Synthethic CDOs - An introduction, evaluation and risk estimation

by

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[1]Kommunal Landspensjonskasse

Summary

In the last decade we have seen a tremendous growth in the issuance of derivatives linked to the credit risk of an individual, or a pool of obligors. While some have been around for a while, some are relatively new. One such derivative is called Synthetic CDO which is the topic of this thesis. The investors in these derivatives are protection buyers, or protection sellers of credit risk. Being a market instrument, the premium demanded for taking such risks is subject to market expectation about the present and future "credit health" of one or more companies. In the light of history, we seek to quantify the risk of actually experiencing the events triggering the contingent payment in such a derivative. This history, and the corresponding empirical event probabilities, are inherent in credit ratings issued by specialized rating companies. We use these probabilities in a multivariate rating transition model to capture these risks in the setting of synthetic CDOs. Also, investors are not only exposed to the risk of losses following such credit events, but also to the risk that the value of their position will change due to market expectation and general supply/demand factors. We therfore use market history on these instruments along with a standard pricing model to forcast a future distribution of market prices that are subject to the occurrence of such credit events. Aggregating all these factors, we calculate a *Profit & Loss* distribution for such an investment on a one-year horizon.

Contents

Chapter 1

Introduction

The risks associated with the value of finacial obligations decreasing due to unexpected events, is called credit risk. Such events could be failure to pay a coupon on a bond, miss a payment on a loan, bankruptcy, or having your status downgraded in a credit rating system. In the last decade several instruments have emerged in order to handle such risks, not only as a way of providing protection against these risks or for hedging purposes, but also to meet different investors appetite when it comes to maximizing future returns. Common derivatives offered are Credit Default Swaps (CDS), n-th to default CDS, Collateralized Debt Obligations (with various types of exposures, loans, bonds, cds), CDO^2 (CDO‘s on CDO‘s) and even forward starting contracts and options on some of these are possible. Although some of these instruments have existed for a while, one has over the last years witnessed a rapid growth in the credit derivatives market, mainly due to the standardization documentation and the introduction of indices like the iTraxx family and DJ CDX family (North American)[1]. Both indices consist of the most liquid names in terms of CDS contracts, with a new index version launched every 6 months, and these indices again having subindices and standard traded contracts and maturities. The ISDA[2] documentation consist of standard terms and what defines a "credit event/default". Therefore we now avoid many of the legal disputes experienced in the earlier years of the market. To get a feel for the size and growth of the market, ISDA‘s 2006 mid-year market survey estimated the notional amount of credit default swaps to be $26.0 trillion, being $17.1 trillion 6 months earlier. On the CDO side, global issuance averaged between 40-50000 $MM quarterly from 2004 to the end of 2005, before booming to about 190000 $MM quarterly issuance, from Q4 2006 and into 2007. While writing this thesis, the CDO market has taken a dramatic downturn to about 11$MM in Q1 2008, which one safely can attribute the ongoing credit crisis originating from the US sub prime mortgage market.

In this thesis we examine a particular type of credit derivative, the synthetic CDO. We will introduce what has been the standard model for valuing these derivatives and try to explain and quantify some of the risks connected to such an invest-

[1] Information about both indices can be found at www.markit.com

[2] ISDA- International Swaps And Derivatives Association www.isda.org

ment. The remainder of this chapter gives an introduction to some common credit derivatives, Chapter 2 gives an introduction to the pricing model, Chapter 3 discuss some risk factors and models to quantify them, and finally in Chapter 4 we provide numerical examples before concluding in Chapter 5.

1.1 Credit Default Swaps (CDS)

The CDS market is big and is considered to be the major risk-transferring instrument developed in the past few years. The contracts allows market participants to transfer credit risk on both individual credits and a portfolios of credits. We give short descriptions of some of these instruments below. An excellent introduction to some of the credit derivatives in this paper and their applications can be found in Bomfim (2004) and Chaplin (2005).

1.1.1 Single name CDS

The single-name CDS is one of the most basic credit derivatives and involves two parties, a protection buyer and a protection seller. The structure can easily be compared to that of an insurance contract. Here the contract can be written on an entity or corporate name, where the the insured (protection buyer) pays a premium to the insurer (protection seller) in the absence of the events which is specified in the contract. Figure 1.1 illustrates a CDS contract. The protection buyer pays a premium called the swap premium or the spread, until the maturity of the contract provided there are no defaults. In the case of a default/credit event the contract terminates and two things happen: The protection buyer pays the accrued premium from the last payment and up to the credit event, and receives the difference between the underlying entity's face value and the recovered value. Much in line with the insurance analogy: When damaging a car the insurance company covers the cost of repairing it, not the cost of buying a new car (provided it can be repaired). Usually the annual premiums are expressed in basispoints (0.01%) and payed quarterly.

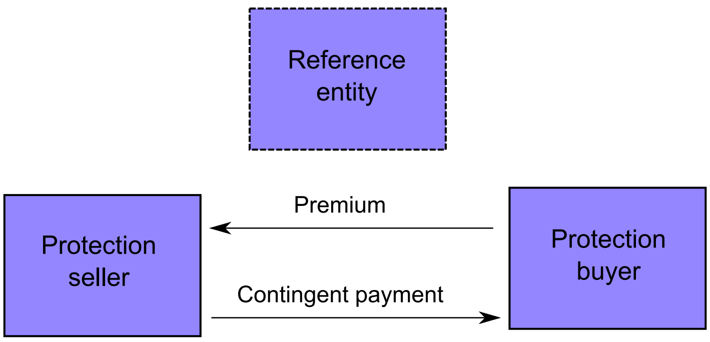

Figure 1.1: Illustration of a Credit Default Swap

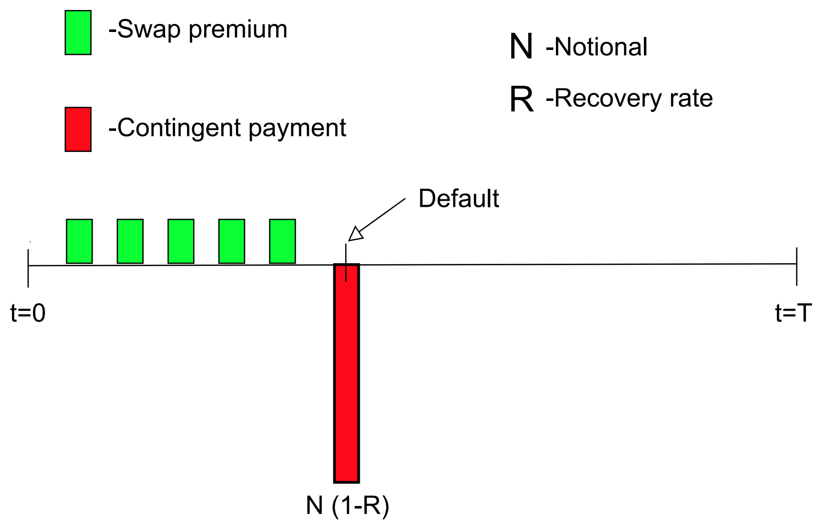

Figure 1.2: The typical cash flow of a CDS. The protection seller receives a premium in the absence of a credit event. If the event occurs, the protection buyer receives the recovered value.

Entities or corporates which the contract is written on is commonly referenced to a defaultable bond. Settlement can be cash or physical, implying the protection seller either pays an amount of cash or delivers an amount of the firm's defaulted bonds, physical delivery is common when dealing with traded debt. The following example will serve as an illustration:

For some reason company A has doubts about company B's ability to fulfill their obligations, and therefore decides to enter a CDS contract with bank C. Company A buys protection on company B from bank C on a notional of 1 billion € and pays a swap premium of say 50 bps per annum (500 €). Furthermore let us assume the recovery rate to be 40%. In the case of company B defaulting, company A receieves 1 billion * (1-0.4) = 600 000 € from bank C. Figure 1.2 gives the general idea. Though a simple example, it illustrates some of the major benefits of these instruments, namely the fact that no asset need to be transfered between parties of the contract, and the reference entity need not even know about the existing contract. The contract allows a credit risky asset (i.e a bond) to be transformed into a credit risk-free asset by purchasing default protection referenced to this credit.

1.1.2 Basket Default Swap

A basket default swap is much like the single-name CDS, but here the contract is written on a portfolio of reference entities. The simplest structures are first-to-default, second-to-default- and n-th to-default swaps. In a first-to-default basket, the contract terminates when the first entity in the portfolio defaults, and in an n-th to-default swap the contract terminates after n defaults. The latter means that swap premiums are still payed during the first $n-1$ defaults. The same benefits of the single-name CDS applies to the basket swap. Picture a bank sitting on a diversified pool of large loans. If the bank wanted to reduce its required economic capital, it could instead of selling loans and risking valuable client relationships, enter into a basket default swap and thereby free up capital.

1.1.3 Index CDS

Like the basket default swap, the index CDS offers protection on a portfolio of entities, here a standardized index of CDSs‘. The mechanics are however different from that of the basket or single-name swap. In the case of the single-name and basket swaps the contract terminates when the reference entity (or the n-th) defaults. In an index CDS the defaulted entity is removed from the pool, but premiums continue to be payed on a reduced notional. Figure 1.3 gives a stylized example. The portfolio consists of 100 equally weighted names, and the protection buyer wishes to purchase default protection on the lot. Let us further assume that the recovered amount given a loss is zero, and that the total notional bought protection on is 100 billion €. In this setting, one default results in a reduction on the notional of 1 billion €. When a firm defaults the protection seller pays the buyer 1 billion €, and receives a reduced coupon until the end of the contract, or the next default, causing a further reduction on the cash flow.

Today there are two families of standardized indices, already mentioned: the Dow Jones CDX and the International Index Company iTraxx. The Dow Jones CDX consist of entities in North America and emerging markets while iTraxx consist of entities from Europe and Asia. They are both standardized in terms of composition procedure, premium payment and maturity. Most actively traded are the the Dow Jones CDX NA IG[3] and the iTraxx Europe, with the 5 year contract being most common. Contracts with 3-,7- and 10 year maturities are also available. The composition of both indices are 125 equally weighted investment grade[4] names, and is renewed every 6 months with the launch of a new "series". The new series contains the majority of the names of the previous series, with some new names replacing names taken out. The start date of the new series is often referred to as the roll date. After the roll date (March and September), the index will be "on-the-run" for the next 6 months until the launch of the next series. The composition of each series remains static in its lifetime if no default occurs. Deals related to series 5 for example, may be rolled into series 6 (on market terms) when the next series is introduced, or the series 5-deal may be held to maturity. The iTraxx Europe index consist of a diversified pool of names from the sectors Autos, Consumer, Energy, Financial, Industrial and TMT[5], and will be given more attention in the remainder of the thesis. Contracts available on the iTraxx Europe are futures, options on the spread movement, and standardized tranches which will be discussed in the next section.

[3] North American Investment Grade

[4] By investment grade, one refers to obligors with rating Baa or better in Moody's scale.

[5] TMT- telecommunications, media and technology

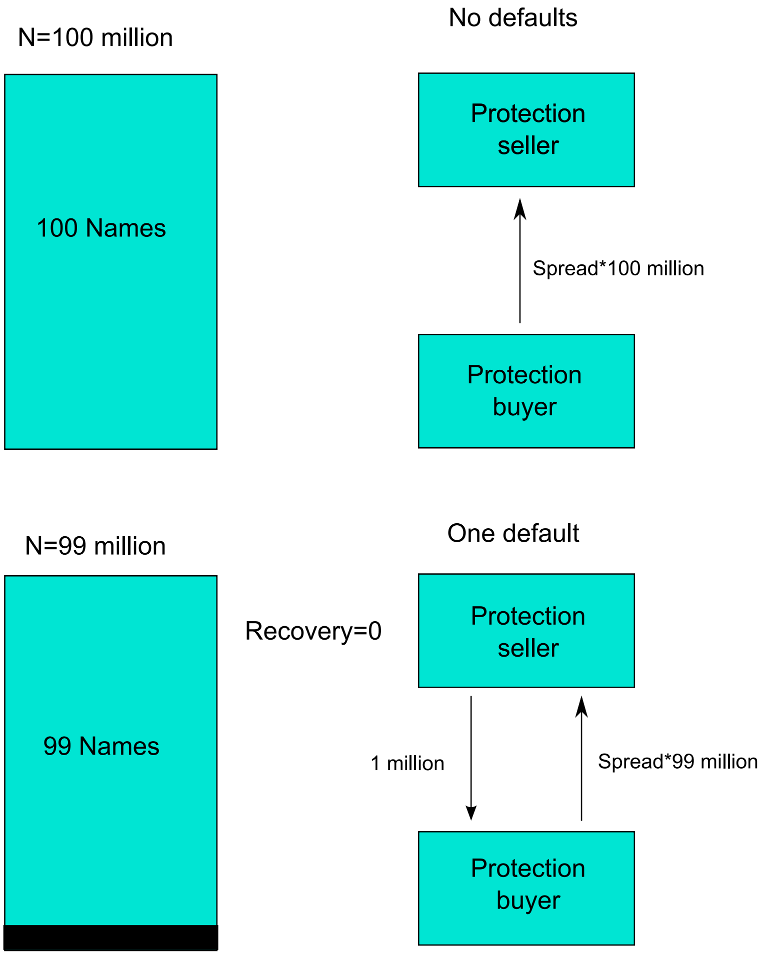

Figure 1.3: Illustration of a stylized index CDS.

1.2 Collateralized Debt Obligations (CDO)

Insurance companies have bought protection on layers of their loss distributions for many years. In these layer contracts, re-insurance companies sell protection on the reference-companys' losses above a certain barrier. A CDO is basically a portfolio of debt instruments where the losses are sliced into such layers.

1.2.1 The general case

In the credit-risk terminology such layers are called tranches and is the defining feature of a CDO. A CDO can be backed by a pool of many kinds of debt obligations such as bonds, loans, CDSs', other CDO's to mention a few. Normally the one that initiate the CDO (i.e a bank), called sponsor, creates a company that is responsible for the administration of the CDO. This company, called the special purpose vehicle (SPV) works as an independent entity, so that the investors (buyers of tranches/protection sellers) are isolated from the credit risk of the sponsor. To obtain the desired exposure, the SPV can buy a pool of debt instrument or synthesize a deal by entering CDSs'. If the SPV buys the pool, the CDO is called a cash CDO. When the pool is made up of bonds, it's called a collateralized bond obligation(CBO), when it is made up of loans, it is called a collateralized loan obligation (CLO). A bank may in this way securitize or transfer risk on some of its loans by initiating a CLO. On the other hand, if the SPV gains credit exposure through CDSs', it's called a synthetic CDO. To explain the mechanics of the CDO, we provide an example similar to that in Logstaff & Rajan (2006).

Assume that a financial institution sets up a portfolio consisting of 100 separate bonds with rating BBB, each with a market value of 1 million. The issuer sells claims against the cash flows generated by the portfolio. These claims are termed CDO tranches and vary in credit risk from very high (equity tranche, or first-loss piece) to very low (senior tranche, super-senior). The equity tranche is responsible for the first losses up to 3% of the total notional, in return the holder of this tranche earns a higher spread than the holder(s) of the other tranches. The next tranche absorbs losses from 3% and up to 7% and so on. This structure is often called the waterfall of the CDO, which specifies the attachment- and detachment points of the tranches. Let us assume that the spread on the equity tranche is 2500 bps, if there are no defaults the investor earns a high coupon during the lifetime of the deal. On the other hand, if one of the firms in the portfolio defaults, the investor looses 1/3 of his investment (again assuming zero recovery), and if three firms default his entire investment is gone. Comparing this to the index CDS example (Figure 1.3), where one default translates into one percent loss, the riskyness of the equity tranche becomes clear. Because the equity tranche is extremly risky (some losses are expected) the sponsor usually holds the equity tranche and the SPV sells the other tranches to investors.

The mechanics of the CDO should now be clear, and possible some of its benefits. The ability to tailormake different tranches to meet different investors risk appetite makes it a very versatile product. Figure 1.4 shows one possible structure. Some tranche exposures can be in swap-format, and some may be in a funded format, meaning that the SPV issues bond-like notes called credit-linked notes (CLN[6]). Usually tranches are given a rating from agencies like Standard&Poors, Moody's or Fitch, thus a tranche investment (i.e mezzanine) with rating BBB, should in theory be comparable to a BBB bond, although history has proven otherwise.

1.2.2 Index tranches

The mechanics of index tranches are the same as the synthetic CDO. It is not strictly speaking a synthetic CDO because, like the index CDS, it's not funded by the sale of CDSs'(see Hull & White (2004)). One simply uses CDO technology to slice the index CDS into tranches. Both the DJ CDX NA IG and iTraxx Europe offers 5 standardized tranches. The tranche structure of iTraxx Europe is 0-3%, 3-6%, 6-9%, 9-12% and 12-22%. Table 1.1 (found in Kalemanova et al. (2005)) shows some market quotes for the iTraxx Europe tranches. Note that the equity tranche is quoted differently than the others. The equity tranche pays a fixed spread of 500 basis points per annum in addition to an upfront payment. The market quote is the upfront percentage payment. The other tranche quotes referes to the annual spread paid to investors, and is paid quarterly.

Index	0-3%	3-6%	6-9%	9-12%	12-22%
32 bps	23.53%	62.75 bps	18 bps	9.25 bps	3.75 bps

Table 1.1: iTraxx Europe (5 year) index and corresponding tranche quotes, on April 12, 2006.

[6]A CLN is most easily compared to a coupon paying bond. The investor pays the notional amount, receives spread and interest rate quarterly and in the absence of default, receives back the notional + interest at maturity

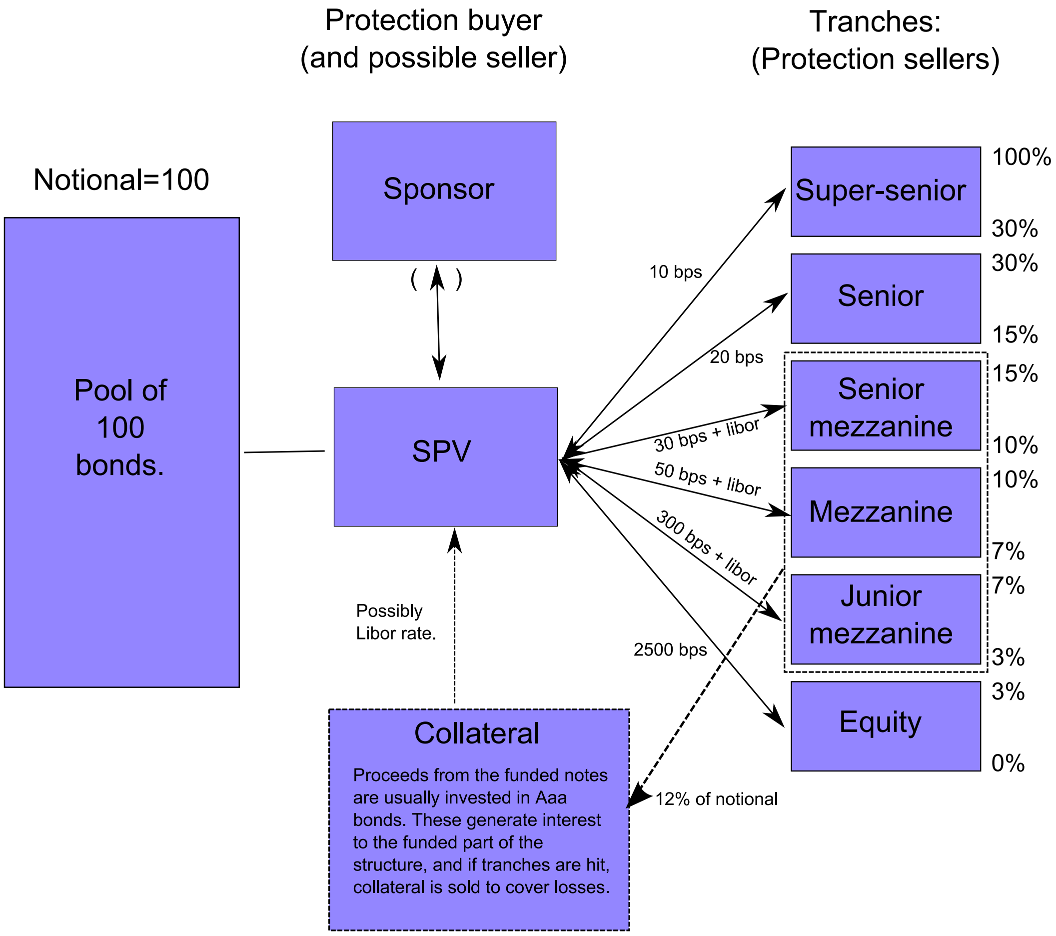

Figure 1.4: Illustration of a stylized CDO.

1.2.3 Summary

It is easily understood that in order to price the instruments introduced in this chapter, one should have a model that describes default behavior. With several entities, we need a model that describes joint default behavior. Imagine the equity investor in the CDO example and assume that it is a 5 year contract. In the case of one default, his 3 million investment is reduced to 2 million. However, the realized loss he experiences over the whole lifetime of the deal will depend greatly upon when the loss occurred. If the default happens at the end of year one, he will experience one year of coupons on 3 million, and four years with coupons on 2 million. If the default happens in the last year of the contract, he would get coupon on 3 million for four years, and then a smaller coupon in the last year. If the investor could choose amongst the two, the latter is certainly the lesser of two evils. So, the timing of default is an important issue: Do the defaults tend to cluster, or do they appear to arrive independently of one and another in time? We introduce the standard model for valuation in the next chapter.

Chapter 2

Pricing Models

2.1 Overview

Credit risk pricing models mainly belong to one of the following two categories:

Structural models. This theory first introduced by Merton (1974) is often called an option-based theory. The aim of this approach is to model the process leading to default. In this model default occurs when the value of the firm‘s assets drops below the face value of the debt. In the original model, the assumption on the asset value process led one directly into the pricing framework of Black&Scholes, and one could price credit risk by considering the face value of debt as the strike in an european call option. In this model default could only happen at the maturity of the debt. Todays structural models have come a long way since then, mainly by introducing other models for the asset value process. Introducing jumps in the process allowed sudden events that could lead to default, which was crucial in order to match credit spreads seen in the market. Though theoretically appealing, one of the drawbacks of this theory is that a company‘s current financial status is only shared to the public at certain times a year. Also, all company debt is not directly observable in the market.

Reduced form models. In this approach one attempts to model the time of default itself, commonly through the use of of an intensity process. A good reference book is Schönbucher (2003). The approach used in this thesis is a simple version of an intensity based model.

