A REMARK ON BANKRUPTCY AND DEFAULT 
IN THE MERON PROBLEM*

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Abstract

When portfolio choice in discontinuous asset price models is considered, the possibility of a jump to 0 is often overlooked. We solve the Merton problem for a (modified) HARA agent in a geometric lévy market where the market coefficients can change simultaneously with the jumps in the prices. We show that fund separation does only to some extent carry over, as agents with same exponent and different intertemporal trade-offs may no longer have the same mutual fund in the presence of such changes.

Key words: Merton problem, bankruptcy and default, geometric lévy motion, portfolio separation, incomplete markets.
JEL classification: G11, D81, D52.

0 Introduction

This paper considers the Merton optimal consumption-portfolio problem, which is well known in the case with Brownian driving noise. When direct utility is of hyperbolic absolute risk aversion (HARA) type, the qualitative results carry over even if one admits other distributions than the Gaussian (see [1] and [4]). When generalizing from the continuous Brownian case to the discontinuous case where the assets are driven by lévy processes, a feature is often overlooked, namely the fact that an asset, hence an investment opportunity may disappear at the same time as a sudden jump in its price. Indeed, the usual transformation via self-financing yields a wealth process (1) below, does implicitly assume that if an asset jumps to 0, it is immediately reborn with some nonzero value and its original dynamics. The reason, of course, is that one cannot substitute the value invested in a certain asset for the number of assets held times its value when the latter is finite, which one does to arrive at (1).

Two fund separation is the property that there exists two funds such that the agent can do equally from investing in these two funds as from the entire market. The cases with HARA and exponential utility admit so-called monetary separation, i.e. if a safe asset exists, it may be taken as one of the funds; hence the other risky portfolio is frequently referred to as the

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mutual fund. In the discrete time setting, [2] is the standard reference for a characterization of the class of preferences admitting separation regardless of distribution; they did however assume the utility functions to meet smoothness conditions not satisfied by the HARA modifications this paper is confined to for the parameter range \( \gamma \geq 1 \) (see below.) We shall see that the mutual fund property to some extent carries over to the case with bankruptcy or other sudden changes in the model parameters. We remark that as long as these changes are independent of the noise driving the assets, things are even simpler; this paper’s raison d’être is to allow for dependence between the driving noise and the future development of the coefficients of the market. Bankruptcy, for example, means that both does an asset’s value vanish, and so does the investment opportunity.

The market model with bankruptcy and default will be presented in Section 1; a family of modified HARA preferences and the optimization problem will be given in Section 2. Section 3 will solve the problem. In a one period discrete model, it is relatively easy to find necessary conditions for two fund separation. The continuous case is more difficult, as the indirect utility function may not be easy to find. A case without separation is therefore included as an Appendix. Throughout this paper, boldface symbols will denote vectors.

1 The model

Assume as given a filtered probability space \((\Omega, \mathcal{G}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) satisfying the usual conditions. We will implicitly assume all processes to be \(\mathcal{F}_t\)-predictable. First, define a market state process \(\Theta\) driven by a \((\mathcal{F}_t)\) Poisson random measure \(N\) only:

\[
d\Theta(s) = \int (\nu(s, \Theta(s^-), \varpi) - \Theta(s^-)) N(ds, d\varpi),
\]

i.e. that a jump will bring the market dynamics from state \(\Theta\) to a new state \(\nu\). The market consists of safe asset \(S_0\), which by discounting can be assumed constant, and a vector \(S = (S_1, \ldots, S_n)^T\) of risky assets following

\[
dS(s) = \text{Diag}(S(s^-)) \, dA
\]

where \(A\) is a Lévy process with Lévy-Khintchine representation

\[
dA = \mu \, dt + dW + \int \eta(\varpi) \, N(ds, d\varpi)
\]

where

\[
dN = dN - \bar{\chi} \, dq \, ds = dN - \bar{\chi} \, E[N(1, d\varpi)] \, ds
\]

and \(\bar{\chi} = \chi(\eta, 1).\) \(W\) is a \((\mathcal{F}_t)\) Wiener process with covariance matrix \(R\). The total wealth invested in the risky assets at time \(s\) at will then be

\[
Y(s) = \xi_0(s) S_0(s) + \xi^T(s) S(s),
\]

where \(\xi_i\) is the number held of asset \(i\). The mathematical definition of a self financing portfolio, motivated from a discrete time approximation, is that the number \(\xi_0\) of monetary units is reserved to satisfy

\[
dY(s) = \xi^T(s^-) \, dS(s) := v^T(s^-) \, dA(s)
\]  \quad (sf)
so that $v_i := \xi_i S_i$ is the market value of the agent’s holding in asset $i$. Formula (sf) has the interpretation that the instantaneous changes in wealth are solely due to changes in the market values $S$ (recall that we have assumed $dS_0 = 0$.) However, in our setting the portfolio shall also finance consumption at rate $c(s)$, so that the wealth process has the dynamics

$$dY(s) = v^T(s^-) dA(s) - c(s) \, ds \quad \text{(instead of the dynamics (sf)).}$$

