EXPLORING & EXTENDING THE BLACK-SCHOLES FORMULA

by

ØYVIND WEFALD ANDERSEN

THESIS

for the degree of

Master of Mathematics

Faculty of Mathematics and Natural Sciences
University of Oslo

April 2010

Preface

This thesis constitutes the written part of my mathematics Master, and its seeds were sown in the ripe spring soil of 2007. I had spent a number of preceding years away from the world of academics, trotting the continents of the actual world, creating and releasing words and music, and playing cards on a professional level.

Now, as exciting and adventurous as that kind of life may be, impending adulthood began to make its voice heard, and my need for monetary stability started to become more urgent than my need for adventure. So in 2007 the decision was made to go back to school and finish my formal education, and I once again found myself wandering the corridors of the Department of Mathematics at the University of Oslo. This time on the hunt for a professor who would want to guide me through the academic jungle.

Since my reasons for becoming a student once again were slightly on the pragmatic side, I felt that it was important to find a subject for my thesis that was somewhat concrete. Something that could be applied to the physical world out there, even though it for formal reasons needed to be at least partly within the boundaries of pure mathematics. But as I started my advisor safari, it soon became clear that this would be a difficult task.

I arranged meetings with several of the professors residing in the Tower of Abel (pun most likely intended), and although I was always greeted in a friendly manner, my request for something 'more concrete' was not. I (obviously) already knew that these elusive and enigmatic creatures known as mathematicians tend to be quite immersed in the realm of the abstract, so this didn't really come as much of a surprise. But since you're sitting there right now reading the preface of a mathematics thesis written by yours truly, it all must have worked out in the end. So what happened? Did I set aside my wish for something a little less pure than what the professors I had visited were working on? Well... I considered it. But then I found room 1027.

Room 1027 was where a professor by the name of Tom Lindstrøm had his nest, and I recognized his name as the author of the textbook used in the first mathematics course I ever took at the university level, back in 1997. Mr. Lindstrøm was also a member of approximately nine million different committees (he claims that this number is somewhat overestimated, but I attribute his objection to modesty), had his nest located within the domains of the Centre of Mathematics for Applications, as opposed to the pure mathematics I was trying to avoid, and he even had his own fan group on Facebook. Obviously, I went ahead and set up a meeting with him.

Sitting there in the aforementioned 1027, surrounded by the vast number of notes and theses and books and other objects-made-from-a-combination-of-paper-and-cryptic-symbols that this Lindstrøm creature had used when builing his nest, I explained what kind of thesis I was looking to write. He suggested that stochastic analysis, with its immediate applications to finance, would be a reasonable way to go. I of course had no idea what stochastic analysis was, I had taken zero courses in finance and I barely knew anything about probability theory, but something about it just felt right. And after we ended the meeting with an additional hour of non-math related chatter, this something felt even more right. So instead of continuing my hunt, I simply asked Lindstrøm right then and there if he was interested in being my advisor. His reply was something along the lines of 'Well... I wasn't really supposed to have any master students this semester... (dramatic pause) But I'll make an exception. Welcome to CMA.'. It had begun...

Over the course of the next two years, I was both a teacher and a student, thinking that teaching calculus to freshmen would be a good way to get back into the mathematical mindset. Then, in the spring of 2009, after completing the required theoretical courses in measure theory, stochastic analysis and finance, I finally started writing the actual thesis, which I am now going to walk you through.

The first section of the thesis presents the classic Black-Scholes formula, derived by solving partial differential equations and doing probabilistic calculations. The details of these calculations are normally omitted from textbooks on the subject, so it felt like a good idea to include them here, as a reference.

Section two uses Girsanov's Theorem to find the equivalent martingale measure for the Black-Scholes market model, and with the help of this measure presents an alternative way to arrive at the Black-Scholes formula.

In section three, we investigate what happens to the option price if the parameters of the model, especially the volatility, are changed. Through a series of MATLAB simulations, culminating in an animated movie, it is demonstrated that the classic method of calculating the greek *vega* should be approached with a great deal of caution.

The fourth section of the thesis leaves behind the safety of continuity and introduces Itô-Lévy processes and a suitable version of the Itô formula to go along with them. Inspired by the financial crisis, particular attention is given to the Poisson process, which is introduced into our market model in an attempt to simulate the possibility of sudden (discontinuous) market falls.

Section five starts off with a discussion on EMMs and how they relate to the notion of market completeness. An equivalent martingale measure for the (discontinuous) market model we used in section four is calculated, and later on used to find the option price, similarly to what was done for the (continuous) Black-Scholes market in section two.

And so ends the tale of what came before, and the preview of what is still to come. The only item left on the preface menu, before flipping the page and getting down to business, is a slice of gratitude. I would like to profoundly thank my parents, Bjørg & Torfinn Andersen, for their neverending support, both financially and in every other way possible and impossible. My advisor, professor Tom Lindstrøm, deserves massive kudos for being outstandingly flexible and forthcoming throughout the whole learning and writing process. Thanks also go out to mr. Paul C. Kettler, who taught me the basics of LaTeX and provided valuable insights, and to the administrative personnel at the Mathematics Department at the University of Oslo, especially Mathias Barra. Live & Prosper. Takk.

Oslo, April 2010 *Øyvind Wefald Andersen*

EXPLORING & EXTENDING
THE BLACK-SCHOLES FORMULA

1. Introduction

We start by considering a traditional Black-Scholes market model on a filtered probability space$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where we have a risk-free investment (also called a bond)

$$S_t = S_0 e^{rt} \ , \tag{1.1}$$

and a risky investment X_t, believed to obey the stochastic differential equation

$$dX_t = \alpha X_t \, dt + \sigma X_t \, dB_t \ , \quad X_0 = x_0 \tag{1.2}$$

or, equivalently

$$X_t = x_0 + \int_0^t \alpha X_s \, ds + \int_0^t \sigma X_s \, dB_s \ . \tag{1.3}$$

Here, α, σ, and r are positive constants, B_t is a real-valued Brownian motion, and $t \geq 0$. σ is commonly referred to as the *volatility.*

Next we introduce the concept of a *European call option*:

Definition 1.1. A *European call option* is a contract that gives the right (but not the obligation) to buy at time T a stock at price K, which is fixed when the contract is signed.

The time T is called the *maturity*, and K is the *strike price.*

If $X_T \geq K$, this option enables its owner to buy the stock at price K and then sell it immediately at price X_T. The difference $X_T - K$ is the realized gain. If $X_T < K$, the gain is zero. In other words, the value of the option is given by $(X_T - K)_+ = max\{X_T - K, 0\}$.

Definition 1.2. Assuming there are no opportunities for arbitrage (NAO) in the market, if $\theta = (\mu, \beta)$ is a portfolio that finances the random variable Z, then $\pi(Z)_t := \mu_t S_t + \beta_t X_t$ is the *implicit price* of Z at time t.

We would like to obtain a financing strategy for the random variable $Z = g(X_T) = (X_T - K)_+$, and thus find its implicit price $\pi(Z)_t$. This leads us to what is known as the Black-Scholes formula.

1.1. **The Black-Scholes Formula.**

Let $C^{1,2}([0,T]\times\mathbb{R}_+,\mathbb{R})$ be the set of functions f from $[0,T]\times\mathbb{R}_+$ into $\mathbb{R}$, of class C^1 with respect to t and C^2 with respect to x.

We suppose that there exists $p\in C^{1,2}([0,T]\times\mathbb{R}_+,\mathbb{R})$ such that

$$\begin{aligned}\pi(Z)_t &= p(t,X_t), \qquad && t<T\\ g(x) &= p(T,x), && x\in\mathbb{R}_+ \ .\end{aligned}$$

Let $Y_t = p(t,X_t)$. Itô's formula then gives us

$$\begin{aligned}dY_t = &\left(\alpha X_t\frac{\partial p}{\partial x}(t,X_t)+\frac{\partial p}{\partial t}(t,X_t)+\frac{1}{2}\sigma^2(X_t)^2\frac{\partial^2 p}{\partial x^2}(t,X_t)\right)dt\\ &+\sigma X_t\frac{\partial p}{\partial x}(t,X_t)\,dB_t \ .\end{aligned}\tag{1.4}$$

Letting $\mathfrak{L}$ denote the infinitesimal generator of the diffusion $V_t=(S_t,X_t)$, $t\geq 0$, defined on $C^{1,2}([0,T]\times\mathbb{R}_+,\mathbb{R})$ by

$$\mathfrak{L}p=\alpha x\frac{\partial p}{\partial x}+\frac{\partial p}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 p}{\partial x^2} \quad ,$$

equation (1.4) can be rewritten as

$$dY_t=\mathfrak{L}p(t,X_t)\,dt+\sigma X_t\frac{\partial p}{\partial x}(t,X_t)\,dB_t \ .\tag{1.5}$$

As previously mentioned, a strategy θ that finances Z is represented by a portfolio (μ,β) where μ is the number of bonds, and β the number of risky assets held.

Thus, if $V_t=(S_t,X_t)$ we have, since the strategy is self-financing,

$$\begin{aligned}\theta_t\cdot V_t &= \mu_tS_t+\beta_tX_t\\ &=\theta_0\cdot V_0+\int_0^t\theta_s\ dV_s\\ &=\mu_0S_0+\beta_0X_0+\int_0^t\mu_s\ dS_s+\int_0^t\beta_s\ dX_s\\ &=p(t,X_t)=Y_t \ ,\end{aligned}\tag{1.6}$$

so another expression for dY_t is

$$\begin{aligned}dY_t &= \mu_t\,dS_t+\beta_t\,dX_t\\ &= r\mu_tS_t\,dt+\beta_t(\alpha X_t\,dt+\sigma X_t\,dB_t)\\ &= (r\mu_tS_t+\alpha\beta_tX_t)\,dt+\sigma\beta_tX_t\,dB_t \ .\end{aligned}\tag{1.7}$$

Comparing (1.5) and (1.7), and identifying the coefficients of the dt and dB_t terms, we obtain

$$\mathfrak{L}p(t, X_t) = r\mu_t S_t + \alpha\beta_t X_t \tag{1.8}$$

and

$$\sigma X_t \frac{\partial p}{\partial x}(t, X_t) = \sigma\beta_t X_t \ . \tag{1.9}$$

From this we conclude that

$$\beta_t = \frac{\partial p}{\partial x}(t, X_t)$$

and

$$\mu_t = (S_t)^{-1}\left(p(t, X_t) - X_t\frac{\partial p}{\partial x}(t, X_t)\right) \ .$$

We have thus obtained a financing strategy for Z as a function of its implicit price. Substitution into (1.8) yields

$$\mathfrak{L}p(t, X_t) = r\left[p(t, X_t) - X_t\frac{\partial p}{\partial x}(t, X_t)\right] + \alpha\frac{\partial p}{\partial x}(t, X_t)X_t \ .$$

After replacing $\mathfrak{L}p$ with its full expression and then simplifying, this last equality can be written

$$rX_t\frac{\partial p}{\partial x}(t, X_t) + \frac{\partial p}{\partial t}(t, X_t) + \frac{1}{2}\sigma^2(X_t)^2\frac{\partial^2 p}{\partial x^2}(t, X_t) = rp(t, X_t) \tag{1.10}$$

for $t \in [0, T]$; $P - a.s.$, and with

$$p(T, X_T) = g(X_T) \ a.s. \ .$$

Note that α does not appear in this equation.

We summarize these results as a theorem:

Theorem 1.3. *Let $\{S_t\}_{t\geq 0}$ be the price of a bond*

$$dS_t = rS_t dt$$

and let $\{X_t\}_{t\geq 0}$ be the price of a risky investment satisfying

$$dX_t = \alpha X_t dt + \sigma X_t dB_t \ .$$

Furthermore, let $Z = g(X_T)$ be a positive random variable, with $\pi(Z)_t$ as its implicit price. We assume that there exists $p \in C^{1,2}([0, T] \times \mathbb{R}_+, \mathbb{R})$ such that

$$\pi(Z)_t = p(t, X_t), \qquad t < T$$
$$g(x) = p(T, x), \qquad x \in \mathbb{R}_+ \ .$$

Then p satisfies the parabolic equation

$$rx\frac{\partial p}{\partial x}(t,x)+\frac{\partial p}{\partial t}(t,x)+\frac{1}{2}\sigma^2x^2\frac{\partial^2 p}{\partial x^2}(t,x)=rp(t,x),\ \ x>0,\ \ t\in]0,T[\tag{1.11}$$

with boundary condition $p(T,x)=g(x),\ x>0$.

A strategy θ that finances Z is given by $\theta=(\mu,\beta)$ where

$$\mu_t=(S_t)^{-1}\left(p(t,X_t)-X_t\frac{\partial p}{\partial x}(t,X_t)\right)$$
$$\beta_t=\frac{\partial p}{\partial x}(t,X_t)\ .$$

1.2. **Obtaining an explicit solution.**

The next step on our adventurous journey is to obtain an explicit solution to equation (1.11). This is done as follows:

Let x and t be fixed, and let ${Z_s}^{x,t}$ be the process indexed by s, $(t\le s\le T)$, and defined by

$${Z_s}^{x,t}=x+r\int_t^s {Z_u}^{x,t}\,du+\sigma\int_t^s {Z_u}^{x,t}\,dB_u\ .$$

${Z_s}^{x,t}$ is initialized at point x at time t, so ${Z_t}^{x,t}=x$. After time t, the process has the same dynamics as $dZ_u=rZ_u\,du+\sigma Z_u\,dB_u$.

Now we apply the following result, known as the Feynman-Kac formula:

Proposition 1.4. *For a positive-valued function $g\in C^2(\mathbb{R})$ such that g, $g\prime$ and $g\prime\prime$ are all piecewise Lipschitz, the function*

$$p(t,x):=E\left[e^{-r(T-t)}g({Z_T}^{x,t})\right] \tag{1.12}$$

is the unique Lipschitz solution to (1.11). *(A function g is Lipschitz on $\mathbb{R}$ if there exists $k>0$ such that $|g(x)-g(y)|\le k|x-y|$ for all x,y.)*

Proof. See Varadhan [18], Krylov [10] or Rogers and Williams [16] □

We now take a closer look at the process Z_t satisfying

$$dZ_t=rZ_t\,dt+\sigma Z_t\,dB_t\ .$$

It is possible to show that Z_t takes strictly positive values, as long as $Z_0>0$. This enables us to define $Y_t=\ln\,Z_t$ where ln denotes the natural logarithm. Using Itô's formula, we have

$$\begin{aligned} dY_t &= \frac{1}{Z_t} r Z_t\, dt + \frac{1}{Z_t}\sigma Z_t\, dB_t - \frac{1}{2}\frac{1}{Z_t^2}(\sigma Z_t)^2\, dt \\ &= \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma\, dB_t \end{aligned}$$

or, written in integral form

$$\begin{aligned} Y(t) &= Y_0 + \int_0^t \left(r - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma\, dB_s \\ &= ln\, Z_0 + \left(r - \frac{1}{2}\sigma^2\right) t + \sigma B_t \ . \end{aligned}$$

Now we recall the following result from elementary probability theory:

Proposition 1.5. *Suppose that the random variable X is normally distributed with mean μ and variance Λ. Then $Y = aX + b$ is also normally distributed, with mean $a\mu + b$ and variance $a^2\Lambda$.*

Proof. See Gut [8], page 124. □

We know that B_t is normally distributed with mean 0 and variance t, and so it follows that Y_t is also normally distributed, with mean $ln\, Z_0 + \left(r - \frac{1}{2}\sigma^2\right) t$ and variance $\sigma^2 t$.

