On Backward Stochastic Partial Differential Equations

Bernt Øksendal\textsuperscript{1,2} Tusheng Zhang\textsuperscript{3}

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Abstract

We prove an existence and uniqueness result for a general class of backward stochastic partial differential equations. This is a type of equations which appear as adjoint equations in the maximum principle approach to optimal control of systems described by stochastic partial differential equations.

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1 Introduction

Let \( B_t, t \geq 0 \) be an \( m \)-dimensional Brownian motion on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\). Fix \( T > 0 \) and let \( \eta(\omega) \) be an \( \mathcal{F}_T \)-measurable random variable. Let

\[
 b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times m} \to \mathbb{R}^n
\]

be a given vector field. Consider the problem to find two \( \mathcal{F}_t \)-adapted processes \( p(t) \in \mathbb{R}^n, q(t) \in \mathbb{R}^{n \times m} \) such that

\[
\begin{align*}
 dp(t) & = b(t, p(t), q(t))dt + q(t)dB_t, t \in (0, T) \\
p(T) & = \eta \quad a.s.
\end{align*}
\]

This is a backward stochastic (ordinary) differential equation (BSDE). It is called backward because it is the terminal value \( p(T) = \eta \) that is given, not the initial value \( p(0) \). Still \( p(t) \) is required to be \( \mathcal{F}_t \)-adapted. In general this is only possible if we also are free to choose \( q(t) \) (in an \( \mathcal{F}_t \)-adapted way).

\textsuperscript{1} Department of Mathematics, University of Oslo, P.O.Box 1053 Blindern, N-0316 Oslo, Norway. Email: oksendal@math.uio.no

\textsuperscript{2} Norwegian School of Economics and Business Administration, Helleveien 30, N-5045 Bergen, Norway

\textsuperscript{3} Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, England, U.K. Email: t.zhang@maths.manchester.ac.uk
The theory of BSDEs is now well developed. See e.g. [EPQ], [MY], [PP] and [YZ] and the references therein.

There are many applications of this theory. Examples include the following:

(i) The problem of finding a replicating portfolio of a given contingent claim in a complete financial market can be transformed into a problem of solving a BSDE.

(ii) The maximum principle method for solving a stochastic control problem involves a BSDE for the adjoint processes $p(t), q(t)$.

For more information about these and other applications of BSDEs we refer to [EPQ] and [YZ] and references therein.

The purpose of this paper is to study backward stochastic partial differential equations (BSPDEs). They are defined in a similar way as BSDEs, but with the basic equation being a stochastic partial differential equation rather than a stochastic ordinary differential equation. More precisely, we will study a class of BSPDEs which includes the following:

\[
\begin{align*}
dY(t, x) &= AY(t, x)dt + b(t, x, Y(t, x), Z(t, x))dt \\
&\quad + Z(t, x)dB_t, (t, x) \in (0, T) \times \mathbb{R}^n \\
Y(T, x) &= \phi(\omega, x)
\end{align*}
\]

(1.3)

(1.4)

Here $dY(t, x)$ denotes the Itô differential with respect to $t$, while $A$ is a partial differential operator with respect to $x$.

The function $b : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given and so is the terminal value function $\phi(\omega, x)$. We assume that $\phi(\omega, x)$ is $\mathcal{F}_T$-measurable for all $x$ and that

\[
E\left[ \int_{\mathbb{R}^n} \phi(\omega, x)^2 dx \right] < \infty,
\]

(1.5)

where $E$ denotes expectation with respect to $P$. We are seeking the two processes $Y(t, x)$ and $Z(t, x)$ such that (1.3) and (1.4) hold. The processes $Y(t, x)$ and $Z(t, x)$ are assumed to be $\mathcal{F}_t$-adapted, i.e., $Y(t, x)$ and $Z(t, x)$ are $\mathcal{F}_t$-measurable for all $x \in \mathbb{R}^n$ and we also require that

\[
E\left[ \int_{\mathbb{R}^n} \int_0^T \{Y(t, x)^2 + Z(t, x)^2\} dt dx \right] < \infty.
\]

(1.6)

Equations of this type are of interest because they appear as adjoint equations in a maximum principle approach to optimal control of stochastic partial differential equations. See [B1] and [O] for details.