2.2 CDO pricing ingredients

In order to derive a pricing model, some notation is needed. Let

$$
\begin{array}{lcl}
N & = & \text{Number of obligors} \\
D(0,t) & = & \text{Discount factor} \\
\tau_i & = & \text{Default time of obligor}\, i \\
Q_i(t) & = & \text{Risk neutral default probability of obligor}\, i \\
S_i(t) & = & 1 - Q_i(t) = \text{Risk neutral survival probability of obligor}\, i \\
A_i & = & \text{Notional exposure to obligor}\, i \\
R_i & = & \text{Recovery rate of obligor}\, i \\
L(t) & = & \text{Cumulative loss on portfolio notional at time}\, t \\
a & = & \text{Tranche attachment point} \\
d & = & \text{Tranche detachment point} \\
L_{a,d}(t) & = & \text{Cumulative loss on tranche at time}\, t \\
EL_{a,d}(t) & = & \text{Expected cumulative loss on tranche notional at time}\, t
\end{array}
$$

To begin with, note that we can express the total portfolio loss as

$$
L(t) \;=\; \sum_{i=1}^{N} (1 - R_i) A_i I_{\{\tau_i < t\}}, \tag{2.1}
$$

where $I_{\{\tau_i<t\}}$ is the indicator function. The term $(1 - R_i)$ is often called *Loss Given Default (LGD)* and expresses the fraction of the notional A_i lost in the case of a default. R_i can be modeled explicitly, or estimated from historical default data. We will consider the case of equal recovery rates and equal notionals for all entities. The recovery assumption will be of great help when we later on extract risk neutral default probabilities from market data. Equal recovery rates is not a necessary assumption, but it makes computations simpler. Recoveries could i.e be random as in Anderson & Sidenius (2004). However, with the assumptions made, we need not keep track of who defaulted, since every default results in the same amount of loss. The cumulative loss can now be expressed as a fraction of the total portfolio notional:

$$
L(t) \;=\; \frac{1}{N}(1 - R) \sum_{i=1}^{N} I_{\{\tau_i < t\}}. \tag{2.2}
$$

In this setting the loss amount can only take on the values $\frac{k(1-R)}{N}, \quad k = 0, 1 \ldots N$. It is easy to see that the loss distribution and the distribution of the number of defaults denoted $\pi_k(t)$, now become exchangeable, that is

$$
\mathbb{P}\left(L(t) = \frac{k(1-R)}{N}\right) \;=\; \mathbb{P}\left(\sum_{i=1}^{N} I_{\{\tau_i<t\}} = k\right) \;=\; \pi_k(t). \tag{2.3}
$$

As can be viewed in the right panel of Figure 2.1, we seek the cumulative tranche loss as a function of portfolio loss, which we express in the following way:

$$L_{a,d}(t) = \begin{cases} 0 & , \quad L(t) \leq a \\ L(t) - a & , \quad a \leq L(t) \leq d \\ d - a & , \quad L(t) \geq d. \end{cases}$$

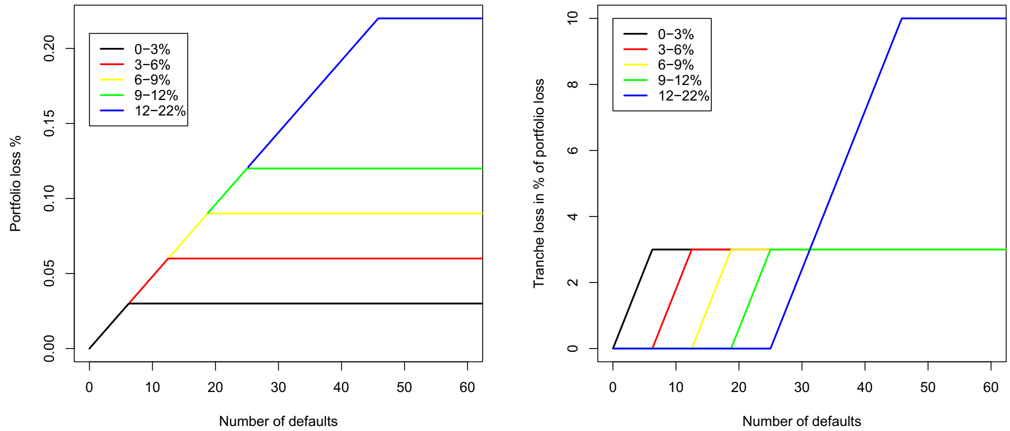

Figure 2.1: Left: Illustration of tranche capital structure. Right: Tranche losses in % of portfolio notional as a function of defaults.

With these expressions at hand the key ingredient of a general pricing model, namely the expected loss of a tranche, can be written in a more compact form:

$$\begin{aligned} EL_{a,d}(t) &= E\left[L_{a,d}(t)\right] \\ &= \frac{1}{d-a}\sum_{k=1}^{N} \max\left\{\min(L(t), d) - a, 0\right\} \mathbb{P}\left(L(t) = \frac{k(1-R)}{N}\right) \\ &= \frac{1}{d-a}\sum_{k=1}^{N} \max\left\{\min(L(t), d) - a, 0\right\} \pi_k(t), \end{aligned} \tag{2.4}$$

where we multiply by the term $\frac{1}{d-a}$ to get the loss expressed in tranche notional and not portfolio notional. If the total portfolio was 100 billion €, than a 0-3% tranche notional would be 3 billion €.

2.2.1 The general setup

A CDO tranche basicly consists of two parts, a premium part and a default part. The premium part, often referred to as premium leg, expresses the expected value of the cash flows the protection seller receives during the lifetime of the contract. We denote the premium payment times by $t_1 < t_2 < .... < t_n = T$, where T is the length of the contract. By assuming independence between the non-stochastic riskless discount rate and default probabilities, the expected present value of the premium leg becomes

$$PL = \sum_{i=1}^{t_n} S_{a,d}\,\Delta t_i\,D(0,t_i)\,\left[1 - EL_{a,d}(t_i)\right],$$

where $\Delta t_i = t_i - t_{i-1}$ denotes the payment frequency, $S_{a,d}$ the tranche premium/spread, $D(0,t_i)$ the discount factor, and the term $1 - EL_{a,d}(t_i)$ the remaining fraction of tranche notional which premiums are received on. Similarly, the default leg expresses the expected discounted cash flows of the payments made by the protection seller in the event of defaults (see appendix A.1):

$$DL = \int_0^T D(0,t)\,dEL_{a,d}(t) \approx \sum_{i=1}^{t_n \times k} D(0,t_{i-1})\,\left[EL_{a,d}(t_i) - EL_{a,d}(t_{i-1})\right].$$

The fair value of a CDO is such that, at the time of inception the contract has zero value. This amounts to choosing a spread S such that $PL(S) - DL = 0$, hence

$$S_{a,d}^{fair} = \frac{\sum_{i=1}^{t_n \times k} D(0,t_{i-1})\,\left[EL_{a,d}(t_i) - EL_{a,d}(t_{i-1})\right]}{\sum_{i=1}^{t_n} \Delta t_i\,D(0,t_i)\,\left[1 - EL_{a,d}(t_i)\right]}.$$

The market practice for the equity tranche is a bit different, as mentioned earlier. Usually the holder of the equity piece receives an upfront payment and a predefined fixed running spread. In this case, the fair premium is the upfront payment that makes the value of the CDO with predefined running spread, zero. The upfront payment is usually quoted in percent(%), see Section 1.2.2, and equals

$$Up_{0,d}^{fair} = \sum_{i=1}^{t_n \times k} D(0,t_{i-1})\,\left[EL_{0,d}(t_i) - EL_{0,d}(t_{i-1})\right] - S_{0,d}^{fix} \sum_{i=1}^{t_n} \Delta t_i\,D(0,t_i)\,\left[1 - EL_{0,d}(t_i)\right].$$

2.3 Termstructure of default probabilities

The termstructure of default probabilities, often called a credit curve, is a fundamental part of the pricing model. The slope and curvature gives information about the risk, or price of risk with respect to time. There are numerous methods of constructing a credit curve. One approach could be using a structural model, i.e Merton and its extensions. Another approach is using historical default rates from rating agencies‘ databases, like the ones shown in figure 2.2[1]. However, the last option is not suitable for pricing, since the market prices default risk differently than implied by historical measures. By that we mean that historical default rates can differ substantially from default rates implied from market data, as will be seen in

[1]Includes bond and loan issuers rated as of January 1 of each year. Annual default study found at www.moodys.com.

Section 2.3.2. A good exposition on how to fit a curve to historical data can be found in Bluhm et al. (2003) and Bluhm & Ludger (2006). In this thesis we will be using market data (i.e defaultable bonds, CDS) to estimate the implicit default term structure, by means of a inhomogeneous Poisson process. We will not present the mathematical background validating the use of point and jump processes in a credit risk context here, but accept the fact that it is a commonly used approach [2]. Again we refer to the book by Schönbucher (2003) for a treatment of the topic, with similar content found in McNeil et al. (2005).

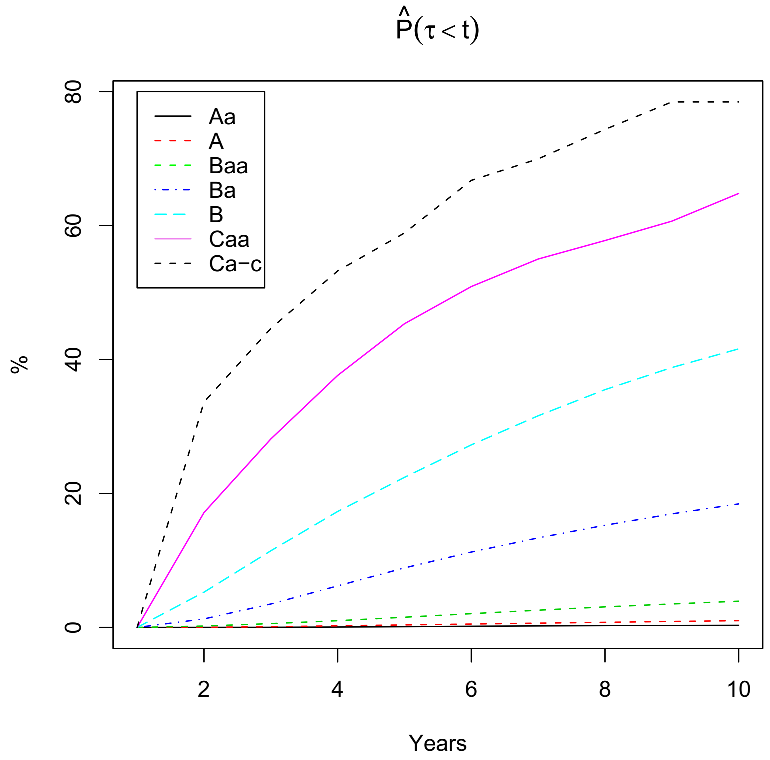

Figure 2.2: Average Cumulative Issuer-Weighted Global Default Rates by Letter Rating, 1983-2006. *Source Moody's.*

2.3.1 Inhomogeneous Poisson process and hazard rates

Definition 2.3.1. *Let the random variable τ be a stopping time (i.e time of default), with distribution function $F(t)$. Assume that $F(t) < 1$ for all t, and that $F(t)$ has density $f(t)$. The function $h(t) := f(t)/(1 - F(t))$ is called the hazard rate of τ.*

The hazard rate $h(t)$ can be interpreted as the instantaneous chance of default, given survival up to time t (see Appendix A.2). An intuitive justification for working

[2]See O'Kane & Turnbull (2003) for a presentation of a standard CDS model.

with hazard rates, can be made by looking at Figure 2.2 again. Notice that the curves assigned to sub investment grade ratings (Ba and below) have a tendency to slow down their growth. The opposite effect is most often experienced with investment grade ratings. The effect might be explained by considering that a lot of sub investment grade firms are "do-or-die" firms: They either make it, or not. Conditional on survival, bad credits today tend to become better credits over time. Therefore working with hazard rates or forward hazard rates we have the ability to accommodate such effects by letting the hazard rate change accordingly through time.

Definition 2.3.2. (Schönbucher (2003)) *An inhomogeneous Poisson process with intensity function $\lambda(t) > 0$ is a non-decreasing, integer-valued process with initial value $N(0) = 0$ whose increments are independent and satisfy*

$$P\left[N(T) - N(t) = n\right] = \frac{1}{n!}\left(\int_t^T \lambda(s)\, ds\right)^n \exp\left\{-\int_t^T \lambda(s)\, ds\right\}. \tag{2.5}$$

If we now consider the time of default as the first jump of a inhomogeneous poisson process, we get the survival probabilities as a consequence of (2.5) by

$$P\left[N(T) - N(0) = 0\right] = \exp\left\{-\int_0^T \lambda(s)\, ds\right\}. \tag{2.6}$$

By utilizing such a framework, we can reach any termstructure of hazard rates by suitable choice of the intensity function. In the following section we are going to calibrate the intensity function to market data by assuming it is a deterministic function of time. In this setting, the intensity function and the conditional/forward hazard rates coincide.

2.3.2 Calibration to market data

The goal of this section is to extract the risk neutral, or implied default probabilities of market data. The reason for extracting these probabilities, is that they serve as building-blocks in more complex multiname products, such as n-th to default baskets and CDOs. Their use will become clear when we return to the loss distribution and CDO pricing model in Section 2.4.2. The method is called "bootstrapping"[3] and works in the same way as deriving forward rates from bonds or swap rates. The method can be found in O'Kane & Turnbull (2003) but we present it for the sake of completeness.

This method utilizes a simple model for the valuation of CDSs'. We begin by noting that a CDS like the CDO tranches, can be seen as consisting of two parts. The risk neutral expectation[4] of the contingent payments made to the protection

[3]Not to be confused with it's statistical brother in name (see Efron & Tibshirani (1994)).

[4]We use the common term risk neutral default probabilities when speaking of default probabilities inferred from market prices. Perhaps arbitrage-free probabilities is a more suiting term.

seller, must equal the expectation of the contingent payments made by the protection buyer at the time of inception. If we ignore the premium accrued and day-count conventions, we have that the following must hold:

$$S\sum_{i=1}^{T=t_n} \Delta t_i\, D(0,t_i)\, S(t_i) \;=\; (1-R)\int_0^T D(0,t)\, dQ(t). \tag{2.7}$$

Again we can approximate the integral by a sum, where a monthly discretization is sufficiently accurate according to O'Kane & Turnbull (2003). Note that the term on the left hand side which we multiply the spread S with, is called *Risky Annuity* and is the marginal present value of the protection leg. The next assumptions commonly made are a fixed recovery rate and that the hazard rate term structure is either piecewise flat, piecewise linear or a mixture of the two. With the discount factors[5], CDS contracts of different maturities, commonly 1, 3, 5, 7 and 10 years, the assumption of a piecewise flat hazard rates, we can solve for the hazard rates in an iterative fashion. Let t_i and t_n denote $\frac{i}{12}$ and $\frac{n}{12}$ respectively and S_i the spread of the CDS with maturity i year(s). Then one finds the hazard rates using the following algorithm:

- Set T=12 months and input the spread for maturity 1Y in (2.7), along with discount factors and recovery rate.
- Use a numerical root finding algorithm to obtain $\lambda_{0,1}$. That is, find $\lambda_{0,1}$ such that

$$\sum_{i=1}^{12} D(0,t_i)(e^{-\lambda_{0,1}t_{i-1}} - e^{-\lambda_{0,1}t_i}) - \frac{S_1}{1-R}\sum_{n=3,6,9,12} \Delta t_n D(0,t_n)e^{-\lambda_{0,1}t_n} = \quad 0.$$

- Set T=36 months and input the spread for maturity 3Y, use probabilities obtained from previous step for t $\in$ [0,1] and find $\lambda_{1,3}$ by solving

$$\sum_{i=1}^{12} D(0,t_i)(e^{-\lambda_{0,1}\,t_{i-1}} - e^{-\lambda_{0,1}\,t_i}) + \sum_{i=1}^{24} D(0,t_{i+12})\, e^{-\lambda_{0,1}}(e^{-\lambda_{1,3}\,t_{i-1}} - e^{-\lambda_{1,3}\,t_i}) -$$

$$\frac{S_3}{1-R}\left(\sum_{n=3,6,9,12} \Delta t_n D(0,t_n)e^{-\lambda_{0,1}\,t_n} + \sum_{n=3,6,..,24} \Delta t_n D(0,t_{n+12})e^{-\lambda_{0,1}-\lambda_{1,3}\,t_n}\right) = 0.$$

- Repeat the procedure with the rest of the contracts, increasing in maturity.

For t longer than the longest maturity it is common to assume a flat hazard rate. In this setting the hazard rate function becomes a step function, and in our example the survival probabilities is given by

[5] I.e derived by Euribor and Euro swap rates by bootstrapping.

$$S(t) = \begin{cases} \exp\{-\lambda_{0,1}\, t\} & , \quad 0 < t \leq 1 \\ \exp\{-\lambda_{0,1} - \lambda_{1,3}(t-1)\} & , \quad 1 < t \leq 3 \\ \exp\{-\lambda_{0,1} - 2\lambda_{1,3} - \lambda_{3,5}(t-3)\} & , \quad 3 < t \leq 5 \\ \exp\{-\lambda_{0,1} - 2\lambda_{1,3} - 2\lambda_{3,5} - \lambda_{5,7}(t-5)\} & , \quad 5 < t \leq 7 \\ \exp\{-\lambda_{0,1} - 2\lambda_{1,3} - 2\lambda_{3,5} - 2\lambda_{5,7} - \lambda_{7,10}(t-7)\} & , \quad t > 7. \end{cases}$$

Let us provide an example. Table 2.1 contains mid CDS quotes for a European company named *Gaz De France*, and is a member on the iTraxx Europe series 7 index. *Gaz De France* was an Aa rated issuer by Moody's classification of credit-worthiness, in the considered time period. Figure 2.3 shows the result of applying the bootstrapping procedure, and Figure 2.4 the corresponding hazard rates. For the sake of comparison we have included the "Aa" termstructure found in Figure 2.2. One can observe several things from these two figures. First of all one can see the increasing forward hazard rates, which is typical for investment grade names, as previously argued. Secondly, in this case one can see a substantial difference between the termstructure of default probabilities implicit in the market instrument, and the termstructure of observed or historical default rates. The difference might not look so dramatic if we had chosen another company. In fact, there is no rule saying that two companies with the same quoted spread levels need to be in the same rating category, although they will generally not differ much, rating or spread-wise.

CDS maturity	T=1	T=3	T=5	T=7	T=10	Rating
Gaz De France	2.4 bps	4.7 bps	7.1 bps	10.6 bps	14.9 bps	Aa

Table 2.1: CDS mid quotes on 27.06.07 and rating obtained from Bloomberg.

As for any traded market instrument, a CDS spread is a result of market expectation. Factors like supply and demand, risk aversion, or a general view of future market environment, affects prices. Recently one has observed the impact of the "crisis" originating from the U.S sub prime mortgage market, on other markets. In the European credit markets one has witnessed a general widening of spreads, and overall volatility has increased. Liquidity might also be an issue, although the names on the iTraxx are among the most liquid. Fear of getting negative "mark-to-market" values because of liquidity reasons, or a simply panic-like behavior due to increasing credit risk, will cause protection sellers to demand a higher spread, if willing at all. Many investors wanting to exit at once, will drive spreads. The point is that, if we can not quantify these other factors in a valuation model, the market implied default probabilities will also contain other factors than the market view of default only.

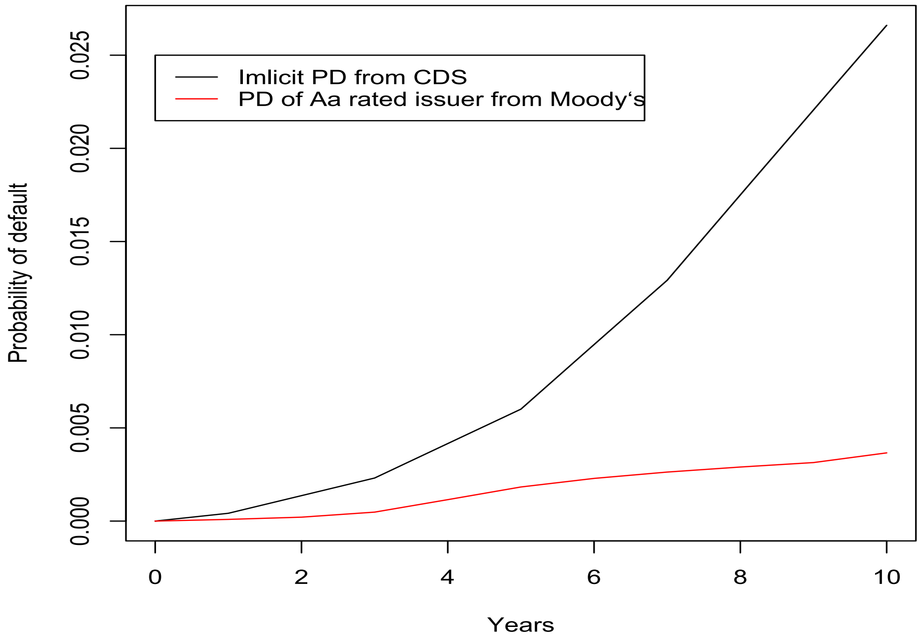

Figure 2.3: Termstructure of default probabilities of the company Gaz De France implied from CDS contracts as of 27.06.07, along with observed defaults of Aa rated obligor by Moody‘s.

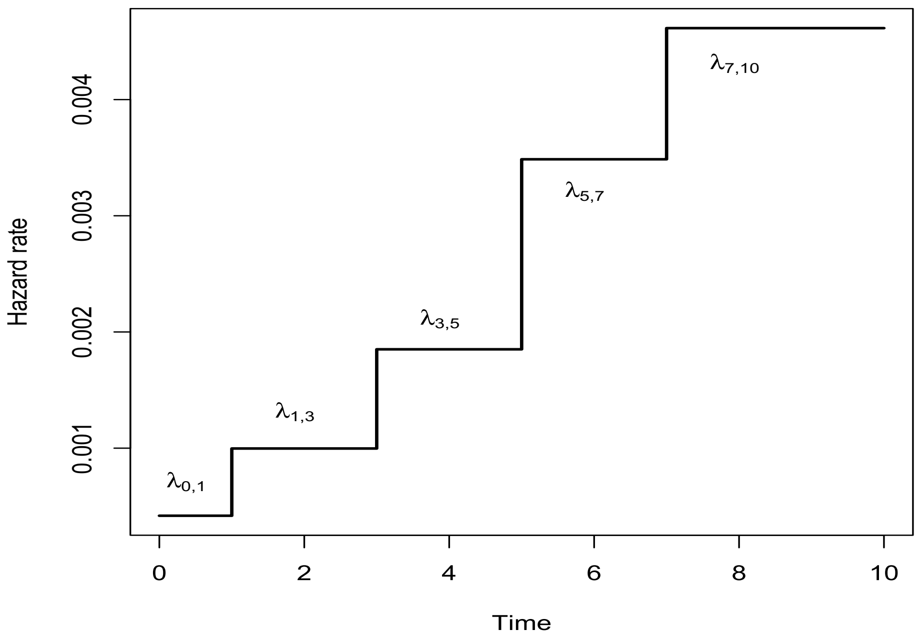

Figure 2.4: Corresponding hazard rates in model implied from CDSs, Gaz De France

2.4 Default dependency

As mentioned at the end of Chapter 1 the concept of dependent default times is an important matter in multiname credit risk. One way of specifying a multivariate default time distribution is through a copula. We quickly review the concept of copulae, and then present the market standard one-factor gaussian copula model.

2.4.1 Copulae

Definition 2.4.1 (Copulae). *A d-dimensional copula is a distribution function on* $[0,1]^d$ *with standard uniform marginal distributions.*

Theorem 2.4.2 (Sklar 1959). *Let F be a joint distribution function with margins $F_1, ..., F_d$. Then there exists a copula C:* $[0,1]^d \to [0,1]$ *such that, for all $x_1, ..., x_d \in \mathbb{R}^d$,*

$$F(x_1, ..., x_d) \;=\; C\left(F_1(x_1), ..., F_d(x_d)\right). \tag{2.8}$$

If the margins are continuous, then C is unique.

A consequence of Sklar's theorem is that we can imply a copula from a multivariate distribution, by evaluating (2.8) at the arguments $x_i = F^{-1}(u_i)$, and then use it with arbitrarily chosen marginals to obtain a fully valid multivariate distribution. Copulae residing in this class are called implicit copulae.

Example 2.4.3 (Gaussian Copula). Recall the multivariate normal distribution $\Phi_d(0, \mathsf{P})$ with expectation zero and correlation matrix P:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\mathsf{P}|^{1/2}} \exp\left\{-\frac{1}{2}\mathbf{x}^T\mathsf{P}^{-1}\mathbf{x}\right\}. \tag{2.9}$$

We obtain the copula by evaluating the multivariate normal distribution at the arguments $x_i = F^{-1}(u_i)$ such that $F_d\left(F_1^{-1}(u_1), ..., F_d^{-1}(u_d)\right) \;=\; C(u_1, ..., u_d)$.

$$\begin{aligned} C_{\mathsf{P}}^{gauss}(u_1, ..., u_d) &= \Phi_{d,0,\mathsf{P}}\left(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_d)\right) \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_d)} f(\mathbf{x}|0,\mathsf{P})\,dx_1 \cdots dx_d. \end{aligned} \tag{2.10}$$

We can now *couple* this copula with our marginal default time distributions and obtain what is called a meta-gaussian distribution

$$\begin{aligned} P\left(\tau_1 < t, ..., \tau_d < t\right) &= C_{\mathsf{P}}^{gauss}\left(Q_1(t), ..., Q_d(t)\right) \\ &= \int_{-\infty}^{\Phi^{-1}(Q_1(t))} \cdots \int_{-\infty}^{\Phi^{-1}(Q_d(t))} f(\mathbf{x}|0,\mathsf{P})\,dx_1 \cdots dx_d. \end{aligned} \tag{2.11}$$

We can view this framework as a mapping of quantiles. The quantile of a normally distributed random variable is mapped to the quantile of the default time distribution. In general the point $x_i = x$ is transformed to $t_i = t$ where $t = Q_i^{-1}(\Phi(x))$. The copula thus expresses the joint probability of these quantile mappings. The latter relation is useful if we want to simulate dependent default times. A simple algorithm:

- Draw a random vector $\mathbf{x}$ of multivariate normal numbers.
- Set $u_i = \Phi(x_i)$ for all i.
- Recover the time of default by setting $t_i = Q_i^{-1}(u_i)$ for all i.
- Repeat desired number of times.

