Minimal variance hedging for fractional Brownian motion

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August 13, 2003

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Abstract

We discuss the extension to the multi-dimensional case of the Wick-Itô integral with respect to fractional Brownian motion, introduced by [6] in the 1-dimensional case. We prove a multi-dimensional Itô type isometry for such integrals, which is used in the proof of the multi-dimensional Itô formula. The results are applied to study the problem of minimal variance hedging in a market driven by fractional Brownian motions.

AMS 2000 subject classifications: Primary 60HXX; Secondary 91B28.
Key words and phrases: Fractional Brownian motion, fractional Itô isometry, minimal variance hedging in incomplete markets.

1 Introduction

In the following we let \( H = (H_1, H_2, \ldots, H_m) \) be an \( m \)-dimensional Hurst vector with components \( H_i \in (\frac{1}{2}, 1) \) for \( i = 1, 2, \ldots, m \), and we let \( B^{(H)}(t) = (B_1^{(H)}(t), \ldots, B_m^{(H)}(t)) \) be an \( m \)-dimensional fractional Brownian motion (fBM) with Hurst parameter \( H \). This means that \( B^{(H)}(t) = B^{(H)}(t, \omega); \ t \in \mathbb{R}, \ \omega \in \Omega \) is a continuous Gaussian stochastic process on a filtered probability space \( (\Omega, \mathcal{F}^{(H)}, \mu) \) with mean

\[
\mathbb{E}[B^{(H)}(t)] = 0 = B^{(H)}(0) \quad \text{for all } t
\]  

(1.1)

and covariance

\[
\mathbb{E}[B_i^{(H)}(s)B_j^{(H)}(t)] = \frac{1}{2} \left( |s|^{2H_i} + |t|^{2H_i} - |s-t|^{2H_i} \right) \delta_{ij}
\]  

(1.2)

where

\[
\delta_{ij} = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j; \quad i \leq i, j \leq m ,
\end{cases}
\]

where \( \mathbb{E} = \mathbb{E}_\mu \) denotes the expectation with respect to the probability law \( \mu \) of \( B^{(H)}(\cdot) \).

In other words, \( B^{(H)}(t) \) consists of \( m \) independent 1-dimensional fractional Brownian motions with Hurst parameters \( H_1, \ldots, H_m \), respectively. If \( H_i = \frac{1}{2} \) for all \( i \), then \( B^{(H)}(t) \) coincides with classical Brownian motion \( B(t) \). We refer to [11], [13] and [18] for more information about 1-dimensional fBM. Because of its properties (persistence/antipersistence and self-similarity) fBM has been suggested as a useful mathematical tool in many applications, including finance [10]. For example, these features of fBM seem to appear in the log-returns of stocks [18], in weather derivative models [3] and in electricity prices in a liberated electricity market [20].

In view of this it is of interest to develop a powerful calculus for fBM. Unfortunately, fBM is not a semimartingale nor a Markov process (unless \( H_i = \frac{1}{2} \) for all \( i \)), so these theories cannot be applied to fBM. However, if \( H_i > \frac{1}{2} \) then the paths have zero quadratic variation and it is therefore possible to define a pathwise integral, denoted by

\[
\int_{\mathbb{R}} f(t, \omega) \delta B^{(H)}(t),
\]
by a classical result of Young from 1936. See [12] and the references therein. This integral will obey Stratonovich type (i.e. “deterministic”) integration rules. Typically the expectation of such integrals is not 0 and it is known ([12], [15], [16], [19]) that the use of these integrals in finance will give markets with arbitrage, even in the most basic cases. In fact, this unpleasant situation (from a modelling point of view) occurs whenever we use an integration theory with Stratonovich integration rules in the generation of wealth from a portfolio. See e.g. the simple examples of [4] and [19].

Because of this — and for several other reasons — it is natural to try other types of integration with respect to \( fBm \). Let \( \mathcal{L}^{1,2}_\phi \) be the set of (measurable) processes \( f(\cdot, \cdot) : \mathbb{R} \times \Omega \rightarrow \mathbb{R} \) such that \( \|f\|_{\mathcal{L}^{1,2}_\phi} < \infty \), where

\[
\|f\|_{\mathcal{L}^{1,2}_\phi}^2 := \mathbb{E}\left[ \int_{\mathbb{R}} \int_{\mathbb{R}} |f(s,t)| \phi(s,t) ds \, dt + \left( \int_{\mathbb{R}} D^\phi_t f(t) dt \right)^2 \right].
\]  

