INFINITE DIMENSIONAL ANALYSIS OF PURE JUMP LÉVY PROCESSES ON THE POISSON SPACE

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ABSTRACT. We develop a white noise calculus for pure jump Lévy processes on Poisson space. This theory covers the treatment of Lévy processes of unbounded variation. The starting point of the theory is a novel construction of a distribution space. This space inherits many of the nice properties of the classical Schwartz space, but differs severely in its behaviour at zero. We apply Minlos' theorem to this space and get a white noise measure on this space which satisfies the first condition of analyticity and which is non-degenerate. Furthermore we obtain generalized Charlier polynomials for all pure jump Lévy processes. We introduce Kondratiev test function and distribution spaces, the $\mathcal{S}$−transform and Wick product. We proceed to establish a differential calculus by using a transfer principle on Poisson spaces.

1. INTRODUCTION

The main objective of this paper is a suitable white noise framework for pure jump Lévy processes. There are several papers dealing with white noise analysis and pure jump Lévy processes (see e.g. [14], [13]), but as far as we know none of them present a framework which is suitable for all pure jump Lévy processes. Some restrictions, typically integrability conditions, are put on the Lévy measure. This paper presents a framework which works for all pure jump Lévy processes.

A pure jump Lévy process L with no drift is a martingale with independent and stationary increments, continuous in probability and with no Brownian motion part. The characteristic function of such a process is given by the Lévy-Khintchine formula in terms of a measure ν called the Lévy measure of the Lévy process. Hence pure jump Lévy processes with no drift can be characterized as Lévy processes with characteristic triplet $(0, 0, \nu)$ (see for instance [17] for more details). The Poisson space is a natural space for dealing with Lévy processes for several reasons. One of them is that Lévy processes in general do not possess the chaos representation property with respect to the Lévy process itself. However, every square integrable functional of the path of a pure jump Lévy process has according to [11] a chaos representation with respect to Poisson random measures. By viewing a Lévy process as an element in an appropriate Poisson space, we therefore obtain a more tractable framework. In addition, every functional on the Poisson space can be given by a chaos expansion in terms of generalized Charlier polynomials.

A usual starting point in white noise analysis is to apply the Bochner-Minlos theorem which gives the existence of a probability measure on the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. It

turns out that $\mathcal{S}'(\mathbb{R}^d)$ is not the most appropriate for dealing with pure jump Lévy processes since this choice would require restrictive conditions to be put on the Lévy measure. This is due to the fact that the Lévy measure in general has a singularity at zero. Section two is therefore devoted to the construction of a nuclear algebra $\tilde{\mathcal{S}}(X)$ which is a variation of the Schwartz space on the space X, but which is more suitable for our purpose. In fact our variant of the Schwartz space is topologically isomorphic to the Schwartz space modulo a certain subspace depending on the Lévy measure. Within this framework we show that any Lévy measure has a Radon-Nikodym derivative with respect to the Lebesgue measure in a generalized sense. We denote this derivative by $\dot{\nu}$. The Bochner-Minlos theorem is then used to prove the existence of a probability measure μ_λ with Poissonian characteristic functional with intensity $\lambda = \nu \times m$ (m being the Lebesgue measure) such that

$$(1.1) \qquad \int_{\tilde{\mathcal{S}}'(x)} e^{i<\omega,\phi>} d\mu_\lambda(\omega) = \exp\left(\int_X \left(e^{i\phi(x)} - 1\right) d\lambda(x)\right)$$

for all test functions ϕ in the Schwartz space $\tilde{\mathcal{S}}(X)$. The continuity of the functional on the righthand side of (1.1) follows from the existence of the generalized Radon-Nikodym derivative of the Lévy measure with respect to the Lebesgue measure. By using an idea of Us [23], we can prove that μ_λ satisfies the first condition of analyticity. Furthermore, we show that the measure μ_λ is non-degenerate in the sense of [12]. We then have all we need in order to have a well defined system of orthogonal generalized Charlier polynomials. The construction and existence of such polynomials is the topic of section 3. This construction is similar to the constructions in [1] and [13].

We proceed in section 4 by extending the chaos expansion in terms of Charlier polynomials treated in section 3. This is done by a Kondratiev type of construction of stochastic test functions and stochastic distributions. Our construction corresponds to the $(\mathcal{S}(J))^{-1}$ distribution space in [23].

For all Lévy measures we therefore have that the process L given by

$$L_t := < \omega - \dot{\nu}, z1_{[0,t]} > = \int_0^t \int_{\mathbb{R}_0} z\big(\omega(y,z) - \dot{\nu}(y,z)\big) dzdy$$

is well defined. The function $z1_{[0,t]}$ may not be in $L^2(\lambda)$. In this case L_t is not in $L^2(\mu_\lambda)$. Hovewer L is a well defined process in the stochastic distributional sense. From the definition of μ_λ given by (1.1) and the Lévy-Khintchine formula it follows that the stochastic process $t \mapsto L_t$ is a pure jump Lévy process with no drift and Lévy measure ν.

In section 5 we define the $\mathcal{S}$-transform and the Wick product. This is included since both the $\mathcal{S}$-transform and the Wick product are useful tools. Even though our definition of the $\mathcal{S}$-transform is slightly different from the transformation in [1] we show by a little argument that the relation between the $\mathcal{S}$-transform, the Kondratiev distributions and holomorphic functions proved in [1] is valid for our definition of the $\mathcal{S}$-transform as well.

The measure μ_λ does not admit a satisfactory construction of a differential calculus on $\tilde{\mathcal{S}}'(X)$. In section 6 we show how it is possible to circumvent this problem by transporting analytical structures from configuration spaces using a unitary isomorphism in a similar fashion as in [13]. This yields a Poisson measure π_L on the configuration space $(\Gamma, \mathcal{B}(\Gamma))$ such that $L^2(\Gamma, \pi_\lambda)$ is unitary isomorphic to $L^2(\mu_\lambda)$.

Section 7 deals with the Poissonian gradient and Skorohod integration. We start by proving that the Poissonian gradient ∇^P and the operator D defined via its action on chaos expansions

in terms of Charlier polynomials are essentially equal. The Skorohod integral is then defined by its action on the chaos expansions of parametrised families of stochastic distributions. We show that Skorohod integration is the dual of the Poissonian gradient ∇^P. Hence Skorohod integration is equal to $(\nabla^P)^*$ and we can link the Skorohod integral with results for $(\nabla^P)^*$ in [13]. By using the duality between ∇^P and the Skorohod integral, we can also prove relations between $\mathcal{S}$-tranformation, ordinary derivation and the Skorohod integral. Finally, we prove a relationship between the Skorohod integral, the Wick product and the Lebesgue integral.

2. Construction of the nuclear spaces

First recall the following well known properties about the Schwartz space and its dual, the space of tempered distributions. Define the operator

$$A := \frac{1}{2}\left(-\frac{d^2}{dx^2} + (x^2 - 1) \right)$$

For each $p \in \mathbb{N}$, introduce the norm

$$|f|_p := \left| (A^{\otimes k} + 1)^p f \right|_0$$

where $|\cdot|_0$ is the norm of $L^2(\mathbb{R}^k)$. Define the Hilbert spaces $\mathcal{S}_p(\mathbb{R}^k)$ as the completion of $\mathcal{S}(\mathbb{R}^k)$ with respect to the norm $|\cdot|_p$ and denote by $\mathcal{S}_{-p}(\mathbb{R}^k)$ its dual. Then the Schwartz space $\mathcal{S}(\mathbb{R}^k)$ is the projective limit of $\mathcal{S}_p(\mathbb{R}^k)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^k)$ the inductive limit of $\mathcal{S}_{-p}(\mathbb{R}^k)$.

Let ξ_n denote the n'th Hermite function (see for instance [18, pp. 142]). The set of Hermite functions $\{\xi_n\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R})$. The Hermite functions are closely related to the Schwartz space in the following way: Let $f = \sum_{n=0}^{\infty} a_n \xi_n$, then

$$A^p f = \sum_{n=0}^{\infty} a_n n^p \xi_n$$

and hence $|f|_p^2 = \sum_{n=0}^{\infty} a_n^2 (1+n)^{2p}$. One can from this identity derive the N-representation theorem for $\mathcal{S}(\mathbb{R}^k)$, which states that the Schwartz space $\mathcal{S}(\mathbb{R}^k)$, is topologically isomorphic to the following space: Let s_k be the set of multisequences $\{a_\alpha\}_{\alpha \in \mathbb{N}^k}$ such that

$$(2.1) \qquad \qquad \sup_{\alpha \in \mathbb{N}^k} |a_\alpha| |\alpha|^p < \infty \text{ for each } p \in \mathbb{N}$$

We equip s_k with the topology generated by the seminorms

$$\left\| \{a_\alpha\}_{\alpha \in \mathbb{N}^k} \right\|_\beta^2 = \sum_\alpha (1+\alpha)^{2\beta} |a_\alpha|^2$$

where $\beta \in \mathbb{N}^k$ and $(1+\alpha)^{2\beta} := \prod_{i=1}^{k} (1+\alpha_i)^{2\beta_i}$ (see [18, pp. 143] for more details). In order to ease notation we make the following conventions:

$$X := \mathbb{R}^d \times \mathbb{R}_0 \qquad \text{for some } d \in \mathbb{N}$$

where

$$\mathbb{R}_0 := \mathbb{R} \setminus \{0\} \qquad \mathbb{R}_- := (-\infty, 0) \qquad \mathbb{R}_+ := (0, \infty)$$

We will now define a set of orthonormal functions by a tranformation of the Hermite functions. For each $n \in \mathbb{N}$ define the functions $\tilde{\xi}_n$ by

$$\tilde{\xi}_n(x) := \frac{\xi_n(-\frac{1}{x})}{x}$$

where ξ_n is the n'th Hermite function. Notice that by a change of variable we have,

$$\int_{\mathbb{R}} \xi_n(x)\xi_m(x)dx = \int_{\mathbb{R}_0} \xi_n\left(-\frac{1}{y}\right)\xi_m\left(-\frac{1}{y}\right)\frac{1}{y^2}dy$$
$$= \int_{\mathbb{R}_0} \tilde{\xi}_n(y)\tilde{\xi}_m(y)dy$$

from which we see that $\{\tilde{\xi}_n\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{R}_0)$ since the Hermite functions form an orthonormal basis for $L^2(\mathbb{R})$.

