DEFORMATION THEORY OF RANK 1 MAXIMAL COHEN-MACAULAY MODULES ON HYPERSURFACE SINGULARITIES AND THE SCANDINAVIAN COMPLEX

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Dedicated to Professor Tor Gulliksen

ABSTRACT. We show that the deformation functor of a MCM-module $M = \text{coker}(\phi)$ over the hypersurface singularity $\text{det}(\phi)$ is given by deformations of the presenting matrix which keeps the determinant constant. A simplified expression for an edge map in the canonical 5-term exact sequence to a change of rings spectral sequence is obtained, including the tangent and obstruction spaces ($H^1$ and $H^2$). We relate the edge map to the Scandinavian complex $\mathcal{S}$ of $\phi$ which yields relations between the homology of $\mathcal{S}$ and $H^i$ for $i = 1, 2$. This gives (infinitesimal) rigidity and non-rigidity results and a dimension estimate for the formally (mini-)versal formal hull $H$ of the deformation functor.

INTRODUCTION

The local moduli problem in algebraic geometry is to find the local rings of the moduli space. The idea is that the algebraic geometric object, e.g. an $A$-module $M$, corresponding to a closed point is containing all information about the infinitesimal neighbourhoods of any sort of moduli space it may occur in. No a priori knowledge about these spaces is necessary. To make this claim precise one introduces the deformation functor: Fix a field $k$ and suppose $A$ is a $k$-algebra and let $\text{Art}_k$ be the category of local Artinian $k$-algebras $R$ with residue field $k$ such that the composition $k \to R \to k$ is the identity and morphisms are maps of local $k$-algebras. Then the deformation functor of $M$ is

$$\text{Def}_M : \text{Art}_k \longrightarrow \text{Sets}$$

where $\text{Def}_M(R)$ is the set of equivalence classes of liftings (or deformations) of $M$ to $R$. A lifting of $M$ to $R$ is an $A \otimes R$-module $M_R$, flat as $R$-module and an $A \otimes k$-$R$-linear map $\pi : M_R \to M$ with $\pi \otimes k : M_R \otimes k \to M$. Two liftings are equivalent if they are isomorphic above $M$. Maps are induced by tensorisation. Let more generally $F : \text{Art}_k \longrightarrow \text{Sets}$ be a covariant functor with $F(k)$ a one element set. M. Schlessinger [15] formulated a sufficient and necessary set of criteria for the existence of a complete local ring $H$ called a (pro-representable) hull, and a formal versal family $\{M_n\}_{n=1}^\infty$, which is a projective system with $M_n = (M_n, \pi_n)$, $M_n \in F(H/m_H^{n+1})$ and $\pi_n : M_n \to M_{n-1}$, such that the induced map

$$\text{Hom}_{k-\text{alg}, k}(H, -) \longrightarrow F$$

is an isomorphism or weaker, is a tangential isomorphism and is formally smooth, a strong inductive surjectivity condition. Most deformation functors satisfy these latter conditions, except possibly for finitedimensionality of the Zariski tangent space of $F$. But Schlessinger did not give any effective way of constructing the hull. The only known general tool to find $H$ only from $M$ is the existence and the computability of a natural obstruction class.
Definition 1. A small lifting situation is a surjective map \( \pi : R \to S \) in \( \text{Art}_k \) where \( \ker(\pi) \) is contained in the socle of \( R \), i.e. \( m_R \cdot \ker(\pi) = 0 \), and a lifting \( M_S \) of \( M \) to \( S \).

The obstruction class is then an element \( o = o(\pi, M_S) \in H^2 \otimes \ker(\pi) \) where \( H^2 \) is the second cohomology group of the object \( M \). If \( F = \text{Def}_M \) then \( o = o_A(\pi, M_S) \) and \( H^2 = \text{Ext}^2_A(M, M) \). The obstruction class is natural with respect to morphisms of the lifting situation. There exists a lifting of \( M_S \) to \( R \) (or a prolongation of the deformation \( M_S \) to the “thicker” Artinian neighbourhood Spec \( R \)) if and only if this obstruction class is zero. The obstruction class has been constructed for many deformation functors, e.g. \cite{9, 10, 13}, for axiomatic approaches see \cite{1, 4, 8}.

The starting point of the construction of the hull \( H \), in the case of \( \text{Def}_M \), is the “universal extension” \( M_1 \) of \( M \)

\[
M_1 : 0 \to M \otimes_k \text{Ext}^1_A(M, M) \to M_1 \to M \to 0
\]

where the extension is given by the image of the identity map under the canonical isomorphism;

\[
id \in \text{End}_k(\text{Ext}^1_A(M, M)) = \text{Ext}^1_A(M, M) \otimes_k \text{Ext}^1_A(M, M)
\]

If \( H_1 = k[\text{Ext}^1_A(M, M)] = k \oplus \text{Ext}^1_A(M, M) \) we naturally get the equivalence class \([M_1] \in \text{Def}_M(H_1)\). The construction of \( H \) then proceeds through successive “prolongations” of \( M_1 \) to thicker Artinian \( k \)-algebras through small lifting situations, at each step calculating the obstruction. If this is done correctly one obtains powerseries in \( T^1 \) of minimal degree \( \geq 2 \), one (possibly “0”) for each cotangential “generator” in \( T^2 \), where \( T \) is the completion of the free \( k \)-algebra which has \( \text{Ext}^1_A(M, M) \) as Zariski tangent space for \( i = 1, 2 \). This defines an obstruction map \( o^A : T^2 \to T^1 \) such that \( H = T^1 \otimes_k k \), see \cite{14, 13, 8}. An estimate for the Krull dimension of \( H \) follows:

\[
dim_k \text{Ext}^1_A(M, M) \geq \dim_k(\text{Def}_M) \geq \dim_k \text{Ext}^1_A(M, M) - \dim_k \text{Ext}^2_A(M, M)
\]

with the first estimate as equality iff \( H \) is smooth and the second estimate as equality iff the obstruction powerseries defines a regular sequence and “all” the second cohomology is “hit” by obstructions, see also \cite{11}.