Since the logarithm of Z_t is normally distributed, the distribution of Z_t is given the name *lognormal*.

Now, if the initial point in time is t, then the logarithm of $Z_s^{x,t}$ is distributed according to $N\left(ln\, Z_t^{x,t} + (r - \frac{1}{2}\sigma^2)(s-t)\ ,\ \sigma^2(s-t)\right)$.

Alternatively, $Z = e^U$ where U is normally distributed.

This enables us to find an expression for the solution given in (1.12):

$$\begin{aligned} E\left[e^{-r(T-t)} g({Z_T}^{x,t})\right] &= e^{-r(T-t)} E\left[g({Z_T}^{x,t})\right] \\ &= e^{-r(T-t)} \int_{-\infty}^{+\infty} g(e^u) f_{T-t}(u)\, du \ , \end{aligned}$$

where $f_{T-t}(u)$ is the probability density function of the normal distribution with mean

$$m = ln\, x + (r - \frac{1}{2}\sigma^2)(T-t)$$

and variance $\sigma^2(T-t)$.

When the function g has an explicit form, it is possible to develop these calculations further. Let us see where we end up in the case of the European call option, $g(x) = (x-K)_+$:

$$\begin{aligned} p(t,x) &= e^{-r(T-t)} \int_{-\infty}^{+\infty} g(e^u) f_{T-t}(u)\, du \\ &= e^{-r(T-t)} \int_{ln\,K}^{+\infty} e^u f_{T-t}(u)\, du - Ke^{-r(T-t)} \int_{ln\,K}^{+\infty} f_{T-t}(u)\, du \\ &= I_1 - Ke^{-r(T-t)} I_2 \end{aligned}$$

We calculate I_1 and I_2 separately:

$$\begin{aligned} I_1 &= e^{-r(T-t)} \int_{ln\,K}^{+\infty} e^u f_{T-t}(u)\, du \\ &= e^{-r(T-t)} \int_{ln\,K}^{+\infty} e^u \frac{1}{\sigma\sqrt{2\pi(T-t)}}\; e^{-\frac{(u-m)^2}{2\sigma^2(T-t)}}\; du \\ &= e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{ln\,K}^{+\infty} \frac{1}{\sigma\sqrt{T-t}}\; e^{\frac{2u\sigma^2(T-t)-u^2+2um-m^2}{2\sigma^2(T-t)}}\; du \\ &= e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{ln\,K}^{+\infty} \frac{1}{\sigma\sqrt{T-t}}\; e^{\frac{-(u^2-2u(m+\sigma^2(T-t))+m^2)}{2\sigma^2(T-t)}}\; du \\ &= e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{ln\,K}^{+\infty} \frac{1}{\sigma\sqrt{T-t}}\; e^{\frac{-(u^2-2u(m+\sigma^2(T-t))+(m+\sigma^2(T-t))^2-(m+\sigma^2(T-t))^2+m^2)}{2\sigma^2(T-t)}}\; du \\ &= e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} \int_{ln\,K}^{+\infty} \frac{1}{\sigma\sqrt{T-t}}\; e^{\frac{-(u-(m+\sigma^2(T-t)))^2}{2\sigma^2(T-t)}}\; e^{\frac{-m^2+(m+\sigma^2(T-t))^2}{2\sigma^2(T-t)}}\; du \end{aligned}$$

Then we make the simplification

$$\begin{aligned} \frac{-m^2+(m+\sigma^2(T-t))^2}{2\sigma^2(T-t)} &= \frac{2m\sigma^2(T-t)+(\sigma^2(T-t))^2}{2\sigma^2(T-t)} \\ &= m + \frac{1}{2}\sigma^2(T-t) = ln\,x + r(T-t)\;, \end{aligned}$$

so that the second exponential factor under the integral sign reduces to $xe^{r(T-t)}$, giving us

$$\begin{aligned} I_1 &= x \frac{1}{\sqrt{2\pi}} \int_{ln\,K}^{+\infty} \frac{1}{\sigma\sqrt{T-t}}\; e^{\frac{-(u-(m+\sigma^2(T-t)))^2}{2\sigma^2(T-t)}}\; du \\ &= x \frac{1}{\sqrt{2\pi}} \int_{ln\,K}^{+\infty} \frac{1}{\sigma\sqrt{T-t}}\; e^{-\frac{1}{2}\left(\frac{u-(ln\;x+(r+\frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}}\right)^2}\; du\;\;. \end{aligned}$$

Next we make the change of variables $v = \frac{u-(ln\;x+(r+\frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}}$,
so $\frac{dv}{du} = \frac{1}{\sigma\sqrt{T-t}}$ or $du = \sigma\sqrt{T-t}\;dv$.

The upper limit of integration is still $+\infty$, but the lower limit needs to be changed:

$$u = ln\,K \Leftrightarrow v = \frac{ln\,K - (ln\,x + (r + \frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}}$$

We now introduce the cumulative distribution function ϕ, given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\,du = \frac{1}{\sqrt{2\pi}} \int_{-x}^{+\infty} e^{-u^2/2}\,du$$

and use it to complete our calculation of I_1:

$$\begin{aligned} I_1 &= x\frac{1}{\sqrt{2\pi}} \int_{\left(\frac{ln\,K-(ln\,x+(r+\frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}}\right)}^{+\infty} e^{-v^2/2}\,dv \\ &= x\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\left(\frac{ln\,x+(r+\frac{1}{2}\sigma^2)(T-t)-ln\,K}{\sigma\sqrt{T-t}}\right)} e^{-v^2/2}\,dv \\ &= x \cdot \phi\left(\frac{ln\left(\frac{x}{K}\right) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}})\right) \end{aligned}$$

Then it's time for I_2:

$$\begin{aligned} I_2 &= \int_{ln\,K}^{+\infty} f_{T-t}(u)\,du \\ &= \int_{ln\,K}^{+\infty} \frac{1}{\sigma\sqrt{2\pi(T-t)}}\, e^{-\frac{(u-m)^2}{2\sigma^2(T-t)}}\,du \\ &= \frac{1}{\sqrt{2\pi}} \int_{ln\,K}^{+\infty} \frac{1}{\sigma\sqrt{T-t}}\, e^{-\frac{1}{2}\left(\frac{u-m}{\sigma\sqrt{T-t}}\right)^2}\,du \end{aligned}$$

We once again make a change of variables, this time $v = \frac{u-m}{\sigma\sqrt{T-t}}$, still giving us $\frac{dv}{du} = \frac{1}{\sigma\sqrt{T-t}}$ or $du = \sigma\sqrt{T-t}\,dv$.

Similarly to what we saw in the calculation of I_1, the upper limit of integration is still $+\infty$, but the lower limit needs to be changed:

$$u = ln\,K \Leftrightarrow v = \frac{ln\,K - (ln\,x + (r - \frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}}$$

Continuing our calculations, we get

$$
\begin{aligned}
I_2 &= \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{\ln K-(\ln x+(r-\frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}}\right)}^{+\infty} e^{-v^2/2}\,dv \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\left(\frac{\ln x+(r-\frac{1}{2}\sigma^2)(T-t)-\ln K}{\sigma\sqrt{T-t}}\right)} e^{-v^2/2}\,dv \\
&= \phi\left(\frac{\ln\left(\frac{x}{K}\right)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}
$$

We summarize this as a theorem:

Theorem 1.6. ***(The Black-Scholes Formula)***
The price of a European call is given by

$$p(0,x) = x\phi(d_1) - Ke^{-rT}\phi(d_2)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\left(\frac{x}{K}\right)+T\left(r+\frac{\sigma^2}{2}\right)\right), \quad d_2 = d_1 - \sigma\sqrt{T}\ .$$

We also have

$$p(t,x) = x\phi(d_1(t)) - Ke^{-r(T-t)}\phi(d_2(t))$$

where

$$d_1(t) = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{x}{K}\right)+(T-t)\left(r+\frac{\sigma^2}{2}\right)\right)$$

and

$$d_2(t) = \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{x}{K}\right)+(T-t)\left(r-\frac{\sigma^2}{2}\right)\right) = d_1(t) - \sigma\sqrt{T-t}\ .$$

2. Using equivalent martingale measures

Let us begin by defining a very relevant concept:

Definition 2.1. An *equivalent martingale measure* Q (EMM for short) is a probability measure that is equivalent to P and such that, under Q, the discounted stock price $\tilde{X}_t := \frac{X_t}{S_t}$ is a martingale. Such a measure is also called a *risk-neutral measure.*

Our work in the previous section was done under the assumption of no opportunities for arbitrage (NAO) in the market. This is often replaced with the assumption of the existence of an equivalent martingale measure Q. As long as we confine ourselves to working in discrete time, these two assumptions turn out to be the same. In continuous time, however, additional technicalities that go beyond the scope of this text need to be introduced if we want to use these two assumptions interchangeably.

(A short discussion on this subject can be found in Dana and Jeanblanc [4], page 91-92. For a more detailed investigation, see Müller [12], Dalang et al. [3], Morton [11], Delbaen and Schachermayer [5],[6], Kabanov [9] or Xia and Yan [19].)

In order to find an equivalent martingale measure for our market model, we will make use of the following version of the Girsanov theorem:

Theorem 2.2. ***(Girsanov's Theorem)*** *Let $\{L_t\}_{t\geq 0}$ be the process defined by*

$$L_t = exp\left(\int_0^t h(s)\, dB_s - \frac{1}{2}\int_0^t h^2(s)\, ds\right)\ , \tag{2.1}$$

where $\{h(s)\}_{0\leq s\leq T}$ is an adapted bounded process.

The process $\{L_t\}_{t\geq 0}$ is the unique solution to

$$dL_t = L_t h_t\, dB_t\ , \qquad L_0 = 1\ ,$$

is a martingale, and satisfies $E\left(L_t\right) = 1,\ \forall t \in [0,T]$.

Let Q be the probability measure defined on $(\Omega, \mathcal{F}_T)$ by $Q(A) = E_P(1_A\, L_T)$. Under Q, the process B^* defined by

$$B^*_t = B_t - \int_0^t h(s)\, ds$$

is a Brownian motion.

Proof. A more general version of this theorem is proved in Øksendal [13], page 162-165. $\square$

Now, let $S_0 = 1$, so that $S_t = e^{rt}$. The discounted stock price is then given by

$$\tilde{X}_t = \frac{X_t}{S_t} = e^{-rt} X_t\ .$$

We use Itô's formula with $g(t,x) = x \cdot e^{-rt}$ and get

$$\begin{aligned} d\tilde{X}_t &= \frac{\partial g}{\partial t}(t, X_t)\, dt + \frac{\partial g}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)\, (dX_t)^2 \\ &= -rX_t e^{-rt}\, dt + e^{-rt}\left(\alpha X_t\, dt + \sigma X_t\, dB_t\right) \\ &= \tilde{X}_t\left((\alpha - r)\, dt + \sigma\, dB_t\right)\ . \end{aligned}$$

Let $\{L_t\}$ be the process satisfying $dL_t = -(\alpha - r)\sigma^{-1} L_t\, dB_t$ with $L_0 = 1$. Girsanov's theorem (with $h_t = -(\alpha - r)\sigma^{-1}$) shows that if we define Q on $\mathcal{F}_T$ by $Q(A) = E_P(1_A\, L_T)$ or, equivalently $\frac{dQ}{dP} = L_T$, then the process B^*_t given by

$$B^*_t = B_t + \int_0^t (\alpha - r)\sigma^{-1} ds = B_t + (\alpha - r)\sigma^{-1} t$$

is a brownian motion under Q. We also have that

$$
\begin{aligned}
d\tilde{X}_t &= \tilde{X}_t(\alpha - r)\,dt + \tilde{X}_t\sigma\,dB_t \\
&= \tilde{X}_t(\alpha - r)\,dt + \tilde{X}_t\sigma\left(dB_t^* - (\alpha - r)\sigma^{-1}dt\right) \\
&= \tilde{X}_t\sigma\,dB_t^* \ .
\end{aligned}
$$

The first part of Girsanov's theorem ensures that the solution to $dX_t = X_t h_t\,dB_t$ (for any Brownian motion) is a martingale as long as h_t is bounded, so we see from the above calculations that $\tilde{X}_t$ is a Q-martingale. Furthermore, Q is equivalent to P. Thus we have found the equivalent martingale measure we were looking for.

Under Q, the risky investment X_t obeys the stochastic differential equation

$$
\begin{aligned}
dX_t &= \alpha X_t\,dt + \sigma X_t\,dB_t \\
&= \alpha X_t\,dt + \sigma X_t\left(dB_t^* - (\alpha - r)\sigma^{-1}dt\right) \\
&= X_t\left(r\,dt + \sigma\,dB_t^*\right)
\end{aligned}
$$

We proceed by constructing a portfolio like the one in equation (1.6),

$$
Y_t = \mu_t S_t + \beta_t X_t \ .
$$

Under our new probability measure Q, we have that

$$
\begin{aligned}
dY_t &= \mu_t\,dS_t + \beta_t\,dX_t \\
&= \mu_t r S_t\,dt + \beta_t X_t\left(r\,dt + \sigma\,dB_t^*\right) \\
&= Y_t r\,dt + \beta_t X_t \sigma\,dB_t^* \\
&= Y_t r\,dt + dM_t \ ,
\end{aligned}
$$

where $\{M_t\}_{t\geq 0}$ is defined by $dM_t = \beta_t X_t \sigma\,dB_t^*$. Under appropriate integrability conditions, $\{M_t\}_{t\geq 0}$ is a stochastic integral, and thus a martingale.