Let $\alpha \in \mathbb{N}^k$ and set $\tilde{\xi}_\alpha = \prod_{i=1}^{k} \tilde{\xi}_{\alpha_i}$. Define the space $\mathcal{S}(\mathbb{R}_0^k)$ by

$$\mathcal{S}\left(\mathbb{R}_0^k\right) := \left\{ \sum_{\alpha \in \mathbb{N}^k} a_\alpha \tilde{\xi}_\alpha : \sup_\alpha |a_\alpha||\alpha|^m < \infty \text{ for each } m \in \mathbb{N} \right\}$$

For each $\beta \in \mathbb{N}^k$ introduce the norm

$$\|f\|_\beta^2 := \sum_{\alpha \in \mathbb{N}^k}^{\infty} (1+\alpha)^{2\beta}|a_\alpha|^2$$

For each $p \in \mathbb{N}$ we define the spaces $\mathcal{S}_p\left(\mathbb{R}_0^k\right)$ as the completion of $\mathcal{S}(\mathbb{R}_0^k)$ with respect to the norm $\|\cdot\|_\beta$ with $\beta = p$. It is straightforward to check that $\mathcal{S}_{-p}(\mathbb{R}_0^k)$ is the dual of $\mathcal{S}_p(\mathbb{R}_0^k)$. We equip $\mathcal{S}(\mathbb{R}_0^k)$ with the projective topology generated by the norms $\|\cdot\|_\beta$, and define $\mathcal{S}'(\mathbb{R}_0^k)$ as the inductive limit of $\mathcal{S}_{-p}(\mathbb{R}_0^k)$. It follows that $\mathcal{S}'(\mathbb{R}_0^k)$ is the dual of $\mathcal{S}(\mathbb{R}_0^k)$. Notice that the unitary operator $T : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}_0)$ given by

$$(2.2) \qquad\qquad T\left(\sum_\alpha a_\alpha \xi_\alpha\right) := \sum_\alpha a_\alpha \tilde{\xi}_\alpha$$

is a topological isomorphism between $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R}_0)$.

For $p \in \mathbb{Z}$ we define the spaces $\mathcal{S}_p(X)$,

$$\mathcal{S}_p(X^k) = \mathcal{S}_p(\mathbb{R}^{kd}) \otimes \mathcal{S}_p(\mathbb{R}_0^k)$$

Then obviously, $\mathcal{S}_{-p}(X)$ is the dual of $\mathcal{S}_p(X)$. Define the test function space $\mathcal{S}(X)$ by

$$\mathcal{S}(X) = \bigcap_{p \in \mathbb{N}} S_p(X)$$

equipped with the projective limit topology, and the corresponding distribution space $\mathcal{S}'(X)$ by

$$\mathcal{S}'(X) = \bigcup_{p \in \mathbb{N}} \mathcal{S}_{-p}(X)$$

with the inductive limit topology. From the construction of $\mathcal{S}(X^k)$, $\mathcal{S}'(X^k)$, $\mathcal{S}(\mathbb{R}_0^k)$ and $\mathcal{S}'(\mathbb{R}_0^k)$ we have that $\mathcal{S}(\mathbb{R}_0^k)$ is topologically isomorphic to $\mathcal{S}(\mathbb{R}^k)$ and $\mathcal{S}(\mathbb{R}_0^k)$ topologically isomorphic

to $\mathcal{S}(\mathbb{R}^{2k})$. Therefore, $\mathcal{S}'(\mathbb{R}_0^k)$ is topologically isomorphic to $\mathcal{S}'(\mathbb{R}^k)$ and $\mathcal{S}'(X^k)$ topologically isomorphic to $\mathcal{S}'(\mathbb{R}^{k(d+1)})$. For the rest of the paper, $\|\cdot\|_p$ will denote the norm on $\mathcal{S}_p(X)$.

Lemma 2.1. *Let ν be a Lévy measure. Then there exists a constant $C < \infty$ such that*

$$\int_{\mathbb{R}_0} \big|\, \tilde{\xi}_n(z)\,\big|\, \nu(dz) < C(n+1)^2$$

Proof. We need the following upper bound for the Hermite functions ξ_n (See [22, p. 26]).

$$|\xi_n(x)| \leq M(2n+1)^{-\frac{1}{12}} \quad , x^2 \leq 2(2n+1)$$
$$\leq Me^{-\gamma x^2} \quad , x^2 > 2(2n+1)$$

for positive constants M and γ. Thus we obtain,

$$\int_0^\infty \big|\, \tilde{\xi}_n(z)\,\big|\, \nu(dz) \leq \int_0^{\frac{1}{\sqrt{2(2n+1)}}} \left| \frac{\tilde{\xi}_n(z)}{z^2} \right| z^2 \nu(dz) + \int_{\frac{1}{\sqrt{2(2n+1)}}}^\infty \left| \frac{\tilde{\xi}_n(z)}{1 \wedge z^2} \right| 1 \wedge z^2 \nu(dz)$$

$$\leq M \int_0^{\frac{1}{\sqrt{2(2n+1)}}} \big(e^{-\gamma(\frac{1}{z})^2} z^{-3} \big) z^2 \nu(dz)$$

$$+ M \int_{\frac{1}{\sqrt{2(2n+1)}}}^\infty (2n+1)^{-\frac{1}{12}} 2^{\frac{3}{2}} (2n+1)^{\frac{3}{2}} (1 \wedge z^2) \nu(dz)$$

$$\leq C_1 \left(\int_0^1 z^2 \nu(dz) + (2n+1)^{\frac{17}{12}} \int_0^\infty (1 \wedge z^2) \nu(dz) \right)$$

$$\leq C(n+1)^2$$

for some positive constants $C_1 < \infty$ and $C < \infty$ since $\int_{\mathbb{R}_0} 1 \wedge z^2 \nu(dz) < \infty$. One gets analogously the same bound for the negative halfline. $\square$

Lemma 2.2. *Convergence in $\mathcal{S}(\mathbb{R}_0)$ implies pointwise convergence.*

Proof. Since this is true for the Schwartz space $\mathcal{S}(\mathbb{R})$ and since $\mathcal{S}(\mathbb{R}_0) = T\big(\mathcal{S}(\mathbb{R})\big)$ under the isomorphism T given by (2.2) the result follows. $\square$

Proposition 2.3. *Let ν be a Lévy measure. Then there exists an element denoted by $\dot{\nu}$ in $\mathcal{S}'(\mathbb{R}_0)$ such that for all $\phi \in \mathcal{S}(\mathbb{R}_0)$*

$$< \dot{\nu}, \phi >= \int_{\mathbb{R}_0} \phi(z) \nu(dz)$$

Proof. Define the functional $F(\phi) = \int_{\mathbb{R}_0} \phi(z) \nu(dz)$. We want to prove that F is a linear functional on $\mathcal{S}(\mathbb{R}_0)$. Let $\phi_k = \sum_{n=1}^\infty a_n^k \tilde{\xi}_n$ be a sequence of functions in $\mathcal{S}(\mathbb{R}_0)$ converging to $\phi = \sum_{n=1}^\infty a_n \tilde{\xi}_n$ in $\mathcal{S}(\mathbb{R}_0)$. This means that

$$\sum_{n=1}^\infty (1+n)^{2\beta} |a_n - a_n^k|^2 \to 0 \quad \text{as } k \to \infty$$

for all $\beta \in \mathbb{N}$. By Lemma 2.1, Lemma 2.2 and the Cauchy-Schwartz inequality,

$$\big|\, F(\phi) - F(\phi_k)\,\big| \leq \int_{\mathbb{R}_0} \big|\, \phi(z) - \phi_k(z)\,\big|\, \nu(dz)$$

$$= \int_{\mathbb{R}_0} \left| \sum_{n=1}^{\infty} (a_n - a_n^k) \tilde{\xi}_n(z) \right| \nu(dz)$$

$$\leq \sum_{n=1}^{\infty} |a_n - a_n^k| \int_{\mathbb{R}_0} \left| \tilde{\xi}_n(z) \right| \nu(dz)$$

$$\leq \sum_{n=1}^{\infty} |a_n - a_n^k| C(1+n)^2$$

$$(2.3) \qquad \leq C \left(\sum_{n=1}^{\infty} (1+n)^{2p+4} |a_n - a_n^k|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} (1+n)^{-2p} \right)^{\frac{1}{2}}$$

Now, $\sum_{n=1}^{\infty} (1+n)^{-2p}$ is finite for $p \geq 1$. The continuity of F then follows from (2.3) since clearly $F(\phi_k)$ converges to $F(\phi)$ whenever ϕ_k converges to ϕ in $\mathcal{S}(\mathbb{R}_0)$. So, F is a linear functional on $\mathcal{S}(\mathbb{R}_0)$ and hence $F \in \mathcal{S}'(\mathbb{R}_0)$. The claimed result follows. $\qquad \square$

Corollary 2.4. *Let ν be a Lévy measure. There exists an element denoted by $1 \otimes \dot{\nu}$ in $\mathcal{S}'(X)$ such that*

$$< 1 \otimes \dot{\nu}, \phi > = \int_X \phi(y, z) \nu(dz) dy$$

for all $\phi \in \mathcal{S}(X)$, where dy denotes the Lebesgue measure on $\mathbb{R}^d$.

Proof. It is well known that $1 \in \mathcal{S}'(\mathbb{R})$. The claimed result therefore follows from Proposition 2.3. $\qquad \square$

The notation $\dot{\nu}$ is used to indicate that $\dot{\nu}$ is the Radon-Nikodym derivative of ν with respect to the Lebesgue measure in a generalized sense. With a slight abuse of notation we will frequently just write $\dot{\nu}$ instead of $1 \otimes \dot{\nu}$. We define the measure λ on X by

$$d\lambda(y, z) := \nu(dz) dy$$

where dy denotes the Lebesgue measure on $\mathbb{R}^d$. By Corollary 2.4, the generalized Radon-Nikodym derivative of λ with respect to the Lebesgue measure on X is $1 \otimes \dot{\nu}$. Later we will need the following:

Lemma 2.5. $\mathcal{S}(\mathbb{R}_0)$ *is dense in* $L^2(\nu)$.

