approx 6.1$$

The formula of Kontsevich can to a certain degree be generalized to account for plane curves of any genus g . References are [Cap] and [CH].

DEFINITION 2.2.2. Let $\alpha = (\alpha_1, \dots, \alpha_h)$ and $\beta = (\beta_1, \dots, \beta_k)$ be finite strings of non-negative integers, henceforth referred to as *multiplicity strings*. By $|\alpha|$ and $|\beta|$ we mean the sum of the respective strings' components. Fix $|\alpha|$ general points on a fixed line $L \subset \mathbb{P}^2$, $\{p_j^{(i)}\}_{1 \leq j \leq \alpha_i}$. Assume further that $\sum i\alpha_i + \sum i\beta_i = d$. We define the *generalized Severi variety* $V(\delta, d)[\alpha, \beta]$ as the closure of the locus of reduced (but possibly reducible) plane curves C of degree d and having δ nodes (and therefore of genus $g = \binom{d-1}{2} - \delta$) which meet the following requirements (note that Bézout's theorem is fulfilled):

- L is not contained in C ;
- L meets C with order i , $1 \leq i \leq h$, in the points $p_j^{(i)}$, $1 \leq j \leq \alpha_i$;
- L meets C with order i in β_i non-fixed points, $1 \leq i \leq k$.

It can be shown that each irreducible component of $V(\delta, d)[\alpha, \beta]$ has the expected dimension

$$r(\delta, d)[\alpha, \beta] := \dim V(\delta, d)[\alpha, \beta] = \frac{d(d+3)}{2} - \delta - \sum i\alpha_i - \sum (i-1)\beta_i = 2d + g - 1 + |\beta|$$

Also, generally speaking, a closed non-singular point in $V(\delta, d)[\alpha, \beta]$ parametrizes a curve that has only nodes as singularities, that is smooth along L and meets the intersection requirements above.

Now let $N(\delta, d)[\alpha, \beta]$ denote the degree of $V(\delta, d)[\alpha, \beta]$. We use the standard notations $\alpha! = \prod \alpha_i!$ and $\binom{\alpha}{\alpha'} = \prod \binom{\alpha_i}{\alpha'_i}$ for multiplicity strings α and α' . If $S = \{s_1, s_2, \dots\}$ is an ordered set, we define $S^\alpha = \prod s_i^{\alpha_i}$. Let $\epsilon^{(j)}$ denote the family of

integers indexed by \mathbb{N} with a 1 in the j th position and with zeros everywhere else.

THEOREM 2.2.3 ([CH], Theorem 1.1) Caporaso–Harris. *Let $\Lambda(\alpha, \beta, \delta)$ denote the set of triples $(\alpha', \beta', \delta')$ consisting of two multiplicity strings α', β' and one integer δ' , such that we have $\alpha' \leq \alpha, \beta' \geq \beta, \delta' \leq \delta$ and $\delta - \delta' + |\beta' - \beta| = d - 1$. We have the following recursive formula for $N(\delta, d)[\alpha, \beta]$,*

$$N(\delta, d)[\alpha, \beta] = \sum_{j: \beta_j > 0} jN(\delta, d)[\alpha + \epsilon^{(j)}, \beta - \epsilon^{(j)}] + \sum_{\Lambda(\alpha, \beta, \delta)} \mathbb{N}^{\beta' - \beta} \binom{\alpha}{\alpha'} \binom{\beta'}{\beta} N(\delta', d-1)[\alpha', \beta']$$

Note that one could add the requirement that $\sum i\alpha'_i + \sum i\beta'_i = d - 1$ in $\Gamma(\alpha, \beta, \delta)$ in order for Bézout's theorem to be fulfilled. Taking $\alpha = 0$ and $\beta = (d, 0, \dots)$, we obtain the degree of the closure of the variety of all (possibly reducible) plane curves having degree d and δ nodes. The classical Severi variety, however, parametrizes the irreducible curves only, and we get its degree by subtracting the excessive degree of the locus of reducible curves, known recursively. It is, in fact, possible to obtain a formula for irreducible curves only, but it is somewhat more complicated, so the simplest way is to use the formula above and subtract the excess.

EXAMPLE 2.2.4. The following example is taken from [CH], pp. 348–349. We will calculate the degree of the Severi variety of quartics with three nodes, assuming known the degrees of the generalized Severi varieties parametrizing cubics that satisfy certain tangency conditions. We write $(\delta, d, \alpha, \beta)$ for $N(\delta, d)[\alpha, \beta]$ and suppress any zeroes at the end of the sequences α and β , as well as the parentheses around sequences α, β of length 1, for the sake of simplifying the notation. Now, if we intersect the variety $V(3, 4)[0, 4]$ with five successive hyperplanes of the form H_p (i.e. hyperplanes corresponding to a general point p on the fixed line L) we get (whenever components appear of which we supposedly know the contribution to the degree, we write this in angle brackets):

$$\begin{aligned} (3, 4, 0, 4) &= (3, 4, 1, 3) \\ &= (3, 4, 2, 2) \\ &\quad + (0, 3, 0, 3) \langle 1 \rangle \\ (3, 4, 2, 2) &= (3, 4, 3, 1) \\ &\quad + 3(1, 3, 0, 3) \langle 3 \times 12 = 36 \rangle \\ &\quad + 2(0, 3, 1, 2) \langle 2 \times 1 = 2 \rangle \\ (3, 4, 3, 1) &= (3, 4, 4, 0) \\ &\quad + 3(2, 3, 0, 3) \langle 3 \times 21 = 63 \rangle \\ &\quad + 2(1, 3, 0, (1, 1)) \langle 2 \times 36 = 72 \rangle \\ &\quad + 6(1, 3, 1, 2) \langle 6 \times 12 = 72 \rangle \\ &\quad + 3(0, 3, 2, 1) \langle 3 \times 1 = 3 \rangle \\ (3, 4, 4, 0) &= (3, 3, 0, 3) \langle 15 \rangle \\ &\quad + 4(2, 3, 1, 3) \langle 4 \times 21 = 84 \rangle \end{aligned}$$

$$\begin{aligned}
&+2(2, 3, 0, (1, 1))\langle 2 \times 30 = 60 \rangle \\
&+6(1, 3, 2, 1)\langle 6 \times 12 = 72 \rangle \\
&+8(1, 3, 1, (0, 1))\langle 8 \times 16 = 128 \rangle \\
&+3(1, 3, 0, (0, 0, 1))\langle 3 \times 21 = 63 \rangle \\
&+4(0, 3, 3, 0)\langle 4 \times 1 = 4 \rangle
\end{aligned}$$

Adding it all up we get $(3, 4, 0, 4) = 675$, but the Severi variety $V(3, 4)[0, 4]$ actually has two irreducible components of dimension 11, one that corresponds to the classical variety parametrizing *irreducible 3-nodal quartics*, and one that parametrizes curves equal to the *reducible union of a line and a cubic*, this one having degree $\binom{11}{2} = 55$. Thus the classical Severi variety has degree $675 - 55 = 620$.

REMARK 2.2.5. The results above by Caporaso–Harris make it possible to give theoretical expressions for the universal polynomials $B_1(q), B_2(q)$ in higher degrees than what the theorem of Kleiman and Piene guarantees. This made it possible for Göttsche to calculate the coefficients of these power series up to degree 28. We also get a confirmation of the expressions of the $p_\mu(\delta) = 3^\delta q_\mu(\delta)$ for all $\mu \leq 28$.

In [Vak] Ravi Vakil presents a generalization of the formula of Caporaso–Harris to account for the problem on any Hirzebruch surface. His paper translates the degeneration methods of Caporaso–Harris into the language of stable maps, and concludes with the same formula, only replacing the line L with the exceptional divisor E .

It could be interesting to relate the recursive formula of Caporaso–Harris to a differential equation (much as we will do in Chapter 3, when showing that the recursive formula of Kontsevich is equivalent to a differential equation involving the generating function of the numbers $N_0(d)$). In [Get], Getzler observes the following: Letting z be a variable and $u = (u_1, u_2, \dots)$ and $v = (v_1, v_2, \dots)$ be sets of variables, we may define the following generating function:

$$G(z, u, v) = \sum_{\delta, d, \alpha, \beta} \frac{u^\alpha}{\alpha!} v^\beta N(\delta, d)[\alpha, \beta] \frac{z^{r(\delta, d)[\alpha, \beta]}}{r(\delta, d)[\alpha, \beta]}.$$

PROPOSITION 2.2.6 ([Get], pp. 19–20) Getzler. *The recursive formula of Caporaso–Harris is equivalent to the following differential equation:*

$$\frac{\partial G}{\partial z} = \sum_{k=0}^{\infty} k v_k \frac{\partial G}{\partial u_k} + \text{Res}_{t=0} \exp \left(\sum_{k=0}^{\infty} \frac{u_k}{t^k} + \sum_{k=0}^{\infty} k t^k \frac{\partial}{\partial u_k} \right) G,$$

where $\text{Res}_{t=0}$ is the residue with respect to the variable t , meaning the coefficient of t^{-1} when the exponential is expanded.

2.3 Node polynomials and the degeneration of \mathbb{P}^2

We have seen how the theorem of Kleiman–Piene (and more generally, Göttsche’s second conjecture) yields the expression of $N(\delta, d)$ for low enough values of δ com-

pared to d . In addition to the recursive methods examined above, there are also other approaches to the enumeration of nodal, plane curves. One in particular has had some success for low values of δ ; first used by Ran, it was later brought further by his student Choi. It is based on a particular method of degenerating the complex projective plane into a union of two surfaces, a top and a bottom component, intersecting transversally along an exceptional line. This method allows us to establish a recursive, enumerative procedure in which the degree of the considered curve is diminished for each step.

References are [Ran1], [Ran2], [Cho1] and [Cho2], but it should be noted that these articles are not always very clear. What follows should be considered an attempt to clarify *some* of the ideas and to highlight the most important results as well as the main ideas of the proofs, which do not seem to be entirely correct at all times. For instance, Choi's proof of our Proposition 2.3.3 includes an induction on the *variable* d of a polynomial, while the statement he attempts to prove concerns the *degree* of this polynomial.

Below we will detail the process of degenerating \mathbb{P}^2 to a reducible surface. We will follow the outline of Stephanie Yang in [Yan], Section 3.1.

DEFINITION 2.3.1. Put $V = \mathbb{C} \times \mathbb{P}^2$ with the two projections p_1, p_2 to \mathbb{C} and \mathbb{P}^2 respectively. Also, put $V_t = \{t\} \times \mathbb{P}^2, t \in \mathbb{C}$. Blowing up a point $(0, p)$ in $\mathbb{C} \times \mathbb{P}^2$ we get a 3-fold X with a birational morphism $f : X \rightarrow V$ and, composing with the projections, maps $\pi_1 : X \rightarrow \mathbb{C}$ and $\pi_2 : X \rightarrow \mathbb{P}^2$. The morphism $\pi_1 : X \rightarrow \mathbb{C}$ is a flat family of surfaces over \mathbb{C} .

We let $X_t = \pi_1^{-1}(t)$, such that if $t \neq 0, X_t = V_t \cong \mathbb{P}^2$ whereas $X_0 = \mathbb{P} \cup_R \mathbb{F}$ where $\mathbb{P} \cong \mathbb{P}^2$ is the *exceptional divisor* of the blowing-up and $\mathbb{F} \cong \mathbb{F}_1$ is a Hirzebruch space, isomorphic to \mathbb{P}^2 blown up at a point. They *intersect transversally* along a divisor $R = \mathbb{P} \cap \mathbb{F}$ which is a general line L in \mathbb{P} and the exceptional divisor E on \mathbb{F} . The following diagram gives a summary of the situation:

$$\begin{array}{ccc}
 \mathbb{P} \cup_R \mathbb{F} = X_0 & \longrightarrow & X \\
 \downarrow & & \downarrow f \\
 & & \mathbb{C} \times \mathbb{P}^2 \\
 & & \downarrow p_1 \\
 0 & \longrightarrow & \mathbb{C} \\
 & & \searrow p_2 \\
 & & \mathbb{P}^2
 \end{array}$$

π_2 (arrow from X to \mathbb{P}^2)
 p_2 (arrow from $\mathbb{C} \times \mathbb{P}^2$ to \mathbb{P}^2)

Let H denote the pullback to \mathbb{F} of the class of a general line in \mathbb{P}^2 (by the birational blowing-up transformation defining \mathbb{F}). Then $\text{Pic}(\mathbb{F})$ is freely generated by the two divisors H and E . We have $\text{Pic}(X_0) = \text{Pic}(\mathbb{P}) \times \text{Pic}(\mathbb{F})$. So giving a line bundle Ξ on X_0 is equivalent to giving a line bundle $\Xi_{\mathbb{P}}$ on \mathbb{P} and a line bundle $\Xi_{\mathbb{F}}$ on \mathbb{F} having same restriction on R . This implies that $\Xi_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}^2}(d)$ and $\Xi_{\mathbb{F}} = \mathcal{O}_{\mathbb{F}_1}(cH - dE)$ for some values c, d , since $E^2 = -1$. This line bundle on X_0 is denoted by $\Xi(c, d)$.

Define $\Psi(c, d) = \pi_2^* \mathcal{O}_{\mathbb{P}^2}(c) \otimes \mathcal{O}_X(-d\mathbb{P})$. This bundle restricts to $\mathcal{O}_{\mathbb{P}^2}(d)$ on X_t and to $\Xi(c, d)$ on X_0 , and further to $\mathcal{O}_{\mathbb{P}^2}(d)$ on \mathbb{P} and to $\mathcal{O}_{\mathbb{F}_1}(cH - dE)$ on \mathbb{F} , since the restriction of $\mathcal{O}_X(\mathbb{P})$ to \mathbb{P} is $\mathcal{O}_{\mathbb{P}^2}(-1)$, and its restriction to \mathbb{F} is $\mathcal{O}_{\mathbb{F}_1}(E)$.

Next we need to know what happens to curves in \mathbb{P}^2 when the plane degenerates to X_0 .

DEFINITION 2.3.2. Consider a projective surface S . Let $\mathcal{L}_S^d(m_1^{k_1}, \dots, m_s^{k_s})$ denote the linear system of curves of degree d passing through $\sum k_i$ fixed general points, k_i of which have multiplicity m_i . Let l_1, \dots, l_s be another sequence of positive integers such that $l_i \leq k_i$ for all i . If we have $\sum k_i$ general points in the reducible fiber X_0 , with l_i of the m_i -fold points in \mathbb{F} and the remaining in \mathbb{P} , the points may be considered as limits of a family of multiple points in general position in nearby fibers X_t . Now let $\mathcal{L}_0^{(a,b)}$ be the linear system of divisors in $|\Psi(a, b)|$ which vanish at these multiple points in X_0 . Then this linear system restricts to the components of X_0 as follows:

$$\begin{aligned} \mathcal{L}_{\mathbb{P}} &\cong \mathcal{L}_{\mathbb{P}}^b(m_1^{k_1-l_1}, \dots, m_s^{k_s-l_s}) \\ \mathcal{L}_{\mathbb{F}} &\cong \mathcal{L}_{\mathbb{F}}^a(m_1^{l_1}, \dots, m_s^{l_s}, b) \end{aligned}$$

(the second equation comes from blowing down the (-1) -curve E) and we see that $\mathcal{L}_0^{(a,b)}$ can be considered as the flat limit of systems $\mathcal{L}_{\mathbb{P}^2}^a(m_1^{k_1}, \dots, m_s^{k_s})$ on X_t as t approaches 0. We say that $\mathcal{L}_0^{(a,b)}$ is obtained from $\mathcal{L}_{\mathbb{P}^2}^a(m_1^{k_1}, \dots, m_s^{k_s})$ by an (a, b) -degeneration. Returning to the Severi problem of nodal curves in \mathbb{P}^2 , consider a curve C in the linear system of degree d curves passing through $k_1 = d(d+3)/2$ simple general points on \mathbb{P}^2 and having, in addition, δ nodes. Using the above ideas and doing a $(d, d-1)$ -degeneration while placing $l_1 = d+1$ of the restriction points in \mathbb{F} and the rest in \mathbb{P} , we obtain curves $C_{\mathbb{P}}$ of degree $d-1$ passing through $\binom{d+1}{2} - 1 - \delta$ points in \mathbb{P} and $C_{\mathbb{F}}$ through $d+1$ points in \mathbb{F} .

More specifically Ran shows (under the assumption $\delta < d$) in [Ran1], 3 (but it is more clearly stated in [Cho2], 2.2), that a general curve C in the linear system of degree d curves having δ nodes in \mathbb{P}^2 may $(d, d-1)$ -degenerate in a way such that:

1. $C \rightarrow C_{\mathbb{P}} \cup C_{\mathbb{F}}$;
2. $C_{\mathbb{P}}$ is a nodal curve with δ_1 nodes in the divisor class $|(d-1)L|$ on \mathbb{P} ;
3. $C_{\mathbb{F}}$ is a nodal curve with δ_2 nodes in the divisor class $|dH - (d-1)E|$ on \mathbb{F} ;
4. $D = C_{\mathbb{P}} \cap R = C_{\mathbb{F}} \cap R$ is a divisor on $\mathbb{P}^1 \cong E$;
5. $C_{\mathbb{P}}$ and $C_{\mathbb{F}}$ are both smooth near E ;
6. if the divisor D has r_i points of multiplicity i (we say it has type π , where π is the multiplicity string (r_1, \dots, r_n)), then we have $\delta_1 + \delta_2 + \sum_{i=1}^n (i-1)r_i = \delta$ (and of course $\sum ir_i = d-1$). Said otherwise, the δ original nodes may

degenerate into nodes on the surfaces or intersection conditions on the axis R . We let Γ_δ denote the set of triples $(\delta_1, \delta_2, \pi)$ such that the equations above hold.

The crucial point is that the curve $C_{\mathbb{P}}$ has lower degree than C , allowing us to establish a recursive procedure. Also, the curve $C_{\mathbb{F}}$ is of a very special form ([Cho1], 3.2):

$$C_{\mathbb{F}} = C_{\mathbb{F},0} + \sum_{i=1}^{\delta_2} R_i,$$

where $C_{\mathbb{F},0}$ is a *smooth rational curve* and the R_i are *distinct rulings* on the ruled surface \mathbb{F} , intersecting this rational curve in the nodes of $C_{\mathbb{F}}$ (which is a reducible curve).