Using Itô's formula with $g(t,x) = x \cdot e^{-rt}$ on Y_t, we end up with

$$
\begin{aligned}
d\left(e^{-rt}Y_t\right) &= -rY_t e^{-rt}\,dt + e^{-rt}\left(Y_t r\,dt + \beta_t X_t \sigma\,dB_t^*\right) \\
&= e^{-rt}\beta_t X_t \sigma\,dB_t^* \\
&= e^{-rt}\,dM_t \ .
\end{aligned}
$$

This means that the process $\{e^{-rt}Y_t\}_{t\geq 0}$ is also a martingale. Therefore we have, from the definition of a martingale, that

$$
e^{-rt}Y_t = E_Q\left[e^{-rT}Y_T \mid \mathcal{F}_t\right] \ ,
$$

and since e^{rt} is a deterministic function, we end up with

$$Y_t = p(t, X_t) = e^{rt} \cdot E_Q\left[e^{-rT} Y_T \mid \mathcal{F}_t\right] = E_Q\left[e^{-r(T-t)} g(X_T) \mid \mathcal{F}_t\right] \ .$$

Theorem 2.3. *The implicit price of $g(X_T)$ is given by*

$$p(t, X_t) = E_Q\left[e^{-r(T-t)} g(X_T) \mid \mathcal{F}_t\right] \tag{2.2}$$

where Q is the equivalent martingale measure to P.

In particular, at time $t = 0$ we have

$$p(0, x) = E_Q\left[e^{-rT} g(X_T)\right] = E_P\left[e^{-rT} g(Z_T)\right]$$

where $\{Z_t\}_{t \geq 0}$ satisfies $dZ_t = Z_t\,(r\,dt + \sigma\,dB_t)$ with $Z_0 = x$, giving us the same formula as in equation (1.12).

For a general time t, we proceed as follows:

From equation (2.2) we know that

$$p(t, X_t) = E_Q\left[e^{-r(T-t)} g(X_T) \mid \mathcal{F}_t\right] \ .$$

Here, X_t is the solution of the stochastic differential equation

$$dX_t = rX_t\,dt + \sigma X_t\,dB_t^* \ , \ \ X(0) = X_0$$

where B_t^* is a Brownian motion under Q. The process X_t is an Itô diffusion, so the Markov property applies, giving us

$$p(t, X_t) = E_Q^{X(t,\omega)}\left[e^{-r(T-t)} g(X_{T-t})\right] \ .$$

This means that the two functions

$$x \mapsto p(t, x) \quad \text{and} \quad x \mapsto E_Q^x\left[e^{-r(T-t)} g(X_{T-t})\right]$$

are equal when we substitute $x = X_t$, so the functions themselves have to be equal:

$$p(t, x) = E_Q^x\left[e^{-r(T-t)} g(X_{T-t})\right]$$

In the above expression, X_t is the solution of the stochastic differential equation

$$dX_t = rX_t\,dt + \sigma X_t\,dB_t^* \ , \ \ X_0 = x \ .$$

Since this equation has a unique solution and is time-homogenous, the solution X_{T-t} at time $T - t$ will have the same distribution as the solution of

$$dZ_s^{x,t} = rZ_s^{x,t}\,ds + \sigma Z_s^{x,t}\,dB_s \ , \ \ Z_t^{x,t} = x$$

at time T. We end up with

$$p(t,x) = E_Q^x\left[e^{-r(T-t)}g(X_{T-t})\right] = E_P\left[e^{-r(T-t)}g\left(Z^{x,t}(T)\right)\right] \ ,$$

which is the same result as in equation (1.12).

We can also recover the partial differential equation from Theorem 1.3 by applying Itô's formula to $p(t, X_t)$:

$$\begin{aligned} dp(t,X_t) &= \frac{\partial p}{\partial t}(t,X_t)\,dt + \frac{\partial p}{\partial x}(t,X_t)\,dX_t + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(t,X_t)\cdot(dX_t)^2 \\ &= \frac{\partial p}{\partial t}(t,X_t)\,dt + rX_t\frac{\partial p}{\partial x}(t,X_t)\,dt + \sigma X_t\frac{\partial p}{\partial x}(t,X_t)\,dB_t^* \\ &\quad + \frac{1}{2}\left(\sigma X_t\right)^2\frac{\partial^2 p}{\partial x^2}(t,X_t)\,dt \end{aligned}$$

It was shown earlier that $dp(t, X_t) = dY_t = Y_t r\, dt + dM_t$ where $\{M_t\}_{t\geq 0}$ is a martingale. By setting the coefficient of the dB_t^*-term equal to zero, we therefore obtain

$$r\cdot p(t,X_t)\,dt = \frac{\partial p}{\partial t}(t,X_t)\,dt + rX_t\frac{\partial p}{\partial x}(t,X_t)\,dt + \frac{1}{2}\left(\sigma X_t\right)^2\frac{\partial^2 p}{\partial x^2}(t,X_t)\,dt \ .$$

This implies that p satisfies the partial differential equation

$$rp(t,x) = rx\frac{\partial p}{\partial x}(t,x) + \frac{\partial p}{\partial t}(t,x) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 p}{\partial x^2}(t,x) \ ,$$

which is the same as equation (1.11).

3. Parameter changes

Now that we have the basic framework established, we turn to the question of what happens to the call price if the risky investment actually follows a different model than the one assumed to be true in (1.2).

Here's our new equation:

$$dY_t = \tilde{\alpha}Y_t dt + \tilde{\sigma}Y_t dB_t \ , \quad Y_0 = x_0 \tag{3.1}$$

As previously noted, α does not appear in the Black-Scholes Formula, so changing α to $\tilde{\alpha}$ does not have any impact on the call price. To see what happens if we change σ to $\tilde{\sigma}$, we first try differentiating the call price with regard to σ, keeping x and t fixed:

$$\begin{aligned}
\frac{\partial p}{\partial \sigma} &= \frac{x}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(d_1(t))^2} \left(\frac{\sigma^2(T-t) - \left(ln\left(\frac{x}{K}\right) + (T-t)(r + \frac{\sigma^2}{2}) \right)}{\sigma^2 \sqrt{T-t}} \right) \\
&\quad - Ke^{-r(T-t)} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(d_2(t))^2} \left(\frac{-\sigma^2(T-t) - \left(ln\left(\frac{x}{K}\right) + (T-t)(r - \frac{\sigma^2}{2}) \right)}{\sigma^2 \sqrt{T-t}} \right) \\
&= \frac{x}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(d_1(t))^2} \left(\sqrt{T-t} - \frac{d_1(t)}{\sigma} \right) \\
&\quad - Ke^{-r(T-t)} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(d_1(t) - \sigma\sqrt{T-t})^2} \left(-\sqrt{T-t} - \frac{1}{\sigma} \left(d_1(t) - \sigma\sqrt{T-t} \right) \right) \\
&= \frac{x}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(d_1(t))^2} \left(\sqrt{T-t} - \frac{d_1(t)}{\sigma} \right) \\
&\quad - Ke^{-r(T-t)} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(d_1(t)^2 - 2d_1(t)\sigma\sqrt{T-t} + \sigma^2(T-t) \right)} \left(-\frac{1}{\sigma}\, d_1(t) \right) \\
&= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(d_1(t))^2} \left(x \left(\sqrt{T-t} - \frac{d_1(t)}{\sigma} \right) + K\, \frac{d_1(t)}{\sigma}\, e^{-r(T-t)} e^{\left(d_1(t)\sigma\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t) \right)} \right) \\
&= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(d_1(t))^2} \left(x \left(\sqrt{T-t} - \frac{d_1(t)}{\sigma} \right) + K\, \frac{d_1(t)}{\sigma}\, e^{-r(T-t)} e^{\left(ln\left(\frac{x}{K}\right) + (T-t)(r + \frac{\sigma^2}{2}) - \frac{1}{2}\sigma^2(T-t) \right)} \right) \\
&= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(d_1(t))^2} \left(x \left(\sqrt{T-t} - \frac{d_1(t)}{\sigma} \right) + K\, \frac{d_1(t)}{\sigma}\, \frac{x}{K}\, e^{-r(T-t) + r(T-t)} \right) \\
&= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(d_1(t))^2} \left(x\sqrt{T-t} - \frac{x\, d_1(t)}{\sigma} + \frac{x\, d_1(t)}{\sigma} \right) \\
&= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(d_1(t))^2} \cdot x\sqrt{T-t}
\end{aligned}$$

This quantity is one of the so-called *Greeks*. It is usually referred to as *vega*, and is denoted by ν. We summarize:

Proposition 3.1. *Vega, the derivative of the call price with regard to the volatility, is given by*

$$\nu = \frac{\partial p}{\partial \sigma} = \frac{x\sqrt{T-t}}{\sqrt{2\pi}}\; e^{-\frac{1}{2}\left(\frac{ln\left(\frac{x}{K}\right) + (T-t)\left(r + \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}} \right)^2} \tag{3.2}$$

We observe that *vega* is always positive. This means that when the volatility σ increases, so does the call price. Intuitively we would expect that the greater the fluctuations in the price of the underlying stock, the more a buyer is prepared to pay for the option. Since σ represents these fluctuations, we see that this intuition is consistent with the preceding result.

However, in getting to this conclusion, we simplified matters quite a bit. Namely by fixing the stock price x while letting σ change at will. But if we take another look at equation (1.2) we see that if σ is changed, then so is the stock price. And we have no guarantee that this change is a negligible one.

In order to investigate this further, we take a look at the explicit solution to the equation that models the stock price:

Proposition 3.2. *The stochastic differential equation given by*

$$dX_t = \alpha X_t \, dt + \sigma X_t \, dB_t, \quad X_0 = x_0 \tag{3.3}$$

or, equivalently

$$X_t = x_0 + \int_0^t \alpha X_s \, ds + \int_0^t \sigma X_s \, dB_s \tag{3.4}$$

has the solution

$$\begin{aligned} X_t &= x_0 \cdot exp\left(\int_0^t \sigma \, dB_s + \int_0^t (\alpha - \frac{1}{2}\sigma^2) \, ds\right) \\ &= x_0 \cdot exp\left(\sigma B_t + (\alpha - \frac{1}{2}\sigma^2)t\right) \quad . \end{aligned} \tag{3.5}$$

Proof. Let $g(t,x) = x_0 \cdot exp\left(\sigma x + (\alpha - \frac{1}{2}\sigma^2)t\right)$. Using Itô's formula, we see that $X_t = g(t, B_t)$ is an Itô process, and that

$$\begin{aligned} dX_t &= \frac{\partial g}{\partial t}(t, B_t) \, dt + \frac{\partial g}{\partial x}(t, B_t) \, dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, B_t) \, dt \\ &= X_t(\alpha - \frac{1}{2}\sigma^2) \, dt + X_t \sigma \, dB_t + \frac{1}{2} X_t \sigma^2 \, dt \\ &= \alpha X_t \, dt + \sigma X_t \, dB_t \end{aligned}$$

□

Using this result, we can rewrite the expression for the European call price as follows:

(3.6)

$$\begin{aligned} d_1(t, X_t) &= \frac{1}{\sigma\sqrt{T-t}}\left(ln\left(\frac{x_0 \cdot exp(\sigma B_t + \left(\alpha - \frac{1}{2}\sigma^2\right) t)}{K}\right) + (T-t)\left(r + \frac{\sigma^2}{2}\right)\right) \\ &= \frac{1}{\sigma\sqrt{T-t}}\left(ln\left(\frac{x_0}{K}\right) + \sigma B_t + \left(\alpha - \frac{1}{2}\sigma^2\right) t + (T-t)\left(r + \frac{\sigma^2}{2}\right)\right) \\ &= \frac{1}{\sigma\sqrt{T-t}}\left(ln\left(\frac{x_0}{K}\right) + \sigma B_t + \left(\alpha - r - \sigma^2\right) t + T\left(r + \frac{\sigma^2}{2}\right)\right) \end{aligned}$$

(3.7)

$$\begin{aligned} d_2(t, X_t) &= \frac{1}{\sigma\sqrt{T-t}}\left(ln\left(\frac{x_0 \cdot exp(\sigma B_t + \left(\alpha - \frac{1}{2}\sigma^2\right) t)}{K}\right) + (T-t)\left(r - \frac{\sigma^2}{2}\right)\right) \\ &= \frac{1}{\sigma\sqrt{T-t}}\left(ln\left(\frac{x_0}{K}\right) + \sigma B_t + \left(\alpha - \frac{1}{2}\sigma^2\right) t + (T-t)\left(r - \frac{\sigma^2}{2}\right)\right) \\ &= \frac{1}{\sigma\sqrt{T-t}}\left(ln\left(\frac{x_0}{K}\right) + \sigma B_t + (\alpha - r)t + T\left(r - \frac{\sigma^2}{2}\right)\right) \end{aligned}$$

(3.8)

$$p(t, X_t) = x_0 \cdot exp\left(\sigma B_t + \left(\alpha - \frac{1}{2}\sigma^2\right) t\right) \phi(d_1(t, X_t)) - Ke^{-r(T-t)}\phi(d_2(t, X_t))$$

3.1. **Simulating the call price in MATLAB.**

In an attempt to gain some useful information about the behaviour of the call price when the parameters α and σ are changed, we use (3.8) as the basis for a MATLAB simulation. Since the value of this expression depends on the value of the Brownian motion B_t, we start by simulating B_t on the interval $[0, T]$. This is done using the following code, where the input parameter `delta_t` is the time increment:

```
% browniansim.m
% Simulates Brownian motion B_t for 0<=t<=T and plots it
% input: (delta_t,T)
function brown=browniansim(delta_t,T)
t=0:delta_t:T;
brown(1)=0;
for i=2:length(t)
    brown(i)=brown(i-1)+sqrt(t(i))*randn;
end
plot(t,brown)
```

An example of Brownian motion made by `browniansim.m` is shown in figure 1.

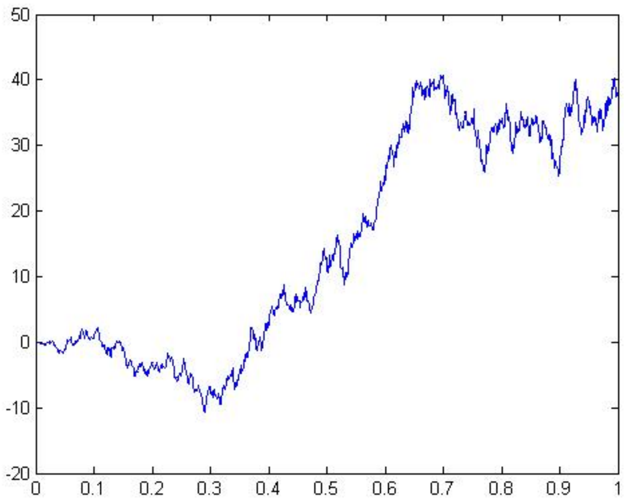

FIGURE 1. Brownian motion.