Note that other copulae might be used for this purpose. A good introduction to copulae with examples related to credit risk can be found McNeil et al. (2005), and a bit more extensive introduction can be found in Cherubini et al. (2004).

2.4.2 Loss distribution and one-factor Gaussian copula

While the evaluation of (2.11) might require high dimensional integration, the market has adopted another way of specifying the marginals X_i of the multivariate normal distribution. Letting X_i have a factor structure, reduces the number of integrations needed to just one. The idea behind factor models is to assume all the obligors are influenced by the same sources of uncertainty, or same set of factors. There could be one or more systematic factors influencing the credit environment, and in addition one idiosyncratic effect. Let the X_i be the sum of two independent variables, one representing a common factor, and the other the idiosyncratic:

$$X_i = a_i Y + \sqrt{1 - a_i^2} Z_i. \tag{2.12}$$

Where $-1 \leq a_i \leq 1$ and both Y and Z_i are i.i.d $N(0, 1)$ variables.[6] Notice that in (2.12) that there is nothing new to the distributional assumptions on the random vector $\mathbf{X}$, it still follows a multivariate normal distribution, but with correlation matrix given by $corr(X_i, X_j) = a_i a_j$ (Appendix A.3). Factor models are convenient as they can be used to describe dependency amongst credits using a "credit-vs-common factors" analysis instead of a pairwise analysis.[7] The a_i are sometimes called *loading factors* because one can think of them as a way of describing the influence the systematic factors have on an obligors credit risk. I.e, one would expect that firms with high creditworthiness or high rating to be more affected by a downturn in the

[6] One may encounter different parameterizations of the model, this one is taken from Hull & White (2004)

[7] Apart form the difficulty of estimating default correlation, in that they are rare events, one would in the iTraxx Europe case have to estimate N(N-1)/2=7750 pairwise correlations.

economy, and less affected by idiosyncratic risk. While several papers have different ways of explaining factor models, we will simply look at it as a different way of specifying the copula model, that will aid us in computing (2.11). Let us see why by looking at the gaussian copula again:

$$P\left(X_1 \leq \Phi^{-1}(u_1), \ldots, X_d \leq \Phi^{-1}(u_d)\right). \tag{2.13}$$

We want to compute the probability in the meta-gaussian default time distribution

$$P\left(X_1 \leq \Phi^{-1}\left[Q_1(t)\right], \ldots, X_d \leq \Phi^{-1}\left[Q_d(t)\right]\right). \tag{2.14}$$

Using the relation (2.12), we substitute X_i and get:

$$P\left(a_1 Y + \sqrt{1-a_1^2} Z_1 \leq \Phi^{-1}\left[Q_1(t)\right], \ldots, a_d Y + \sqrt{1-a_d^2} Z_d \leq \Phi^{-1}\left[Q_d(t)\right]\right)$$

$$= P\left(Z_1 \leq \frac{\Phi^{-1}\left[Q_1(t)\right] - a_1 Y}{\sqrt{1-a_1^2}}, \ldots, Z_d \leq \frac{\Phi^{-1}\left[Q_d(t)\right] - a_d Y}{\sqrt{1-a_d^2}}\right). \tag{2.15}$$

Conditioning on the common factor $Y = y$ gives independence, and the conditional probability of (2.15) reduces to the product of d normally distributed variables:

$$P\left(\tau_1 < t, ..., \tau_d < t | \, Y = y\right) = \prod_{i=1}^{d} \Phi\left(\frac{\Phi^{-1}\left(Q_i(t)\right) - a_i y}{\sqrt{1-a_i^2}}\right). \tag{2.16}$$

Integrating out the common factor Y yields the unconditional distribution:

$$P\left(\tau_1 < t, ..., \tau_d < t\right) = \int_{-\infty}^{\infty} \prod_{i=1}^{d} \Phi\left(\frac{\Phi^{-1}\left(Q_i(t)\right) - a_i y}{\sqrt{1-a_i^2}}\right) \phi(y) \, dy. \tag{2.17}$$

The last assumption made in these markets, is that all correlations are equal, or $a_i = a$ for all firms. Though unrealistic to believe that all dependence can be summarized in one parameter, we will stick with it for now and explain its use in the next section. From this point on we can choose three paths to relate our default distribution to the loss. The first approximation is often called *LHP*-model, short for *large homogeneous portfolio*. By assuming that we have a homogeneous portfolio and letting the number of obligors become large ($d \to \infty$), we get an analytical solution to the problem. We consider two other options. One assuming that all entities have the same probability of default (homogeneous portfolio-*HP*), and another assuming we have a heterogeneous portfolio. Here heterogeneous in the sense that each obligor have his/hers own probability of default. These are for obvious reasons called *semi-analytical* models.

Let us first consider the conditional distribution (2.16). Conditioning upon the common factor, the defaults in the portfolio become independent events. We let the probability that entity i defaults before time t be denoted by

$$p_i(t|y) = \Phi\left(\frac{\Phi^{-1}\left(Q_i(t)\right) - a_i y}{\sqrt{1-a_i^2}}\right). \tag{2.18}$$

Homogeneous portfolio: When all $p_i(t|y)$ are equal, the conditional distribution of the number of defaults becomes *Binomial* $(N, p(t|y))$. We get the unconditional distribution by integrating out the common factor. In this way we get the last unknown quantity p_k needed in the evaluation of expected tranche loss (2.4) by:

$$P(\# \text{ of defaults} = \text{k}) \;=\; \int_{-\infty}^{\infty} \binom{N}{k} p(t|y)^k \left(1 - p(t|y)\right)^{N-k} \phi(y)\, dy. \tag{2.19}$$

Heterogeneous portfolio: If we assume a heterogeneous portfolio, we are no longer in the binomial setting since the probabilities are unequal. There is still $\binom{N}{k}$ ways we can order k defaults, but now each ordering results in a different probability since the individual probabilities differ. In order to compute the conditional probabilities we choose a numerical method found in Hull & White (2004) and Zhen (2006).

Let $\pi(k|Y)$ be the conditional probability that k out of the N entities will default before t, then $\pi(0|Y) = \prod_{i=1}^{N} S_i(t|Y)$, where $S_i(t|Y)$ is the conditional survival probability of entity i. This case is trivial, since there is only one way we can get zero defaults. For $k > 0$ we have that:

$$\pi(k|Y) \;=\; \pi(0|Y) \sum_{j_1 < \cdots < j_k} \omega_{j_1} \cdots \omega_{j_k}, \tag{2.20}$$

where

$$\omega_i \;=\; \frac{1 - S_i(t|Y)}{S_i(t|Y)} \;=\; \frac{p_i(t|Y)}{S_i(t|Y)}. \tag{2.21}$$

Newton-Girard Formula: If we let $u_k = \sum_{i=1}^{k} \omega_i^k$ be the sum of the k-th power of the variables and $v_k = \sum_{j_1 < \cdots < j_k} \omega_{j_1} \cdots \omega_{j_k}$ be symmetric polynomials of order k, then

$$u_k - v_1 u_{k-1} + v_2 u_{k-2} - \cdots + (-1)^{k-1} v_{k-1} u_1 + (-1)^k k v_k \;=\; 0, \tag{2.22}$$

for $1 \leq k \leq N$

By using *Newton-Girard Formula* we can compute v_k from u_k in a recursive manner, and get the conditional probabilities of k defaults by $\pi(k|Y) = \pi(0|Y) v_k$. As usual the unconditional probabilities require integrating out the common factor Y since

$$\pi(k) \;=\; \int_{-\infty}^{\infty} \pi(k|Y) \phi(y) dy. \tag{2.23}$$

This is only one of many numerical methods one can employ. Usually, the methods are based on fast fourier transforms or some recursive scheme. A popular recursive method can be found in Anderson et al. (2003), and a very similar one called *probability bucketing* can be found in Hull & White (2004). For a comparison of several methods the reader might have a look at the PhD thesis by Ma (2007).

2.5 Valuation of iTraxx tranches

As mentioned, the iTraxx family offers a broad exposure to the credit market through a variety of products. After briefly discussing the role of correlation and the *base correlation* concept, we turn our attention to valuation of index tranches using the framework introduced in Section 2.4.2.

The Itraxx tranches differs from ordinary tailor made synthetic CDO's in that they are highly liquid, where the unfunded format has proved more liquid than the funded format, i.e issued notes. As in any liquid market, prices are mainly driven by supply and demand factors and not models. This does not mean that models are useless, as they play an important role in hedging, or more generally to extract market information. Generally a more complex model will be able to answer more questions than a simple one. In the case of the homogeneous model in our setting, we can look at sensitivities on the price with respect to changes in the correlation parameter, the uniform spread or default probability and the recovery level. With a heterogeneous model we can answer the same questions but at a more granular level, i.e what is the effect of a spread widening on just one, or a few names?

The correlation parameter can be viewed as an implied parameter, it is not in any way a result of estimation on historical data. The correlation parameter is the answer to the question: Given todays market prices, what is the level of correlation in the gaussian copula model that reproduces these prices? As the gaussian copula model is not capable of reproducing all tranche prices with just one correlation figure, we have to solve for a different correlation value for each tranche in question. The single implied correlation parameter corresponding to a tranche price is called *compound correlation*, and Figure 2.5 illustrates a well known observation, namely the existence of a *"correlation smile"*. By this we mean that the correlation value drops when moving from the equity tranche to the lower mezzanine tranche, and increases again with subordination. This itself does perhaps not pose a problem when only considering the standard tranches of the iTraxx, but compound correlation has its sets of problems. First of all, there has been occasions when no correlation value would solve for a price, i.e the known "correlation crisis" in 2005 when Ford and GM had their debts downgraded. Secondly, there could be two values giving the same price in the case of mezzanine tranches. Last but perhaps most important, is that there is no good way of telling what the correlation should be when valuing bespoke tranches, or tranches with non-standard attachment and detachment points. Seeking to overcome some of these issues, JPMorgan introduced the concept of *base correlation* which has become a market standard quoting mechanism.

2.5.1 Base correlation

Whereas one speaks of a *correlation smile* in the *compound correlation* case, one refers to the strictly increasing *base correlations* as a *correlation skew*. Instead of viewing a tranche as a function of the one loss distribution resolving from one correlation parameter, one views a tranche as a difference of base tranches. The base

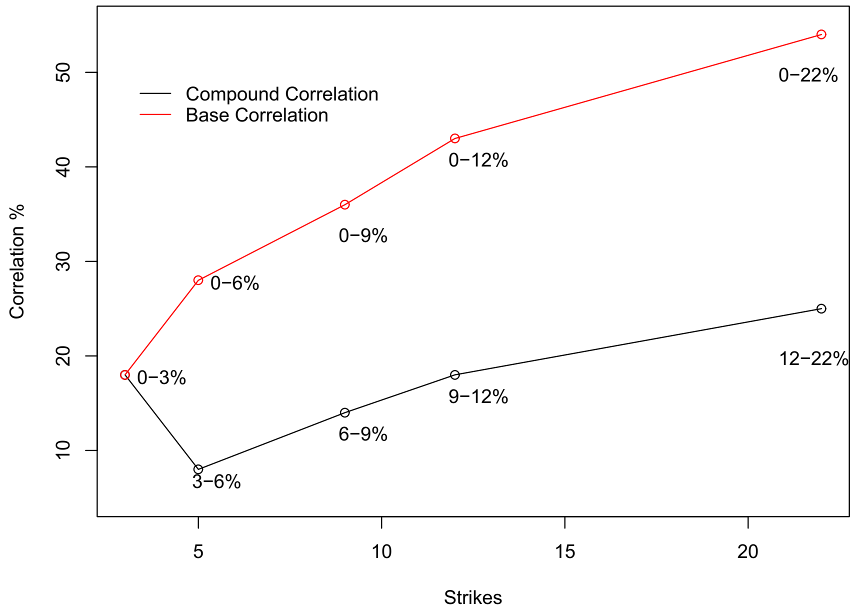

Figure 2.5: Compound vs. base correlation

tranches is in a sense a series of equity tranches. The only difference in the valuation is how we calculate the expected tranche loss $EL_{a,d}(t)$. If we denote by ρ_a and ρ_d the *compound correlation* of a equity tranche with attachment and detachment [0,a%] and [0,d%] respectively we rewrite $EL_{a,d}(t)$ as

$$EL_{a,d}(t) = \frac{dEL_{0,d,\rho_d}(t) - aEL_{0,a,\rho_a}(t)}{d-a}. \tag{2.24}$$

Here we can see that the expected loss of a tranche can be viewed as the expected loss of a equity tranche with detachment point $d(> a)$, reduced by the loss of a equity tranche with detachment point a. The whole *base correlation* curve can then be obtained in an iterative manner. We start with the tranche with lowest subordination, here the *compound correlation* and *base correlation* coincide. We then search for the correlation parameter ρ_d in (2.24) using the spread corresponding to the $(a, d\%)$-tranche. With this correlation parameter we continue to search for the the next correlation parameter using the the spread of the next tranche, until we have solved for all the tranches.

Since the *base correlations* are increasing with detachment points we get a more sensible way of valuing non standard tranches. For example, if we were to value a tranche with attachment 2% and detachment point 5% we would interpolate the *base correlation* curve to find the corresponding correlation values for the two base tranches. This procedure is of course not flawless, as the slope of the curve comes in to play. In the case of a perfectly linear curve, simple linear interpolation will be a sensible choice, but if there is some curvature linear interpolation might not be so sensible. One might consider other alternatives and in any way check if the prices given by the model admits arbitrage opportunities.

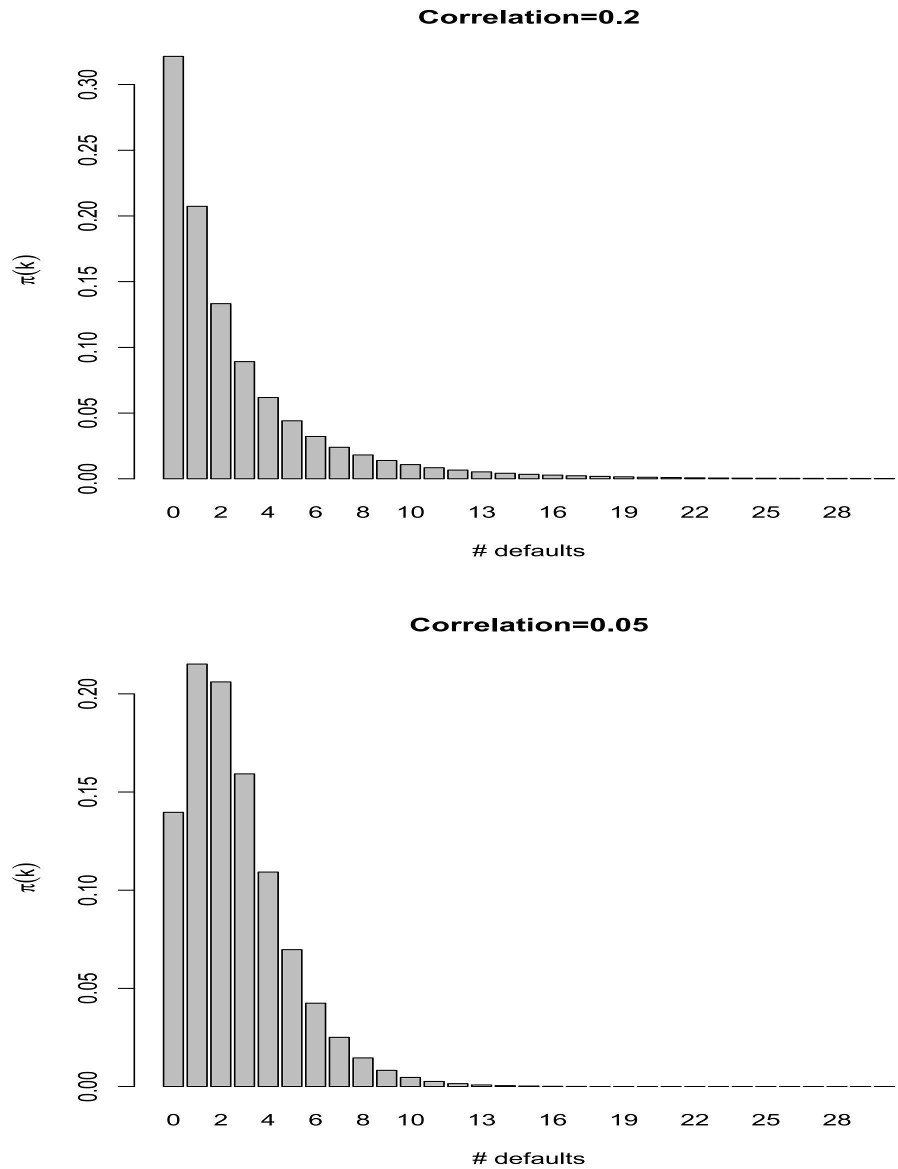

Figure 2.6: Loss distribution at terminal time 5 years, for two different correlation values.

What other information than simply being a model parameter might we infer from the correlation values? Let us take a look at how *compound correlation* influences our loss distribution. Figure 2.6 shows two loss distributions from our homogeneous model with two different correlations. We see that increasing correlation values shifts mass to the tail, implying there is a higher probability for many defaults. As more mass is shifted to the tail, the expected loss of an equity tranche holder reduces, and therefore the spread of the tranche also reduces. This has the opposite effect on senior tranches. Remembering the factor-structure in equation 2.12, increasing correlation means that we are putting more weight on the systematic factor. In this setting the market can view the correlation parameter as an indication of the state of the overall credit environment, where high correlation values indicate a bad overall credit environment, and low correlation values have the opposite effect.

2.5.2 Homogeneous vs. heterogeneous loss distributions

In order to compute a loss distribution one has to consider the recovery rate. While it is possible to take a view on each obligor, i.e based on historical default rates for sector and perhaps rating, one will usually encounter a recovery rate of 40% in most academic papers treating the iTraxx. Recovery rate studies on corporate bonds can be found at *www.moodys.com*, where one can read that the recovery rates calculated from year to year exhibit a mean reversion towards a long-term mean of roughly 40%, which in a way justify using such a value. In the case of the iTraxx index, market makers are given the opportunity to send in their views on the recovery rate along with the preferred list of names to be included in the index. Before each roll, the recovery rate is rounded to the nearest 5% before posted on the web-site. Taking a look at the iTraxx presentation found at *www.itraxx.com*, one can see that all indices except *subordinated financials* exhibit a recovery rate of 40%, where the latter is 20%. This means that every protection seller or buyer is exposed to recovery risk unless hedged, as the true recoverd value can not be known in advance. The recovery assumption is nontheless a very practical and necessary assumption as it is a key ingredient in models. Deals between two parties are made and perhaps unwound, using the same recovery rate assumption. The parties might still have very different views on the true recovery rate. We will be using a recovery rate of 40% in all valuation models.

Before calculating any prices, we investigate the dataset we had available for pricing. The data consist of end-of-day mid-quotes for all iTraxx constituents on June 27, 2007. The quotes are for both the 3 year and 5 year contract. It may be argued that one should have more maturities when considering a non-flat termstructure, but one should also bear in mind liquidity as an issue, as many CDS contracts only have liquidity at the 5 year contract. Another solution when few maturities are available, is using a linear hazard rate function as mentioned in Section 2.3.2, or some other form of interpolation to make the termstructure smoother. We have not investigated the effect in this thesis.

Figure 2.7 shows the distribution of spreads, with the majority of contracts lying within the interval [0,50] bps. In Table 2.2 we have included some summary statistics of the single names along with the iTraxx index quote. Note that the iTraxx index quote is not the same as the average of the individual CDSs. It is actually closer to a weighted average of the single name spreads, where the survival

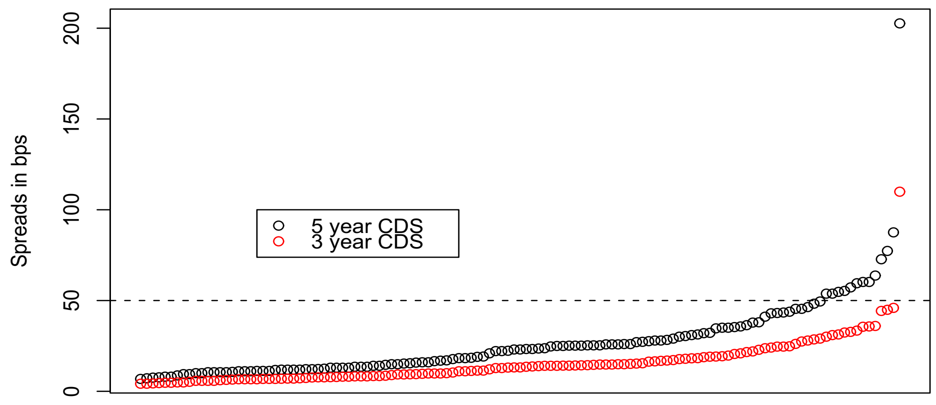

Figure 2.7: Sorted single name CDS spreads on iTraxx series 7, on June 27, 2007. *Source Bloomberg.*

	Index	Min	1st Quartile	Median	Mean	3rd Quartile	Max
3Y	14.233	4.20	7.79	13.17	15.70	19.06	110.00
5Y	24.806	6.90	13.00	23.10	27.11	32.20	202.70

Table 2.2: Summary statistics of all single name CDSs' along with the index spread

probabilities act as weights. Taking into account the termstructure of survival/default probabilities and interest rates in this weighted average, yields what is called the *theoretical spread* of the index. What do we mean my this? The *Risky Annuities* mentioned in 2.3.2 acts as weights. Denoting S_i and Ra_i, the spread and *Risky Annuity* of name i respectfully, we get the following formula:

$$Index\ theoretical\ spread = \sum_i \frac{S_i\, Ra_i}{\sum_j Ra_j}. \tag{2.25}$$

The interested reader will find some illustrative examples and explanation for this relationship in the technical report from HVB-Felsenheimer et al. (2004). The difference between the *theoretical spread* and the quoted index spread is known as *basis-to-theoretical*. This basis is generally larger in volatile markets and smaller in calmer markets. One possible explanation for the existence of such a basis is that the index being a more liquid instrument than single names, will contain a smaller liquidity premium. It can also be argued that credit trends are more quickly seen on the index. If there is a worsening in the overall credit environment, it is more likely to be seen on the index level before a general widening of single name spreads, thus creating a lag-effect. In a homogeneous pool setting one usually uses the index spread for calculating the uniform default probabilities to be used in the model. Since there exists a basis, the expected portfolio loss will not be the same in the two models. Because the index CDS is a much more liquid instrument than the single name constituents, and therefore believed to be a better overall measure of expected portfolio loss, one usually adjust single names to the index when pricing index tranches in a heterogeneous model. One way of doing this, is as mentioned in Beinstein et al. (2005). One computes the ratio of the index spread over the theoretical value, and multiply all single name spreads by this value. If considering a non-flat curve, one adjust all maturities as well. In this setting both the slope an level of single names is being adjusted. We provide an example using our data from June 27, 2007.

	Index	Theoretical	Ratio	EL Index	EL Port	EL Adjusted Port
3	14.233	15.662	0.90876	0.4239%	0.46637%	0.424%
5	24.806	26.95486	0.9202792	1.2673%	1.3745%	1.26751%

Table 2.3: Single name adjustment

The portfolio expected loss in Table 2.3 is obtained by using equation 2.2 and the linearity of the expectation:

$$\mathbb{E}\left(L(t)\right) = \mathbb{E}\left(\sum_{i=1}^{N} \frac{1-R}{N} I_{\{\tau_i < t\}}\right) = \sum_{i=1}^{N} \frac{1-R}{N}\, \mathbb{P}(\tau_i < t). \qquad (2.26)$$

We see that after adjustment, the expected loss of the portfolio and the index coincide. The small difference is due to numerical implementation. We visualize the effect of moving from a homogeneous to a heterogeneous pool in Figure 2.8, where only probabilities for the first 30 defaults are included. Clearly the homogeneous pool approximation exhibit a greater probability of zero defaults, and corresponding lower probabilities for the first few defaults than the heterogeneous model. This is perhaps to be expected when looking at the spread distribution in Figure 2.7. The few names with high spreads should intuitively have a greater probability of defaulting in the considered time horizon of 5 years, than the average. The simplifying homogeneous approximation is not able to account for such effects leading to the deviations observed in the loss distributions.

