THE MODULAR ISOMORPHISM PROBLEM

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ABSTRACT. We show that the isomorphism class of a finite \( p \)-group \( G \) is determined by \( \mathbb{F}_p G \), the group algebra of \( G \) over the field of \( p \) elements, hence solving the modular isomorphism problem.

1. Introduction

The modular isomorphism problem goes back to the 1950s and asks whether the isomorphism class of a finite \( p \)-group \( G \) can be determined by its modular group algebra \( \mathbb{F}_p G \) where \( \mathbb{F}_p \) is the field of \( p \) elements. In [1] and [6] we considered this problem from a cohomological point of view. Using the classical Massey product structure on \( H^1(G, \mathbb{F}_p) \) and \( H^2(G, \mathbb{F}_p) \) we could say something towards a solution, and in [2] we discussed the limitations of the methods we used.

This last work was inspired by the work done by Laudal in non-commutative geometry [5], and going back to this framework, and more specifically, deformation theory, we are now able to give a positive answer to the modular isomorphism problem. This is done in section 4, where we show that the isomorphism class of a finite \( p \)-group \( G \) is in fact determined by its modular group algebra \( \mathbb{F}_p G \) (theorem 4.3).

Quillen showed that the restricted graded Lie algebra of a finite \( p \)-group \( G \) is determined by \( \mathbb{F}_p G \), see [8]. The question is how do we get from a graded object to a filtered object? The idea is to use obstruction theory. Both \( G \) and \( \mathbb{F}_p G \) have one simple module, \( \mathbb{F}_p \). Also, both the group and the group algebra have an obstruction theory, and so in both cases we can treat the obstructions for deforming this simple, trivial module.

In theorem 2.9, we show that \( G \) is in a sense the formal moduli of \( \mathbb{F}_p \) in the category of pro-\( p \) groups. Section 3 treats the algebra-situation, in particular we have from theorem 3.7 that \( \mathbb{F}_p G \) is the formal moduli of \( \mathbb{F}_p \) in the category of complete local \( \mathbb{F}_p \)-algebras with one simple module. Since we can show that the obstructions (which are unique) needed in the group-construction

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coincide with the obstructions needed in the algebra-construction, our main result (theorem 4.3) will follow.

2. Obstruction theory for pro-\(p\) groups

Let \(G\) be a finite \(p\)-group and let \(K\) be a finite dimensional vector space over \(\mathbb{F}_p\) on which \(G\) acts trivially. We call such a \(K\) an elementary module of \(G\).

**Definition 2.1.** We define

\[
H^n(G, K) = \text{Ext}_{\mathbb{F}_p G}^n(\mathbb{F}_p, K),
\]

where \(\text{Ext}_{\mathbb{F}_p G}^n(\mathbb{F}_p, K)\) is the \(n\)-th right-derived functor of \(\text{Hom}_{\mathbb{F}_p G}(-, K)\).

The cohomology groups \(H^n(G, K)\) are independent of our choice of resolution of \(\mathbb{F}_p\). The bar resolution \(B_\bullet\), for example, is the simplicial way of doing group cohomology. This resolution is considered to be more of a theoretical tool rather than a practical tool, hence we will be using other resolutions when it comes to calculations. However, the fact that \(H^2(G, K)\) is the space of obstructions for lifting certain group homomorphisms can best be seen using the bar resolution:

**Definition 2.2.** The dual bar resolution \(B^\bullet(G, K)\) is the cocomplex

\[
\cdots \rightarrow \text{Maps}(G^n, K) \xrightarrow{\delta^n} \text{Maps}(G^{n+1}, K) \rightarrow \cdots
\]

with \(\delta^n(\phi)(g_1, \ldots, g_{n+1}) \) equal to

\[
\phi(g_2, \ldots, g_{n+1}) + \sum_{i=1}^n (-1)^i \phi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \phi(g_1, \ldots, g_n)
\]

where \(\phi\) is a map from \(G^n\) (as a set) to \(K\).

In particular, \(\delta^1 \phi(g_1, g_2) = \phi(g_2) - \phi(g_1 g_2) + \phi(g_1)\) and

\[
\delta^2 \phi(g_1, g_2, g_3) = \phi(g_2, g_3) - \phi(g_1 g_2, g_3) + \phi(g_1, g_2 g_3) - \phi(g_1, g_2).
\]

We have \(H^n(G, K) \simeq H^n(B^\bullet(G, K), \delta)\).

Our first lemma relates to the following lifting-situation. Let

\[
D : 1 \rightarrow K_D \xrightarrow{i} H_1 \xrightarrow{\pi} H_2 \rightarrow 1
\]

be a diagram where \(H_1, H_2\) are groups and the short-exact sequence is a central extension, i.e. \(i(K_D) \subset Z(H_1)\) with \(K_D\) elementary \((K_D \simeq \mathbb{F}_p^m, \text{some } m)\) and written multiplicatively.
Definition 2.3. We shall let $D$ be the obvious category of all such diagrams (2.3) for a fixed group $G$. So an object $D$ in $D$ is a central extension of groups with elementary kernel $K_D \simeq F_p^m$, some $m$, and a homomorphism $\phi$ from $G$ to the third group in the extension.

Lemma 2.4. To each diagram in $D$ one associates a specific cohomology class $obs_G(\phi) \in H^2(G, K_D)$ such that $obs_G(\phi) = 0$ iff there exists a group homomorphism $\phi'$ making the diagram (2.3) commutative.

Proof. Let $\sigma$ be a set theoretic section of $\pi$, $\sigma : H_2 \xrightarrow{\gamma} H_1, g_1, g_2 \in G$ and consider the map $\psi$ defined by

$$\psi(g_1, g_2) = \sigma(\phi(g_1 g_2)) \sigma(\phi(g_2))^{-1} \sigma(\phi(g_1))^{-1}. \quad (2.4)$$

Since $\pi(\psi(g_1, g_2)) = 1$, $\psi \in Maps(G \times G, K_D)$. Moreover, using (2.2), we check that $(\delta^2 \psi)(g_1, g_2, g_3) = 1$, and so we can let $obs_G(\phi) = [\psi]$. We note that a different choice $\tilde{\sigma}$ for the section in (2.4) will give a $\tilde{\psi}$ with $\psi - \tilde{\psi} = \delta^1 (\tilde{\sigma} \circ \phi - \sigma \circ \phi)$, so the cohomology class $obs_G(\phi) \in H^2(G, K_D)$ is well defined.

Assume $obs_G(\phi) = 0$. Then $\psi \in im \delta^1$, and so there exists $\alpha \in Maps(G, K_D)$ such that $\delta^1 \alpha = \psi$. Define

$$\phi'(g) = \sigma(\phi(g)) \alpha(g).$$

Then since we have a central extension, we find that $\phi'$ is a group homomorphism, and $\pi(\phi'(g)) = \phi(g)$.

Conversely, suppose we have $\phi'$ such that the diagram (2.3) commutes. We need to show that $obs_G(\phi)$ is 0 as an element of $H^2(G, K_D)$, i.e. it is “$\delta^1$ of something”. So we have to find an element $\gamma \in Maps(G, K_D)$ such that $\delta^1 \gamma = \psi$. Let $\gamma(g) = \phi'(g) \sigma(\phi(g))^{-1}$. Then we can check that $\pi(\gamma(g)) = 1$, so $\gamma \in Maps(G, K_D)$, and $\psi(g_1, g_2) = \gamma(g_2) \gamma(g_1 g_2) \gamma(g_1)^{-1}$, which was what we wanted.

\[\square\]

An important consequence is that the cohomology class $obs_G(\phi)$ is functorial: Let us consider the commutative diagram

\[
\begin{array}{c}
1 \xrightarrow{\psi} K' \xrightarrow{\xi} H'_1 \xrightarrow{\eta} H'_2 \xrightarrow{\theta} 1 \\
1 \xrightarrow{\phi} K \xrightarrow{\pi} H_1 \xrightarrow{\phi'} H_2 \xrightarrow{\phi''} 1
\end{array}
\]
where the short-exact sequences are central extensions, $K' \cong \mathbb{F}_p$, some $l$, $H_1', H_2'$ are groups and $\zeta, \eta, \theta$ are homomorphisms. Hence we have a morphism between two objects in $\mathcal{D}$.