In [6] a Wick-Itô type of integral is constructed, denoted by

\[
\int_{\mathbb{R}} f(t, \omega) dB^{(H)}(t),
\]

where \( B^{(H)}(t) \) is a 1-dimensional \( fBm \) with \( H \in \left( \frac{1}{2}, 1 \right) \). This integral exists as an element of \( L^2(\mu) \) for all (measurable) processes \( f(t, \omega) \) such that \( \|f\|_{\mathcal{L}^{1,2}_\phi} < \infty \). Here, and in the following,

\[
\phi(s,t) = \phi_H(s,t) = H(2H-1)|s-t|^{2H-2}; \quad (s,t) \in \mathbb{R}^2; \quad \frac{1}{2} < H < 1
\]  

and

\[
D^\phi_t F = \int_{\mathbb{R}} \phi(s,t) D_s F \, ds
\]

denotes the Malliavin \( \phi \)-derivative of \( F \) (see [6, Definition 3.4]). If \( f(t, \omega) \) is a step process of the form

\[
f(t, \omega) = \sum_{i=1}^{n} f_i(\omega) \mathcal{X}_{[t_i, t_{i+1})}(t), \quad \text{where} \quad t_1 < t_2 < \cdots < t_{n+1},
\]

and \( \|f\|_{\mathcal{L}^{1,2}_\phi} < \infty \), then the integral is defined by

\[
\int_{\mathbb{R}} f(t, \omega) dB^{(H)}(t) = \sum_{i=1}^{n} f_i(\omega) \circ (B^{(H)}(t_{i+1}) - B^{(H)}(t_i)),
\]

where \( \circ \) denotes the Wick product. We have the following basic properties of the Wick-Itô integral:

\[
\mathbb{E}\left[ \int_{\mathbb{R}} f(t, \omega) dB^{(H)}(t) \right] = 0 \quad \text{for all} \quad f \in \mathcal{L}^{1,2}_\phi
\]  

\[
\mathbb{E}\left[ \left( \int_{\mathbb{R}} f(t, \omega) dB^{(H)}(t) \right) \left( \int_{\mathbb{R}} g(t, \omega) dB^{(H)}(t) \right) \right] = \langle f, g \rangle_{\mathcal{L}^{1,2}_\phi} \quad \text{for all} \quad f, g \in \mathcal{L}^{1,2}_\phi
\]  

\[
\langle f, g \rangle_{\mathcal{L}^{1,2}_\phi} = \mathbb{E}\left[ \int_{\mathbb{R}} \int_{\mathbb{R}} |f(s)g(t)| \phi(s,t) ds \, dt + \left( \int_{\mathbb{R}} D^\phi_s f(t) dt \right) \cdot \left( \int_{\mathbb{R}} D^\phi_t g(t) dt \right) \right].
\]

This Wick-Itô fractional calculus was subsequently extended to a white noise setting and applied to finance in [9]. Later this white noise theory was generalized to all $H \in (0,1)$ by [7].

All the above papers [6], [9] and [7] only deal with the 1-dimensional case. In Section 2 of this paper we discuss the extension of this integral to the $m$-dimensional case, i.e. we discuss the integral

$$\int_{\mathbb{R}} f(t, \omega) dB^{(H)}(t) = \sum_{i=1}^{m} \int_{\mathbb{R}} f_i(t, \omega) dB_i^{(H)}(t) \quad \text{for } f = (f_1, \ldots, f_m) \in \mathcal{L}^{1,2}_\phi(m)$$

where $B^{(H)}(t) = (B_1^{(H)}(t), \ldots, B_m^{(H)}(t))$ is $m$-dimensional $fBm$, $\phi = (\phi_{H_1}, \ldots, \phi_{H_m})$ and $\mathcal{L}^{1,2}_\phi(m)$ is the corresponding class of integrands (see (2.5) below). We prove the $m$-dimensional analogue of the isometry (1.9), which turns out to have some unexpected features (see Theorem 2.1). By combining the multi-dimensional fractional Itô formula (Theorem 2.6) with Theorem 2.1 we obtain another fractional Itô isometry (Theorem 2.7). Finally, we end Section 2 by proving a fractional integration by parts formula (Theorem 2.9 and Theorem 2.10).

In Section 3 we apply the above results to study the problem of minimal variance hedging in a (possibly incomplete) market driven by $m$-dimensional $fBm$. Here we use fractional mathematical market model introduced by [9] and by [7]. For classical Brownian motions (and semimartingales) this problem has been studied by many researchers. See for example the survey [17] and the references therein. It turns out that for $fBm$ this problem is even harder than in the classical case and in this paper we concentrate on a special case in order to get more specific results.

## 2 Multi-dimensional Wick-Itô integration with respect to $fBm$

Let $B^{(H)}(t) = (B_1^{(H)}(t), \ldots, B_m^{(H)}(t)); t \in \mathbb{R}, \omega \in \Omega$ be $m$-dimensional $fBm$ with Hurst vector $H = (H_1, \ldots, H_m) \in (\frac{1}{2},1)^m$, as in Section 1. Since the $B_k^{(H)}(\cdot)$ are independent, we may regard $\Omega$ as a product $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m$ of identical copies $\Omega_k$ of some $\tilde{\Omega}$ and write $\omega = (\omega_1, \ldots, \omega_m) \in \Omega$.