(1)

We shall assume that the Lévy jumps will a.s. not change the sign of $Y$; then each portfolio weight must be upper bounded if the corresponding asset may jump downwards and nonpositive (nonnegative) if the jumps are not lower (upper) bounded. By scaling with a constant, we may apart from assets in which the zero position is the only admissible, and they are without loss of generality left out of the model, assume that no asset will jump by a factor (strictly) less than $-1$. Let us now relax the Lévy process assumption of stationary increments and allow for $\mu, \eta$ and the covariance matrix $R$ to depend on both time and $\Theta$; actually, it may be useful to take

$$\Theta(s) = (\mu(s), \eta(s, \cdot), R(s)).$$

(2)

For convenience, we may assume that the number of assets do not change (a disappearing asset will get zero dynamics and may be identified with the safe asset). For simplicity, we assume that the intensity of jumps in $\Theta$, namely $q(\{\omega; \nu(t, \vartheta, \omega) \neq \vartheta\})$, is (uniformly) bounded, a condition which should admit generalizations.

2 The preferences.

The HARA family of utility functions wrt. $y$ consist of the functions for which the Arrow-Pratt measure of absolute risk aversion is a hyperbola in $y$. These are the translated power functions

$$\frac{1 - \gamma}{\gamma} \frac{y + \beta}{1 - \gamma}$$

where $0 \neq \gamma \neq 1$, and in addition the translated logarithm

$$\log(y + \beta) = \lim_{\gamma \to 0} \frac{1 - \gamma}{\gamma} (\frac{y + \beta}{1 - \gamma} - 1)$$

to correspond to $\gamma = 0$. Strictly speaking, the HARA class has an additional positive scaling parameter – which without loss of generality can be put equal to one in our setting, as we are soon to introduce a discounting factor.

For $\frac{\gamma + \beta}{1 - \gamma} < 0$, the utility function is not very interesting ($\gamma > 1$) or not possibly not even defined ($\gamma < 1$). To cope with this, we can modify the utility function\footnote{The reader may check that we may perform the calculations with the HARA utility unmodified for $\gamma > 1.$} by putting it equal to a constant when $\frac{\gamma + \beta}{1 - \gamma} < 0$. Now if $\gamma < 1$, then this constant has to be minus infinity for the utility function to be concave; otherwise, it is intuitively obvious that some infinite position is optimal, and it is not difficult to show that there will be such an optimal strategy to the problem we are about to pose (the case where we stop the process the moment of bankruptcy corresponds to zero utility). We skip the details. If $\gamma > 1$, the only constant extension which
makes utility nondecreasing and concave, is to put utility equal to zero when \( \frac{1}{\gamma - \gamma} \) < 0. We shall therefore choose to work with the utility functions

\[
\bar{\Upsilon}_\beta(y) = \begin{cases} 
\frac{1}{\gamma} (\frac{y^{\gamma}}{\gamma - 1})^{\gamma} & \text{if } 0 \neq \gamma \neq 1 \text{ and } \frac{y^{\gamma}}{\gamma - 1} \geq 0 \\
\log(y + \beta) & \text{if } \gamma = 0 \text{ and } \frac{y^{\gamma}}{\gamma - 1} \geq 0 \\
-\infty & \text{if } \gamma < 1 \text{ and } \frac{y^{\gamma}}{\gamma - 1} < 0 \\
0 & \text{if } \gamma > 1 \text{ and } \frac{y^{\gamma}}{\gamma - 1} < 0 
\end{cases}
\]  

(3a)

with \( \log 0 = -0^\gamma = -\infty \) if \( \gamma < 0 \). This truncated construction also permits

\[
\bar{\Upsilon}(y) = \max(0, y + \beta) \text{ if } \gamma = 1.
\]  

(3b)

The calculations will then be a simplified version of the case \( \gamma > 1 \), only a bit notationally inconvenient, and will be left to the reader. We note that the cases \( \gamma \geq 1 \) are not contained in [2] because they are not \( C^2 \) as assumed therein.

Let \( \Delta \geq 0 \) be the discount term. We will consider the finite horizon problem

\[
\Phi(t, y, \theta) := \sup E^{t, y, \theta} \left[ \int_t^T \Delta(s, \Theta(s)) \cdot \bar{\Upsilon}_\beta(c(s)) \, ds + \bar{\Delta}(\Theta(T)) \cdot \bar{\Upsilon}_\beta(Y(T)) \right]
\]  

(4)

where \( E^{t, y, \theta} \) denotes expectation with respect to the probability law \( P^{t, y, \theta} \) of the processes starting at \( \Theta(t) = \theta, Y(t) = y \). We impose the restriction that

\[
c(s) = 0 \text{ whenever } \Delta(s) = 0.
\]

Without loss of generality, we can therefore assume that

\[
\beta(s) = 0 \text{ whenever } \Delta(s) = 0.
\]

We will have to assume further conditions which will be stated in Proposition 1; we may for example wish to forbid short sale. A convenient class of conditions is given in (6).