Table 2.4 shows some iTraxx quotes which we calibrate the two models to, and obtain the base correlation curves shown in Figure 2.9. Observe that the base correlations for the heterogeneous case are higher than in the homogeneous setting. Thinking in terms of equity tranches, we saw that that the probabilities for experiencing the first few defaults were higher in the heterogeneous setting. Thus using the same correlation in both models, causes a higher premium in the heterogeneous

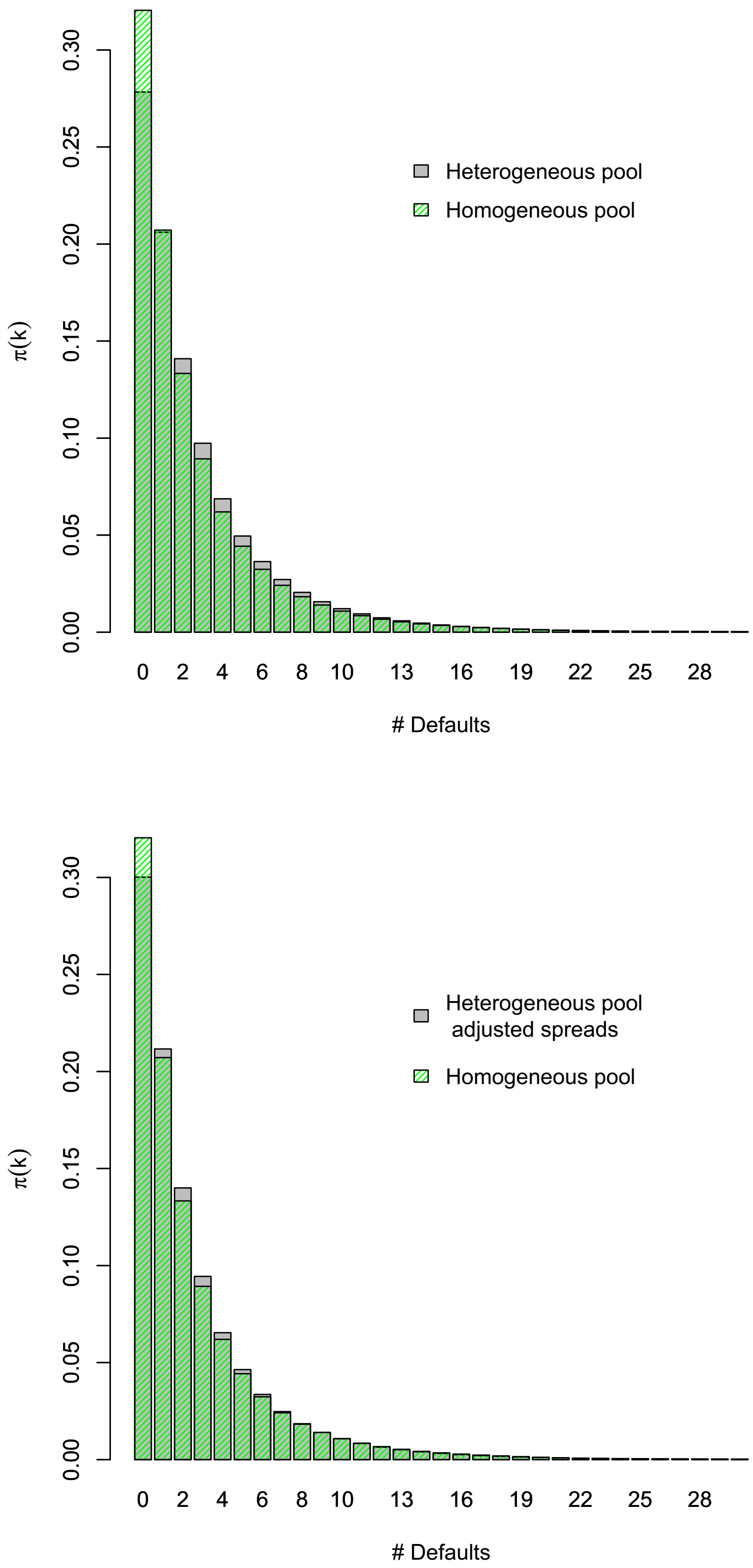

Figure 2.8: Comparison between of loss distributions from homogeneous and heterogeneous pool using a correlation equal to 0.2.

model. To lower the spread of the tranche we increase the correlation parameter. As mentioned, this causes probability mass to be shifted to the tail, and therefore reduce the expected loss of the tranche.

Tranche	0-3% Upfront	0-3%	3-6%	6-9%	9-12%	12-22%
	11.875%	500	63	16.25	7.25	3.375

Table 2.4: iTraxx tranche quotes on June 27, 2007 obtained from MorganMarkets. All quotes except equity tranche are in basispoints.

As shown, the base correlations are highly model dependent, emphasising that one should be careful when comparing published correlation numbers. Since different banks will have different implementations or variations of the model, and not to mention the source of data, the implied correlation will vary. This further supports the idea of correlation acting more as a model parameter than a precise measure of default dependency. Nevertheless, it is a well established framework and will continue to be used until better models are developed. Base correlations has other uses than simply reproducing already observable prices. We are presented with a bigger challenge when asked to value a bespoke CDO. If the CDO has near identical characteristics as our index tranches, we may argue that the prices in the synthetic CDO should be given by the same base correlation curve inherit in the iTraxx tranches. What we want is parameters that produce prices in our bespoke CDO that the market would be willing to buy/sell protection for. We want a price for our bespoke CDO that is fair in a market sense. The challenge then presents itself in mapping the iTraxx base curve into the custom pool. While there exists several mapping procedures, some of them explained in Baheti & Morgan (2007), we do not know of any standard procedure.

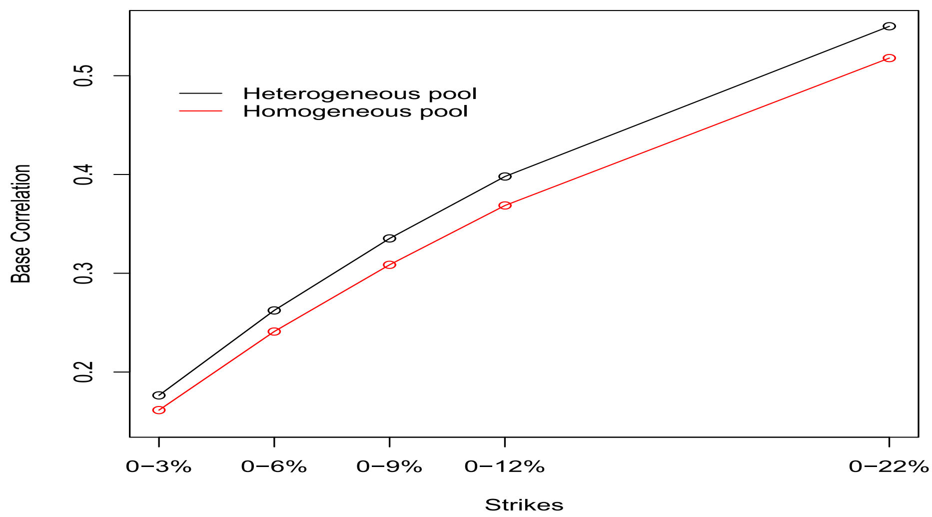

Figure 2.9: Base correlation curves obtained for for two models.

We end this presentation of the market standard model with some remarks. First of all the standard model is static, it does not take into account any evolution of the termstructure of default, nor does it contain any dynamics on the dependence. As such, it cannot price forward starting contracts or options. The poor fitting capabilities to all tranches simultaneously is considered a weakness, and much research have been aimed at finding a dependence structure that allows for a better fit. Popular paths of research in this static framework has been varying the dependency structure by using alternative copulae, or altering the distributions of the factors, with the aim of eliminating the observed base correlation skew.

Chapter 3

Risk

As with any financial instrument, there is no reward without taking risk. Structured credit investors face mainly two sources of risk:

Default risk Obviously defaults contribute directly to losses. Depending on subordination, one or more defaults will reduce part of our tranche notional. If the exposure is in the swap format, we must pay the lost amount, and if our exposure is through a note (funded format), we will receive back the principal amount less the loss at maturity.

Market risk This risk component arises from the financial movement of the underlying factors, and affects us as the market value of our position can change over time. Possible sources affecting a CDO, can be risk arising from changes in interest rates, spread movements, liquidity, and rating movements if applicable. In the most simple case consider selling protection on a single name for the market spread of 100 bps per year. The day after the trade the spread on that name widens to 150 bps. If for some reason you need to exit the position, you may exit the position with your dealer, or perhaps buy protection on that same name for the remainder of the original contract. In the latter case you would be receiving 100 bps, and paying 150 bps - clearly a negative mark-to-market value.

In this chapter we seek a method to quantify some of these sources. For the market risk component we will only model spread risk explicitly through a simplified model. For the default risk component we will rely on information based on historical data, published by rating agencies. The evolution of rating changes and ultimately default will be modeled in the framework of continuous time Markov chains, and for that reason we will rely heavily on the work of Israel et al. (2001)(IRW) and Jarrow et al. (1997)(JLT).

3.1 Rating transition model

3.1.1 Rating migration

Rating agencies such as *Moody's*, *Standard&Poors* and *Fitch* have made it their business to assess the credit quality of obligors. For many years they have been assigning ratings based on financial and non-financial information. These ratings represent the agencies views on the company's ability and willingness to meet its financial obligations. The common master scale consist of letters, i.e Aaa, Aa, A, Baa, Ba, B, Caa, C (*Moody's*) and AAA, AA, AA, BBB, BB, B, C (*S&P* and Fitch), where Aaa and AAA in the two systems represent the best rating an obligor can achieve. Both systems also use modifiers to brake ratings into a finer scale. *Moody's* uses numbers 1, 2 and 3 i.e Aa1, and *Standard&Poors* uses + and - , i.e AA+. By forming cohorts or groups of obligors with same rating and perhaps other characteristics, rating agencies can track the evolution of credit quality within these groups through time. By using such history one can estimate default probabilities for the different rating categories, and rating up/downgrade, also called *rating migration.* While the major databases are only accessible to paying customers, some research are made available to the public.

Before utilizing any framework it is natural to ask if the rating agency definition of default deviates from the credit events we are seeking to model (ISDA definitions). In Moody's (2007) we find the following definitions of default:

- A missed or delayed disbursement of interest and/or principal.
- Filing for bankruptcy, administration, legal receivership, or other legal blocks (perhaps regulators) to the timely payment of interest and/or principal.
- A distressed exchange occurs when: (i) the issuer offers debt holders a new security or package of securities that amount to a diminished financial obligation (such as preferred or common stock, or debt with a lower coupon or par amount); or (ii) the exchange had the apparent purpose of helping the borrower avoid imminent default.

We will not state the ISDA definitions, they are treated thoroughly in JPMorgan (2006), but they are basically the same, meaning that they both seek to cover any event leading to an economic loss, thus gaining confidence in using rating agency data.

The defining feature of a Markov chain, is its transition matrix. A rating migration matrix completely summarises changes in credit ratings over a given time horizon. The cells of the published transition matrices are discrete-time estimates of rating migration probabilities. They show the rate of rating changes measured at two points in time. The agencies publish these matrices annually, both the matrix spanning only last years migrations, and the updated rating matrix averaged over

From/To	Aaa	Aa	A	Baa	Ba	B	Caa-C	Defaults	WR
Aaa	83.9	12.5	0.4	0	0.1	0	0	0	3.1
Aa	1.2	85.4	9.2	0.5	0	0	0	0	3.6
A	0.1	3.8	84.6	6.2	0.5	0.1	0	0	4.5
Baa	0	0.5	8.3	78.7	5	1.7	0.7	0.2	4.9
Ba	0	0	1	9.9	65.7	10.9	1.1	1	10.5
B	0	0	0.6	0.4	7.7	68.2	8.6	4.3	9.5
Caa-C	0	0.6	0.1	0	0.7	13.2	52.5	23.2	6.7

Table 3.1: Broad letter grade one-year average rating transition matrix for Europe 1985-2006. Rates in %. *Source: Moody's.*

several years. Table 3.1 shows the matrix presented in Moody's (2007), where one also can find how the rates are calculated.

As usual, the *(i,j)*-th entry represents the probability of migrating from state i to j within the given time horizon. The extra column named *WR* contains withdrawn ratings. Withdrawn ratings are caused by debt maturing, failure to cooperate or for some other reason like merging with another company etc. Though rating agencies are likely to continue to observe these entities in case of re-entry or default, information about them are not publicly available so we use the common approach of treating the *WR*-category as non informing and simply add the values back in to the columns proportional to column entries. Also in the case of rows not summing exactly to one, the remainder of the sum is distributed using the same approach. The resulting transition matrix can be viewed in Table 3.2, where we have added the row *Default* with zero entries except the last column, indicating that default is an absorbing state.

From/To	Aaa	Aa	A	Baa	Ba	B	Caa-C	Default
Aaa	86.584	12.900	0.413	0.000	0.103	0.000	0.000	0.000
Aa	1.246	88.681	9.553	0.519	0.000	0.000	0.000	0.000
A	0.105	3.987	88.772	6.506	0.525	0.105	0.000	0.000
Baa	0.000	0.526	8.728	82.755	5.258	1.788	0.736	0.210
Ba	0.000	0.000	1.116	11.049	73.326	12.165	1.228	1.116
B	0.000	0.000	0.668	0.445	8.575	75.947	9.577	4.788
Caa-C	0.000	0.664	0.111	0.000	0.775	14.618	58.140	25.692
Default	0.000	0.000	0.000	0.000	0.000	0.000	0.000	100.000

Table 3.2: Transition matrix adjusted for WR.

There are some features worth commenting. The matrix clearly exhibit dominance along the diagonal, indicating a high likelihood of staying in the same state, or migrating to a near state. Also notice the many zero entries which contradicts intuition. However small the probability may be, there should be a positive probability of default, even in the highest credit classes. Clearly the zero entries is a result of not observing the events, not that the event is impossible.

As stated in Standard&Poors (2007), by assuming that the rating transition rates are stable and follow a first order Markov process, cumulative default rates and rating migrations can be projected for any number of years in the future. In Jarrow et al. (1997) the same assumptions are made, but the chain is modeled in continuous time. We are considering a stochastic process $\kappa(\omega, t)$, on the finite state space S = {Aaa, Aa, A, Baa, Ba, B, Caa-C, D}. Here the rating function $\kappa : \Omega \times$ [0,T] $\rightarrow$ S, and we are now working under the actual or physical probability measure $\mathbb{P}$. We fist define some features of the Markov chain in our setting:

Definition 3.1.1. The (8×8) transition matrix for the period [t,T] under $\mathbb{P}$ is written as

$$P(t,T) = \begin{pmatrix} p_{1,1}(t,T) & p_{1,2}(t,T) & \dots & p_{1,8}(t,T) \\ p_{2,1}(t,T) & p_{2,2}(t,T) & \dots & p_{2,8}(t,T) \\ \vdots & \vdots & \ddots & \vdots \\ p_{7,1}(t,T) & p_{7,2}(t,T) & \dots & p_{7,8}(t,T) \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

where $p_{h,j}(t,T) \geq 0$ for all $h, j,\ h \neq j$ and $p_{h,h} = 1 - \sum_{j=1, j\neq h}^{8} p_{h,j}(t,T)$.

Definition 3.1.2. For all h, the default time τ_h is defined as

$$\tau_h = \inf s \geq t : \kappa(s) = 8, \tag{3.2}$$

which is the first time the company hits the state of default 8, assumed to be absorbing.

By assuming the Markov property, conditioning on all the information given up to time t yields no extra information, or the future distribution of states depend only on the current state. If we let $\mathcal{F}_t$ represent the entire history[1] of the process we have the relation

$$\mathbb{P}\left(\kappa(T) = j | \mathcal{F}_t\right) = \mathbb{P}\left(\kappa(T) = j | \kappa(t)\,\right), \forall\, j\ \in \text{S}. \tag{3.3}$$

The time-homogeneity assumption implies that the transition probabilities depend only on the length of the time interval, not which interval. If we are considering a discrete time Markov chain with given one-period transition matrix, all one-period matrices must coincide, and the n-period transition matrix is simply the n-th fold of the one-period matrix.

[1]Formally stochastic proscesses are collections of random variables, for instance indexed by time. $\mathcal{F}_t$ is the sigma algebra which $\kappa(t)$ is a random variable with respect to.

Definition 3.1.3. The continuous time-homogeneous Markov chain $\{\kappa(t) : 0 \leq t \leq \tau\}$ is specified in terms of its (8×8) constant infinitesimal generator, or intensity matrix defined as

$$Q = \begin{pmatrix} -q_{1,1} & q_{1,2} & \cdots & \cdots & q_{1,8} \\ q_{2,1} & -q_{2,2} & \cdots & \cdots & q_{2,8} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{7,1} & q_{7,2} & \cdots & \cdots & q_{7,8} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.4}$$

where $q_{h,j} \geq 0$ for all h, j, $h \neq j$, and $q_{h,h} = -\sum_{j=1, j\neq h}^{8} q_{h,j}$ for h =1,2...,8.

In (3.4) the off-diagonal elements are the constant intensities of jumping to rating j from h, whereas the diagonal elements are the constant intensities of moving away from h. In our finite state Markov chain setting, the (8×8) rating transition matrix for κ of the $T - t$ period is

$$P(t, T) = e^{Q(T-t)} = \sum_{n=0}^{\infty} \frac{[(T-t)Q]^n}{n!}, \tag{3.5}$$

showing again that the probabilities in the time-homogeneous setting only depend on the length of the time interval. In our setting it would be convenient to model transitions over arbitrary time horizons, but a problem arises in that our given empirical transition matrix might have entries such that (3.5) does not hold. This is also known as the embedding problem, the embedding of a discrete time Markov chain into a continuous time Markov chain.

Israel et al. (2001) identified a set of conditions for the existence of an exact generator for a transition matrix. We restate the theorem here:

Theorem 3.1.4 (IRW)**.** *Let P be a transition matrix, and suppose that*

a) det(P) $\leq$ 0, or

b) det(P) $> \prod_i p_{i,i}$, or

c) there are states i and j such that j is accessible from i, but $p_{i,j}$=0.

Then there does not exist an exact generator for P.

Computing the determinant and the product of the diagonal entries in Table (3.2), we get values of 0.1693 for the determinant and 0.1826 for the product, so part a) and b) of Theorem 3.1.4 does not apply. However, by Part c) we do not have an exact generator, since we observe that $p_{1,3} > 0$ ($Aaa \rightarrow A$) and $p_{3,4} > 0$ ($A \rightarrow Baa$), but $p_{1,4} = 0$ ($Aaa \rightarrow Baa$). We may hope to find an approximate generator $\tilde{Q}$ such that $P \approx e^{\tilde{Q}}$. Both IRW and JLT provide suggestions as to how to compute an approximation. We summarize some results from IRW into a theorem, also stated in this form in Bluhm & Ludger (2006). The proof is found in IRW:

Theorem 3.1.5 (Bluhm 06)**.** *If a migration matrix P is strictly diagonal dominant, i.e $p_{i,i} > \frac{1}{2}$ for every i, then the log-expansion*

$$\tilde{Q}_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{(P-I)^k}{k} \quad n \in \mathbb{N} \tag{3.6}$$

converges to a matrix $\tilde{Q}$ satisfying

1. $\sum_j \tilde{q}_{ij}$ *for every* i
2. $e^{\tilde{Q}} = P$

The convergence $\tilde{Q}_n \to Q$ is geometrically fast.

The problem with this method, is that the matrix $\tilde{Q}$ is not guaranteed to have non-negative off-diagonal entries. This poses a problem in that given sufficiently small $t > 0$, $P_t = e^{t\tilde{Q}}$ will also have negative off-diagonal entries, which means that P_t is not a proper Markov transition matrix. IRW note that these entries will usually be quite small and propose to correct the problem by replacing the negative off-diagonal entries with zeros and distribute the negative part along the rows of the generator matrix proportionally to their absolute values. As in IRW let

$$G_i = |\tilde{q}_{ii}| + \sum_{j \neq i} \max(\tilde{q_{ij}}, 0); \quad B_i = \sum_{j \neq i} \max(-\tilde{q}_{ij}, 0) \tag{3.7}$$

be the "good" and "bad" totals for each row i, and then set

$$q_{ij} = \begin{cases} 0, & i \neq j \text{ and } \tilde{q}_{ij} < 0 \\ \tilde{q_{ij}} - B_i|\tilde{q_{ij}}|/G_i, & \text{otherwise if } G_i > 0 \\ \tilde{q_{ij}}, & \text{otherwise if } G_i = 0. \end{cases} \tag{3.8}$$

The adjustment guarantees that $\tilde{Q}$ is a valid generator. For illustrative purposes we show the result of the log-expansion $\tilde{Q}$ and $\tilde{Q}_{IRW}$ (the adjusted generator) in Table 3.3.

Another method of obtaining an approximate generator for the transition matrix given is given in JLT. They obtain an approximate generator by assuming that there is never more than one transition per year, and give the following relationship:

$$q_{ii} = \log p_{ii}; \qquad q_{ij} = p_{ij} \frac{\log(p_{ii})}{p_{ii} - 1} \quad (i \neq j), \tag{3.9}$$

where we no longer encounter the problem of negative off-diagonal elements. The proof is found in JLT.

As shown, we get the one-year transition matrix by computing the matrix exponential of the generators. $\tilde{P}_{IRW}$ and $\tilde{P}_{JLT}$ are rounded to 6 decimal points. With two possible solutions to the embedding problem, it is natural to ask which

$$\tilde{Q} = \begin{pmatrix} -0.14510 & 0.14744 & -0.00325 & -0.00031 & 0.00133 & -0.00009 & -0.00000 & -0.00000 \\ 0.01418 & -0.12358 & 0.10786 & 0.00195 & -0.00038 & -0.00004 & 0.00000 & 0.00000 \\ 0.00088 & 0.04484 & -0.12530 & 0.07586 & 0.00394 & 0.00023 & -0.00044 & -0.00002 \\ -0.00008 & 0.00382 & 0.10170 & -0.19781 & 0.06663 & 0.01647 & 0.00893 & 0.00030 \\ 0.00000 & -0.00053 & 0.00598 & 0.14249 & -0.32478 & 0.16270 & 0.00580 & 0.00831 \\ 0.00000 & -0.00076 & 0.00771 & -0.00266 & 0.11616 & -0.29917 & 0.14510 & 0.03362 \\ -0.00007 & 0.00929 & 0.00013 & -0.00026 & -0.00152 & 0.22216 & -0.55987 & 0.33015 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \end{pmatrix}$$

$$\tilde{Q}_{IRW} = \begin{pmatrix} -0.14692 & 0.14560 & 0.00000 & 0.00000 & 0.00132 & 0.00000 & 0.00000 & 0.00000 \\ 0.01416 & -0.12380 & 0.10767 & 0.00196 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\ 0.00088 & 0.04476 & -0.12554 & 0.07572 & 0.00394 & 0.00024 & 0.00000 & 0.00000 \\ 0.00000 & 0.00383 & 0.10169 & -0.19786 & 0.06663 & 0.01647 & 0.00894 & 0.00031 \\ 0.00000 & 0.00000 & 0.00598 & 0.14238 & -0.32505 & 0.16257 & 0.00580 & 0.00831 \\ 0.00000 & 0.00000 & 0.00767 & 0.00000 & 0.11550 & -0.30088 & 0.14428 & 0.03343 \\ 0.00000 & 0.00928 & 0.00013 & 0.00000 & 0.00000 & 0.22180 & -0.56081 & 0.32960 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \end{pmatrix}$$

Table 3.3: Generator obtained by log-expansion(top), and then adjusted(bottom).

$$\tilde{P}_{IRW} = \begin{pmatrix} 0.864261 & 0.127321 & 0.006888 & 0.000369 & 0.001062 & 0.000086 & 0.000008 & 0.000007 \\ 0.012427 & 0.886602 & 0.095370 & 0.005201 & 0.000313 & 0.000059 & 0.000022 & 0.000006 \\ 0.001049 & 0.039794 & 0.887505 & 0.064929 & 0.005244 & 0.001083 & 0.000317 & 0.000080 \\ 0.000071 & 0.005278 & 0.087246 & 0.827527 & 0.052557 & 0.017854 & 0.007366 & 0.002100 \\ 0.000009 & 0.000476 & 0.011176 & 0.110551 & 0.733023 & 0.121426 & 0.012213 & 0.011127 \\ 0.000008 & 0.000646 & 0.006783 & 0.006531 & 0.085312 & 0.758077 & 0.095101 & 0.047542 \\ 0.000051 & 0.006697 & 0.001127 & 0.000469 & 0.008685 & 0.145821 & 0.580785 & 0.256365 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{pmatrix}$$

$$\tilde{P}_{JLT} = \begin{pmatrix} 0.866646 & 0.121586 & 0.010099 & 0.000659 & 0.000919 & 0.000072 & 0.000010 & 0.000009 \\ 0.011650 & 0.889548 & 0.090405 & 0.007786 & 0.000432 & 0.000132 & 0.000030 & 0.000016 \\ 0.001225 & 0.037850 & 0.892566 & 0.059740 & 0.006212 & 0.001888 & 0.000325 & 0.000194 \\ 0.000087 & 0.006758 & 0.083019 & 0.833433 & 0.046211 & 0.019475 & 0.006848 & 0.004170 \\ 0.000011 & 0.000675 & 0.016039 & 0.101307 & 0.741447 & 0.108276 & 0.015168 & 0.017076 \\ 0.000006 & 0.000521 & 0.007271 & 0.009212 & 0.074221 & 0.772068 & 0.073958 & 0.062742 \\ 0.000045 & 0.006290 & 0.002030 & 0.001173 & 0.013045 & 0.127469 & 0.588116 & 0.261832 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{pmatrix}$$

Table 3.4: The transition matrix obtained by using approximations IRW and JLT for the generator.

one is the best approximation. IRW suggest using a distance measure such as the L^1-norm of $P - exp(\tilde{Q})$ to see which distance of the two is smallest. Computing the sum of the absolute value of the differences between the empirical matrix and the approximations, we obtain in the two cases

$$\text{norm}[P - \exp(\tilde{Q}_{IRW})] = 0.01822539; \qquad \text{norm}[P - \exp(\tilde{Q}_{JLT})] = 0.2138069,$$

which clearly favors the IRW method. As shown in Figure 3.1, using the homogeneous continuous time Markov chain, we have all the termstructures of default for our rating categories. Migration to other states could also be an important, i.e some cash flow CDO‘s have waterfall structures triggered by rating actions, diverting cash flow in the case of downgrade etc.