The next portion of MATLAB code, `callprice.m`, takes t (a fixed point in time), r, α, σ, T, K and x_0 as input parameters. It then uses `browniansim.m` to get a value for B_t and plots a graph of the call price, given in equation (3.8), as a function of α and σ while keeping the other input parameters fixed:

```
% callprice.m
% Calculates the call price
% input: (t,r,alpha,sigma,T,K,x0)
function p=callprice(t,r,alpha,sigma,T,K,x0)
% B contains values of the Brownian motion B_t, 0<=t<=T
B=browniansim(0.001,T);
% B_t contains the particular value of the Brownian motion
% that will be used to calculate the call price later on
B_t=B(round(1+t*1000))
sigma_axis=(sigma*(2/3)):0.001:(sigma*(4/3));
alpha_axis=(alpha*(2/3)):0.001:(alpha*(4/3));
[X,Y]=meshgrid(sigma_axis,alpha_axis);
d1=(log(x0/K)+X*B_t+(Y-r-X.^2)*t+T*(r+0.5*X.^2))./(X*sqrt(T-t));
d2=(log(x0/K)+X*B_t+(Y-r)*t+T*(r-0.5*X.^2))./(X*sqrt(T-t));
p=x0*exp(X*B_t+(Y-0.5*X.^2)*t).*normcdf(d1,0,1)
-K*exp(-r*(T-t))*normcdf(d2,0,1);
colormap(pink)
surfc(X,Y,p)
xlabel('sigma')
ylabel('alpha')
shading flat
```

Before we use `callprice.m` to produce a graph, let us pause for a moment and reflect on what we expect this graph to look like, based on our available information thus far. Equation (3.2) tells us that the call price should be an increasing function of σ, and we know that α does not appear in the Black-Scholes Formula at all. In other words, the graph should be increasing along the σ-axis and have level curves in the form of lines parallell to the α-axis.

Another question that deserves some attention, is what values to choose for the various input parameters in `callprice.m`. Our explorations start by using $t = 0.5$, $r = 0.02$, $\alpha = 0.03$, $\sigma = 0.15$, $T = 1$, $K = 10$ and $x_0 = 10$. These values were chosen based on advice given by mr. Paul C. Kettler, (who in addition to working for the Department of Mathematics at the University of Oslo, used to run a stock broker firm) to represent a fairly realistic market model.

For additional information about the graph, we also differentiate the call price a second time with regard to σ:

$$\frac{\partial^2 p}{\partial \sigma^2} = \frac{x\sqrt{T-t}}{\sqrt{2\pi}}\, e^{-\frac{1}{2}d_1^2} \cdot \frac{\partial}{\partial \sigma}\left(-\frac{1}{2}d_1^2\right) = \frac{x\sqrt{T-t}}{\sqrt{2\pi}}\, e^{-\frac{1}{2}d_1^2} \cdot (-d_1) \cdot \frac{\partial d_1}{\partial \sigma}$$

Then we use this expression, along with the input parameter values to determine the expected curvature properties of the graph of p:

$$\frac{\partial d_1}{\partial \sigma} = \frac{\sigma^2(T-t) - \left(ln\left(\frac{x}{K}\right) + (T-t)\left(r + \frac{\sigma^2}{2}\right)\right)}{\sigma^2\sqrt{T-t}}$$

In this calculation, we assume that x is fixed, so that $x = x_0 = 10 = K$. This means that the logarithm term disappears, and we end up with

$$\frac{\partial d_1}{\partial \sigma} = \frac{(T-t)\left(\sigma^2 - r - \frac{\sigma^2}{2}\right)}{\sigma^2\sqrt{T-t}} = \frac{\sqrt{T-t}\left(\frac{\sigma^2}{2} - r\right)}{\sigma^2} .$$

Since d_1 is positive with this choice of input parameters, we have that

$$sgn\left(\frac{\partial^2 p}{\partial \sigma^2}\right) = -sgn\left(\frac{\partial d_1}{\partial \sigma}\right) = -sgn\left(\frac{\sigma^2}{2} - r\right) = sgn\left(r - \frac{\sigma^2}{2}\right)$$

In other words, p viewed as a function of σ should be convex for $\sigma < \sqrt{2r} = 0.2$.

Now we have a good idea of what we would expect the graph of the call price to look like, given that the error introduced by keeping the stock price fixed while varying σ and α is a negligible one. So without further ado, here is the first graph produced by `callprice.m`:

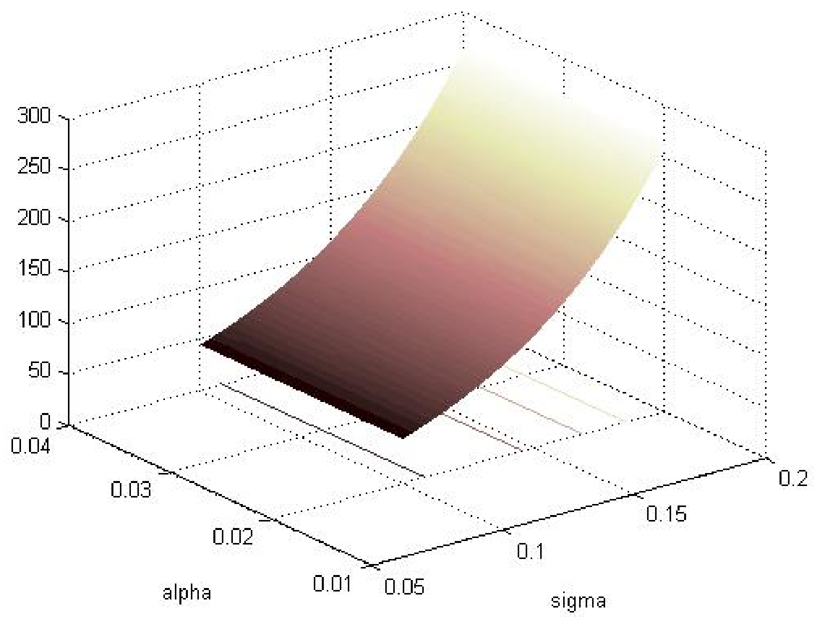

FIGURE 2. The first graph produced by callprice.m.

We see that the graph in figure 2 is increasing and convex along the σ-axis and has level curves in the form of lines parallell to the α-axis, which was what we expected. But we are not finished with this investigation yet. Here, the value of B_t ended up being 16.5. But for each different value of B_t we get a new graph, and that new graph may exhibit qualitatively different behaviour. So let us run `callprice.m` a few more times and see what happens:

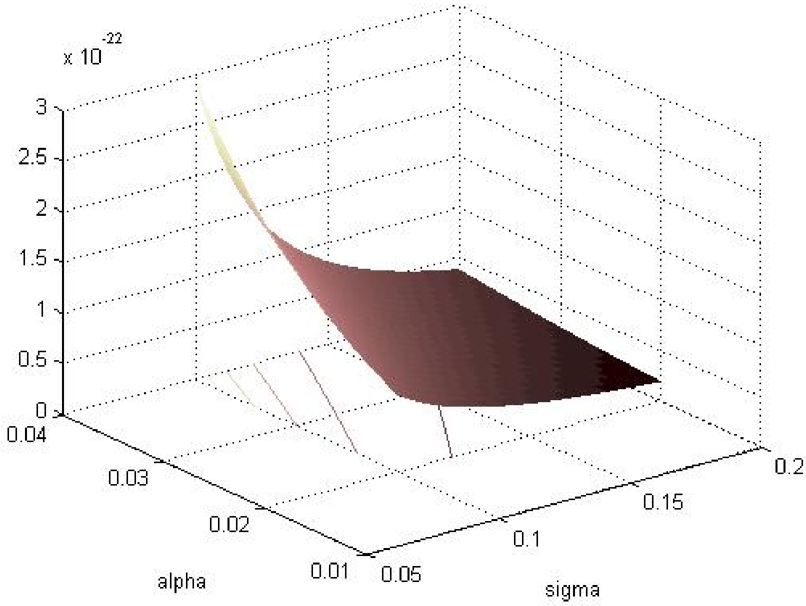

FIGURE 3. Graph produced by callprice.m with $B_t = -6.87$.

Figures 3, 4 and 5 depict graphs that are qualitatively very different from the one shown in figure 2, neither being increasing along the σ-axis or having parallell lines as level curves. Looking at the scale on the vertical axes, it is possible that this is just some kind of anomaly created by MATLAB when the call price approaches zero. But it is also possible that these graphs are trying to tell us that a fair bit of caution should be applied when using (3.2) to predict changes in the call price based on changes in the volatility.

More experimentation is definitely needed before reaching any form of conclusion. Luckily for you, the reader, this experimentation has already taken place, and you will not be bothered with all the details. But a summary of what has been done behind the scenes should be in order:

To start things off, `callprice.m` was called upon quite a few times, while varying some of the input parameters. More precisely, $K \in \{4, 7, 10, 13, 16\}$, $t \in \{0.1, 0.75, 0.99\}$ and $r \in \{0.02, 0.1, 1\}$.

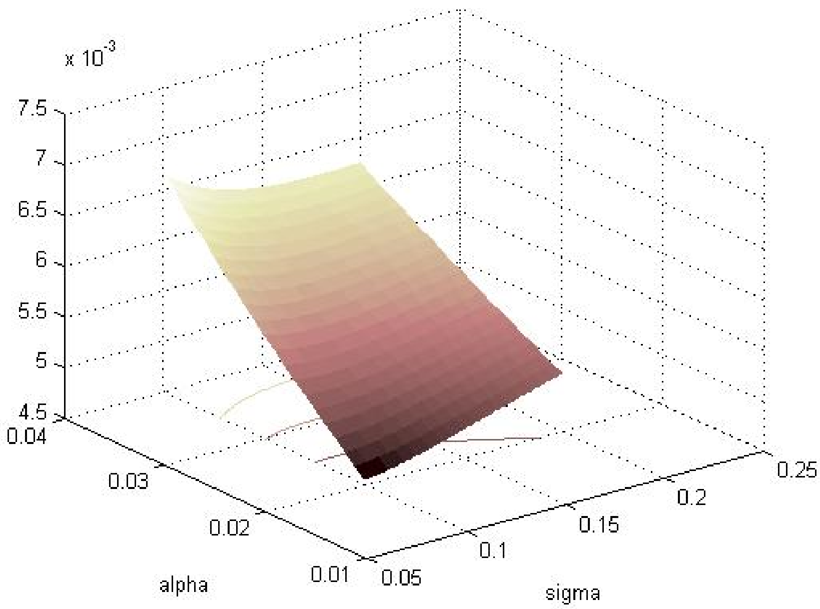

FIGURE 4. Graph produced by callprice.m with $B_t = -1.64$.

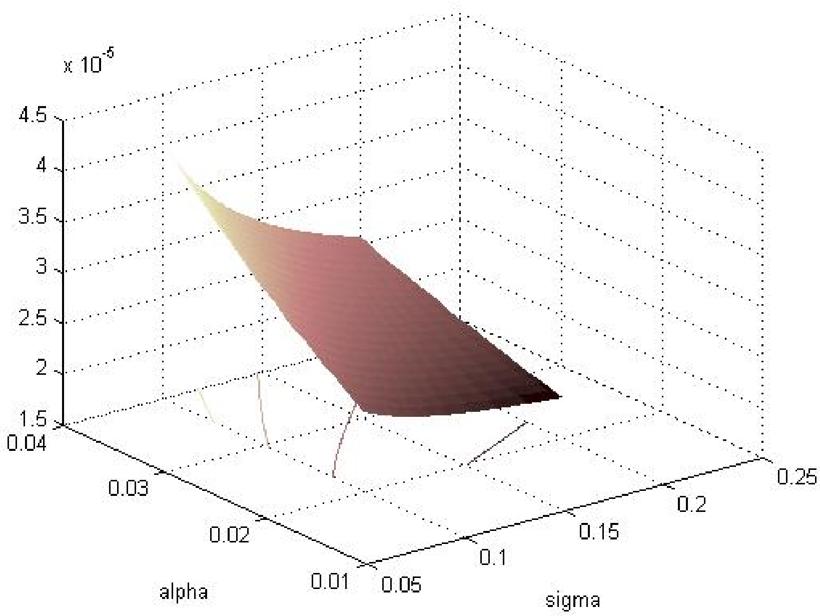

FIGURE 5. Graph produced by callprice.m with $B_t = -2.73$.

All the different combinations of these values were used to make at least 50 graphs each. For fairly obvious reasons, these graphs are not included here as figures. But if you (still referring to the reader) should happen to be in a particularly sceptical or curious state of mind while reading this, then by all means feel free to use `callprice.m` to have your computer draw all of the graphs for you, in the comfort of your own office.

What became apparent after looking through the graphs, was that the qualitative shapes shown in figures 3,4 and 5 appeared quite a few times, and not just with values close to zero on the vertical axis. Furthermore, it seemed like the graphs were in accordance with that shown in figure 2 as long as the value of B_t was positive and above a certain level L^+, which varied with the particular input parameters used. Similarly, if the value of B_t was negative and beneath a certain level L^-, then the graphs would again look like the one in figure 2, but with reversed orientation, so that they were decreasing along the σ-axis. If $B_t \in (L^-, L^+)$, then the graphs would take on various unusual shapes, like for instance the one shown in figure 4.