**Example 1.1** Consider the following BSPDE:

\[
\begin{align*}
dY(t, x) &= -\frac{1}{2} \Delta Y(t, x) dt + Z(t, x) dB_t, (t, x) \in (0, T) \times \mathbb{R}^n \\
Y(T, x) &= \phi(\omega, x)
\end{align*}
\]

(1.7)

(1.8)
Here $\Delta Y(t, x) = \sum_{i=1}^{n} \frac{\partial^2 Y(t,x)}{\partial x_i^2}$ is the Laplacian with respect to $x$ applied to $Y$, and $\phi(\omega, x)$ satisfies $E[\int_{\mathbb{R}^n} \phi(\omega, x)^2 dx] < \infty$.

In this simple case, we are able to find the solution explicitly. We first use the Itô representation theorem to write, for almost all $x$,

$$\phi(\omega, x) = h(x) + \int_0^T g(s, x, \omega) dB_s$$

(1.9)

where

$$h(x) = E[\phi(\cdot, x)],$$

(1.10)

$g(s, x, \cdot)$ is $\mathcal{F}_s$-measurable for all $s, x$ and

$$E\left[\int_{\mathbb{R}^n} \int_0^T g(s, x, \cdot)^2 ds dx\right] < \infty.$$

(1.11)

Let

$$Q_tf(x) = \int_{\mathbb{R}^n} (2\pi t)^{-\frac{n}{2}} f(y) \exp\left(-\frac{|x-y|^2}{2t}\right) dy, \quad t > 0$$

(1.12)

be the transition operator for Brownian motion defined for all measurable $f : \mathbb{R}^n \to \mathbb{R}$ such that the integral converges. Then it is well known that

$$\frac{\partial}{\partial t} (Q_tf(x)) = \frac{1}{2} \Delta (Q_tf(x))$$

(1.13)

Now define

$$Y(t, x) = Q_{T-t} \left( \int_0^t g(s, \cdot, \omega) dB_s + h(\cdot) \right)$$

(1.14)

Then

$$dY(t, x) = \left[ \int_0^t -\frac{1}{2} \Delta (Q_{T-t} g(s, \cdot, \omega))(x) dB_s - \frac{1}{2} \Delta Q_{T-t} h(\cdot)(x) \right] dt$$

(1.15)

$$+ (Q_{T-t} g(t, \cdot, \omega))(x) dB_t$$

(1.16)

$$= -\frac{1}{2} \Delta Y(t, x) dt + Z(t, x) dB_t,$$

(1.17)

where

$$Z(t, x) = (Q_{T-t} g(t, \cdot, \omega))(x).$$

(1.18)

Hence the processes $Y(t, x), Z(t, x)$ given by (1.14) and (1.18) solve the BSPDE (1.7)–(1.8).
In the general case it is not possible to find explicit solutions of a BSPDE. However, in Section 3 we will prove an existence and uniqueness result for a general class of such equations. We will achieve this by regarding the BSPDE of type (1.3)–(1.4) as a special case of a backward stochastic evolution equation for Hilbert space valued processes. This, in turn, is studied by taking finite dimensional projections and then taking the limit. This is the well known Galerkin approximation method which has been used by several authors in other connections. See e.g. [B1], [B2] and [P]. We also refer readers to [PZ] for the general theory of stochastic evolution equations on Hilbert spaces.

The rest of the paper is organized as follows: In Section 2 we give the precise framework. The main result and its proof are given in Section 3.

2 Framework

Let $V$, $H$ be two separable Hilbert spaces such that $V$ is continuously, densely imbedded in $H$. Identifying $H$ with its dual we have

$$V \subset H \cong H^* \subset V^*,$$

where $V^*$ stands for the topological dual of $V$. Let $A$ be a bounded linear operator from $V$ to $V^*$ satisfying the following coercivity hypothesis: There exist constants $\alpha > 0$ and $\lambda \geq 0$ such that

$$2\langle Au, u \rangle + \lambda ||u||^2_H \geq \alpha ||u||^2_V \quad \text{for all } u \in V,$$

where $\langle Au, u \rangle = Au(u)$ denotes the action of $Au \in V^*$ on $u \in V$.