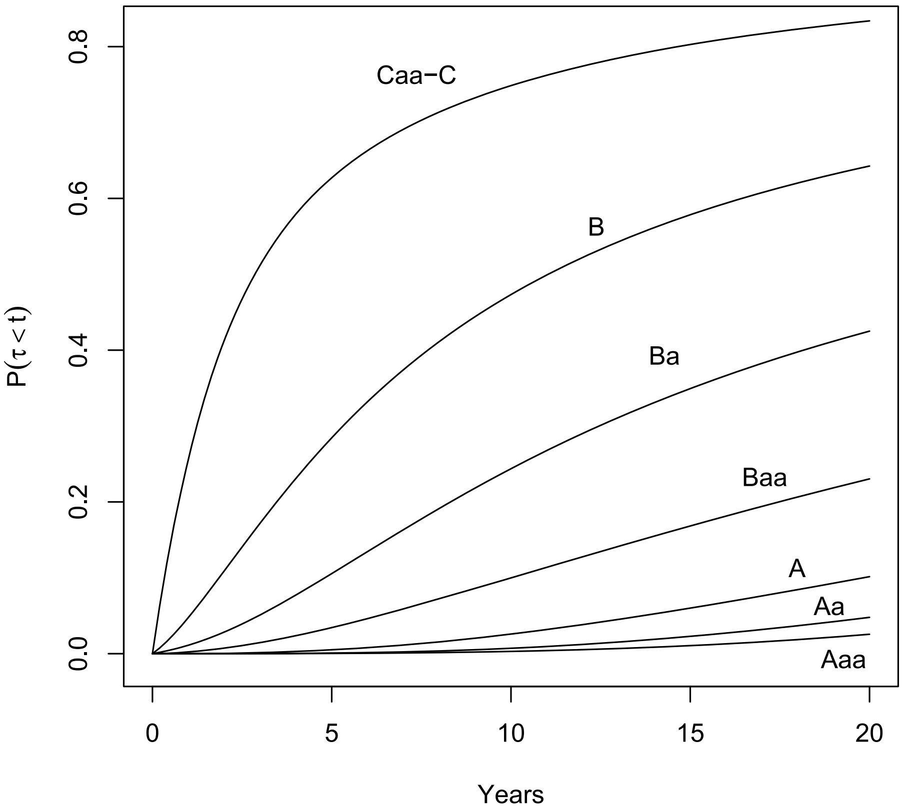

Figure 3.1: Termstructure of default in continuous time homogeneous Markov chain.

3.1.2 Default adjustment

Though very appealing, the time homogeneity has one drawback which we address in this section. The termstructure of default implied in the model does not take into account any aging effect. That it, all the one-period transition matrices are the same provided that the periods are of th same length. As addressed in 2.3, both risk-neutral and historical default probabilities exhibit termstructure effects such as increasing marginal default probabilities for investment grade names, and marginally declining default probabilities for sub-investment grade names (conditional on survival). A way of achieving this, is by scaling the generator matrix appropriately. The general idea is found in JLT, where the generator is scaled to match market implied termstructures in order to price bonds and bond options. The method used in this thesis is an explicit procedure found in Bluhm & Ludger (2006), and here we seek to match historical default probabilities.

The adjustments done to the generator, is simply scaling the rows of the matrix. By Definition 3.1.3, we see that scaling the rows conserves the properties of the generator. We can achieve this by suitably choosing a diagonal matrix to multiply the generator with. Formally let $\Psi(t)$ denote the diagonal matrix with elements

$$\psi_{ii}(t) = t\phi_{\alpha_i,\beta_i}(t), \qquad (i = 1, ...8,\ t \geq 0) \tag{3.10}$$

where

$$\phi_{ij}(t) = \begin{cases} 0 & ,\text{if } i \neq j \\ \phi_{\alpha_i,\beta_i}(t) & \text{if } i = j \end{cases} \tag{3.11}$$

and the function $\phi_{\alpha,\beta}(t)$ is defined as

$$\phi_{\alpha,\beta} : [0, \infty) \to [0, \infty), t \to \phi_{\alpha,\beta}(t) = \frac{(1 - e^{-\alpha t})t^{\beta-1}}{1 - e^{-\alpha}} \tag{3.12}$$

for non-negative constants α and β. Since we are interested in matching observed default frequencies, the problem amounts to choosing vectors of α_i's and β_i's , such that the default entries [i,8] of the t-year transition matrix matches the t-year observed default frequencies as closely as possible. As a function to optimize, we have chosen simply to minimize the sum of these squared deviations. As an example of the method, Figure 3.2 shows the historical default frequencies for Europe as in Moody's (2007) along with the time homogeneous and time in-homogeneous cumulative default probabilities. Finally we can reach our goal of simulating portfolio rating transitions, by utilizing the copula framework and Markov chain theory. We use the fact that in a continuous time Markov chain, the transition matrix over time interval [0,t] can be written as

$$P(0, t) = P(0, t - 1)P(t - 1, t) \tag{3.13}$$

and use this relation to obtain transition matrices for arbitrarily small intervals. We start by defining rating thresholds which corresponds to the event of staying in the same rating or migrating to another state at the end of the intervals. The thresholds are obtained by partitioning the unit interval [0,1] into 8 sub-intervals corresponding to the rating classes. We define these threshold as the cumulative sums of the matrix rows, that is

$$
\begin{aligned}
a_{h1} &= \big[0\,,\, p_{h1}(t-1,t)\big] \\
a_{h2} &= \big(p_{h1}(t-1,t)\,,\, \sum_{j=1}^{2} p_{hj}(t-1,t)\big] \\
a_{h3} &= \big(\sum_{j=1}^{2} p_{hj}(t-1,t)\,,\, \sum_{j=1}^{3} p_{hj}(t-1,t)\big] \\
a_{h4} &= \big(\sum_{j=1}^{3} p_{hj}(t-1,t)\,,\, \sum_{j=1}^{4} p_{hj}(t-1,t)\big] \\
a_{h5} &= \big(\sum_{j=1}^{4} p_{hj}(t-1,t)\,,\, \sum_{j=1}^{5} p_{hj}(t-1,t)\big] \\
a_{h6} &= \big(\sum_{j=1}^{5} p_{hj}(t-1,t)\,,\, \sum_{j=1}^{6} p_{hj}(t-1,t)\big] \\
a_{h7} &= \big(\sum_{j=1}^{6} p_{hj}(t-1,t)\,,\, \sum_{j=1}^{7} p_{hj}(t-1,t)\big] \\
a_{h8} &= \big(\sum_{j=1}^{6} p_{hj}(t-1,t)\,,\, 1\big],
\end{aligned} \tag{3.14}
$$

for all rating classes h. The Markov chain can thus be simulated by drawing uniform variates and finding the corresponding interval. E.g if the sampled uniform number was 0.5, we would conclude that a firm with initial rating Aaa would stay in Aaa according to Table 3.2. For multivariate rating transitions, we rely again on the concept of copulae. From Chapter 2.4 we have that a copula is a multivariate distribution of uniform variates, and we can draw a vector of uniform variates corresponding to the size of the portfolio and find the appropriate interval for all names.

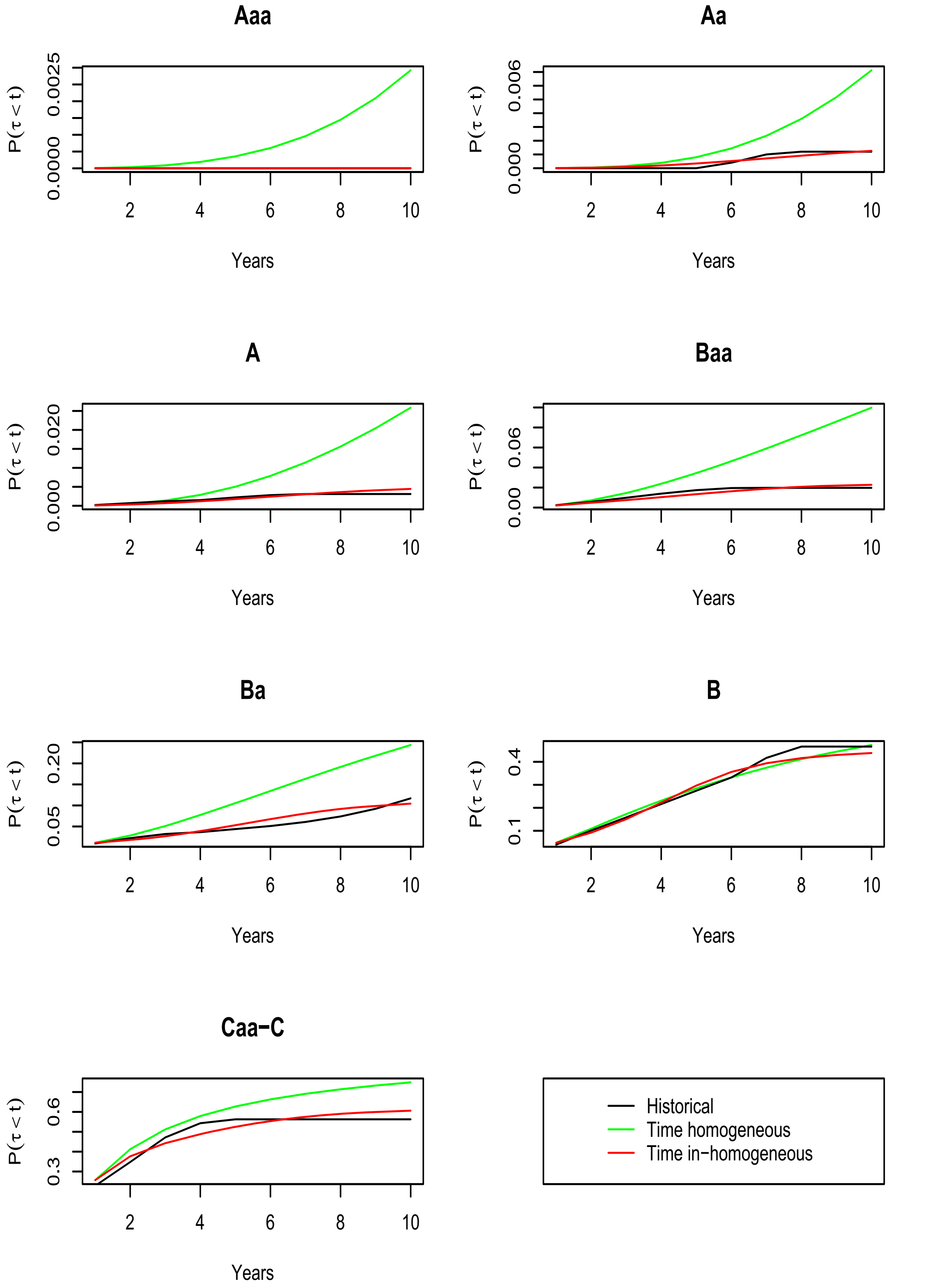

Figure 3.2: Cumulative probability of default with, and without adjustment of generator. European default data only.

3.2 Recovery risk

Although we have taken recovery to be a fixed fraction for valuation purposes, we choose to incorporate it as a random factor in this risk assessment. In real life, the recovered value of defaulted debt is not predetermined, but random and dependent on the seniority of debt, and perhaps subject to country specific legislative issues. In line with most commercial credit portfolio models, or credit Value-at-Risk models such as J.P. Morgan‘s *CreditMetrics*®, Credit Suisse‘s *CreditRisk+*®, McKinsey‘s *CreditPortfolio View*®, KMV’s *CreditPortfolioManager*® and Kamakura’s *RiskManager*®, we treat the recovery rate independently of the probability of default. As is the general case in *CreditMetrics*® we model the recovery rate as a stochastic variable using a beta distribution. As shown in Figure 3.3, the beta distribution is a flexible two-parameter distribution with support on the unit interval which is convenient for modeling recovery rates. The probability density function is given as:

$$
\begin{aligned}
f(x) &= \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\, x^{a-1}(1-x)^{b-1}, \text{ where} \\
\Gamma(z) &= \int_0^\infty t^{z-1}e^{-t}\,dt,
\end{aligned}
\tag{3.15}
$$

Usually the distribution parameters a and b are estimated by the method of moments, which in Renault & Scaillet (2003) was not significantly different from maximum likelihood estimates. The authors challenge the beta assumption on the fact that empirical recovery rates sometimes exhibit bimodal densities, however they show that the assumption of beta distribution compared to their beta kernel density estimator, is benign with respect to credit VaR when considering quantiles less than 99%.

Remark 3.2.1. It is worth mentioning that although the modeling framework presented here share the features of traditional credit portfolio models, a lot of research in recent years have focused on conditioning default probabilities, ratings migrations and recovery rates on the business cycle, and on the interdependencies between default rates and recovery rates. The idea is that during economic downturns default rates increase and recovery rates decline, thus amplifying the risk of a portfolio during recessions. The reasons for omitting these models in this thesis is twofold, the added complexity of model, and the data required to reestimate and calibrate these models. Most of the research done in this area is conducted using the large databases of information gathered by the rating agencies over the years, which is not publically available. This does not mean that traditional models are useless, cycle-effects are much more pronounced in non-investment grade credits, and we are only considering investment grade credits in this thesis. The interested reader might find the works of Edward I. Altman, Rüdiger Frey, Alexander Mc.Neil, Til Schuermann, Jon Frey, Stuart Trunbull, David Lando and their co-authors useful.

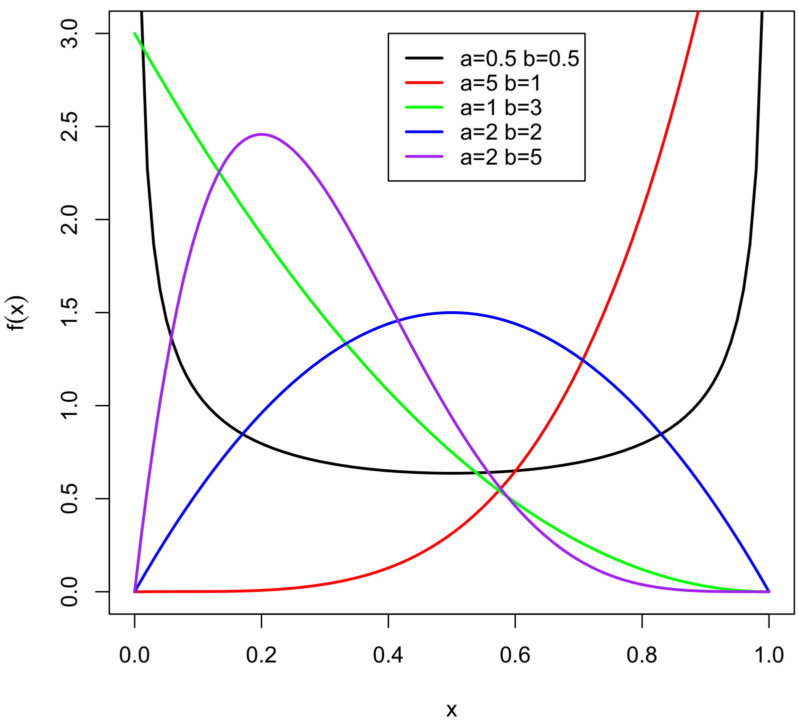

Figure 3.3: The Beta density function with various parameters.

3.3 Spread risk

As mentioned in the beginning of this chapter, a synthetic CDO tranche is subject to mark-to-market movements as the underlying risk factors move over the life of the transaction. While the previous sections sought to model the risk of default only, default risk is only part of the picture when it comes to CDOs. A pure default analysis might be sufficient for a HTM[2] or buy-and-hold investor, but if you want to track the value of your position through time, or might consider offsetting your position, it must be done at market terms. In this thesis we will consider the case of a long position on a 5 year tranche and exiting the position one year later. This means that we are initially selling protection on a tranche, and at a later time buying protection on the same tranche which may or may not have experienced defaults. You can offset the the initial contract in two ways:

- Buy protection on the same tranche and receive or pay the difference in spreads for the remaining duration of the original contract. This cash flow may be reduced if the tranche is hit by defaults, and is in this sense risky. E.g consider a long position of 100bps, and then a short position after one year for the market spread of 80 bps. In the case of no defaults, you will be receiving 20bps on you tranche notional for 4 years.

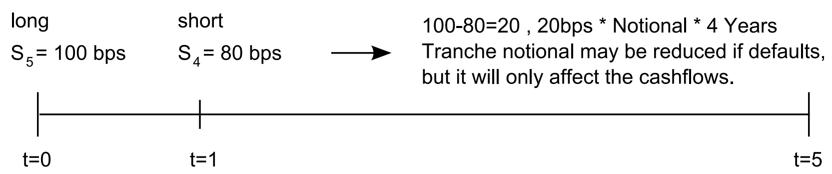

Figure 3.4: Note that the notional may also have been reduced before the short position. This position will be more profitable than an unwind if there is no defaults, or defaults happen late.

- Exit the position by doing a PV calculation of the position mentioned above. That is discount the cash flows with interest rate and expected tranche loss, or put in another way, multiply the spread difference by with "tranche risky annuity". For the equity tranche the MTM is simply the change in upfront payment, as the running spread is fixed at 500bps and the long-short position will cancel each other.

In order to do a PV calculation of our 5 year position in one year, we need to know the fair spread of a 4 year contract in one year time. This is of course not known in advance, but we can build a probability distribution of future tranche spreads by simulating the factors involved in pricing. If we consider a fixed interest rate, the factors that are involved in the valuation model of Chapter 2, is either single

[2]HTM held-to-maturity securities are those securities that an investor intends to hold and is able to hold until maturity. Unrealized gains or losses are not shown on the balance sheet, reflected in report or reflected in reported net worth.

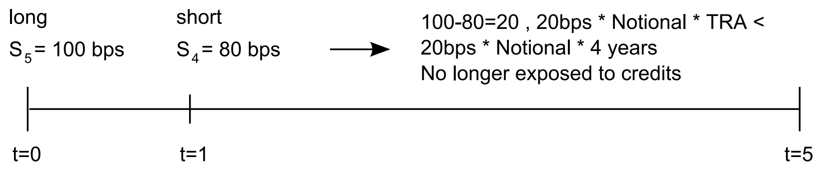

Figure 3.5: The unwind will be more profitable if the offset position above experience defaults, or a tranche wipe-out at an early time.

names spreads or portfolio spread, and base correlations. One way of achieving this, is by fitting a suited time series model for the spread, and do a 1 year forecast for either the uniform spread or on all names, along with base correlation curves.

Since we are working with the iTraxx we can choose an alternative route. The composition between rolls are very similar, and since we have a long history of index spreads along with base correlations, we can do a non-parametric bootstrap of the sets of index level and base correlations. We are then restricting the possible outcomes to the ones already observed in history, but if there should be any relations between the index level and base correlation curve, it will be preserved in this method. To get the spread level of a 4 year contract, we simply interpolate linearly between the quoted 3 and 5 year levels. As for the base correlation curve, we will use the 5 year levels as a proxy for the 4 year levels. As stated in Garcia & Goossens (2007) there is no widely accepted standard approach for interpolating a base correlation surface for a nonstandard attachment point or for a nonstandard maturity. Further they found no uniform behavior across maturity for a given attachment point of the base correlation surface in the gaussian copula model. If one had sufficient quotes for 4 year deal, a slightly better approach would be to back out correlations and use these when sampling. In any case, ignoring the change in base correlations will not yield a realistic picture of the spread variation of the different tranches, the only free parameter to change would be the uniform spread. Base correlation is also a measure for the relative supply and demand across the capital structure, so shifts over time may be caused by supply/demand factors than a fundamental change in the market‘s view of default correlation.

There is also another factor that investors need to be aware of. As time elapses, the value of their investment will change even if spreads and correlation remain the same. An equity investor will always be exposed to immediate risk of default, but a senior investor will have a safer position through time as there is less time to experience the amount of loss needed to hit his or hers tranche. So in relation to the other tranches, the equity tranche has become more risky than mezzanine and senior tranches. Time decay will therefore have a different impact on the different tranches. These variations will also be portfolio specific, as the expected loss legs depends on the timing of defaults over time, which again depends on the term structure of credit spreads. Figure 3.6 illustrates the point. If time is the only changing parameter, increasing time will shift expected losses to the right, meaning more mass located in more senior tranches. Shorter term gives less expected loss in senior tranches, and therefor reduces price of risk.

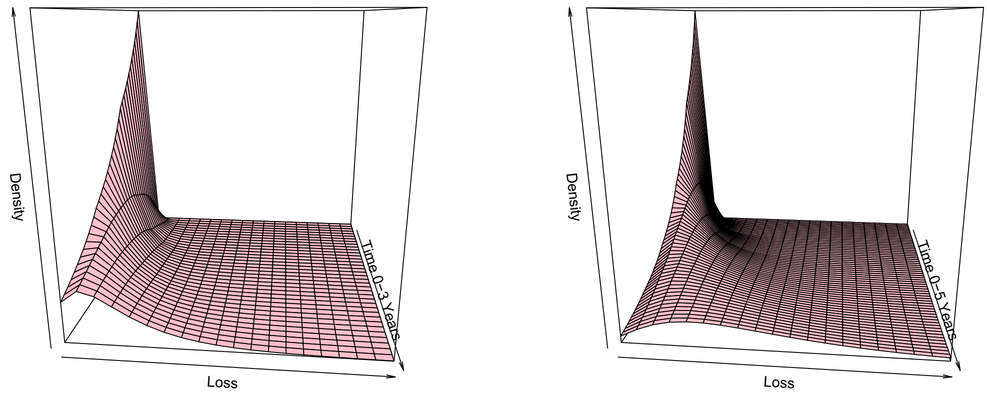

Figure 3.6: Loss surface, or loss density as a function of time. 0-3 years on left, and 0-5 years on right.

3.4 Calculating the P&L distribution

As we now have covered the most important drivers of risk in a CDO investment, we can turn our attention towards calculating the profit and loss distributions for our CDO investment. The last, and positive contributing part to the tranche return is what practitioners call the *carry*. The *carry* is simply the coupons earned through the life of our investment, which may have been reduced due to defaults. In theory all coupons should be calculated with earned interest, but we decide to omit them for all tranches but the equity. In practice the accumulated interest earned would be very small when considering only a short time horizon of one year and the the spread size. The choice of including interest on the equity tranche is based on the substantial size of the upfront payment, i.e 15% of notional. After one year this amount will have increased by the one year risk free rate if no defaults occur. Two scenarios are included in Figure 3.7 and 3.8 that illustrate in more detail the P&L impact for an equity tranche, and a mezzanine/senior tranche.