Let $\mathcal{F} = \mathcal{F}_{\infty}^{(m,H)}$ be the $\sigma$-algebra generated by $\{B_k^{(H)}(\cdot); s \in \mathbb{R}, k = 1, 2, \ldots, m\}$ and let $\mathcal{F}_t = \mathcal{F}_t^{(m,H)}$ be the $\sigma$-algebra generated by $\{B_k^{(H)}(\cdot); 0 \leq s \leq t, k = 1, 2, \ldots, m\}$. If $F : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable, $1 \leq k \leq m$, we set

$$D_{k,t}^\phi F = \int_{\mathbb{R}} \phi_k(s,t) D_{k,t} F \, dt \quad \text{if the integral converges} \quad (2.1)$$

where

$$\phi = (\phi_1, \ldots, \phi_m) \quad (2.2)$$

$$\phi_k(s,t) = \phi_{H_k}(s,t) = H_k(2H_k - 1)|s-t|^{2H_k-2}; \quad (s,t) \in \mathbb{R}^3, \; k = 1, 2, \ldots, m \quad (2.3)$$

and $D_{k,t} F = \frac{\partial F}{\partial \omega_k}(t, \omega)$ is the Malliavin derivative of $F$ with respect to $\omega_k$, at $(t, \omega)$ (if it exists).
Let $\mathcal{B} = \mathcal{B}(\mathbb{R})$ denote the Borel $\sigma$-algebra on $\mathbb{R}$. Similarly to the 1-dimensional case we can define the multi-dimensional fractional Wick-Itô integral

$$
\int_{\mathbb{R}} f(t, \omega) dB^{(H)}(t) = \sum_{k=1}^{m} \int_{\mathbb{R}} f_k(t, \omega) dB^{(H)}_k(t) \in L^2(\mu) \quad (2.4)
$$

for all $\mathcal{B} \times \mathcal{F}$-measurable processes $f(t, \omega) = (f_1(t, \omega), \ldots, f_m(t, \omega)) \in \mathbb{R}^m$ such that

$$
\|f_k\|_{L_{\phi_k}^{1,2}} < \infty \quad \text{for all } k = 1, 2, \ldots, m, \text{ where }
$$

$$
\|f_k\|_{L_{\phi_k}^{1,2}} := \mathbb{E}\left[\left(\int_{\mathbb{R}} f_k(s) f_k(t) \phi_k(s, t) ds \, dt + \left(\int_{\mathbb{R}} D_{\phi_k}^d f_k(t) dt\right)^2\right)\right]. \quad (2.5)
$$

Denote the set of all such $m$-dimensional processes $f$ by $L_{\phi}^{1,2}(m)$. As in the 1-dimensional case we obtain the isometries

$$
\mathbb{E}\left[\left(\int_{\mathbb{R}} f_k dB^{(H)}_k(t)\right)^2\right] = \|f_k\|_{L_{\phi_k}^{1,2}}^2; \quad k = 1, 2, \ldots, m. \quad (2.6)
$$

This is intuitively clear, since we (by independence of $B^{(H)}_1, \ldots, B^{(H)}_m$) can treat the remaining stochastic variables $\omega_1, \ldots, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_m$ as parameters and repeat the 1-dimensional approach in the $\omega_k$ variable. It is also easy to prove (2.6) rigorously by writing $f_k(t, \omega_1, \omega_2, \ldots, \omega_m)$ as a limit of sums of products of functions depending only on $(t, \omega_k)$ and only on $(\omega_1, \ldots, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_m)$, respectively.

In view of this it is clear that if $f = (f_1, \ldots, f_m) \in L_{\phi}^{1,2}(m)$, then the Wick-Itô integral (2.4) is well-defined as an element of $L^2(\mu)$ and by (2.6) we have

$$
\left\|\int_{\mathbb{R}} f dB^{(H)}\right\|_{L^2(\mu)} \leq \sum_{k=1}^{m} \|f_k\|_{L_{\phi_k}^{1,2}}. \quad (2.7)
$$

It is useful to have an explicit expression for the norm on the left hand side of (2.7). The following formula is our main result of this section:

**Theorem 2.1 (Multi-dimensional fractional Wick-Itô Isometry 1)**

Let $f, g \in L_{\phi}^{1,2}(m)$. Then

$$
\mathbb{E}\left[\left(\int_{\mathbb{R}} f dB^{(H)}\right) \cdot \left(\int_{\mathbb{R}} g dB^{(H)}\right)\right] = (f, g)_{L_{\phi}^{1,2}(m)} \quad (2.8)
$$

where

$$
(f, g)_{L_{\phi}^{1,2}(m)} = \mathbb{E}\left[\sum_{k=1}^{m} \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(s) g_k(t) \phi_k(s, t) ds \, dt + \sum_{k=1}^{m} \int_{\mathbb{R}} \int_{\mathbb{R}} D_{\phi_k}^d f_k(t) dt \cdot \left(\int_{\mathbb{R}} D_{\phi_k}^d g_k(t) dt\right)\right]. \quad (2.9)
$$
Remark  Note the crossing of the indices $\ell, k$ of the derivatives and the components $f_k, g_{\ell}$ in the last terms of the right hand side of (2.9).