The most important factor in determining what these graphs would look like, seemed to be the value of B_t obtained from `browniansim.m`. This led to some rewritten MATLAB code, where the value of B_t was no longer randomly determined, but instead included as yet another input parameter:

```
% movie_callprice.m
% Calculates the call price for use in a movie. Here the
% particular value of the Brownian motion to be used in
% the calculations, are given as an additional input
% instead of being simulated.
function p=movie_callprice(t,r,alpha,sigma,T,K,x0,Bt)
B_t=Bt;
sigma_axis=(sigma*(2/3)):0.001:(sigma*(4/3));
alpha_axis=(alpha*(2/3)):0.001:(alpha*(4/3));
[X,Y]=meshgrid(sigma_axis,alpha_axis);
d1=(log(x0/K)+X*B_t+(Y-r-X.^2)*t+T*(r+0.5*X.^2))./(X*sqrt(T-t));
d2=(log(x0/K)+X*B_t+(Y-r)*t+T*(r-0.5*X.^2))./(X*sqrt(T-t));
p=x0*exp(X*B_t+(Y-0.5*X.^2)*t).*normcdf(d1,0,1)-K*exp(-r*(T-t))
*normcdf(d2,0,1);
colormap(pink)
surfc(X,Y,p)
xlabel('sigma')
ylabel('alpha')
shading flat
```

The comments in the above code refer to a movie. What is being shown in that movie is the graph of the call price (with $t = 0.75$, $r = 0.02$, $\alpha = 0.3$, $\sigma = 0.15$, $T = 1$, $K = 4$ and $x_0 = 10$) as the value of B_t is reduced from 0.5 to -0.5 in increments of 0.001. In other words, the movie tries to show the transition from a graph such as in figure 2 to a graph with the reverse orientation, as was discussed previously. To view this movie, run the following m-file in MATLAB: (it calls on `movie_callprice.m`, so make sure you give your copy of MATLAB access to that first)

```
% play_movie.m
for k=1:1000
```

```
    movie_callprice(0.75,0.02,0.03,0.15,1,4,10,0.5-0.001*k);
    xlim([0.1 0.2])
    ylim([0.02 0.04])
    M(k)=getframe;
end
movie(M)
```

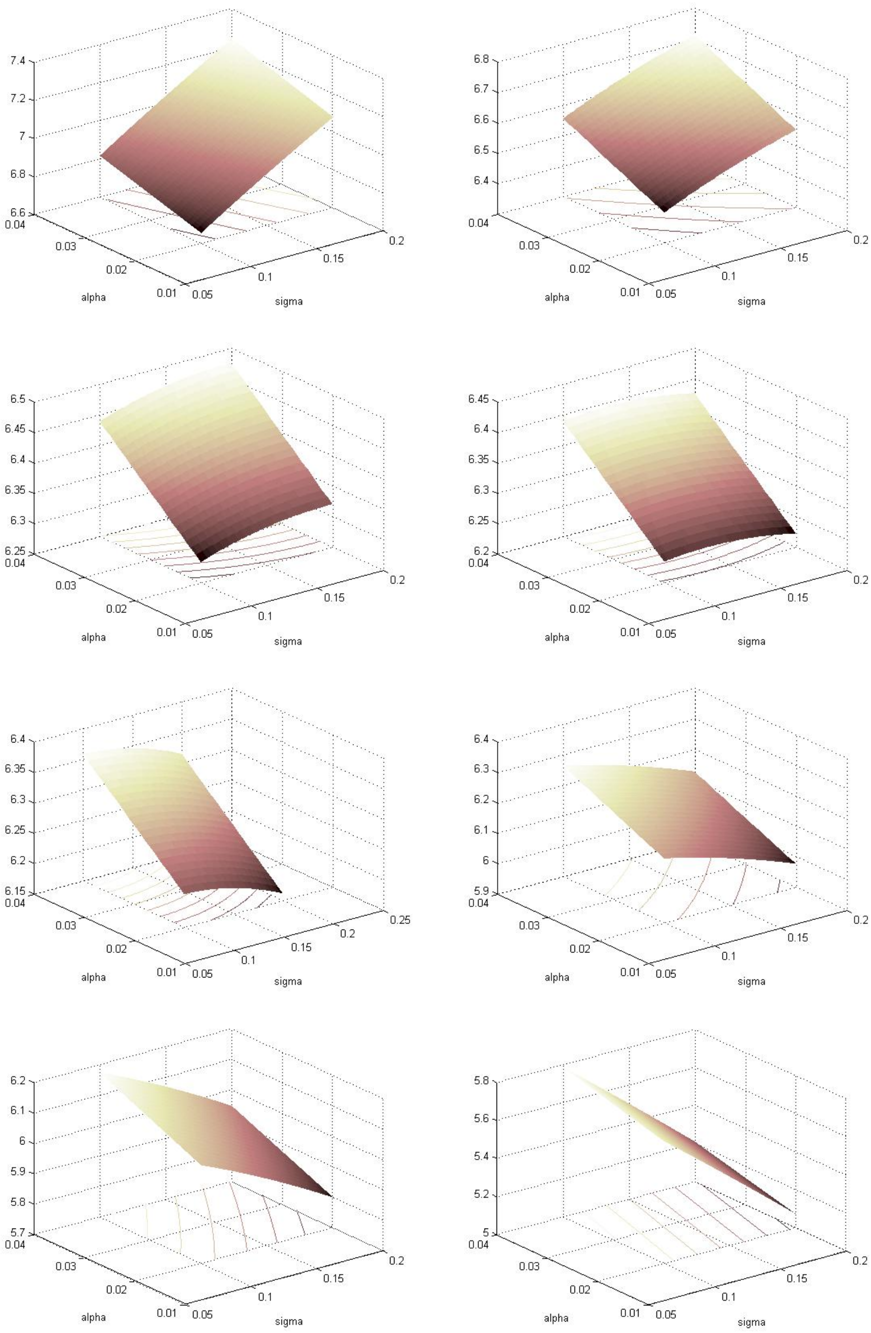

FIGURE 6. A collection of movie snapshots.

Figure 6 shows a sequence of snapshots from the movie. These snapshots are to be read linewise, from the top left to the bottom right, and were selected to demonstrate how the shape of the graph changes when the value of B_t decreases. By looking at the level curves, we can see how the graph rotates 180 degrees and thus changes orientation, which is exactly what we wanted the movie to show. For reference, the values of B_t in these snapshots were (in order) 0.500, 0.260, 0.148, 0.100, 0.070, -0.022, -0.113 and -0.498.

Now seems like a good time to formulate a hypothesis based on our experimental findings so far:

Observation 3.3. *For each collection of input parameters* $(t, r, \alpha, \sigma, T, K, x_0)$ *chosen to represent a realistic market model, there exist numbers* L^+ *and* L^-*, such that the call price is an increasing function of* σ *when* $B_t \geq L^+$*, and a decreasing function of* σ *when* $B_t \leq L^-$*.*

Unfortunately, at this point we only have circumstantial evidence to support this claim.

3.2. **Searching for more evidence.**

In equation (3.2), we differentiated the call price with regard to the volatility, keeping the stock price x fixed. Now we generalize this result by differentiating the expression in equation (3.8) instead. That is, we differentiate the call price without fixing x:

Definition 3.4. $\tilde{\nu}$, the *adjusted vega*, is given by

$$\tilde{\nu} = \frac{\partial \left(p(t, X_t)\right)}{\partial \sigma} \quad . \tag{3.9}$$

Now let us perform this differentiation and see where it leads us:

$$\begin{aligned} \frac{\partial \left(p(t, X_t)\right)}{\partial \sigma} &= x_0 \cdot (B_t - \sigma t)\, e^{\sigma B_t + \left(\alpha - \frac{1}{2}\sigma^2\right)t} \cdot \phi\left(d_1(t, X_t)\right) \\ &+ \frac{x_0}{\sqrt{2\pi}} e^{\sigma B_t + \left(\alpha - \frac{1}{2}\sigma^2\right)t - \frac{1}{2}(d_1(t,X_t))^2} \cdot \frac{\partial}{\partial \sigma}\left(d_1(t, X_t)\right) \\ &- \frac{K}{\sqrt{2\pi}} e^{-r(T-t) - \frac{1}{2}(d_2(t,X_t))^2} \cdot \frac{\partial}{\partial \sigma}\left(d_2(t, X_t)\right) \end{aligned}$$

Here, $d_1(t, X_t)$ and $d_2(t, X_t)$ are as given in equations (3.6) and (3.7), respectively. We differentiate $d_1(t, X_t)$ first:

$$
\begin{aligned}
\frac{\partial}{\partial\sigma}\left(d_1(t,X_t)\right) &= \frac{\left(B_t - 2\sigma t + \sigma T\right)\sigma\sqrt{T-t}}{\sigma^2(T-t)} \\
&\quad - \frac{\left(ln\left(\frac{x_0}{K}\right) + \sigma B_t + \left(\alpha - r - \sigma^2\right)t + T\left(r + \frac{\sigma^2}{2}\right)\right)\sqrt{T-t}}{\sigma^2(T-t)} \\
&= \frac{B_t\sigma - 2\sigma^2 t + \sigma^2 T - ln\left(\frac{x_0}{K}\right) - \sigma B_t + \left(\sigma^2 + r - \alpha\right)t - T\left(r + \frac{\sigma^2}{2}\right)}{\sigma^2\sqrt{T-t}} \\
&= \frac{\sigma^2\left(\frac{T}{2} - t\right) + (r-\alpha)t - rT - ln\left(\frac{x_0}{K}\right)}{\sigma^2\sqrt{T-t}}
\end{aligned}
$$

Then we sink our teeth into $d_2(t, X_t)$:

$$
\begin{aligned}
\frac{\partial}{\partial\sigma}\left(d_2(t,X_t)\right) &= \frac{\left(B_t - \sigma T\right)\sigma\sqrt{T-t}}{\sigma^2(T-t)} \\
&\quad - \frac{\left(ln\left(\frac{x_0}{K}\right) + \sigma B_t + (\alpha - r)t + T\left(r - \frac{\sigma^2}{2}\right)\right)\sqrt{T-t}}{\sigma^2(T-t)} \\
&= \frac{B_t\sigma - \sigma^2 T - ln\left(\frac{x_0}{K}\right) - \sigma B_t - (\alpha - r)t - T\left(r - \frac{\sigma^2}{2}\right)}{\sigma^2\sqrt{T-t}} \\
&= \frac{\sigma^2\left(-\frac{T}{2}\right) + (r-\alpha)t - rT - ln\left(\frac{x_0}{K}\right)}{\sigma^2\sqrt{T-t}}
\end{aligned}
$$

The complexity of these expressions, and the presence of ϕ in particular, suggests that getting information using the standard analytical techniques is going to be a rough ride. So we once again turn to trusty old MATLAB for answers. The following code plots $\tilde{\nu}$ as a function of σ, using $t, r, \alpha, \sigma, T, K, x_0$ and B_t as input parameters:

```
%adjustedvega.m
%plots the graph of the derivative of the call price with
%regard to sigma
function v=adjustedvega(t,r,alpha,sigma,T,K,x0,B_t)
sigma_axis=(sigma*(2/3)):0.001:(sigma*(4/3));
d1=(log(x0/K)+sigma_axis*B_t+(alpha-r-sigma_axis.^2)*t
   +T*(r+0.5*sigma_axis.^2))./(sigma_axis*sqrt(T-t));
d2=(log(x0/K)+sigma_axis*B_t+(alpha-r)*t
   +T*(r-0.5*sigma_axis.^2))./(sigma_axis*sqrt(T-t));
diff_d1=((T/2-t)*sigma_axis.^2+(r-alpha)*t-r*T-log(x0/K))
        ./(sqrt(T-t)*sigma_axis.^2);
diff_d2=((-T/2)*sigma_axis.^2+(r-alpha)*t-r*T-log(x0/K))
        ./(sqrt(T-t)*sigma_axis.^2);
v1=(B_t-sigma_axis*t).*exp(sigma_axis*B_t
   +(alpha-0.5*sigma_axis.^2)*t).*normcdf(d1,0,1);
```

```
v2=(1/sqrt(2*pi))*exp(sigma_axis*B_t
   +(alpha-0.5*sigma_axis.^2)*t).*exp(-0.5*d1.^2).*diff_d1;
v3=K*exp(-r*(T-t))*(1/sqrt(2*pi))*exp(-0.5*d2.^2).*diff_d2;
v=x0*(v1+v2)-v3;
plot(sigma_axis,v)
xlabel('sigma')
ylabel('adjusted vega')
```

In the following visual presentation, each of the movie frames from figure 6 (along with two reference graphs depicting the call price when $B_t \geq L^+$ and when $B_t \leq L^-$) is shown together with its corresponding $\tilde{\nu}$. So all graphs are made using the parameter values $t = 0.75$, $r = 0.02$, $\alpha = 0.3$, $\sigma = 0.15$, $T = 1$, $K = 4$ and $x_0 = 10$. The value of B_t is as given in the captions.