We can now try to lift $\theta \circ \phi$ to $H_1'$, and by lemma 2.4 we find $\text{obs}_G(\theta \circ \phi)$, the obstruction for doing this. The following corollary states the functoriality.

**Corollary 2.5.** The map $\zeta_* : H^2(G, K) \to H^2(G, K')$ induced by $\zeta$ maps $\text{obs}_G(\phi)$ to $\text{obs}_G(\theta \circ \phi)$.

**Proof.** Let $\sigma'$ be a section from $H_2'$ to $H_1'$ (as sets). Referring to the proof of lemma 2.4, we get $\psi' \in \text{Maps}(G \times G, K')$ with

$$
\psi'(g_1, g_2) = \sigma'(\theta(\phi(g_1))) \sigma'(\theta(\phi(g_2)))^{-1} \sigma'(\theta(\phi(g_1)))^{-1}
$$

and $\text{obs}_G(\theta \circ \phi) = [\psi']$.

On the other hand, $\zeta_*(\psi) = \zeta \circ \psi$. If we pick a section $\bar{\sigma}$ from $H_2'$ to $H_1'$ such that $\eta \circ \sigma = \bar{\sigma} \circ \theta$ then $\bar{\beta} := (\bar{\sigma} \circ \theta \circ \phi - \sigma' \circ \theta \circ \phi) \in \text{Maps}(G, K')$. Furthermore, we see that $\zeta \circ \psi = \psi' = \delta \bar{\beta}$, hence $[\zeta \circ \psi] = [\psi']$ and the result follows.

$\square$

So to every diagram $D$ in $\mathcal{D}$ with kernel $K_D$ and a $\phi$ from $G$ to the third group in the extension, there exists a cohomology class $\text{obs}_G(\phi) \in H^2(G, K_D)$ which is functorial with respect to $D$. We can therefore think of $H^2(G, -)$ as a functor from the category $\mathcal{D}$ to the category of vector spaces over $\mathbb{F}_p$.

Now, more generally, for each object $D$ in $\mathcal{D}$,

$$
1 \rightarrow K_D \xrightarrow{i} H_1 \xrightarrow{\pi} H_2 \rightarrow 1
$$

suppose $o_D \in H^2(G, K_D)$ is such that

**OG1:** $o_D$ is functorial, i.e. if $D \rightarrow D'$ is a morphism in $\mathcal{D}$, then $H^2(G, K_D) \rightarrow H^2(G, K_{D'})$ maps $o_D$ to $o_{D'}$;

**OG2:** $o_D = 0$ if and only if $G \xrightarrow{\phi} H_2$ lifts to $H_1$.

We make the following definition.

**Definition 2.6.** A map $o$ defined on the set of objects of $\mathcal{D}$ mapping $D \in \text{ob}(\mathcal{D})$ to an element $o_D \in H^2(G, K_D)$ satisfying **OG1** and **OG2** will be called an obstruction.

In lemma 2.4, we have constructed such an obstruction independent upon the choices made.
Now observe that
\begin{equation}
H^2(G, K_D) \simeq H^2(G, \mathbb{F}_p) \otimes_{\mathbb{F}_p} K_D \simeq \text{Hom}_{\mathbb{F}_p}(H^2(G, \mathbb{F}_p)^*, K_D),
\end{equation}
hence the obstructions are characterised up to automorphisms of $H^2(G, \mathbb{F}_p)^*$ and $K_D$. However, $H^2(G, -)$ and $\text{Hom}_{\mathbb{F}_p}(H^2(G, \mathbb{F}_p)^*, -)$ are canonically isomorphic as functors. This gives us the universal property of these obstructions:

**Lemma 2.7.** Let $o$ and $o'$ be two obstructions. For every $D \in \text{ob}(D)$, consider $o_D, o'_D \in H^2(G, K_D)$. Then $\text{im} o_D = \text{im} o'_D$ as subvector spaces in $K_D$.

**Proof.** Consider a diagram $D$ in $D$ with $K_D, H_1$ and $H_2$ making up the central extension. Since $o$ is an obstruction, we can lift $\phi$ to $H_1/\text{im} o_D$:

\[
\begin{array}{c}
1 \longrightarrow K_D/\text{im} o_D \longrightarrow H_1/\text{im} o_D \longrightarrow H_2 \longrightarrow 1 \\
\downarrow \zeta \quad \downarrow \pi \quad \downarrow \eta \quad \downarrow \phi \quad \downarrow 1 \\
1 \longrightarrow K_D \longrightarrow H_1 \longrightarrow H_2 \longrightarrow 1
\end{array}
\]

Now, $o'_D$ is also an obstruction for lifting $\phi$, and $\text{im} o'_D$ is therefore sent to 0 in $K_D/\text{im} o_D$ by functoriality. Hence $o'_D \subseteq \text{im} o_D$, and by symmetry we get equality.

\[\square\]

From this result we see that it is the image of the obstruction that is the interesting thing, and so this will be called the obstruction from now on.

Now let $T^i$ be the free pro-$p$ group on a dual basis for $H^i(G, \mathbb{F}_p)$, $i = 1, 2$. In the next theorem we construct a morphism $o_G$ between $T^2$ and $T^1$ with $G \simeq T^1/[\text{im} o_G]$. Here we use $\langle \rangle$ to denote the normal closure of a subset as a closed normal subgroup in $T^1$. We construct the morphism as an inverse limit of a certain coherent sequence of morphisms from $T^2$ to quotients of $T^1$ using the results above at each level. The quotients involve a filtration $\{F_i(T^1)\}_{i\geq 1}$ of $T^1$, and we choose to use the $\mathcal{M}$-series, also known as the Brauer-Jennings-Zassenhaus series, as our filtration:

**Definition 2.8.** We define the $\mathcal{M}$-series of a pro-$p$ group $G$ inductively by
\begin{align}
\mathcal{M}_1(G) &= G \\
\mathcal{M}_i(G) &= \langle [\mathcal{M}_{i-1}(G), G], \mathcal{M}_{i/p}(G)^{(p)} \rangle
\end{align}
where $(i/p)$ is the least integer $\geq i/p$ and $\mathcal{M}_i(G)^{(p)}$ denotes the set of $p$-th powers in $\mathcal{M}_i(G)$. 
We note that in the sequel when we speak about a subgroup of a pro-$p$ group we shall always mean the topological closure of the subgroup. Moreover, we note that $T^i$ is the projective limit of the quotients $T^i/M_n(T^i)$.

The successive quotients $M_i(G)/M_{i+1}(G)$ are vector spaces over $\mathbb{F}_p$. The series $\{M_i(G)\}_i$ is not the fastest descending series with this property. However, it is the fastest descending series satisfying the following two properties:

\begin{align*}
(2.8) \quad [M_n(G),M_m(G)] &\leq M_{n+m}(G), \\
(2.9) \quad M_{n}(G)^p &\leq M_{np}(G).
\end{align*}

Hence it is the “restricted mod $p$ lower central series”, see [3].

We are now ready to prove our first theorem.

**Theorem 2.9.** Let $G$ be a finite $p$-group and let $T^i$ be the free pro-$p$ group on a dual basis for $H^i(G,\mathbb{F}_p)$, $i = 1, 2$. There exists an obstruction morphism $o_G : T^2 \longrightarrow T^1$ with $G \simeq T^1/(\text{im}o_G)$.