In practice it is difficult to calculate \( H \) in this way, the degree of success will depend on how explicit and simple one can represent the involved cohomology. In the present paper we will consider a class of modules where the cohomology has a particularly nice and explicit representation. Our results fit into a general change of rings framework which we now briefly describe.

Let \( B \) be a \( k \)-algebra quotient of \( A \) and \( J = \ker(A \to B) \) and assume \( M \) is a \( B \)-module as \( A \)-module, i.e. that \( J \subseteq \text{Ann}_A(M) \). In \cite{8} we construct a new obstruction class \( o = o_A(\pi, M_S) \) for a lifting situation were \( o_A(\pi, M_S) = 0 \). We have

\[
o_B(\pi, M_S) = 0 \iff o_A(\pi, M_S) = 0 = o_A(\pi, M_S).
\]

It turns out that one with these two classes can construct two obstruction maps which defines the hull of \( \text{Def}_M^B \), the deformation functor of \( M \) as \( B \)-module. The \( J \)-obstruction class resides in the cokernel of a natural map

\[
\partial_B : \text{Ext}^1_B(M, M) \to \text{Hom}_A(J, \text{End}_A(M))
\]

where the \( \partial_B \) is induced by the pullback along any homotopy killing the action of \( J \) on an \( A \)-free resolution of \( M \), hence \( o_A(\pi, M_S) \in \text{coker}(\partial_B) \otimes_k \ker(\pi) \). Moreover the tangent space of \( \text{Def}_M^B \) is ker \( \partial_B \), see \cite[Chap. 1]{8}. The \( \partial_B \) is also the second
non-trivial map in the 5-term exact sequence
\[ 0 \rightarrow \text{Ext}^3_B(M, N) \rightarrow \text{Ext}^3_A(M, N) \rightarrow \text{Hom}_A(J, \text{Hom}_A(M, N)) \]
\[ \rightarrow \text{Ext}^4_B(M, N) \rightarrow \text{Ext}^4_A(M, N) \]
of a change of rings spectral sequence
\[ E_2^{pq} = \text{Ext}^p_A(M, \text{Ext}^q_B(B, N)) \Rightarrow \text{Ext}^n_A(M, N) \],
see [8, Chap. 4] and Lemma 1. Let \( T_A^3, T_B^2 \) and \( T^1 \) be the completion of free \( k \)-algebras with Zariski tangent spaces the image of the natural map \( \text{Ext}^2_B(M, M) \rightarrow \text{Ext}^2_A(M, M) \), \( \text{coker } \partial_3 \) and \( \text{ker } \partial_3 \) respectively. In the case these cohomology groups are not finite but countably dimensional as \( k \)-vector spaces one has to introduce a suitable topology allowing proper duality, and a compatible topology in which the \( T \)'s are complete, see [13, 8]. We have

**Theorem 1** ([8]). \( \text{Def}^B_M \) is a functor with two obstructions in
\[ \text{im} \left( \text{Ext}^2_B(M, M) \rightarrow \text{Ext}^2_A(M, M) \right) \text{ and } \text{coker } \partial_3 \]
and with tangent space \( \text{ker } \partial_3 \) such that if these spaces have countable \( k \)-dimension, there are (continuous) obstruction maps
\[ o_A : T_A^3 \rightarrow T^1 \text{ and } o_B : T_B^3 \rightarrow T^1 \]
for the obstructions \( o_A \) and \( o_B \). In particular the hull is given as
\[ H \cong (T^1 \otimes_{T_A^3} k) \otimes_{T_B^3} k. \]

This theorem implies that one can find the hull for \( \text{Def}^B_M \) calculating the obstructions as cup- and generalised Massey products entirely within an \( A \)-free complex, see [8, Thm. 3.1].

For explicit non-trivial calculations of obstructions (given by cup-products) for the Hilbert functor of space curves see [18, 5]. A. Siqveland gave the local equations for the compactified Jacobian of the \( E_6 \) curve singularity and found the degeneracy diagram of the rank 1 torsion free modules in [16] by calculating the obstruction maps, the Massey product algorithms are given in [17]. Similar ideas have recently been used by A. C. Borge [2] to define a new class of \( p \)-groups for which the modular isomorphism problem can be solved.

**Deforming matrix factorisations**

**Definition 2** ([3]). A *matrix factorisation* (mf) of an element \( f \in A \) is a pair \((\phi, \psi)\) of \( A \)-linear maps of free modules \( \phi : F \rightarrow G \), \( \psi : G \rightarrow F \) with \( \phi \psi = f \cdot \text{id}_G \) and \( \psi \phi = f \cdot \text{id}_F \).