The \( \Theta \)-dependence in the bequest function seems natural, as the bequest function may represent future investment optimization problems (beyond time \( T \)). The \( \Theta \)-dependence in running utility has not the same intuitive interpretation, but does not complicate the below calculations.

## 3 Finding the optimal strategy

If \( \gamma > 1 \), then for large enough \( y \), namely \( y \geq -b \) with \( b \) given by

\[
-b + b(t) = \int_t^T -c(s) \, ds = \int_t^T \beta(s) \, ds
\]  

(5)

one can use the zero portfolio, consume \( c = -\beta \) (which yields maximum direct utility) and still end up with \( Y(T) \geq -\beta \) (and maximum terminal utility). On the other hand, if \( Y(t) < -b(t) \), then there is no portfolio (satisfying the tameness conditions in Proposition 1) for which
\( Y(T) \geq -\bar{\beta} \) a.s. Therefore, \( \Phi(t, y, \vartheta) = 0 \) if \( y \geq -b(t) \) with \( b \) as in (5). Arguing the same way for \( \gamma < 1 \), we arrive at the following property:

\[
\Phi(t, -b, \vartheta) = \begin{cases} 
0 & \text{if } \gamma > 0 \\
-\infty & \text{if } \gamma < 0
\end{cases}
\]

\[
\Phi(t, y, \vartheta) = \begin{cases} 
0 & \text{if } \gamma > 1 \\
-\infty & \text{if } \gamma < 1 \quad \text{if } \frac{y + b}{1 - \gamma} < 0.
\end{cases}
\]

Therefore, it is no restriction to assume

\[
v = \left(\frac{y + b}{1 - \gamma}\right)^+ \cdot f
\]

if \( 0 \leq \gamma \neq 1 \) (and \( v = (y + b)^+ \cdot f \) in the truncated linear case). If \( f \) is unrestricted, it will turn out that the optimal \( f^* \) does not depend on \( b \) nor \( y \). This also holds if the following kind of restriction applies:

\[
f \in \Gamma \quad \text{for some set } \Gamma \text{ not depending on } y.
\]

The requirement that \( \Gamma \) should not depend on \( y \) seems quite artificial in its general ad hoc formulation, but it covers forbidding short sale (i.e. \( \Gamma \) is the first orthant.) Furthermore, in the CRRA case (\( b = 0 \)) any restriction on portfolio weights satisfy (6), and we may in particular forbid both short sale and borrowing.

We will consider candidates \( \phi = \phi_\epsilon \) for the value function:

\[
\phi = \phi_\epsilon(t, y, \vartheta) = \begin{cases} 
D(t, \vartheta)Y_\gamma^{y, b(t)}(y) + g(t, \vartheta) \cdot \chi_{\{\gamma = 0\}} & \text{if } \frac{y + b}{1 - \gamma} > 0 \\
0 \text{ or } \infty & \text{as above}
\end{cases}
\]

where \( D \) is supposed to be positive. The case \( \frac{y + b}{1 - \gamma} \leq 0 \) is already solved, so assume \( \frac{y + b}{1 - \gamma} > 0 \).

In the following, \( \dot{D} \) will denote \( \frac{\partial D(t, \vartheta)}{\partial t} \) and so forth. Put

\[
V = V_\epsilon(s) = \phi_\epsilon(s, \Theta(s), Y(s)).
\]

Using the feedback control \( (c, \frac{y + b}{1 - \gamma}) + f \), we get

\[
\Delta(s, \Theta) \cdot \gamma^\gamma(c) \, ds + dV
\]

\[= \left[ \Delta \frac{1 - \gamma}{\gamma} \left( \frac{c + \beta}{1 - \gamma} \right)^+ \gamma - cV \frac{\gamma}{y + b + \epsilon} \right] ds
\]

\[+ V \left[ \frac{\dot{D}(s, \Theta)}{D(s, \Theta)} + \hat{b} \frac{\gamma}{y + b + \epsilon} \right] ds
\]

\[+ V \left[ \frac{\gamma}{1 - \gamma} (f^T \mu - \frac{1}{2} f^T R f)
\]

\[+ \int \left( \frac{D(s, \nu(\Theta(s^-)))}{D(s, \Theta(s^-))} \right) ((1 + \frac{f^T \eta}{1 - \gamma})^+ \gamma - 1 - \frac{\gamma}{1 - \gamma} f^T \eta) \, dq \right] ds
\]

\[+ V \left[ \frac{\gamma}{1 - \gamma} f^T dW + \int \left( \frac{D(s, \nu(\Theta(s^-)))}{D(s, \Theta(s^-))} ((1 + \frac{f^T \eta}{1 - \gamma})^+ \gamma - 1) \, dN \right]
\]
where $d\tilde{N} = dN - dq \, ds$ and where if $\gamma < 1$, we only consider $f$ such that