Remark that $A$ is generally not bounded as an operator from $H$ into $H$. Let $K$ be another separable Hilbert space. Let $\{B_t, t \geq 0\}$ be a cylindrical Brownian motion with covariance space $K$ on a probability space $(\Omega, \mathcal{F}, P)$, i.e., for any $k \in K, \langle B_t, k \rangle$ is a real valued-Brownian motion with $E[\langle B_t, k \rangle^2] = t||k||_K^2$. Denote by $\mathcal{F}_t = \sigma(B_s, s \leq t)$ the natural, completed filtration generated by $\{B_t, t \geq 0\}$. Recall that a linear operator $S$ from $K$ into $H$ is called Hilbert-Schmidt if $\sum_{i=1}^{\infty} ||Sk_i||^2_H < \infty$ for some orthonormal basis $\{k_i, i \geq 1\}$ of $K$. $L_2(K, H)$ will denote the Hilbert space of Hilbert-Schmidt operators from $K$ into $H$ equipped with the inner product $\langle S_1, S_2 \rangle_{L_2(K, H)} = \sum_{i=1}^{\infty} \langle S_1k_i, S_2k_i \rangle_H$. Let $b(t, y, z, \omega)$ be a measurable mapping from $[0, T] \times H \times L_2(K, H) \times \Omega$ into $H$ such that $b(t, y, z, \omega)$ is $\mathcal{F}_t$-adapted, i.e., $b(t, y, z, \cdot)$ is $\mathcal{F}_t$-measurable for all $t, y, z$. Suppose we are given an $\mathcal{F}_T$-measurable, $H$-valued random variable $\phi(\omega)$. We are looking for two $\mathcal{F}_t$-adapted processes $Y_t, Z_t$ with values in $H$ and $L_2(K, H)$, respectively, such that the following backward stochastic evolution equation holds:

$$dY_t = AY_t dt + b(t, Y_t, Z_t) dt + Z_t dB_t, \quad t \in (0, T)$$

$$Y_T = \phi(\omega) \text{ a.s.}$$
From now on we assume that the following, (2.5) and (2.6), hold:
There exists a constant \( c < \infty \) such that
\[
|b(t, y_1, z_1)(\omega) - b(t, y_2, z_2)(\omega)|_H \leq c(|y_1 - y_2|_H + |z_1 - z_2|_{L_2(K, H)})
\]
for all \((t, y, z) \in [0, T] \times H \times L_2(K, H)\).
\[
E[\int_0^T |b(t, 0, 0)|^2_H dt] < \infty
\]

3 Results

We now state and prove the main result of this paper.

**Theorem 3.1** Assume that \( E[|\phi|^2_H] < \infty \). Then there exists an unique \( H \times L_2(K, H) \)-valued progressively measurable process \((Y_t, Z_t)\) such that

(i) \( E[\int_0^T |Y_t|^2_H dt] < \infty, E[\int_0^T |Z_t|^2_{L_2(K, H)} dt] < \infty \).

(ii) \( \phi = Y_t + \int_t^T AY_s ds + \int_t^T b(s, Y_s, Z_s) ds + \int_t^T Z_s dB_s; \quad 0 \leq t \leq T \).

**Proof.** We will complete the proof by three steps.

**Step 1.** Assume that \( b(t, y, z, \omega) = b(t, \omega) \) is independent of \( y \) and \( z \), and \( E[\int_0^T |b(t)|^2_H dt] < \infty \).

Existence of solution.
Set \( D(A) = \{v; v \in V, Av \in H\} \). Then \( D(A) \) is a dense subspace of \( H \).
Thus we can choose and fix an orthonormal basis \( \{e_1, \ldots, e_n, \ldots\} \) of \( H \) such that \( e_i \in D(A) \). Set \( V_n = span(e_1, e_2, \ldots, e_n) \). Denote by \( P_n \) the projection operator from \( H \) into \( V_n \). Put \( A_n = P_n A \). Then \( A_n \) is a bounded linear operator from \( V_n \) to \( V_n \). For the cylindrical Brownian motion \( B_t \), it is well known that the following decomposition holds:
\[
B_t = \sum_{i=1}^{\infty} \beta^i_k k_i
\]
where \( \{k_1, k_2, \ldots, k_i, \ldots\} \) is an orthonormal basis of \( K \), and \( \beta^i_k, i = 1, 2, 3, \ldots \) are independent standard Brownian motions. Set \( B^n_t = (\beta^i_k, \ldots, \beta^n_k) \). Define \( \mathcal{F}^n_t = \sigma(B^n_s, s \leq t) \) completed by the probability measure \( P \), and put \( \phi_n = E[P_n \phi(\mathcal{F}^n_t)] \) and \( b_n(t) = E[P_n b(t) | \mathcal{F}^n_t] \). Consider the following backward stochastic differential equation on the finite dimensional space \( V_n \):
\[
dY^n_t = A_n Y^n_t dt + b_n(t) dt + Z^n_t dB^n_t; \quad t < T
\]
\[
Y^n_T = \phi_n(\omega) \quad \text{a.s.}
\]
As \( A_n \) is a bounded linear operator from \( V_n \) to \( V_n \), it follows by the results of Pardoux and Peng [PP] that (3.2)–(3.3) admits a unique, continuous,
the adapted solution \((Y_t^n, Z_t^n)\), where \(Y_t^n \in \mathbb{R}^n \cong V_n, Z_t^n \in \mathbb{R}^n \otimes \mathbb{R}^n \cong L_2(K_n, V_n)\), \(K_n = \text{span}(k_1, k_2, ..., k_n)\). Next we are going to show that the sequence \((Y_t^n, Z_t^n)\) admits a convergent subsequence. Using Itô’s formula, we find that