Proof. Choose a compact interval $[k_1, k_2] \subseteq \mathbb{R}_+$ or $[k_1, k_2] \subseteq \mathbb{R}_-$. Define the algebra,

$$\mathcal{A} := \left\{ \bigcup_{n=1}^{N} I_n : I_n \subset [k_1, k_2] \text{ interval} \right\}$$

We have that $\sigma(\mathcal{A}) = \mathcal{B}([k_1, k_2])$. For each interval $I_n \subset [k_1, k_2]$, the function 1_{I_n} can be approximated by an element of $\mathcal{S}(\mathbb{R}_0)$. For all $U \in \mathcal{B}([k_1, k_2])$ and $\epsilon > 0$ there exists an $A \in \mathcal{A}$ such that

$$\nu \big|_{[k_1, k_2]} (A \triangle U) < \epsilon$$

where $\triangle$ denotes the symmetric difference (see [2]). If we apply the above to $K_n := [-n, -\frac{1}{n}] \cup [\frac{1}{n}, n]$ the proof follows since K_n converges monotonically to $\mathbb{R}_0$. $\qquad \square$

Denote by $L^2(X^n, \lambda)$ the space of all functions on X^n which are square integrable with respect to $\lambda^{\times n}$. We let $(\cdot, \cdot)_\lambda$ denote the innerproduct on $L^2(X, \lambda)$ and $|\cdot|_\lambda$ the corresponding norm on this space.

Proposition 2.6. *The triplet*

$$\mathcal{S}(X) \hookrightarrow L^2(\lambda) \hookrightarrow \mathcal{S}'(X)$$

is a Gel'fand triplet.

Proof. By the proof of Lemma 2.1 and the arguments of Lemma 2.3 one verifies that the scalar product $(\cdot, \cdot)_\lambda$ of $L^2(\lambda)$, restricted to $\mathcal{S}(X) \times \mathcal{S}(X)$, is continuous. Since $\mathcal{S}(X)$ is dense in $L^2(\lambda)$ the proof follows. $\qquad \square$

Now we introduce our space $\tilde{\mathcal{S}}(X)$ which will serve as our starting point for the construction of the white noise measure.

Definition 2.7. We define the space $\tilde{\mathcal{S}}(X)$ as follows:

$$\tilde{\mathcal{S}}(X) := \mathcal{S}(X)/\mathcal{N}_\lambda$$

where

$$\mathcal{N}_\lambda := \left\{ \phi \in \mathcal{S}(X) : \|\phi\|_{L^2(\lambda)} = 0 \right\}$$

Note that $\mathcal{N}_\lambda$ is a closed ideal in $\mathcal{S}(X)$ (see Proposition 2.6). Let $\tilde{\mathcal{S}}'(X)$ denote the topological dual of $\tilde{\mathcal{S}}(X)$.

Further, for all $p \in \mathbb{N}$ define the norms:

(2.4) $$\left\| \hat{\phi} \right\|_{p,\lambda} := \inf_{\psi \in \mathcal{N}_\lambda} \left\| \phi + \psi \right\|_p$$

Theorem 2.8. *The space $\tilde{\mathcal{S}}(X)$ in Definition 2.7 is a nuclear algebra with a compatible system of norms given by (2.4). Moreover the Cauchy-Bunjakowski inequality holds, that is for all q there exists an M_q such that for all $\phi, \psi \in \tilde{\mathcal{S}}(X)$ we have*

$$\left\| \phi \psi \right\|_{q,\lambda} \le M_q \left\| \phi \right\|_{q,\lambda} \left\| \psi \right\|_{q,\lambda}$$

Proof. The first statement of the proof follows from [8, p.72]. Note that $\mathcal{S}(X)$ is a nuclear algebra since it is topologically isomorphic to the classical Schwartz space. As for the Cauchy-BunjaKowski intequality let us choose $\phi, \psi \in \mathcal{S}(X)$ and $\rho_1, \rho_2 \in \mathcal{N}_\lambda$. Then we have

$$\left\| (\phi + \rho_1)(\psi + \rho_2) \right\|_q \le M_q \left\| \phi + \rho_1 \right\|_q \left\| \psi + \rho_2 \right\|_q$$

and

$$\left\| (\phi + \rho_1)(\psi + \rho_2) \right\|_q = \left\| \phi \psi + \rho_3 \right\|_q \ge \left\| \hat{\phi} \hat{\psi} \right\|_{q,\lambda}$$

where $\rho_3 \in \mathcal{N}_\lambda$. The result follows. $\qquad \square$

Remark 2.9. The space $\tilde{\mathcal{S}}'(X)$ is isomorphic to the orthogonal complement of $\mathcal{N}_\lambda$ (see e.g. [20]).

Lemma 2.10. *The functional $L(\hat{\phi}) := \int_X \phi(x)d\lambda(x)$ satisfies the inequality*

$$\big| L(\hat{\phi}) \big| \leq M_p \big\| \hat{\phi} \big\|_{p,\lambda}$$

for all $p \geq p_0$ which yields the continuity of the functional L.

Proof. Let $\psi \in \mathcal{N}_\lambda$. Then by Corollary 2.4 we have

$$\big| L(\hat{\phi}) \big| = \left| \int_X \phi + \psi d\lambda(x) \right| \leq M_p \big\| \phi + \psi \big\|_p$$

The result follows by taking the infimum over all $\psi \in \mathcal{N}_\lambda$. $\square$

Theorem 2.11. *Let ν be a Lévy measure. There exists a probability measure μ_λ on $\tilde{\mathcal{S}}'(X)$ such that for all $\phi \in \tilde{\mathcal{S}}(X)$,*

$$(2.5) \qquad \int_{\tilde{\mathcal{S}}'(X)} e^{i<\omega,\phi>} d\mu_\lambda(\omega) = \exp\left(\int_X \big(e^{i\phi} - 1 \big) d\lambda \right)$$

Moreover, there exists a $p_0 \in \mathbb{N}$ such that $\dot{\nu} \in \tilde{\mathcal{S}}_{-p_0}(X)$ and a natural number $q_0 > p_0$ such that the imbedding operator $\tilde{\mathcal{S}}_{q_0}(X) \hookrightarrow \tilde{\mathcal{S}}_{p_0}(X)$ is Hilbert-Schmidt and

$$\mu_\lambda\big(\tilde{\mathcal{S}}_{-q_0}(X) \big) = 1$$

Proof. Consider the functional Φ given by

$$\Phi(\phi) := \exp\left(\int_X \big(e^{i\phi(x)} - 1 \big) d\lambda(x) \right)$$

Obviously $\Phi(0) = 1$ and since Φ is the Fourier transform of Poissonian variables it follows that Φ is positive definite. We need to show that Φ is continuous. Obviously it is enough to show that $F(\phi) = \ln(\Phi(\phi))$ is continuous. We have that

$$|F(\phi) - F(\phi_k)| = \left| \int_X \big(e^{i\phi(x)} - 1 \big) d\lambda(x) - \int_X \big(e^{i\phi_k(x)} - 1 \big) d\lambda(x) \right|$$

$$= \left| \int_X \big(e^{i\phi(x)} - e^{i\phi_k(x)} \big) d\lambda(x) \right|$$

$$(2.6) \qquad \leq \int_X \big| \phi(x) - \phi_k(x) \big| d\lambda(x)$$

The continuity of F on $\tilde{\mathcal{S}}(X)$ then follows from (2.6) and Lemma 2.10. By the Bochner-Minlos theorem for conuclear spaces (see for instance [10, thm. 1.1, pp. 2]) the result follows. $\square$

From now on we will let p_0 and q_0 denote the numbers described in Theorem 2.11.

Lemma 2.12 (1. condition of analyticity). *μ_λ satisfies the first condition of analyticity, that is there exists $\epsilon > 0$ such that*

$$\int_{\tilde{\mathcal{S}}'(X)} \exp\big(\epsilon \|\omega\|_{-q_0,\lambda} \big) d\mu_\lambda(\omega) < \infty$$

Proof. The proof follows the argument of [23, Lemma 3]. Introduce the moment functions of μ_λ, which by a criterion of Cramer [4] can be expressed by

$$M_n(\phi) := \int_{\tilde{\mathcal{S}}'(X)} <\omega, \phi>^n d\mu_\lambda(\omega) = \frac{d^n L(t\phi)}{dt^n}\Big|_{t=0}$$

for every $\phi \in \tilde{\mathcal{S}}(X)$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Denote by Λ_n^k the set

$$\Lambda_n^k := \Big\{(\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k : \alpha_i \geq 1, \sum_{i=1}^k \alpha_i = n\Big\}$$

We can then deduce the following expression for M_n,

$$(2.7) \qquad M_n(\phi) = \sum_{k=1}^n \frac{n!}{k!} \sum_{\alpha \in \Lambda_n^k} \prod_{j=1}^k \frac{<\dot\nu, \phi^{\alpha_j}>}{\alpha_j!}$$

We have that

$$(2.8) \qquad |<\dot\nu, \phi>| \leq \|\dot\nu\|_{-p_0,\lambda}\|\phi\|_{p_0,\lambda} < \infty$$

By Theorem 2.8 there exists a constant C_{p_0} such that

$$(2.9) \qquad \|\phi\psi\|_{p_0,\lambda} \leq C_{p_0}\|\phi\|_{p_0,\lambda}\|\psi\|_{p_0,\lambda} \quad \text{for all } \phi, \psi \in \tilde{\mathcal{S}}(X)$$

By the inequalities (2.8) and (2.9) we obtain from (2.7) that

$$|M_n(\phi)| \leq \sum_{k=1}^n \frac{n!}{k!} \sum_{\alpha \in \Lambda_n^k} \prod_{j=1}^k \frac{\|\dot\nu\|_{-p_0,\lambda}\|\phi^{\alpha_j}\|_{p_0,\lambda}}{\alpha_j!}$$

$$\leq \sum_{k=1}^n \frac{n!}{k!} \sum_{\alpha \in \Lambda_n^k} \prod_{j=1}^k \frac{\|\dot\nu\|_{-p_0,\lambda}}{\alpha_j!} C_{p_0}^n \|\phi\|_{p_0,\lambda}^n$$

$$= F_n\big(\|\dot\nu\|_{-p_0,\lambda}\big) C_{p_0}^n \|\phi\|_{p_0,\lambda}^n$$

where $F_n(x)$ is the n'th moment of the Poisson distribution with intensity x. We know that for the Poisson distribution with intensity parameter $x = \|\dot\nu\|_{-p_0,\lambda} < \infty$ there exists a constant C_x such that for all $n \in \mathbb{N}$,

$$|F_n(\|\dot\nu\|_{-p_0,\lambda})| \leq n! C_{\|\dot\nu\|_{-p_0,\lambda}}^n$$

Hence there is a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $\phi \in \tilde{\mathcal{S}}(X)$,

$$|M_n(\phi)| \leq n! C^n \|\phi\|_{p_0,\lambda}^n$$

from which the claimed result follows (see [12, Lemma 3]). $\square$

3. Chaos expansion and orthogonal polynomials

Let Y, Z be linear topological spaces. For a mapping $f : Y \to Z$ we define the derivative in the weak sense at the point $y \in Y$ in the direction $\phi \in Y$ by

$$\nabla_\phi f(y) = \frac{d}{d\epsilon} f(y + \epsilon\phi)\big|_{\epsilon=0}$$

We say that this mapping is differentiable along a subspace $\Lambda \subset Y$ if the derivative $\nabla_\phi f(y)$ exists for any $\phi \in \Lambda$ and $\nabla_\phi f(y) = f'(y)\phi$ where $f'(y)$ is a linear continuous mapping of Λ into