**Proof.** In the proof we will use the notation $T^1_n$ and $G_n$ for $T^1/M_n(T^1)$ and $G/M_n(G)$ respectively.

We note that $\text{dim}_{\mathbb{F}_p} H^1(G,\mathbb{F}_p)$ is the minimal number of generators for $G$ and that $\text{dim}_{\mathbb{F}_p} H^2(G,\mathbb{F}_p)$ is the minimal number of pro-$p$ relations for $G$.

The obstruction calculus is written down in some detail in [6], but then using the lower central $p$-series. However, for our purposes, things turn out nicer by using the $\mathcal{M}$-series. We will have a look at the construction here to see the analogy with the algebra-construction in the next section. It will also become clear what properties of a filtration we will need.

For the first step, we note that $G_2 \simeq T^1_2$ and consider the following diagram:

\[
\begin{array}{ccccccc}
1 & \longrightarrow & \mathcal{M}_2(T^1)/\mathcal{M}_3(T^1) & \overset{\pi_1}{\longrightarrow} & T^1_3 & \longrightarrow & 1 \\
& & & & \downarrow \cong & \downarrow \sim & \\
& & & & G_2 & \longrightarrow & 1 \\
& & & & \phi_1 & \downarrow & \\
& & & & G & \longrightarrow & 1
\end{array}
\]

We now have a diagram as in (2.3), and by lemma 2.4 we have a unique cohomology class $\text{obs}_G(\phi_1) \in H^2(G,\mathcal{M}_2(T^1)/\mathcal{M}_3(T^1))$ such that $\text{obs}_G(\phi_1) = 0$ iff there exists a lifting $\phi_2$ of $\phi_1$ to $T^1_3$.

Now,

\[
H^2(G,\mathcal{M}_2(T^1)/\mathcal{M}_3(T^1)) \simeq H^2(G,\mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{M}_2(T^1)/\mathcal{M}_3(T^1)
\]

\[
\simeq \text{Hom}_{\mathbb{F}_p}(H^2(G,\mathbb{F}_p)^*,\mathcal{M}_2(T^1)/\mathcal{M}_3(T^1))
\]

\[
\simeq \text{Mor}(T^2,\mathcal{M}_2(T^1)/\mathcal{M}_3(T^1)) \subseteq \text{Mor}(T^2,T^1_3),
\]
so let \( o_2 \) be \( ob_G(\phi_1) \) as an element in \( Mor(T^2, T^1_3) \) (note that up to now, we have made no choices). Pick a basis \( \{\eta_1, \ldots, \eta_r\} \) for \( H^2(G, \mathbb{F}_p) \) and let \( \eta^*_1, \ldots, \eta^*_r \) be the corresponding generators for \( T^2 \). Then \( im o_2 \) is the (normal) subgroup of \( T^1_3 \) generated by the images of \( \eta^*_1, \ldots, \eta^*_r \) under \( ob_G(\phi_1) \) and \( \iota : \mathcal{M}_2(T^1)/\mathcal{M}_3(T^1) \longrightarrow T^1_3 \). We shall refer to \( im o_2 \) as the obstructions on the first level, but note that they really lie in \( K_2 := \mathcal{M}_2(T^1)/\mathcal{M}_3(T^1) \).

By lemma 2.4, \( im o_2 \) is the minimal normal subgroup of \( T^1_3 \) containing the elements that need to be reduced to 1 in order to get a lifting of \( \phi_1 \). In other words, the universal functorial property of these obstructions implies that \( T^1/(\mathcal{M}_3(T^1), im o_2) \) is the largest quotient of \( T^1_3 \) making a lifting of \( \phi_1 \) possible. So for all other quotients \( T^1_3/N \) and a map \( G \longrightarrow T^1_3/N \) lifting \( \phi_1 \) there exists a map such that the following diagram becomes commutative:

\[
\begin{array}{ccc}
T^1_3/im o_2 & \longrightarrow & G \\
\phi_2 \downarrow & & \downarrow \\
T^1_3/N & \longrightarrow & G.
\end{array}
\]

The universal property implies that the lifting \( \phi_2 \) (which is a surjection by construction) is initial. Moreover, it also gives us the isomorphism \( G_3 \simeq T^1_3/im o_2 \):

\[
\begin{array}{cccccccc}
1 & \longrightarrow & \mathcal{M}_2(T^1)/\mathcal{M}_3(T^1) & \longrightarrow & T^1_3 & \longrightarrow & T^1_2 & \longrightarrow & 1 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& \mathcal{M}_2(T^1)/\langle \mathcal{M}_3(T^1), im o_2 \rangle & \longrightarrow & T^1_3/im o_2 & \longrightarrow & G & \cong & G_3 \\
\end{array}
\]

To see that \( \psi \) is an isomorphism, we observe that \( \phi_2 \) factorises via \( G_3 \) since \( \mathcal{M}_3(G) \) is 1 in \( T^1_3/im o_2 \) and we see from the next diagram that \( \psi \) has an
inverse map by the universal property of the obstruction:

\[
\begin{array}{ccc}
T^1 / i m o_2 & \xrightarrow{\phi_2} & T^1 \\
\psi \uparrow & \Downarrow \exists! & \Downarrow G & \Downarrow T^1 \\
G_3 & \xrightarrow{} & T^1_3 & \xrightarrow{} & T^1_2
\end{array}
\]

We have now seen most of the first non-trivial step in the construction of \( o_G \): So far we have found

1) a unique map: \( o_2 : T^2 \longrightarrow T^1 / \mathcal{M}_3(T^1) \) and

2) the lifting: \( \phi_2 : G \longrightarrow T^1 / \langle \mathcal{M}_3(T^1), i m o_2 \rangle \) inducing

3) the isomorphism: \( G / \mathcal{M}_3(G) \cong T^1 / \langle \mathcal{M}_3(T^1), i m o_2 \rangle \).

And

4) we have seen that the obstructions on this level are elements in

\( H^2(G, K_2) \) where \( K_2 := \mathcal{M}_2(G) / \mathcal{M}_3(G) \).

Our aim is to find a coherent sequence of maps \( o_1 \) is trivial

\[ \{ o_n : T^2 \longrightarrow T^1 / \mathcal{M}_{n+1}(T^1) \}_n \geq 1, \]

and also a coherent sequence of liftings

\[ \{ \phi_n : G \longrightarrow T^1 / \langle \mathcal{M}_{n+1}(T^1), i m o_n \rangle \}_n \geq 1 \]

inducing isomorphisms \( G_{n+1} \cong T^1_{n+1} / i m o_n \). We note here that we can have

\( G_i \cong G_{i+1} \) with \( G_i \neq \{ 1 \} \), since the \( \mathcal{M} \)-series depends on \( p \), hence we need to lift more times than we would have if we had used the lower central \( p \)-series for example.

Anyway, to finish the first step of the construction, we need to see the coherence between the first and second step. So, before we do the general step, let us see what happens when we want to lift \( \phi_2 \).

We start with the diagram

\[
\begin{array}{cccc}
1 & \longrightarrow & \langle \mathcal{M}_3(T^1), i m o_2 \rangle / \mathcal{M}_4(T^1) & \longrightarrow T^1_4 & \longrightarrow T^1_3 / i m o_2 & \longrightarrow 1 \\
 & & \uparrow \phi_2 & & \Downarrow \phi_2 & & \\
 & & G, & & & &
\end{array}
\]

but this is not a diagram to which we can apply lemma 2.4. We have first to make the short-exact sequence into a central extension. We want the kernel to be a vector space over \( \mathbb{F}_p \), hence the kernel must be

\[
K_3 := \langle \mathcal{M}_3(T^1), i m o_2 \rangle / \langle \mathcal{M}_4(T^1), [T^1, i m o_2], (i m o_2)^p \rangle,
\]
which means we want to lift $\phi_2$ to $T^1_1/[T^1,imo_2],(imo_2)^p)$. From lemma 2.4 we get a unique element $obs_G(\phi_2) \in H^2(G,K_3) \simeq Mor(T^2,K_3)$ which has to be 0 in order to get a lifting of $\phi_2$. Hence the obstructions on the second level will lie in $K_3$.