\[
E[|Y_t^n|_{H}^2] = E[|\phi|_{H}^2] - 2E[\int_t^T \langle Y_s^n, P_n AY_s^n \rangle ds] - 2E[\int_t^T b_n(s) > ds] - E[\int_t^T |Z_s^n|_{L_2(K_n, V_n)}^2] \quad (3.4)
\]

where \(|Z_s^n|_{L_2(K_n, V_n)}^2 = \sum_{i,j=1}^n (Z_s^n(i,j))^2\) stands for the Hilbert-Schmidt norm. It follows from (2.2) that

\[
E[|Y_t^n|_{H}^2] \leq E[|\phi|_{H}^2] - \alpha E[\int_t^T ||Y_s^n||_{V}^2 ds] + \lambda E[\int_t^T |Y_s^n|_{H}^2 ds] + E[\int_t^T |Y_s^n|_{H}^2 ds] + E[\int_t^T |b_n(s)|_{H}^2 ds] - E[\int_t^T |Z_s^n|_{L_2(K_n, V_n)}^2] \quad (3.5)
\]

Hence,

\[
E[|Y_t^n|_{H}^2] + \alpha E[\int_t^T ||Y_s^n||_{V}^2 ds] + E[\int_t^T |Z_s^n|_{L_2(K, H)}^2 ds] \leq E[|\phi|_{H}^2] + (\lambda + 1)E[\int_t^T |Y_s^n|_{H}^2 ds] + E[\int_t^T |b(s)|_{H}^2 ds]
\]

where \(Z_s^n = Z_s^\theta \bar{P}_n\), and \(\bar{P}_n\) is the projection from \(K\) into \(K_n = \text{span}(k_1, ..., k_n)\). Therefore,

\[
E[|Y_t^n|_{H}^2] \leq E[|\phi|_{H}^2] + (\lambda + 1)E[\int_t^T |Y_s^n|_{H}^2 ds] + E[\int_t^T |b(s)|_{H}^2 ds] \quad (3.6)
\]

Set \(Y_t^n = \int_t^T |Y_s^n|_{H}^2\). Then (3.6) implies that

\[
-\frac{d(e^{(\lambda + 1)tY_t^n})}{dt} \leq e^{(\lambda + 1)t}(E[|\phi|_{H}^2] + E[\int_t^T |b(s)|_{H}^2 ds]) \quad (3.7)
\]

Hence,

\[
\int_0^T E[|Y_s^n|_{H}^2] ds \leq C(E[|\phi|_{H}^2] + E[\int_0^T |b(s)|_{H}^2 ds]), \quad (3.8)
\]

where \(C\) is an appropriate constant. This together with (3.5) yields that

\[
\sup_n \int_0^T E[|Y_s^n|_{H}^2] ds < \infty \quad (3.9)
\]

\[
\sup_n \int_0^T E[|Y_s^n|_{V}^2] ds < \infty \quad (3.10)
\]