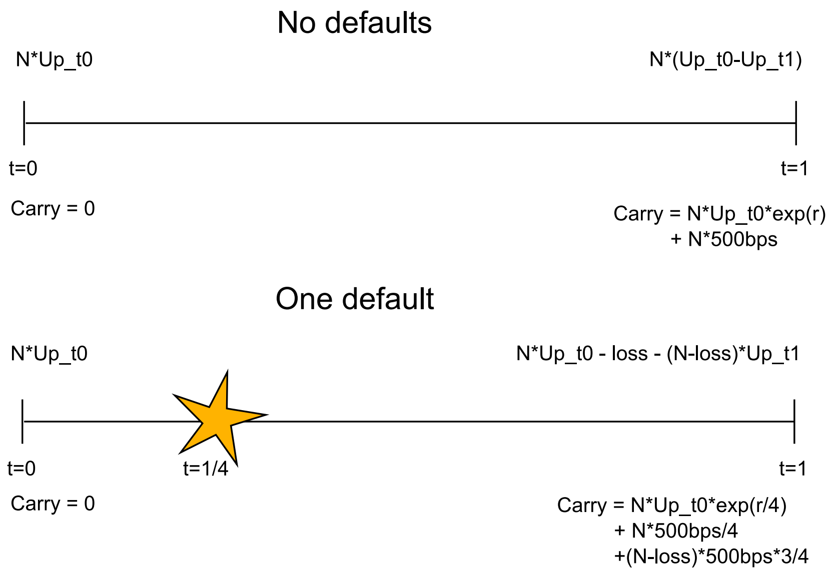

Figure 3.7: The Equity tranche. The 500 bps are distributed over the year quarterly. Keep in mind that the upfront premium calculated in the case of one default will be based on a thinner tranche (N-loss).

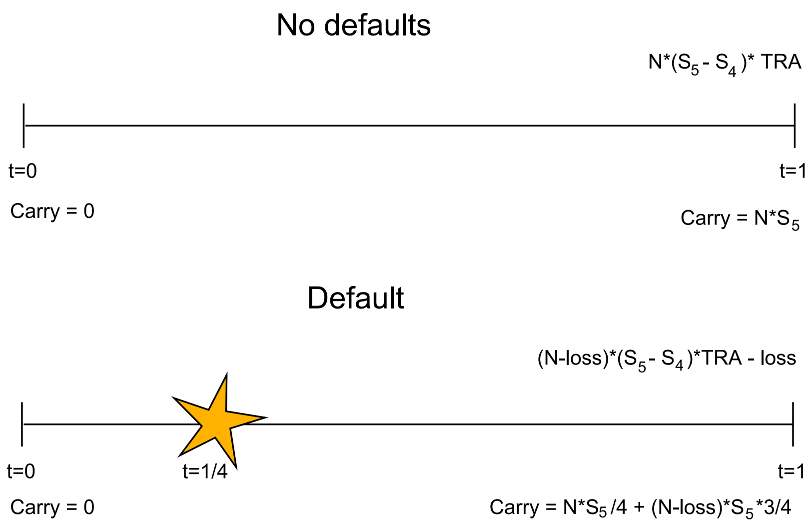

Figure 3.8: The Mezzanine/senior tranche(s). No upfront premium, but a running spread payed quarterly. The spread and tranche risky annuity will be calculated based on a thinner tranche if defaults occured.

Chapter 4

Numerical example

4.1 Data

The data used in this thesis was obtained from *Bloomberg* and *MorganMarkets*[1]. The data consist of company ratings and time series of single name CDSs for all names on the iTraxx Series 7. Table 4.1 shows ratings in *Moody's* scale along with business sector, where sectors was found in the presentational material on *www.markit.com*. Along with these, we also had time series of tranche and index quotes from March 2005 to April 2008, see Figure 4.3. Historical default rates and transition matrix where found in *Moody's* corporate default and recovery rates report of 2008. The report contains empirical estimates for the most recent year, and also estimates that are averaged over a larger time horizon. In this thesis we use averaged values from 1970 to 2007 based on global data. The methodology and details regarding how these are calculated can be found in Moody's (2008). Before presenting any results, we address some details concerning the data and what modifications that were done in order to apply the theory introduced.

Ratings The ratings-dataset we had access to contained ratings with modifiers. These modifiors are discarded in this study. This removes some of the granularity of the portfolio, but on the positive side, the empirical estimates of broad letter default rates and rating migrations are based on more observations. This should yield less uncertain estimates. Also, one can observe in Moody's (2008) page 49, that the estimated migration matrix for ratings with modifiers exhibit non-monotonicity with respect to rating. One would for instance not expect an Aa2 rated firm to have higher default rates than an Aa3 rated firm. While this contradicts the nature of ratings, it is not unthinkable to get such empirical estimates when dealing with few observations.

[1] Kindly provided to us by Mr. Sigurd Ugland of KLP and Mr. Halvor Hoddevik of Arctic Securities

Company	Rating	Broad rating	Sector
01) ABN Ambro Bank NV	Aa2	Aa	Financials
02) Accor SA	Baa	Baa	Consumer
03) Adecco SA	Baa	Baa	Industrials
04) Aegon NV	A1	A	Financials
05) Electrolux AB	Baa2	Baa	Consumer
06) Volvo AB	A3	A	Autos
07) Akzo Nobel NV	A3 *-	A	Industrials
08) Alliance Boots PLC	B2 *-	B	Consumer
09) Allianz SE	Aa3	Aa	Financials
10) Altadis SA	Baa1 *-	Baa	Consumer
11) Arcelor Finance SCA	Baa3	Baa	Industrials
12) Assicurazioni Genera...	A1	A	Financials
13) Aviva PLC	A1	A	Financials
14) AXA SA	A2	A	Financials
15) Banca Monte dei Pasc...	Aa3	Aa	Financials
16) Banca Popolare Itali...	A3	A	Financials
17) Banco Bilbao Vizcaya...	Aa1	Aa	Financials
18) Banco Comercial Port...	Aa3	Aa	Financials
19) Banco Espirito Santo...	Aa3	Aa	Financials
20) Banco Santander Cent...	Aa1	Aa	Financials
21) Barclays Bank PLC	Aa2	Aa	Financials
22) Bayer AG	A3	A	Industrials
23) Bayerische Motoren W...	A1	A	Autos
24) Bertelsmann AG	Baa1	Baa	TMT
25) BNP Paribas	Aa1	Aa	Financials
26) British American Tob...	Baa1	Baa	Consumer
27) British Telecommunic...	Baa1	Baa	TMT
28) Cadbury Schweppes PL...	Baa2 *	Baa	Consumer
29) Capitalia SpA	A1 *+	Aa	Financials
30) Carrefour SA	A2	A	Consumer
31) Casino Guichard Perr...	Baa3	Baa	Consumer
32) Centrica PLC	A3	A	Energy
33) Ciba Specialty Chemi...	Baa2	Baa	Industrials
34) Commerzbank AG	Aa3	Aa	Financials
35) Cie de Saint-Gobain	Baa1	Baa	Industrials
36) Compagnie Financiere...	Baa	Baa	Industrials
37) Compass Group PLC	Baa2	Baa	Consumer
38) Continental AG	Baa1 *-	Baa	Autos
39) DaimlerChrysler AG	Baa1	Baa	Autos
40) Deutsche Bank AG	Aa1	Aa	Financials
41) Deutsche Lufthansa A...	Baa3	Baa	Consumer
42) Deutsche Telekom AG	A3	A	TMT
43) Diageo PLC	A3	A	Consumer
44) DSG International PL...	Baa2	Baa	Consumer
45) E.ON AG	A2	A	Energy
46) Edison SpA	Baa2	Baa	Energy
47) Energias de Portugal...	A2 *-	A	Energy
48) Electricite de Franc...	Aa1	Aa	Energy
49) EnBW Energie Baden-W...	A2	A	Energy
50) Endesa SA	A3 *-	A	Energy
51) Enel SpA	A1 *-	A	Energy
52) European Aeronautic ...	A1	A	Industrials
53) Experian Finance PLC	Baa1	Baa	Consumer
54) Fortum Oyj	A2	A	Energy
55) France Telecom SA	A3	A	TMT
56) Gallaher Group PLC	A2 *+	A	Consumer
57) Gas Natural SDG SA	A1	A	Energy
58) Gaz de France SA	Aa1 *-	Aa	Energy
59) GKN Holdings PLC	Baa3	Baa	Autos
60) Glencore Internation...	Baa3	Baa	Industrials
61) Groupe Auchan SA	A2	A	Consumer
62) Hannover Rueckversic...	Aa3	Aa	Financials
63) Hanson PLC	Baa3	Baa	Industrials
64) Hellenic Telecommuni...	Baa1	Baa	TMT
65) Henkel KGaA	A2	A	Consumer

66) Iberdrola SA	A2 *-	A	Energy
67) Imperial Chemical In...	Baa2 *+	Baa	Industrials
68) Imperial Tobacco Gro...	Baa2 *-	Baa	Consumer
69) Intesa Sanpaolo SpA	Aa2	Aa	Financials
70) Kingfisher PLC	Baa3	Baa	Consumer
71) Koninklijke DSM NV	A3	A	Consumer
72) Royal KPN NV	Baa2	Baa	TMT
73) Koninklijke Philips ...	A3	A	Consumer
74) Lafarge SA	Baa2	Baa	Industrials
75) Linde AG	Baa1	Baa	Industrials
76) LVMH Moet Hennessy L...	A3	A	Consumer
77) Marks & Spencer PLC	Baa2	Baa	Consumer
78) Metro AG	Baa2	Baa	Consumer
79) Muenchener Rueckvers...	Aa3	Aa	Financials
80) National Grid PLC	Baa1	A	Energy
81) Pearson PLC	Baa1	Baa	TMT
82) Peugeot SA	Baa1	Baa	Autos
83) PPR	Baa3	A	Consumer
84) Publicis Groupe	Baa1	Baa	TMT
85) Reed Elsevier PLC	A3	A	TMT
86) Renault SA	Baa1	Baa	Autos
87) Repsol YPF SA	Baa2 *-	Baa	Energy
88) Reuters Group PLC	Baa1	A	TMT
89) Royal & Sun Alliance...	A3	A	Financials
90) RWE AG	A1	A	Energy
91) Safeway Ltd	Baa2	Baa	Consumer
92) Sanofi-Aventis	A1	A	Industrials
93) Siemens AG	Aa3	Aa	Industrials
94) Sodexho Alliance SA	Baa1	Baa	Consumer
95) Solvay SA	A2	A	Industrials
96) STMicroelectronics N...	A3	A	TMT
97) Stora Enso Oyj	Baa3	Baa	Industrials
98) Suez SA	A3 *+	A	Energy
99) Svenska Cellulosa AB	Baa1	Aa	Consumer
100) Swiss Reinsurance	Aa2	Aa	Financials
101) Tate & Lyle PLC	Baa2	Baa	Consumer
102) Telecom Italia SpA	Baa2	Baa	TMT
103) Telefonica SA	Baa1	Baa	TMT
104) Telekom Austria AG	A3	A	TMT
105) Telenor ASA	A2	A	TMT
106) TeliaSonera AB	A2	A	TMT
107) Tesco PLC	A1	A	Consumer
108) Royal Bank of Scotla...	Aaa	Aaa	Financials
109) Thomson	Baa3	Baa	Consumer
110) ThyssenKrupp AG	Baa2	Baa	Industrials
111) UniCredito Italiano ...	Aa2 *-	Aa	Financials
112) Unilever NV	A1	A	Consumer
113) Union Fenosa SA	Baa1	Baa	Energy
114) United Utilities PLC	A3 *-	A	Energy
115) UPM-Kymmene Oyj	Baa2	Baa	Industrials
116) Valeo SA	Baa2	Baa	Autos
117) Vattenfall AB	A2	A	Energy
118) Veolia Environnement	A3	A	Energy
119) Vinci SA	Baa1	Baa	Industrials
120) Vivendi	Baa2	Baa	TMT
121) Vodafone Group PLC	Baa1	A	TMT
122) Volkswagen AG	A3	A	Autos
123) Wolters Kluwer NV	Baa1	Baa	TMT
124) WPP 2005 Ltd	Baa2	Baa	TMT
125) Zurich Insurance Co	A3	Aa	Financials

Table 4.1: Names on the iTraxx Series 7 along with ratings. "*+"' means positive outlook "*-" means negative outlook. Third column is ratings after removal of modyfiers as explained in Section 4.1, and final column is sector.

In doing this simplification, it is natural to check if we have made any major changes to the portfolio characteristics. An important measure is portfolio expected loss. After the adjustment, the expected loss was about 2% higher. In order to bring them a little closer to each other a few changes was made to the data. If a company had a 1-modifyer and positive outlook[2], it was upgraded. Also if the company had a 1-modifyer and FITCH and S&P had a higher rating, the company was upgraded. Figures 4.1 and 4.2 show how the ratings are distributed before and after removal of the modifiers. Some of the increased risk in the broad letter portfolio could also be caused by the relatively fewer number Baa3 companies compared to Baa1 and Baa2.

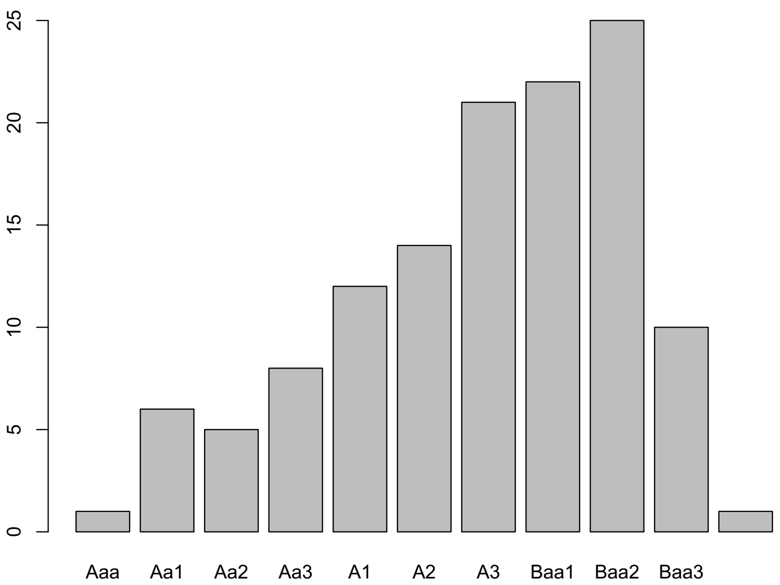

Figure 4.1: Distribution of ratings with modifiers for iTraxx Europe Main series 7, as of June 2007.

Market spreads Since the value of a position is always subject to market terms, the timing of entering and exiting a position will always play a crucial role on the return on the investment. Since we are bootstrapping previous market spreads, also called historical simulation (see Hull (2006), page 438), a glance at history might provide an intuition on the direction of the results. Figure 4.3 shows historical levels for the period for which we had a complete set available. As can be seen, the set includes the ongoing turmoil experienced since the summer of 2007. The spread levels in this period are much higher than the previous years. Sampling from this period will then likely result in large negative MtM's if ones position was entered before this period. In Chapter 2, we quoted spreads from June 27, 2007, which we

[2]A positive outlook means that a company is on review for an upgrade, but it is not surely going to be upgraded.

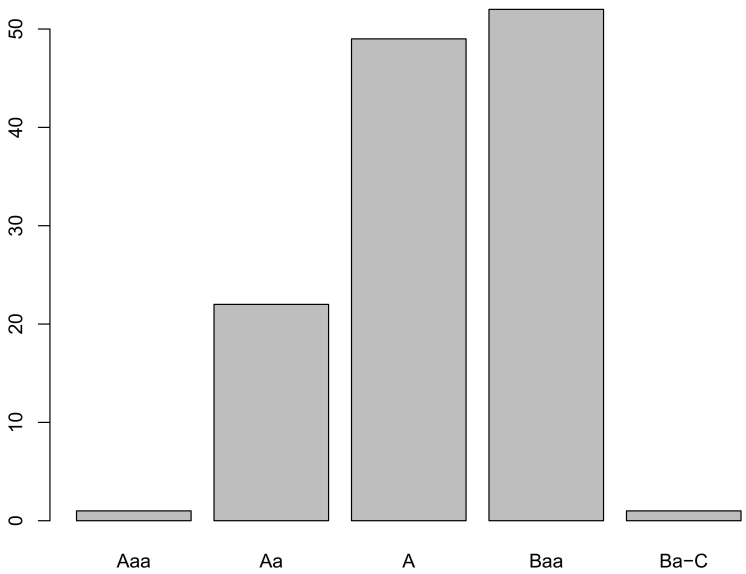

Figure 4.2: Ratings without modifiers.

will continue to use in the example of this chapter. Taking a look at the historical levels, spring/early summer 2007 was a time of narrow spreads even if you exclude recent times, hence without knowing the impact of time decay, the results should yield negative MTM's in most cases. The reader should not make the mistake of concluding that this means that this asset class performs poorly in general, but view this as a result of entering into a deal at a time when the market was very calm. A deal analysis based on tranches from June 27, 2007, should in reality just be based on market data up to this date, but we have included this last period of data to see to what extent the carry, or coupons earned, outbalance the negative MtM's.

Rating transition dependency We have not found any public research on joint rating migrations, where dependencies have been estimated using corporate bond history. In our example we are using a gaussian copula with correlation matrix estimated from arithmetic returns on single name CDS spreads. This might seem as an arbitrary choice, but many commercial credit portfolio models employ such assumptions. In some structural, or firm value models, equity correlation is used as a proxy for firm value correlation directly, or extracted after mapping equity returns to asset/firm value returns. CreditMetrics uses this approach (see JPMorgan (1997)). CDS spreads, although saturated by risk premiums and sometimes lack of liquidity, are nonetheless an expression of a company's relative credit risk. The effect of estimating correlations with CDS data versus equity data have not been explored in this thesis, and to our knowledge, not in any other paper. Further, since we are simulating over periods of quarterly length, quarterly data have been used in estimation of the covariance matrix.

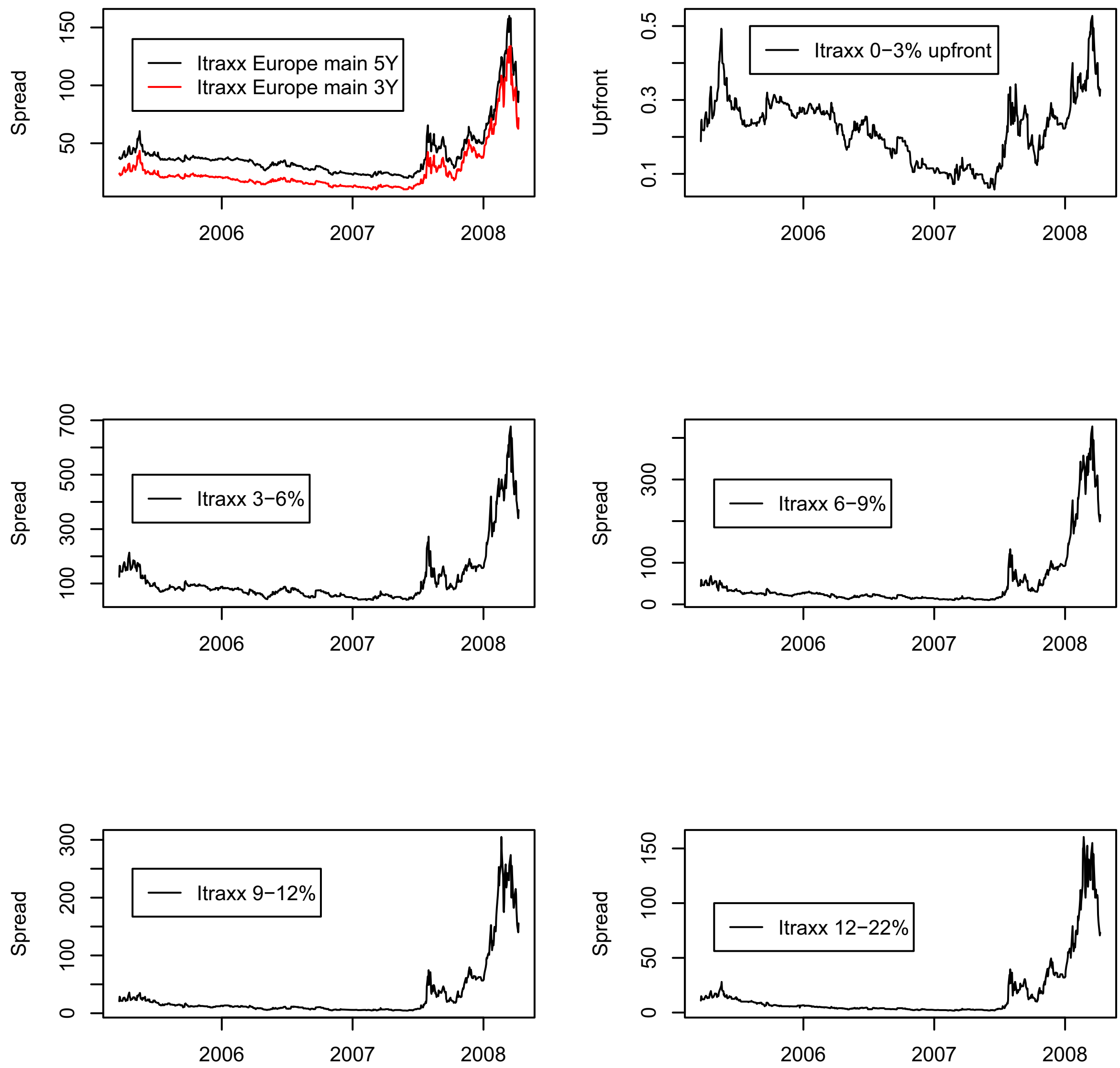

Figure 4.3: Historical levels for Itraxx Europe main index, and tranches. March 2005 - April 2008

Recovery rates For the recovery rates, we use beta distributions. We add some more heterogeneity to the portfolio by separating companies by industry. We estimate the parameters by the method of moments using historical means and standard deviations as found in Renault & Scaillet (2003). All sector names except *Industrials* corresponded to the iTraxx sector names in this paper. For *Industrials*, we averaged the values of *Chemicals*, *High tech* and *Building*. Table 4.2 shows the values and parameters that we have used.

Sector	Mean	Std.dev	$\hat{\alpha}$	$\hat{\beta}$
TMT	0.2473	0.0753	7.871282	23.95760
Energy	0.4556	0.2561	1.267325	1.514337
Financials	0.2970	0.2463	0.725208	1.716570
Autos	0.4298	0.2136	1.878847	2.492598
Industrials	0.4349	0.2433	1.370498	1.781158
Consumer	0.3680	0.2121	1.534529	2.635387

Table 4.2: Mean and standard deviation of recovery rates by sector, and corresponding α and β parameters.

4.2 Losses

In our example, we will be using the euro denoted zero curve[3] from June 27, 2007, and unit exposure to all names such that tranche sizes are as given in Table 4.3. There is no requirement on the size of the exposure as far as we know. For the sake of comparison we have made them a function of the pool notional. So that the tranche notional equals the (pool notional $\times$ tranche width).

	Pool	0-3%	3-6%	6-9%	9-12%	12-22%
Notional	125	3.75	3.75	3.75	3.75	12.5

Table 4.3: Itraxx tranche notionals with unit exposure to every name.

The experiment was carried out using 10000 Monte Carlo runs. The computations were done using `R` on a laptop with 3GB of RAM. Due to the high dimensionality of problem, memory allocation was an issue, and therefore it was not feasible to carry out many more runs. We first compute the defaults by simulating rating transitions in quarterly steps. When a company defaults, a random number from the beta distribution corresponding to that company's sector is drawn, and losses are calculated. We then compute the carry for all the tranches as a function of losses experienced in the pool. Figure 4.4 shows histograms of the number of defaults for the Itraxx series 7 portfolio after one and five years respectively.

[3] The interest rate yield curve derived from Euribor and euro denoted futures and swaps, kindly provided by Mr. Bjørn Bakken.