Y. Let us introduce the differential operator ∇ for this linear mapping, that is $\nabla_\phi = < \nabla, \phi >$. For higher order differentials we introduce the notation:

$$\nabla_{\phi_1} \cdots \nabla_{\phi_n} = < \nabla, \phi_1 > \cdots < \nabla, \phi_n > = < \nabla \otimes \cdots \otimes \nabla, \phi_n \otimes \cdots \otimes \phi_n >$$

For $\phi \in \tilde{\mathcal{S}}_{q_0}(X)$ introduce the function $e(\phi) = e(\phi, \omega)$ given by:

$$e(\phi, \omega) := \exp\big(< \omega, \phi > - < \dot{\nu}, e^\phi - 1 > \big)$$

According to [1] it follows from the first condition of analyticity of μ_λ given by Lemma 2.12 that $e(\phi)$ is analytic in a neighborhood of zero on the space $\tilde{\mathcal{S}}_{q_0}(X)$. Set $l(x) := \ln(1 + x)$. The function l is analytic in a neighborhood of zero and $l(0) = 0$. Hence it follows that the function $\tilde{e}(\phi)$ given by

$$\tilde{e}(\phi, \omega) := \big(e(\cdot, \omega) \circ l \big)(\phi) = \exp\big(< \omega, \ln(1 + \phi) > - < \dot{\nu}, \phi > \big)$$

is analytic as a function of $\phi \in \tilde{\mathcal{S}}_{q_0}(X)$ where $\phi(x) > -1$ for all $x \in X$. For $\phi_1, \ldots, \phi_n \in \tilde{\mathcal{S}}(X)$ and $\omega \in \tilde{\mathcal{S}}'(X)$ define

$$P(\phi_1, \ldots, \phi_n; \omega) := \nabla^\theta_{\phi_1} \cdots \nabla^\theta_{\phi_n} \tilde{e}(\theta, \omega)|_{\theta=0}$$

Then from the analyticity at zero of $\tilde{e}(\theta, \omega)$ we obtain the following expansion:

$$\tilde{e}(t\theta, \omega) = \sum_{n=0}^\infty \frac{t^n}{n!} P(\theta, \ldots, \theta; \omega)$$

which is convergent in a neighborhood of zero. We have that

$$P(\phi_1, \ldots, \phi_n; \omega) = < \nabla^\theta \otimes \cdots \otimes \nabla^\theta \tilde{e}(\theta, \omega)|_{\theta=0}, \phi_1 \otimes \cdots \otimes \phi_n >$$
$$= < P_n(\omega), \phi_1 \otimes \cdots \otimes \phi_n >$$

where

$$(3.1) \qquad\qquad P_n(\omega) := \nabla^\theta \otimes \cdots \otimes \nabla^\theta \tilde{e}(\theta, \omega)|_{\theta=0}$$

We see that $< P_n(\omega), \phi^{(n)} >$ is well defined for all functions $\phi^{(n)} \in \tilde{\mathcal{S}}(X)^{\hat{\otimes}n}$, where $\tilde{\mathcal{S}}(X)^{\hat{\otimes}n}$ is the symmetric tensor product of $\tilde{\mathcal{S}}(X)$ with itself taken n times. This can be seen to be equal to all functions $f \in \tilde{\mathcal{S}}(X^n)$ such that $f = f(x_1, \ldots, x_n)$ is symmetric $\lambda^{\times n}$-almost everywhere in the variables $x_1, \ldots, x_n$ where $x_1, \ldots, x_n \in X$ (see (3.6) for more details). From [1] and [13] we have that

$$\left\{ < P_n(\omega), \phi^{(n)} >: n \in \mathbb{N} \cup \{0\}, \phi^{(n)} \in \tilde{\mathcal{S}}(X)^{\hat{\otimes}n} \right\}$$

forms a total set in $L^2(\mu_\lambda)$ and

$$(3.2) \qquad\qquad \tilde{e}(\phi, \omega) = \sum_{n=0}^\infty \frac{1}{n!} < P_n(\omega), \phi^{\otimes n} >$$

From the form of $\tilde{e}(\cdot)$ we see that the polynomials P_n are generalized Charlier polynomials. In order to emphasize this we will from now on write $C_n(\cdot) := P_n(\cdot)$. Moreover, from the relation between e and $\tilde{e}$ we get that

$$(3.3) \qquad\qquad e(\phi, \omega) = \sum_{n=0}^\infty \frac{1}{n!} < C_n(\omega), (e^\phi - 1)^{\otimes n} >$$

Proposition 3.1. *For any $m, n \in \mathbb{N}$, $\phi^{(n)} \in \tilde{\mathcal{S}}(X)^{\hat{\otimes}n}$ and $\psi^{(m)} \in \tilde{\mathcal{S}}(X)^{\hat{\otimes}m}$ we have*

$$\int_{\tilde{\mathcal{S}}'(X)} < C_n(\omega), \phi^{(n)} > < C_m(\omega), \psi^{(m)} > d\mu_\lambda(\omega) = \delta_{n,m} n! \big(\phi^{(n)}, \psi^{(n)} \big)_\lambda$$

Proof. The proof is similar to the proof of [13, Prop. 2.3]. Choose $\phi, \psi \in \tilde{\mathcal{S}}(X)$ and $z_1, z_2 \in \mathbb{C}$. Then

$$\int_{\tilde{\mathcal{S}}'(X)} \tilde{e}(z_1\phi, \omega) \tilde{e}(z_2\psi, \omega) d\mu_\lambda(\omega)$$

$$= \exp\big(- < \dot{\nu}, z_1\phi + z_2\psi > \big) \int_{\tilde{\mathcal{S}}'(X)} \exp\Big(< \omega, \ln(1 + z_1\phi) + \ln(1 + z_2\psi) > \Big) d\mu_\lambda(\omega)$$

$$= \exp\big(- < \dot{\nu}, z_1\phi + z_2\psi > \big) \int_{\tilde{\mathcal{S}}'(X)} \exp\Big(< \omega, \ln\big((1 + z_1\phi)(1 + z_2\psi)\big) > \Big) d\mu_\lambda(\omega)$$

$$= \exp\big(z_1 z_2 < \dot{\nu}, \phi\psi > \big)$$

$$= \exp\big(z_1 z_2 (\phi, \psi)_\lambda \big)$$

$$(3.4) \qquad = \sum_{n=0}^{\infty} \frac{z_1 z_2}{n!} \big(\phi^{\otimes n}, \psi^{\otimes n} \big)_\lambda$$

On the other hand we obtain by using (3.2),

$$\int_{\tilde{\mathcal{S}}'(X)} \tilde{e}(z_1\phi, \omega) \tilde{e}(z_2\psi, \omega) d\mu_\lambda(\omega)$$

$$(3.5) \qquad = \sum_{n,m=0}^{\infty} \frac{z_1^n z_2^m}{n! m!} \int_{\tilde{\mathcal{S}}'(X)} < C_n(\omega), \phi^{\otimes n} > < C_m(\omega), \psi^{\otimes m} > d\mu_\lambda(\omega)$$

By comparing the coefficients in the expressions (3.4) and (3.5) we see that the result holds for $\phi^{(n)} = (z_1\phi)^{\otimes n}$ and $\psi^{(m)} = (z_2\psi)^{\otimes m}$. The proof then follows by using the linearity and the polarization identity to extend the result to $\phi^{(n)} \in \tilde{\mathcal{S}}(X)^{\hat{\otimes}n}$ and $\psi^{(m)} \in \tilde{\mathcal{S}}(X)^{\hat{\otimes}m}$. $\qquad\square$

Definition 3.2. A function $F : \tilde{\mathcal{S}}'(X) \to \mathbb{C}$ of the form

$$F(\omega) = \sum_{n=0}^{N} < \omega^{\otimes n}, \phi^{(n)} >, \qquad \omega \in \tilde{\mathcal{S}}(X), N \in \mathbb{N}$$

is called a continuous polynomial iff $\phi^{(n)} \in \tilde{\mathcal{S}}_{\mathbb{C}}(X)^{\hat{\otimes}n}$ (the complexification of $\tilde{\mathcal{S}}(X)^{\hat{\otimes}n}$). We denote the space of continuous polynomials on $\tilde{\mathcal{S}}(X)$ by $\mathcal{P}(\tilde{\mathcal{S}}'(X))$.

Corollary 3.3 (non-degeneracy of μ_λ). *For all $F \in \mathcal{P}(\tilde{\mathcal{S}}'(X))$ with $F = 0$ μ_λ-almost everywhere we have $F(\omega) = 0$ for all $\omega \in \tilde{\mathcal{S}}'(X)$.*

Proof. It can be shown that each continuous polynomial F is representable in the form

$$F(\omega) = \sum_{n=0}^{N} < C_n(\omega), \hat{\phi}_n >$$

where $\hat{\phi}_n \in \tilde{\mathcal{S}}(X)^{\hat{\otimes}n}$. Take without loss of generality $F(\omega) = < C_n(\omega), \hat{\phi}_n >$. Then by Proposition 3.1

$$0 = \mathbb{E}_{\mu_\lambda}\left[< C_n(\omega), \hat{\phi}_n >^2 \right] = \int_X \phi_n^2(x) d\lambda(x)$$

Hence, $\phi_n \in \mathcal{N}_\lambda$, which implies that $\hat{\phi}_n$ is the null element in $\tilde{\mathcal{S}}(X)$. Therefore F is identically zero. $\square$

A probability measure with the property in Corollary 3.3 is called *non-degenerate* (see [12]). We will make use of this property for the construction of stochastic test function and distribution spaces.