\[
\sup_n \int_0^T E[|Z_s^n|_{L_2(K, H)}^2] ds < \infty \quad (3.11)
\]
For a separable Hilbert space $L$, we denote by $M^2([0, T], L)$ the Hilbert space of progressively measurable, square integrable, $L$-valued processes equipped with the inner product $<a, b>_M = E[\int_0^T <a_t, b_t>_L dt]$. By the weak compactness of a Hilbert space, it follows from (3.10) and (3.11) that a subsequence $\{n_k, k \geq 1\}$ can be selected so that $Y_{n_k}^n, k \geq 1$ converges weakly to some limit $Y$ in $M^2([0, T], V)$, and $\tilde{Z}_{n_k}^n, k \geq 1$ converges weakly to some limit $Z$ in $M^2([0, T], L_2(K, H))$. Let us prove that (a version of) $(Y, Z)$ is a solution to the backward stochastic evolution equation (2.3) and (2.4). For $n \geq i \geq 1$, we have that

$$d\langle Y^n_t, e_i \rangle = \langle P_n AY^n_t, e_i \rangle dt + \langle h_n(t), e_i \rangle dt + \langle \tilde{Z}_n^i dB_t, e_i \rangle$$

$$= \langle AY^n_t, e_i \rangle dt + \langle h_n(t), e_i \rangle dt + \langle \tilde{Z}_n^i dB_t, e_i \rangle$$

(3.12)

Let $h(t)$ be an absolutely continuous function from $[0, T]$ to $\mathbb{R}$ with $h'(\cdot) \in L^2([0, T])$ and $h(0) = 0$. By the Itô formula,

$$\langle Y^n_T, e_i \rangle h(T) = \int_0^T h(t)\langle AY^n_t, e_i \rangle dt + \int_0^T h(t)\langle b_n(t), e_i \rangle dt$$

$$+ \int_0^T h(t) d\left( \int_0^t Z^n_s dB_s, e_i \right) + \int_0^T \langle Y^n_t, e_i \rangle h'(t) dt.$$  

(3.13)

Replacing $n$ by $n_k$ in (3.16) and letting $k \to \infty$ to obtain

$$\langle \phi, e_i \rangle h(T)$$

$$= \int_0^T h(t)\langle AY_t, e_i \rangle dt + \int_0^T h(t)\langle b(t), e_i \rangle dt$$

$$+ \int_0^T h(t) d\left( \int_0^t Z_s dB_s, e_i \right) + \int_0^T \langle Y_t, e_i \rangle h'(t) dt.$$  

(3.17)

From (3.16) to (3.19), we have used the fact that the linear mapping $G$ from $M^2([0, T], L_2(K, H))$ into $L^2(\Omega)$ defined by

$$G(Z) = \int_0^T h(t) d\left( \int_0^t Z_s dB_s, e_i \right) = \sum_{j=1}^{\infty} \int_0^T h(t)\langle Z_t(k_j), e_i \rangle d\beta_t^j$$

is continuous. So, the convergence of (3.16) to (3.19) takes place weakly in $L^2(\Omega)$. Fix $t \in (0, T)$ and choose, for $n \geq 1$,

$$h_n(s) = \begin{cases} 1, & s \geq t + \frac{1}{2n}, \\ 1 - \frac{1}{n}(t + \frac{1}{2n} - s), & t - \frac{1}{2n} \leq s \leq t + \frac{1}{2n}, \\ 0, & s \leq t - \frac{1}{2n} \end{cases}$$

With $h(\cdot)$ replaced by $h_n(\cdot)$ in (3.19), it follows that

$$\langle \phi, e_i \rangle = \int_0^T h_n(s)\langle AY_s, e_i \rangle ds + \int_0^T h_n(s)\langle b(s), e_i \rangle ds$$

$$+ \int_0^T h_n(s)d\left( \int_0^s Z_u dB_u, e_i \right) + \frac{1}{n} \int_{t - \frac{1}{2n}}^{t + \frac{1}{2n}} \langle Y_s, e_i \rangle ds.$$  

(3.20)

(3.21)
Sending $n$ to infinity in (3.21) we get that
\[
\langle \phi, e_i \rangle = \int_t^T \langle AY_s, e_i \rangle ds + \int_t^T \langle b(s), e_i \rangle ds + \int_t^T d\langle \int_0^s Z_u dB_u, e_i \rangle + \langle Y_t, e_i \rangle.
\]
for almost all $t \in [0, T]$ (with respect to Lebesgue measure).

As $i$ is arbitrary, this implies that
\[
\phi = \int_t^T AY_s ds + \int_t^T b(s)ds + \int_t^T Z_s dB_s + Y_t.
\]
for almost all $t \in [0, T]$ (with respect to Lebesgue measure).