To prove Theorem 2.1 we proceed as in [6], but with the appropriate modifications:

In the 1-dimensional case, let $L^2_{\phi_k}$ be the set of deterministic functions $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$
(\alpha, \alpha)_{\phi_k} := |\alpha|^2_{\phi_k} := \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha(s)\alpha(t)\phi_k(s, t)ds \, dt < \infty .
$$

If $\alpha \in L^2_{\phi_k}$ then clearly $\alpha \in \mathcal{L}^{1,2}_{\phi_k}$. Hence we can define the Wick (or Doleans-Dale) exponential

$$
\mathcal{E}(\alpha) = \exp \left( \int_{\mathbb{R}} \alpha(t)dB_k^{(H)}(t) \right) = \exp \left( \int_{\mathbb{R}} \alpha(t)dB_k^{(H)}(t) - \frac{1}{2} |\alpha|^2_{\phi_k} \right).
$$

See e.g. [6, (3.1)] or [9, Example 3.10].

Similarly, in the multidimensional case we put $\phi = (\phi_1, \ldots, \phi_m)$ and we let $L^2_\phi$ be the set of all deterministic functions $\alpha = (\alpha_1, \ldots, \alpha_m) : \mathbb{R} \to \mathbb{R}^m$ such that $\alpha_k \in L^2_{\phi_k}$ for $k = 1, \ldots, m$. If $\alpha \in L^2_\phi$ we define the corresponding Wick exponential

$$
\mathcal{E}(\alpha) = \exp \left( \int_{\mathbb{R}} \alpha(t)dB^{(H)}(t) \right) = \exp \left( \sum_{k=1}^m \int_{\mathbb{R}} \alpha_k(t)dB_k^{(H)}(t) \right)
$$

$$
= \exp \left( \sum_{k=1}^m \int_{\mathbb{R}} \alpha_k(t)dB_k^{(H)}(t) - \frac{1}{2} |\alpha|^2_\phi \right),
$$

where

$$
|\alpha|^2_\phi = \sum_{k=1}^m \int_{\mathbb{R}} \alpha_k(s)\alpha_k(t)\phi_k(s, t)ds \, dt = \sum_{k=1}^m |\alpha|^2_{\phi_k} .
$$

Let $\mathcal{E}$ be the linear span of all $\mathcal{E}(\alpha)$; $\alpha \in L^2_\phi$. Then we have

**Theorem 2.2 ([6, Theorem 3.1])**

$\mathcal{E}$ is a dense subset of $L^p(F, \mu)$, for all $p \geq 1$.

and

**Theorem 2.3 ([6, Theorem 3.2])**

Let $g_i = (g_{i1}, \ldots, g_{im}) \in L^2_\phi$ for $i = 1, 2, \ldots, n$ such that

$$
|g_{ik} - g_{jk}|_{\phi_k} \neq 0 \quad \text{if} \quad i \neq j, \ k = 1, \ldots, m .
$$

Then $\mathcal{E}(g_1), \ldots, \mathcal{E}(g_n)$ are linearly independent in $L^2(F, \mu)$.

If $F \in L^2(F, \mu)$ and $g_k \in L^2_{\phi_k}$ we put, as in [6],

$$
D_{k,\phi}(g_k) F = \int_{\mathbb{R}} D_{k,t}^{\phi} F \cdot g_k(t)dt .
$$

We list some useful differentiation and Wick product rules. The proofs are similar to the 1-dimensional case and are omitted.
Lemma 2.4  Let $f = (f_1, \ldots, f_m) \in L^2_{\phi}$, $g = (g_1, \ldots, g_m) \in L^2_{\phi}$. Then

(i) $D_{k, \phi(g_k)} \left( \sum_{i=1}^{m} f_i dB_i^H \right) = (f_k, g_k)_{\phi_k}$, $k = 1, \ldots, m$,

where

$$
(f_k, g_k)_{\phi_k} = \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(s) g_k(t) \phi_k(s, t) \, ds \, dt ; \quad k = 1, \ldots, m ,
$$

(ii) $D^0_{k, \phi} \left( \sum_{i=1}^{m} f_i dB_i^H \right) = \int_{\mathbb{R}} f_k(u) \phi_k(s, u) \, du ; \quad k = 1, \ldots, m ,

(iii) $D_{k, \phi(g_k)} \mathcal{E}(f) = \mathcal{E}(f) \cdot (f_k, g_k)_{\phi_k}$ ; $k = 1, \ldots, m$ ,

(iv) $D^0_{k, \phi} \mathcal{E}(f) = \mathcal{E}(f) \cdot \int_{\mathbb{R}} f_k(u) \phi_k(s, u) \, du ; \quad k = 1, \ldots, m$ ,

(v) $\mathcal{E}(f) \circ \mathcal{E}(g) = \mathcal{E}(f + g)$

(vi) $F \circ \int_{\mathbb{R}} g_k dB_k^H = F \cdot \int_{\mathbb{R}} g_k dB_k^H - D_{k, \phi(g_k)} F , \quad k = 1, \ldots, m$ ,

provided that $F \in L^2(\mathcal{F}, \mu)$ and $D_{k, \phi(g_k)} F \in L^2(\mathcal{F}, \mu)$.