$$1 - \gamma + f^T \eta \geq 0 \quad q \text{-a.e.}$$

as the others yield minus infinity. The $c^*$ maximizing the right hand side of (8) with respect to $c$, satisfies

$$c^* = \left( \frac{D}{\Delta} \right)^\frac{1}{1-\gamma} (y + b + \epsilon) \chi_\Delta - \beta$$

where $\chi_\Delta(s) = \chi(s) \Delta(s) \geq 0$, and thus

$$\Delta \frac{1 - \gamma}{\gamma} (\gamma + b + \epsilon) + (\hat{b} - c^*) V \frac{\gamma}{y + b + \epsilon} = V (1 - \gamma) \left( \frac{D}{\Delta} \right)^\frac{1}{1-\gamma} \chi_\Delta$$

since $\hat{b} = \beta$ by (5). Therefore, if $D$ satisfies the inequality

$$0 \geq \hat{D}(t, \vartheta) \frac{1 - \gamma}{\gamma} + D(t, \vartheta) \Delta(t, \vartheta) \frac{1 - \gamma^2}{\gamma} \left( \frac{D(t, \vartheta)}{\Delta(t, \vartheta)} \right)^\frac{1}{1-\gamma} \chi_\Delta + D(t, \vartheta) \left(f^T \mu - \frac{1}{2} f^T R f\right)$$

$$+ \frac{1 - \gamma}{\gamma} \int \left( D(t, \nu(\vartheta)) (1 + f^T \eta) \right) \chi_\Delta + \frac{1 - \gamma}{\gamma} \left( D(t, \vartheta) f^T \eta \right) \eta \, dq$$

for all $f$, then

$$\Delta(s, \Theta) \gamma^\gamma(c) \, ds + dV \leq V \left[ \frac{\gamma}{1 - \gamma} f^T dW + \int \left( \frac{D(s, \nu(\Theta(s^{-})))}{D(s, \Theta(s^{-}))} (1 + f^T \eta) \right) \chi_\Delta \right] \, d\tilde{N}$$

with equality if both $c = c^*$ and (10) holds with equality for $f^*$ maximizing its right hand side.

Similarly, for log utility we get an optimal consumption $c^*$ given by (9) with $\gamma = 0$, and the following for $D$, $g$:

$$0 = \hat{D}(t, \vartheta) + \int \left( D(t, \nu(\vartheta)) - D(t, \vartheta) \right) \, dq + \Delta(t, \vartheta)$$

$$0 \geq \hat{g}(t, \vartheta) + \int \left( g(t, \nu(\vartheta)) - g(t, \vartheta) \right) \, dq + \Delta(t, \vartheta) \left( \log(\Delta(t, \vartheta)) - \log(D(t, \vartheta)) - 1 \right)$$

$$+ D(t, \vartheta) \left( f^T \mu - \frac{1}{2} f^T R f \right) + \int \left( D(t, \nu(\vartheta)) \log(1 + f^T \eta) - D(t, \vartheta) \chi_\Delta f^T \eta \right) \, dq$$

for all $f$; $f^T \eta \geq -1$ (here, $0 \log 0 := 0$). If (12) holds, then

$$\Delta(s, \Theta) \log(c + \beta) \, ds + dV \leq D(s, \Theta(s)) f^T dW$$

$$+ \int \left( D(s, \nu(\Theta(s^{-})) \log(1 + f^T \eta) + g(s, \nu(\Theta(s^{-})) - g(s, \Theta(s^{-})) \right) \, d\tilde{N}$$

with equality if both $c = c^*$ and (12b) holds with equality for $f^*$ maximizing its right hand side.

We may now proceed to prove that the solution suggested is optimal under certain conditions:

**PROPOSITION 1.**

Consider problem (4), where the supremum is taken among all consumption-investment strategies $(c, \nu) = (c(s), (\frac{Y(s) + b(s)}{1 - \gamma})^+ \cdot f)$ satisfying:
a). (1) has a unique weak solution.

b). \( c(s) = 0 \) whenever \( \Delta(s) = 0 \).

c). \( f \) is bounded, and \( f = 0 \) whenever \( Y(s) \) is less than some lower bound. Furthermore, restrictions of the form (6) may apply.

d). For all \( f \) such that \( 1 - \gamma + f^T \eta \geq 0 \) q-a.s., and for all \( f \) if \( \gamma > 1 \),
\[
\int \left( \left( \frac{(1 + f^T \eta)}{1 - \gamma} \right)^\gamma - 1 \right)^2 \, dq < \infty \quad \text{for } \gamma \neq 0 \quad (14a)
\]
\[
\int \left( \log(1 + f^T \eta) \right)^2 \, dq < \infty \quad \text{for } \gamma = 0 . \quad (14b)
\]

Assume furthermore that there exist bounded \( D \) satisfying (10) (if \( \gamma \neq 0 \)) or \( D, g \) satisfying (12) (if \( \gamma = 0 \)), with \( D > 0 \) and
\[
D(T, \cdot) \geq \tilde{\Delta}, \quad g(T, \cdot) \geq 0 . \quad (15)
\]

Then \( \Phi \leq \phi_0 \) given by (7) with \( b \) given by (5).