For functions $f : X^n \to \mathbb{R}$ define the symmetrization $\tilde{f}$ of f by

(3.6)
$$\tilde{f}(x_1, \ldots, x_n) := \frac{1}{n!} \sum_\sigma f(x_{\sigma_1}, \ldots, x_{\sigma_n})$$

for all permutations σ of $\{1, \ldots, n\}$. We call a function $f : X^n \to \mathbb{R}$ symmetric if $\tilde{f} = f$, that is it is equal to its symmetrization. Denote by $L_s^2(X^n, \lambda)$ the space of all symmetric functions on X^n which are square integrable with respect to $\lambda^{\times n}$. Let f_n be a function in $L_s^2(X^n, \lambda)$. By Lemma 2.5 we can find a sequence of functions $f_n^{(i)} \in \tilde{\mathcal{S}}(X)^{\hat{\otimes}n}$ such that $f_n^{(i)}$ converges to f_n in $L_s^2(X^n, \lambda)$ as i tends to infinity. We can therefore define $< C_n(\omega), f_n >$ by

$$\langle C_n(\omega), f_n \rangle = \lim_{i \to \infty} \langle C_n(\omega), f_n^{(i)} \rangle \qquad (\text{limit in } L^2(\mu_\lambda))$$

Note that this limit exists by Proposition 3.1. Moreover, let $f_n^{(j)} \in \tilde{\mathcal{S}}(X)^{\hat{\otimes}n}$ be another sequence converging to f_n. Then by Proposition 3.1,

$$\int_{\tilde{\mathcal{S}}'(X)} \left(\langle C_n(\omega), f_n^{(i)} \rangle - \langle C_n(\omega), f_n^{(j)} \rangle \right)^2 d\mu_\lambda(\omega) = n! \left| f_n^{(i)} - f_n^{(j)} \right|_\lambda^2$$

which shows that the definition is well defined. Notice also that from the definition of C_1 we have that

$$\nabla_\phi \tilde{e}(\eta, \omega) \big|_{\eta=0} = \frac{d}{d\epsilon} \exp\left(< \omega, \ln(1 + \eta + \epsilon\phi) > - < \dot{\nu}, \eta + \epsilon\phi > \right) \big|_{\epsilon=0, \eta=0}$$
$$= < \omega - \dot{\nu}, \phi >$$

From which we see that $C_1(\omega) = \omega - \dot{\nu}$. By Proposition 3.1 we therefore obtain the familiar isometry:

Lemma 3.4. *Let $f \in L^2(X, \lambda)$. Then*

$$\int_{\tilde{\mathcal{S}}'(X)} < \omega - \dot{\nu}, f >^2 d\mu_\lambda(\omega) = |f|_\lambda^2$$

For any Borel sets $\Lambda_1 \subset \mathbb{R}$ and $\Lambda_2 \subset \mathbb{R}_0$ with $0 \notin \text{cl}(\Lambda_2)$ we can define the random measures $N(\Lambda_1, \Lambda_2) := < \omega, 1_{\Lambda_1 \times \Lambda_2} >$ and $\tilde{N}(\Lambda_1, \Lambda_2) := < \omega - \dot{\nu}, 1_{\Lambda_1 \times \Lambda_2} >$. From the characteristic function of μ_λ it is clear that N is a Poisson random measure and $\tilde{N}$ is the corresponding compensated Poisson random measure. Moreover, the compensator of $N(\Lambda_1, \Lambda_2)$ is given by $< \dot{\nu}, 1_{\Lambda_1 \times \Lambda_2} >$ which is equal to $\lambda(\Lambda_1, \Lambda_2)$. We have therefore justified the following identity:

(3.7)
$$\int_X \phi(t, z) \tilde{N}(dz, dt) = < \omega - \dot{\nu}, \phi >$$

for all $\phi \in L^2(X, \lambda)$. Thus in a generalized sense, the compensated Poisson random measure $\tilde{N}$ has a Radon-Nikodym derivative with respect to the Lebesgue measure on X which is given by $\omega - \dot{\nu}$.

4. Stochastic test and distribution functions

In this section we will define spaces of test functions and distributions as pairs of dual spaces with respect to the innerproduct $(\cdot, \cdot)_\lambda$ and stochastic test functions and distributions as pairs of dual spaces with respect to the innerproduct on $L^2(\mu_\lambda)$. By Theorem 2.8,

$$(4.1) \qquad (\phi, \phi)_\lambda = <\dot{\nu}, \phi^2> \leq \|\dot{\nu}\|_{-q_0, \lambda} \|\phi^2\|_{q_0, \lambda} \leq M_{q_0} \|\dot{\nu}\|_{-q_0, \lambda} \|\phi\|_{q_0, \lambda}^2$$

From (4.1) it follows that $\tilde{\mathcal{S}}_p(X)$ is contained in $L^2(X, \lambda)$ for all $p \geq q_0$. For each natural number $p \geq q_0$ define the space $\mathcal{S}_{-p, \lambda}(X)$ as the completion of $\tilde{\mathcal{S}}(X)$ with respect to the norm:

$$(4.2) \qquad |\phi|_{-p, \lambda} := \|\dot{\nu}\phi\|_{-p, \lambda}$$

We have the inclusions, $\mathcal{S}_{-p, \lambda}(X) \subset \mathcal{S}_{-(p+1), \lambda}(X)$. Define the space $\mathcal{S}'_\lambda(X)$ as the inductive limit of $\mathcal{S}_{-p, \lambda}(X)$.

Lemma 4.1. *For every natural number $p \geq q_0$, $\mathcal{S}_{-p, \lambda}(X)$ is the $L^2(\lambda)$ dual of $\tilde{\mathcal{S}}_p(X)$.*

Proof. By the inequality

$$\left| (\phi, \psi)_\lambda \right| = \left| <\dot{\nu}\phi, \psi> \right| \leq \left\| \dot{\nu}\phi \right\|_{-p, \lambda} \left\| \psi \right\|_{p, \lambda} = \left| \phi \right|_{-p, \lambda} \left\| \psi \right\|_{p, \lambda}$$

it follows that the $L^2(X, \lambda)$ dual of $\tilde{\mathcal{S}}_p(X)$ consists of all elements ϕ for which $|\phi|_{-p, \lambda}$ is finite. Since this by definition is $\mathcal{S}_{-p, \lambda}(X)$ the result follows. □

We will now contruct the Kondratiev type stochastic test function space and the Kondratiev stochastic distribution space. Let $f \in \mathcal{P}(\tilde{\mathcal{S}}'(X))$. Then by Corollary 3.3 f has a unique representation:

$$f(\omega) = \sum_{n=0}^N < C_n(\omega), f_n >, \quad f_n \in \tilde{\mathcal{S}}(X)^{\hat{\otimes}n}$$

For any natural number $p \geq q_0$ define the Hilbert space $(\mathcal{S})_p^1$ as the completion of $\mathcal{P} := \mathcal{P}(\tilde{\mathcal{S}}'(X))$ with respect to the norm

$$\|f\|_{p,1,K}^2 = \sum_{n=0}^\infty (n!)^2 \|f_n\|_{p, \lambda}^2 \quad , f \in \mathcal{P}$$

with corresponding inner product given by:

$$((f, g))_{p,1,K} = \sum_{n=0}^\infty (n!)^2 (f_n, g_n)_p$$

where $(\cdot, \cdot)_p$ denotes the innerproduct on $\tilde{\mathcal{S}}_p(X)^{\hat{\otimes}n}$. Obviously $(\mathcal{S})_{p+1}^1 \subset (\mathcal{S})_p^1$. We define the space $(\mathcal{S})^1$ as the projective limit of $(\mathcal{S})_p^1$. By [1, Thm. 4], $(\mathcal{S})^1$ is a nuclear Fréchet space which is densely topologically embedded in $L^2(\mu_\lambda)$.

For every natural number $p \geq q_0$ define the space $(\mathcal{S})_{-p}^{-1}$ as the $L^2(\mu_\lambda)$ dual of $(\mathcal{S})_p^1$. We have the inclusion $(\mathcal{S})_{-p}^{-1} \subset (\mathcal{S})_{-(p+1)}^{-1}$. Denote by $(\mathcal{S})^{-1}$ the inductive limit of $(\mathcal{S})_{-p}^{-1}$, which is

equal to the dual of $(\mathcal{S})^1$. Let $<< \cdot, \cdot >>$ denote the dual pairing between $(\mathcal{S})^1$ and $(\mathcal{S})^{-1}$. Moreover, for any $F(\omega) = \sum_{n=0}^{\infty} < C_n(\omega), F_n >$ with kernels $F_n \in \mathcal{S}'_\lambda(X)^{\hat{\otimes} n}$ define the norm:

$$\|F\|^2_{-p,-1,K} := \sum_{n=0}^{\infty} |F_n|^2_{-p,\lambda}$$

Lemma 4.2. *$F \in (\mathcal{S})^{-1}$ iff F admits an expansion $F(\omega) = \sum_{n=0}^{\infty} < C_n(\omega), F_n >$ where $F_n \in \mathcal{S}'_\lambda(X)^{\hat{\otimes} n}$ and $\|F\|_{-p,-1,K} < \infty$ for some $p > q_0$. If $f(\omega) = \sum_{n=0}^{\infty} < C_n(\omega), f_n >\in (\mathcal{S})^1$, then*

$$(4.3) \qquad\qquad << F, f >> = \sum_{n=0}^{\infty} n! (F_n, f_n)_\lambda$$

Moreover, $<< \cdot, \cdot >>$ is an extension of the innerproduct on $L^2(\mu_\lambda)$.