For $t \in [0, T]$, define
\[
\hat{Y}_t = \phi - \int_t^T AY_s ds - \int_t^T b(s)ds - \int_t^T Z_s dB_s
\]
Then we see that $(\hat{Y}_t, Z_t)$ also satisfies (ii) in the Theorem 3.1 with $Y$ replaced by $\hat{Y}$ for all $t \in [0, T]$. Hence, $(\hat{Y}_t, Z_t)$ is a solution to the equations (2.3) and (2.4).

**Uniqueness:**
Let $(Y_t, Z_t)$ and $(\tilde{Y}_t, \tilde{Z}_t)$ be two solutions of the equation (2.3). Then
\[
\int_t^T A(Y_s - \tilde{Y}_s)ds + \int_t^T (Z_s - \tilde{Z}_s)dB_s + (Y_t - \tilde{Y}_t) = 0
\]
Applying Itô’s formula, we get
\[
0 = |Y_t - \tilde{Y}_t|^2_H + 2 \int_t^T \langle Y_s - \tilde{Y}_s, dM_s \rangle + 2 \int_t^T \langle A(Y_s - \tilde{Y}_s), Y_s - \tilde{Y}_s \rangle ds + \int_t^T |Z_s - \tilde{Z}_s|^2_{L^2(K,H)} ds
\]
where $M_t = \int_0^t (Z_s - \tilde{Z}_s)dB_s$. By (2.2), we get that
\[
E[|Y_t - \tilde{Y}_t|^2_H] = -2 \int_t^T E[\langle A(Y_s - \tilde{Y}_s), Y_s - \tilde{Y}_s \rangle] ds - E[\int_t^T |Z_s - \tilde{Z}_s|^2_{L^2(K,H)} ds]
\leq -\alpha \int_t^T E[|Y_s - \tilde{Y}_s|_V^2] ds + \lambda \int_t^T E[|Y_s - \tilde{Y}_s|_H^2] ds
\leq \lambda \int_t^T E[|Y_s - \tilde{Y}_s|_H^2] ds.
\]
By a Gronwall type inequality, it follows that $E[|Y_t - \tilde{Y}_t|^2_H] = 0$, which proves the uniqueness.
Step 2. Assume that \( b(t, y, z)(\omega) = b(t, z)(\omega) \) is independent of \( y \).

Set \( Z_t^0 = 0 \). Denote by \((Y^n_t, Z^n_t)\) the unique solution of the backward stochastic evolution equation:

\[
\begin{align*}
    dY^n_t &= AY^n_t dt + b(t, Z^n_t)dt + Z^n_t dB_t \\
    Y^n_T &= \phi(\omega).
\end{align*}
\]

(3.28)

(3.29)

The existence of such a solution \((Y^n_t, Z^n_t)\) has been proved in step 1. Putting \( M^n_t = \int_0^t Z^n_s dB_s \), and by Itô’s formula we get that

\[
0 = |Y^{n+1}_T - Y^n_T|^2_H
\]

\[
= |Y^{n+1}_t - Y^n_t|^2_H + 2 \int_t^T \langle A(Y^{n+1}_s - Y^n_s), Y^{n+1}_s - Y^n_s \rangle ds
\]

\[
+ 2 \int_t^T \langle b(t, Z^n_s) - b(t, Z^n_{s-}), Y^{n+1}_s - Y^n_s \rangle ds
\]

\[
+ 2 \int_t^T \langle Y^{n+1}_s - Y^n_s, d(M^{n+1}_s - M^n_s) \rangle + \int_t^T |Z^{n+1}_s - Z^n_s|^2_{L_2(K, H)} ds
\]

(3.30)

(3.31)

(3.32)

(3.33)

In virtue of (2.2), for \( \varepsilon > 0 \),

\[
E[|Y^{n+1}_t - Y^n_t|^2_H] + E[\int_t^T |Z^{n+1}_s - Z^n_s|^2_{L_2(K, H)} ds]
\]

\[
= -2E[\int_t^T \langle A(Y^{n+1}_s - Y^n_s), Y^{n+1}_s - Y^n_s \rangle ds]
\]

\[
- 2E[\int_t^T \langle b(t, Z^n_s) - b(t, Z^n_{s-}), Y^{n+1}_s - Y^n_s \rangle ds]
\]

\[
\leq \lambda E[\int_t^T |Y^{n+1}_s - Y^n_s|^2_H ds] - \alpha E[\int_t^T ||Y^{n+1}_s - Y^n_s||^2_H ds]
\]

\[
+ \varepsilon E[\int_t^T |b(t, Z^n_s) - b(t, Z^n_{s-})|^2_H ds] + \frac{1}{\varepsilon} E[\int_t^T |Y^{n+1}_s - Y^n_s|^2_H ds]
\]