Let \( B = A/(f) \) then \( M = \text{coker } \phi \) is a \( B \)-module as \( A \)-module since \( f \) annihilates \( M \). If \( (f)/(f^2) \) is free as \( B \)-module then the following 2-periodic complex of free \( B \)-modules
\begin{equation}
\begin{array}{ccccccc}
\text{G} & \xrightarrow{\phi} & F & \xrightarrow{\psi} & \text{G} & \xrightarrow{\phi} & F & \xrightarrow{\psi} & \ldots
\end{array}
\end{equation}
is a free resolution of \( M \) where \( \text{G} \xrightarrow{\phi} F \ldots = [G \xrightarrow{\phi} F \ldots] \otimes_A B \), [3, Prop. 5.1]. Notice that any \( A \)-linear map \( \alpha : M \rightarrow M \) defines a map \( L_2 := \text{G} \rightarrow L_0 := \text{G} \), which gives a cocycle in the complex computing \( \text{Ext} \), and therefore defines an element \( \pi \in \text{Ext}^2_B(M, M) \). If \( A \) is Noetherian and \( G \) is of finite rank, then \( \text{rk } F = \text{rk } G \), [3, Prop. 5.3]. We will assume the rank(s) of \( G \) and \( F \) to be finite. Define a *regular* element in \( A \) to be an \( f \in A \) with \( \text{ker}(f) = 0 \) and \( \text{coker}(f) \neq 0 \) where \( f \cdot \) is the multiplication with \( f \)-map on \( A \).
**Example 1.** Let $M$ be a finitely generated maximal Cohen-Macaulay (MCM) module over a local Noetherian ring $B$. If there is a local regular ring $A$ and a regular element $f \in A$ with $A/(f) \cong B$ then there is a mf $(\phi, \psi)$ of $f$ with coker $\phi \cong M$ and (2) is a $B$-free resolution of $M$: Let $\phi : F \to G$ be a finite $A$-free presentation of $M$, we may assume $\phi$ injective by the Auslander-Buchsbaum theorem. Multiplication with $f$ on the presentation as complex is homotopic to zero since $f$ annihilates $M$. Hence there is a $\psi : G \to F$ with $\psi \psi = f \cdot \text{id}_G$. But then $(\phi, \psi)$ is a mf of $f$ by
\[
\psi \psi = f \cdot \text{id}_G \phi = \phi \cdot f \cdot \text{id}_F \implies \psi \psi = f \cdot \text{id}_F
\]
since $\phi$ is injective.

On the other hand, if $(\phi, \psi)$ is a mf with rk$G < \infty$ of the regular element $f$ in the local Cohen-Macaulay ring $A$ then $M = \ker \phi$ is a MCM module over $A/(f)$: Since $f$ is regular, both $\phi$ and $\psi$ are injective, hence pd$_A(M) = 1 =$ depth $A -$ depth $M$ and depth $M = $ depth $A/(f) = \dim A/(f)$.

Our main result is a sharpened form of Theorem 1 for modules given by certain matrix factorisations. Let $T^i(X)$ be the completion of the free $k$-algebra with Zariski tangent space $X$ (of countable dimension) in the proper topology, see [8, Chap. 2] and [13, Chap. 4]. Let grade$I = n$ if a maximal $A$-regular sequence in $I$ has length $n$.

**Theorem 2.** Suppose $A$ is a Noetherian $k$-algebra, $\phi : F \to G$ a homomorphism of free $A$-modules of equal rank $g$ with det$(\phi) = f$ and $f \in A$ a regular element. Set $M = \ker \phi$ and $B = A/(f)$ and assume Ann$_A M = (f)$. Then the deformation functor Def$_M^B$ has tangent space Def$_M^B(k[\varepsilon]) \cong \text{Ext}^1_B(M, M) \cong H_1(\phi)$, where $S = S(\phi)$ is the Scandinavian Complex of $\phi$. The first of the two obstruction maps $o^\phi$ is trivial while the other $o^J$ for $J = (f)$ factors through the quotient
\[
\begin{array}{ccc}
T^2(\text{Ext}^2_B(M, M)) & \xrightarrow{o^J} & T^1(H_1(S)) = T^1
\end{array}
\]
induced by a natural inclusion $A/I_{g-1}(\phi) \hookrightarrow \text{Ext}^2_B(M, M)$, hence the hull is given as
\[
H \cong T^1 \otimes_{T^2} k.
\]
Moreover the tangent and obstruction space of Def$_M^B$ have finite dimension if $\dim_k A/I_{g-1}(\phi) < \infty$ and then
\[
(3) \quad \dim_k H_1(S) \geq \dim_{\text{null}} H \geq \dim_k H_1(S) - \dim_k A/I_{g-1}(\phi) = \dim_k \text{Ext}^1_B(M, M) - \dim_k \text{Ext}^2_B(M, M) + \dim_k H_2(S)
\]
In particular $H_2(S) \neq 0$ if grade$I_{g-1}(\phi) = 2$ and Def$_M^B$ is infinitesimally rigid if grade$I_{g-1}(\phi) = 4$.

The proof of Theorem 2 takes up the rest of the paper. The first obstruction map in Theorem 1 is trivial here since Ext$_A^2(M, M) = 0$. One obstruction map $o^* : T^2 \to T^1$ as $o^J$ above for a functor $F$ with a natural obstruction class $o_i$ in some $H^2$ should, in addition to being continuous, satisfy the following: Fix a formal versal family $\{M_i\}$, $M_i \in F(H_i)$ where $H = \lim H_i$ is a hull for $F$. For any lifting situation as in Definition 1 and map $\sigma : H_i \to S$ in Art$_k$ with $\sigma_* M_i = M_S$ (which
we know exists by versality) we should have that \( o_\tau(\pi, M_S) \in H^2 \otimes_k I \) is the adjoint to \((o^*)^* \theta : H^2 \rightarrow I\) in the commutative diagram

\[
\begin{array}{cccc}
H^2 & \stackrel{(o^*)^* \theta}{\longrightarrow} & T^2 & \stackrel{o^*}{\longrightarrow} & T^1 & \stackrel{\theta}{\longrightarrow} & R \\
& & \downarrow & & \downarrow & & \\
& & H_i & \stackrel{\sigma}{\longrightarrow} & S & & \\
\end{array}
\]

where \( \theta \) is continuous and lifts \( \sigma \). In [8, Chap. 2] we give an axiomatic treatment of a functor with \( n \) obstructions and prove the existence of \( n \) obstruction maps.