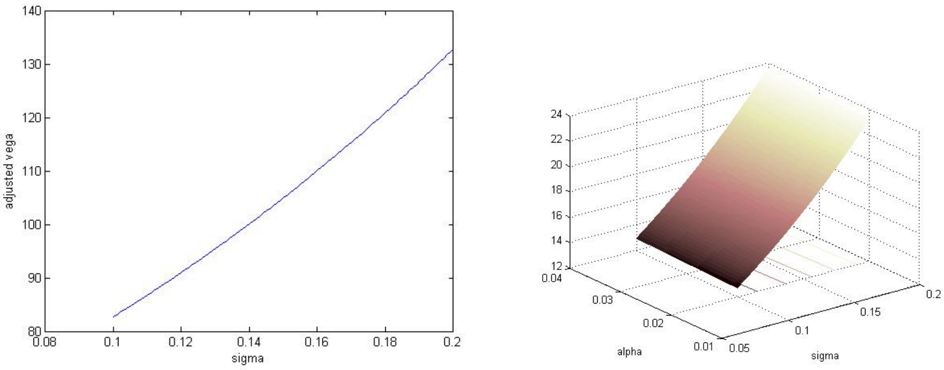

FIGURE 7. $B_t = 5$

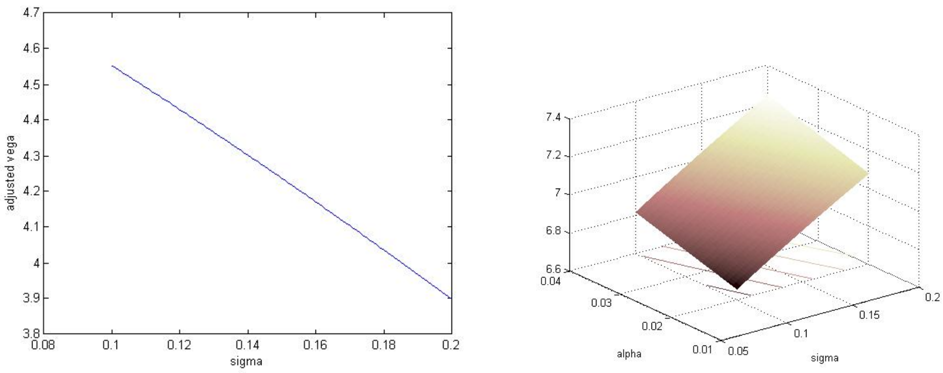

FIGURE 8. $B_t = 0.500$

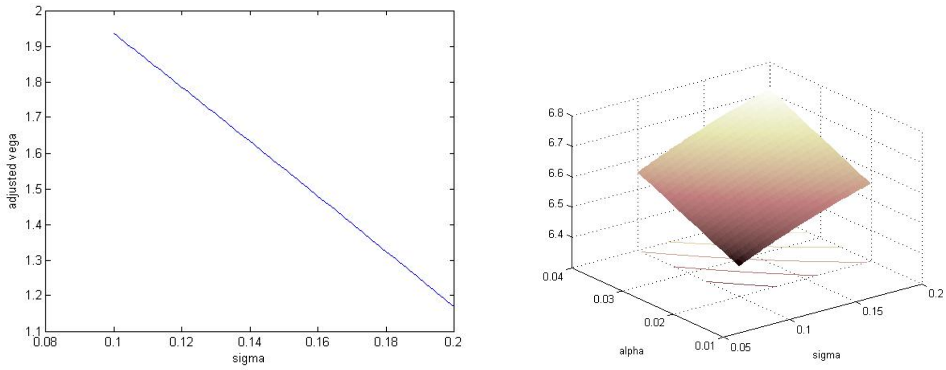

FIGURE 9. $B_t = 0.260$

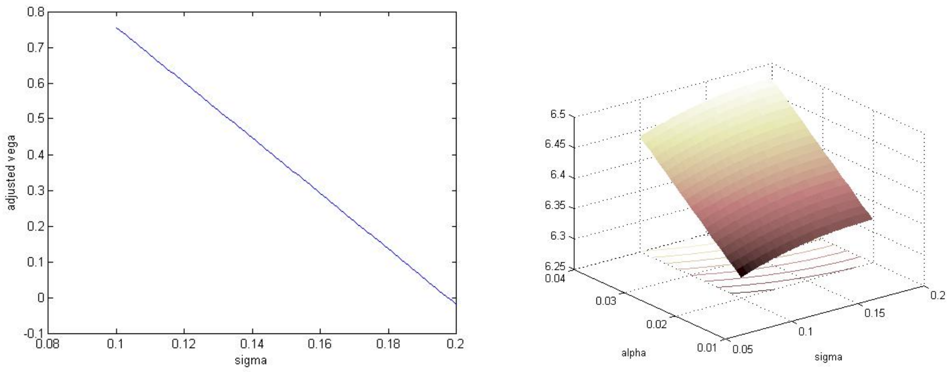

FIGURE 10. $B_t = 0.148$

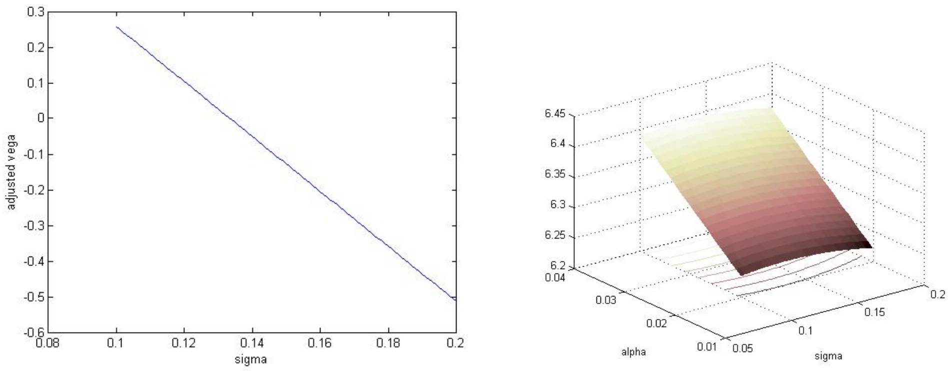

FIGURE 11. $B_t = 0.100$

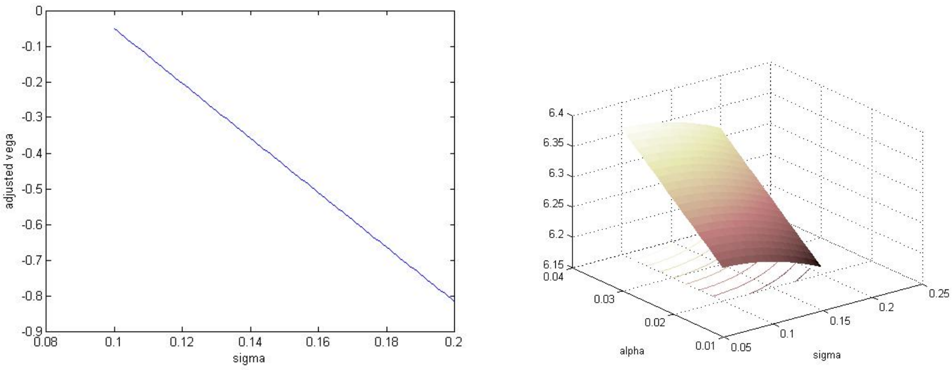

FIGURE 12. $B_t = 0.070$

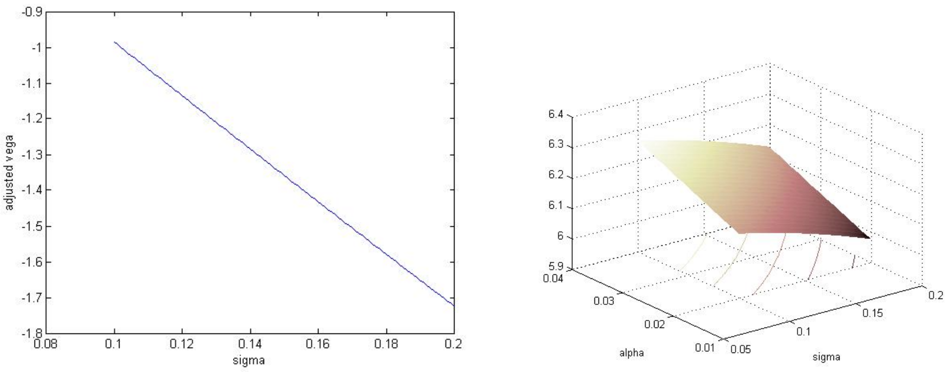

FIGURE 13. $B_t = -0.022$

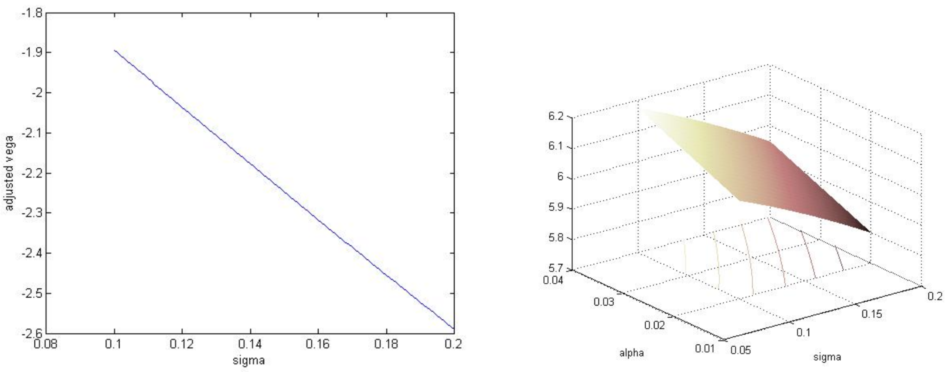

FIGURE 14. $B_t = -0.113$

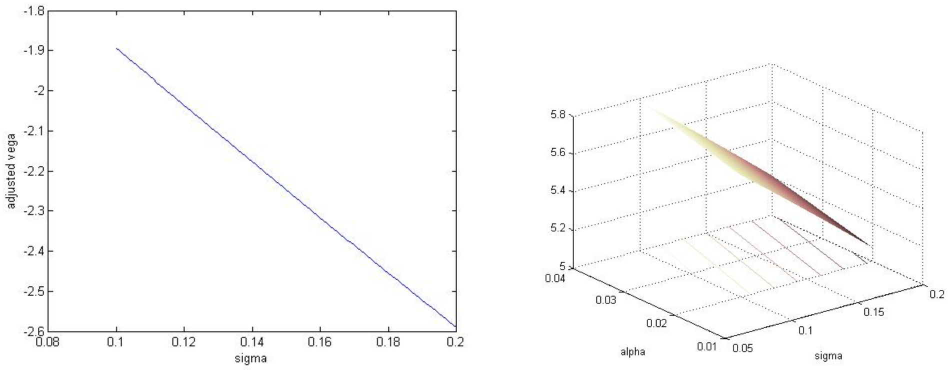

FIGURE 15. $B_t = -0.498$

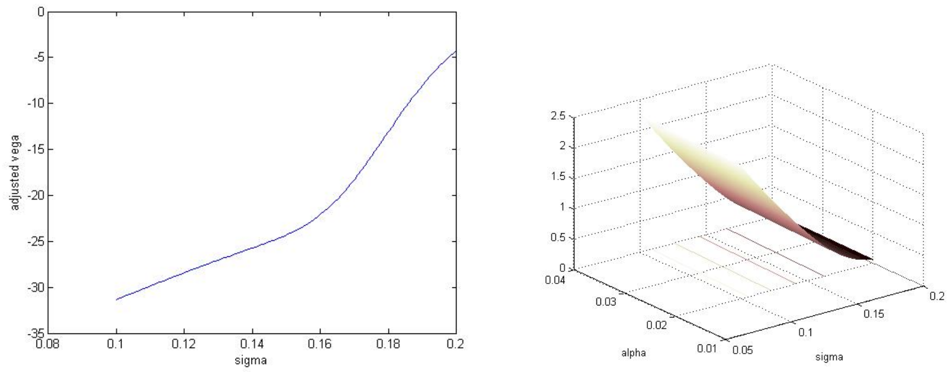

FIGURE 16. $B_t = -5$

Observe that $\tilde{\nu}$ is positive in figures 7 through 10 and negative in figures 12 through 16. Since the figures are shown in order of decreasing B_t-values, this means that if we choose $L^+ = 0.148$ and $L^- = 0.070$, the call price is an increasing function of σ for $B_t \geq L^+$ and a decreasing function of σ for $B_t \leq L^-$. So the claim put forth in observation (3.3) is (experimentally) verified.

We would also like to know at what value of B_t the graph of the call price changes orientation. So `adjustedvega.m` was modified in order to plot $\tilde{\nu}$ as a function of B_t, this time keeping σ fixed. Figure 17 shows the result of running this modified code, appropriately named `adjustedadjustedvega.m`, once again using the parameter values $t = 0.75$, $r = 0.02$, $\alpha = 0.3$, $\sigma = 0.15$, $T = 1$, $K = 4$ and $x_0 = 10$. The additional parameters `Bt_min` and `Bt_max` were chosen to be -0.5 and 0.5, respectively.

```
%adjustedadjustedvega.m
%plots the graph of the derivative of the call price
%with regard to sigma,
%using B_t as the variable, fixing sigma.
```

```
%B_t varies from Bt_min to Bt_max
function
v=adjustedadjustedvega(t,r,alpha,sigma,T,K,x0,Bt_min,Bt_max)
Bt_axis = Bt_min:0.001:Bt_max;
d1=(log(x0/K)+sigma*Bt_axis+(alpha-r-sigma^2)*t
   +T*(r+0.5*sigma^2))/(sigma*sqrt(T-t));
d2=(log(x0/K)+sigma*Bt_axis+(alpha-r)*t
   +T*(r-0.5*sigma^2))/(sigma*sqrt(T-t));
diff_d1=((T/2-t)*sigma^2+(r-alpha)*t
        -r*T-log(x0/K))/(sqrt(T-t)*sigma^2);
diff_d2=((-T/2)*sigma^2+(r-alpha)*t
        -r*T-log(x0/K))/(sqrt(T-t)*sigma^2);
v1=(Bt_axis-sigma*t).*exp(sigma*Bt_axis
   +(alpha-0.5*sigma^2)*t).*normcdf(d1,0,1);
v2=(1/sqrt(2*pi))*exp(sigma*Bt_axis+(alpha-0.5*sigma^2)*t)
   .*exp(-0.5*d1.^2)*diff_d1;
v3=K*exp(-r*(T-t))*(1/sqrt(2*pi))*exp(-0.5*d2.^2)*diff_d2;
v=x0*(v1+v2)-v3;
plot(Bt_axis,v)
xlabel('B_t')
ylabel('adjusted vega')
```

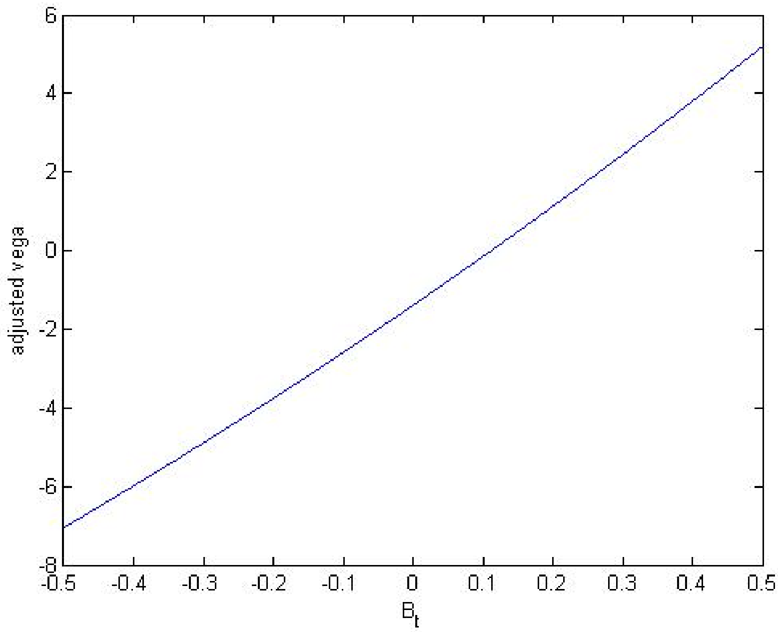

FIGURE 17. Graph produced by `adjustedadjustedvega.m`.

We see from figure 17 that $\tilde{\nu} = 0$ when B_t is approximately 0.11, so that is where the graph of the call price changes orientation, using these specific input parameters. However, we would like an answer that is a bit more accurate than that, so the following lines of MATLAB code was added to `adjustedadjustedvega.m`:

```
%these additional lines of code find the approximate
%value of B_t for which the adjusted vega equals zero:
min_value=min(abs(v));
found=0;
counter=0;
  while
    found == 0
    counter=counter+1;
    if abs(v(counter)) == min_value
      found=1;
    end
  end
root=Bt_min+(counter-1)*0.001
```

This is a simple algorithm that finds the smallest root (in case there are more than one) of the graph shown in figure 17. Using it gave us this answer:

```
>> adjustedadjustedvega(0.75,0.02,0.3,0.15,1,4,10,-0.5,0.5);

root =
    0.1120
```

Since we know how B_t is distributed, this information lets us calculate with what probability the call price will decrease, given a slightly greater market volatility than first anticipated. This gives a distinct advantage compared to only using the traditional method of calculating the vega, where the situation of a decreasing call price with increasing volatility never comes up.

4. Modelling falls in the market

In this section we investigate what happens if equation (1.2) is correct, apart from the possibility of a sudden fall in the market. Let us start by defining a few new concepts:

Definition 4.1. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space. An $\mathcal{F}_t$-adapted process $\{\eta(t)\}_{t\geq 0} = \{\eta_t\}_{t\geq 0} \subseteq \mathbb{R}$ with $\eta_0 = 0$ a.s. is called a *Lévy process* if η_t is continuous in probability and has stationary, independent increments.

Theorem 4.2. *Let $\{\eta_t\}$ be a Lévy process. Then η_t has a cadlag (right continuous with left limits) version which is also a Lévy process.*

Proof. See Protter [15] or Sato [17]. □

Due to this result, we will assume that the Lévy processes we work with are cadlag.