For each of these possible degenerations (i.e. for each appropriate δ_1, δ_2 and multiplicity string $\pi = (r_1, \dots, r_n)$) we get a certain number of possible configurations in the upper part (\mathbb{P}) and in the lower part (\mathbb{F}). For instance, the number of possible configurations in \mathbb{P} is equal to the degree of the Severi variety parametrizing δ_1 -nodal curves of degree $d - 1$ on \mathbb{P}^2 intersecting the general line L in a divisor of type (r_1, \dots, r_n) and passing through an appropriate number of general points. Similarly, the number of possible configurations in the lower part equals the degree of the Severi variety parametrizing the curves of given nodal number, in the appropriate divisor class and with the correct intersection type with E .

The total number of possible configurations for the degenerated curve in this case is the product of the degrees of the upper and the lower Severi varieties, $V(\delta_1, \delta_2, \pi)$ and $V'(\delta_1, \delta_2, \pi)$. Summing all these numbers (for all possible $(\delta_1, \delta_2, \pi) \in \Gamma_\delta$) we get the number $N(\delta, d)$. So

$$N(\delta, d) = \sum_{(\delta_1, \delta_2, \pi) \in \Gamma_\delta} \deg V(\delta_1, \delta_2, \pi) \cdot \deg V'(\delta_1, \delta_2, \pi).$$

We may also consider the locus of nodal curves intersecting a general line in a divisor of fixed type: Fix integers $0 \leq \delta < d$ and a finite sequence of non-negative integers $(\beta_2, \dots, \beta_n)$ such that $\sum_{i=2}^n i\beta_i \leq d$. Let $\beta_1(d) = d - \sum_{i=2}^n i\beta_i$ and $\beta(d) = (\beta_1(d), \beta_2, \dots, \beta_n)$. Let $N(\delta, d)[0, \beta(d)]$ (same notation as in the recursion of Caporaso–Harris) denote the number of curves in \mathbb{P}^2 of degree d having δ nodes, such that the intersection with a fixed general line L' in \mathbb{P}^2 defines a *divisor of type* $\beta(d)$ (i.e. β_i intersection points of multiplicity i , $1 \leq i \leq n$) and such that it passes through $\binom{d+2}{2} - 1 - \delta - \sum_{i=1}^n (i-1)\beta_i$ general points. It follows from the generalized formula of Kontsevich (proved by Caporaso–Harris) referred to above that $N(\delta, d)[0, \beta(d)]$ is a polynomial in d .

PROPOSITION 2.3.3 ([Cho1], Proposition 4.2). *The degree of the polynomial $N(\delta, d)[0, \beta(d)]$ is $2\delta + \sum_{i=2}^n \beta_i$.*

Proof. We will only sketch the main ideas, and refer to [Cho1] and [Cho2] for the details. The proof is done by induction on δ . If $\delta = 0$ we are considering smooth curves of degree d intersecting L' in a divisor of type $\beta(d)$. It follows from Corollary 2.3 in [Cho1] that the number of such curves is given by $m(\beta(d))n(\beta(d))$ where, for a multiplicity string $\pi = (l_1, \dots, l_n)$,

$$m(\pi) = \prod_{i=1}^n i^{l_i} \text{ and } n(\pi) = \frac{(\sum_{i=1}^n l_i)!}{l_1! \dots l_n!}.$$

But then we get a number of curves equal to

$$\begin{aligned} N(0, d)[0, \beta(d)] &= \prod_{i=2}^n i^{\beta_i} \cdot \frac{(d - \sum_{i=2}^n i\beta_i + \sum_{i=2}^n \beta_i)!}{(d - \sum_{i=2}^n i\beta_i)! \beta_2! \dots \beta_n!} \\ &= \frac{\prod_{i=2}^n i^{\beta_i}}{\prod_{i=2}^n \beta_i!} \underbrace{\left(d - \sum_{i=2}^n (i-1)\beta_i \right) \left(d - \sum_{i=2}^n (i-1)\beta_i - 1 \right) \dots}_{\sum_{i=2}^n \beta_i \text{ factors}} \end{aligned}$$

It follows that we have a polynomial in d of degree $\sum_{i=2}^n \beta_i$, as we wished.

For the inductive step, we degenerate \mathbb{P}^2 to $\mathbb{P} \cup \mathbb{F}$ and distribute the points so that $n_1 = \binom{d+1}{2} - 1 - \delta$ points are placed in \mathbb{P} , $n_2 = (d+1) - \sum_{i=1}^n (i-1)\beta_i$ points are placed in \mathbb{F} , and such that we degenerate the tangency conditions with L' to \mathbb{F} (that is, we require that the curve $C_{\mathbb{F}}$ intersect a general line in \mathbb{F} in a divisor of type $\beta(d)$). This results in a number of different possible configurations, depending on the distribution of nodes between $C_{\mathbb{P}}$ and $C_{\mathbb{F}}$ — each configuration corresponds to a Severi polynomial which is the product of the Severi polynomial of the upper part ($C_{\mathbb{P}}$) and the Severi polynomial of the lower part ($C_{\mathbb{F}}$). We want to show that the degree of each such polynomial is less than or equal to (and, in at least one case, equal to) $2\delta + \sum_{i=2}^n \beta_i$.

There are two possibilities: $\delta_1 = \delta$ or $\delta_1 < \delta$. In the first case we get the total Severi polynomial $m(\beta(d))n(\beta(d)) \cdot N(\delta, d-1)[0, (d-1)]$, which is a polynomial in d of degree $2\delta + \sum_{i=2}^n \beta_i$ (same argument as above, plus a general result on standard Severi varieties). If $\delta_1 < \delta$ and $C_{\mathbb{P}}$ intersects R in a divisor of type $\beta' = (\beta'_1, \dots, \beta'_m)$, with the requirement

$$\delta_1 + \sum_{i=1}^m (i-1)\beta'_i = \delta - \delta_2,$$

we have, by the inductive assumption, that the Severi polynomial of this upper part has degree $2\delta_1 + \sum_{i=2}^m \beta'_i$.

For the lower part, the crucial point is that a possible curve $C_{\mathbb{F}}$ has a simple configuration, as we have seen. It is a smooth rational curve with δ_2 rulings defining its nodes; it intersects the axis $R = E$ in a divisor of type β' , it intersects a general

line in \mathbb{F} in a divisor of type β , and it passes through n_2 general points on \mathbb{F} , where $n_2 = (d+1) - \sum_{i=1}^k (i-1)\beta_i$. The degree of the Severi variety parametrizing such curves is $\delta_2 + \sum_{i=2}^n \beta_i$.

The degree of the total Severi polynomial is thus less than or equal to $\delta_2 + \sum_{i=2}^n \beta_i + 2\delta_1 + \sum_{i=2}^m \beta'_i$. From the equation $\delta_1 + \sum_{i=1}^m (i-1)\beta'_i = \delta - \delta_2$ we get:

$$\begin{aligned} \delta_2 + \sum_{i=2}^n \beta_i + 2\delta_1 + \sum_{i=2}^m \beta'_i &\leq \delta_2 + \sum_{i=2}^n \beta_i + \delta_1 + \underbrace{\delta_1 + \sum_{i=1}^m (i-1)\beta'_i}_{\delta - \delta_2} \\ &= \delta + \delta_1 + \sum_{i=2}^n \beta_i \leq 2\delta + \sum_{i=2}^n \beta_i \end{aligned} \quad (*)$$

with equality only for the case where $C_{\mathbb{P}}$ has δ nodes and $C_{\mathbb{F}}$ is a smooth rational curve. This completes the proof. \square

PROPOSITION 2.3.4 ([Cho1], Corollary 4.3 and 4.5). *Consider the Severi polynomials $N(\delta, d) = \sum_{\mu=0}^{2\delta} p_{\mu}(\delta)d^{2\delta-\mu}$, which are of degree 2δ by the proposition above. Then we have (1) $p_0(\delta) = \frac{3^{\delta}}{\delta!}$ and (2) $p_1(\delta) = -\frac{2 \cdot 3^{\delta}}{(\delta-1)!}$.*

Proof. (1) is based on the recursive formula $p_0(\delta) \cdot 2\delta = 6p_0(\delta-1)$. In the proof of the proposition above we saw that we had equality only for one configuration, the one where $C_{\mathbb{P}}$ has δ nodes and degree $d-1$, whereas $C_{\mathbb{F}}$ is a smooth rational curve. The Severi polynomial of this configuration is thus $N(\delta, d-1)$.

The next step is to consider the configurations where the degree of the Severi polynomial is one less. By equation (*) there are only two such: the one where $C_{\mathbb{P}}$ has $\delta-1$ nodes and $C_{\mathbb{F}}$ has one ruling, and the one where $C_{\mathbb{P}}$ has $\delta-1$ nodes, intersecting R in a divisor of type $(d-3, 1)$, and $C_{\mathbb{F}}$ is a smooth rational curve. The coefficient of degree $2\delta-1$ of the polynomial of the first component is $2p_0(\delta-1)$, and for the second component it is $4p_0(\delta-1)$, so the sum is $6p_0(\delta-1)$ and we get the recursion formula $N(\delta, d) = N(\delta, d-1) + 6p_0(\delta-1)d^{2\delta-1} + \dots$ where the remaining terms have lower degrees. Consider terms of degree $2\delta-1$ on both sides, we get:

$$p_1(\delta)d^{2\delta-1} = -p_0(\delta) \cdot 2\delta d^{2\delta-1} + p_1(\delta)d^{2\delta-1} + 6p_0(\delta-1)d^{2\delta-1}$$

which yields $p_0(\delta) \cdot 2\delta = 6p_0(\delta-1)$.

The proof of (2) is somewhat similar; here we have to consider the components having a polynomial of degree two less than the degree of the complete Severi polynomial. Counting contributions, we end up with the formula

$$N(\delta, d) = N(\delta, d-1) + 6p_2(\delta)d^{2\delta-1} - 9p_1(\delta-1) \cdot (\delta-2)d^{2\delta-2} + \dots$$

and identifying coefficients like above, we get $N(\delta, d) = p_0(\delta)d^{2\delta} - 2\delta p_0(\delta)d^{2\delta-1} + \dots$ so the result follows immediately. \square

CHAPTER 3

KONTSEVICH'S FORMULA FOR RATIONAL PLANE CURVES

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In this chapter we explore in detail the recursive formula of Kontsevich. Two different proofs are considered; the first is based purely on intersection theory on an appropriate moduli space, the second is based on a concept from modern physics, quantum cohomology. In the following sections we will therefore establish a series of definitions and results necessary to formally define the quantum product. It is the associativity of this product that implies the validity of Kontsevich's formula. We will outline most of the ideas in their generality, but since we are mainly interested in the case of \mathbb{P}^2 we will concentrate on this (much simpler) case. Main references are [Alu] and [KV]. Another source of inspiration is [Katz].

3.1 Moduli spaces for stable maps

DEFINITION 3.1.1. An n -pointed quasi-stable curve of genus g , for which we will use the notation (C, p_1, \dots, p_n) , is a projective, reduced, connected and at worst nodal curve C with $h^1(C, \mathcal{O}_C) = g$, together with n distinct smooth points. We say that two n -pointed quasi-stable curves of same genus, (C, p_1, \dots, p_n) and (C', p'_1, \dots, p'_n) , are isomorphic if there is an isomorphism of curves $\varphi : C \rightarrow C'$ with $\varphi(p_i) = p'_i$.

A *family* \mathcal{C} of quasi-stable curves of genus g over a scheme S is a flat, projective map $\mathcal{C} \rightarrow S$ together with n sections $p_i, 1 \leq i \leq n : S \rightarrow \mathcal{C}$, such that each geometric fiber $(\mathcal{C}_s, p_1(s), \dots, p_n(s))$ is quasi-stable. Two such families are said to be isomorphic if there is an isomorphism $\varphi : \mathcal{C} \rightarrow \mathcal{C}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\varphi} & \mathcal{C}' \\ p_i \updownarrow & & p'_i \updownarrow \\ S & \xrightarrow{=} & S \end{array}$$

DEFINITION 3.1.2. Let X be a scheme. A *Kontsevich stable map* of an n -pointed quasi-stable curve (having irreducible components of which the genus may vary) to X is a pair consisting of a quasi-stable curve (C, p_1, \dots, p_n) as above and a map $f : C \rightarrow X$ such that

1. all smooth, irreducible components of genus 0 which are contracted to points in X have at least 3 special points, and
2. all irreducible components of genus 1 which are contracted to points have at least 1 special point,

where a special point is either one of the marked points p_i or an intersection of the component with the closure of its complement (a singularity). The definition implies that every $f : C \rightarrow X$ will have a *finite automorphism group* $\text{Aut}(f)$. Here the notion of isomorphism is the natural one, i.e. two stable maps $f : C \rightarrow X$ and $f' : C' \rightarrow X$ are identified as isomorphic if there is an isomorphism of quasi-stable curves $g : C \rightarrow C'$ (restricting to isomorphisms on components) such that $f = f' \circ g$.

REMARK 3.1.3. The definitions above make sense in a very general context. Originally, we were interested in rational curves on \mathbb{P}^2 ; more specifically, we will follow the approach of moduli theory. At its most basic level, the idea is to put the family of objects we are studying (rational curves) in bijection with points of an algebraic variety M (a *moduli space*) and represent each condition on these objects as a subvariety. Thus we are reducing our counting problem to an intersection theoretical question on M .

As we study rational curves passing through certain points we would naturally first consider smooth n -pointed projective rational curves (C, p_1, \dots, p_n) with the notions of isomorphism and families as introduced above. There is a *fine moduli space* $M_{0,n}$ for classifying such curves up to isomorphism (see Appendix A for the definition of moduli spaces); that is, there exists a universal family $U_{0,n} \rightarrow M_{0,n}$ such that every family $\mathcal{C} \rightarrow S$ of projective smooth rational curves with n disjoint sections is induced by pulling back along a unique morphism $S \rightarrow M_{0,n}$.

As we want to do intersection theory on this moduli space, we need a compactification of $M_{0,n}$: it turns out that the appropriate way to do this is to allow curves

that break — this is what is meant by the notion of stability. So we define a *tree of projective lines* as a connected curve without any closed circuits, such that each irreducible component (each *twig*) is isomorphic to \mathbb{P}^1 and the intersection points are ordinary nodes. A *stable n -pointed rational curve* (for $n \geq 3$) is then a tree of projective lines with n distinct marked, smooth points (henceforth referred to as marks) such that every twig has at least 3 special points. So we may redefine a (Kontsevich) stable map of an n -pointed quasi-stable rational curve to be a map which contracts only stable components. Also note that the source curve of a (Kontsevich) stable map need not be stable itself.

For all $n \geq 3$ there exists a smooth projective variety $\overline{M}_{0,n}$ which is a fine moduli space for the classification of stable n -pointed rational curves, which is a compactification of $M_{0,n}$ and of which this moduli space forms an open dense subset.

For the classification of stable maps up to isomorphism we have the following results (see [Alu], paragraph 3, Theorem 2):

THEOREM 3.1.4. *Let X be a projective algebraic scheme and let $\beta \in A_1X$. There exists a compactified projective (coarse) moduli space, $\overline{M}_{g,n}(X, \beta)$, which parametrizes stable maps (C, p_1, \dots, p_n, f) , (C being a curve of genus g) satisfying $f_*[C] = \beta$.*

THEOREM 3.1.5. *Let X be a smooth projective convex variety (i.e. such that all maps $f : \mathbb{P}^1 \rightarrow X$ satisfy $H^1(\mathbb{P}^1, f^*T_X) = 0$) and let $\beta \in A_1X$. Then $\overline{M}_{0,n}(X, \beta)$ is a pure-dimensional variety, of dimension*

$$\dim(X) + \int_{\beta} c_1(T_X) + n - 3.$$

The closed points corresponding to irreducible curves form a dense open subset of $\overline{M}_{0,n}(X, \beta)$, noted $M_{0,n}(X, \beta)$.

We will use the notation $\overline{M}_{0,n}(\mathbb{P}^r, d)$ for the moduli space parametrizing isomorphism classes of stable n -pointed maps of degree d from trees of \mathbb{P}^1 s to \mathbb{P}^r . This is a projective, normal, irreducible variety, containing a smooth open dense subvariety $\overline{M}_{0,n}^*(\mathbb{P}^r, d)$ which is a fine moduli space for stable n -pointed maps of degree d to \mathbb{P}^r having *trivial automorphism group*. Also note that the dimension formula above gives

$$\dim \overline{M}_{0,n}(\mathbb{P}^r, d) = rd + r + d + n - 3.$$

EXAMPLE 3.1.6. We have $\overline{M}_{0,0}(\mathbb{P}^r, 1) = \mathbb{G}(1, r)$, the Grassmannian of lines in \mathbb{P}^r , as there are no marked points, and thus no reducible curves can map stably in the above sense. On the other hand, $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ recovers the space of complete conics: a general element is a smooth conic, which can degenerate to a line pair parametrized by a pair of intersecting lines. If the two components of the domain map to the same \mathbb{P}^1 we get a double line with one marked point, and from double

covers we get double lines with two marked points.

DEFINITION 3.1.7. We need the definition (in the general setting) of the *evaluation maps* $\rho_i, 1 \leq i \leq n$. They are morphisms from $\overline{M}_{0,n}(X, \beta)$ to X , generalizing the natural concept from the case of $\overline{M}_{0,n}(\mathbb{P}^r, d)$:

$$\begin{aligned} \rho_i : \quad \overline{M}_{0,n}(\mathbb{P}^r, d) &\rightarrow \mathbb{P}^r \\ (C, p_1, \dots, p_n, \mu) &\mapsto \mu(p_i) \end{aligned}$$

These maps are flat morphisms. We may take their product to get a *total evaluation map* $\underline{\rho} : \mu \mapsto (\mu(p_1), \dots, \mu(p_n))$ (which is not, in general, flat).