**Example 2.** In the \( 2 \times 2 \)-case the Koszul complex \( K(x_{ij}) \) and \( S(\phi) \) for \( \phi = (x_{ij}) \) are isomorphic and \( I_{i-1}(\phi) = I(\phi) = (x_{ij}) \) so

- grade \( I(\phi) = 4 \iff K(x_{ij}) \cong S(\phi) \) are acyclic.
- grade \( I(\phi) = 3 \iff H_i(K) \cong H_i(S) \neq 0 \) and \( H_i(K) \cong H_i(S) = 0 \) for \( i \geq 2 \).
  - If \( (\underline{x}) = (x_{ij}) \) and grade\((\underline{x}) = 3 \) we have \( H_i(S) = \ker(A/(\underline{x}) \rightarrow A/(\underline{x})) \) by the mapping cone sequence of the Koszul complex. In particular
   \[ H_1(S) = A/I(\phi) \text{ if } x_{i,j} \in (\underline{x}) \]
  - grade \( I(\phi) = 2 \iff H_2(K) \cong H_2(S) \neq 0 \) and \( H_2(K) \cong H_2(S) = 0 \) for \( i \geq 3 \).
  - If \( (x_{ij}) = (x_1, \ldots, x_4) \) such that \( (x_1, x_2) \) is a regular sequence then \( H_2(S) \cong ((x_1, x_2) : (x_1, \ldots, x_4))/(x_1, x_2) \).
  In particular if \( (x_3, x_4) \subseteq (x_1, x_2) \) then \( H_2(S) = A/I(\phi) \).

To a matrix factorisation \((\phi, \psi)\) of \( f \in A \) where \( A \) is a \( k \)-algebra there is a deformation functor \( \text{Def}^A_{(\phi, \psi)} \) of equivalence classes of liftings \((\tilde{\phi}, \tilde{\psi})\) of \((\phi, \psi)\) as mf, i.e. \( \text{Def}^A_{(\phi, \psi)}(R) \) is equivalence classes of commutative diagrams

\[
\begin{array}{cccc}
G \otimes_k R & \phi \longrightarrow & F \otimes_k R & \psi \longrightarrow & G \otimes_k R \\
\downarrow & & \downarrow & & \downarrow \\
G & \phi \longrightarrow & F & \psi \longrightarrow & G \\
\end{array}
\]

such that \( \tilde{\phi} \tilde{\psi} = f \cdot \text{id}_{G \otimes_k R} \). If \( B = A/(f) \) there is a map

\[
\text{Def}^A_{(\phi, \psi)} \rightarrow \text{Def}^B_{M}
\]

by taking the cokernel of \( \tilde{\phi} \) which gives an \( R \)-flat module since \( \phi \) and hence \( \tilde{\phi} \) is injective. Moreover \( M_R = \text{coker}(\tilde{\phi}) \) is a \( B \otimes R \)-module as \( A \otimes R \)-module since it is annihilated by \( f \) which follows from the relation \( \tilde{\phi} \tilde{\psi} = f \cdot \text{id}_{G \otimes_k R} \). The map is a tangential isomorphism and smooth since we more generally can construct the hull of \( \text{Def}^B_M \) from lifting an \( A \)-free resolution \((F_*, d_*)\) of \( M \) and a map \( \psi : E \otimes_A F_0 \rightarrow F_1 \) satisfying \( d_0 \psi = (f_1, \ldots, f_r) \otimes_A F_0 \) where \( E \cong A^r \rightarrow J = (f_1, \ldots, f_r) \), see [8]. Hence the map (5) is an isomorphism since it clearly is injective.

If \( f \) is a regular element one has that \( \psi \), if it exists, is uniquely determined by \( \phi \), see [3, Prop. 5.5]. In fact if \( \phi^a \) is the adjoint map of \( \phi \) then \( \det(\phi^a) \cdot \psi = f \cdot \phi^a \) which uniquely determines \( \psi \) since \( \det(\phi) \) is regular, op. cit. Let us therefore define another deformation functor \( \text{Def}^A_{(\phi, \det \phi)} \) of deformations of \( \phi \) with fixed determinant. If \( f \) is regular, we have

\[
\text{Def}^A_{(\phi, \det \phi)} \rightarrow \text{Def}^A_{(\phi, \psi)}
\]

The following result is essential for the proof of our Theorem 2.
Proposition 1. Let \((\phi, \psi)\) be a matrix factorisation of \(\det \phi = f \in A\) where \(A\) is a \(k\)-algebra. Set \(M = \ker \phi\), \(B = A/(f)\) and assume \(f\) is regular in \(A\) and \(\text{Ann}_A M = (f)\). Then the natural maps
\[
\text{Def}_M^A(\det \phi) \longrightarrow \text{Def}_M^A(\phi, \psi) \longrightarrow \text{Def}_M^B
\]
are isomorphisms. In particular there exists a versal lifting \(\tilde{\phi} = \varprojlim \phi_n\) of \(\phi\) to the hull \(H = \varprojlim H_n\) of \(\text{Def}_M^B\) such that \(\det \tilde{\phi} = f\).

Proof. We already have that the last map is an isomorphism and the first map is injective, hence we only have to show surjectivity of the first map. We proceed by induction on the length of \(R\), the beginning is trivial. Assume \(\pi : R \rightarrow S\) is surjective in \(\text{Art}_k\) and \(m_R : I = 0\) where \(I = \ker \pi\). Assume we have a lifting \((\phi_S, \psi_S)\) of \((\phi, \psi)\) to \(S\) with \(\phi_S \psi_S = f \cdot \text{id}_{G \otimes S}\) and \(\det \phi_S = \det \phi = f\). Given a further lifting \((\tilde{\phi}, \tilde{\psi})\) of \((\phi_S, \psi_S)\) to \(R\) with \(\tilde{\phi} \psi = f \cdot \text{id}_{G \otimes R}\). Set \(\tilde{M} = \ker \tilde{\phi}\). Then \(\det \tilde{\phi} = \det \phi_S + u = f + u\) with \(u = \sum a_i \otimes u_i \in A \otimes I\). We get \(a_i \in \text{Ann}_A M\) since \(\det \phi \in \text{Ann} \tilde{M}\) and \(f \in \text{Ann} \tilde{M}\) implies \(u \in \text{Ann} \tilde{M}\). But \(\text{Ann}_A M = (f)\) by assumption, hence \(u = bf\) and we can modify \((\tilde{\phi}, \tilde{\psi})\) to \((\tilde{\phi}', \tilde{\psi}')\) where \(\tilde{\phi}' = \tilde{\phi} - b c_{11} \phi\) and \(\tilde{\psi}' = \tilde{\psi} + b \psi e_{11}\) where the endomorphism \(e_{11}\) is given by a matrix with 1 in the upper left corner and zeroes elsewhere. Then \((\tilde{\phi}', \tilde{\psi}')\) is a matrix factorisation of \(f\) equivalent to \((\tilde{\phi}, \tilde{\psi})\) with \(\det \tilde{\phi}' = \det \tilde{\phi} - b \cdot \text{tr}(e_{11} \phi \phi^\ast) = f + u - bf = f\) where \(\text{tr}\) is the trace.