Let $o(\phi_2)$ be $obs_G(\phi_2)$ as an element in $Mor(T^2,T^1_1/[T^1,imo_2],(imo_2)^p)$ and consider the following diagram where the four arrows $\sim \sim$ denote two short-exact sequences:

(2.12)

$\begin{array}{c}
T^2 \xrightarrow{\phi_2} K_2 \\
\phi_3 \downarrow \quad \quad \pi_2 \\
K_3 \sim \sim T^1_4/[T^1,imo_2],(imo_2)^p \\
\phi_3 \downarrow \quad \quad \pi_2 \\
T^1_4/[T^1,imo_2],(imo_2)^p,imo(\phi_2) \\
A \sim B \\
\phi_3 \downarrow \quad \quad \pi_2 \\
T^1_4/imo_3 \\
G_4 \xleftarrow{G} \\
\end{array}$

Recall that $imo_2$ really lies in $\mathcal{M}_2(T^1)/\mathcal{M}_3(T^1)$, and so $\mu$ exists as a factorization of $\pi_2$. It also factors $\pi_2$, and induces a map $\overline{\pi} : K_3 \longrightarrow K_2$, which again induces a map $\overline{\pi} : H^2(G,K_3) \longrightarrow H^2(G,K_2)$. The coherence is therefore coming from the functoriality of the obstructions, which says that the map $\overline{\pi}$ sends $obs_G(\phi_2)$ to $obs_G(\phi_1)$.

For the isomorphisms $A$ and $B$ in the diagram, we have by lemma 2.4 a universal property for $imo(\phi_2)$ similar to the one for $imo_2$, with the quotient $T^1_4/[T^1,imo_2],(imo_2)^p,imo(\phi_2)$ being the maximal factor group of $T^1_4/[T^1,imo_2],(imo_2)^p$ making a lifting of $\phi_2$ possible. So the isomorphism $A$ follows by a similar argument as the one we used to establish the isomorphism $\psi$ in diagram (2.10).
Pick an $o_3$ such that $\rho_*(o_3) = o(\phi_2)$ where $\rho_*$ is the morphism

$$\rho_* : \text{Mor}(T^2, T^1) \longrightarrow \text{Mor}(T^2, T^1/[\langle T^1, imo_2 \rangle, (imo_2)^p])$$

induced by $\rho$ in the diagram above. By construction, $\rho_*(o_3) = o_2$, and by the properties (2.8) and (2.9) of the $\mathcal{M}$-series, we observe that

$$([T^1, imo_2], (imo_2)^p) \subseteq imo_3,$$

therefore there is a canonical map

$$T^1_4/[T^1, imo_2], (imo_2)^p \longrightarrow T^1_4/imo_3$$

inducing the map $B$. Moreover $imo_3$ maps to $imo(\phi_2)$ in $K_3$, so we have that $B$ is an isomorphism.

We have now seen the first non-trivial step of the construction $o_G$ in detail, and the construction continues: When we have reached $o_{n-1}$, we will have done it in such a way that $imo_{n-1}$ will contain all the obstructions $imo_k$, $k < n - 1$.

Hence, for lifting $\phi_{n-1}$, we have the diagram

$$1 \longrightarrow \langle\mathcal{M}_n(T^1), imo_{n-1}\rangle/\mathcal{M}_{n+1}(T^1) \longrightarrow T^1_{n+1} \longrightarrow T^1_n/imo_{n-1} \longrightarrow 1$$

and, similarly as for the first step, the obstructions on this level will lie in

$$K_n := \langle\mathcal{M}_n(T^1), imo_{n-1}\rangle/\langle\mathcal{M}_{n+1}(T^1), [T^1, imo_{n-1}], (imo_{n-1})^p\rangle.$$

As before, we get an element $o(\phi_{n-1})$, and again, we can combine the subgroups $\langle[T^1, imo_{n-1}], (imo_{n-1})^p\rangle$ and $imo(\phi_{n-1})$ to get $imo_n$ and a lifting

$$\phi_n : G \longrightarrow T^1_{n+1}/imo_n,$$

inducing an isomorphism $G_{n+1} \simeq T^1_{n+1}/imo_n$.

To finish, the map $o_G : T^2 \longrightarrow T^1$ will be the inverse limit of the system

$$\{o_n : T^2 \longrightarrow T^1/\mathcal{M}_{n+1}(T^1)\}_{n \geq 1},$$

induce isomorphisms

$$G_n := G/\mathcal{M}_{n+1}(G) \longrightarrow T^1/\langle\mathcal{M}_{n+1}(T^1), imo_n\rangle$$

for all $n \geq 1$, and so $G \simeq T^1/\langle imo_G \rangle$.

$\Box$
3. Obstruction theory for complete local $\mathbb{F}_p$-algebras with one simple module

We will now be dealing with various ideals, and so if $S$ is a subset of an algebra $A$, we shall let $(S)$ denote the two-sided ideal in $A$ generated by $S$.

We are interested in the situation where the group and the group algebra have one simple module. Then, as we will see, we get a correspondence between the group and the group algebra via the obstruction theory for the two objects.

There exists a non-commutative deformation theory for families of modules on $k$-algebras ($k$ a field), see [4]. In particular, the theory will apply to the case of $\mathbb{F}_p$, the only simple $\mathbb{F}_p G$-module for a finite $p$-group $G$. This is the case we are interested in in this paper.

In the previous section we let $T^i$ denote the free pro-$p$ group on a dual basis for $H^i(G, \mathbb{F}_p)$, $i = 1, 2$. We will now be interested in the group algebra of these groups, and so we shall let $T^i_{\mathbb{F}_p G}$ denote the obvious completion of the tensor algebra of the dual vector space of $Ext^i_{\mathbb{F}_p G}(\mathbb{F}_p, \mathbb{F}_p)$ over $\mathbb{F}_p$, $i = 1, 2$. If we let $E$ denote $Ext^i_{\mathbb{F}_p G}(\mathbb{F}_p, \mathbb{F}_p)$, we have

$$T^i_{\mathbb{F}_p G} \simeq \mathbb{F}_p \oplus E^* \oplus (E^* \otimes E^*) \oplus (E^* \otimes E^* \otimes E^*) \oplus \cdots$$

The theory says that there exists an obstruction morphism

$$\sigma_{\mathbb{F}_p G} : T^2_{\mathbb{F}_p G} \to T^1_{\mathbb{F}_p G}$$

with $T^1_{\mathbb{F}_p G} \simeq T^1_{\mathbb{F}_p G}/(im \sigma_{\mathbb{F}_p G})$, where $\overline{T}$ denotes the closure of the two-sided ideal $I$ in the complete algebra. The closure-notation will normally be omitted. In the language of deformation theory, this says that $\mathbb{F}_p G$ is “the formal moduli” of the trivial module $\mathbb{F}_p$. Similarly, we have seen in theorem 2.9 that $G$ is the formal moduli of $\mathbb{F}_p$ when we are in the category of pro-$p$ groups. Whereas the construction of the obstruction morphism in the previous section used a filtration of $T^1$ by its $M$-series, we will now filter $T^1_{\mathbb{F}_p G}$ by powers of its maximal ideal $m$, generated by $E^*$, $(m/m^2)^* = E$.