(vii) $\mathbb{E}[\mathcal{E}(f) \cdot \mathcal{E}(g)] = \exp(f, g)_{\phi}$.

We now turn to the multi-dimensional case. We will prove

Lemma 2.5  Suppose $\alpha_k \in L^2_{\phi_k}$, $\beta_k \in L^2_{\phi_k}$, $D_{k, \phi(\beta_k)} F \in L^2(\mu)$ and $D_{k, \phi(\alpha_k)} G \in L^2(\mu)$. Then

$$
\mathbb{E} \left[ \left( F \circ \int_{\mathbb{R}} \alpha_k dB_k^H \right) \cdot \left( G \circ \int_{\mathbb{R}} \beta_k dB_k^H \right) \right] = \mathbb{E} \left[ \left( D_{k, \phi(\beta_k)} F \right) \cdot \left( D_{k, \phi(\alpha_k)} G \right) + \delta_{k\ell} FG(\alpha_k, \beta_k)_{\phi_k} \right] ,
$$

where

$$
\delta_{k\ell} = \begin{cases} 
1 & \text{if } k = \ell \\
0 & \text{otherwise}
\end{cases}
$$

PROOF.  We adapt the argument in [6] to the multi-dimensional case:

First note that by a density argument we may assume that

$$
F = \mathcal{E}(f) = \exp \left\{ \int_{\mathbb{R}} f(t) dB^H(t) - \frac{1}{2} |f|_{\phi}^2 \right\}
$$

and

$$
G = \mathcal{E}(g) = \exp \left\{ \int_{\mathbb{R}} g(t) dB^H(t) - \frac{1}{2} |g|_{\phi}^2 \right\}
$$

for some $f \in L^2_{\phi}$, $g \in L^2_{\phi}$.
Choose \( \delta = (\delta_1, \ldots, \delta_m) \in \mathbb{R}^m \), \( \gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{R}^m \) and put \( \delta \times f = (\delta_1 f_1, \ldots, \delta_m f_m) \) and \( \gamma \times g = (\gamma_1 g_1, \ldots, \gamma_m g_m) \). Then by Lemma 2.4

\[
\mathbb{E}[(\mathcal{E}(f) \circ \mathcal{E}(\delta \times \alpha)) \cdot (\mathcal{E}(g) \circ \mathcal{E}(\gamma \times \beta))] = \mathbb{E}[\mathcal{E}(f + \delta \times \alpha) \cdot \mathcal{E}(g + \gamma \times \beta)] = \exp(f + \delta \times \alpha, g + \gamma \times \beta)\phi
\]

(2.18)

\[
\exp \left\{ \sum_{i=1}^{m} \int_{\mathbb{R}} \int_{\mathbb{R}} (f_i + \delta_i \alpha_i)(s)(g_i + \gamma_i \beta_i)(t) \phi(t, s) ds dt \right\}. 
\]

(2.19)

We now compute the double derivatives

\[
\frac{\partial^2}{\partial \delta_k \partial \gamma_\ell}
\]

of (2.18) and (2.19) at \( \delta = \gamma = 0 \). We distinguish between two cases:

**Case 1** \( k \neq \ell \)

Then if we differentiate (2.18) we get

\[
\frac{\partial^2}{\partial \delta_k \partial \gamma_\ell} \mathbb{E}[(\mathcal{E}(f) \circ \mathcal{E}(\delta \times \alpha)) \cdot (\mathcal{E}(g) \circ \mathcal{E}(\gamma \times \beta))]_{\delta, \gamma = 0}
\]

\[
= \frac{\partial}{\partial \gamma_\ell} \mathbb{E} \left[ (\mathcal{E}(f) \circ \mathcal{E}(\delta \times \alpha)) \cdot \left( \int_{\mathbb{R}} \alpha_k dB_h^{(H)} \right) \cdot (\mathcal{E}(g) \circ \mathcal{E}(\gamma \times \beta)) \right]_{\delta, \gamma = 0}
\]

\[
= \mathbb{E} \left[ (\mathcal{E}(f) \circ \int_{\mathbb{R}} \alpha_k dB_h^{(H)}) \cdot (\mathcal{E}(g) \circ \int_{\mathbb{R}} \beta_\ell dB_\ell^{(H)}) \right].
\]

(2.20)