The result says that we only have to lift \(\phi\) and solve the equation \(\det \tilde{\phi} = f\) to find the obstructions. If \(\det \tilde{\phi} = f\) then \(\tilde{\psi} = (\tilde{\phi})^\ast\), the adjoint of \(\tilde{\phi}\).

If \(B = A/(f)\) is a domain, i.e. \(f\) is prime, the rank of \(M = \ker \phi\) is the dimension of the \(K\)-vector space \(K \otimes_B M\) where \(K = K(B)\) is the field of fractions of \(B\). If \((\phi, \psi)\) is a mf of \(f\) which is regular and prime then \(\det \phi = x \cdot f^r\) with \(x \notin \mathfrak{m}\) and \(r = \text{rk}_B M\), [3, Prop. 5.6]. Hence in this case the \(B\)-module \(M\) of Proposition 1 has rank one. In particular all maximal Cohen-Macaulay modules of rank one on irreducible hypersurface singularities are subsumed by the proposition.

**AN EDGE MAP AS A TRACE**

**Lemma 1.** Let \(A \rightarrow B\) be a ring homomorphism and \(N\) an \(A\)- and a \(B\)-module respectively. Then there is a first quadrant cohomological spectral sequence
\[
E_2^{pq} = \text{Ext}_B^p(M, \text{Ext}_A^q(B, N)) \Rightarrow \text{Ext}_A^p(M, N).
\]
In particular there is a canonical 5-term exact sequence which in the case \(B = A/(f)\) and \(N = M = \ker \phi\) for a matrix factorisation \((\phi, \psi)\) of \(f \in A\) where \(f\) is regular reduces to the 4-term exact sequence
\[
0 \rightarrow \text{Ext}_B^1(M, M) \rightarrow \text{Ext}_A^1(M, M) \xrightarrow{\partial_1} \text{End}_A(M) \xrightarrow{d_2} \text{Ext}_B^2(M, M) \rightarrow 0
\]
where \(\partial_1 = \psi^\ast\) is the pullback of cocycles along \(\psi\) and \(d_2\) is the differential of the spectral sequence. Moreover, \(d_2\) is the map sending the endomorphism \(\alpha \in \text{End}_A(M)\) to \(\overline{x}\), defined after (2).

Proof. Let \(F = F \rightarrow M\) be a \(B\)-projective resolution of \(M\) and \(N \rightarrow I = I\) an \(A\)-injective resolution of \(N\). Then the \(I\)-filtration of \(\text{Hom}_B(F, \text{Hom}_A(M, I))\) gives \(H^p F q \cong \text{Hom}_A(M, \text{Hom}_A(B, F))\) for \(p = 0\) and \(0\) for \(p > 0\), since \(\text{Hom}_A(B, F)\) is \(B\)-injective. We get \(\text{Hom}_A(M, \text{Hom}_A(B, I)) = \text{Hom}_A(M, I)\) by adjointness so the spectral sequence collapses at stage 2 to a single row with
\[
\text{Ext}_A^p(M, N) \cong H^p \text{Hom}_A(M, \text{Hom}_A(B, I)),
\]
the total cohomology. The \(I\)-filtration gives \(H^0 \text{Ext}_B^0(M, N) = \text{Hom}_B(F, \text{Ext}_A^0(B, N))\) thus \(H^0 \text{Ext}_B^0(M, N) = \text{Ext}_B^0(M, \text{Ext}_A^0(B, N))\). The 5-term exact sequence is the standard one.
0 → \text{Ext}^1_{B}(M, M) → \text{Ext}^1_{A}(M, M) \xrightarrow{\text{tr}(-\phi^a)} B \rightarrow A/I_{g-1}(\phi) \rightarrow 0
\quad 0 → \text{Ext}^2_{B}(M, M) \xrightarrow{\partial_2} \text{End}_{B}(M) \xrightarrow{\text{tr} \circ \text{id}} \text{Ext}^2_{A}(M, M) \rightarrow 0

In particular \(\partial_2 = (\phi^a)^*\) is induced by \(\text{tr}(-\phi^a)\text{id}_M\) where \(\phi^a\) is the adjoint of \(\phi\) and \(\text{tr}\) is the trace.