We do not include the construction of the moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ here. However, we include a study of its boundary, needed later: This boundary is formed by maps having reducible curves as domains. We will concentrate on the *boundary divisors* defined as follows (note: by the *degree* of a morphism $\mu : \mathbb{P}^1 \rightarrow \mathbb{P}^r$ we mean the degree of the image cycle $\mu_*[\mathbb{P}^1]$ — for instance, a constant map has degree 0):

DEFINITION 3.1.8. *Boundary divisors.* Consider a finite set $S = \{p_1, \dots, p_n\}$ and a positive integer d . A *d-weighted partition* of S is a partition $A \cup B = S$ together with a partition $d_A + d_B = d$ with $d_A, d_B \geq 0$ integers. For each such partition satisfying the condition $\#A \geq 2$ if $d_A = 0$ and $\#B \geq 2$ if $d_B = 0$ there exists an irreducible divisor in $\overline{M}_{0,n}(\mathbb{P}^r, d)$ called a *boundary divisor*, noted $D(A, B; d_A, d_B)$, such that a general point on this divisor represents a Kontsevich stable map μ whose domain is a tree with two twigs, $C = C_A \cup C_B$, the points of A in C_A and vica versa for B , the restriction of μ to C_A being a map of degree d_A and the restriction of μ to C_B being a map of degree d_B .

The union of the boundary divisors forms, up to a finite quotient, a normal crossing divisor. We have a natural gluing morphism

$$\overline{M}_{0,A \cup \{x\}}(\mathbb{P}^r, d_A) \times_{\mathbb{P}^r} \overline{M}_{0,B \cup \{x\}}(\mathbb{P}^r, d_B) \rightarrow D(A, B; d_A, d_B).$$

Indeed, if we consider the evaluation maps at the marked point x

$$\begin{aligned} \rho_x^A : \overline{M}_{0,A \cup \{x\}}(\mathbb{P}^r, d_A) &\rightarrow \mathbb{P}^r \\ \rho_x^B : \overline{M}_{0,B \cup \{x\}}(\mathbb{P}^r, d_B) &\rightarrow \mathbb{P}^r \end{aligned}$$

then the fiber product above is simply $(\rho_x^A \times \rho_x^B)^{-1}(\Delta)$, where Δ is the diagonal in $\mathbb{P}^r \times \mathbb{P}^r$, i.e. the images of x by the maps must be the same.

Define the divisor $D(ij|kl) = \sum D(A, B; d_A, d_B)$, the sum being taken over all d -weighted partitions of $S = \{p_1, \dots, p_n\}$ such that $i, j \in A$ and $k, l \in B$. We have the fundamental relation of rational equivalence (see [KV], 2.7.5):

$$D(ij|kl) \equiv D(ik|jl) \equiv D(il|jk).$$

3.2 Counting rational curves using moduli spaces

3.2.1 Classical approach

Let us first consider the enumeration problem of rational curves for low degrees; these results were known classically, and are nicely outlined in [Itz], pp. 255–257. We denote by $N(d) = N_0(d)$ the number of rational curves of degree d passing through $3d - 1$ general fixed points in \mathbb{P}^2 . For instance, there is clearly a single line through two points and a single conic through five points, therefore $N(1) = N(2) = 1$.

Now if $d = 3$, we are looking for uninodal cubics through eight general points. A *general cubic* depends on nine parameters, so through the given eight points we find a pencil of cubics (that is, a linear family over \mathbb{P}^1) of the form $\lambda_0 f_0 + \lambda_1 f_1 = 0$, where f_0, f_1 are two cubic homogenous polynomials in x, y, z intersecting at nine points, the eight given ones and a ninth (unassigned), common to all curves in the pencil. Among these cubics, the rational ones will have one simple node, this occurring for pairs (λ_0, λ_1) such that

$$\lambda_0 \frac{\partial f_0}{\partial x} + \lambda_1 \frac{\partial f_1}{\partial x} = \lambda_0 \frac{\partial f_0}{\partial y} + \lambda_1 \frac{\partial f_1}{\partial y} = \lambda_0 \frac{\partial f_0}{\partial z} + \lambda_1 \frac{\partial f_1}{\partial z} = 0.$$

Provided that

$$\begin{vmatrix} \frac{\partial f_0}{\partial x} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_0}{\partial y} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_0}{\partial z} & \frac{\partial f_1}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial f_0}{\partial y} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_0}{\partial z} & \frac{\partial f_1}{\partial z} \end{vmatrix} = 0,$$

each solution of the system above yields a value of $(\lambda_0 : \lambda_1)$. But (by Bézout's theorem) the two quartics corresponding to the zero sets of the determinants intersect in 16 points, 4 of which correspond to $\frac{\partial f_0}{\partial z} = \frac{\partial f_1}{\partial z} = 0$ and must be removed, so we get $N(3) = 12$. Of course, this argument can be generalized for curves of degree d with one node, giving the number $3(d - 1)^2$, but for higher number of nodes the procedure quickly gets more complicated.

It turns out that a somewhat more efficient approach (at least for treating the general case) is to use the theory of moduli spaces established above. This also allows us to get a proof of Kontsevich's formula. We will start out by illustrating the technique on the specific case of irreducible rational quartics through 11 points in \mathbb{P}^2 . The ideas apply quite easily to the general case.

3.2.2 Enumerating rational quartics through 11 points in \mathbb{P}^2

We will show that the number $N(4)$ of rational quartics passing through 11 points in general position in the projective plane is 620. For this, we consider the moduli space $\overline{M}_{0,12}(\mathbb{P}^2, 4)$ of dimension 23, as introduced above. Let $m_1, m_2, p_1, \dots, p_{10}$ be marks with corresponding evaluation maps ρ_{m_i}, ρ_{p_j} , and fix lines L_1, L_2 and points Q_1, \dots, Q_{10} in general position in \mathbb{P}^2 . Then it can be shown ([KV], 3.4.3) that

$$Y = \rho_{m_1}^{-1}(L_1) \cap \rho_{m_2}^{-1}(L_2) \cap \rho_{p_1}^{-1}(Q_1) \cap \dots \cap \rho_{p_{10}}^{-1}(Q_{10}) \subset \overline{M}_{0,9}(\mathbb{P}^2, 3)$$

is a curve (it has codimension 22 if the points and lines are chosen generically in \mathbb{P}^2) intersecting each boundary divisor transversally and contained in $M_{0,12}^*(\mathbb{P}^2, 4)$. We know from the fundamental relation of boundary divisors that

$$Y \cap D(m_1, m_2 | p_1, p_2) \equiv Y \cap D(m_1, p_1 | m_2, p_2),$$

and this will yield a relation allowing us to compute $N(4)$ (provided that we know that $N(1) = N(2) = 1$ and $N(3) = 12$). We start by examining the left hand side of the equivalence. This divisor has 1280 irreducible components; indeed,

$$D(m_1, p_1 | m_2, p_2) = \sum D(A, B; d_A, d_B),$$

the sum being taken over all A, B, d_A, d_B such that $A \cup B = \{m_1, m_2, p_1, \dots, p_{10}\}$ and $d_A + d_B = 4$, but with m_1 and m_2 in A and p_1 and p_2 in B . This leaves us with the distribution of p_3, \dots, p_{10} among A and B : there are 256 ordered such distributions, which must be multiplied by the 5 partitions $d_A + d_B = 4$. Since Y intersect these divisors transversally, the result follows. We now examine each possible $D(A, B; d_A, d_B)$ according to these 5 partitions:

If $d_B = 0$ then a map in $D(A, B; d_A, d_B)$ maps C_B to a point (since the degree of its restriction to this twig is 0). But this twig has at least two marked points which must map to distinct points Q_i if Y intersects this divisor; this is a contradiction. So there is no contribution to the intersection. Now if $d_A = 0$ then a map in a divisor $D(A, B; d_A, d_B)$ which contributes to the intersection must send C_A to a point $z \in L_1 \cap L_2$, because the mark m_1 is mapped to L_1 and m_2 is mapped to L_2 . Also, C_B maps to a rational quartic passing through the 11 points z, Q_1, \dots, Q_{10} ; there are $N(4)$ such curves, so the number $N(4)$ intervenes as an additive factor of the degree of $Y \cap D(m_1, m_2 | p_1, p_2)$.

If $d_A = 1$, we only get a contribution to the intersection if we put 2 of the 8 remaining marks on C_A and 6 on C_B . Indeed, a map in $D(A, B; 1, 3)$ sends C_A to a line and C_B to a cubic passing through Q_1 and Q_2 if we are on Y . But if we put more than 2 marks on C_A , at least three of the points Q_i would be collinear (contradicting the general position requirement) and if we put more than 6 on C_B its image cubic would pass through at least 9 of the points Q_i , again contradicting generality. This means there are $\binom{8}{2} = 28$ possible ways of distributing the remaining marks in such a way that we get a contribution to the intersection with Y , so here we are considering 28 different components of the intersection. For each component, there is only $N(1) = 1$ choice for the image line $\mu(C_A)$ and $N(3) = 12$ choices for the image cubic $\mu(C_B)$; we must also know how the partial maps μ_{C_A} and μ_{C_B} glue together; there are $d_A d_B = 3$ ways to choose a point from $\mu(C_A) \cap \mu(C_B)$ (Bézout's theorem), so finally, the total contribution from the 28 divisors with $d_A = 1$ amounts to $28 \cdot N_1 \cdot 3 \cdot N_3 = 1008$.

For the situation $d_A = 2$, we only get a contribution when 5 of the 8 remaining marks are placed on C_A (because $d_A = d_B = 2$ so the image of both twigs are conics, which are not supposed to pass through more than 5 of the chosen points), so we are

considering $\binom{8}{5} = 56$ irreducible components. There is $N(2) = 1$ choice for $\mu(C_A)$ and $N(2) = 1$ for $\mu(C_B)$, and this means we have fixed the marks p_i once and for all. However, there are 2 choices for the mark m_1 on C_A , it can be one of the inverse images of the points of $L_1 \cap \mu(C_A)$. The same goes for m_2 . In addition, there are $d_A d_B = 4$ ways to glue the partial morphisms together by Bézout's theorem, so the total contribution of this component is $\binom{8}{5} \cdot 2^2 \cdot 4 = 896$.

If $d_A = 3$ and $d_B = 1$ we must put all 8 remaining marks on C_A for generality reasons — only one irreducible component is considered. There are $N(3) = 12$ choices for $\mu(C_A)$ and $N(1) = 1$ for $\mu(C_B)$. There are 3 choices for each of the marks m_1 and m_2 and $d_A d_B = 3$ ways to glue the partial morphisms. So the contribution here is $12 \cdot 3^2 \cdot 3 = 324$. So finally, we have

$$\deg\left(Y \cap D(m_1, m_2 | p_1, p_2)\right) = N(4) + 1008 + 896 + 324 = N(4) + 2228.$$

Now consider the intersection $Y \cap D(m_1, p_1 | m_2, p_2)$. Here there is no contribution from boundary divisors with $d_A = 0$ or $d_B = 0$; the situation is symmetric, so consider for instance $d_A = 0$. Then the restriction of a map $\mu \in D(A, B; 0, 4)$ to C_A has degree 0, so it maps this twig to a point, but this point must be Q_1 , and we would get $Q_1 \in L_1$, contradicting generality.

If $d_A = 1$, we must, for reasons of generality (as above) place exactly one more marked point on C_A , the image being a line: this gives us 8 choices, and for each choice, the p -marks are then determined, and C_A maps to the unique line through two points in \mathbb{P}^2 , C_B to one of the 12 cubics through 8 general points. There remains, however, $d_A = 1$ choice for m_1 and $d_B = 3$ choices for m_2 , as well as $d_A d_B = 3$ choices for gluing together the partial morphisms, yielding a total of $12 \cdot 8 \cdot 3 \cdot 3 = 864$ contributions. The situation being symmetric, we also get a contribution of 864 to the degree of the intersection with $d_B = 1$. For the situation $d_A = d_B = 2$ we must put 4 of the remaining marks on C_A and 4 on C_B . This gives $\binom{8}{4} = 70$ choices. Counting the remaining choices for m_1, m_2 and the gluing of the morphisms, we get at total of $70 \cdot 16 = 1120$. Therefore

$$\deg\left(Y \cap D(m_1, p_1 | m_2, p_2)\right) = 864 \cdot 2 + 1120 = 2848$$

By the equivalence established as the fundamental relation, $N(4) + 2228 = 2848$, so $N(4) = 620$.

3.2.3 First proof of Kontsevich's formula

Basically, we will follow the same ideas as above, although in some greater generality (this is based on [KV], Theorem 3.3.1). We will prove the following form of Kontsevich's formula

$$N(d) + \sum_{d_A + d_B = d} \binom{3d - 4}{3d_A - 1} d_A^2 N(d_A) N(d_B) d_A d_B$$

$$= \sum_{d_A+d_B=d} \binom{3d-4}{3d_A-2} d_A N(d_A) \cdot d_B N(d_B) d_A d_B$$

Introduce the number $n = 3d$ and consider the moduli space $\overline{M}_{0,n}(\mathbb{P}^2, d)$ with marks $m_1, m_2, p_1, \dots, p_{n-2}$ and corresponding evaluation maps. Let L_1, L_2 be lines and Q_1, \dots, Q_{n-2} points in general position in \mathbb{P}^2 , and consider, as previously,

$$Y = \rho_{m_1}^{-1}(L_1) \cap \rho_{m_2}^{-1}(L_2) \bigcap_{i=1}^{n-2} \rho_{p_i}^{-1}(Q_i).$$

It can be shown that the points and lines can be chosen in such a way that Y is a curve in $\overline{M}_{0,n}(\mathbb{P}^2, d)$ contained in $M_{0,n}^*(\mathbb{P}^2, d)$ and *intersecting each boundary divisor transversally*. The equality we wish to show follows from the equivalence of divisors

$$Y \cap D(m_1, m_2 | p_1, p_2) \equiv Y \cap D(m_1, p_1 | m_2, p_2).$$

We start by examining the left hand side, more precisely the contribution of each $Y \cap D(A, B; d_A, d_B)$ to the intersection. We get no contribution if $d_B = 0$, because then the twig C_B would be contracted to a point in \mathbb{P}^2 , but it contains marks mapping to distinct points (p_1 and p_2 map to Q_1 and Q_2). If $d_A = 0$ we get a contribution (by a reasoning similar to the one above) if and only if all the remaining $3d - 4$ marks fall on the B -twig. But the number of ways to draw a rational curve of degree d through the $3d - 1$ points z, Q_1, \dots, Q_{3d-2} (where $\{z\} = L_1 \cap L_2$) is by definition $N(d)$, so this number intervenes as an additive factor in the degree of the left hand side.

Now suppose both d_A and $d_B > 0$. We only get a contribution to the intersection if $3d_A - 1$ marks fall on the twig C_A ; if we place more marks we get a contradiction on the general position of the points that fall on the rational curve $\mu(C_A)$ for a stable map $(C_A \cup C_B, \dots, \mu)$, if we place less marks we get the same problem for $\mu(C_B)$. Now there are $\binom{3d-4}{3d_A-1}$ ways to choose these marks, so there are $\binom{3d-4}{3d_A-1}$ irreducible components to consider. There are also $N(d_A)$ ways to draw the image of C_A and $N(d_B)$ ways to draw the image of C_B . These choices determine the distribution of the p -marks, but there remains a choice for the mark m_1 on the intersection $\mu(C_A) \cap L_1$; by Bézout's theorem, there are d_A intersection points, so d_A choices for the image of the mark m_1 . Similarly, there are d_A choices for the image of the mark m_2 . Now, for the gluing of the two partial morphisms, there are $d_A d_B$ points in the intersection of the curves $\mu(C_A) \cap \mu(C_B)$. So we get

$$\deg(Y \cap D(m_1, m_2 | p_1, p_2)) = N(d) + \sum_{d_A+d_B=d} \binom{3d-4}{3d_A-1} d_A^2 N(d_A) N(d_B) d_A d_B.$$

The reasoning for the right hand side is similar. There is no contribution if $d_A = 0$ or $d_B = 0$, as this would give $Q_1 \in L_1$ or $Q_2 \in L_2$, contradicting generality. When both partial degrees are > 0 we must, again for generality reasons, put $3d_A - 2$ remaining marks on C_A : this gives $\binom{3d-4}{3d_A-2}$ components, and for each one of these we have $N(d_A)$ choices for the drawing of the image curve $\mu(C_A)$ and $N(d_B)$ choices

for $\mu(C_B)$. There are $d_A = \#(L_1 \cap \mu(C_A))$ choices for the mapping of m_1 , d_B for m_2 , and $d_A d_B$ for the gluing morphism. Finally,

$$\deg\left(Y \cap D(m_1, p_1 | m_2, p_2)\right) = \sum_{d_A+d_B=d} \binom{3d-4}{3d_A-2} d_A N(d_A) \cdot d_B N(d_B) d_A d_B.$$

The equivalence of the divisors gives the equality of the two degrees, that is, Kontsevich’s formula.

3.3 Gromov–Witten invariants

DEFINITION 3.3.1. *Gromov–Witten invariants.* Let $\beta \in A_1 X$ and $\gamma_i \in A^* X$ for $1 \leq i \leq n$. Recall the evaluation maps $\rho_i, 1 \leq i \leq n$, defined on the moduli space $\overline{M}_{0,n}(X, \beta)$. The *Gromov–Witten invariants* are simply the *intersection numbers*

$$I_\beta(\gamma_1, \dots, \gamma_n) = \int_{\overline{M}_{0,n}(X, \beta)} \rho_1^*(\gamma_1) \cup \dots \cup \rho_n^*(\gamma_n) \in \mathbb{Z}.$$

The idea is that if the γ_i are the *cohomology classes of subvarieties* Y_i of X in general position, then $I_\beta(\gamma_1, \dots, \gamma_n)$ should count the (possibly virtual) number of irreducible rational curves C in X , having *homology class* β and intersecting all the Y_i . Note that $I_\beta(\gamma_1, \dots, \gamma_n)$ could be written $I_\beta(\gamma_1 \cdots \gamma_n)$ as it is invariant under permutation of the γ_i .