Definition 4.3. The *jump* of η_t at $t \geq 0$ is defined by

$$\Delta\eta_t = \eta_t - \eta_{t^-} \ . \tag{4.1}$$

Definition 4.4. Let $\mathbf{B}_0$ be the family of Borel sets $U \subset \mathbb{R}$ whose closure $\bar{U}$ does not contain 0. For $U \in \mathbf{B}_0$ we define

$$N(t,U) = N(t,U,\omega) = \sum_{s:\,0<s\leq t} \mathcal{X}_U(\Delta\eta_s) \ . \tag{4.2}$$

So $N(t,U)$ is the number of jumps of size $\Delta\eta_s \in U$ which occur before or at time t. $N(t,U)$ is called the *Poisson random measure (or jump measure)* of $\eta(\cdot)$. It is written in differential form as $N(dt,dz)$.

It is well-known that the Brownian motion B_t has stationary and independent increments (see for example Øksendal [13]), and therefore is a Lévy process. But since Brownian motion has continuous paths, we need a different kind of process in order to construct a market model that allows these sudden market falls, modelled as discontinuous jumps. Enter the *Poisson process*:

Definition 4.5. A *Poisson process* $\eta(t)$ with intensity $\lambda > 0$ is a Lévy process taking values in $\mathbb{N} \cup \{0\}$, such that

$$P\left[\eta(t) = n\right] = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \ ; \qquad n = 0, 1, 2, ... \tag{4.3}$$

There are several other (equivalent) ways to define Poisson processes. This next definition will be useful to us later on:

Definition 4.6. A *Poisson process* is a stochastic process $\{\eta(t)\}_{t\geq 0}$, with independent, stationary, Poisson-distributed increments and with $\eta(0) = 0$. In other words,

(i) the increments $\{\eta(t_k) - \eta(t_{k-1})\}_{1\leq k\leq n}$ are independent random variables for all $0 \leq t_0 \leq t_1 \leq \cdots \leq t_{n-1} \leq t_n$ and all n.

(ii) $\eta(0) = 0$ and there exists $\lambda > 0$ such that

$$\eta(t) - \eta(s) \in Po\left(\lambda(t-s)\right), \quad \text{for } 0 \leq s < t \ .$$

The constant λ is called the *intensity* of the process.

With the help of the Poisson process, we can now try to present a model that takes the possibility of sudden market falls into account, by replacing equation (1.2) with

$$dZ(t) = \alpha Z(t^-)\,dt + \sigma Z(t^-)\,dB_t - \gamma Z(t^-)\,d\eta_t \ , \quad Z_0 = x_0 \ , \tag{4.4}$$

where $0 < \gamma < 1$ and η is a standard Poisson process with intensity λ and jumps of size 1. Note that the reason for using $Z(t^-)$ instead of plain old Z_t is that we now allow discontinuities in our processes. Since we want the Poisson process to represent that the market suddenly drops in value by a certain percentage, we want this to be a percentage of the market value *before* the drop, not after.

Before attempting to solve equation (4.4), we investigate how to integrate with respect to a Poisson process:

Let η be a Poisson process with intensity λ. This process only has jumps of size 1, and so the stochastic measure $N(t, U, \omega)$ is given by

$$N(t, U, \omega) = \eta(t, \omega)\mathcal{X}_U(1) \ .$$

(The Poisson process η counts the number of jumps, and the characteristic function $\mathcal{X}_U$ checks if U contains 1, the only valid jump size.) This means that

$$\int g(z)\, N(t, dz, \omega) = g(1)\eta(t, \omega) \ . \tag{4.5}$$

We can also view N as a stochastic measure on $[0, \infty) \times \mathbb{R}$. This measure is generated by

$$N\left((s, t] \times U, \omega\right) = \left(\eta(t) - \eta(s)\right)\mathcal{X}_U(1) \ ,$$

and we have

$$\int g(t, z)\, N(dt, dz, \omega) = \int g(t, 1)\, d\eta_t = \sum g(t, 1)\Delta\eta_t \ . \tag{4.6}$$

When attempting to solve equation (4.4), we set $g(t, z)$ equal to the constant $-\gamma$ and get

$$\begin{aligned} dZ(t) &= Z(t^-)\left[\alpha\, dt + \sigma\, dB_t - \gamma\, d\eta_t\right] \\ &= Z(t^-)\left[\alpha\, dt + \sigma\, dB_t - \int_{\mathbb{R}} \gamma\, N(dt, dz)\right] \ . \end{aligned} \tag{4.7}$$

The process $Z(t)$ given by the stochastic differential equation (4.7) is an example of an *Itô-Lévy process.* We will use this term to describe stochastic integrals of the form

(4.8)
$$Z(t) = Z(0) + \int_0^t \alpha(s, \omega)\, ds + \int_0^t \sigma(s, \omega)\, dB_s + \int_0^t \int_{\mathbb{R}} \gamma(s, z, \omega)\, N(ds, dz)$$

or, equivalently

$$dZ(t) = \alpha(t)\,dt + \sigma(t)\,dB_t + \int_{\mathbb{R}} \gamma(t,z)\,N(dt,dz)\ , \tag{4.9}$$

where the integrands satisfy the appropriate conditions for the integrals to exist.

When finding an explicit solution to equation (1.2), we made use of Itô's formula. Does a similar formula exist for Itô-Lévy processes? If $f:\mathbb{R}^2 \to \mathbb{R}$ is a C^2 function, is the process $Y(t) := f(t, Z(t))$ again an Itô-Lévy process and if so, how do we represent it in the form given in equation (4.9)?

To answer this we make a heuristic argument where we let $Z^{(c)}(t)$ be the continuous part of $Z(t)$. In other words, $Z^{(c)}(t)$ is obtained by removing the jumps from $Z(t)$. Then an increment in $Y(t)$ comes from an increment in $Z^{(c)}(t)$ plus the jumps (coming from $N(\cdot,\cdot)$). So in view of the classical Itô formula, a natural guess would be that

$$\begin{aligned} dY(t) = {} & \frac{\partial f}{\partial t}(t,Z(t))\,dt + \frac{\partial f}{\partial x}(t,Z(t))\,dZ^{(c)}(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,Z(t))\cdot\left(dZ^{(c)}(t)\right)^2 \\ & + \int_{\mathbb{R}} \left(f\left(t, Z(t^-)+\gamma(t,z)\right) - f\left(t, Z(t^-)\right)\right) N(dt,dz)\ . \end{aligned}$$

It is possible to prove that this guess is correct, so we end up with the following result:

Theorem 4.7. ***(The Itô formula for Itô-Lévy processes)***
Suppose that $Z(t) \in \mathbb{R}$ is an Itô-Lévy process of the form

$$dZ(t) = \alpha(t,\omega)\,dt + \sigma(t,\omega)\,dB_t + \int_{\mathbb{R}} \gamma(t,z,\omega)\,N(dt,dz)\ . \tag{4.10}$$

Let $f \in C^2\left(\mathbb{R}^2\right)$ and define $Y(t) = f(t,Z(t))$. Then $Y(t)$ is also an Itô-Lévy process and

$$\begin{aligned} dY(t) = {} & \frac{\partial f}{\partial t}(t,Z(t))\,dt + \frac{\partial f}{\partial x}(t,Z(t))\left[\alpha(t,\omega)\,dt + \sigma(t,\omega)\,dB_t\right] \\ & + \frac{1}{2}\sigma^2(t,\omega)\frac{\partial^2 f}{\partial x^2}(t,Z(t))\,dt \\ & + \int_{\mathbb{R}} \left(f\left(t, Z(t^-)+\gamma(t,z)\right) - f\left(t, Z(t^-)\right)\right) N(dt,dz)\ . \end{aligned} \tag{4.11}$$

Proof. See Bensoussan and Lions [2], Applebaum [1] or Protter [15]. □

We now have all the necessary tools to obtain an explicit solution to equation (4.4), so let's get down to business:

The stochastic differential equation we wish to solve is

$$dZ(t) = Z(t^-)\left[\alpha\,dt + \sigma\,dB_t - \int_{\mathbb{R}} \gamma\,N(dt,dz)\right]\ , \tag{4.12}$$

where α, σ and γ are constants, $0 < \gamma < 1$. We start by rewriting this equation as

$$\frac{dZ(t)}{Z\left(t^{-}\right)} = \alpha\, dt + \sigma\, dB_t + \int_{\mathbb{R}} (-\gamma)\, N(dt, dz) \ .$$

Then we define $Y(t) = ln\, Z(t)$ and use the Itô formula for Itô-Lévy processes:

$$\begin{aligned} dY(t) = \frac{1}{Z(t)} \big(Z(t^{-})\alpha\, dt + Z(t^{-})\sigma\, dB_t\big) + \frac{1}{2} Z(t^{-})^2 \cdot \sigma^2 \cdot (-1) \cdot \frac{1}{Z(t)^2}\, dt \\ + \int_{\mathbb{R}} ln\left(Z(t^{-}) - \gamma Z(t^{-})\right) - ln\left(Z(t^{-})\right) N(dt, dz) \end{aligned}$$

Since the first two terms of this expression only deal with the continuous part of $Z(t)$, we have that $Z(t) = Z(t^{-})$ in these terms. We keep this in mind when simplifying:

$$\begin{aligned} dY(t) &= \frac{1}{Z(t)} \left(Z(t)\alpha\, dt + Z(t)\sigma\, dB_t\right) + \frac{1}{2} Z(t)^2 \cdot \sigma^2 \cdot (-1) \cdot \frac{1}{Z(t)^2}\, dt \\ &\quad + \int_{\mathbb{R}} ln\left(Z(t^{-}) - \gamma Z(t^{-})\right) - ln\left(Z(t^{-})\right) N(dt, dz) \\ &= \alpha\, dt + \sigma\, dB_t - \frac{1}{2}\sigma^2 dt + \int_{\mathbb{R}} ln\left(\frac{Z(t^{-}) - \gamma Z(t^{-})}{Z(t^{-})}\right) N(dt, dz) \\ &= \left(\alpha - \frac{1}{2}\sigma^2\right) dt + \sigma\, dB_t + \int_{\mathbb{R}} ln\,(1-\gamma)\, N(dt, dz) \end{aligned}$$

or, equivalently

$$Y(t) = Y(0) + \left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B_t + \int_0^t \int_{\mathbb{R}} ln\,(1-\gamma)\, N(ds, dz)$$

Since we defined $Y(t)$ to be the logarithm of $Z(t)$, we get

$$\begin{aligned} Z(t) &= x_0 \cdot exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B_t + \int_0^t \int_{\mathbb{R}} ln\,(1-\gamma)\, N(ds, dz)\right) \\ &= x_0 \cdot exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma B_t + \sum_{s=0}^{t} ln\,(1-\gamma)\, \Delta\eta_s\right) \ . \end{aligned} \tag{4.13}$$

We summarize these results:

Proposition 4.8. *The stochastic differential equation given by*

$$dZ(t) = \alpha Z(t^{-})\, dt + \sigma Z(t^{-})\, dB_t - \gamma Z(t^{-})\, d\eta_t \ , \quad Z_0 = x_0 \ , \tag{4.14}$$

where $0 < \gamma < 1$ and η is a standard Poisson process with intensity λ and jumps of size 1, has the solution

$$Z(t) = x_0 \cdot exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma B_t + \sum_{s=0}^{t} ln\,(1-\gamma)\,\Delta\eta_s\right). \tag{4.15}$$

5. The return of the equivalent martingale measures

We start this section with a theorem describing a *complete market*:

Theorem 5.1. ***(Complete financial markets)***
For a financial market $V = (S_t, X_t)$ where there are no arbitrage opportunities (NAO), the following two statements are equivalent:

(i) *There is exactly one equivalent martingale measure Q.*

(ii) *Each $f \in L^\infty(\Omega, \mathcal{F}, P)$ may be represented as $f = a + (\theta \cdot V)_T$ for some $a \in \mathbb{R}$ and portfolio θ.*

In this case, $a = E_Q[f]$, the stochastic integral $\theta \cdot V$ is unique, and we have that

$$E_Q\left[f \mid \mathcal{F}_t\right] = E_Q[f] + (\theta \cdot V)_t\,, \quad t \in [0,T]\,. \tag{5.1}$$

Proof. See Delbaen and Schachermayer [7]. The proof presented therein assumes a finite probability space Ω and is done is discrete time, but the result can be generalized. □

For a given $f \in L^\infty(\Omega, \mathcal{F}, P)$, the constant a in Theorem 5.1 is called an *arbitrage-free price*, if in addition to the market V, the introduction of the contingent claim f at price a does not create an opportunity for arbitrage.

In section 2 we worked with a complete market, so the equivalent martingale measure we found was unique. However, after introducing the possible market falls by adding a Poisson process to the model in section 4, our market is no longer necessarily complete. This means that there may be several equivalent martingale measures to be found, each possibly leading to a different suggestion regarding how to correctly price the call option.

Our next theorem makes this discussion a bit more precise:

Theorem 5.2. *Assume that the market $V_t = (S_t, X_t)$ satisfies (NAO) and let $f \in L^\infty(\Omega, \mathcal{F}, P)$. Define*

$$\underline{\pi}(f) = \inf\left\{\, E_Q[f] \mid Q \textit{ is an equivalent martingale measure}\,\right\} \tag{5.2}$$

and

$$\overline{\pi}(f) = \sup\left\{\, E_Q[f] \mid Q \textit{ is an equivalent martingale measure}\,\right\} \tag{5.3}$$

Either $\underline{\pi}(f) = \overline{\pi}(f)$*, in which case* f *is attainable at price* $\pi(f) := \underline{\pi}(f) = \overline{\pi}(f)$*, meaning that* $f = \pi(f) + (\theta \cdot V)_T$ *for some portfolio* θ*, and therefore* $\pi(f)$ *is the unique aribitrage-free price for* f*.*

Or $\underline{\pi}(f) < \overline{\pi}(f)$*, in which case*

$$]\underline{\pi}(f), \overline{\pi}(f)[= \{ E_Q[f] \mid Q \textit{ is an equivalent martingale measure} \}$$

and a *is an arbitrage-free price for* f *if and only if* $a \in]\underline{\pi}(f), \overline{\pi}(f)[$ *.*

Proof. A proof assuming discrete time and finite Ω can be found in Delbaen and Schachermayer [7]. Once again, the result can be generalized. □

Now let us start the hunt for an equivalent martingale measure that fits the market model we used in section 4. For your convenience, the equations describing that model are repeated here:

$$S_t = S_0 e^{rt} \tag{5.4}$$

$$dZ(t) = Z(t^-) \left[\alpha \, dt + \sigma \, dB_t - \int_{\mathbb{R}} \gamma \, N(dt, dz) \right] \tag{5.5}$$

Once again, we let $S_0 = 1$, so that $S_t = e^{rt}$. The discounted stock price is now given by

$$\tilde{Z}_t = \frac{Z_t}{S_t} = e^{-rt} Z_t \ .$$

We use Itô's formula for Itô-Lévy processes (theorem 4.7) with $f(t, x) = x \cdot e^{-rt}$ and get

$$\begin{aligned} d\tilde{Z}(t) &= -rZ(t)e^{-rt} \, dt + e^{-rt} \left[\alpha Z(t^-) \, dt + \sigma Z(t^-) \, dB_t \right] \\ &\quad + \int_{\mathbb{R}} e^{-rt} \left(Z(t^-) - \gamma Z(t^-) \right) - e^{-rt} Z(t^-) \, N(dt, dz) \\ &= e^{-rt} Z(t^-) \left[(\alpha - r) \, dt + \sigma \, dB_t + \int_{\mathbb{R}} -\gamma \, N(dt, dz) \right] \ . \end{aligned}$$

This means that the stochastic differential equation describing the discounted stock price is

$$\begin{aligned} d\tilde{Z}(t) &= \tilde{Z}(t^-) \left[(\alpha - r) \, dt + \sigma \, dB_t + \int_{\mathbb{R}} -\gamma \, N(dt, dz) \right] \\ &= \tilde{Z}(t^-) \left[(\alpha - r) \, dt + \sigma \, dB_t - \gamma \, d\eta_t \right] \ , \end{aligned} \tag{5.6}$$

where η is still a standard Poisson process with intensity λ and jumps of size 1.