\textbf{Proof.} The image of \(\text{tr}(-\phi^a) : \text{Hom}_{A}(F, G) → A, \xi → \text{tr}(\xi \phi^a)\), is \(I_1(\phi^a) = I_{g-1}(\phi)\). To get the commutativity of the central square in the diagram, we have to show that \((\phi^a)^* = \text{tr}(-\phi^a)\text{id}_M\). Let \(e_{ij}\) be the \(g \times g\)-matrix with 1 in \(ij\)-position and 0 elsewhere. It is sufficient to find a \(g \times g\)-matrix \(\phi_{ij}\) with

\begin{equation}
e_{ij} \phi^a + \phi \phi_{ij} = c_{ij} \cdot \text{id}_G
\end{equation}

where \(c_{ij}\) is the \(ij\)-cofactor; \(c_{ij} = (-1)^{i+j} m_{ij}(\phi)\) where \(m_{ij}(\phi)\) is the \(ij\)-minor of \(\phi\). We have \(\text{tr}(e_{ij} \phi^a) = c_{ij}\) and since any \(\xi \in \text{Hom}_{A}(F, G)\) is an \(A\)-linear combination of \(e_{ij}\)'s, we get \(\partial_2(\xi) = \text{tr}(\xi \phi^a)\text{id}_M\). If \(m_{ik}(\phi_{ij})\) is the \(ik\)-minor of the matrix \(\phi_{ij}\)
obtained from \( \phi \) by deleting the \( l \)-th row and \( j \)-th column, we define \( \phi^{ij} \) by

\[
\phi^{ij} = (-1) \delta^{ij}
\]

Then the \( kl \)-entry of \( \phi^{ij} \) is

\[
(\phi^{ij})_{kl} = \begin{cases} 
0 & k \neq i \neq l \neq k \quad \text{(Laplace relation)} \\
-c_{ij} & k = i \neq l \quad \text{(Laplace expansion of } \det \phi_{ij}) \\
c_{ij} & k = l \neq i \\
0 & l = i
\end{cases}
\]

hence (8) follows and the rest follows from Lemma 1. \( \square \)

**The Scandinavian Complex of a matrix factorisation**

To \( \phi \) there is a functorial complex due to T. Gulliksen and O. Negård, the “Scandinavian Complex” \( S(\phi) \), which approximates an \( A \)-free resolution of \( A/I_{g-1}(\phi) \), see [6]. We define the complex for a general matrix factorisation.

**Definition 3.** Let \( G \xleftarrow{\phi} F \) be a matrix factorisation of \( f \in A \) where \( A \) is a ring. Then there is a complex \( S(\phi, \psi) \) of free \( A \)-modules;

\[
A \xleftarrow{d_1} \text{Hom}(F, G) \xleftarrow{d_2} E \xleftarrow{d_3} \text{Hom}(G, F) \xleftarrow{d_4} A,
\]

functorial for maps of the matrix factorisation. \( E \) is the middle cohomology of the split monad

\[
A \xleftarrow{d_1} \text{End}(G) \oplus \text{End}(F) \xleftarrow{d_1} A
\]

where \( i(a) = (aI, aI), j(p_0, p_1) = \text{tr}(p_0) - \text{tr}(p_1) \). The differentials are \( d_1(\xi) = \text{tr}(\xi \psi) \), \( d_2 \) and \( d_3 \) are induced by the differentials in the Yoneda complex,\n
\[
0 \leftarrow \text{Hom}(F, G) \leftarrow d_2 \text{End}(G) \oplus \text{End}(F) \leftarrow d_3 \text{Hom}(G, F) \leftarrow 0,
\]

i.e. \( d_2(p_0, p_1) = \phi p_1 - p_0 \phi \) and \( d_3(\tau) = (\phi \tau, \tau \phi) \), and \( d_4(\tau) = r \psi \).

**Remark 1.** In the case \( \psi = \phi^a \) we have \( S(\phi, \psi) = S(\phi) \). We will only be concerned with this case. Gulliksen and Negård proved that \( S(\phi) \) is a free resolution of \( A/I_{g-1}(\phi) \) if grade \( I_{g-1}(\phi) \) has the maximal value which is 4.

**Corollary 1.** With assumptions as in Proposition 2,

\[
\ker(\partial_*) \cong \text{Ext}^1_H(M, M) \cong H_1(S)
\]

where \( \partial_* = (\phi^a)^* \) is the pullback map in (7) and \( S = S(\phi) \) is the Scandinavian complex of \( \phi \).

**Proof.** By (8), for each \( \xi \) there is a \( \phi^* \) with \( \xi \phi^a + \phi \phi^* = (\sum r_{ij} c_{ij}) \cdot \text{id}_G \) if \( \xi = \sum r_{ij} c_{ij} \). But clearly \( \text{tr}(\xi \phi^a) = \sum r_{ij} c_{ij} \) hence every class \( [\xi] \in \text{Ext}_H^1(M, M) \), i.e. with \( \partial_*([\xi]) = 0 \), may be represented by a cocycle with \( \text{tr}(\xi \phi^a) = x f \) for some \( x \in A \). Set \( \xi = \xi^* - \xi e_{11} \phi \), then \( \xi \in Z_{\partial_*} := \ker d_1 \) and \( [\xi] = [\xi^*] \in \text{Ext}_H^1(M, M) \). Let \( B_{\text{tr}} = B \cap Z_{\partial_*} \) where \( B \) is the set of Yoneda coboundaries.

Since \( (\phi p_1 - p_0 \phi) \phi^a = \phi p_1 \phi^a - f p_0 \) we get \( \text{tr}((\phi p_1 - p_0 \phi) \phi^a) = f(\text{tr} p_1 - \text{tr} p_0) \) which equals zero iff \( \text{tr} p_1 = \text{tr} p_0 \), i.e. \( B_{\text{tr}} = \text{im}(d_2) \) and \( \text{Ext}_H^1(M, M) \cong H_1(S) \). By Lemma 1, \( \text{Ext}_H^1(M, M) \cong \ker(\partial_*) \). \( \square \)
We say that an $A$-module $M$ is infinitesimally rigid if $\text{Ext}^1_A(M, M) = 0$. Whenever the deformation functor of $M$ is defined, e.g. if $A$ is a $k$-algebra, every lifting $M_R$ of $M$ to $R$ in $\text{Art}_k$ is equivalent to the trivial lifting $M \otimes R$ of $M$ iff $M$ is infinitesimally rigid.

**Corollary 2.** In addition to the assumptions in Proposition 2 assume $A$ is Noetherian. Then $M$ is infinitesimally rigid as $B$-module if $\text{grade} I_{g-1}(\phi) = 4$, the maximal value. Moreover $M$ is not infinitesimally rigid as $B$-module if $\text{grade} I_{g-1}(\phi) = 3$.