REMARK 3.3.2. At this point, we seem to have deviated quite a bit from the original counting problem: we were interested in rational curves, not maps or marks. The setting is the following: Let $X = \mathbb{P}^r$ (later on, we will have $r = 2$) and consider X^n with projections $\tau_i, 1 \leq i \leq n$. If $\Gamma_i, 1 \leq i \leq n$, are irreducible subvarieties of X then $\underline{\Gamma} = \Gamma_1 \times \dots \times \Gamma_n = \bigcap_{i=1}^n \tau_i^{-1}(\Gamma_i)$ is a subvariety of X^n . Also, if \overline{M} denotes $\overline{M}_{0,n}(X, d)$, we have evaluation maps $\rho_i : \overline{M} \rightarrow X$, the product of which gives a morphism $\underline{\rho} : \overline{M} \rightarrow X^n$ such that the following diagram commutes for all i :

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\underline{\rho}} & X^n \\ \rho_i \downarrow & \swarrow \tau_i & \\ X & & \end{array}$$

Note that $\rho_i^{-1}(\Gamma_i)$ consists of all stable maps μ such that $\mu(p_i) \in \Gamma_i$. Let $k_i = \text{codim}(\Gamma_i, \mathbb{P}^r)$. Then, since ρ_i is a flat morphism, we also have

$$\text{codim}(\rho_i^{-1}(\Gamma_i), \overline{M}) = k_i.$$

On the other hand, the map $\underline{\rho}$ is not necessarily flat, so the locus $\underline{\rho}^{-1}(\underline{\Gamma}) = \bigcap_{i=1}^n \rho_i^{-1}(\Gamma_i)$ consisting of maps μ such that for all $i, \mu(p_i) \in \Gamma_i$, is not necessarily of codimension $\sum k_i$. However, if $\sum k_i = \dim \overline{M}$, then the *expected dimension* of the intersection of the inverse images is 0. We have the following proposition:

RESULT A ([KV], Proposition 3.4.3). If the subvarieties $\Gamma_i, 1 \leq i \leq n$ are chosen generically in \mathbb{P}^r and $\sum \text{codim}(\Gamma_i, \mathbb{P}^r) = \dim \bar{M}$, then $\underline{\rho}^{-1}(\underline{\Gamma})$ consists of a finite number of reduced points, supported in the locus $M^* \subset \bar{M}$ of maps with smooth source and without automorphisms. This number of points is the degree of the cycle $[\underline{\rho}^{-1}(\underline{\Gamma})]$, i.e. $\int_{\bar{M}} [\underline{\rho}^{-1}(\underline{\Gamma})]$.

Stable maps in this inverse image send each p_i to Γ_i , so the image curve by such a map intersects each Γ_i . Therefore, the enumerative problem we are considering is the one of counting *how many rational curves meet all the Γ_i* . The problem, of course, is that the curves could intersect the Γ_i 's in more than one point. Such a rational curve would then correspond to at least two n -pointed stable maps with $\mu(p_i) \in \Gamma_i$ (following the different ways of putting the marks on the same curve in the moduli space). We need some way to circumvent this. The following result gives us the necessary guarantee:

RESULT B ([KV], Lemma 3.5.3). If $\Gamma_1, \dots, \Gamma_n \subset \mathbb{P}^r$ are chosen generically and their codimensions add up to $\dim \bar{M}_{0,n}(\mathbb{P}^r, d)$, then for every map $\mu \in \underline{\rho}^{-1}(\underline{\Gamma})$ we have $\mu^{-1}(\mu(p_i)) = \{p_i\}$ with multiplicity 1, for all $1 \leq i \leq n$. It follows that if $\Gamma_1, \dots, \Gamma_{3d-1}$ are general points in \mathbb{P}^2 , the number of stable maps such that the mark p_i maps to Γ_i is equal to the number $N(d)$ of degree d rational curves passing through the Γ_i .

Combining Results A and B we see that for generic points Γ_i in \mathbb{P}^2 giving the product $\underline{\Gamma}$ we have

$$N(d) = \int_{\bar{M}_{0,3d-1}(\mathbb{P}^2, d)} [\underline{\rho}^{-1}(\underline{\Gamma})].$$

Let us now translate this to the language of Gromov–Witten invariants. Let $X = \mathbb{P}^r$ and denote by $\gamma_i \in A^*X$ the class corresponding to $[\Gamma_i] \in A_*X$ via Poincaré duality. Then $\underline{\gamma} = \gamma_1 \times \dots \times \gamma_n = \bigcup_{i=1}^n \tau_i^*(\gamma_i) \in A^*(X^n)$ corresponds to $[\underline{\Gamma}] \in A_*(X^n)$. Instead of intersecting cycles $[\rho_i^{-1}(\Gamma_i)]$ in the moduli space we can take the product of cohomology classes, $\rho_i^*(\gamma_i)$, i.e. consider $\underline{\rho}^*(\underline{\gamma})$.

In the case of $X = \mathbb{P}^r$, the Gromov–Witten invariant of degree d associated with the classes $\gamma_i \in A^*(\mathbb{P}^r), 1 \leq i \leq n$, is $I_\beta(\gamma_1, \dots, \gamma_n) = \int_{\bar{M}_{0,n}(\mathbb{P}^r, d)} \underline{\rho}^*(\underline{\gamma})$, where $\beta = dH^1$. This number equals 0 unless $\sum \text{codim}(\gamma_i) = \dim \bar{M}$. Let us emphasize the *enumerative meaning* of the Gromov–Witten invariants in this special case (as it follows from the remarks made above):

PROPOSITION 3.3.3. *Let $\beta = dH^1$ where H^1 is the class of a hypersurface, and let $\gamma_1, \dots, \gamma_n \in A^*(\mathbb{P}^r)$ denote homogenous classes of codimension ≥ 2 such that $\sum \text{codim}(\gamma_i) = \dim \bar{M}_{0,n}(\mathbb{P}^r, d)$. If $\Gamma_i, 1 \leq i \leq n$, are generically chosen subvarieties of \mathbb{P}^r with $[\Gamma_i] = \gamma_i \cap [\mathbb{P}^r]$, then $I_\beta(\gamma_1 \cdots \gamma_n)$ is equal to the number of rational curves having degree d and that are incident to all the Γ_i . In particular, for $\mathbb{P}^2, I_\beta(H^2 \cdots H^2) = N(d)$ is the number of rational curves of degree d passing*

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through $3d - 1$ general points (here H denotes the class of a hyperplane in \mathbb{P}^2).

LEMMA 3.3.4 ([Alu], paragraph 5). *It can be shown that we have the following general results for Gromov–Witten invariants:*

$$I_0(\gamma_1 \cdots \gamma_n) = \begin{cases} 0 & \text{if } n > 3 \\ \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3 & \text{if } n = 3 \end{cases} \quad (3.1)$$

and

$$I_\beta(1, \gamma_2 \cdots \gamma_n) = \begin{cases} 0 & \text{if } n > 3 \text{ or } \beta \neq 0 \\ \int_X \gamma_2 \cup \gamma_3 & \text{if } n = 3 \text{ and } \beta = 0 \end{cases} \quad (3.2)$$

Also, if $\gamma_1 \in A^1 X$ then we have a recursive formula:

$$I_\beta(\gamma_1 \gamma_2 \cdots \gamma_n) = \left(\int_\beta \gamma_1 \right) I_\beta(\gamma_2 \cdots \gamma_n) \quad (3.3)$$

3.4 Quantum cohomology and a second proof of Kontsevich's formula

We want to construct the *generating function* for the Gromov–Witten invariants. In the case of \mathbb{P}^r these depend on d and the classes γ_i , $1 \leq i \leq n$, varying freely in $A^*(\mathbb{P}^r)$. However, the *linearity* of the Gromov–Witten invariants allows us to only use classes belonging to a basis for $A^*(\mathbb{P}^r)$, that is, $\{1, H, H^2, \dots, H^{r-1}, H^r\}$ where H is the class of a hypersurface. Define the *total Gromov–Witten invariant*

$$I(\gamma_1 \cdots \gamma_n) = \sum_{d=0}^{\infty} I_{dH^1}(\gamma_1 \cdots \gamma_n).$$

This infinite sum consists, in fact, of only one term. Indeed, we may, by linearity, assume that the classes are homogenous, having codimensions c_i . We work in $\overline{M}_{0,n}(\mathbb{P}^r, d)$, a space of dimension $rd + r + d + n - 3$. So we have $I_{dH^1}(\gamma_1 \cdots \gamma_n) \neq 0$ if and only if

$$\sum c_i = \dim \overline{M} \Leftrightarrow d = \frac{\sum c_i - r - n + 3}{r + 1}.$$

Since $I(\gamma_1 \cdots \gamma_n)$ is independent of the order of the γ_i , we may regroup the classes such that we get total Gromov–Witten invariants

$$I\left((H^0)^{\bullet a_0} (H^1)^{\bullet a_1} \dots (H^r)^{\bullet a_r}\right),$$

that is, we are indexing by \mathbb{N}^{r+1} . We use the symbol \bullet to emphasize that we are not considering the cup product of the classes. The Gromov–Witten potential is defined as the generating function over \mathbb{N}^{r+1} for these invariants:

$$\Phi(x_0, \dots, x_r) = \sum_{a_0, \dots, a_r} \frac{x_0^{a_0} \cdots x_r^{a_r}}{a_0! \cdots a_r!} I\left((H^0)^{\bullet a_0} (H^1)^{\bullet a_1} \dots (H^r)^{\bullet a_r}\right).$$

Let $\mathbf{x} = (x_0, \dots, x_r)$, $\mathbf{a} = (a_0, \dots, a_r)$ and $\mathbf{H} = (H^0, \dots, H^r)$, then we can introduce the somewhat more compact form

$$\Phi(\mathbf{x}) = \sum_{\mathbf{a} \in \mathbb{N}^{r+1}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} I(\mathbf{H}^{\mathbf{a}}),$$

and we see that the third partial derivatives are given by

$$\Phi_{ijk}(\mathbf{x}) = \sum_{\mathbf{a}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} I(\mathbf{H}^{\mathbf{a}} H^i H^j H^k) \in \mathbb{Q}[[\mathbf{x}]].$$

Also, introducing $\gamma = \sum_{i=0}^r x_i H^i$ we get (formally)

$$\Phi = I(\exp(\gamma)) = \sum_{n=0}^{\infty} \frac{1}{n!} I(\gamma^{\bullet n}).$$

DEFINITION 3.4.1. We define the *quantum product* $*$ by setting $H^i * H^j = \sum_{e+f=r} \Phi_{ije}(\mathbf{x}) H^f$, considered as an element of $A^*(\mathbb{P}^r) \otimes_{\mathbb{Z}} \mathbb{Q}[[\mathbf{x}]]$, and by extending by linearity. Note that since the Φ_{ijk} are symmetric in the indexes, the quantum product is immediately seen to be *commutative*. The identity is H^0 . More importantly the quantum product is *associative*, i.e. for all i, j, k , we have

$$(H^i * H^j) * H^k = H^i * (H^j * H^k).$$

To prove Kontsevich's formula, let $X = \mathbb{P}^2$. Then H^0, H^1, H^2 are the classes of X , of a line and of a point. Put $\beta = dH^1, \delta = \frac{1}{2}(d-1)(d-2)$, then we have seen (Proposition 3.3.3 together with the remarks made concerning the relation between the degree and the number of nodes of a rational curve) that $I_{\beta}((H^2)^{n_2})$ is the number $N(d)$ of δ -nodal rational plane curves of degree d passing through n_2 general points, equal to 0 unless $n_2 = 3d - 1$. The potential

$$\Phi(\gamma) = \sum_{n=0}^{\infty} \frac{1}{n!} I(\gamma^{\bullet n}) = \Phi^{cl} + \Gamma$$

actually splits into two parts, the classical Φ^{cl} , whose contribution comes from $d = 0$, and a quantum part Γ , containing enumerative information about curves which are not contracted to points; $\Gamma = \sum_{n=0}^{\infty} \frac{1}{n!} I_+(\gamma^{\bullet n})$, where $I_+ = \sum_{d>0} I_d$. There is a similar decomposition of the quantum product, which we will establish below:

Examining Φ^{cl} with the use of Lemma 3.3.4, part (3.1), we have that the only nonzero Gromov–Witten invariants for \mathbb{P}^r in degree 0 are those with three marks, and

$$I_0(\gamma_1 \gamma_2 \gamma_3) = \int_{M_{0,3}(\mathbb{P}^r, 0)} (\gamma_1 \cup \gamma_2 \cup \gamma_3),$$

which is again equal to 0 unless $\sum \text{codim } \gamma_i = r$. Putting $\gamma = \sum_{i=0}^r x_i H^i$ we get

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$$\Phi^{cl} = \frac{1}{3!} I_0 \left(\left(\sum_{i=0}^r x_i H^i \right)^3 \right) = \frac{1}{3!} I_0 \left(\sum_{i,j,k} x_i x_j x_k H^i H^j H^k \right),$$

so that $\Phi_{ijk}^{cl} = I_0(H^i H^j H^k)$. Since $I_0(H^i H^j H^e) = \int_{\mathbb{P}^r} H^i \cup H^j \cup H^e = 0$ if $i+j+e \neq r$ and 1 otherwise, we have

$$H^i \cup H^j = H^{i+j} = \sum_{e+f=r} I_0(H^i H^j H^e) H^f,$$

which gives $\sum_{e+f=r} \Phi_{ije}^{cl} H^f$. So we get the following decomposition of the quantum product:

$$\begin{aligned} H^i * H^j &= \sum_{e+f=r} \Phi_{ije} H^f = \sum_{e+f=r} \left(\Phi_{ije}^{cl} + \Gamma_{ije} \right) H^f \\ &= \sum_{e+f=r} \left(I_0(H^i H^j H^e) + \Gamma_{ije} \right) H^f = (H^i \cup H^j) + \sum_{e+f=r} \Gamma_{ije} H^f \end{aligned}$$

Note that $\Gamma_{ije} = 0$ if i, j or $e = 0$, as it follows from Lemma 3.3.4, part (3.2).

PROPOSITION 3.4.2 Kontsevich's formula. *Let $N(d)$ denote the number of rational curves passing through $3d - 1$ points in general position in \mathbb{P}^2 , then the following recursive formula holds:*

$$\begin{aligned} N(d) &+ \sum_{d_A+d_B=d} \frac{(3d-4)!}{(3d_A-1)!(3d_B-3)!} d_A^3 N(d_A) d_B N(d_B) \\ &= \sum_{d_A+d_B=d} \frac{(3d-4)!}{(3d_A-2)!(3d_B-2)!} d_A^2 N(d_A) d_B^2 N(d_B). \end{aligned}$$

Proof. In \mathbb{P}^2 we have:

$$\begin{aligned} H^1 * H^1 &= H^2 + \Gamma_{111} H^1 + \Gamma_{112} H^0 \\ H^1 * H^2 &= \Gamma_{121} H^1 + \Gamma_{112} H^0 \\ H^2 * H^2 &= \Gamma_{221} H^1 + \Gamma_{222} H^0 \end{aligned}$$

Writing down the associativity relation $(H^1 * H^1) * H^2 = H^1 * (H^1 * H^2)$ and expanding this we get the following differential equation

$$\Gamma_{222} + \Gamma_{111} \Gamma_{122} = \Gamma_{112} \Gamma_{112}$$

Now $\Gamma_{ijk} = \sum_{n=0}^{\infty} \frac{1}{n!} I_+(\gamma^{\bullet n} H^i H^j H^k) = \sum_{\mathbf{a}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} I_+(\mathbf{H}^{\mathbf{a}} H^i H^j H^k)$. We have γ of the form $x_0 H^0 + x_1 H^1 + x_2 H^2$, but using Lemma 3.3.4, part (3.2), we can actually reduce to $x_0 = x_1 = 0$. Indeed, the presence of a fundamental class H^0 annihilates

the Gromov–Witten invariant in all degrees but 0, so we only get a contribution if $a_0 = 0$. The same lemma, part (3.3), shows the Gromov–Witten invariants containing a factor H^1 are determined by those without (since $\int_{dH^1} H^1 = d$), so the reduction makes sense.

Put $x_2 = x$, then $\Gamma_{ijk} = \sum_{n=0}^{\infty} \frac{x^n}{n!} I_+ \left((H^2)^{\bullet n} H^i H^j H^k \right)$ is the generating function for the numbers $I_+ \left((H^2)^{\bullet n} H^i H^j H^k \right)$. Then, identifying coefficients, the differential equation above corresponds to a recursive formula:

$$\begin{aligned} I_+ \left((H^2)^{\bullet n} H^2 H^2 H^2 \right) &+ \sum_{n_A+n_B=n} \frac{n!}{n_A! n_B!} I_+ \left((H^2)^{\bullet n_A} H^1 H^1 H^1 \right) I_+ \left((H^2)^{\bullet n_B} H^1 H^2 H^2 \right) \\ &= \sum_{n_A+n_B=n} \frac{n!}{n_A! n_B!} I_+ \left((H^2)^{\bullet n_A} H^1 H^1 H^2 \right) I_+ \left((H^2)^{\bullet n_B} H^1 H^2 H^2 \right). \end{aligned}$$

We need to understand the numbers $I_+ \left((H^2)^{\bullet n} H^i H^j H^k \right)$. Each of them is a sum over values $d > 0$, but we have already seen that they are zero except for compatible values of d and n . We have $n + 3$ marks, so the space on which we are working is $\overline{M}_{0,n+3}(\mathbb{P}^2, d)$, of dimension $3d + 2 + n$. So in order to get a non-zero value, the sum of the codimensions of the classes, i.e. $2n + i + j + k$, must be equal to $3d + 2 + n$, giving $n = 3d + 2 - i - j - k$. Also, from Lemma 3.3.4, the factors H^1 can be moved outside, becoming a multiplicative factor d . So for example (with $n_B = 3d_B + 2 - 1 - 2 = 3d_B - 3$),

$$\begin{aligned} I_+ \left((H^2)^{\bullet n_B} H^1 H^2 H^2 \right) &= I_{d_B} \left((H^2)^{\bullet 3d_B-3} H^1 H^2 H^2 \right) \\ &= d_B I_{d_B} \left((H^2)^{\bullet 3d_B-1} \right) = d_B N(d_B). \end{aligned}$$

Doing this for all the terms, we get exactly Kontsevich's formula. □

CHAPTER 4

CONFIGURATION SPACES AND NODE POLYNOMIALS

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The aim of this chapter is to provide the main ideas of an original proof of our main theorem, which is a partial generalization of the theorem of Kleiman–Piene and thus provides the first of two steps towards a complete proof of Göttsche’s first conjecture. More precisely, we will show that for a sufficiently ample linear system $|\mathcal{L}|$ of curves on a projective surface S , the number of r -nodal curves in \mathcal{L} is given — up to a factor $r!$ — by the r th Bell polynomial in r enumerative classes on $Y = \mathbb{P}(H^0(S, \mathcal{L}))$. It remains to show that these classes can be expressed as linear polynomials in the four basic Chern numbers ∂, k, s, x introduced in Chapter 1.