(3.34)

(3.35)

(3.36)

(3.37)

(3.38)

Choose \( \varepsilon < \frac{1}{2c} \), where \( c \) is the Lipschitz constant in (2.5). It follows from (3.38) that

\[
E[|Y^{n+1}_t - Y^n_t|^2_H] + E[\int_t^T |Z^{n+1}_s - Z^n_s|^2_{L_2(K, H)} ds] + \alpha E[\int_t^T ||Y^{n+1}_s - Y^n_s||^2_H ds]
\]

\[
\leq (\lambda + \frac{1}{\varepsilon}) E[\int_t^T |Y^{n+1}_s - Y^n_s|^2_H ds] + \frac{1}{2} E[\int_t^T |Z^{n+1}_s - Z^n_s|^2_{L_2(K, H)} ds]
\]

(3.39)
Hence,

\[- \frac{d}{dt}(e^{(\lambda + \frac{1}{2})t} E[\int_t^T |Y_s^{n+1} - Y_s^n|^2_H ds]) \]

(3.40)

\[+ e^{(\lambda + \frac{1}{2})t} E[\int_t^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds] \]

(3.41)

\[+ \alpha e^{(\lambda + \frac{1}{2})t} E[\int_t^T |\int_s^t Z_s^n - Z_s^{n-1}|^2_{L_2(K,H)} ds] \]

(3.42)

\[\leq \frac{1}{2} e^{(\lambda + \frac{1}{2})t} E[\int_t^T |Z_s^n - Z_s^{n-1}|^2_{L_2(K,H)} ds] \]

(3.43)

From here, following a similar proof as in [PP] we will show that \((Y^n, Z^n)\) converges to some limit \((Y, Z)\) in the product space of \(M^2([0,T], V)\) and \(M^2([0,T], L_2(K, H))\).

Let \(\beta = \lambda + \frac{1}{2}\). Integrating both sides in (3.43) we get that

\[E[\int_0^T |Y_s^{n+1} - Y_s^n|^2_H ds] + E[\int_0^T \int_t^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds | e^{\beta t} dt \]

\[+ \alpha \int_0^T E[\int_t^T |\int_s^t Z_s^n - Y_s^n|^2_T ds] e^{\beta t} dt \]

\[\leq \frac{1}{2} \int_0^T E[\int_t^T |Z_s^n - Z_s^{n-1}|^2_{L_2(K,H)} ds] e^{\beta t} dt \]

(3.44)

In particular,

\[E[\int_0^T \int_t^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds | e^{\beta t} dt \]

\[\leq \frac{1}{2} \int_0^T E[\int_t^T |Z_s^n - Z_s^{n-1}|^2_{L_2(K,H)} ds] e^{\beta t} dt \]

(3.45)

This implies that

\[E[\int_0^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds] e^{\beta t} dt \leq \left(\frac{1}{2}\right)^n C \]

for some constant \(C\). Thus, it follows from (3.44) that

\[E[\int_0^T |Y_s^{n+1} - Y_s^n|^2_H ds] \leq \left(\frac{1}{2}\right)^n C \]

(3.46)

Hence, we conclude from (3.39) that

\[E[\int_0^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds] \leq \left(\frac{1}{2}\right)^n C \beta + \frac{1}{2} E[\int_0^T |Z_s^n - Z_s^{n-1}|^2_{L_2(K,H)} ds] \]

(3.47)
Using the above inequality repeatedly gives

\[ E[\int_0^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds] \leq (\frac{1}{2})^n \left( nC\beta + E[\int_0^T |Z_s^1|^2_{L_2(K,H)} ds] \right) \]  

(3.48)

Combining (3.39) and (3.47) we have that

\[ E[\int_0^T ||Y_s^{n+1} - Y_s^n||_V^2 ds] \leq \left( \frac{1}{2} \right)^n \left\{ (n + 1)C\beta + E[\int_0^T |Z_s^1|^2_{L_2(K,H)} ds] \right\} \]  

(3.49)