*Proof.* By [6, Théorème 1], $c + q = 4$ where $c = \text{grade} I_{g-1}(\phi)$ and $q$ is maximal with $H_q(S(\phi)) \neq 0$. The result then follows from Corollary 1. □

There is also a relation between the obstruction group and the second homology group of the Scandinavian complex which we may obtain from the commutative diagram in Proposition 2:

**Corollary 3.** With assumptions as in Proposition 2, there is an exact sequence

\[
0 \to \text{Ext}^1_B(M, M) \to \text{Ext}^1_A(M, M) \xrightarrow{\text{tr}(\phi^a)} B \to \text{Ext}^2_B(M, M) \to H_2(S) \to 0
\]

where $\phi^a$ is the adjoint to $\phi$ and $S = S(\phi)$ is the Scandinavian complex of $\phi$. In the case $A$ is Noetherian and grade $I_{g-1}(\phi) \geq 3$ we have

\[
\text{coker}(\partial) \cong \text{Ext}^2_B(M, M) \cong A/I_{g-1}(\phi).
\]

*Proof.* The map $B \to \text{Ext}^2_B(M, M)$ is the composition

\[
B \to A/I_{g-1}(\phi) \to \text{Ext}^2_B(M, M),
\]

see Proposition 2. The map $B \to \text{End}_B(M)$ is injective since $\text{Ann}_A M = (f)$ and hence also $A/I_{g-1}(\phi) \to \text{Ext}^2_B(M, M)$ is injective and thus (9) is exact at $B$. The map $\text{Ext}^2_B(M, M) \to H_2(S)$ is defined via the natural $d_2 : \text{End}_B(M) \to \text{Ext}^2_B(M, M)$ in Lemma 1 and the second non-trivial map $h$ in the sequence

\[
0 \to A/\text{Ann}_A M \to \text{End}_B(M) \xrightarrow{h} H_2(S) \to 0
\]

which is defined and short exact for any matrix factorisation $(\phi, \psi)$ of a regular $f \in A$ and with $S = S(\phi, \psi)$. If $\rho_0 : G \to G$ and $\rho_1 : F \to F$ represent an endomorphism $[\rho]$ i.e. $\rho_0 \phi = \phi \rho_1$, we have $\psi \rho_0 \phi = \psi \rho_1 = f \rho_1$ which implies $f \text{tr}(\rho_1) = f \text{tr}(\rho_0)$ and hence $\text{tr}(\rho_1) = \text{tr}(\rho_0)$. Let $h([\rho])$ be defined as the class represented by $(\rho_0, \rho_1)$ in $S_2$ which clearly is independent of representatives by the definition of $d_2$ as the Yoneda differential essentially. Surjectivity of $h$ is immediate since $d_2$ is induced by the Yoneda differential. Exactness in the middle: Assume $h([\rho]) = 0$ then $(\rho_0, \rho_1) = (\phi \tau, \tau \phi) + (a \cdot \text{id}_G, a \cdot \text{id}_F)$ for some $\tau \in \text{Hom}_A(G, F)$ and $a \in A$, but then $[\rho] = a \cdot \text{id}_M$. The exactness at $\text{Ext}^2_B(M, M)$ follows because the cokernels of $B \to \text{End}_B(M)$ and of $A/I_{g-1}(\phi) \to \text{Ext}^2_B(M, M)$ are isomorphic to $H_2(S)$. The remaining follows from Proposition 2 and the argument in Corollary 2. □

Let us call an affine scheme $X = \text{Spec}(B)$ over a field $k$ a hypersurface singularity if $B \cong A/(f)$ where $f \in A$ is regular and $A$ is a regular local Noetherian $k$-algebra.

**Corollary 4.** With assumptions as in Proposition 2 and $A$ a Noetherian ring, we have the following assertions:

\[
l(A/I_{g-1}(\phi)) < \infty \quad \Longrightarrow \quad l(\text{Ext}^i_B(M, M)) < \infty \quad \text{for} \quad i \geq 1
\]

\[
l(\text{Ext}^2_B(M, M)) < \infty \quad \Longrightarrow \quad l(A/I_{g-1}(\phi)) < \infty.
\]

In particular, if the deformation functor of a rank 1 maximal Cohen-Macaulay module over an irreducible hypersurface singularity $X$ has finite dimensional obstruction
space then $\dim X \leq 3$. If in addition $\dim X = 3$ then the module is infinitesimally rigid.

Proof. If $A$ is Noetherian and the rank of $G$ and $F$ is finite the homology modules $H_i(S)$ are finitely generated. By Lemme 2 and Lemme 4 in [6] $I_g^{-1}(\phi)\cdot H_1(S) = 0$ for $i = 1, 2$, the result then follows from Corollary 1 and from the short exact sequence

$$0 \longrightarrow A / I_g^{-1}(\phi) \longrightarrow \text{Ext}^2_\mathbb{Z}(M, M) \longrightarrow H_2(S) \longrightarrow 0$$

derived from the exact sequence (9) in Corollary 3. For the last part see Corollary 2 and remarks after Proposition 1.

Remark 2. If we are mainly interested in modules with non-trivial deformation functors where the tangent and obstruction space is finite dimensional, Corollary 4 says that there are not too many places to look for such MCM-modules of rank 1 on hypersurface singularities. But this certainly is not the case for higher rank MCMs. To a mf $(\phi, \psi)$ of $f$, H. Knörrer defined a mf $\left( \left( \begin{array}{cc} e \cdot \text{id} & \phi \\ \psi & -u \cdot \text{id} \end{array} \right) , \left( \begin{array}{cc} u \cdot \text{id} & \phi \\ -v \cdot \text{id} & \psi \end{array} \right) \right)$ of $f+uv$ which induces a functor of MCMs modulo stable equivalence, see [12]. In [7] we proved that this functor gives a natural transformation of the corresponding deformation functors inducing isomorphisms of the tangent and obstruction spaces (see also [8, Thm. 7.4.18]). Hence starting with a rank 1 MCM in the “interesting” range, applying the Knörrer functor one or more times, produces MCMs in higher dimensions with the same deformation theory. But these new MCMs naturally can not be of rank 1. In fact $\text{rk}K(M) = \text{rk}G$ where $K$ is the Knörrer functor.