4.1 Warming up: Counting 2-nodal curves in \mathbb{P}^2

Let S denote the complex projective plane \mathbb{P}^2 . Let $d \geq 2$ be an integer and consider the linear system associated with degree d curves on S , which form a projective space

$$Y = \mathbb{P}(H^0(S, \mathcal{O}_S(d))) \cong \mathbb{P}^{\binom{d+2}{2}-1}$$

Consider $\mathcal{D} \subset S \times Y$ consisting of pairs (x, y) such that x is a point on the curve $D_y \subset S$ corresponding to $y \in Y$, and let $X \subset \mathcal{D}$ be the closure of the set of pairs $(x, y) \in S \times Y$ such that x is a node on D_y . Letting γ denote the projection from $S \times Y$ to Y we have the following situation:

$$X \hookrightarrow \mathcal{D} \hookrightarrow S \times Y \xrightarrow{\gamma} Y.$$

Let $f \in \mathbb{C}[x_0, x_1, x_2, a_{ijk} | i + j + k = d]$ be the homogenous polynomial of degree d in x_0, x_1 and x_2 , and in degree 1 in the a_{ijk} :

$$\sum_{i+j+k=d} a_{ijk} x_0^i x_1^j x_2^k.$$

Then $\mathcal{D} = Z(f)$ is a hypersurface in $S \times Y$, whereas (recall the Euler identity):

$$X = \left\{ \left((x_0 : x_1 : x_2), (a_{ijk}, i + j + k = d) \right), \frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0 \right\},$$

so X is a complete intersection of three hypersurfaces in $S \times Y$. Thus we have $\text{cod}(X, \mathcal{D}) = 2$ and $\text{cod}(X, S \times Y) = 3$. Let $\widetilde{\mathcal{L}}$ denote the sheaf on $S \times Y$ defined by $\widetilde{\mathcal{L}} = \mathcal{L} \boxtimes \mathcal{O}_Y(1)$. Considering the following diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ & & \Omega_{S \times Y/Y}^1 \otimes \widetilde{\mathcal{L}} \\ & & \downarrow \\ \mathcal{O}_{S \times Y} & \xrightarrow{s'} & \mathcal{P}_{S \times Y/Y}^1(\widetilde{\mathcal{L}}) \\ & \searrow s & \downarrow \\ & & \widetilde{\mathcal{L}} \\ & & \downarrow \\ & & 0 \end{array}$$

where the section s corresponds to $\mathcal{D} \hookrightarrow S \times Y$ and $s' = 0$ corresponds to X , we see that the rational equivalence class of X in $F = S \times Y$ is

$$x = c_3(\mathcal{P}_{S \times Y/Y}^1(\widetilde{\mathcal{L}})).$$

2-nodal curves correspond to the variety which is the closure of $(X \times_Y X) \setminus \Delta$ in $F^2 = (S \times Y) \times_Y (S \times Y) \cong S^2 \times Y$ where Δ denotes the diagonal (corresponding to coinciding nodes), and we wish to find the pushdown to Y of the rational equivalence class of this subvariety in $A^*(F^2)$. If p_i is the projection from $(S \times Y) \times_Y (S \times Y)$ to the i th factor $S \times Y, i \in \{1, 2\}$, we have

$$[X^2 \setminus \Delta] = p_1^* x \cdot p_2^* x - B_\Delta \in A^*(F^2),$$

where B_Δ is a class supported on the diagonal. This class is the sum of the equivalence of the diagonal Δ for the intersection product $p_1^* x \cdot p_2^* x$ and the class of an

embedded component which appears when we take the closure of $X^2 \setminus \Delta$ in F^2 , and which corresponds to the locus of cuspidal curves (of singularity type A_2 instead of A_1^2). In his article [Kaz], Kazarian gives enumerative polynomials for this sort of embedded components; specifically, we have $S_{A_2}(d) = 12d^2 - 36d + 24$.

The calculation of the equivalence of the diagonal is an exercise in the use of Proposition 9.1.1 in [Ful], which we have included in Appendix A.2.8. In our case the calculation takes place on F^2 , and we get a class in $A_m(\Delta)$ where $m = \dim(F^2) - \sum_{i=1}^2 \text{codim}(p_i^{-1}X, F^2) = 4 + \dim Y - 2 \cdot 3 = \dim Y - 2$ (recall that X has codimension 3 in $F = S \times Y$, so X^r has codimension $3r$ in $F^r = S^r \times Y$). If we instead consider the corresponding class in $A^*(\Delta)$ we get a class in $A^n(\Delta)$ where $n = \dim F^2 - m = 6$. However, since $\Delta \cong X \subset F$, it will be more practical to calculate the equivalence as a class supported on $X \subset F$, so we get a class in $A^4(X)$. More precisely, we have:

$$(p_1^*x \cdot p_2^*x)^{\Delta \cong X} = \left\{ c(p_1^*N_X F) c(p_2^*N_X F) c(N_\Delta F^2)^{-1} \cap [X] \right\}^4.$$

Now, $\Delta \subset \Delta_F \subset F^2$ are two regular imbeddings, the normal bundle of the first being $N_X F/Y \cong \mathcal{P}_S^1(\widetilde{\mathcal{L}})$ and the one of the second being $T_{S \times Y/Y} \cong T_S$, therefore:

$$(p_1^*x \cdot p_2^*x)^{\Delta \cong X} = \left\{ c\left(\mathcal{P}_S^1(\widetilde{\mathcal{L}})\right) c(T_S)^{-1} \cap [X] \right\}^4.$$

But $c(T_S) = c(T_{\mathbb{P}^2}) = (1+H)^3$ where H denotes the class of a hyperplane in \mathbb{P}^2 , with $H^3 = 0$. We get $c(T_S)^{-1} = 1 - 3H + 6H^2$. On the other hand, the exact sequence

$$0 \rightarrow \Omega_S^1 \otimes \widetilde{\mathcal{L}} \rightarrow \mathcal{P}_S^1(\widetilde{\mathcal{L}}) \rightarrow \widetilde{\mathcal{L}} \rightarrow 0$$

yields $c(\mathcal{P}_S^1(\widetilde{\mathcal{L}})) = c(\Omega_S^1 \otimes \widetilde{\mathcal{L}})c(\widetilde{\mathcal{L}})$, with $c(\widetilde{\mathcal{L}}) = d\bar{H} + \bar{H}'$ if we let H' denote the class of a hypersurface in Y and let the bar symbolize pullback to $F = S \times Y$. Considering Chern polynomials we have

$$\begin{aligned} c_t(\Omega_S^1 \otimes \widetilde{\mathcal{L}}) &= \sum_{i=0}^2 t^i (1 + tc_1(\widetilde{\mathcal{L}}))^{2-i} c_i(\Omega_S^1) \\ &= (1 + t(dH + H'))^2 + t(1 + t(dH + H'))(-3H) + t^2 3H^2. \end{aligned}$$

Also, we have $[X] = x = c_3(\mathcal{P}_S^1(\widetilde{\mathcal{L}}))$ which is equal to

$$(dH + H')^3 + (dH + H')(-3H) + (dH + H')3H^2.$$

We want the degree 4 part of the polynomial in H, H' given as

$$c(\Omega_S^1 \otimes \widetilde{\mathcal{L}})c(\widetilde{\mathcal{L}})c(T_S)^{-1} \cap [X],$$

but when pushing down to Y , only the $H^2 H'^2$ survives, and a simple calculation gives the coefficient of this to be

$$E_2 = 18d^2 - 45d + 27$$

We see that $E_2 + 2S_{A_2} = 18d^2 - 45d + 27 + 2(12d^2 - 36d + 24) = 42d^2 - 117d + 75$ (note the multiplicity 2), which is, as expected, the polynomial $-a_2$ introduced in Chapter 1, with ∂, k, s, x replaced by their values on \mathbb{P}^2 . On the other hand, the pushdown to Y of the intersection product $p_1^*x \cdot p_2^*x$ is equal to $a_1^2H'^2$ where $a_1H' = \gamma_*x = (3d^2 - 6d + 3)H'$ (we recognize the polynomial from Chapter 1).

Now it would be natural to assume that a similar procedure would yield the polynomial a_3 for 3-nodal curves, i.e. we should get a_3 by calculating the equivalences of the various diagonals in X^3 for the intersection product $(p_1^*x \cdot p_2^*x \cdot p_3^*x)$ and adding the equivalences for the embedded components which appear when we take the closure in F^3 of X^3 minus its diagonals. This, however, is not as simple as it appears. Indeed, the four diagonals in X^3 are not well separated, so one must take care not to count the same things more than once. To avoid this problem (and to proceed correctly for a higher number r of nodes), we need to pass to a compactification of the configuration space of X^r (or of F^r), where the diagonals become separated.

4.2 A compactification of configuration spaces

In this section we recall some important notions and results which will be used in our proof.

DEFINITION 4.2.1. Let F denote a non-singular algebraic variety (it makes sense to speak of the configuration space of something singular, but some of the proofs below demand a non-singular object). Let $r \geq 1$ be an integer and denote by $[r]$ the set of integers $\{1, \dots, r\}$. A *diagonal* in F^r (the fibered product over some scheme Y) is a subvariety of the form

$$\Delta_I = \{(p_1, \dots, p_r) \in F^r \mid p_i = p_j, \forall i, j \in I\}$$

for some $I \subseteq [r]$ with at least two elements. A *polydiagonal* is the intersection of diagonals; as such, it corresponds to a partition π of $[r]$ with at least one *essential block* (i.e. a block with at least two elements): we denote by $L_{[r]}$ the set of such *relevant partitions* of $[r]$. We will occasionally need to include the partition of $[r]$ into 1-blocks only; let $\overline{L}_{[r]}$ denote the set of partitions including this one. For a partition π of $[r]$, $\rho(\pi)$ denotes its number of blocks (including 1-blocks), and $s_i(\pi)$ denotes the number of blocks of size i . A diagonal in F^r corresponds to a partition with only one essential block. The polydiagonal in F^r corresponding to $\pi \in L_{[r]}$ is denoted by $\Delta_\pi^{(r)}$.

DEFINITION 4.2.2. The *configuration space* of F with respect to r points is defined by

$$\mathbb{F}(F, r) = F^r \setminus \bigcup_I \Delta_I \text{ where } I \subset [r], \#I \geq 2.$$

There are several ways to compactify this space, resulting in varieties having different properties. The first intersection theoretically interesting compactification was

done by Fulton and MacPherson in [FM]. It turns out that for our purpose the ideal compactification is *Ulyanov's polydiagonal compactification* $F\langle r \rangle$. There are several ways to construct this variety; we will concentrate on two of those, an inductive procedure introduced by Ulyanov and the viewpoint of so-called *wonderful compactifications*. We will start by the latter, referring to [Uly] and [Li] for more details. Some proofs will be included because they illustrate properties of the constructions which are important to us.

DEFINITION 4.2.3. Let V be a variety. An *arrangement* \mathcal{S} of subvarieties of V is a finite collection of non-singular subvarieties such that all non-empty scheme-theoretic intersections of subvarieties in \mathcal{S} are again in \mathcal{S} . A subset $\mathcal{G} \subseteq \mathcal{S}$ is called a *building set* of \mathcal{S} if $\forall S \in \mathcal{S}$ the minimal elements in $\{G \in \mathcal{G}, G \supset S\}$ intersect transversally and their intersection is S . If this is the case, these minimal elements are referred to as the \mathcal{G} -factors of \mathcal{S} .

More generally, a finite set \mathcal{G} of non-singular subvarieties of F is called a building set if the set of all possible intersections of collections of subvarieties from \mathcal{G} forms an arrangement \mathcal{S} having \mathcal{G} as a building set.

Given a building set \mathcal{G} of V we define the wonderful compactification $V_{\mathcal{G}}$ of the arrangement \mathcal{S} induced by \mathcal{G} as the *closure of the image* of the locally closed embedding

$$V^{\circ} \hookrightarrow \prod_{G \in \mathcal{G}} Bl_G V$$

where $V^{\circ} = V \setminus \bigcup_{G \in \mathcal{G}} G$.

EXAMPLE 4.2.4. Take $V = F^r$ and let \mathcal{G} be the building set consisting of all polydiagonals. Then the arrangement induced by \mathcal{G} is equal to \mathcal{G} , and the wonderful compactification resulting from blowing up each polydiagonal is Ulyanov's polydiagonal compactification, $F\langle r \rangle$. The configuration space being irreducible, its closure is as well, so $F\langle r \rangle$ is an *irreducible variety*. The complement of the configuration space in $F\langle r \rangle$ is a *normal crossing divisor*; more precisely, for each $\pi \in L_{[r]}$ there exists a *non-singular divisor* $D_{\pi}^{(r)} \subset F\langle r \rangle$, the union of which forms $F\langle r \rangle \setminus \mathbb{F}(F, r)$ and such that these divisors meet *transversally*.

REMARK 4.2.5. Ulyanov originally constructed his compactification $F\langle r \rangle$ by a sequence of blow-ups, consisting of $r - 1$ stages. The first stage is the blowup of the small diagonal in F^r . The k th stage, for $1 < k < r$, consists of the blowup of the disjoint union of the *previous stage proper transforms* $Y_{\pi}^{(k-1)}$ of $\Delta_{\pi}^{(r)}$ for all partitions π of $[r]$ into exactly k blocks. This method of construction shows that there exists a canonical proper morphism from $F\langle r \rangle$ to F^r , $\varphi^{(r)}$.

THEOREM 4.2.6 ([Uly], Theorem 1). *The sequence of blowups above results in a smooth compactification (assuming F is non-singular) of the configuration space*

$\mathbb{F}(F, r)$.

Proof. Let Y_k be the scheme obtained at stage k , so that we have

$$F\langle r \rangle = Y_{r-1} \longrightarrow Y_{r-2} \longrightarrow \dots \longrightarrow Y_1 \longrightarrow Y_0 = F^r$$

So $Y_\pi^{(0)}$ is simply the polydiagonal $\Delta_\pi^{(r)}$, while $Y_\pi^{(k)} \subset Y_k$ is the proper transform of $Y_\pi^{(k-1)}$ if $\rho(\pi) \neq k$, while $Y_\pi^{(\rho(\pi))}$ is the part of the exceptional divisor in Y_k over $Y_\pi^{(\rho(\pi)-1)}$. What we need to show is that the centers of the simultaneous blowups performed at a certain stage are indeed disjoint (as a result of the actions performed at earlier stages).

The procedure is such that we obtain, after each stage $1 \leq k \leq r-1$, a certain number of copies of $F\langle k+1 \rangle \subset Y_k$; more precisely, if $\pi \in L_{[r]}$ is such that $\rho(\pi) = k+1$ then $F\langle k+1 \rangle \cong Y_\pi^{(k)} \subset Y_k$. So we will do a proof by induction on k , which stops for $F\langle k+1 \rangle$ after stage k , but continues for $F\langle n \rangle$ for all $n > k+1$. We will need the following results ([Uly], Lemma 1):

LEMMA A. *Suppose W is a smooth algebraic variety with smooth subvarieties U, V intersecting cleanly (i.e. $T = U \cap V$ is a disjoint union of non-singular subvarieties and if $\mathcal{I}_U, \mathcal{I}_V$ are the ideal sheaves defining U and V , then $\mathcal{I}_U + \mathcal{I}_V = \mathcal{I}_T$). Then (1) the proper transforms of U and V in $\text{Bl}_T W$ are disjoint; furthermore, (2) for each smooth subvariety Z of $U \cap V$, the proper transforms of U and V in $\text{Bl}_Z W$ intersect cleanly.*

LEMMA B. *Two polydiagonals $\Delta_{\pi_i}^{(r)}, i \in \{1, 2\}$, in F^r intersect cleanly if and only if none of them contains the other.*

First consider the first stage, where we blow up the small diagonal in F^r . We must show that the strict transforms of the polydiagonals with two blocks become disjoint after this blowing-up. But given a pair of distinct partitions π_1, π_2 of $[r]$ into two blocks they intersect cleanly by remark B and we have $\Delta_{\pi_1}^{(r)} \cap \Delta_{\pi_2}^{(r)} = \Delta$, the small diagonal of F^r . Then by Lemma A, (1), their proper transforms $Y_{\pi_i}^{(1)}$ in $Y^{(1)}$ are disjoint, so the procedure may continue to the second stage.