To make it clear that we can do this construction without using the group structure, we will do the construction for a complete, local $\mathbb{F}_p$-algebra $A$ with only one simple module (the residue field $\mathbb{F}_p$).

We shall need an algebra-version of lemma 2.4. In order to formulate this we shall have to give some easy (and well-known) results on the Hochschild cohomology of $A$.

**Definition 3.1.** Let $A$ be an associative $k$-algebra, $k$ a field, and let $Q$ be an $A$-bimodule. The Hochschild cochain $C^\bullet(A, Q)$ is given by

$$\cdots \to \text{Hom}_k(A^\otimes n, Q) \xrightarrow{\partial_n} \text{Hom}(A^\otimes (n+1), Q) \to \cdots$$
where the differential is given by the formula
\[
\partial^n \phi(a_1 \otimes \cdots \otimes a_{n+1}) = a_1 \phi(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i \phi(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} \phi(a_1 \otimes \cdots \otimes a_n) a_{n+1}
\]
for \( \phi \in C^n(A, Q) \), \( a_1, \ldots, a_{n+1} \in A \). The cohomology groups of this complex are the Hochschild cohomology groups, denoted \( HH^n(A, Q) \).

We note that if \( M \) and \( N \) are right \( A \)-modules, \( \text{Hom}_k(M, N) \) is an \( A \)-bimodule, and we have the isomorphism
\[
\text{Ext}_A^n(M, N) \simeq HH^n(A, \text{Hom}_k(M, N))
\]
for \( n \geq 0 \).

Now, let \( B_1 \) and \( B_2 \) be local artinian \( \mathbb{F}_p \)-algebras with residue field \( \mathbb{F}_p \), and let \( m_1 \) be the maximal ideal of \( B_1 \). Consider the diagram
\[
\begin{align*}
E : & 0 \longrightarrow V_E \stackrel{i}{\longrightarrow} B_1 \stackrel{\pi}{\longrightarrow} B_2 \longrightarrow 0 \\
& \downarrow \phi' \downarrow \phi \downarrow \\
& A
\end{align*}
\]
where the short-exact sequence is a small extension, i.e. \( m_1 V_E = V_E m_1 = 0 \), thus \( V_E \) is a vector space over \( \mathbb{F}_p \), which has trivial \( A \)-action (i.e. action via \( A \longrightarrow \mathbb{F}_p \)).

**Definition 3.2.** We shall let \( \mathcal{E} \) denote the category of all such diagrams (3.2) for a fixed associative algebra \( A \). So an object \( E \) in \( \mathcal{E} \) is a small extension of local artinian \( \mathbb{F}_p \)-algebras with kernel \( V_E \) (vector space over \( \mathbb{F}_p \)) and a homomorphism from \( A \) to the third algebra in the extension.

**Lemma 3.3.** To each diagram in \( \mathcal{E} \) one associates a specific cohomology class \( \text{obs}_A(\phi) \in \text{Ext}_A^3(\mathbb{F}_p, V_E) \) such that \( \text{obs}_A(\phi) = 0 \) iff there exists an algebra homomorphism \( \phi' \) making the diagram (3.2) commutative.

**Proof.** Let \( \sigma \) be a section from \( B_2 \) to \( B_1 \) as vector spaces over \( \mathbb{F}_p \), \( a_1, a_2 \in A \) and consider the map \( \psi \) defined by
\[
(3.3) \quad \psi(a_1 \otimes a_2) = \sigma(\phi(a_1 a_2)) - \sigma(\phi(a_1)) \sigma(\phi(a_2)).
\]
Then \( \psi \in \text{Hom}_{\mathbb{F}_p}(A \otimes A, V_E) \) since \( \phi \) is a homomorphism. And we check that \( (\partial^2 \psi)(a_1 \otimes a_2 \otimes a_3) = 0 \). Let \( \text{obs}_A(\phi) = [\psi] \in HH^2(A, V_E) \) and observe that this cohomology class is independent upon the choice of \( \sigma \).

Assume \( \text{obs}_A(\phi) = 0 \). Then \( \psi \in \text{im} \partial^1 \), and so there exists \( \alpha \in \text{Hom}_{\mathbb{F}_p}(A, V_E) \) such that \( \partial^1 \alpha = \psi \). Define
\[
\phi'(a) = \sigma(\phi(a)) + \alpha(a).
\]
Then $\phi'$ is an algebra homomorphism since we have a small extension, and $\pi(\phi'(a)) = \phi(a)$.

Conversely, suppose we have $\phi'$ such that the diagram (3.2) commutes. We need to show that $obs_A(\phi)$ is 0 as an element in $Ext^2_A(\mathbb{F}_p, V_E)$, i.e. we must show that we have a $\gamma \in Hom_{\mathbb{F}_p}(A, V_E)$ such that $\partial^1 \gamma = \psi$. Let $\gamma(a) = \phi'(a) - \sigma(\phi(a))$. Then we can check that $\pi(\gamma(a)) = 0$, so $\gamma \in Hom_{\mathbb{F}_p}(A, V_E)$, and $\psi(a_1 \otimes a_2) = \gamma(a_1) - \gamma(a_1a_2) + \gamma(a_2)$, which was what we wanted.

\[ \square \]

We make a similar definition of an obstruction as in the previous section, and the functoriality and universal property will also be similar:

The cohomology class $obs_A(\phi)$ is also functorial (but now we are in a different category): Let us consider the commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & V' & \overset{\iota'}{\longrightarrow} & B'_1 & \overset{\pi'}{\longrightarrow} & B'_2 & \longrightarrow & 0 \\
\bigg\downarrow \zeta & & \bigg\downarrow \eta & & \bigg\downarrow \phi'' & & \bigg\downarrow \phi & \\
0 & \longrightarrow & V & \overset{\iota}{\longrightarrow} & B_1 & \overset{\pi}{\longrightarrow} & B_2 & \longrightarrow & 0 \\
& & & \bigg\downarrow \phi ' & & \bigg\downarrow \phi & & \\
& & & \bigg\downarrow \phi & & \bigg\downarrow \phi & & \\
& & & \bigg\downarrow A & & \bigg\downarrow \phi & & \\
\end{array}
\]

where the short-exact sequences are small extensions, $V, V'$ are ideals killed by the corresponding maximal ideals, and thus vector spaces over $\mathbb{F}_p$, where $B_1, B_2, B'_1, B'_2$ are algebras, and where $\zeta, \eta, \theta$ are homomorphisms, defining a morphism in the category $\mathcal{E}$.

We can now try to lift $\theta \circ \phi$ to $B'_1$, and by lemma 3.3 we find $obs_A(\theta \circ \phi)$, the obstruction for doing this. The following corollary states the functoriality.

**Corollary 3.4.** The map $\zeta_* : Ext^2_A(\mathbb{F}_p, V) \longrightarrow Ext^2_A(\mathbb{F}_p, V')$ induced by $\zeta$ maps $obs_A(\phi)$ to $obs_A(\theta \circ \phi)$.

**Proof.** Similar to the proof of corollary 2.5.

\[ \square \]

So to every small extension $E$ with kernel $V_E$ and a $\phi$ from $A$ to the third group in $E$, there exists a cohomology class $obs_A(\phi) \in Ext^2_A(\mathbb{F}_p, V_E)$ which is functorial in $E$. As in the case of the functor $H^2(G, -)$, we can now think of $Ext^2_A(\mathbb{F}_p, -)$ as a functor from the category $\mathcal{E}$ to the category of vector spaces over $\mathbb{F}_p$. 
And as above we make a formal generalization. Assume that for each object \( E \) in \( \mathcal{E} \),
\[
0 \longrightarrow V_E \xrightarrow{\iota} B_1 \xrightarrow{\pi} B_2 \longrightarrow 0
\]
the element \( o_E \in Ext_A^2(\mathbb{F}_p, V_E) \) is such that

**OA1:** \( o_E \) is functorial, i.e. if \( E \longrightarrow E' \) is a morphism in \( \mathcal{E} \), then
\[
Ext_A^2(\mathbb{F}_p, V_E) \longrightarrow Ext_A^2(\mathbb{F}_p, V_{E'})
\]
maps \( o_E \) to \( o_{E'} \);

**OA2:** \( o_E = 0 \) if and only if \( A \xrightarrow{\phi} B_2 \) lifts to \( B_1 \).

Then we make the following definition.