The deformation theory of rank 1 MCM modules also subsumes the deformation theory of their syzygies which have rank $g - 1$ by

Lemma 2. If $M$ is a MCM $B = A/(f)$-module, $A$ is a local, regular $k$-algebra and $f$ is regular and prime such that the minimal presentation $\psi : G \rightarrow F$ of $M$ has $\text{rk} F = g$ while $\text{rk}B M = g - 1$, then there is a rank 1 MCM $B$-module $M'$ such that

$$\text{Def}^B_M \cong \text{Def}^B_{M'}.$$  

Proof. To $\psi$ there is a $\phi : F \rightarrow G$ s.t. $(\psi, \phi)$ is a mf of $f$. Since $\det \psi \cdot \det \phi = f^g$ and $\text{rk} M = g - 1$ we may assume $\det \phi = f$. Hence $M' := \operatorname{coker} \phi$ is a MCM $B$-module of rank 1. Now

$$\text{Def}^B_M \cong \text{Def}^A_{(\psi, \phi)} \cong \text{Def}^A_{(\phi, \psi)} \cong \text{Def}^B_{M'}.$$  

Proof of Theorem 2. By Proposition 1, $\text{Def}^B_M \cong \text{Def}^A_{(\phi, \psi)}$ and $\text{Def}^A_{(\psi, \phi)}$ has, by the proof of Proposition 1, the obstruction induced by $\det \tilde{\phi} - f$ where $\tilde{\phi}$ is a lifting of $\phi_S$ (with $\det \phi_S = f$) to $R$ in a small lifting situation, Definition 1. This implies that the obstruction always is in the image of $B$ in (9) hence maps to zero in $H_2(S)$ and therefore naturally is found in $A/I_g^{-1}$ by (11). The tangent space is given by Corollary 2 which gives the upper estimate of the dimension of the hull since the obstruction powerseries has minimal degree greater than or equal to two by construction. At “worst” they give a regular sequence with a maximal number of elements, this gives the lower estimate. The construction of the obstruction map is done as in [14] and [8]. The rest follows from Corollary 2, Corollary 4 and [6, Theorem 1].

Example 3. If $\phi = (x_{ij})$ is the generic $g \times g$-matrix, then $M = \operatorname{coker} \phi$ is a rigid rank 1 MCM $B = k[x_{ij}] / (\det \phi)$-module for all $g \geq 1$. By Lemma 2, $M' = \operatorname{coker}(\phi^g)$ is also a rigid MCM module, but of $B$-rank $g - 1$. Let $g = 2$ then the equation $\text{tr}(a_{ij}) = a_{11} x_{22} - a_{12} x_{21} - a_{21} x_{12} + a_{22} x_{11} = 0$ has solutions generated by Koszul
relations, they are all coboundaries. But if we set $x_{12} = x_{21}$ then $a_{12} + a_{21} = 0$ is a non-trivial solution, and we obtain $\text{Ext}^1_{B}(M, M) \cong k$ generated by $\xi = \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right)$. We try to lift the universal extension $M_1 \in \text{Def}^1_{\text{gr}}(k[\mu]/(\mu^2))$ given by $\phi + \xi u$ to $k[\mu]/(\mu^3)$ and calculate the obstruction given by $\det(\phi + \xi u) = \det \phi + \det(\xi) u^2 = f + u^2$ where $f = \det \phi$. The cup product $\xi \cup \xi = d_2(1_M)$ is non-zero, hence the hull is $H = k[\mu]/(u^2)$, as there can be no further liftings.

**Example 4.** Let $\phi = (x_{ij})$ be the $3 \times 3$-generic, with the restriction that $x_{ii} = 0$ for $i = 1, 2, 3$. Then $\det \phi = x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32}$ and $\text{tr}((a_{ij})^2) = -a_{11}x_{23}x_{32} + a_{12}x_{23}x_{31} + a_{13}x_{21}x_{32} + a_{21}x_{12}x_{33} + a_{22}x_{13}x_{31} + a_{23}x_{13}x_{21} + a_{31}x_{12}x_{23} + a_{32}x_{13}x_{31} - a_{33}x_{12}x_{21}$; but we do not get finite dimensional tangent- and obstruction spaces. Set $x_{13} = x_{21} = x_{32} = y$ then $f = \det \phi = x_{12}x_{23}x_{31} + y^2$ and we get a two dimensional tangent space for the graded deformation functor given by the relation $a_{13} + a_{21} + a_{32} = 0$. Deform $\phi$ to $\widetilde{\phi} = \left( \begin{array}{c} 0 \\ x_{12} \\ y + u + v \\ 0 \\ x_{23} \\ y - v \\ 0 \\ x_{31} \\ y - v \\ 0 \end{array} \right)$. Then $\det \widetilde{\phi} = f - y(u^2 - uv + v^2) + (u + v)uv$ which gives the second order obstruction polynomial $g = u^2 - uv + v^2$ carried by the class of $-y$ in the obstruction space $\mathcal{A}/m^2$, hence $H_2 = k[u, v]/((g) + (u, v)^2)$. The obstruction to lift $M_2$ along $\pi : k[u, v]/((g)(u, v) + (u, v)^4) \to H_2$ is $-y - g + 1 - h$ where $h = (u + v)uv$. The cohomology classes $-y$ and $1$ are independent over $k$ and there are no further obstructions hence $H \cong k[[u, v]]/(u^2 - uv + v^2, (u + v)uv)$.

**References**


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