Now assume that the stage $k-1$ has been performed for some $k \geq 2$. Then the varieties Y_n have been constructed for all $1 \leq n \leq k-1$, as well as all the $F\langle n \rangle$ for $2 \leq n \leq k$, while those for $n > k$ still remain unconstructed. The inductive assumption is that all the proper transforms $Y_\pi^{(k-1)} \subset Y^{(k-1)}$ for $\rho(\pi) = k$ are disjoint. Consider a partition $\pi \in L_{[r]}$ with $\rho(\pi) = k$. By the projection $F\langle k \rangle \rightarrow F^k$, the isomorphism $F^k \cong \Delta_\pi^{(r)} \subset F^r$ pulls back to $F\langle k \rangle \cong Y_\pi^{(k-1)} \subset Y_{k-1}$. By assumption, these subvarieties are all disjoint, so we may blow them up at the same time, each of them pulling back to what is, by definition, $F\langle k+1 \rangle$ in

$$Y_k = \text{Bl}_{\coprod_{\rho(\pi)=k} Y_\pi^{(k-1)}} Y_{k-1}.$$

We must show that $Y_{\pi_1}^{(k)} \cap Y_{\pi_2}^{(k)} = \emptyset$ whenever $\pi_i \in L_{[r]}$ are distinct partitions into $k + 1$ blocks. Such a pair intersects cleanly by Lemma B. Define $\Delta_{\pi_1 \wedge \pi_2}^{(r)}$ to be $\Delta_{\pi_1}^{(r)} \cap \Delta_{\pi_2}^{(r)}$, then $\rho := \rho(\pi_1 \wedge \pi_2) < k + 1$. Using several times Lemma A (2) we see that $Y_{\pi_1}^{(\rho-1)} \cap Y_{\pi_2}^{(\rho-1)} = Y_{\pi_1 \wedge \pi_2}^{(\rho-1)}$ is a clean intersection, but then, by Lemma A (1), the intersection $Y_{\pi_1}^{(\rho)} \cap Y_{\pi_2}^{(\rho)} = \emptyset$. This completes the inductive step. \square

PROPOSITION 4.2.7 ([Uly], Proposition 3). *The two constructions introduced above yield the same compactification of $\mathbb{F}(F, r)$.*

Proof. We wish to show that the variety $F\langle r \rangle$ constructed by stages of blowing-ups is equal to the closure of $\mathbb{F}(F, r)$ in

$$\prod_{\pi \in L_{[r]}} \text{Bl}_{\Delta_{\pi}^{(r)}} F^r.$$

First notice that we might just as well consider the closure in $F^r \times \prod_{\pi \in L_{[r]}} \text{Bl}_{\Delta_{\pi}^{(r)}} F^r$, which will, in fact, be convenient for our proof: We proceed by induction on k , showing that each Y_k in the sequence of blowing-ups is the closure of $\mathbb{F}(F, r)$ in

$$F^r \times \prod_{\rho(\pi) \leq k} \text{Bl}_{\Delta_{\pi}^{(r)}} F^r.$$

Clearly, this is true for $k = 0$, since $Y_0 = F^r$. Assume the result is true for $k - 1$, where $k \geq 1$. We have, by definition, that Y_k is the blowup of Y_{k-1} along

$$\prod_{\rho(\pi)=k} Y_{\pi}^{(k-1)}.$$

Considering *ideal sheaves* instead of subschemes, we blow up along

$$\mathcal{I} \left(\prod_{\rho(\pi)=k} Y_{\pi}^{(k-1)} \right) = \prod_{\rho(\pi)=k} \mathcal{I}(Y_{\pi}^{(k-1)})$$

where $\mathcal{I}(Y_{\pi}^{(k)})$ denotes the ideal sheaf of $Y_{\pi}^{(k)}$ in \mathcal{O}_{Y_k} . If we let $\tau_k : Y_k \rightarrow F^r$ denote the blow-down morphism and $\mathcal{I}_k(\pi)$ denote the ideal sheaf in \mathcal{O}_{Y_k} generated by $\tau_k^*(\mathcal{I}(\Delta_{\pi}^{(r)}))$, then, since $Y_{\pi'}^{(k)}$ is a divisor in Y_k whenever $\rho(\pi') < k$, it follows that $\mathcal{I}_k(\pi) = \mathcal{I}(Y_{\pi}^{(k)}) \cdot \mathcal{I}$ for some invertible ideal sheaf \mathcal{I} . So the ideal sheaf above becomes, after multiplication with some invertible ideal sheaf,

$$\mathcal{I}_{k-1} = \prod_{\rho(\pi)=k} \mathcal{I}_{k-1}(\pi).$$

But blowing up \mathcal{I}_{k-1} is the same as taking the closure of the graph of the rational map from Y_{k-1} to $\prod_{\rho(\pi)=k} \text{Bl}_{\Delta_{\pi}^{(r)}} F^r$, which shows that Y_k is the closure of $\mathbb{F}(F, r)$ in

$$F^r \times \prod_{\rho(\pi) \leq k} \text{Bl}_{\Delta_{\pi}^{(r)}} F^r,$$

as we wanted. \square

4.3 Proof of the main theorem

Let S be a smooth, irreducible projective surface, and consider a linear system of curves $|\mathcal{L}|$ on S . Suppose this system is *sufficiently ample*, so that the considerations below make sense for the relevant values of r , the number of nodes (indeed, a curve in $|\mathcal{L}|$ cannot have an unlimited number of nodes; in particular, we must have at least $\dim |\mathcal{L}| \geq r$). We let Y be $\mathbb{P}(H^0(S, \mathcal{L}))$, so that each point $y \in Y$ corresponds to a curve $D_y \subset S$. H' denotes the class of a hypersurface in Y . The object of this section is the proof of the following theorem:

THEOREM 4.3.1. *There exist classes $a_i \in A^i(Y)$, $i \geq 1$, depending on S and \mathcal{L} , such that for all relevant r (i.e. values of $r \geq 1$ such that \mathcal{L} is sufficiently ample with respect to r), the number $N(r, \mathcal{L})$ of r -nodal curves in the linear system $|\mathcal{L}|$ is given by*

$$N(r, \mathcal{L})H^r = \frac{P_r(a_1, \dots, a_r)}{r!},$$

where P_r is the r th complete Bell polynomial.

Let $\mathcal{D} \subset S \times Y$ be the set of pairs (x, y) such that $x \in D_y$ and consider the closure X of the set of pairs $(x, y) \in S \times Y$ such that $x \in D_y$ is a simple node. Note that while $F = S \times Y$ is a non-singular projective variety, X is not necessarily non-singular, which is why the intersection theory must take place on F . Like before, we have the following situation:

$$X \hookrightarrow \mathcal{D} \hookrightarrow F = S \times Y \xrightarrow{\gamma} Y$$

Considering r -nodal curves in $|\mathcal{L}|$ now amounts to studying the closure of

$$\underbrace{(X \times_Y X \times_Y \dots \times_Y X)}_{r \text{ factors}} \setminus \{\Delta_\pi^{(r)}(X), \pi \in L_{[r]}\}$$

(we must exclude the diagonals to avoid coinciding nodes) in $F \times_Y F \times_Y \dots \times_Y F$. (We will use the notation F^r for this product, keeping in mind that what we are considering are fibered products over Y , so $F^r \cong S^r \times Y$). More precisely, we are interested in the pushdown to Y of the rational equivalence class in $A_*(F^r)$ of this variety (cf. Remark 4.3.5).

The problem on F^r is that the polydiagonals are not well separated, in the sense that they intersect in a very inconvenient way. To simplify our efforts (considerably), the calculations must take place on the polydiagonal compactification $F\langle r \rangle$, where the complement of the configuration space is a normal crossing divisor, with one divisor for each polydiagonal in F^r , intersecting each other transversally in a somewhat more convenient way (see [Uly] for more on this):

We say that a collection of partitions $(\pi_i)_{i \in I}$ of $[r]$ form a *chain* if they form a totally ordered subset of $\overline{L_{[r]}}$ for the *partial ordering* \preceq defined by $\pi_1 \preceq \pi_2$ whenever

each block of π_2 is contained in a block of π_1 . We will use the same notation for divisors, i.e. write $D_{\pi_1}^{(r)} \preceq D_{\pi_2}^{(r)}$ and refer to the first divisor as *smaller* than the second.

PROPOSITION 4.3.2 ([Uly], Proposition 1). *An intersection $\bigcap_{i=1}^k D_{\pi_i}^{(r)}$ of polydiagonal divisors in $F\langle r \rangle$ is non-empty if and only if the partitions π_i form a chain.*

Let $x \in A_*(F)$ be the rational equivalence of X and consider the natural embeddings $\mathbb{F}(X, r) \hookrightarrow \mathbb{F}(F, r) \hookrightarrow F\langle r \rangle$. Write (note the alternating signs):

$$C_r = [\mathbb{F}(X, r)] = \prod_{k=1}^r \theta_k^{(r)*} x + \sum_{\pi \in L_{[r]}} (-1)^{r+\rho(\pi)} B_{\pi}^{(r)} \in A_*(F\langle r \rangle),$$

where $B_{\pi}^{(r)}$ is a class supported on the divisor $D_{\pi}^{(r)}$. Here a great difficulty appears. Most likely, the appropriate definition of $B_{\pi}^{(r)}$ is the following: let $\mathfrak{C}_{\pi}^{(r)}$ denote the set of connected component of $\bigcap_{k=1}^r |X_k^{(r)}| \cap |D_{\pi}^{(r)}|$ (where $X_k^{(r)}$ denotes $\theta_k^{(r)-1}(X)$) and we "conventionally" consider $\mathfrak{C}_1^{(1)}$ to consist uniquely of $|X|$, with the induced scheme structure from $\bigcap X_k^{(r)}$, then

$$\begin{aligned} B_{\pi}^{(r)} &:= \sum_{Z \in \mathfrak{C}_{\pi}^{(r)}} (X_1^{(r)} \cdot \dots \cdot X_r^{(r)})^Z \\ &= \sum_{Z \in \mathfrak{C}_{\pi}^{(r)}} \left\{ \prod_{k=1}^r c(N_k^{(r)}|Z) \cap s(Z, F\langle r \rangle) \right\}_{m_r}, \end{aligned}$$

where $N_k^{(r)}$ is the normal bundle of $X_k^{(r)}$ in $F\langle r \rangle$ and

$$m_r = \dim F\langle r \rangle - \sum_{k=1}^r \operatorname{codim} (X_k^{(r)}, F\langle r \rangle).$$

The problem is that even though the divisors $D_{\pi}^{(r)}$ are more separated than the polydiagonals $\Delta_{\pi}^{(r)}$, they still intersect (transversally) for chains of partitions. This means we *could* get connected components belonging to different divisors, but which *still intersect*. If so, we risk counting the same contributions to certain equivalences more than once. Even so, we clearly see the importance of working on $F\langle r \rangle$ instead of F^r ; on F^r , even greater care must be taken to avoid counting several times the same equivalences, because the connected components similar to the ones considered above are *not* in any way well separated. The situation on $F\langle r \rangle$ is less simple than one could hope for, but at least the intersections are transversal, and non-empty only for chains of partitions, which makes the entire problem more manageable. To solve the problem in this case, one could introduce, for each $0 \leq l \leq r - 1 - \rho(\pi)$, a set $\mathfrak{C}_{\pi, l}^r$ consisting of connected components of intersections of the form

$$\bigcap_{k=1}^r |X_k^{(r)}| \cap |D_{\pi}^{(r)}| \cap \bigcap_{s=1}^l |D_{\pi_s}^{(r)}|$$

for chains $\pi \prec \pi_1 \prec \dots \prec \pi_l$. Then $B_\pi^{(r)}$ should be defined as

$$\begin{aligned} B_\pi^{(r)} &= \sum_{l=0}^{r-1-\rho(\pi)} (-1)^l \sum_{Z \in \mathfrak{C}_{\pi,l}^{(r)}} (X_1^{(r)} \cdots X_r^{(r)})^Z \\ &= \sum_{l=0}^{r-1-\rho(\pi)} (-1)^l \sum_{Z \in \mathfrak{C}_{\pi,l}^{(r)}} \left\{ \prod_{k=1}^r c(N_k^{(r)}|Z) \cap s(Z, F\langle r \rangle) \right\}_{m_r} \end{aligned}$$

Unfortunately, this creates a huge obstacle in the proof of Proposition 4.3.4. We will therefore assume that the connected components belonging to different divisors *do not intersect*. Although we have not been able to prove this is the case, the fact should intervene that we are working with linear systems which are sufficiently ample with respect to the number of nodes considered, making this simplification justifiable.

In the following lemma we will restrict ourselves to *simple partitions* of $[r]$, which are partitions obtained by writing down the numbers from 1 to r and then adding symbols $|$ in such a way the smallest blocks appear first (for instance $12|34|567$ is simple, while $2|45|13$ and $24|1|35$ are not). The reason for doing this restriction is to avoid unnecessarily complicated notations when it comes to the projections from the $F\langle i \rangle$ to F^r (see below). We lose nothing by doing this; indeed, everything is symmetric in the F 's, and we are ultimately interested in pushing down the $B_\pi^{(r)}$ to Y through a single F , so everything that matters is the shape of the partition π , which may therefore be taken to be simple.

LEMMA 4.3.3. *Let $\pi \in L_{[r]}$ be a simple partition. There exists a birational proper morphism $\varphi_\pi^{(r)}$ from $F\langle r \rangle$ to $\prod_i F\langle i \rangle^{s_i(\pi)}$ such that the following diagram commutes:*

$$\begin{array}{ccc} F\langle r \rangle & \xrightarrow{\varphi_\pi^{(r)}} & \prod_i F\langle i \rangle^{s_i(\pi)} \\ \downarrow \varphi^{(r)} & \swarrow & \prod (\varphi^{(i)})^{s_i(\pi)} \\ F^r & & \end{array}$$

(and such that $D_\pi^{(r)}$ is mapped to $\prod_i (D_{1\dots i}^{(i)})^{s_i(\pi)}$).

Proof. Note that such a result is trivial if $\pi = 12\dots r$, by taking $\varphi_\pi^{(r)}$ to be the identity. In the other cases, it suffices to construct, for each $1 \leq i < r$ such that there is at least one block in π of size i , and each $1 \leq j \leq s_i(\pi)$, a proper morphism $\varphi_{\pi,i,j}^{(r)} : F\langle r \rangle \rightarrow F\langle i \rangle$ such that the composition with $\varphi^{(i)}$ yields the same as composing $\varphi^{(r)}$ with the projection p_{ij} from F^r to the F^i which appears in F^r according to the blocks of the divisor π . (For instance, if $\pi = 1|23|45$, then we have projections $p_{11} : F^5 \rightarrow F, p_{21} : F^5 \rightarrow F^2$ and $p_{22} : F^5 \rightarrow F^2$).

The existence of such a morphism results from the universal property of blow-ups. If we let $\mathcal{I}_{\pi'}^{(i)}$ denote the sheaf of ideals on F^i corresponding to the closed subscheme $\Delta_{\pi'}^{(i)}$, we must simply check that $(p_{ij} \circ \varphi^{(r)})^{-1} \mathcal{I}^{(i)} \cdot \mathcal{O}_{F\langle r \rangle}$ is an invertible sheaf, where $\mathcal{I}^{(i)}$ is the sheaf of ideals $\prod_{\pi' \in \mathcal{L}_{[i]}} \mathcal{I}_{\pi'}^{(i)}$. If π' is a partition of $[i]$ then p_{ij} induces a partition $\tilde{\pi}'$ of $[r]$ by adding 1-blocks. (For instance, consider the following example: $\pi = 1|234|567$ with projections p_{11}, p_{31} and p_{32} from F^6 to F, F^3, F^3 , and $\pi' = 1|23$ a partition associated to the first factor F^3 . Then the partition on $[7]$ induced by p_{31} is $1|2|34|5|6|7$.) We have:

$$\begin{aligned}
(p_{ij} \circ \varphi^{(r)})^{-1} \mathcal{I}^{(i)} \cdot \mathcal{O}_{F\langle r \rangle} &= \varphi^{(r)-1} p_{ij}^{-1} \mathcal{I}^{(i)} \cdot \mathcal{O}_{F\langle r \rangle} \\
&= \varphi^{(r)-1} p_{ij}^{-1} \prod_{\pi' \in \mathcal{L}_{[i]}} \mathcal{I}_{\pi'}^{(i)} \cdot \mathcal{O}_{F\langle r \rangle} \\
&= \prod_{\pi' \in \mathcal{L}_{[i]}} \varphi^{(r)-1} (p_{ij}^{-1} \mathcal{I}_{\pi'}^{(i)} \cdot \mathcal{O}_{F^r}) \cdot \mathcal{O}_{F\langle r \rangle} \\
&= \prod_{\pi' \in \mathcal{L}_{[i]}} \varphi^{(r)-1} \mathcal{I}_{\tilde{\pi}'}^{(r)} \cdot \mathcal{O}_{F\langle r \rangle}.
\end{aligned}$$

Here we use that if $f : X \rightarrow Y$ is a morphism of schemes and Z is a closed subscheme of Y defined by a sheaf of ideals \mathcal{I} , then $f^{-1}(Z)$ is defined by the inverse image ideal sheaf $f^{-1}(\mathcal{I}) \cdot \mathcal{O}_X$. So if $\mathcal{I}_{\pi}^{(i)}$ defines $\Delta_{\pi}^{(i)}$ on F^i then $p_{ij}^{-1} \mathcal{I}_{\pi}^{(i)} \cdot \mathcal{O}_{F^r}$ defines $\Delta_{\tilde{\pi}}^{(r)}$ and as such must be equal to $\mathcal{I}_{\tilde{\pi}}^{(r)}$.

The last expression obtained above is a product of invertible sheaves, thus an invertible sheaf, which is what we wanted. \square

As a birational mapping, the surjective morphism $\varphi_{\pi}^{(r)}$ is a degree 1 map from $F\langle r \rangle$ to $\prod F\langle i \rangle^{s_i(\pi)}$. Here we should perhaps explain a bit more naively what the morphism $\varphi_{\pi}^{(r)}$ does. Recall that $F\langle r \rangle$ contains $\mathbb{F}(F, r)$ as well a divisor $D_{\pi'}^{(r)}$ for each $\pi' \in L_{[r]}$. On $\mathbb{F}(F, r)$ the morphism $\varphi_{\pi}^{(r)}$ sends a point (x_1, \dots, x_r) to its natural image in $\prod_i \mathbb{F}(F, i)^{s_i(\pi)}$. Divisors corresponding to partitions $\pi' \succeq \pi$ are mapped, in a way that respects the blocks, to products of divisors in $\prod_i F\langle i \rangle^{s_i(\pi)}$, with the convention that $D_{1|2|\dots|i}^{(i)} = F\langle i \rangle$. When it comes to divisors which are not comparable (by \preceq) to π , they are partially mapped into $\prod \mathbb{F}(F, i)^{s_i(\pi)}$, but as it turns out, we are not really too concerned about the details of this.