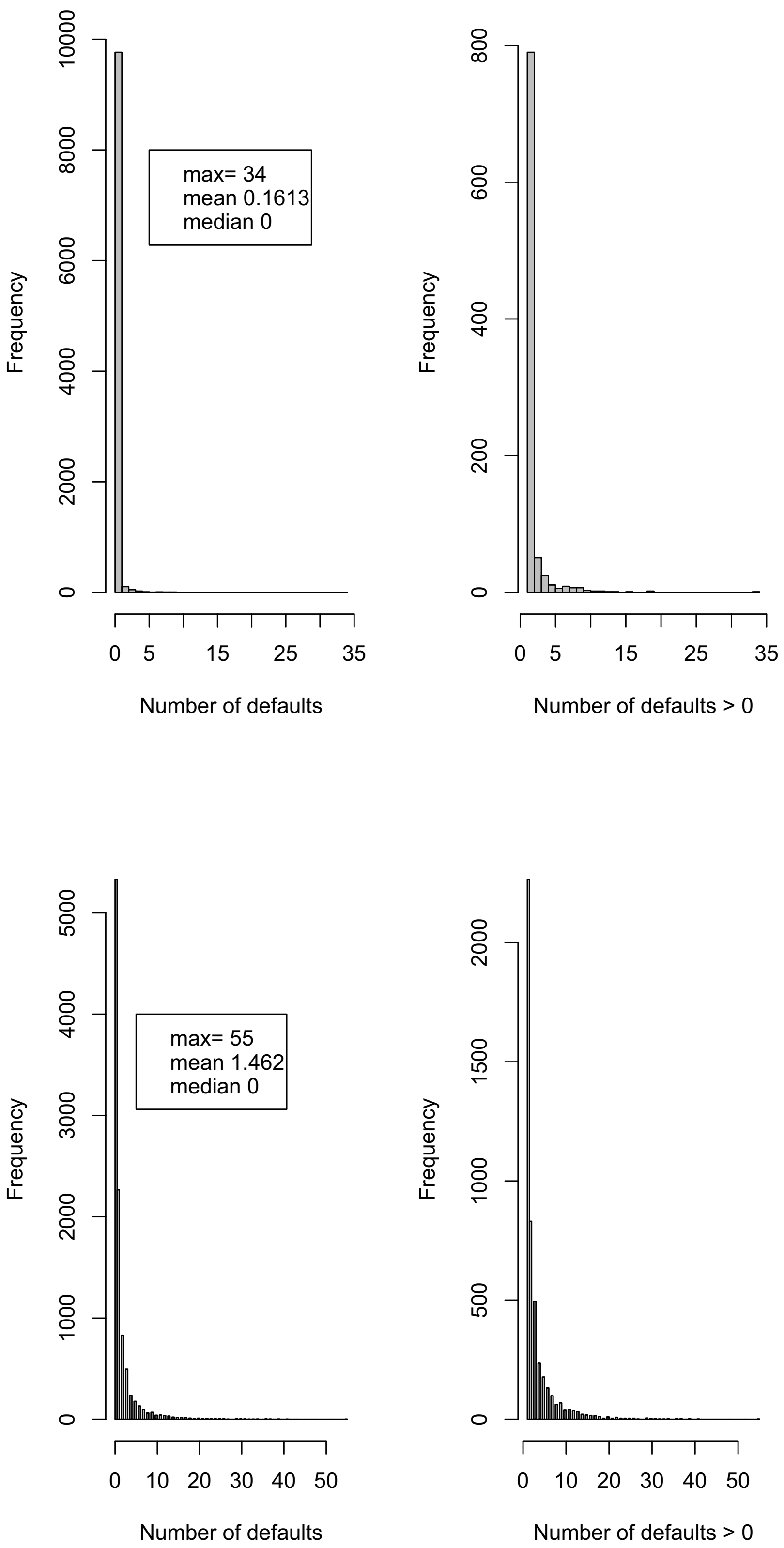

Figure 4.4: Histogram of number of defaults after 1 year (top) and 5 years (bottom).

As expected when dealing with low credit risk portfolios, defaults rarely happen. The tails of the loss distribution are in the gaussian copula attributed to the level of correlation only, as discussed in Chapter 2. Any statistic based on large quantiles is of course very uncertain due to the large amounts of simulations needed to get a decent estimate. It is however worth noting how rare these events are, and therefore how well protected the very senior tranches of the capital structure are. That is, if one believes history to be a good prediction of the future.

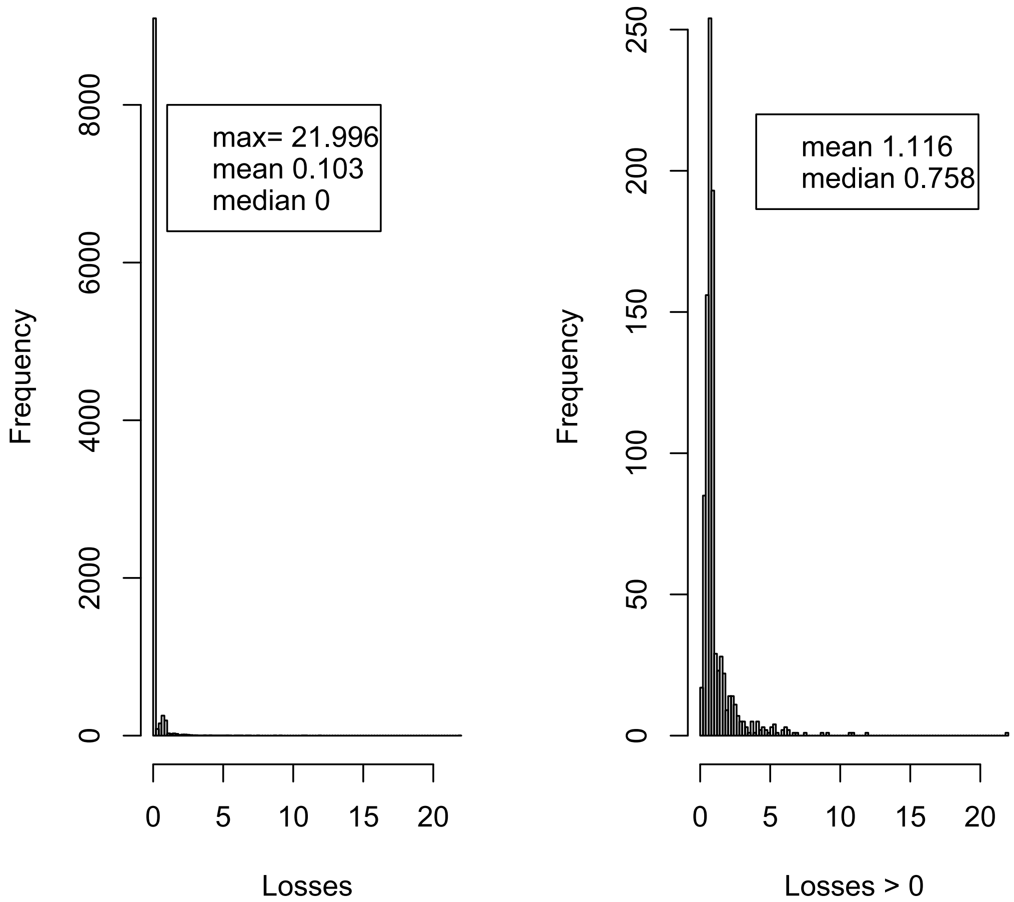

Figure 4.5: Losses after 1 year. Pool notional is 125.

Figures 4.5 and 4.6 shows histograms of losses experienced by those defaults. Because of the high probability of zero defaults, large losses are not visible in the histogram. Based on the simulations, we can also address some portfolio risk measures such as Value-at-Risk (VaR) and Expected Shortfall (ES). VaR is a quantile in our loss distribution, and expresses what the losses are at some probability level. ES on the other hand, is a conditional expectation, which tells us what losses to expect given that losses exceed a given quantile.

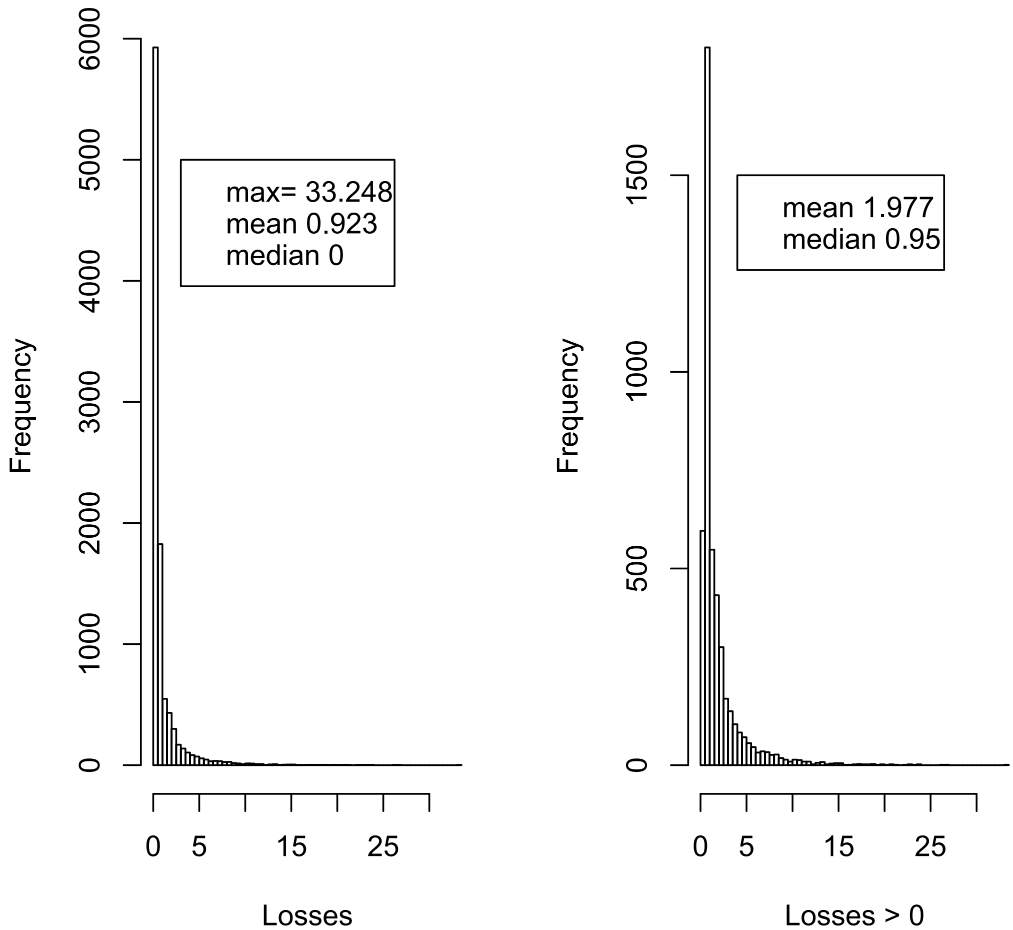

Figure 4.6: Losses after 5 years. Pool notional is 125.

We define VaR and ES as

$$VaR_\alpha\left(L_t\right) = \inf\{x \geq 0 \mid \mathbb{P}\left(L_t \leq x\right) \geq \alpha\} \tag{4.1}$$

and

$$ES_\alpha(L_t) = \mathbb{E}[L_t \mid L_t \geq VaR_\alpha\left(L_t\right)], \tag{4.2}$$

where L_t denotes the loss variable.

When dealing with simulations, it is an easy task to compute these quantities. We sort the loss vector and pick out the 95% quantile. The ES is then simply the mean of the outcomes exceeding this value, in our case 500 values. Figure 4.7 reports these measures of risk, and also shows some limitations of VaR. In general one should be aware of summing up risk in single quantities like VaR. In our toy example, calculating VaR at the 95% level gave a value of 4.3 units of notional. This measure of risk tells us that with probability 95% losses will not exeed 4.3 within the 5 year term. This value recides in the junior tranche with 3.75 units of notional subordinated by the equity tranche. However, if the losses were greater than 4.3 units of notional, we actually expect a loss of approximately 8 units more than VaR! That is, because of the fatness of the tails, when breaching the 95 % level we actually expect to get a loss amount that wipes out the three first tranches and takes a bite of the fourth tranche. Looking at VaR alone when considering worst case scenarios, might give the illusion of a much more optimistic picture than might be experienced.

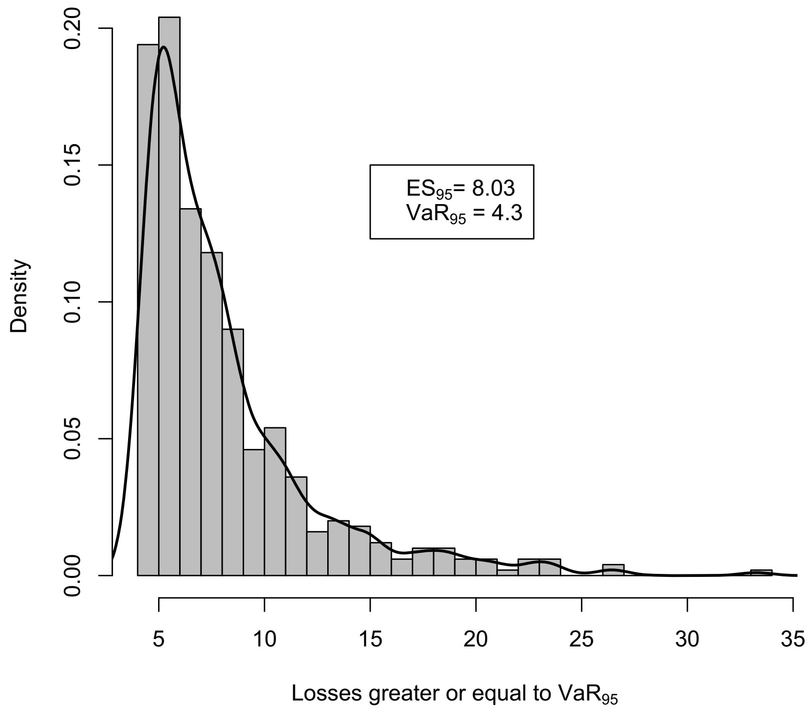

Figure 4.7: Histogram and gaussian kernel density estimator of losses greater than or equal to Value-at-Risk 95 % level for the 5 year term.

In Table 4.4 we show the hitting probabilities of the various tranches, that is the probabilities

$$PD_{tranche} = \mathbb{P}(L(t) \geq \alpha), \tag{4.3}$$

where $L(t)$ is the portfolio loss variable at time t, and α is the tranche attachment point. We estimate the quantity by computing the relative frequency of losses that are large enough to reduce the notional of the tranche in question. We denote by $L^{(i)}(t)$ the loss in the i-th scenario and let $I(\cdot)$ be the indicator function. An estimate is then given by:

$$\widehat{PD}_{tranche} = \frac{1}{n} \sum_{i=1}^{n} I(L^{(i)}(t) \geq \alpha). \tag{4.4}$$

The hitting probabilities express the probability of being affected by losses, or loosely speaking the probability of default for a tranche. The effect of subordination is large for tranches senior to equity with very low hitting probabilities. On the other hand, the experiment shows how risky the equity tranche is, with nearly a 50% chance of beeing hit with a loss. No losses has occurred in the iTraxx Europe pool to this day, but the last years have been good times. It will be interesting to see if any of the iTraxx constituents will default in the years to come, especially in view of the ongoing credit crisis.

Tranche	0-3%	3-6%	6-9%	9-12%	12-22%
Hitting probabilities	46.7%	6.1%	2%	0.77%	0.35%

Table 4.4: Hitting probabilities of tranches within 5 years.

4.3 Spreads and P&L

The previous section focused on a pure notional loss analysis for the whole term of a 5 year contract. We will now focus on a one year perspective and therefore also consider risks attached to market value.

As mentioned in Section 3.3, we need the spread of a 4 year contract in order to calculate MtM values after one year. Since we have no history of tranches with maturity 4 years, the 4 year spreads we calculate are model-implied. Using the interest rate curve of June 27, 2007, we first calibrate the pricing model to the available history. This is done by first calculating the termstructure of default probabilities (market implied) using index spread level for the maturities of 3 and 5 years, and secondly by extracting the base correlation curve corresponding to the tranche quotes. Unfortunately the calibration of the 12-22% senior tranche proved to be very difficult. In order to get convergence in the root searching algorithm used, a feasible interval of base correlations had to be supplied. For the 12-22% tranche, this interval turned out to be very narrow, making the calculations a very time consuming manual task. We do not know if this is an issue with the model itself, or due to the implementation, but for this reason we decided to omit this tranche in further analysis. Using the simulated losses of Section 4.2 we calculate a possible future spread as follows:

- For every simulated loss scenario, sample one historic set of uniform default probabilities and base correlations.
  - If there are no losses in the scenario, calculate tranche spreads using sampled set, but with a 4 year term.
  - If losses has occurred in scenario, calculate new attachment and detachment points and find corresponding base correlations by interpolation. Remove defaulted name(s) from pool, and calculate new tranche spreads for a 4 year term.

For the interpolation and extrapolation on the base correlation curve, we fitted a cubic spline with the help of the function `splinefunc` in the software package `R`. While interpolation is straight forward, extrapolation poses some challenges. Since we have no quotes for tranches with detachment points less than 3%, it is hard to say what a correct value should be. In lack of better options, we set the base correlation at detachment point 0% to be a 0.01. The extrapolation performed by the spline function is then linear from the 3% point down to 0%.

In Figure 4.8 we present histograms of 4 years spreads obtained by historical simulation. These are based on market data from March 2005 and up to June 27, 2007. These histograms do not include spreads based on default scenarios, but are provided along with historic market spreads for the 5 year contract to show the effect of a shorter term. As should be the case, spread levels are shifted downwards. What should be noted, is that although time decay seemingly removes some of the negative MtM risk using this history, there is still a very large variability in spread levels that ultimately could cause a negative position. One should also keep in mind that the considered time period is in itself a calm period. If faced with spread levels seen this last year (following summer 2007), time decay effects would not help much.

Figure 4.9 have several features that deserve commenting. Here we have included scenarios where losses has occurred. Black colored points are spreads without losses, green colored marks are spreads where one or more losses in the pool has occurred, but still below respective tranche attachment point. Red marks are tranche spreads, when losses have affected the tranche directly by reduction of notional or a total wipe-out(red marks at zero). When looking at these plots, one should keep in mind the figures of the previous section. The median pool notional loss over a one year time horizon was roughly 0.8. In the majority of outcomes, one will therefore not expect a dramatic change in spreads for tranches senior to equity. Subordination is still substantial. We stress the fact that this is a result of the models employed in this thesis. A sudden event might cause a major reaction in overall spread levels on a shorter term. This was experienced with the unexpected downgrading of Ford and GM's debt in May 2005, when spreads widened. However, the spreads quickly fell back to their normal levels, and if we ignore supply/demand factors for a moment, spreads will generally reflect market *expectation* of credit risk. Our model simply does not account for *unexpected* events.

Analysing spreads as in Figure 4.8 and 4.9, highlights a source of risk that easily can be overlooked. Speaking of spread risk without taking into account spread changes as a result of defaults, might understate the spread risk. Volatility or standard deviation is arguably not a good measure of risk, especially when dealing with highly skewed distributions, but we give a comparison to highlight the issue. Table 4.3 shows the standard deviation of the model implied 4 year spread with, and without default scenarios. The effect is substantial.

	0-3%	3-6%	6-9%	9-12%
Without defaults	6.65 %	20.3 bps	6.3 bps	4.4 bps
With defaults	7.75 %	137.4 bps	54.3 bps	11.6 bps

Table 4.5: Standard deviation of 4 year spreads without defaults (top), and with defaults (bottom).

Based on the simulations we can calculate a *Profit & Loss* distribution. In Table 2.4 we quoted some iTraxx tranches from June 27, 2007. Comparing these with the distribution of 4 year spreads should give a rough guide to the sign of MtM values. If the 4 year spread is higher than the spread quoted, then you get a negative MtM value. For all tranches but equity, it looks as if most outcomes will end with a positive sign. For equity it is more uncertain, and also the time effect seems to be of lesser importance, which is intuitive as a first loss piece is always exposed to immediate default risk no matter how benign or grim the credit environment is. In view of Section 3.4 we take into account both losses, carry and MtM values we can calculate a P&L distribution that is relative to notional exposure. Such a calculation is done and shown in Figure 4.10, where we have used the quoted values of June 27, 2007 as an example. This calculation is of course subject to the data at hand and spreads at initiation, which where low and therefore yields low expected returns. All distributions but the one for the equity tranche exhibit extremly fat tails. This is as expected since most default outcomes hit the equity tranche. Another observation can be made by looking at the 5% quantiles. The size of the quantile tells us that these losses are not based on pool defaults. Arguably the equity quantile could be attributed to a default if the recovery was very high.

An unescapable issue with these kinds of risk assessments, is what history to use. In the calculations done so far, we used history from March 2005 to June 27, 2007. We also had history for the following year, and including these spread levels, will dramatically change the downside risk. Looking at Figure 4.3 again, tells us that the levels in this period is many times higher than the history from March 2005 up to summer 2007. Since this accounts for roughly a quarter of the data we had available, the probability of getting large negative MtM will increase substantially. Arguably, downturn cycles of this magnitude will not appear at such a high rate, i.e one in four years, but it would be unrealistic to claim that the levels now witnessed

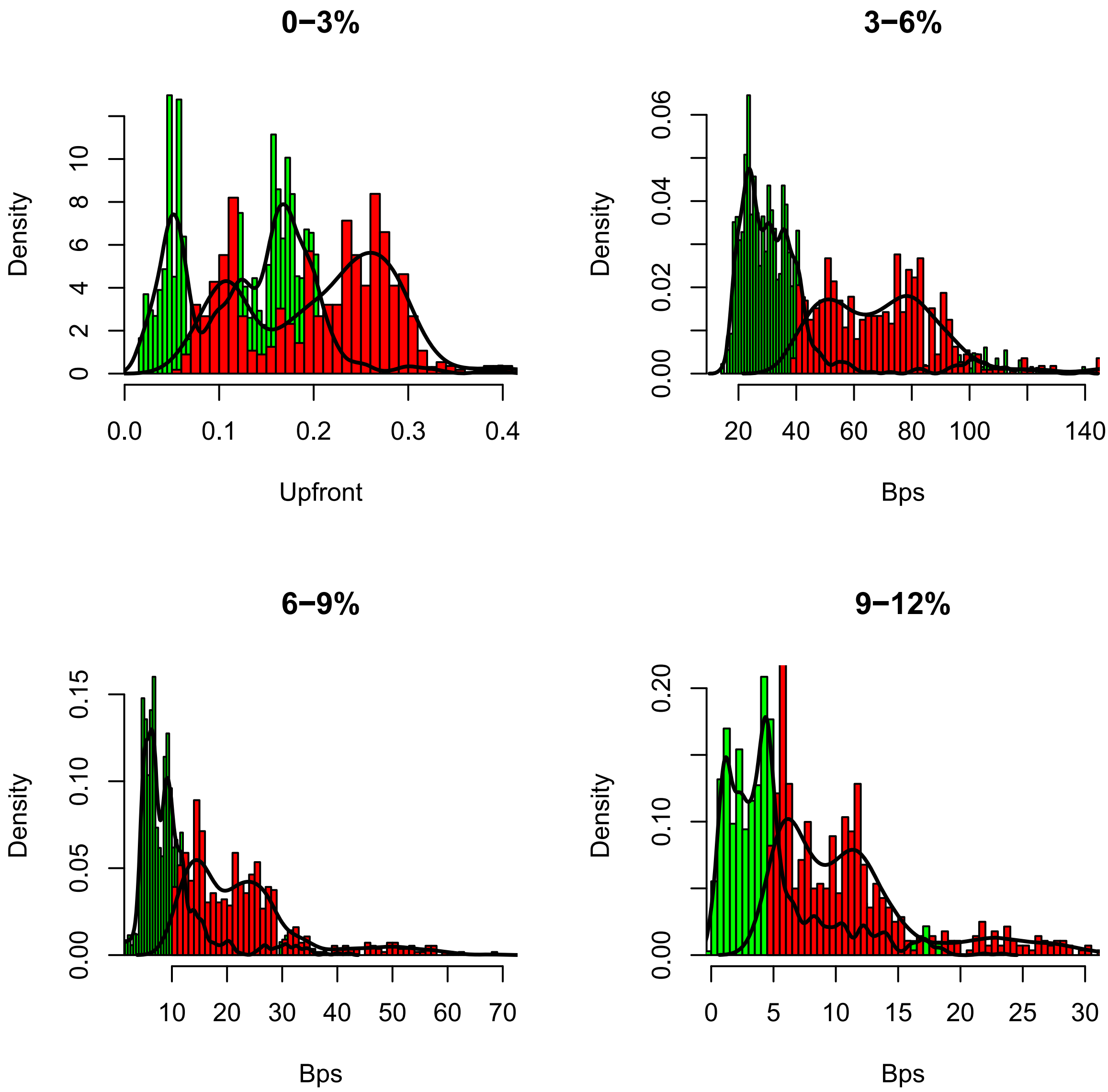

Figure 4.8: Histogram of 4 year model spreads (green), and 5 year market spreads (red).