On the other hand, if we differentiate (2.19) we get

\[
\frac{\partial^2}{\partial \delta_k \partial \gamma_\ell} \left[ \exp(f + \delta \times \alpha, g + \gamma \times \beta)\phi \right]_{\delta, \gamma = 0}
\]

\[
= \frac{\partial}{\partial \gamma_\ell} \left[ \exp(f + \delta \times \alpha, g + \gamma \times \beta)\phi \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha_k (s)(g_k + \gamma_k \beta_k)(t) \phi(t, s) ds dt \right]_{\delta, \gamma = 0}
\]

\[
= \exp(f, g)\phi \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha_k (s) g_k (t) \phi(t, s) ds dt \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} \beta_\ell (s) f_\ell (t) \phi(t, s) ds dt
\]

\[
= \exp(f, g)\phi \cdot (\alpha_k, g_k)\phi \cdot (\beta_\ell, f_\ell)\phi
\]

\[
= \mathbb{E}[\mathcal{E}(f) \cdot (\beta_\ell, f_\ell)\phi_k \cdot \mathcal{E}(g) \cdot (\alpha_k, g_k)\phi_k]
\]

\[
= \mathbb{E} [D_{\ell, \phi(\alpha_k)} \mathcal{E}(f) \cdot D_{k, \phi(\alpha_k)} \mathcal{E}(g)].
\]

(2.21)

This proves (2.17) in this case.

**Case 2** \( k = \ell \).

In this case, if we differentiate (2.18) we get

\[
\frac{\partial^2}{\partial \delta_k \partial \gamma_k} \mathbb{E}[(\mathcal{E}(f) \circ \mathcal{E}(\delta \times \alpha)) \cdot (\mathcal{E}(g) \circ \mathcal{E}(\gamma \times \beta))]_{\delta, \gamma = 0}
\]

\[
= \frac{\partial}{\partial \gamma_k} \mathbb{E} \left[ (\mathcal{E}(f) \circ \mathcal{E}(\delta \times \alpha)) \cdot \left( \int_{\mathbb{R}} \alpha_k dB_h^{(H)} \right) \cdot (\mathcal{E}(g) \circ \mathcal{E}(\gamma \times \beta)) \right]_{\delta, \gamma = 0}
\]

\[
= \mathbb{E} \left[ (\mathcal{E}(f) \circ \int_{\mathbb{R}} \alpha_k dB_h^{(H)}) \cdot (\mathcal{E}(g) \circ \int_{\mathbb{R}} \beta_k dB_k^{(H)}) \right].
\]

(2.22)
On the other hand, if we differentiate (2.19) we get
\[
\frac{\partial^2}{\partial \delta_k \partial \gamma_k} \left[ \exp(f + \delta \times \alpha, g + \gamma \times \beta) \phi \right]_{\delta = \gamma = 0} \\
= \frac{\partial}{\partial \gamma_k} \left[ \exp(f + \delta \times \alpha, g + \gamma \times \beta) \phi \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha_k(s) (g_k + \gamma_k \beta_k)(t) \phi_k(s, t) ds \ dt \right]_{\delta = \gamma = 0} \\
= \exp(f, g) \cdot \left[ (\alpha_k, g_k) \phi \cdot (\beta_k, f_k \phi_k + \int \int_{\mathbb{R}} \alpha_k(s) \beta_k(t) \phi_k(s, t) ds \ dt \right] \\
= \mathbb{E} \left[ D_k, \phi(\beta_k) \mathcal{E}(f) \cdot D_k, \phi(\alpha_k) \mathcal{E}(g) + \mathcal{E}(f) \mathcal{E}(g)(\alpha_k, \beta_k) \phi_k \right]. \tag{2.23}
\]
This proves (2.17) also for Case 2 and the proof of Lemma 2.5 is complete. \hfill \Box

We are now ready to prove Theorem 2.1:

PROOF. We may consider \( \int_{\mathbb{R}} f_k(t) dB_k(t) \) as the limit of sums of the form
\[
\sum_{i=1}^{N} f_k(t_i) \circ (B_k^{(H)}(t_{i+1}) - B_k^{(H)}(t_i))
\]
when \( \Delta t_i = t_{i+1} - t_i \to 0 \), \( t_1 < t_2 < \cdots < t_N \), \( N = 2, 3, \ldots \) Hence \( \mathbb{E} \left[ \left( \int_{\mathbb{R}} f dB^H \right)^2 \right] = \)
\[
\mathbb{E} \left[ \left( \sum_{k=1}^{m} \int_{\mathbb{R}} f_k dB_k^{(H)} \right)^2 \right]
\]
is the limit of sums of the form
\[
\sum_{i,j,k,l} \mathbb{E} \left[ (f_k(t_i) \circ (B_k^{(H)}(t_{i+1}) - B_k^{(H)}(t_i))) \cdot (f_l(t_j) \circ (B_l^{(H)}(t_{j+1}) - B_l^{(H)}(t_j))) \right],
\]
which by Lemma 2.5 is equal to
\[
\sum_{i,j,k,l} \mathbb{E} \left[ \left( \int_{t_i}^{t_{i+1}} D_{i,l,t} f_k(t_i) dt \right) \cdot \left( \int_{t_j}^{t_{j+1}} D_{j,l,t} f_l(t_j) dt \right) + \delta_{k,l} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} f_k(t_i) f_k(t_j) \phi_k(s, t) ds dt \right].
\]
When \( \Delta t_i \to 0 \) this converges to
\[
\mathbb{E} \left[ \sum_{k,l=1}^{m} \left( \int_{\mathbb{R}} D_{i,l,t} f_k(t) dt \right) \cdot \left( \int_{\mathbb{R}} D_{j,l,t} f_l(t) dt \right) + \sum_{k=1}^{m} \int_{\mathbb{R}} f_k(t) \phi_k(s, t) ds dt \right]. \tag{2.24}
\]
This proves (2.9) when \( f = g \). By polarization the proof of Theorem 2.1 is complete. \hfill \Box

Using Theorem 2.1 we can now proceed as in the 1-dimensional case ([6, Theorem 4.3]), with appropriate modifications, and obtain a fractional multi-dimensional Itô formula. We omit the proof.