It follows now from (3.48) and (3.49) that the sequence \((Y_t^n, Z_t^n), \ n \geq 1\) converges in \(M^2([0, T], V) \times M^2([0, T], L_2(K, H))\) to some limit \((\bar{Y}_t, \bar{Z}_t)\). Letting \(n \to \infty\) in (3.28), we see that \((\bar{Y}_t, \bar{Z}_t)\) satisfies

\[ Y_t + \int_t^T AY_s ds + \int_t^T b(s, Z_s) ds + \int_t^T Z_s dB_s = \phi \]  

(3.50)

i.e., \((\bar{Y}_t, \bar{Z}_t)\) is a solution to equation (2.3).

**Uniqueness**

Let \((\bar{Y}_t, \bar{Z}_t), (\bar{\bar{Y}}_t, \bar{\bar{Z}}_t)\) be two solutions to (3.50). By Itô’s formula, as in (3.27) we have

\[ E[|Y_t - \bar{\bar{Y}}_t|_H^2] + E[\int_t^T |Z_s - \bar{\bar{Z}}_s|_{L_2(K,H)}^2 ds] \]  

(3.51)

\[ = -2E[\int_t^T < A(Y_s - \bar{\bar{Y}}_s), Y_s - \bar{\bar{Y}}_s > ds] \]  

(3.52)

\[ - 2E[\int_t^T (b(t, Z_t) - b(t, \bar{\bar{Z}}_t), Y_s - \bar{\bar{Y}}_s) ds] \]  

(3.53)

\[ \leq \lambda E[\int_t^T |Y_s - \bar{\bar{Y}}_s|^2 ds] - \alpha E[\int_t^T ||Y_s - \bar{\bar{Y}}_s||_V^2 ds] \]  

(3.54)

\[ + \frac{1}{2} E[\int_t^T |Z_t - \bar{\bar{Z}}_t|_{L_2(K,H)}^2 ds] + cE[\int_t^T |Y_s - \bar{\bar{Y}}_s|_{H'}^2 ds] \]  

(3.55)

Consequently,

\[ E[|Y_t - \bar{\bar{Y}}_t|_H^2] \leq (\lambda + c) E[\int_t^T |Y_s - \bar{\bar{Y}}_s|^2_{H'} ds] \]  

(3.56)

By Gronwall’s inequality,

\[ Y_t = \bar{\bar{Y}}_t \]

which further implies \(Z_t = \bar{\bar{Z}}_t\) by (3.55).
Step 3. General case $b(t, x, y)(\omega)$. Let $Y_t^0 = 0$. Define, for $n \geq 1$, $(Y_t^{n+1}, Z_t^{n+1})$ to be the solution of the equation:

$$
dY_t^{n+1} = AY_t^{n+1}dt + b(t, Y_t^n, Z_t^n)dt + Z_t^{n+1}dB_t \quad (3.57)
$$

$$
Y_T^{n+1} = \phi \quad (3.58)
$$

The existence of $(Y_t^{n+1}, Z_t^{n+1})$ is contained in step 2.

Similarly as in step 2 we can show that $(Y_t^{n+1}, Z_t^{n+1})$ converges to some limit $(Y_t, Z_t)$, and moreover $(Y_T, Z_T)$ is the unique solution to equation (2.3).

We omit the details.

Example 3.2 Let $H = L^2(\mathbb{R}^d)$, and set

$$
V = H^2_{2}(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d); \nabla u \in L^2(\mathbb{R}^d \rightarrow \mathbb{R}^d)\}
$$

Denote by $a(x) = (a_{ij}(x))$ a matrix-valued function on $\mathbb{R}^d$ satisfying the uniform ellipticity condition:

$$
\frac{1}{c}I_d \leq a(x) \leq cI_d \quad \text{for some constant} \quad c \in (0, \infty).
$$

Let $f(x)$ be a vector field on $\mathbb{R}^d$ with $f \in L^p(\mathbb{R}^d)$ for some $p > d$. Define

$$
Au = -div(a(x)\nabla u(x)) + f(x) \cdot \nabla u(x)
$$

Then (2.2) is fulfilled for $(H, V, A)$. Thus, for any choice of cylindrical Brownian motion $B$, any drift coefficient $b(t, y, z, \omega)$ satisfying (2.5) and (2.6) and terminal random variable $\phi$, the main result in Section 3 applies.

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References