We know from section 2 how to make the continuous part of equation (5.6) into a martingale. Unfortunately, the jump term of said equation complicates matters, since η_t is not a martingale. However, by introducing what is known as a compensated Poisson process, we can still find an equivalent martingale measure for this market.

Let us start by writing the Poisson process $\eta(t)$ as

$$\eta(t) = \xi t + [\eta(t) - \xi t] = \xi t + \tilde{\eta}(t) \ ,$$

where ξ is a constant. Our goal now is to choose a value for ξ so that the *compensated Poisson process* $\tilde{\eta}(t) = \eta(t) - \xi t$ becomes a martingale. In other words, we want the conditional expectation $E[\,\tilde{\eta}(t) \mid \mathcal{F}_s]$ to equal $\tilde{\eta}(s)$ for $t \geq s$.

We have that

$$\begin{aligned} E[\,\tilde{\eta}(t) \mid \mathcal{F}_s] &= E[\,\tilde{\eta}(t) - \tilde{\eta}(s) + \tilde{\eta}(s) \mid \mathcal{F}_s] \\ &= E[\,\tilde{\eta}(t) - \tilde{\eta}(s) \mid \mathcal{F}_s] + E[\,\tilde{\eta}(s) \mid \mathcal{F}_s] \ , \end{aligned}$$

and since $E[\,\tilde{\eta}(s) \mid \mathcal{F}_s] = \tilde{\eta}(s)$, we end up with $\tilde{\eta}$ being a martingale as long as

$$E[\,\tilde{\eta}(t) - \tilde{\eta}(s) \mid \mathcal{F}_s] = [\,(\eta(t) - \eta(s)) - \xi(t-s) \mid \mathcal{F}_s] = 0 \ .$$

According to Definition 4.5, $\eta(t) - \eta(s)$ is Poisson-distributed with mean $\lambda(t-s)$, where λ is the intensity of η. Therefore, if we choose $\xi = \lambda$, our compensated Poisson process $\tilde{\eta}$ will be a martingale.

Proposition 5.3. *Let η be a Poisson process with intensity λ. Then the compensated Poisson process $\tilde{\eta}$, defined by*

$$\tilde{\eta}(t) = \eta(t) - \lambda t \ , \tag{5.7}$$

is a martingale under P.

We rewrite equation (5.6), using the compensated Poisson process:

$$\begin{aligned} d\tilde{Z}(t) &= \tilde{Z}(t^-)\,[(\alpha - r)\,dt + \sigma\,dB_t - \gamma\,d\eta_t] \\ &= \tilde{Z}(t^-)\,[(\alpha - r)\,dt + \sigma\,dB_t - \gamma\,(\lambda\,dt + d\tilde{\eta}_t)] \\ &= \tilde{Z}(t^-)\,[(\alpha - r - \lambda\gamma)\,dt + \sigma\,dB_t - \gamma\,d\tilde{\eta}_t] \end{aligned} \tag{5.8}$$

Then we use the Girsanov theorem on the continuous part of the process described in equation (5.8), by letting $\{L_t\}$ be the process satisfying

$$dL_t = -(\alpha - r - \lambda\gamma)\sigma^{-1} L_t\,dB_t \ . \tag{5.9}$$

Define Q on $\mathcal{F}_T$ by $\frac{dQ}{dP} = L_T$. Then, still according to the Girsanov theorem, the process B_t^* given by

$$B_t^* = B_t + \int_0^t (\alpha - r - \lambda\gamma)\sigma^{-1} ds = B_t + (\alpha - r - \lambda\gamma)\sigma^{-1} t$$

is a brownian motion under Q. Furthermore,

$$\begin{aligned} d\tilde{Z}(t) &= \tilde{Z}(t^-)\left[(\alpha - r - \lambda\gamma)\, dt + \sigma\, dB_t - \gamma\, d\tilde{\eta}_t\right] \\ &= \tilde{Z}(t^-)\left[(\alpha - r - \lambda\gamma)\, dt + \sigma\left(dB_t^* - (\alpha - r - \lambda\gamma)\sigma^{-1} dt\right) - \gamma\, d\tilde{\eta}_t\right] \\ &= \tilde{Z}(t^-)\left[\sigma\, dB_t^* - \gamma\, d\tilde{\eta}_t\right] \end{aligned}$$

or, equivalently

$$\tilde{Z}(t) = \tilde{Z}(0) + \int_0^t \sigma \tilde{Z}(s^-)\, dB_s^* + \int_0^t (-\gamma)\tilde{Z}(s^-)\, d\tilde{\eta}_s \ .$$

We observe that $\tilde{Z}$ is the sum of two stochastic integrals, each with regard to a martingale. This implies that the integrals themselves are also martingales, and a sum of two martingales is yet again a martingale. All in all we conclude that $\tilde{Z}$ is a martingale under Q, so that Q is an equivalent martingale measure for the market described by equations (5.4) and (5.5).

Proposition 5.4. *Let $\{L_t\}_{t\geq 0}$ be the process defined by*

$$L_t = exp\left(\int_0^t h(s)\, dB_s - \frac{1}{2}\int_0^t h^2(s)\, ds\right) ,$$

where $h(s) = -(\alpha - r - \lambda\gamma)\sigma^{-1}$, and define Q on $\mathcal{F}_T$ by $\frac{dQ}{dP} = L_T$. Then Q is an equivalent martingale measure for the market described by the stochastic differential equations

$$S_t = S_0 e^{rt}$$

and

$$dZ(t) = Z(t^-)\left[\alpha\, dt + \sigma\, dB_t - \int_{\mathbb{R}} \gamma\, N(dt, dz)\right] \ .$$

Now that we have found an equivalent martingale measure Q, we proceed in a manner similar to what was done in section 2, in an attempt to find an expression for the call price, or implicit price of $g(Z_T)$, for this market. First we express the stock price Z in terms of the compensated Poisson process $\tilde{\eta}$ and our newfound Brownian motion dB_t^*:

$$\begin{aligned} dZ(t) &= Z(t^-)\left[\alpha\, dt + \sigma\, dB_t - \gamma\, d\eta_t\right] \\ &= Z(t^-)\left[\alpha\, dt + \sigma\left(dB_t^* - (\alpha - r - \lambda\gamma)\sigma^{-1} dt\right) - \gamma\left(\lambda\, dt + d\tilde{\eta}_t\right)\right] \\ &= Z(t^-)\left[r\, dt + \sigma\, dB_t^* - \gamma\, d\tilde{\eta}_t\right] \end{aligned}$$

Like in section 2, we then proceed by constructing a portfolio

$$Y_t = \mu_t S_t + \beta_t Z_t \ .$$

Under Q, we have that

$$\begin{aligned} dY_t &= \mu_t \, dS_t + \beta_t \, dZ_t \\ &= \mu_t r S_t \, dt + \beta_t Z(t^-) \left[r \, dt + \sigma \, dB_t^* - \gamma \, d\tilde{\eta}_t \right] \\ &= r \left(\mu_t S_t + \beta_t Z_t \right) dt + \beta_t Z(t^-) \left[\sigma \, dB_t^* - \gamma \, d\tilde{\eta}_t \right] \\ &= r Y_t \, dt + dM_t \ , \end{aligned}$$

where $\{M_t\}_{t \geq 0}$ is defined by

$$dM_t = \beta_t Z(t^-) \left[\sigma \, dB_t^* - \gamma \, d\tilde{\eta}_t \right] \ .$$

Under appropriate integrability conditions, $\{M_t\}_{t \geq 0}$ is the sum of two stochastic integrals, each with regard to a martingale. This means that M itself is also a martingale.

We use Itô's formula for Itô-Lévy processes (theorem (4.7)) on Y_t with $f(t, x) = x \cdot e^{-rt}$, and get

$$\begin{aligned} d\left(e^{-rt} Y_t\right) &= -r Y_t e^{-rt} \, dt + e^{-rt} \left(r Y_t \, dt + \beta_t Z(t^-) \sigma \, dB_t^* \right) \\ &\quad + \int_{\mathbb{R}} e^{-rt} \left(Y(t^-) - \gamma \beta_t Z(t^-) \right) - e^{-rt} Y(t^-) \, N(dt, dz) \\ &= e^{-rt} \beta_t Z(t^-) \sigma \, dB_t^* + \int_{\mathbb{R}} -\gamma \beta_t e^{-rt} Z(t^-) \, N(dt, dz) \\ &= e^{-rt} \beta_t Z(t^-) \left[\sigma \, dB_t^* - \gamma \, d\tilde{\eta}_t \right] \\ &= e^{-rt} dM_t \ , \end{aligned}$$

where $\int_{\mathbb{R}} (\cdot) \, N(dt, dz)$ refers to integration with regard to the compensated Poisson process $\tilde{\eta}$. From this we see that the process $\{e^{-rt} Y_t\}_{t \geq 0}$ is a martingale. So, from the definition of a martingale, we get

$$e^{-rt} Y_t = E_Q \left[e^{-rT} Y_T \mid \mathcal{F}_t \right] \ ,$$

leading us to the same result as that given in theorem (2.3), namely that the implicit price of $g(Z_T)$ is given by

$$(5.10) \quad Y_t = p(t, Z_t) = e^{rt} \cdot E_Q \left[e^{-rT} Y_T \mid \mathcal{F}_t \right] = E_Q \left[e^{-r(T-t)} g(Z_T) \mid \mathcal{F}_t \right] \ .$$

By using the same arguments as in section 2, we can show that

$$(5.11) \qquad p(t, x) = E_P \left[e^{-r(T-t)} g\left(W^{x,t}(T) \right) \right] \ ,$$

where $W^{x,t}$ is the solution of the stochastic differential equation

$$dW^{x,t}(s) = W^{x,t}(s^-) \left[r \, ds + \sigma \, dB_s - \gamma \, d\tilde{\eta}_s \right] \ , \ W^{x,t}(t) = x \ .$$

According to proposition (4.8) we have

$$W^{x,t}(s) = x \cdot exp\left(\left(r - \frac{1}{2}\sigma^2\right)(s-t) + \sigma(B_s - B_t) + \sum_{u=t}^{s} ln\,(1-\gamma)\,\Delta\tilde{\eta}_u\right) \quad ,$$

and substitution into equation (5.11) yields

$$p(t,x) = E_P\left[e^{-r(T-t)} g\left(xe^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma(B_T-B_t)+\sum_{u=t}^{T} ln\,(1-\gamma)\,\Delta\tilde{\eta}_u}\right)\right] \quad .$$

In order to arrive at an explicit formula for p when $g(z) = (z-K)_+$, we perform probabilistic calculations like those in section 1 for each of the possible number of jumps $\tilde{\eta}$ can make during the time interval $[t,T]$. Each of these calculations will lead to an expression similar to the Black-Scholes formula, and each of these resulting expressions will be valid with a certain probability corresponding to the probability of the number of jumps that occured. The formula we end up with will be quite complicated, so it will most likely have little practical use.

References

[1] D. Applebaum: *Lévy Processes and Stochastic Calculus*, Cambridge University Press, Cambridge, 2003.

[2] A. Bensoussan, J.L. Lions: *Impulse Control and Quasi-Variational Inequalities*, Gauthiers-Villars, Paris, 1984.

[3] R.C. Dalang, A. Morton, W. Willinger: *Equivalent martingale measures and no arbitrage in stochastic securities market models*, Stochastics and Stochastics Reports 29, 185-202, 1989.

[4] R.-A. Dana, M. Jeanblanc: *Financial Markets in Continuous Time*, Springer-Verlag, Berlin Heidelberg, 2003.

[5] F. Delbaen, W. Schachermayer: *A general version of the fundamental theorem of asset pricing*, Mathematische Annalen 300, 463-520, 1993.

[6] F. Delbaen, W. Schachermayer: *Arbitrage and free-lunch with bounded risk for unbounded continuous processes*, Mathematical Finance 4, 343-348, 1994.

[7] F. Delbaen, W. Schachermayer: *The Mathematics of Arbitrage*, Springer-Verlag, Berlin Heidelberg, 2006.

[8] A. Gut: *An Intermediate Course in Probability*, Springer-Verlag, New York, 1995.

[9] Y. Kabanov: *Arbitrage theory. E.Jouini, J.Cvitanic, M.Musiela (eds): Option Pricing, Interest Rates and Risk Management*, Cambridge University Press, 3-42, Cambridge, 2001.

[10] N. Krylov: *Controlled Diffusion Processes*, Springer-Verlag, Berlin Heidelberg, 1980.

[11] A.J. Morton: *Arbitrage and Martingales*, Ph.D. Thesis, Cornell University, 1989.

[12] S. Müller, *Arbitrage Pricing of Contingent Claims, Lecture Notes in Economics and Mathematical Systems 254*, Springer-Verlag, Berlin Heidelberg, 1987.

[13] B. Øksendal: *Stochastic Differential Equations - An Introduction with Applications, 6th edition*, Springer-Verlag, Berlin Heidelberg, 2007.

[14] B. Øksendal, A. Sulem: *Applied Stochastic Control of Jump Diffusions*, Springer-Verlag, Berlin Heidelberg, 2005.

[15] P. Protter: *Stochastic Integration and Differential Equations, 2nd edition*, Springer-Verlag, 2003.

[16] L.C.G. Rogers, D. Williams: *Diffusions, Markov Processes and Martingales*, Wiley, New York, 1988.

[17] K. Sato: *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, 1999.

[18] S.R.S. Varadhan: *Lectures on Diffusion Problems and Partial Differential Equations*, Tata Institute of Fundamental Research, Bombay, 1980.

[19] J. Xia, J.-A. Yan: *Some remarks on arbitrage pricing theory. In: J.Yong (ed): International Conference on Mathematical Finance: Recent Developments in Mathematical Finance*, World Scientific, 218-227, Singapore, 2001.