Proof. Let $F(f) = << F, f >>$ where $<< \cdot, \cdot >>$ is given by (4.3). Assume first that $F, f \in L^2(\mu_\lambda)$. Then

$$<< F, f >> = \sum_{n=0}^{\infty} n! (F_n, f_n)_\lambda = \mathbb{E}[Ff]$$

Hence $<< \cdot, \cdot >>$ is an extension of the innerproduct on $L^2(\mu_\lambda)$. By the Cauchy-Schwartz inequality,

$$
\begin{aligned}
\left| << F, f >> \right| &\leq \sum_{n=0}^{\infty} n! \left| < \dot{\nu}^{\hat{\otimes} n} F_n, f_n > \right| \\
&\leq \sum_{n=0}^{\infty} \left\| \dot{\nu}^{\hat{\otimes} n} F_n \right\|_{-p,\lambda} n! \left\| f_n \right\|_{p,\lambda} \\
&\leq \left(\sum_{n=0}^{\infty} \left| F_n \right|^2_{-p,\lambda} \right)^{1/2} \left(\sum_{n=0}^{\infty} (n!)^2 \left\| f_n \right\|^2_{p,\lambda} \right)^{1/2} \\
&= \left\| F \right\|_{-p,-1,K} \left\| f \right\|_{p,1,K}
\end{aligned}
$$

Hence any $F = \sum_{n=0}^{\infty} < C_n(\omega), F_n >$ with $\|F\|_{-p,-1,K} < \infty$ for some p belongs to $(\mathcal{S})^{-1}$. Choose an $f_n \in \tilde{\mathcal{S}}(X)^{\hat{\otimes} n}$ and $F \in (\mathcal{S})^{-1}$. Then it follows from Lemma 4.1 that

$$\left\langle \left\langle F, < C_n(\cdot), f_n > \right\rangle \right\rangle = n! < \dot{\nu}^{\hat{\otimes} n} F_n, f_n >$$

for some $F_n \in \mathcal{S}'_\lambda(X)^{\hat{\otimes} n}$. Hence, any $F \in (\mathcal{S})^{-1}$ has an expansion $F = \sum_{n=0}^{\infty} < C_n(\cdot), F_n >$ with $F_n \in \mathcal{S}'_\lambda(X)^{\hat{\otimes} n}$. $\qquad\square$

Remark 4.3. Our system of generalized Charlier polynomials coincides with the generalized Appell systems in [12], which provides in the same manner stochastic test function and distribution spaces (compare [12, Example 27]).

5. The $\mathcal{S}$-transform and Wick products

Let $F \in (\mathcal{S})^{-1}$. Then there exists a natural number $p(F) > 0$ such that $F \in (\mathcal{S})^{-1}_{-p(F)}$. Denote by U_p the set

$$U_p := \left\{ \phi \in \tilde{\mathcal{S}}(X) : \|\phi\|_{p,\lambda} < 1 \right\}$$

For any $\phi \in \tilde{\mathcal{S}}(X)$, it follows from the chaos expansion of $\tilde{e}(\phi)$ given by equation (3.2) that

$$(5.1) \qquad \left\| \tilde{e}(\phi, \omega) \right\|^2_{p,1,K} = \sum_{n=0}^{\infty} \|\phi\|^{2n}_{p,\lambda}$$

From (5.1) we see that $\tilde{e}(\phi) \in (\mathcal{S})^1_p$ iff $\phi \in U_p$.

Definition 5.1. Let $F \in (\mathcal{S})^{-1}_{-p}$ and $\xi \in U_p$. We define the $\mathcal{S}$ transform of F by:

$$\mathcal{S}(F)(\xi) := \left\langle\!\left\langle F(\omega), \tilde{e}(\xi, \omega) \right\rangle\!\right\rangle$$

Note that since $F \in (\mathcal{S})^{-1}_{-p}$ and $\tilde{e}(\xi) \in (\mathcal{S})^1_p$ it follows that $|\mathcal{S}(F)(\xi)| < \infty$.

Lemma 5.2. Let $F = \sum_{n=0}^{\infty} < C_n(\omega), F_n > \in (\mathcal{S})^{-1}_{-p}$. Then

$$\mathcal{S}(F)(\xi) = \sum_{n=0}^{\infty} \left(F_n, \xi^{\otimes n} \right)_{\lambda} \quad , \xi \in U_p$$

Proof. Let $\xi \in U_p$. By Proposition 3.1 and the chaos expansion of $\tilde{e}(\xi)$ given by (3.2),

$$\mathcal{S}(F)(\xi) = \left\langle\!\left\langle \sum_{n=0}^{\infty} < C_n(\omega), F_n >, \sum_{m=0}^{\infty} < C_m(\omega), \frac{1}{m!}\xi^{\otimes m} > \right\rangle\!\right\rangle = \sum_{n=0}^{\infty} (F_n, \xi^{\otimes n})_{\lambda}$$

$\square$

Denote by l^{-1} the inverse of the function l. We have that $l^{-1}(x) = e^x - 1$. Define another transform $\mathcal{S}_P(F)$ by:

$$\mathcal{S}_P(F)(\xi) := \left\langle\!\left\langle F(\omega), e(\xi, \omega) \right\rangle\!\right\rangle$$

By Proposition 3.1 and the chaos expansion of $e(\xi)$ given by (3.3) it follows that for ξ such that $e^{\xi} - 1 \in U_p$ and $F = \sum_{n=0}^{\infty} < C_n(\omega), F_n > \in (\mathcal{S})^{-1}_{-p}$,

$$(5.2) \qquad \mathcal{S}_P(F)(\xi) = \sum_{n=0}^{\infty} \left(F_n, (e^{\xi} - 1)^{\otimes n} \right)_{\lambda} = \left(\mathcal{S}(F) \circ l^{-1} \right)(\xi)$$

Denote by $\mathrm{Hol}(\theta_0)$ the vector space of all functions which are holomorphic in a neighborhood of θ_0 (See [1], [5] and [16] for more details). The function l^{-1} is obviously analytic in a neighborhood of zero. A characterization of the image of $(\mathcal{S})^{-1}$ under the $\mathcal{S}_P$-transform was proved in [1]. By equation (5.2) and the analyticity of l^{-1} we deduce that the characterization of the image of the $\mathcal{S}_P$-transform in [1] also is valid for the $\mathcal{S}$-transform, hence by [1, Thm. 5]:

Theorem 5.3. *If $F \in (\mathcal{S})^{-1}$ then $\mathcal{S}(F) \in \mathrm{Hol}(0)$. Conversely, if $G \in \mathrm{Hol}(0)$ there is a uniquely defined distribution $F \in (\mathcal{S})^{-1}$ such that $\mathcal{S}(F) = G$ on some neighborhood of zero in $(\mathcal{S})^1_{\mathbb{C}}$ (the complexification of $(\mathcal{S})^1$).*

Let $\mathcal{U} = \text{Hol}(0)$, the image of $(\mathcal{S})^{-1}$ under the $\mathcal{S}$-transform. Then by Theorem 5.3 we have that the $\mathcal{S}$-transform is an isomorphism between $(\mathcal{S})^{-1}$ and $\mathcal{U}$. Note also that if $f, g \in \text{Hol}(0)$ then $fg \in \text{Hol}(0)$. Hence the following definition is well defined,

Definition 5.4. Let $F, G \in (\mathcal{S})^{-1}$. We define the wick product, denoted by $F \diamond G$, of F and G by

$$F \diamond G := \mathcal{S}^{-1}\big(\mathcal{S}(F)\mathcal{S}(G)\big)$$

Note that it follows directly from the properties of the $\mathcal{S}$-transform and the definition of the Wick product that Wick multiplication is a continuous operation.

Proposition 5.5. Let $F(\omega) = \sum_{n=0}^{\infty} < C_n(\omega), F_n >$ and $G(\omega) = \sum_{n=0}^{\infty} < C_n(\omega), G_n >$ be elements in $(\mathcal{S})^{-1}$. Then $F \diamond G \in (\mathcal{S})^{-1}$ and

$$\big(F \diamond G\big)(\omega) = \sum_{k=0}^{\infty} \bigg(\sum_{n+m=k} \big\langle C_k(\omega), F_n \hat{\otimes} G_m \big\rangle \bigg)$$

Proof. By Theorem 5.3 we have that $\mathcal{S}(F)$ and $\mathcal{S}(G)$ both are in $\text{Hol}(0)$. The product of two holomorphic functions is a holomorphic function. Hence $\mathcal{S}(F)\mathcal{S}(G) \in \text{Hol}(0)$. Hence by Theorem 5.3, $F \diamond G \in (\mathcal{S})^{-1}$. By Lemma 5.2,

$$\mathcal{S}\bigg(\sum_{k=0}^{\infty} \sum_{n+m=k} < C_k(\omega), F_n \hat{\otimes} G_m > \bigg)(\xi) = \sum_{k=0}^{\infty} \sum_{n+m=k} \big(F_n \hat{\otimes} G_m, \xi^{\otimes k}\big)_\lambda$$

$$= \sum_{k=0}^{\infty} \sum_{n+m=k} \big(F_n, \xi^{\otimes n}\big)_\lambda \big(G_m, \xi^{\otimes m}\big)_\lambda$$

$$= \sum_{n,m=0}^{\infty} \big(F_n, \xi^{\otimes n}\big)_\lambda \big(G_m, \xi^{\otimes m}\big)_\lambda$$

$$= \mathcal{S}\big(F(\omega)\big)(\xi) \cdot \mathcal{S}\big(G(\omega)\big)(\xi)$$

and the result follows.   □

For $F \in (\mathcal{S})^{-1}$, we set $F^{\diamond n} := F \diamond \cdots \diamond F$ (the Wick product taken n times). Moreover, we define the Wick exponential of F, denoted $\exp^\diamond(F)$, by

$$\exp^\diamond(F) := \sum_{n=0}^{\infty} \frac{1}{n!} F^{\diamond n}$$

whenever $\sum_{n=0}^{\infty} \frac{1}{n!} F^{\diamond n} \in (\mathcal{S})^{-1}$. It is easy to check that if $\phi \in \mathcal{S}'_\lambda(X)$ then $< C_1(\omega), \phi > \in (\mathcal{S})^{-1}$ and

$$< C_1(\omega), \phi >^{\diamond n} = < C_n(\omega), \phi^{\otimes n} >$$

By the chaos expansion (3.2) of $\tilde{e}(\phi, \omega)$ we therefore have that

(5.3) $$\tilde{e}(\phi, \omega) = \exp^\diamond\big(< \omega - \dot{\nu}, \phi >\big)$$

for all $\phi \in \mathcal{S}'_\lambda(X)$ such that $\phi(x) > -1$ for all $x \in X$.

6. A UNITARY ISOMORPHISM AND CONFIGURATION SPACES

In this section we want to establish a unitary isomorphism between $L^2(\mu_\lambda)$ and the classical Poisson space with intensity λ defined via the configuration space. This isomorphism can be used to transfer analytical structures from Poisson spaces to $L^2(\mu_\lambda)$.