**Theorem 2.6 (The fractional multi-dimensional Itô formula)**

Let \( X(t) = (X_1(t), \ldots, X_n(t)) \), with
\[
dX_i(t) = \sum_{j=1}^{m} \sigma_{ij}(t, \omega) dB_j^{(H)}(t); \\
\text{where } \sigma_i = (\sigma_{i1}, \ldots, \sigma_{im}) \in \mathcal{L}^{1, 2}_{\phi}(m); \quad 1 \leq i \leq n. \tag{2.25}
\]
Suppose that for all \( j = 1, \ldots, m \) there exists \( \theta_j > 1 - H_j \) such that
\[
\sup_i \mathbb{E}[(\sigma_{ij}(u) - \sigma_{ij}(v))^2] \leq C |u - v|^\theta_j \quad \text{if} \quad |u - v| < \delta
\]  
(2.26)

where \( \delta > 0 \) is a constant. Moreover, suppose that
\[
\lim_{\substack{0 \leq u, v \leq t \\ k, \ell \to 0}} \{ \sup_{i, j, k} \mathbb{E}[(D^\phi_{k, u} \{\sigma_{ij}(u) - \sigma_{ij}(v)\})^2] \} = 0.
\]  
(2.27)

Let \( f \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n) \) with bounded second order derivatives with respect to \( x \). Then, for \( t > 0 \),
\[
f(t, X(t)) = f(0, X(0)) + \int_0^t \frac{\partial f}{\partial s}(s, X(s))ds + \int_0^t \sum_{i=1}^n \frac{\partial f}{\partial x_i}(s, X(s))dX_i(s) \\
+ \int_0^t \left\{ \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X(s)) \sum_{k=1}^m \sigma_{ik}(s)D^\phi_{k, s}(X_j(s)) \right\}ds \\
= f(0, X(0)) + \int_0^t \frac{\partial f}{\partial s}(s, X(s))ds + \sum_{j=1}^m \int_0^t \left[ \sum_{i=1}^n \frac{\partial f}{\partial x_i}(s, X(s))\sigma_{ij}(s, \omega) \right]dB^{(H)}_j(s) \\
+ \int_0^t \text{Tr} [\Lambda^T(s)f_{xx}(s, X(s))]ds.
\]  
(2.28)

Here \( \Lambda = [\Lambda_{ij}] \in \mathbb{R}^{n \times m} \) with
\[
\Lambda_{ij}(s) = \sum_{k=1}^m \sigma_{ik}D^\phi_{k, s}(X_j(s)); \quad 1 \leq i \leq n, \quad 1 \leq j \leq m,
\]  
(2.30)

\[
f_{xx} = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{1 \leq i, j \leq n} \in \mathbb{R}^{n \times n}
\]  
(2.31)

and \((\cdot)^T\) denotes matrix transposed, \(\text{Tr}[\cdot]\) denotes matrix trace.

If we combine Theorem 2.6 with Theorem 2.1 we get the following result, which also may be regarded as a fractional Itô isometry:

**Theorem 2.7 (Fractional Itô isometry II)**

Suppose \( f = (f_1, \ldots, f_m) \in \mathcal{L}^{1,2}_\phi(m) \). Then, for \( T > 0 \),
\[
\mathbb{E}\left[ \left( \int_0^T D^\phi_{t, \ell} f_\ell(t) dt \right) \cdot \left( \int_0^T D^\phi_{t, \ell} f_\ell(t) dt \right) \right] \\
= \mathbb{E}\left[ \int_0^T \left\{ f_\ell(t) \int_0^t D^\phi_{t, \ell} f_\ell(s) dB^{(H)}_\ell(s) + f_\ell(t) \int_0^T D^\phi_{t, \ell} f_\ell(s) dB^{(H)}_\ell(s) \right\} dt \right]
\]  
(2.32)
PROOF. By the Itô formula (Theorem 2.6) we have
\[
\mathbb{E}\left[ \left( \int_0^T f_k dB^{(H)}_k \right) \cdot \left( \int_0^T f_l dB^{(H)}_l \right) \right] \\
= \mathbb{E}\left[ \int_0^T \left\{ f_k(t) D_{k,t}^\phi \left( \int_0^t f_l(s) dB^{(H)}_l(s) \right) + f_k(t) \right\} dt \right] \\
= \mathbb{E}\left[ \int_0^T \left\{ f_k(t) \int_0^t D_{k,t}^\phi f_l(s) dB^{(H)}_l(s) + f_k(t) \int_0^t D_{l,t}^\phi f_k(s) dB^{(H)}_k(s) \right\} dt \right] \\
+ \delta_k \mathbb{E}\left[ \int_0^T \int_0^t \left\{ f_k(t) f_k(s) + f_k(t) f_k(s) \right\} \phi_k(s, t) ds dt \right],
\] (2.33)
where we have used that, for \( u > 0 \),
\[
D_{k,t}^\phi \left( \int_0^u f_l(s) dB^{(H)}_l(s) \right) = \int_0^u D_{k,t}^\phi f_l(s) dB^{(H)}_l(s) + \delta_k \int_0^u f_k(s) \phi_k(t, s) ds.
\] (2.34)
(See [6, Theorem 4.2].)