Suppose in addition that \( c^* \) is given by (9), and that (10) (if \( \gamma \neq 0 \)) or (12b) (if \( \gamma = 0 \)) holds with equality for the \( f^* = f^*(\theta) \) maximizing the right hand side, (15) holds with equality and
\[
E^{t, \theta} \left[ \int_0^T f^*(\Theta(s))^T R f^*(\Theta(s)) \, ds \right] < \infty . \quad (16)
\]

Then \( \Phi = \phi_0 \) and \( (c^*, (\frac{(Y(s)+b(s))}{1-\gamma})^+ \cdot f^*) \) is optimal.

Proof. Consider the process \( V_c(s) \). For \( \gamma \leq 0 \) let \( \epsilon > 0 \) be given and note that for \( y < -b \), then \( c \) and \( f \) are both zero; For \( \gamma > 0 \), use \( \epsilon = 0 \). Thus if we start at \( y > -b \), \( V_c(s) \) is lower bounded if \( \gamma < 1 \). Define
\[
\tau_M = T \land \inf \{ s \geq t; |V_c(s)| \geq M \} . \quad (17)
\]

By the boundedness of \( f \) and \( D \),
\[
V_c(t) \geq E[V_c(\tau_M) + \int_0^{\tau_M} \Delta \cdot \Upsilon^*(c) \, ds]
\]

Let \( M \to \infty \). By lower boundedness and Fatou's lemma, we get \( V_c(t) \geq \Phi \) and thus \( V_0(t) \geq \Phi \) if \( \gamma < 1 \). The same conclusion holds for \( \gamma > 1 \); though \( V \) is not necessarily lower bounded, \( c \) above and (14a) suffice.

To prove optimality, consider \( (c^*, (\frac{(Y(s)+b(s))}{1-\gamma})^+ \cdot f^*) \). The case of log utility is easy, so consider the case \( \gamma \neq 0 \). Notice that
\[
\Delta \cdot \Upsilon(c^*) = \chi_\Delta \left( \frac{D}{\Delta} \right)^{\frac{1}{\gamma-1}} V \quad (18)
\]
and therefore, \( V \) is a geometric process, since \( f^* \) depends only on \( V \) through \( D \). Solving the equation (10) for \( V \), we get
\[
V(T) \cdot \exp \left\{ \int_0^T \frac{\chi_\Delta \left( \frac{D(s, \Theta(s))}{\Delta(s, \Theta(s))} \right)^{\frac{1}{\gamma-1}}}{V(t) \cdot X(T)} \right\} = V(t) \cdot X(T)
\]
where \( X(t) = 1 \) and
\[
dX = X_\cdot \left( \int_t^T \gamma \frac{f^*}{1 - \gamma} \, dW + \int \frac{D(s, \nu(\Theta(s^-)))}{D(s, \Theta(s^-))} \left( (1 + \frac{f^*}{1 - \gamma})^\gamma - 1 \right) \, d\mathbb{N} \right)
\]
has zero expectation by (14a), (16). So the right hand side is a martingale. Differentiating, we get that the right hand side of (11) — which now holds with equality — has zero expectation. It follows that \( V_0(t) = \Phi \).

\[\square\]

Remark. Obviously the boundedness assumptions on \( D, g \) and \( f \) may be weakened. Note also that the optimal \( f^* \) does not satisfy condition (e) of Proposition 1 for \( \gamma > 1 \), and must then be obtained as a limiting case. We remark, though that in view of [3], these “tameness conditions” (which have the economic interpretation of excluding “doubling strategies”) on the portfolio may not just be dropped.

\[\triangle\]

**Corollary**
The fund \( f^* \) does not depend on \( b \), and is thus a mutual fund for all agents with \( (\Delta, \overline{\Delta}) \) common (up to the obvious multiplicative constant). If \( \Theta \) is a.s. constant, then \( f^* \) does not depend on \( \Delta \), i.e. it is a mutual fund for all agents with (modified) HARA utility.

However, if \( \Theta \) is stochastic, then there exist counterexamples where the agents do not share \( D(s, \nu)/D(s, \vartheta) \) and thus neither \( f^* \); we will prove this at the end, as it will utilize calculations from the below Example.