We need the definition of a configuration space. Similarly to [13], we introduce the configuration space Γ over $X = \mathbb{R}^d \times \mathbb{R}_0$ by

$$\Gamma := \big\{ \gamma \subset X : card(\gamma \cap K) < \infty \text{ for any compact } K \subset X \big\}.$$

Denote by ϵ_x the Dirac measure in the point $x \in X$. The correspondence

$$(6.1) \qquad \Gamma \ni \gamma \longmapsto d\gamma := \sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}_p(X)$$

provides a one-to-one mapping Φ from Γ into the space of positive integer valued measures $\mathcal{M}_p(X)$ over $\mathcal{B}(X)$. We endow Γ as a closed subset of $\mathcal{M}_p(X)$ with the relative vague topology. That is, a sequence of measures σ_n converges to σ in Γ iff for any $f \in C_c(X)$ (i.e. the space of continuous functions with compact support) we have

$$\int_X f(x)d\sigma_n(x) \longrightarrow \int_X f(x)d\sigma(x) \text{ for } n \longrightarrow \infty.$$

Then for any $f \in C_c(X)$ the continuous functionals

$$(6.2) \qquad \Gamma \ni \gamma \longmapsto < \gamma, f >:= \int_X f(x)d\gamma(x) = \sum_{x \in \gamma} f(x)$$

induce the topology of Γ.

We need the following key observation regarding the support of μ_λ.

Proposition 6.1.

$$(6.3) \qquad \mu_\lambda \left(\left\{ \sum_{x \in \gamma} \varepsilon_x \in \tilde{\mathcal{S}}'(X) : \gamma \in \Gamma \right\} \right) = 1$$

where the Dirac measures ε_x in x are naturally identified with the corresponding delta functions in $\tilde{\mathcal{S}}'(X)$.

Proof. Relation (6.3) follows from the path properties of Poisson processes (see e.g. [15]). $\qquad \square$

Further we define the *Poisson measure* π_λ on the Borel σ-algebra $\mathcal{B}(\Gamma)$ with intensity measure λ as follows.

Definition 6.2. The Laplace transform of π_λ is given by

$$(6.4) \qquad l_{\pi_\lambda}(\varphi) = \int_\Gamma \exp\big(< \gamma, \varphi >\big)d\pi_\lambda(\gamma) = \exp\left(\int_X (e^{\varphi(x)} - 1)d\lambda(x) \right)$$

where $\varphi \in \tilde{\mathcal{S}}(X)$. The existence of π_λ follows from Proposition 6.1 and the identification in relation (6.1).

Taking into account that we have

$$\int_{\tilde{\mathcal{S}}'(X)} e^{<\omega,\phi>}d\mu_\lambda(\omega) =: l_{\mu_\lambda}(\varphi) = l_{\pi_\lambda}(\varphi)$$

for all $\varphi \in \tilde{\mathcal{S}}(X)$, we conclude that the measure μ_λ is the image of π_λ under the mapping $\Phi : \Gamma \longrightarrow \Phi(\Gamma) =: \Omega$ in (6.1), that is for all $B \in \mathcal{B}(\tilde{\mathcal{S}}'(X))$,

$$(6.5) \qquad \qquad \mu_\lambda(B) = \mu_\lambda(B \cap \Omega) = \pi_\lambda(\Phi^{-1}(B \cap \Omega))$$

Relation (6.5) together with the change of variable formula for the Lebesgue integral yields then that for all $f \in L^1(\Gamma, \pi_\lambda)$ the function $f \circ \Phi^{-1}$ is in $L^1(\Omega, \mu_\lambda)$ and

$$\int_\Gamma f(\gamma) d\pi_\lambda(\gamma) = \int_\Omega f \circ \Phi^{-1}(\omega) d\mu_\lambda(\omega)$$

Thus we proved the following result.

Theorem 6.3. *The map $U_\Phi : L^2(\mu_\lambda) \longrightarrow L^2(\pi_\lambda)$ given by*

$$(6.6) \qquad \qquad g \longmapsto g \circ \Phi$$

is a unitary isomorphism.

Remark 6.4. It is important to note that the measure μ_λ does not admit the construction of a satisfactory differential calculus on $\tilde{\mathcal{S}}'(X)$. Since $\mu_\lambda(\xi + \varphi) \perp \mu_\lambda(\xi)$ (see e.g. [7]), integration by parts and adjoint operators are not available. We overcome this circumstance by using the unitary isomorphism of Theorem 6.3 to transport analytical structures from $L^2(\pi_\lambda)$ to $L^2(\mu_\lambda)$ (compare [13]). We will make use of this principle in Section 7, where a Poissonian gradient for Lévy processes is introduced.

7. The Poissonian gradient and the Skorohod integral

We start with the definition of the set $\mathbb{D} \subset L^2(\mu_\lambda)$,

$$\mathbb{D} := \left\{ f(\omega) = \sum_{n=0}^\infty < C_n(\omega), f_n > : \sum_{n=0}^\infty nn! |f_n|_\lambda^2 < \infty \right\}$$

Define the linear operator $D : \mathbb{D} \mapsto L^2(\lambda \times \mu_\lambda)$, by

$$(7.1) \qquad \qquad D_x f(\omega) := \sum_{n=1}^\infty \big\langle C_{n-1}(\omega), n f_n(\cdot, x) \big\rangle$$

for $: f(\omega) = \sum_{n=0}^\infty < C_n(\omega), f_n > \in \mathbb{D}$, where $f_n(\cdot, x)$ is the function f_n with the last argument $x = (t_1, \ldots, t_d, z) \in X$ held fixed. It can be seen from a direct calculation that

$$\|Df\|_{L^2(\lambda \times \mu_\lambda)}^2 = \sum_{n=1}^\infty nn! |f_n|_\lambda^2 < \infty$$

Thus, $Df \in L^2(\lambda \times \mu_\lambda)$ whenever $f \in \mathbb{D}$. Note that $(\mathcal{S})^1 \subseteq \mathbb{D}$. Define another linear operator called the Poissonian gradient, denoted by ∇^P:

$$\big(\nabla^P f\big)(\gamma, x) = f(\gamma + \epsilon_x) - f(\gamma), \quad \gamma \in \Gamma, x \in X$$

for all variables of the form $f(\gamma) = g\big(< \gamma, \phi_1 >, \ldots, < \gamma, \phi_N > \big)$ with $g \in C_b^\infty(\mathbb{R}^N)$ and $\phi_1, \ldots, \phi_N \in \tilde{\mathcal{S}}(X)$. Notice that ∇^P is defined on $L^2(\pi_\lambda)$. We define the operator $\tilde{\nabla}^P$ on $L^2(\mu_\lambda)$ by

$$\tilde{\nabla}^P = U_\Phi^{-1} \nabla^P U_\Phi$$

Lemma 7.1. *The operators D and $\tilde{\nabla}^P$ coincide on a dense subset of $L^2(\mu_\lambda)$, hence the closure of D equals the closure of $\tilde{\nabla}^P$.*

Proof. The chaos expansion (3.2) of $\tilde{e}(\phi, \omega)$ and the definition of D gives

$$
D_x \tilde{e}(\phi, \omega) = D_x \left(\sum_{n=0}^\infty \left\langle C_n(\omega), \frac{\phi^{\otimes n}}{n!} \right\rangle \right)
$$

$$
= \sum_{n=1}^\infty \left\langle C_{n-1}(\omega), \frac{n\phi^{\otimes n-1}\phi(x)}{n!} \right\rangle
$$

$$
= \phi(x) \sum_{n=0}^\infty \left\langle C_n(\omega), \frac{\phi^{\otimes n}}{n!} \right\rangle
$$

$$
(7.2) \qquad = \phi(x)\tilde{e}(\phi, \omega)
$$

On the other hand, $\tilde{\nabla}^P$ applied to $\tilde{e}(\phi, \omega)$ gives

$$
\left(\tilde{\nabla}^P \tilde{e}(\phi, \omega) \right)(x) = \left(U_\Phi^{-1} \nabla^P U_\Phi \tilde{e}(\phi, \omega) \right)(x)
$$

$$
= U_\Phi^{-1} \left(\tilde{e}(\phi, \gamma + \epsilon_x) - \tilde{e}(\phi, \gamma) \right)
$$

$$
= U_\Phi^{-1} \left(\tilde{e}(\phi, \gamma) \left(e^{<\epsilon_x, \ln(1+\phi)>} - 1 \right) \right)
$$

$$
= U_\Phi^{-1} \left(\tilde{e}(\phi, \gamma)\phi(x) \right)
$$

$$
(7.3) \qquad = \tilde{e}(\phi, \omega)\phi(x)
$$

By comparing (7.2) and (7.3) we see that $D_x \tilde{e}(\phi, \omega) = \left(\tilde{\nabla}^P \tilde{e}(\phi, \omega) \right)(x)$ for all $x \in X$ and $\phi \in \tilde{\mathcal{S}}(X)$. Since both operators are closable and the linear span of variables of the form $\tilde{e}(\phi)$ is dense in $L^2(\mu_\lambda)$ the result follows. $\square$

We will now consider generalized random fields $F : X \to (\mathcal{S})^{-1}$. Observe that the chaos expansion of such fields may be written as

$$
F(x) = \sum_{n=0}^\infty \left\langle C_n(\omega), F_n(\cdot, x) \right\rangle
$$

where $F_n(\cdot, x) \in \mathcal{S}'_\lambda(X)^{\hat{\otimes} n}$ for every $x = (t_1, \ldots, t_d, z) \in X$ and $\|F(x)\|_{-p,-1,K} < \infty$ for some natural number $p > 0$. We define $\mathbb{L}$ to be the set of all $F : X \to (\mathcal{S})^{-1}$ such that $\tilde{F}_n \in \mathcal{S}'_\lambda(X)^{\hat{\otimes}(n+1)}$ and

$$
\sum_{n=0}^\infty |\tilde{F}_n|^2_{-p,\lambda} < \infty
$$

for some natural number $p > 0$. Recall that $\tilde{F}_n$ is the symmetrization of F_n.

Definition 7.2. Let $F \in \mathbb{L}$. Define the Skorohod integral by

$$
\delta(F) := \sum_{n=0}^\infty \left\langle C_{n+1}(\omega), \tilde{F}_n \right\rangle
$$

where $\tilde{F}_n$ denotes the symmetrization of F_n.

From the definition of the Skorohod integral conjoint with the assumption on elements of $\mathbb{L}$ it follows that $\delta(F) \in (\mathcal{S})^{-1}$. Obviously, the Skorohod integral is a linear operator.

Note that in [3] it was proved that for predictable integrands, the Skorohod integral coinsides with the usual Ito-type integral with respect to the compensated Poisson random measure $\tilde{N}$.