**Definition 3.5.** A map \( o \) defined on the set of objects of \( \mathcal{E} \) mapping \( E \in \text{ob}(\mathcal{E}) \) to an element \( o_E \in Ext_A^2(\mathbb{F}_p, V_E) \) satisfying **OA1** and **OA2** will be called an obstruction.

In lemma 3.3, we have constructed such an obstruction independent upon the choices made.

Now observe that
\[
Ext_A^2(\mathbb{F}_p, V_E) \cong Ext_A^2(\mathbb{F}_p, \mathbb{F}_p) \otimes_{\mathbb{F}_p} V_E \cong Hom_{\mathbb{F}_p}(Ext_A^2(\mathbb{F}_p, \mathbb{F}_p)^*, V_E).
\]
As \( Ext_A^2(\mathbb{F}_p, -) \) and \( Hom_{\mathbb{F}_p}(Ext_A^2(\mathbb{F}_p, \mathbb{F}_p)^*, -) \) are canonically isomorphic as functors, we get the following universal property for obstructions on \( \mathcal{E} \): They are characterized by their images in \( V_E \).

**Lemma 3.6.** Let \( o \) and \( o' \) be two obstructions. For every \( E \in \text{ob}(\mathcal{E}) \) consider \( o_E, o'_E \in Ext_A^2(\mathbb{F}_p, V_E) \). Then \( im o_E = im o'_E \) as subvector spaces in \( V_E \).

**Proof.** Similar to the proof of lemma 2.7.

\[\square\]

Again, we see that it is the image of the obstruction that is the interesting thing, and so they will also be called obstructions from now on.

We are now ready to state and prove the algebra-version of theorem 2.9:

**Theorem 3.7.** Let \( A \) be a complete local algebra over \( \mathbb{F}_p \) with \( \mathbb{F}_p \) being the only simple module. Let \( T_A^i \) be the tensor algebra of the dual vector space of \( Ext_A^i(\mathbb{F}_p, \mathbb{F}_p) \), \( i = 1, 2 \), over \( \mathbb{F}_p \). There exists an obstruction morphism \( o_A : T_A^2 \longrightarrow T_A^1 \) such that \( A \cong T_A^1/(im o_A) \).

**Proof.** In the proof we will use \( T_A^i \) for \( T_A^i \), \( m_A \) and \( m_A \) will denote the maximal ideals of \( T^1 \) and \( A \) respectively, and \( T_A^1 \) and \( A_n \) will be short for \( T^1/m_A \) and \( A/m_A^2 \) respectively.
We note that $Ext^1_A(\mathbb{F}_p, \mathbb{F}_p) \simeq (m_A/m_A^2)\ast$ and that $A/m_A^2 \simeq T^1/m^2$. Moreover, $T^1$ is the tensor algebra, and so $T^1_0$ is always the universal extension of $T^1_{n-1}$ with $(Ext^1_A(\mathbb{F}_p, \mathbb{F}_p) \ast) \otimes (n-1)$, in particular, $T^1/m^2 \simeq \mathbb{F}_p \oplus Ext^1_A(\mathbb{F}_p, \mathbb{F}_p)\ast$.

We want to lift algebra homomorphisms from $A$ to quotients of $T^1$ involving powers of $m$, and for the first step of the construction we have the following diagram.

$$
\begin{array}{cccccc}
0 & \longrightarrow & m^2/m^3 & \longrightarrow & T^1_0 & \longrightarrow & 0 \\
\downarrow & & & & \pi_1 & & \downarrow \\
A/m^2_A & \longrightarrow & T^1_2 & \longrightarrow & \quad & & \quad \\
\phi_1 & & & & & & \phi_2 \\
A & \quad & \quad & \quad & \quad & \quad & A
\end{array}
$$

For the analogy with the proof of theorem 2.9, it helps to keep the following "correspondence" in mind: $\mathcal{M}_1(T^1_C) = T^2_0$ in the proof of theorem 2.9 corresponds to $m$. $\mathcal{M}_2(T^1_C)$ corresponds to $m^2$, etc. The trivial step in the two constructions is lifting the map $G \longrightarrow 1$ for groups and lifting the map $A \longrightarrow \mathbb{F}_p$ for algebras, both of which are always possible, i.e. there are no obstructions to doing this. The analogy holds by the well-known isomorphism $T^1_C/\mathcal{M}_2(T^1_C) \simeq m/m^2$.

We now have a diagram as in (3.2), and by lemma 3.3 we have a unique cohomology class $obs_A(\phi_1) \in Ext^2_A(\mathbb{F}_p, m^2/m^3)$ such that $obs_A(\phi_1) = 0$ iff there exists a lifting $\phi_2$ of $\phi_1$ to $T^1/m^3$.

Now,

$$
Ext^2_A(\mathbb{F}_p, m^2/m^3) \simeq Ext^2_A(\mathbb{F}_p, \mathbb{F}_p) \otimes_{\mathbb{F}_p} m^2/m^3
$$

$$
\simeq Hom_{\mathbb{F}_p}(Ext^2_A(\mathbb{F}_p, \mathbb{F}_p)\ast, m^2/m^3) \simeq Mor(T^2, m^2/m^3) \subseteq Mor(T^2, T^1_3).
$$

In analogy with the group situation we shall write $imo_n$ for the ideal generated by $o_n(m_T^2)$.

Let $o_2$ be $obs_A(\phi_1)$ as an element in $Mor(T^2, T^1_3)$, hence $imo_2$ is the ideal of $T^1_3$ generated by the images of the generators for $T^2$ under $obs_A(\phi_1)$ and $i:\ m^2/m^3 \longrightarrow T^1_3$. Again, we shall refer to $imo_2$ as the obstructions on the first level, but note that they really lie in $V^2 := m^2/m^3$.

By lemma 3.3, $imo_2$ is the minimal ideal of $T^1_3$ containing the elements that need to be reduced to 0 in order to get a lifting of $\phi_1$. In other words, the universal property of these obstructions implies that $T^1/(m^3 + imo_2)$ is the largest quotient of $T^1_3$ making a lifting of $\phi_1$ possible. So for all other quotients $T^1_3/I$ and a map $A \longrightarrow T^1_3/I$ there exists a map such that the
following diagram becomes commutative:

\[
\begin{array}{ccc}
T^1_3/imo_2 \\
\downarrow \phi_2 \\
T^1_3/I & \longrightarrow & A.
\end{array}
\]

The universal property implies that the lifting \( \phi_2 \) (which is a surjection by construction) is initial. Moreover, it also gives us the isomorphism \( \psi : A_3 \simeq T^1_3/imo_2 \) in the diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & m^2/m^3 & \longrightarrow & T^1_3 & \longrightarrow & T^1_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & m^2/(m^3 + imo_2) & \longrightarrow & T^1_3/imo_2 & \longrightarrow & A \\
\downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \psi & & \downarrow \\
A_3 & \longrightarrow & 0 & \longrightarrow & \psi & & A
\end{array}
\]

Observe first that \( \phi_2 \) factorises via \( A_3 \) since \( m^3_1 \) is 0 in \( T^1_3/imo_2 \), and next that it suffices to show that there exists a natural homomorphism \( T^1_3/imo_2 \xrightarrow{\psi} A_3 \).