This far we have only considered finite time horizon. For \( T = \infty \), consider the following criterion with \( \beta = 0 \), \( \gamma < 1 \):
\[
\Phi(t, y, \vartheta) := \sup \lim_{T \to \infty} \mathbb{E}^{t, y, \vartheta}_w \left[ \int_t^T \Delta(s, \Theta(s)) \cdot \gamma Y(s) \, ds \right], \tag{19}
\]
the supremum being taken over all strategies such that almost surely, \( Y(s) \geq 0 \) for all \( s \geq t \). Then we have the following:

**Proposition 2.**
Consider the problem (19) subject to \( Y(s) \geq 0 \) a.s., with \( \gamma < 1 \), \( \beta = 0 \), optimizing with respect to consumption-investment strategies satisfying (a) — (d) of Proposition 1. Assume that there exist bounded \( D \) satisfying (10) (if \( \gamma \neq 0 \)) or \( D \), \( g \) satisfying (12) (if \( \gamma = 0 \)), with \( D > 0 \). Define \( \phi_0 \) by (7) with \( h = 0 \), and suppose that for \( y > 0 \),
\[
\lim_{M \to \infty} \sup_{\tau_M} \phi_0(\tau_M, Y(\tau_M), \Theta(\tau_M)) \geq 0
\]
with \( \tau_M \) given by (17). Then \( \Phi \leq \phi_0 \).

Suppose in addition that \( c^* \) is given by (9), and that (10) (if \( \gamma \neq 0 \)) or (12b) (if \( \gamma = 0 \)) holds with equality for the \( f^* = f^*(\vartheta) \) maximizing the right hand side and that
\[
\lim_{M \to \infty} \sup_{\tau_M} \mathbb{E}[\phi_0(\tau_M, Y^*(\tau_M), \Theta(\tau_M))] = 0 \tag{20}
\]
with $Y^*$ being the process obtained by using $(c^*, \frac{Y^*}{1-\gamma} f^*)$.

Then $\Phi = \phi_0$ and $(c^*, \frac{Y^*}{1-\gamma} f^*)$ is optimal. If in addition equation (20) holds with $\lim$ instead of $\limsup$, then we can have $\lim$ instead of $\liminf$ in the definition of the value function.

Proof. As in the proof of Proposition 1, we have

$$V_t(t) - E[V_t(\tau_M)] \geq E[\int_t^{\tau_M} \Delta \cdot \mathcal{Y}_{\phi}(c) \, ds]$$

(21)

with equality for $\epsilon = 0$, $c^*$, $f^*$. Take $\liminf$ on both sides. If (20) holds with $\lim$ instead of $\limsup$, we can take $\liminf$ for superoptimality and then $\lim$ for optimality.

With equality, (10) is a generalization of the Bernoulli equation. The following example is a case where one can solve Bernoulli equations inductively:

Example. Assume that there is a total of $n$ assets; assets $1, \ldots, n_0$ are always accessible, and assets $n_0 + 1, \ldots, n$ are bonds, each each being driven by one Poisson jump source $N_i$ of which everything else is independent, and each bond will become inaccessible at first jump in $N_i$, whose intensity we denote $q_i$. By scaling the drift term, we can without loss of generality assume that a bond jumps to 0; also, it is more convenient working with $d\tilde{N}_i$ instead of $dN_i = dN_i - q_i \chi dt$. The bond then gets an adjusted drift term $\tilde{\mu}_i$ which has to be $\geq 1$ in order to avoid arbitrage. The market state can now be represented as $\Theta(s) = (\Theta_{n_0+1}(s), \ldots, \Theta_n(s))$, where $\Theta_i$ is one if the bond is still accessible, zero if not:

$$d\Theta_i(s) = -\Theta_i(s^-) \, d\tilde{N}_i(s)$$

where each $\tilde{N}_i$ has Lévy measure $q_i > 0$. For simplicity, assume $0 \neq \gamma \neq 1$, and consider (10) with equality. For the first $n_0$ assets, the maximization with respect to the vector $f$ consisting of the $n_0$ first $f_i$'s, is independent of $D$:

$$\sup f \left\{ f^\top \mu - \frac{1}{2} f^\top R f + \frac{1-\gamma}{\gamma} \int \left( \left(1 + \frac{f^\top \tilde{\eta}}{1-\gamma} \right)^+ \gamma - 1 - \tilde{\chi} \frac{f^\top \tilde{\eta}}{1-\gamma} \right) dq \right\} =: F(0) \frac{1-\gamma}{\gamma}$$

(22)

where the $\tilde{}$ accents on the vectors and matrix denote the obvious truncation of the dimension. Again, if $\gamma < 1$ the supremum is only taken among the $f$ which satisfy $\frac{f^\top \eta}{1-\gamma} \geq -1$ q-a.e. The last $n - n_0$ assets are each independent of everything, so the portfolio optimization can be carried out for each $i > n_0$. We write $\vartheta \succ \epsilon$ if $\epsilon$ and $\vartheta - \epsilon$ are also vectors of zeros and ones (i.e., the latter has no negative components and all assets accessible with $\vartheta$ are accessible with $\epsilon$.) Writing $e_i$ for the $i$th unit vector, the maximization with respect to $f_i$ becomes:

$$\sup_{f_i} \left\{ D(t, \vartheta)f_i\tilde{\mu}_i + \frac{1-\gamma}{\gamma} q_i \left( D(t, \vartheta - e_i) \left(1 - \frac{f_i}{1-\gamma} \right)^+ \gamma - D(t, \vartheta) \right) \right\}$$