At this point, letting γ be the natural projection from $F = S \times Y$ to Y , we have the following commutative diagram:

$$\begin{array}{ccc}
F\langle r \rangle & \xrightarrow{\varphi_\pi^{(r)}} & \prod_i F\langle i \rangle^{s_i(\pi)} \\
\downarrow \varphi^{(r)} & \searrow \prod_{ij} \varphi^{(i)} & \downarrow \\
& F^r & \\
\downarrow \theta_k^{(r)} & \downarrow p_k^{(r)} & \downarrow \\
& F & \\
& \downarrow \gamma & \\
& Y &
\end{array}$$

PROPOSITION 4.3.4. Define classes $a_i \in A^*(Y)$ for each relevant $i \geq 1$ by

$$a_i = (-1)^{i+1} \gamma_* \theta_{i*}^{(i)} B_{12\dots i}^{(i)} \in A^i(Y).$$

Then, for each relevant partition π of $[r]$, we have

$$(-1)^{r+\rho(\pi)} \gamma_* \theta_{r*}^{(r)} B_\pi^{(r)} = \prod_{i=1}^r a_i^{s_i(\pi)}.$$

Proof. Let us first check the dimensional aspect of this definition-proposition. The class $B_\pi^{(r)}$ supported on $D_\pi^{(r)}$ is defined as a sum of equivalences of components supported on $D_\pi^{(r)}$ for the intersection product $\prod_{k=1}^r \theta_k^{(r)*} x$. As such, we have $B_\pi^{(r)} \in A_{m_r}(F\langle r \rangle)$ where

$$\begin{aligned}
m_r &= \dim F\langle r \rangle - \sum_{k=1}^r \operatorname{codim}(X_k^{(r)}, F\langle r \rangle) \\
&= \dim (S\langle r \rangle \times Y) - \sum_{k=1}^r \operatorname{codim}(X_k^{(r)}, F\langle r \rangle) \\
&= 2r + \dim Y - 3r = \dim Y - r,
\end{aligned}$$

since the morphism $F\langle r \rangle \rightarrow F^r$ is birational and X has codimension 3 in $F = S \times Y$. Since the pushdown preserves dimension, we get $\gamma_* \theta_{r*}^{(r)} B_\pi^{(r)} \in A^r(Y)$ and, applying for $r = i$ and $\pi = 1 \dots i$, $a_i \in A^i(Y)$; So $a_i = U_i(S, \mathcal{L}) H^i$, where H^i denotes the class of a hypersurface in Y and $U_i(S, \mathcal{L}) \in \mathbb{Z}$. See the remark below for the enumerative meaning of this.

Here we may, for symmetry reasons, reduce to the consideration of *simple partitions* π . That is, only the shape of the partition π matters for the final value of the pushdown to Y of $B_\pi^{(r)}$.

By $\varphi_\pi^{(r)}$, the set $\mathfrak{C}_\pi^{(r)}$ corresponds bijectively to the product

$$\prod_i \prod_{j=1}^{s_i(\pi)} \mathfrak{C}_{1\dots i}^{(i)}$$

(recall the "convention" that $\mathfrak{C}_1^{(1)}$ consists simply of $|X|$). Indeed, if \tilde{Z} is a connected component of $\prod_{i,j} (\bigcap_{k=1}^i |X_k^{(i)}| \cap |D_{1\dots i}^{(i)}|)$ then *a priori* it pulls back, through $\varphi_\pi^{(r)}$, to a union of connected components in $\mathfrak{C}_\pi^{(r)}$. But $\varphi_\pi^{(r)}$ has degree 1, so there is only one connected component in the inverse image. Also, a connected component of a product is a product of connected components, so let $Z \in \mathfrak{C}_\pi^{(r)}$ correspond to $\prod_{i,j} Z_{ij}$, where $Z_{ij} \in \mathfrak{C}_{1\dots i}^{(i)}$. Since $\varphi_\pi^{(r)}$ realizes a degree 1 surjective map $F\langle r \rangle \rightarrow \prod F\langle i \rangle^{s_i(\pi)}$ we have, by Proposition 4.2 and Example 4.2.5 in [Ful]:

$$\varphi_{\pi*}^{(r)} s(Z, F\langle r \rangle) = s \left(\prod_{i,j} Z_{ij}, \prod_{i,j} F\langle i \rangle \right) = \prod_{i,j} s(Z_{ij}, F\langle i \rangle).$$

On the other hand, if we consider the partial morphisms $\varphi_{\pi,i,j}^{(r)}$ defining $\varphi_\pi^{(r)}$, then for each i such that $s_i(\pi) \neq 0$ and each $1 \leq j \leq s_i(\pi)$, $1 \leq k \leq i$, we have

$$\varphi_{\pi,i,j}^{(r)*} N_k^{(i)} = N_{\sum_{i' < i} s_{i'}(\pi) i' + (j-1)s_i(\pi) + k}^{(r)}$$

(consider the pullback of the normal bundle of X in F to respectively the appropriate $F\langle i \rangle$ through F^i , and to $F\langle r \rangle$ through the same F^i and then F^r), so by the standard projection formula A.2.4, (3), we get

$$\varphi_{\pi*}^{(r)} B_\pi^{(r)} = \sum_{Z \in \mathfrak{C}_\pi^{(r)}} \left\{ \bigotimes_i \bigotimes_{j=1}^{s_i(\pi)} \prod_{k=1}^i c(N_k^{(i)}|Z_{ij}) \cap s(Z_{ij}, F\langle i \rangle) \right\}_{m_r}.$$

Notice that if $m_i = \dim Y - i$ and we take the exterior product (fibered over Y) of classes in $A_{m_i}(F\langle i \rangle)$ for each i such that $s_i(\pi) \neq 0$ and each $1 \leq j \leq s_i(\pi)$, we get a class of dimension $\dim Y - \sum_i i s_i(\pi) = \dim Y - r = m_r$. So (since summing over \mathfrak{C}_π^r comes down to summing over $\prod_{i,j} \mathfrak{C}_{1\dots i}^{(i)}$) we may write this expression as

$$\bigotimes_i \bigotimes_{j=1}^{s_i(\pi)} \sum_{Z \in \mathfrak{C}_{1\dots i}^{(i)}} \left\{ \prod_{k=1}^i c(N_k^{(i)}|Z) \cap s(Z, F\langle i \rangle) \right\}_{m_i},$$

which shows us that $\varphi_{\pi*}^{(r)} B_\pi^{(r)} = \bigotimes_{i,j} B_{1\dots i}^{(i)}$. Pushing this down to Y (with the appropriate signs) yields $\prod_i a_i^{s_i(\pi)}$ as we wanted. \square

REMARK 4.3.5. *Enumerative meaning.* Above, we have shown that the pushdown to Y of the class representing r -nodal curves, $C_r \in A^*(F\langle r \rangle)$, yields a sum of products of the form $\prod_i a_i^{s_i(\pi)}$ with $\sum i s_i(\pi) = r$. Since each class a_i is of the form $U_i H^i$ for some (unknown) expression U_i , we get that the pushdown of C_r to Y yields a class in $A^r(Y)$. This means that intersecting with $\dim Y - r = N - r$ hypersurfaces, each representing some codimension 1 condition on our curves (such as passing through a general fixed point on S) we get an enumerative expression, which is the number $N_S(r, \mathcal{L})$ we searched for in the first place. We are now ready to prove the main theorem:

THEOREM 4.3.1. *There exist classes $a_i \in A^i(Y)$, $i \geq 1$, depending on S and \mathcal{L} , such that for all relevant r (i.e. values of $r \geq 1$ such that \mathcal{L} is sufficiently ample with respect to r), the number $N(r, \mathcal{L})$ of r -nodal curves in the linear system $|\mathcal{L}|$ is given by*

$$N(r, \mathcal{L})H^r = \frac{P_r(a_1, \dots, a_r)}{r!},$$

where P_r is the r th complete Bell polynomial.

Proof. Since there are $r!$ ways to arrange the r nodes we have

$$N(r, \mathcal{L})H^r = \frac{1}{r!} \gamma_* \theta_{r*}^{(r)} C_r$$

Now $\prod_{k=1}^r \theta_k^{(r)*}(x)$ pushes down to a_1^r , and each $B_\pi^{(r)}$, $\pi \in L_{[r]}$, pushes down to $\prod a_i^{s_i(\pi)}$. This means that

$$N(r, \mathcal{L})H^r = \frac{1}{r!} \underbrace{\sum_{j_1 + \dots + r j_r = r} e_{j_1 \dots j_r} \prod_{i=1}^r a_i^{j_i}}_{P_r(a_1, \dots, a_r)}$$

where $e_{j_1 \dots j_r}$ is the number of polydiagonals with j_i blocks of size i . If we regroup the polydiagonals by their number of blocks and note that polydiagonals with k blocks can have no blocks of size $> r - k + 1$ (indeed, each block must have at least one element, so we would get a number of elements $> (k - 1) \cdot 1 + r - k + 1 = r$, which is impossible), then

$$P_r(a_1, \dots, a_r) = \sum_{k=1}^r \sum_{I_{r,k}} \tilde{e}_{j_1 \dots j_{r-k+1}} \prod_{i=1}^{r-k+1} a_i^{j_i}$$

where $I_{r,k}$ is the set of tuples (j_1, \dots, j_{r-k+1}) such that we have $\sum_{i=1}^{r-k+1} i j_i = r$ and $\sum_{i=1}^{r-k+1} j_i = k$ (so $\sum j_i$ is the number of blocks and $\sum i j_i$ is the number of elements for the corresponding partition). Here, the coefficient $\tilde{e}_{j_1 \dots j_{r-k+1}}$ is the number of polydiagonals with k blocks of which j_i have size i . But this is exactly how the coefficients of the partial Bell polynomials are defined (see Appendix A), so P_r is the r th complete Bell polynomial in the a_i , $1 \leq i \leq r$, which is what we wanted to prove. \square

APPENDIX A

TOPICS FROM ALGEBRA AND ALGEBRAIC GEOMETRY

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A.1 Moduli spaces

We have made some use of the theory of moduli spaces in Chapter 3. The essential problem of moduli theory is the classification of geometric objects (varieties, schemes, bundles, maps...) up to some given equivalence, like isomorphism, birationality etc. The object of this section is to give a short overview of the definitions used. We will make the distinction between fine and coarse moduli spaces.

DEFINITION A.1.1. *Family. Fine moduli space.* By a *family over a base scheme* B we mean a morphism $\Upsilon \rightarrow B$ together with some extra structure (depending on the case studied) such that if $\varphi : B' \rightarrow B$ is a morphism, we have an induced family over B' , denoted by $\varphi^*\Upsilon/B'$.

By a *universal family* for a *moduli problem* we mean a family U/M (where the base M will be referred to as a *fine moduli space*) such that for any family Υ/B there exists a unique morphism $\kappa : B \rightarrow M$ such that $\kappa^*U \cong \Upsilon$ as families over B (\cong denotes some given equivalence, compatible with the pullback operation). So the important property of the fine moduli space M is that for each base B , there is a bijection between the set of equivalence classes of families over B and the set of morphisms $B \rightarrow M$.

DEFINITION A.1.2. *Representable functors.* A functor F is said to be *representable* if it is isomorphic to a *functor of points* for some scheme M , i.e. the

functor $h_M : \mathfrak{Sch} \rightarrow \mathfrak{Set}$ such that $B \mapsto \text{Hom}(B, M)$ and a morphism $\varphi : B' \rightarrow B$ is mapped to the map of sets from $\text{Hom}(B, M)$ to $\text{Hom}(B', M)$ given by $\beta \mapsto \beta \circ \varphi$. If $\mathbf{u} : h_M \xrightarrow{\sim} F$ is an appropriate isomorphism of functors, we say that the pair (M, \mathbf{u}) represents F . However, by Yoneda's lemma the transformation \mathbf{u} can be identified with some $U \in F(M)$, and we also say that (M, U) represents F .

REMARK A.1.3. As an alternative definition of fine moduli spaces, we may describe a moduli problem as a contravariant functor F from the category of schemes \mathfrak{Sch} to the category of sets \mathfrak{Set} , taking B on the equivalence classes of families over B and sending a morphism $\varphi : B' \rightarrow B$ to $\varphi^* : F(B) \rightarrow F(B')$.

We say that a family U/M is a universal family and that M is a fine moduli space for F if the pair (M, U) represents F .

EXAMPLE A.1.4. In the case where $B = \{*\}$ is a point, we get a classification of objects, as a family over $\{*\}$ is just a object, and a morphism $\{*\} \rightarrow M$ is a point in M , i.e. the geometric points of M correspond bijectively to equivalence classes of objects.

DEFINITION A.1.5. A *coarse moduli space* for a moduli functor F is a pair (M, \mathbf{v}) consisting of a scheme M and a transformation $\mathbf{v} : F \rightarrow h_M$ such that (M, \mathbf{v}) is initial among such pairs and the set map $\mathbf{v}_* : F(*) \rightarrow \text{Hom}(*, M)$ is a bijection. By an initial pair we mean a pair (M, \mathbf{v}) such that for any other pair (M', \mathbf{v}') with $\mathbf{v}' : F \rightarrow h_{M'}$ a natural transformation, there is a unique morphism $\psi : M \rightarrow M'$ such that $\mathbf{v}' = \psi \circ \mathbf{v}$.

By definition, a fine moduli space is also a coarse moduli space, while the converse is not true.

A.2 Intersection theory and Chern classes

Modern intersection theory represents the accumulation of ideas and contributions from several important mathematicians over the last centuries. In this section we will simply recall some of the fundamental definitions and the most important results for our purpose. We refer to the first 10 chapters of [Ful] for more details.

DEFINITION A.2.1. Let X denote an algebraic scheme of dimension n (over \mathbb{C} , that is, with a finite morphism to $\text{Spec } \mathbb{C}$). By a *k-cycle* we mean a finite formal sum $\sum n_i [V_i]$ where the V_i are k -dimensional subvarieties of X and $n_i \in \mathbb{Z}$. The free abelian group on the k -dimensional subvarieties of X is called the *group of k-cycles* on X , $Z_k X$. Now if W is a $(k+1)$ -dimensional subvariety of X and $r \in R(W)^*$, the function field of W , i.e. $\mathcal{O}_{W,X}/\mathfrak{m}_{W,X}$, we may define a k -cycle $[\text{div}(r)]$ on X by

$$[\text{div}(r)] = \sum \text{ord}_V(r)[V], \text{ where } \text{ord}_V(r) = l_{\mathcal{O}_{V,X}}(\mathcal{O}_{V,X}/(r)),$$

the sum being over all codimension 1 subvarieties V of W . Note that for a fixed $r \in R(W)^*$ there are only finitely many codimension one subvarieties V of W such

that $\text{ord}_V(r) \neq 0$, so this is indeed a k -cycle. A k -cycle α is *rationaly equivalent* to 0 if it is the sum of finitely many such k -cycles. Since $[\text{div}(r^{-1})] = -[\text{div}(r)]$, cycles rationaly equivalent to 0 form a subgroup of $Z_k X$, the factor group is noted $A_k X$. Let $A_* X$ denote $\bigoplus_{k=0}^n A_k X$.

PROPOSITION A.2.2. If $f : X \rightarrow Y$ is a proper morphism we may define a *push-forward homomorphism* of cycles $f_* : Z_k X \rightarrow Z_k Y$ by

$$f_*[V] = \deg(V/f(V))[f(V)]$$

and extending by linearity. We get induced a homomorphism $f_* : A_k X \rightarrow A_k Y$.

For schemes X which are proper over $\text{Spec } \mathbb{C}$ (complete), the *degree* of a 0-cycle $\alpha = \sum_P n_P [P]$ is

$$\deg(\alpha) = \int_X \alpha = \sum_P n_P = p_* \alpha,$$

where p is the structure morphism $X \rightarrow \text{Spec } \mathbb{C}$. Extend this to a degree homomorphism on $A_* X$ by $\int_X \alpha = 0$ for $\alpha \in A_k X, k > 0$.

Now consider a flat morphism $f : X \rightarrow Y$ of relative dimension m , i.e. such that:

1. for all affine open sets $U \subset Y, U' \subset X$ with $f(U') \subset U$, the induced map $f^* : A(U) \rightarrow A(U')$ makes $A(U')$ into a flat $A(U)$ -module;
2. for all subvarieties V of Y and all irreducible components V' of $f^{-1}(V)$, we have $\dim V' = \dim V + m$.

If V is a subvariety of Y then define $f^*[V] = [f^{-1}(V)]$, where $f^{-1}(V)$ is the inverse image scheme. Then by linearity this extends to *pull-back homomorphisms* $f^* : Z_k Y \rightarrow Z_{k+m} X$, which induce flat pull-backs $f^* : A_k Y \rightarrow A_{k+m} X$.

DEFINITION A.2.3. *Exterior product.* Suppose X, Y are two algebraic schemes over $\text{Spec } \mathbb{C}$; let $X \times Y$ denote their Cartesian fibre product. We define the exterior product $Z_k X \otimes Z_l Y \xrightarrow{\times} Z_{k+l}(X \times Y)$ by $[V] \times [W] = [V \times W]$ and extending bilinearly to general cycles. Then, if $\alpha \sim 0$ or $\beta \sim 0$, it follows that $\alpha \times \beta \sim 0$. Also, if we let $f : X' \rightarrow X$ and $g : Y' \rightarrow Y$ be morphisms and $f \times g$ denote the induced morphism $X' \times Y' \rightarrow X \times Y$, then $f \times g$ is proper if f and g are proper, with

$$(f \times g)_*(\alpha \times \beta) = f_* \alpha \times g_* \beta.$$

We also have a corresponding result for flat morphisms and pullbacks. It follows that we also have exterior products $A_k X \otimes A_l Y \rightarrow A_{k+l}(X \times Y)$ satisfying these properties.

DEFINITION A.2.4. *Chern classes and Segre classes.* If E is a vector bundle of rank $e + 1$ on a scheme X , $P = \mathbb{P}(E)$ is the projective bundle of lines in E and $p : P \rightarrow X$ is the projection to X , let $\mathcal{O}(1)$ denote the canonical line bundle on P and define homomorphisms $A_k X \rightarrow A_{k-i} X$ by

$$\alpha \mapsto s_i(E) \cap \alpha = p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^* \alpha),$$

where the *first Chern class* $c_1(\mathcal{L})$ of a line bundle \mathcal{L} is defined by $c_1(\mathcal{L}) \cap [V] = [C]$ where V is a k -dimensional subvariety of X and C is a Cartier divisor such that $\mathcal{L}|_V \cong \mathcal{O}_V(C)$. This is then extended by linearity to give a homomorphism $\alpha \mapsto c_1(\mathcal{L}) \cap \alpha$ from $Z_k(X)$ to $A_{k-1}(X)$, which again induces a homomorphism $A_k X \rightarrow A_{k-1} X$.