We have now seen most of the first non-trivial step in the construction of \( o_A \): So far we have found

1) a unique map: \( o_2 : T^2 \xrightarrow{} T^1/m^3 \) and
2) the lifting: \( \phi_2 : A \xrightarrow{} T^1/(m^3 + imo_2) \) inducing
3) the isomorphism: \( A_3 \simeq T^1/(m^3 + imo_2) \).

And

4) we have seen that the obstructions on this level are elements in \( \text{Ext}^2_A(\mathbb{F}_p, V_2) \) where \( V_2 := m^2/m^3 \).

Our aim is to find a coherent sequence of maps (\( o_1 \) is trivial) \( \{ o_n : T^2 \xrightarrow{} T^1_{n+1} \}_{n \geq 1} \), and also a coherent sequence of liftings \( \{ \phi_n : A \xrightarrow{} T^1/(m^{n+1} + imo_n) \}_{n \geq 1} \).
inducing the isomorphisms \( \{ A_{n+1} \cong T^1/(\mathfrak{m}^{n+1} + imo_n) \}_{n \geq 1} \). To see the coherence between the first and second step, let us see what happens when we want to lift \( \phi_2 \).

We start with the diagram

\[
0 \longrightarrow (\mathfrak{m}^3 + imo_2)/\mathfrak{m}^4 \longrightarrow T_1 \xrightarrow{\phi_2} T_3/imo_2 \longrightarrow 0 \]

but this is not a diagram to which we can apply lemma 3.3. We make the short-exact sequence into a small extension by defining

\( V_3 := (\mathfrak{m}^3 + imo_2)/(\mathfrak{m}^4 + \mathfrak{m} \cdot imo_2) \) and \( \mathring{T}_4^1 := T^1/(\mathfrak{m}^4 + \mathfrak{m} \cdot imo_2) \),

hence we obtain

\[
0 \longrightarrow V_3 \longrightarrow \mathring{T}_4^1 \longrightarrow T_3^1/imo_2 \longrightarrow 0 \]

which means we want to lift \( \phi_2 \) to \( \mathring{T}_4^1 \). Note that \( V_3 \) is a vector space over \( \mathbb{F}_p \). From lemma 3.3 we get a unique element \( \text{obs}_A(\phi_2) \in \text{Ext}_A^2(\mathbb{F}_p, V_3) \cong \text{Mor}(T^2, V_3) \) which has to be 0 in order to get a lifting of \( \phi_2 \). Hence the obstructions on the second level will lie in \( V_3 \).

Let \( o(\phi_2) \) be \( \text{obs}_A(\phi_2) \) as an element in \( \text{Mor}(T^2, T_3^1/(\mathfrak{m}^4 + \mathfrak{m} \cdot imo_2)) \) and consider the following diagram where the four arrows \( \sim \sim \) denote two
short-exact sequences:

\[
\begin{array}{c}
\xymatrix{ & T^2 \ar[rr]^{o_2} & & V_2 \ar[dd]_{o_3} & \\
& & T_1/(m^4 + m \cdot imo_2) \ar[rr]_\rho & & T_1^{im_2} \\
V_3 \ar[rrrr]_\varpi & \sim & T_1/(m^4 + m \cdot imo_2 + imo(\phi_2)) \ar[rr]_\varphi & & T_1^{im_2}/imo_2 \\
& & C \ar[rr]_\sim & & T_1^{im_2}/imo_3 \\
& & & A_4 \ar[rr]_\phi & & A
}\end{array}
\]

Recall that \( imo_2 \) really lies in \( V_2 := \frac{m^2}{m^3} \), and so \( \mu \) exists as a factorization of \( \pi_2 \). It also factors \( \pi_2 \), and induces a map \( \varpi : V_3 \to V_2 \), which again induces a map \( \varpi_* : \text{Ext}_A^2(F, V_3) \to \text{Ext}_A^2(F, V_2) \). The coherence is therefore coming from the functoriality of the obstructions, which says that the map \( \varpi_* \) sends \( obs_A(\phi_2) \) to \( obs_A(\phi_1) \).

For the isomorphisms \( B \) and \( C \) in the diagram, we have by lemma 3.3 a universal property for \( imo(\phi_2) \) similar to the one for \( imo_2 \), with the quotient \( T_1/(m^4 + m \cdot imo_2 + imo(\phi_2)) \) being the maximal factor group of \( T_1/(m^4 + m \cdot imo_2) \) making a lifting of \( \phi_2 \) possible. So the isomorphism \( C \) follows by a similar argument as for the isomorphism \( \psi \).

To show that \( B \) is an isomorphism, we first choose \( o_3 \) such that \( \rho_*(o_3) = o(\phi_2) \) where \( \rho_* \) is the morphism

\[
\rho_* : \text{Mor}(T^2, T_1^4) \to \text{Mor}(T^2, T_1/(m^4 + m \cdot imo_2))
\]

induced by \( \rho \) in the diagram above. Then, by construction, the map \( B \) is a canonical map. Note that \( m \cdot imo_2 = m \cdot imo(\phi_2) \) modulo \( m^4 \) and \( m \cdot imo_2 \subseteq imo_3 \). Since the map \( T_1/(m^4 + m \cdot imo_2) \to T_1^4/imo_3 \) factorizes via \( T_1/(m^4 + m \cdot imo_2 + imo(\phi_2)) \), we have shown the isomorphism \( B \).

We have now seen the first non-trivial step of the construction \( o_A \) in detail, and the construction continues: When we have reached \( o_{n-1} \), we will have
done it in such a way that $imo_{n-1}$ will contain all the obstructions $imo_k$, $k < n - 1$.

Hence, for lifting $\phi_{n-1}$, we have the diagram

$$
0 \longrightarrow (m^n + imo_{n-1})/m^{n+1} \longrightarrow T^1_{n+1} \longrightarrow T^1_{n}/imo_{n-1} \longrightarrow 0
$$

and, similarly as for the first step, the obstructions on this level will lie in (3.5)

$$
V_n := (m^n + imo_{n-1})/(m^{n+1} + m \cdot imo_{n-1}).
$$

As before, we get an element $o(\phi_{n-1})$, and again, we can combine the ideals $m \cdot imo_{n-1}$ and $imo(\phi_{n-1})$ to get $imo_n$ and a lifting

$$
\phi_n : A \longrightarrow T^1_{n+1}/imo_n.
$$

To finish, the map $o_A : T^2 \longrightarrow T^1$ will be the inverse limit of the system $\{o_n : T^2 \longrightarrow T^1/m^{n+1}\}_{n \geq 1}$, moreover we get that the liftings $\{\phi_n\}_{n \geq 1}$ induce isomorphisms

$$
A/m^{n+1} \cong T^1/(m^{n+1} + imo_n)
$$

for all $n \geq 1$, and so for a complete local $\mathbb{F}_p$-algebra $A$, $A \cong T^1/(imo_A)$.

\[\Box\]

4. The Connection Between Sections 2 and 3

From the proofs of theorems 2.9 and 3.7 we see that there is a clear analogy between the construction of the obstruction morphisms in the group-situation and the algebra-situation. This is not surprising, since the obstruction calculus is categorical.

What will help us now is that there is a connection between the obstructions involved in the two situations when we let $A$ from section 3 be $\mathbb{F}_p G$, the modular group algebra of a finite $p$-group $G$.

Let us first see why we wanted to use the $\mathcal{M}$-series rather than the lower central $p$-series: The $\mathcal{M}$-series was first introduced by Jennings, and in [3] he proves that this series is the same as the modular dimension subgroup-series, i.e. $g \in \mathcal{M}_i(G)$ if and only if $(g - 1) \in (IG)^i$, where $IG$ is the augmentation ideal of $\mathbb{F}_p G$. We know that $\mathbb{F}_p G$ is local with $IG$ being the maximal ideal, so we get the following lemma.