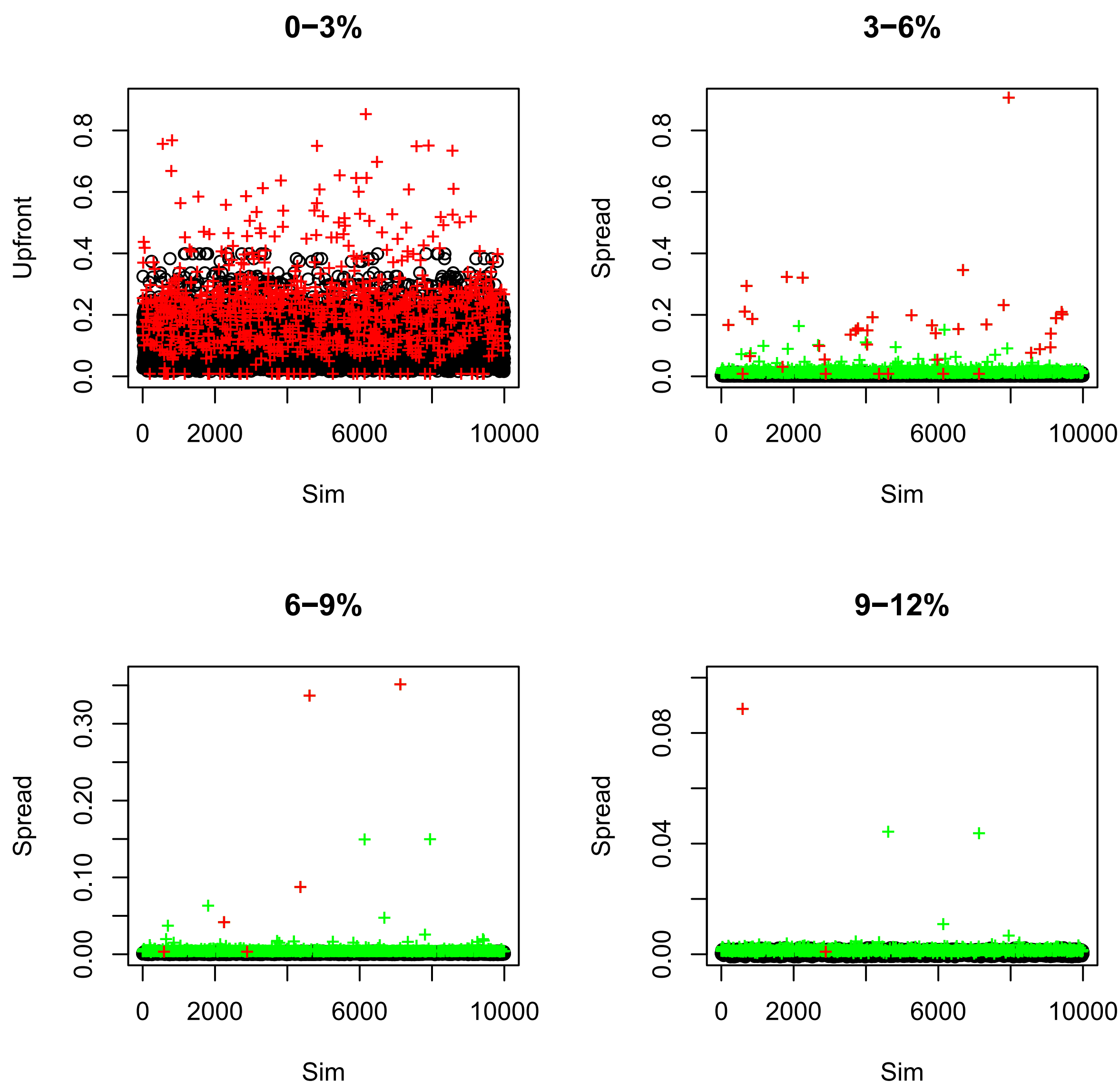

Figure 4.9: 4 year spread levels when default scenarios are included. Red + are results of tranche directly affected by defaults, green +, are tranches affected by loss of subordination (lower tranches are hit).

will not happen again. Although the index tranche market has only been around for about four years, credit spreads have been high in the past, in example late 80's/early 90's, and also at the beginning of this millenium. A possible solution to the problem of the dominating downturn cycle in our data set, could be weighting the sampling a bit. That is, sampling from this period with a smaller probability. We would still get the large outcomes of spreads, but at lower frequency. We have not investigated this effect, where one would also have to tackle the issue of how often such cycles occurr. The effect of simply including this period in the same experiment is shown in Figure 4.11. The results imply a negative expected return, which is caused by by the relative large amount of wide spreads, which in turn yields negative MtM values.

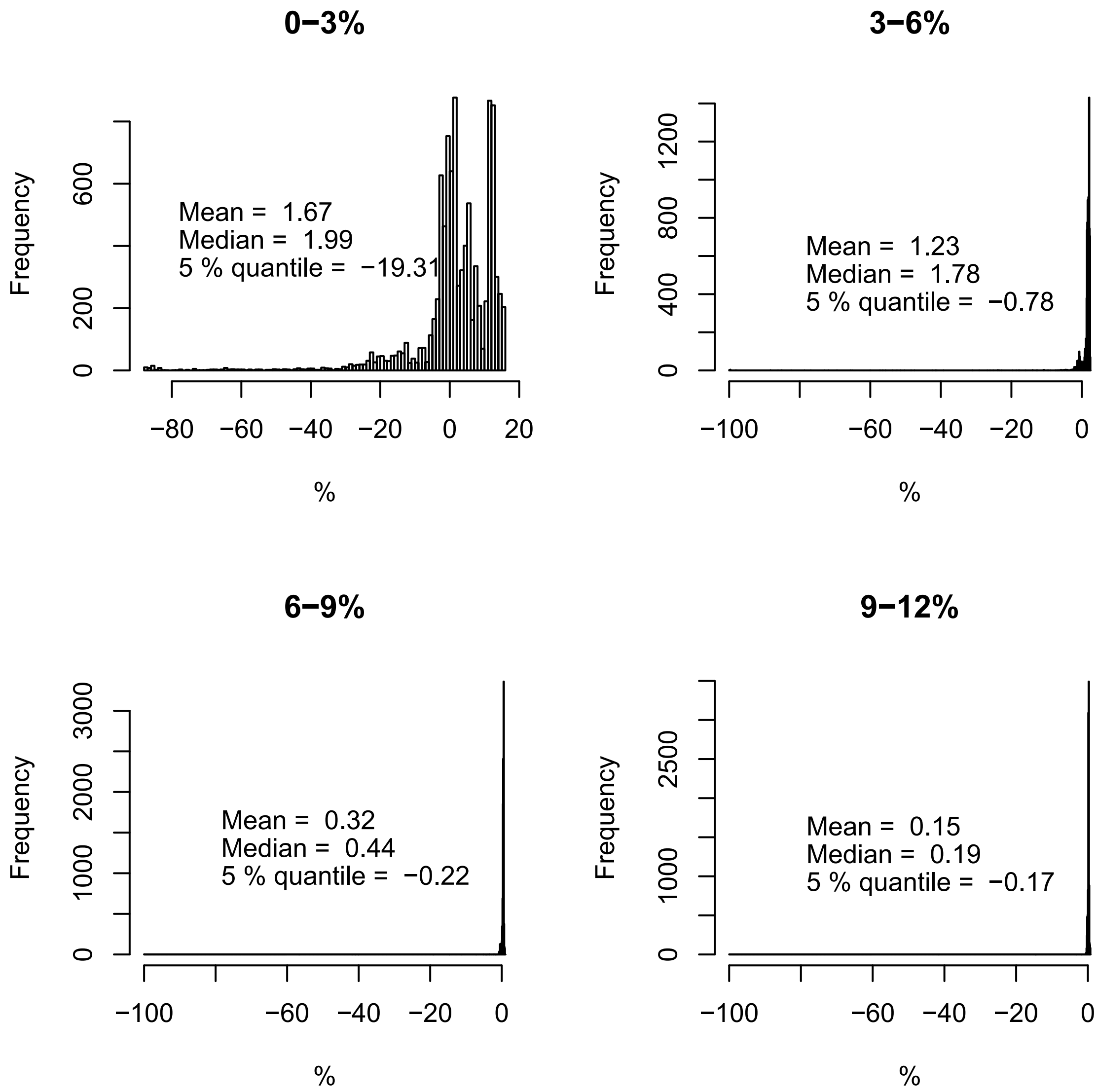

Figure 4.10: Profit & Loss distribution for the four tranches in %. Positive values is a profit, negative values are losses. Spreads are modeled using history from March 2005 to June 2007.

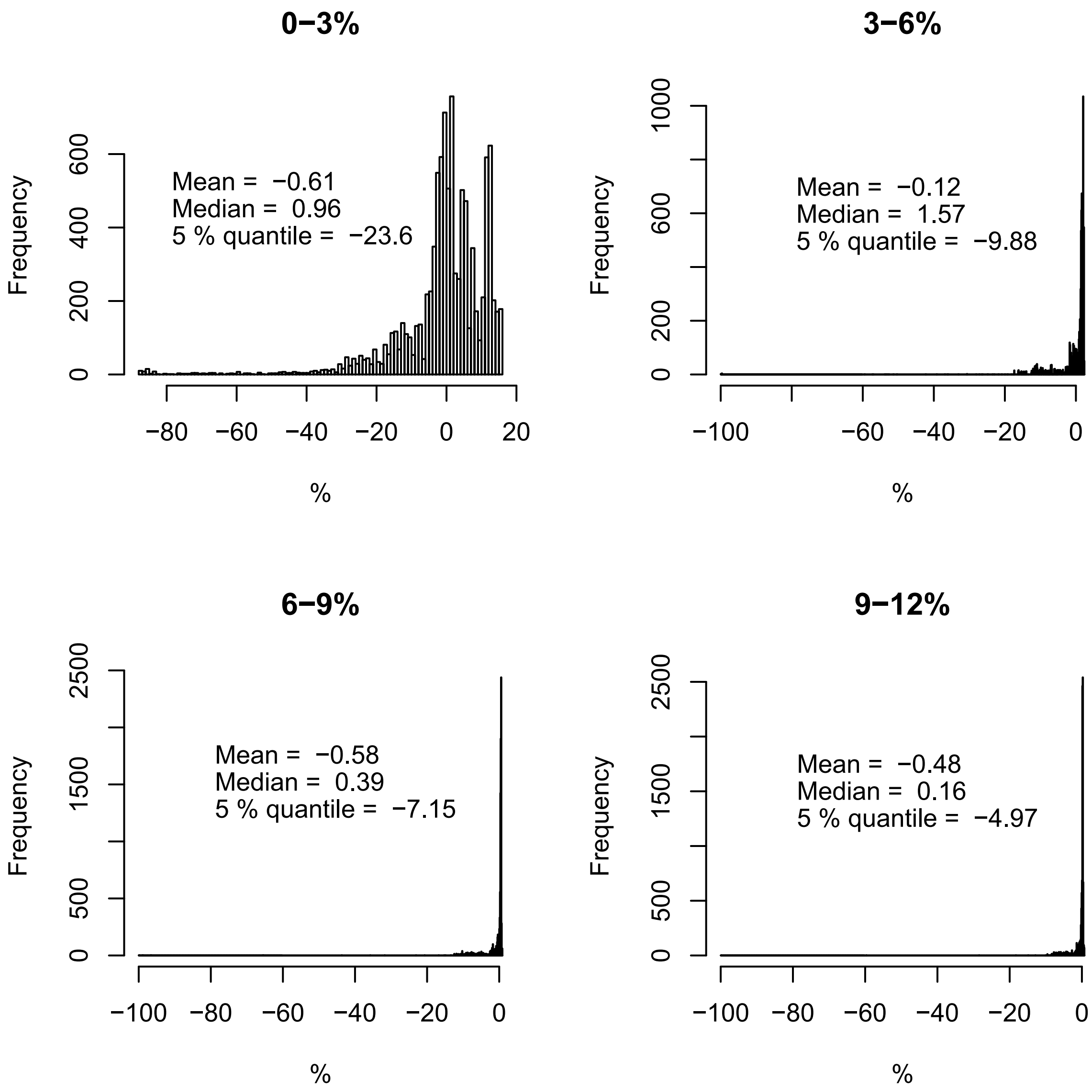

Figure 4.11: Profit & Loss distribution for the four tranches in %. Positive values is a profit, negative values are losses.Spreads are modeled using history from March 2005 to April 2008.

Chapter 5

Conclusion

In this thesis we have introduced some common credit derivatives like CDSs and CDOs. These are derivatives that originally where invented to diversify or manage credit risk. For banks they offered the ability to reduce economic capital requirements by buying protection on some parts of their portfolio, without the need to sell off exposures. Today they are not only used in a risk management setting, but also actively traded by speculative investors, like any other market instrument. As they are market instruments, one of the key interests was to explore the difference between the probabilities of default inherit in the price of a market instrument and those implied by history and published by rating agencies. We showed through an example that they can differ substantially, but also discussed that a part of the difference might be attributed to other factors such as liquidity. This difference can be understood as a risk premium, inherit in the market price. This differs from standard actuarial pricing used in insurance, where the technical premium is calculated according to a model fitted to history, and then possibly added a risk premium. Also market prices must obey the "law" of no arbitrage, which is not applicable for a standard insurance contract, you can not short-sell your insurance policy as far as we know.

As the main topic of the thesis is synthetic CDOs, we wanted to shed some light of these differences in a portfolio instrument, and to explain some of the risks an investor is faced with. Through a multivariate rating transition model, we handled the risk of pure default, and using a standard synthetic CDO pricing model along with historic market data, we forcasted a distribution of market spreads. This was done on a one-year horizon for the iTraxx Europe index, and P&L distributions where calculated. The results obtained through simulation, imply that the main source of risk in terms of return on a short horizons, can be explained by spread risk. The equity tranche is always exposed to immediate risk of default, but all other tranches seems very default remote. A reason for this, is that the constituents on the iTraxx are all investment grade, and looking at history, investment grade corporates rarely default on their obligations. Figure 5.1 shows this history. Keep in mind that this is global history and not defaults in a pool of only 125 names. Figure 5.2 shows this same history for sub investment grade corporates, and here the downturns periods mentioned in Section 4.3 are more apparent. This brings us to the issue of model validity. We argued in Section 3.2 that investment grade corporates are less sensitive to business cycles, and that a good rated company can withstand downturns of the economy for a longer period of time, than a poorly rated company. In such a view, ignoring business cycles when looking at a short horizon might not be a big issue when dealing with investment grade credits. When looking at longer time horizons, the effect of including business cycles in the model might be more pronounced. A lower rated company must finance itself at a higher cost than a better rated company. A downturn in the economy might trigger more downgrades, which again could make the operating environment for an initially investment grade rated company horrible when faced with a difficult market situation. It is therefore very plausible that such spiral effects are amplified with business cycles. A further line of research could therefore be to redo the assessments in this thesis, but including business cycles affecting both market spreads and rating transitions, and possibly at longer time horizons than a single year.

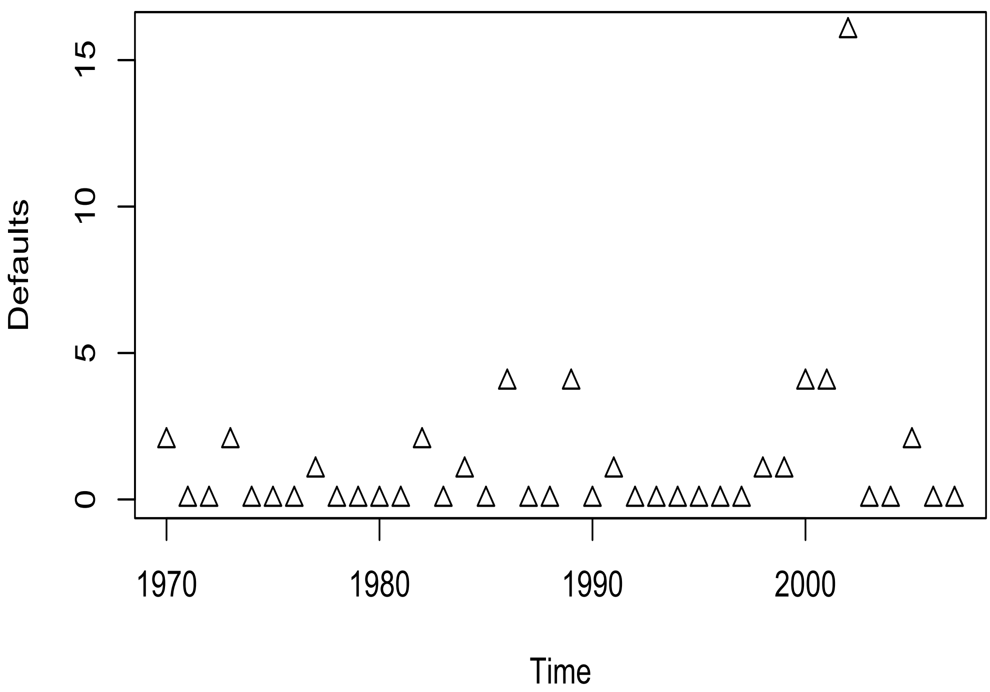

Figure 5.1: Number of defaults globally on investment grade credits since 1970. *Source Moody's (2008)*

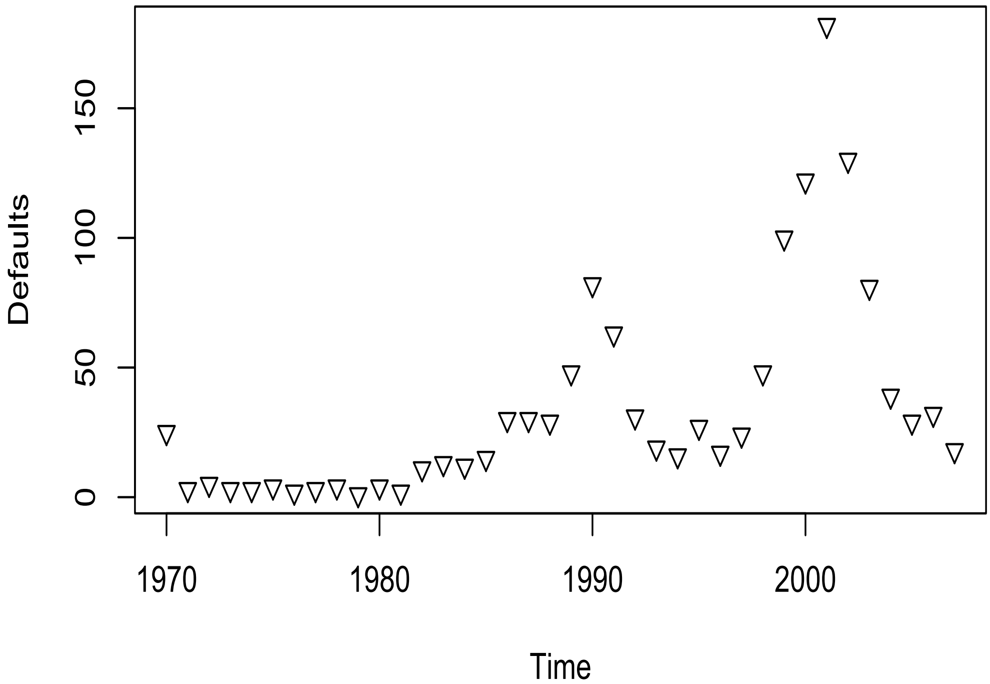

Figure 5.2: Number of defaults globally on sub investment grade credits since 1970.
Source Moody's (2008)

Appendix A

Appendix

A.1 Default leg

First notice that $L(t)$, and hence $L_{a,d}(t)$ is a pure jump process. At every jump of the process there is a payment of $L_{a,d}(t^+) - L_{a,d}(t) > 0$. We assume that the discount function $D(0,t)$ and the tranche loss function $L_{a,d}(t)$ do not share any points of discontinuity so that we can define Stieltjes integrals with respect to $L_{a,d}(t)$. We take a partition of the interval $[0,T]$, $0 = t_0 < t_1 < \ldots < t_n = T$ and consider the Riemann sum

$$\sum_{i=1}^{n} D(0,\xi_{i-1})\left[L_{a,d}(t_i) - L_{a,d}(t_{i-1})\right] \tag{A.1}$$

for $\xi_i \in [t_i, t_{i+1}]$. Then if the sum tends to a fixed number I as $\max(t_{i+1} - t_i) \to 0$, I is called the Stieltjes integral of D with respect to $L_{a,d}(t)$, and denoted

$$\int_0^T D(0,t)dL_{a,d}(t) \tag{A.2}$$

By assuming independence between the deterministic discount function $D(0,t)$and the tranche loss function $L_{a,d}(t)$, we approximate the expectation of the limiting sum, or integral by a discrete sum on a fixed mesh of the partition, meaning $t_i - t_{i-1} = c$ for all i.

$$\begin{aligned}
& \mathbb{E}\left[\sum_{i=1}^{t_n \times k} D(0,t_{i-1})\left(L_{a,d}(t_i) - L_{a,d}(t_{i-1})\right)\right] \\
= & \sum_{i=1}^{t_n \times k} D(0,t_{i-1})\left[EL_{a,d}(t_i) - EL_{a,d}(t_{i-1})\right] \qquad \text{(A.3)}
\end{aligned}$$

where t_n denotes the number of points in the summand of the default leg. I.e if we have a 5 year contract which pays premium quarterly, $t_n = 5 \times 4$. The approximation (A.3) becomes more accurate the higher value of k we choose. If we set $k = 3$ it corresponds to a monthly discretization.

A.2 Hazard Rates

Let the random variable τ be the time of default. Then the instantaneous probability of default, or hazard rate is defined as

$$\begin{aligned} h(t) &= \lim_{\Delta t \to 0} \frac{P\left[\tau \le t + \Delta t | \tau > t\right]}{\Delta t} && \text{(A.4)} \\ &= \lim_{\Delta t \to 0} \frac{P\left[t < \tau \le t + \Delta t\right]}{\Delta t \, P\left[\tau > t\right]} \quad \text{by Bayes} && \text{(A.5)} \\ &= \lim_{\Delta t \to 0} \frac{\int_t^{t+\Delta t} f(s) ds}{\Delta t \, \int_t^{\infty} f(s) ds} = \frac{\frac{\partial}{\partial t} F(t)}{1 - F(t)} = -\frac{\partial}{\partial t} \log\left(1 - F(t)\right). && \text{(A.6)} \end{aligned}$$

By solving the ODE, we get

$$F(t) = 1 - \exp\left(-\int_0^t h(s)\, ds\right) \tag{A.7}$$

and

$$f(t) = h(t)\, \exp\left(-\int_0^t h(s)\, ds\right) \tag{A.8}$$

A.3 Factor Model

To derive det correlations in the factor model, we first note that

$$\begin{aligned} \mathbb{E}(X_i) &= \mathbb{E}(a_i Y + \sqrt{1 - a_i^2} Z_i) = 0, \text{because}\, Y, Z_i \, i.i.d \, N(0,1) && \text{(A.9)} \\ \mathbb{V}\text{ar}(X_i) &= a_i^2 \mathbb{V}\text{ar}(Y) + (1 - a_i^2) \mathbb{V}\text{ar}(Z_i) = 1. && \text{(A.10)} \end{aligned}$$

Since each X_i have unit variance, the correlation is given by

$$\begin{aligned} \mathbb{C}\text{ov}(X_i, X_j) &= a_i a_j \mathbb{V}\text{ar}(Y) + a_i \sqrt{1 - a_j^2}\, \mathbb{C}\text{ov}(Y, Zj) \\ &+ a_j \sqrt{1 - a_i^2}\, \mathbb{C}\text{ov}(Z_i, Y) + \sqrt{1 - a_i^2} \sqrt{1 - a_j^2}\, \mathbb{C}\text{ov}(Z_i, Z_j) \\ &= a_i a_j = \rho_{ij}. && \text{(A.11)} \end{aligned}$$

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