The next result shows that the operator D (by Lemma 7.1 equal to $\tilde{\nabla}^P$) is the dual of δ, the Skorohod integral.

Theorem 7.3. *Let $F \in \mathbb{L}$ and $f \in (\mathcal{S})^1$. Then*

$$(7.4) \qquad \int_X << F(x), D_x f >> d\lambda(x) =<< \delta(F), f >>$$

Proof. Put $F(x) = \sum_{n=0}^\infty < C_n(\omega), F_n(\cdot, x) >$ and $f = \sum_{m=0}^\infty < C_m(\omega), f_m >$ where $F \in \mathbb{L}$ and $f \in (\mathcal{S})^1$. Then $\delta(F) \in (\mathcal{S})^{-1}$. Choose a natural number $p > 0$ such that $\delta(F) \in (\mathcal{S})_{-p}^{-1}$. By the Cauchy-Schwartz inequality,

$$\int_X \left| \left(F_n(\cdot, x), f_{n+1}(\cdot, x) \right)_\lambda \right| d\lambda(x) \le \int_{X^{n+1}} \left| F_n f_{n+1} \right| d\lambda^{\times(n+1)}$$

$$= \int_{X^{n+1}} \left| F_n \dot\nu^{\otimes(n+1)} f_{n+1} \right| dx^{\times(n+1)}$$

$$\le \left\| F_n \dot\nu^{\otimes(n+1)} \right\|_{-p,\lambda} \left\| f_{n+1} \right\|_{p,\lambda}$$

$$(7.5) \qquad = \left| F_n \right|_{-p,\lambda} \left\| f_{n+1} \right\|_p$$

where dx denotes the Lebesgue measure on X. By first changing the order of integration and summation using [6, Thm. 2.15], then inequality (7.5) and finally the Cauchy-Schwartz inequality and the definition of $\delta(F)$, we obtain

$$\int_X \sum_{n=0}^\infty (n+1)! \left| \left(F_n(\cdot, x), f_{n+1}(\cdot, x) \right)_\lambda \right| d\lambda(x)$$

$$= \sum_{n=0}^\infty (n+1)! \int_X \left| \left(F_n(\cdot, x), f_{n+1}(\cdot, x) \right)_\lambda \right| d\lambda(x)$$

$$\le \sum_{n=0}^\infty |F_n|_{-p,\lambda} (n+1)! \|f_n\|_{p,\lambda}$$

$$\le \left(\sum_{n=0}^\infty |F_n|_{-p,\lambda}^2 \right)^{1/2} \left(\sum_{n=0}^\infty \left((n+1)! \right)^2 \|f_n\|_{p,\lambda}^2 \right)^{1/2}$$

$$= \left\| \delta(F) \right\|_{-p,-1,K} \left\| f \right\|_{p,1,K} < \infty$$

Therefore, by the dominated convergence theorem,

$$\int_X << F(x), D_x f >> d\lambda(x)$$

$$= \int_X \sum_{n=0}^\infty (n+1)! \left(F_n(\cdot, x), f_{n+1}(\cdot, x) \right)_\lambda d\lambda(x)$$

$$= \sum_{n=0}^{\infty} (n+1)! \big(F_n, f_{n+1} \big)_{\lambda}$$

$$\overset{(*)}{=} \sum_{n=0}^{\infty} (n+1)! \big(\tilde{F}_n, f_{n+1} \big)_{\lambda}$$

$$= <<\delta(F), f>>$$

where $(*)$ is due to the fact that f_{n+1} is symmetric and hence $\big(F_n, f_{n+1} \big)_{\lambda}$ is equal to $\big(\tilde{F}_n, f_{n+1} \big)_{\lambda}$. The result follows. $\qquad\square$

Notice that if $\delta(F) \in L^2(\mu_{\lambda})$ and $f \in \mathbb{D} \subset L^2(\mu_{\lambda})$, then equation (7.4) reads

$$\mathbb{E} \left[\int_X F(x) D_x f d\lambda(x) \right] = \mathbb{E} \big[\delta(F) f \big]$$

According to Theorem 7.3, $\delta = (\tilde{\nabla}^P)^*$. By using the Mecke identity it was proved in [13] that

$$\big((\nabla^P)^* F \big)(\gamma) = \int_X F(\gamma - \epsilon_x, x) d\gamma(x) - \int_X F(\gamma, x) d\lambda(x)$$

Hence,

$$U_{\Phi} \big(\delta(F) \big)(\gamma) = \int_X F(\gamma - \epsilon_x, x) d\gamma(x) - \int_X F(\gamma, x) d\lambda(x)$$

Corollary 7.4. *Let $F \in \mathbb{L}$. Then $\delta(F) \in (\mathcal{S})_{-p}^{-1}$ for some $p \in \mathbb{N}$ and*

$$\mathcal{S} \big(\delta(F) \big)(\xi) = \int_X \left(\mathcal{S} \big(F(x) \big)(\xi) \cdot \xi(x) \right) d\lambda(x)$$

for all $\xi \in U_p$.

Proof. By equation (7.2), we have that $D_x \tilde{e}(\xi, \omega) = \tilde{e}(\xi, \omega) \xi(x)$. Hence by Theorem 7.3,

$$\mathcal{S} \big(\delta(F) \big)(\xi) = \big\langle \big\langle \delta(F), \tilde{e}(\xi) \big\rangle \big\rangle$$

$$= \int_X \big\langle \big\langle F(x), \tilde{e}(\xi) \xi(x) \big\rangle \big\rangle d\lambda(x)$$

$$= \int_X \mathcal{S} \big(F(x) \big)(\xi) \cdot \xi(x) d\lambda(x)$$

and the result follows. $\qquad\square$

Define a process $W : X \times \Omega \mapsto \mathbb{R}$ by

$$W(x)(\omega) := < C_1(\omega), \epsilon_x >$$

where ϵ_x is the Dirac delta function with mass at the point $x = (t_1, \ldots, t_d, z) \in X$. We then have the following relationship between the Skorohod integral, the Wick product and the Lebesgue integral:

Proposition 7.5. *Let $F \in \mathbb{L}$. Then*

$$\delta(F) = \int_X \left(F(x) \diamond W(x) \right) dx$$

where dx denotes the Lebesgue measure on X.

Proof. By Corollary 7.4 and the definition of the Wick product, we have that

$$
\begin{aligned}
\mathcal{S}\left(\int_X F(x) \diamond W(x)dx\right)(\xi) &= \int_X \mathcal{S}\left(F(x) \diamond W(x)\right)(\xi)dx \\
&= \int_X \mathcal{S}\big(F(x)\big)(\xi) \cdot \mathcal{S}\big(W(x)\big)(\xi)dx \\
&= \int_X \mathcal{S}\big(F(x)\big)(\xi) \cdot \big(\epsilon_x, \xi\big)_\lambda dx \\
&= \int_X \mathcal{S}\big(F(x)\big)(\xi) \cdot \xi(x)(1 \otimes \dot{\nu})(x)dx \\
&= \int_X \mathcal{S}\big(F(x)\big)(\xi) \cdot \xi(x)d\lambda(x) \\
&= \mathcal{S}\big(\delta(F)\big)(\xi)
\end{aligned}
$$

The result then follows from Theorem 5.3. $\qquad\square$

References

[1] S. Albeverio, Y. L. Daletsky, Y. G. Kondratiev and L. Streit, *Non-Gaussian infinite dimensional analysis.* J. Func. Anal., Vol. **138**, 311-350, 1996.

[2] H. Bauer, *Mass- und Integrationstheorie.* de Gruyter, 1990.

[3] F.E. Benth and A. Løkka, Anticipative calculus for Lévy processes and stochastic differential equations. Preprint University of Oslo **6**, 2002.

[4] H. Cramer, *Mathematical methods in statistics.* Princeton University Press, Princeton 1946.

[5] S. Dineen, *Complex analysis in locally convex spaces.* Mathematical studies **57**, North-Holland, 1981.

[6] G. B. Folland, *Real Analysis.* John Wiley and Sons, 1984.

[7] I. M. Gel'fand, M. J. Graev and A. M. Vershik, *Representations of the group of diffeomorphisms.* Russian Math. Surveys **30**, 3-50, 1975.

[8] I. M. Gel'fand and N. Y. Vilenkin, *Generalized Functions.* Vol. **4**. Some Applications of Harmonic Analysis, Academic Press, New York, 1964.

[9] H. Holden, B. Øksendal, J. Ubøe and T. Zhang, *Stochastic Partial Differential Equations.* Birkhäuser, 1996.

[10] T. Hida, H. Kuo, J. Potthoff and L. Streit, *White noise, an infinite dimensional calculus.* Kluwer academic publishers, 1993.

[11] K. Ito, *Spectral type of the shift transformation of differential processes with stationary increments.* Trans. Am. Math. Soc. **81**, 253-263, 1956.

[12] Y. G. Kondratiev, J. L. Silva and L. Streit, *Generalized appell systems.* Methods Funct. Anal. Topology **3**, No. 3, 1997.

[13] Y. G. Kondratiev, J. L. Da Silva, L. Streit and G. F. Us, *Analysis on Poisson and gamma spaces.* Infin. Dimens. Anal. Quantum Probab. Relat. Top. **1**, 91-117, 1998.

[14] Y-J Lee and H-H Shih, *Donsker's delta function of Lévy processes.* Acta Appl. Math. **63**, 219-231, 2000.

[15] K. Matthes, J. Kerstan and J. Mecke, *Infinitely Divisible Point Processes.* Wiley Series in Prob. and Math. Stat., 1978.

[16] L. Nachbin, *Topology on spaces of holomorphic mappings.* Springer-Verlag, 1969.

[17] P. Protter, *Stochastic Integration and Differential Equations.* Springer-Verlag, 1990.

[18] M. Reed and B. Simon, *Methods of modern mathematical physics; functional analysis.* Academic press, 1972.

[19] M. Reed and B. Simon, *Methods of modern mathematical physics; fourier analysis, self-adjointness.* Academic press, 1975.

[20] H. H. Schaefer, *Topological Vector Spaces.* Springer, New York, 1971.

[21] B. Simon, *Functional integration and quantum physics.* Academic press, 1979.

[22] S. Thangavelu, *Lectures on Hermite and Laguerre expansions.* Mathematical Notes **42**, Princeton University Press 1993.

[23] G. F. Us, *Dual Appell Systems in Poissonian Analysis.* Methods Funct. Anal. Topol. **1**, 93-108, 1995.

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