**Lemma 4.1.** Let $G$ be a $p$-group, $\mathbb{F}_p G$ the group algebra with $m$ the maximal ideal. Then the inclusion

$$
\mathcal{M}_i(G)/\mathcal{M}_{i+1}(G) \longrightarrow m^i/m^{i+1}
$$
is a linear map for all $i \geq 1$.

Proof. Using Jennings' result, we can let

$$i(g.M_{i+1}(G)) = (g - 1) + m^{i+1}.$$ 

Since $gh - 1 = (g - 1) + (h - 1) + (g - 1)(h - 1)$, the lemma follows. 

This gives us one connection between the group and the group algebra. We had an obstruction lemma as our starting point in both situations (lemmas 2.4 and 3.3), and in the proofs we used the dual bar resolution and the Hochschild cocomplex respectively. This gives us another connection: We know that $\prod^n G$ is an $\mathbb{F}_p$-basis for $\prod^n \mathbb{F}_p G$, so, referring to definitions 2.2 and 3.1, we have that $B^n(G, V) \simeq C^n(\mathbb{F}_p G, V)$. Furthermore, for a trivial module $V$, the differentials in the two definitions are the same, so

$$H^n(G, V) \simeq HH^n(\mathbb{F}_p G, V)$$

for $n \geq 0$, which can also be seen from definition 2.1 and the isomorphism (3.1).

We will now start comparing the obstruction calculus for $G$ (theorem 2.9) with the obstruction calculus for $\mathbb{F}_p G$ (theorem 3.7). We have seen that the obstructions we get from the categories $\mathcal{D}$ (definition 2.3) and $\mathcal{E}$ (definition 3.2) are functorial. To compare the categories, we make the following observation.

Lemma 4.2. A diagram in $\mathcal{E}$ for $A$ where $A = \mathbb{F}_p G$ induces a diagram in $\mathcal{D}$ for $G$.

Proof. Let

$$0 \to V \to B_1 \xrightarrow{\pi} B_2 \to 0$$

be a diagram in $\mathcal{E}$. Then

$$1 \to 1 + V \to \pi^{-1}(\phi(G)) \xrightarrow{\pi} \phi(G) \to 1$$

is a diagram in $\mathcal{D}$. 

\qed
We are now ready for the main theorem. The essential point is to prove that the reconstruction of the \( p \)-group \( G \) from the obstruction calculus in section 2 can be done using only information from \( \mathbb{F}_p G \).

**Theorem 4.3.** The isomorphism class of a finite \( p \)-group \( G \) is determined by its modular group algebra \( \mathbb{F}_p G \).

**Proof.** We let \( A \) denote \( \mathbb{F}_p G \), and as before we use \( G_n \) and \( T_n^1 \) for \( G/\mathcal{M}_n(G) \) and \( T_1^1/\mathcal{M}_n(T_1^1) \) respectively.

We know that \( G_2 = T_2^1 \) depends only on \( H^1(G, \mathbb{F}_p) = \text{Ext}_A^1(\mathbb{F}_p, \mathbb{F}_p) \). By induction we will show that \( G_n \) can be constructed using obstructions depending only on \( A \). To show this we will make frequent use of the \( \mathcal{M} \)-series.

Assume that \( G_{n-1} \) has been constructed, and consider as in section 2 (see diagram (2.12)), the exact sequences that depend only upon \( G_{n-1} \) and \( A \):

\[
\begin{array}{ccccccc}
1 & \longrightarrow & K'_n & \longrightarrow & T_n^1 & \longrightarrow & G_{n-1} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & K'_n/\mathcal{M}_2(K'_n) & \longrightarrow & T_n^1/\mathcal{M}_2(T_n^1) & \longrightarrow & G_{n-1} & \longrightarrow & 1 \\
\downarrow & \simeq & \downarrow & \simeq & \downarrow & \simeq & \downarrow & \simeq & \downarrow \\
1 & \longrightarrow & K_n & \longrightarrow & G_n' & \longrightarrow & G_{n-1} & \longrightarrow & 1 \\
\end{array}
\]

Then we have \( K_n \subset \mathcal{M}_{n-1}(G_n') \subset \mathcal{M}_n(G_n') = \{1\} \), \( \mathcal{M}_2(K_n) = 1 \).

Note that \( G_{n-1} = G_n'/\mathcal{M}_{n-1}(G_n') \). We have another diagram that only depends upon the structure of \( A \):

(4.4)

\[
\begin{array}{ccccccc}
D : 1 & \longrightarrow & K_n & \longrightarrow & G_n' & \longrightarrow & G_{n-1} & \longrightarrow & 1 \\
\downarrow & \simeq & \downarrow & \simeq & \downarrow & \simeq & \downarrow & \simeq & \downarrow \\
E : 0 & \longrightarrow & V_n & \longrightarrow & \mathbb{F}_p[G_n']/m^n & \longrightarrow & \mathbb{F}_p[G_{n-1}]/m^{n-1} & \longrightarrow & 0 \\
\end{array}
\]
Since $K_n \subset M_{n-1}(G'_n)$, $K_n \subset Z(G'_n)$, \{g - 1\$g \in K_n\} \subset m^{n-1}$, and since $V_n$ is the ideal generated by \{g - 1\$g \in K_n\} in $\mathbb{F}_p[G'_n]$, we have that $m \cdot V_n = 0$ in $B'_n$.

Clearly, $K_n \xrightarrow{\tau} V_n$ where $\tau(g) = g - 1$ is injective. The obstruction $imo_D \subset K_n$ is mapped to $imo_{D'} \subset V_n + 1 := V'_n$ where $D'$ is the diagram

$$D': \quad 1 \longrightarrow V'_n \longrightarrow \pi^{-1}_2(G_{n-1}) \longrightarrow G_{n-1} \longrightarrow 1 \longrightarrow G,$$

and the obstruction $imo_E \subset V_n$ has to be equal to $\tau(imo_{D'})$.

In fact, the morphism $A \longrightarrow B_{n-1}$ can be (minimally) lifted to $A \longrightarrow B'_n/imo_E$, therefore the morphism $G \longrightarrow G_{n-1}$ can be lifted to $G \longrightarrow \pi^{-1}_2(G_{n-1})/(imo_E + 1)$, which gives $imo_{D'} - 1 \subset imo_E$.

Now, the morphism $G \longrightarrow G_{n-1}$ can be lifted to $G \longrightarrow G'_n/imo_D$, therefore $A \longrightarrow B_{n-1}$ can be lifted to

$$A \longrightarrow B'_n/(\iota(imo_D) - 1) = B'_n/(imo_{D'} - 1).$$

Hence $imo_E \subset imo_{D'} - 1$, and so $imo_E = imo_{D'} - 1$.

This means that $\iota(imo_D) = 1 + imo_E$, therefore $imo_D$ is (entirely) determined by the obstruction $imo_E$, therefore

$$G_n = G'_n/imo_D$$

by section 2 (see diagram (2.12)), and we are done.

\[ \Box \]

We note that $\lim_{\leftarrow n} B_n = A$.

As a final remark we note that the last proof also gives us a criterion for when a local (complete) $\mathbb{F}_p$-algebra $A$ with $\mathbb{F}_p$ as the only simple module is the group algebra $\mathbb{F}_pG$ for a $p$-group $G$: the necessary and sufficient conditions are that, inductively, the map $\tau: K_n \longrightarrow V_n$ in the diagram (4.4) is injective and that $imo'_E \subset \tau(K_n) \subset V_n$.

\textbf{References}


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