and the optimal $f_i$ is given by

$$f_i^* = (1-\gamma) \left(1 - \frac{q_i D(t, \vartheta - e_i)}{\tilde{\mu}_i D(t, \vartheta)} \frac{1}{1-\gamma} \right) \vartheta_i.$$
Inserting into (10), we get the following Bernoulli equation for $D(t, \vartheta)$:

$$
0 = \frac{F(\vartheta)}{1-\gamma} + \frac{\dot{D}(t, \vartheta)}{D(t, \vartheta)} + (1-\gamma) \left[ (\Delta(t, \vartheta))^{\frac{1}{1-\gamma}} \chi_\Delta + \sum_{i>n_0} \left( \frac{q_i}{\bar{\mu}_i} D(t, \vartheta - e_i) \right)^{\frac{1}{1-\gamma}} \cdot \bar{\mu}_i \vartheta_i \right] \left( D(t, \vartheta) \right)^{-\frac{1}{1-\gamma}}
$$

with

$$
F(\vartheta) := F(0) + \sum_{i>n_0} (\gamma \bar{\mu}_i - q_i) \vartheta_i.
$$

The solution is given by

$$
D(t, \vartheta) e^{F(\vartheta) t} = \left( A(\vartheta) - \int_0^t e^{F(\vartheta) s} \cdot \left[ (\Delta(s, \vartheta))^{\frac{1}{1-\gamma}} \chi_\Delta + \sum_{i>n_0} \left( \frac{q_i}{\bar{\mu}_i} D(s, \vartheta - e_i) \right)^{\frac{1}{1-\gamma}} \cdot \bar{\mu}_i \vartheta_i \right] ds \right)^{1-\gamma}
$$

(with abuse of notation if $\vartheta_i = 0$) to be solved inductively starting from $\vartheta = 0$. In the special infinite horizon case

$$
\Delta(t, \vartheta) = H(\vartheta) e^{-\delta t}
$$

(take for simplicity $\delta > 0$ independent of $\vartheta$), we find an easy generalization of the version of the Merton problem treated in [4]. We can take $A = 0$, and by induction, it follows that if $\delta > \max_{\epsilon \leq \vartheta} F(\epsilon)$, then

$$
D(t, \vartheta) = K(\vartheta) e^{-\delta t}
$$

(25a)

with $K$ inductively given by

$$
K(\vartheta) = \left( \frac{1-\gamma}{\delta - F(\vartheta)} \left[ (H(\vartheta))^{\frac{1}{1-\gamma}} + \sum_{i>n_0} \left( \frac{q_i}{\bar{\mu}_i} K(\vartheta - e_i) \right)^{\frac{1}{1-\gamma}} \cdot \bar{\mu}_i \vartheta_i \right] \right)^{1-\gamma}. \quad (25b)
$$

To prove optimality, note that by (18) and (13) with equality, it suffices that $\Delta/D$ is bounded away from 0 to conclude that $\mathbb{E}[Z(\tau_M)]$ tends to 0. However, in our case the fraction is constant. For $\gamma \in (0,1)$ it also follows, by solving the problem for a sequence $\delta_n \downarrow F(\vartheta)$ that the value function is $+\infty$ for $\delta \leq F$ (which is impossible for $\gamma < 0$, as $F < 0$ as well). Finally, for $\vartheta = 0$ we have weakened the sufficient conditions for finiteness given in [4], formulae (24) - (26) to an inequality which is also necessary.

To this end, we give the counterexample promised following the Corollary: The presence of market dynamics changes may cause agents with different intertemporal trade-off coefficients $\Delta$ to require different individual funds, even if they do not depend on $\vartheta$ directly:

**Proposition 3.**

Let in the infinite horizon case (19) two agents $j = 1, 2$ have the same $\Upsilon$ (both with $\beta = 0$) but $\Delta$'s given by $\Delta_j(t, \vartheta) = \Delta_j(t) = e^{-\delta_j t}$. Then there is not in general a mutual fund $f$ common to both.
Proof. We give a counterexample: Assume we are in the setting of the above Example with \( n_0 = 1, n = 2 \) so that there are only two risky assets, one of which a bond with default. For the other, the optimal \( f_1^* \) will be determined by (22) and independent of \( \delta \), so the optimal portfolio \( f = (f_1^*, f_2)^T \) is then a mutual fund common to both agents iff \( f_2^* \) does not depend on \( \delta \); we shall see that such \( \delta \)-dependence will occur.

\( \mathcal{A} = \emptyset \) will now be 1 iff the bond is still accessible, and then 0 for all eternity. It is easy to see that neither \( F(0) \) nor \( F(1) \) will depend on \( \delta \), so we skip those calculations, merely noting that we will have finite value if only \( \delta \) is chosen big enough. From (25) (with \( H = 1 \)) we have

\[
K(0) = \left( \frac{1 - \gamma}{\delta - F(0)} \right)^{1-\gamma} \quad \text{and} \quad K(1) = \left( \frac{1 - \gamma}{\delta - F(1)} \left[ 1 + \tilde{b}_2 \left( \frac{q_i}{\mu_i} K(0) \right)^{1-\gamma} \right] \right)^{1-\gamma}.
\]

We then remark that \( D(t, 0)/D(t, 1) = K(1)/K(0) \). Plug this into (23) and see that it does indeed depend on \( \delta \).