The homomorphisms $s_i(E) \cap -$ defined above are referred to as the *Segre classes* of the vector bundle E . To define the Chern classes of E , consider the formal power series in t

$$s_t(E) = \sum_{i=0}^{\infty} s_i(E) t^i = 1 + s_1(E) t + s_2(E) t^2 + \dots$$

and let the *Chern polynomial* be the inverse power series, $c_t(E) = 1 + c_1(E) t + c_2(E) t^2 + \dots$. One can show that this is, in fact, a polynomial, whose terms are known as the *Chern classes* of E . They satisfy the following general properties:

1. for all vector bundles E on X and all $i > \text{rg}(E)$ we have $c_i(E) = 0$;
2. for all bundles E, F on X , all integers i, j and all cycles α on X ,

$$c_i(E) \cap (c_j(E) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha);$$

3. if E is a vector bundle on X , $f : X' \rightarrow X$ is a proper morphism, then for all cycles α on X' and all i

$$f_*(c_i(f^* E) \cap \alpha) = c_i(E) \cap f_* \alpha;$$

4. if E is a vector bundle on X , $f : X' \rightarrow X$ is a flat morphism, then

$$c_i(f^* E) \cap f^* \alpha = f^*(c_i(E) \cap \alpha).$$

METHOD A.2.5. To properly define the intersection of general varieties, the setup is the following: Suppose $i : X \rightarrow Y$ is a closed regular imbedding of codimension d and let V be a k -dimensional scheme with a morphism f to Y . Let $W = f^{-1}(X)$ with the inverse image scheme structure, and let $g : W \rightarrow X$ denote the map induced by f . Then $N = g^* N_X Y$ is a bundle of rank d on W with projection π to W . The ideal sheaf \mathcal{I} defining X in Y generates the ideal sheaf \mathcal{J} of W in V , so we have a surjection:

$$\bigoplus_n f^*(\mathcal{I}^n/\mathcal{I}^{n+1}) \rightarrow \bigoplus_n \mathcal{I}^n/\mathcal{I}^{n+1}$$

which determines a closed imbedding of the normal cone $C = C_W V$ as a subcone of N . Because V has pure dimension k , so does C , and as such determines a k -cycle $[C]$ on N . The *intersection product* of V by X on Y is the unique class in $A_{k-d}(W)$ such that $\pi^*(X \cdot V) = [C] \in A_k(N)$. More precisely,

$$X \cdot V = \{c(N) \cap s(W, V)\}_{k-d}.$$

Here $s(W, V)$ is the Segre class of W in V , defined as $s(W, V) = s(C_W V) \in A_* W$. If we let C_1, \dots, C_r be the irreducible components of the cone C and $Z_i = \pi(C_i)$ be the support of C_i , the varieties Z_1, \dots, Z_r , which are closed subvarieties of W , are called the *distinguished varieties* of $X \cdot V$. Denote by s_i the zero section of N_i and put $\alpha_i = s_i^*[C_i] \in A_{k-d}(Z_i)$. Then $X \cdot V = \sum m_i \alpha_i$. If Z is a distinguished subvariety of W , the sum of the terms $m_i \alpha_i$ with $Z_i = Z$ is called the *equivalence* of Z for $X \cdot V$. For a closed subset S of V the part of $X \cdot V$ supported on S is the class in $A_{k-d} S$ denoted by $(X \cdot V)^S$ which is obtained by adding the equivalence of all distinguished varieties contained in S , i.e.

$$(X \cdot V)^S = \sum_{Z_i \subset S} m_i \alpha_i.$$

DEFINITION A.2.6. *Chow ring.* If Y is a non-singular variety of dimension n , the diagonal imbedding $\delta : Y \rightarrow Y \times Y$ is a regular imbedding. For $x, y \in A_* Y$ the product $x \cdot y \in A_* Y$ is defined by $x \cdot y = \delta^*(x \times y)$. If we put $A^p Y = A_{n-p} Y$ this product makes $A^* Y$ into a commutative ring graded by codimension, with unit $[Y]$. We may use the morphism $\delta^!$ to *refine* this product, i.e. $\delta^!(x \times y) \in A_*(|x| \cap |y|)$. (Recall that for a regular imbedding $i : T \rightarrow S$ of codimension d and a morphism $f : S' \rightarrow S$ which restricts to $g : T' \rightarrow T$, $i^! : Z_k S' \rightarrow A_{k-d} T'$ is defined by

$$i^! \left(\sum n_i [V_i] \right) = \sum n_i (T \cdot V_i),$$

where $X \cdot V_i$ is the intersection product introduced in Method A.2.5. The ring $A^* Y$ obtained above is commonly referred to as the *Chow ring* (or the *intersection ring*) of Y . For proper morphisms of non-singular varieties $f : X \rightarrow Y$ we have $f_*(f^* y \cdot x) = y \cdot f_* x$.

More generally, if $f : X \rightarrow Y$ is a morphism and Y is non-singular, the graph morphism $\gamma_f : X \rightarrow X \times Y$ (defined by $x \mapsto (x, f(x))$) is a regular imbedding of codimension n , and for $x \in A_* X, y \in A^* Y$ we may define the *cap product*

$$x \cdot_f y = f^*(y) \cap x = \gamma_f^*(x \times y) \in A_* X$$

from $A_i Y \otimes A_j X$ into $A_{i+j-n} X$, making $A_* X$ into a graded module over $A^* Y$. If, in addition, X is non-singular, and we put

$$f^*(y) = [X] \cdot_f y$$

we get a homomorphism of graded rings $f^* : A^*Y \rightarrow A^*X$.

For an n -dimensional non-singular variety Y with V, W closed subschemes of pure dimensions k, l , we have that every irreducible component Z of $V \cap W$ has dimension at least $k + l - n$. We say that Z is a *proper component* of V and W if its dimension is equal to this number. If Z is proper, its coefficient in the intersection class $V \cdot W \in A_{k+l-n}(V \cap W)$ is the intersection multiplicity of Z in $V \cdot W$.

DEFINITION A.2.7. Equivalence. If Y is a scheme and $X_i \hookrightarrow Y$ are regularly imbedded subschemes, $1 \leq i \leq r$, and V is a k -dimensional subvariety of Y , it follows from the above that the intersection product $X_1 \cdot \dots \cdot X_r \cdot V$ is a class in $A_m(\bigcap X_i \cap V)$, where $m = \dim V - \sum_{i=1}^r \text{codim}(X_i, Y)$. Now if Z is a connected component of $\bigcap |X_i| \cap |V|$, we let

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z \in A_m(Z)$$

denote the part of the intersection product supported on Z ; this is the *equivalence of Z* for the intersection $X_1 \cdot \dots \cdot X_r \cdot V$. If N_i denotes the restriction of $N_{X_i}Y$ to Z , then we have the following result:

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_i) \cap s(Z, V) \right\}_m.$$

If Z is regularly imbedded in V and its normal bundle is N_ZV then this becomes:

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_i) \cdot c(N_ZV)^{-1} \cap [Z] \right\}_m.$$

If $V = Y$, we write $(X_1 \cdot \dots \cdot X_r)^Z$ instead of $(X_1 \cdot \dots \cdot X_r \cdot V)^Z$.

APPENDIX B

RESULTS FROM NUMBER THEORY

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B.1 Bell polynomials

DEFINITION B.1.1. The *partial Bell polynomials* are defined for all $n \geq 1$ and all $1 \leq k \leq n$, by

$$P_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \dots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \dots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

where we sum over all integers $j_1, \dots, j_{n-k+1} \geq 0$ such that $j_1 + \dots + j_{n-k+1} = k$ and $j_1 + 2j_2 + \dots + (n-k+1)j_{n-k+1} = n$. Combinatorically, the coefficient in front of $x_1^{j_1} x_2^{j_2} \dots x_{n-k+1}^{j_{n-k+1}}$ is interpreted as the number of ways to partition a set of n elements into k blocks where j_1 blocks have 1 element, j_2 have 2 elements etc., the members of the set being indistinguishable.

This allows us to easily see a link to the *Sterling number of the second kind*, $S(n, k)$, which is the number of ways to partition $[n]$ into k blocks: $P_{n,k}(1, \dots, 1) = S(n, k)$. Also, we see that

$$\sum_{k=1}^n P_{n,k}(1, \dots, 1) = \sum_{k=1}^n S(n, k) = \mathfrak{B}(n)$$

is the n th Bell number, equal to the number of partitions of a set of size n .

DEFINITION B.1.2. The *complete (exponential) Bell polynomials* are defined as $P_n(x_1, \dots, x_n) = \sum_{k=1}^n P_{n,k}(x_1, \dots, x_{n-k+1})$. An alternative definition is by the formal identity

$$\sum_{n=0}^{\infty} \frac{P_n t^n}{n!} = \exp \left(\sum_{k=1}^{\infty} \frac{x_k t^k}{k!} \right).$$

This is easily seen to be equivalent to a recursive definition, putting $P_0 = 1$ and

$$\forall n \geq 0, P_{n+1}(x_1, \dots, x_{n+1}) = \sum_{k=0}^n \binom{n}{k} P_{n-k}(x_1, \dots, x_{n-k}) x_{k+1}$$

Proof. Starting with the formal identity, consider the generating function $\phi(t) = \sum \frac{P_n t^n}{n!}$. Differentiating the formal identity $\log \phi(t) = \sum \frac{x_k t^k}{k!}$ we get $\frac{\phi'(t)}{\phi(t)} = \sum \frac{x_k t^{k-1}}{(k-1)!}$. It is then a simple matter of equating the coefficients of $\phi'(t) = \phi(t) \sum \frac{x_k t^{k-1}}{(k-1)!}$.

For the other way, suppose the recursive relation holds, then the generating function of the P_n satisfies the differential equation above, and by equality of the 0th polynomials, the two generating functions are equal. \square

B.2 Quasimodular forms

For more details we refer to the first chapter of [Dia] and to [KZ].

DEFINITION B.2.1. The *modular group* $SL_2(\mathbb{Z})$, generated by the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

acts on the upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} | \text{Im}(\tau) > 0\}$. This allows us to define a *weakly modular form of weight* $k \in \mathbb{Z}$ as a meromorphic function $f : \mathcal{H} \rightarrow \mathbb{C}$ such that

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \text{ for all } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}), \tau \in \mathcal{H}$$

This is equivalent to saying that $f(\tau + 1) = f(\tau)$ and $f(-1/\tau) = \tau^k f(\tau)$.

A modular form is a weakly modular form of some integer weight satisfying certain extra conditions, for which we need the notion of *holomorphy at* ∞ . Let D be the open complex unit disk and $D' = D \setminus \{0\}$, then the \mathbb{Z} -periodic holomorphic map $\tau \mapsto e^{2\pi i \tau} = q$ takes \mathcal{H} to D' . So given f , the function $g : D' \rightarrow \mathbb{C}$ defined by $g(q) = f(\log(q)/(2\pi i))$ is well-defined, and we have $f(\tau) = g(e^{2\pi i \tau})$. If f is holomorphic then g is as well, and we define f to be holomorphic at ∞ if g extends holomorphically to $q = 0$. This means that f has a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n(f) q^n, q = e^{2\pi i \tau},$$

but since $q \rightarrow 0$ if and only if $\text{Im}(\tau) \rightarrow \infty$, in order to check if a holomorphic function $f : \mathcal{H} \rightarrow \mathbb{C}$ is holomorphic at ∞ it is only necessary to check whether $f(\tau)$ is bounded when $\text{Im}(\tau) \rightarrow \infty$.

DEFINITION B.2.2. Let $k \in \mathbb{Z}$. A function $f : \mathcal{H} \rightarrow \mathbb{C}$ is modular of weight k (henceforth called k -modular) if it is holomorphic on \mathcal{H} and at ∞ , and a weakly modular form of weight k . We denote by $\mathcal{M}_k(SL_2(\mathbb{Z}))$ the set of k -modular functions — this can be shown to be a finite dimensional \mathbb{C} -vector space, such that the sum

$$\mathcal{M}(SL_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k(SL_2(\mathbb{Z}))$$

is a graded ring, graded by weight.

DEFINITION B.2.3. Let $Y = 4\pi\text{Im}(\tau)$. Note that a modular form can then be described as growing at most polynomially in $1/Y$ as $Y \rightarrow 0$. Now in addition to holomorphic k -modular forms there are functions $F(\tau)$ satisfying the same transformation properties (with respect to k) and growth condition but belonging to $\mathbb{C}[[q]][Y^{-1}]$ instead of $\mathbb{C}[[q]]$, that is, they have the form

$$F(\tau) = \sum_{m=0}^M f_m(\tau)Y^{-m},$$

where $f_i, 0 \leq i \leq M$, are all holomorphic functions, for some integer $M \geq 0$ which is necessarily $\leq k/2$. Such a function is said to be an almost-holomorphic modular form of weight k and they form a vector space denoted by $\widehat{\mathcal{M}}_k(SL_2(\mathbb{Z}))$. The holomorphic function f_0 obtained as the constant term with respect to $1/Y$ is a *quasimodular form of weight k* : the vector space of such functions is denoted by $\widetilde{\mathcal{M}}_k(SL_2(\mathbb{Z}))$. As in the case of modular forms we get graded rings $\widehat{\mathcal{M}}_*(SL_2(\mathbb{Z}))$ and $\widetilde{\mathcal{M}}_*(SL_2(\mathbb{Z}))$ of almost-holomorphic modular forms and quasimodular forms, together with a natural ring homomorphism $\widehat{\mathcal{M}}_*(SL_2(\mathbb{Z})) \rightarrow \widetilde{\mathcal{M}}_*(SL_2(\mathbb{Z}))$.

EXAMPLE B.2.4. For any $k \geq 2$, let B_k denote the k th Bernoulli number, i.e. the coefficients in the generating function of $\frac{x}{e^x-1}$, such that

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

For k even we define the k th Eisenstein series G_k as

$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \right) q^n.$$

All of these are k -modular forms, except G_2 which is quasimodular. Another example of a modular form is the Ramanujan discriminant function $\Delta(\tau) = q \prod_{m>0} (1 - q^m)^{24}$, which is 12-modular.

PROPOSITION B.2.5. *The ring of quasimodular forms is closed under differentiation, and all derivatives of holomorphic modular forms of G_2 are quasimodular.*

Proof. See [KZ], p. 167.

□

BIBLIOGRAPHY

- [Alu] Aluffi, P. (editor), *Quantum cohomology at the Mittag-Leffler Institute*, Scuola Normale Superiore, 1997.
- [Beau] Beauville, Arnaud, *Complex Algebraic Surfaces*, London Mathematical Society Student Texts, 1996.
- [BL1] Bryan, Jim & Leung, N. C., *Generating functions for the number of curves on abelian surfaces*. *Duke Math. J.* 99 (1999), no. 2, 311–328.
- [BL2] Bryan, Jim & Leung, N. C., *The enumerative geometry of $K3$ surfaces and modular forms*. *J. Amer. Math. Soc.* 13 (2000), no. 2, 371–410 (electronic).
- [Cap] Caporaso, Lucia, *Counting curves on surfaces: a guide to new techniques and results*. *European Congress of Mathematics, Vol. I (Budapest, 1996)*, 136–149, *Progr. Math.*, 168, Birkhuser, Basel, 1998.
- [CH] Caporaso, Lucia & Harris, Joe, *Counting plane curves of any genus*. *Invent. Math.* 131 (1998), no. 2, 345–392.
- [Cho1] Choi, Youngook, *Severi degrees in cogenus 4*, arXiv:alg-geom/9601013v2, 1996.
- [Cho2] Choi, Youngook, *On the degree of a Severi variety*, Unpublished thesis.
- [Dia] Diamond, Fred & Shurman, Jerry, *A First Course in Modular Forms*, Springer, 2005.
- [DI] Di Francesco, P. & Itzykson, C., *Quantum intersection rings in The moduli space of curves* (Texel Island, 1994), 81–148, *Progr. Math.*, 129, Birkhuser Boston, Boston, MA, 1995.
- [Ful] Fulton, William, *Intersection Theory*, Springer-Verlag, 1984.
- [FM] Fulton, William & MacPherson, Robert, *A compactification of configuration spaces*. *Ann. of Math. (2)* 139 (1994), no. 1, 183–225.

- [Get] Getzler, E., *Intersection theory on $\overline{M}_{1,4}$ and elliptic Gromov–Witten invariants*. J. Amer. Math. Soc. 10 (1997), no. 4, 973–998.
- [Got] Göttsche, Lothar, *A Conjectural Generating Function for Numbers of Curves on Surfaces*. Comm. Math. Phys. 196 (1998), no. 3, 523–533.
- [Har] Harris, Joe, *On the Severi Variety*, Invent. Math. 84 (1986), 445–461.
- [Itz] Itzykson, C., *Counting rational curves on rational surfaces*. Internat. J. Modern Phys. B 8 (1994), no. 25-26, 3703–3724.
- [Katz] Katz, Sheldon, *Enumerative Geometry and String Theory*, AMS, 2006.
- [Kaz] Kazarian, M., *Multisingularities, cobordism, and enumerative geometry*, Russian Math. Surveys 58:4 (2003), 665–724.
- [KP1] Kleiman, Steven & Piene, Ragni, *Enumerating singular curves on surfaces*. Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 209–238, Contemp. Math., 241, Amer. Math. Soc., Providence, RI, 1999.
- [KP2] Kleiman, Steven & Piene, Ragni, *Enriques diagrams, infinitely near points, and Hilbert schemes*, arXiv:0905.2169v1, 2009.
- [KV] KV, Joachim & Vainsencher, Israel, *An Invitation to Quantum Cohomology*, Birkhäuser, 2007.
- [KZ] Kaneko, Masanobu & Zagier, Don, *A generalized Jacobi theta function and quasimodular forms in The moduli space of curves* (Texel Island, 1994), 165–172, Progr. Math., 129, Birkhuser Boston, Boston, MA, 1995.
- [Li] Li, Li, *Wonderful compactification of an arrangement of subvarieties*, arXiv:math/0611412v2, 2009.
- [Ran1] Ran, Z., *The degree of a Severi variety*. Bull. Amer. Math. Soc. (N.S.) 17 (1987), no. 1, 125–128.
- [Ran2] Ran, Z., *On the quantum cohomology of the plane, old and new, and a K3 analogue*. Collect. Math. 49 (1998), no. 2-3, 519–526.
- [Uly] Ulyanov, A., *Polydiagonal compactification of configuration spaces*. J. Algebraic Geom. 11 (2002), no. 1, 129–159.
- [Vak] Vakil, Ravi, *Counting curves on rational surfaces*. Manuscripta Math. 102 (2000), no. 1, 53–84.
- [Yan] Yang, Stephanie, *Linear systems in \mathbb{P}^2 with base points of bounded multiplicity*. J. Algebraic Geom. 16 (2007), no. 1, 19–38.