It is straightforward to show similar examples in the finite horizon case. We conclude that the introduction of a random \( \Theta \) is significant to the portfolio separation property.

Appendix: a counterexample

In a single period model where direct and indirect utility are the same, it is easy to see that only a special class of utility functions will admit separation irrespectively of the probability law of the returns. The same holds true in the continuous time setting; the author has not succeeded in finding a (counter)proof in the literature, and chooses to give one for the sake of completeness:

**Proposition 4.**

There are finite horizon problems of the form

\[
\Phi(y) = \sup_v J(v)(y) = \sup_v E^y[\Upsilon(Y(1))]
\]

(without intermediate consumption,) with \( \Upsilon \) is concave and smooth, and such that the optimal \( v^* \) does not separate into two funds independent of wealth.

**Proof.** We shall see that there will be counterexamples of the following form: Let \( \tilde{\Upsilon}(y) \) be HARA, and consider a strictly increasing, concave and \( C^1 \) modification \( \Upsilon \) such that

\[
\Upsilon(y) = \begin{cases} 
\tilde{\Upsilon}, & \text{if } y \leq 1 \\
< \tilde{\Upsilon}, & \text{if } y > 1.
\end{cases}
\]

To prove the claim, it suffices – and simplifies notation – to prove the case \( \beta = 0, \gamma = 2 \) (so \( \tilde{\Upsilon}(y) = \sqrt{2y} \)). Let the market consist of two (nonnegative) assets,

\[
dX_i(t) = bX_i(t^*) (dt - dN_i(t))
\]

where \( E[N_i(1)] = q_i \in (0, 1) \geq b \) such that \( X_i \) are submartingales. We assume the assets are not identical in law, i.e. \( q_1 \neq q_2 \). Let \( \Phi \) and \( J(v) \) correspond to the limiting case \( \Upsilon \neq \tilde{\Upsilon} \); then we have the form \( \Phi(y) = K \sqrt{2y} \) and the (unique) optimal control is given by

\[
b\tilde{\varepsilon}_i = (1 - q_i^2)y.
\]

(27)
Now if
\[
y \leq y_0 := \exp\{-2(q_1^2 - q_2^2)\},
\]
we will almost surely have \( Y(1) \leq 1 \) and therefore for all choices of \( \Upsilon \) as above, \( \Phi(y) = \hat{\Phi}(y) \) with optimal control \( v^* = \hat{v} \). On the other hand we clearly have
\[
J^{(\hat{\nu})}(y) < J^{(\nu)}(y) \quad \text{and thus} \quad \Phi(y) < \hat{\Phi}(y) \quad \text{for} \quad y > y_0.
\]
We want to prove that \( \Phi \) is differentiable at \( y_0 \), which by concavity (which by linearity of the dynamics is inherited from \( \Upsilon \)) will imply continuous differentiability up to some \( y_1 > y_0 \). Consider the control \( \hat{\nu} \), and let \( y = y_0 + \epsilon \) with \( \epsilon > 0 \) so small be so small that if there is at least one jump, \( Y(1) \leq 1 \) almost surely. Let \( \pi \) be the probability that no jump occurs before time 1. Then
\[
J^{(\hat{\nu})}(y_0 + \epsilon) - \hat{\Phi}(y_0 + \epsilon) = \pi \cdot \left( \Upsilon((y_0 + \epsilon)e^{2-v_1-q_2^2}) - \hat{\Upsilon}((y_0 + \epsilon)e^{2-v_1-q_2^2}) \right)
\]
Divide by \( \epsilon \) and let \( \epsilon \downarrow 0 \), we see that the right hand side tends to \( \pi \cdot (\Upsilon'(1) - \hat{\Upsilon}'(1)) = 0 \) by (28). So \( \Phi \) is differentiable at \( y_0 \) as claimed. Therefore, since there is no restriction to assume nonnegative positions, the HJB equation holds for \( y > y_0 \) small enough; \( v_1 \) then satisfies the first order condition
\[
\Phi'(y) = q_1 \Phi'(y) - b v_1 = q_1 K \cdot (2(y - b v_1))^{-1}
\]
which implies that
\[
\frac{v_2}{v_1} = \frac{y(\Phi'(y))^2 - (K q_2)^2}{y(\Phi'(y))^2 - (K q_1)^2}
\]
Assume for contradiction that there is two fund separation; then by (27), the bank may be taken as one of the two funds, hence \( v_2/v_1 \) is constant, which by (30) implies that for \( y > y_0 \) small enough, \( y(\Phi'(y))^2 \) is constant and therefore by continuity, \( \Phi(y) = K \sqrt{2y} = \hat{\Phi}(y) \) on some nonempty interval \((y_0, y_2)\), contradicting (29). \( \square \)

References