On the other hand, the Itô isometry (Theorem 2.1) gives that
\[
\mathbb{E}\left[ \left( \int_0^T f_k dB^{(H)}_k \right) \cdot \left( \int_0^T f_l dB^{(H)}_l \right) \right] \\
= \mathbb{E}\left[ \left( \int_0^T D_{l,t}^\phi f_k(t) dt \right) \cdot \left( \int_0^T D_{k,t}^\phi f_l(t) dt \right) + \delta_k \left| f_k \right|^2_{\phi_k} \right].
\] (2.35)
Comparing (2.33) and (2.35) we get Theorem 2.7.

We end this section by proving a fractional integration by parts formula. First we recall

**Theorem 2.8 (Fractional Girsanov formula)**

Suppose \( \gamma = (\gamma_1, \ldots, \gamma_m) \in (L^2(\mathbb{R}))^m \) and \( \hat{\gamma} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_m) \in L^2_\phi \) are related by
\[
\gamma_k(t) = \int_\mathbb{R} \hat{\gamma}_k(s) \phi_k(s, t) ds; \quad t \in \mathbb{R}, \quad k = 1, \ldots, m.
\] (2.36)

Let \( G \in L^2(\mu) \). Then
\[
\mathbb{E}[G(\omega + \gamma)] = \mathbb{E}[G(\omega) \exp^{\phi}(\langle \omega, \hat{\gamma} \rangle)] = \mathbb{E}\left[G(\omega) \mathcal{E} \left( \int_\mathbb{R} \hat{\gamma} dB^{(H)} \right) \right].
\] (2.37)

For a proof in the 1-dimensional case see e.g. [9, Theorem 3.16]. The proof in the multi-dimensional case is similar.

If \( F \in L^2(\mu) \) and \( \gamma = (\gamma_1, \ldots, \gamma_m) \in (L^2(\mathbb{R}))^m \) the directional derivative of \( F \) in the direction \( \gamma \) is defined by
\[
D_\gamma F(\omega) = \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon \gamma) - F(\omega)}{\varepsilon},
\] (2.38)
provided the limit exists in \( L^2(\mu) \). We say that \( F \) is differentiable if there exists a process \( D_t F(\omega) = (D_{1,t} F(\omega), \ldots, D_{m,t} F(\omega)) \) such that \( D_{k,t} F(\omega) \in L^2(d\mu \times dt) \) for all \( k = 1, \ldots, m \) and
\[
D_\gamma F(\omega) = \int_\mathbb{R} D_t F(\omega) \cdot \gamma(t) dt \quad \text{for all } \gamma \in (L^2(\mathbb{R}))^m.
\] (2.39)
Theorem 2.9 (Fractional integration by parts I)
Let $F, G \in L^2(\mu)$, $\gamma \in (L^2(\mathbb{R}))^m$ and assume that the directional derivatives $D_\gamma F$, $D_\gamma G$ exist. Then

$$
\mathbb{E}[D_\gamma F \cdot G] = \mathbb{E}
\left[ F \cdot G \cdot \int_\mathbb{R} \gamma dB^{(H)} \right] - \mathbb{E}[F \cdot D_\gamma G].
$$

(2.40)

**Proof.** By Theorem 2.8 we have, for all $\varepsilon > 0$,

$$
\mathbb{E}[F(\omega + \varepsilon \gamma)G(\omega)] = \mathbb{E}[F(\omega)G(\omega - \varepsilon \gamma)\exp(\varepsilon(\omega, \gamma))].
$$

Hence

$$
\mathbb{E}[D_\gamma F \cdot G] = \mathbb{E}\left[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{F(\omega + \varepsilon \gamma) - F(\omega)\} G(\omega) \right]
$$

$$
= \mathbb{E}\left[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{F(\omega)[G(\omega - \varepsilon \gamma) \exp(\varepsilon(\omega, \gamma))] - G(\omega)\} \right]
$$

$$
= \mathbb{E}\left[ F(\omega) \frac{d}{d\varepsilon} \left\{ G(\omega - \varepsilon \gamma) \exp\left(\varepsilon \int_\mathbb{R} \gamma dB^{(H)} - \frac{1}{2} \varepsilon^2 |\gamma|^2_{\phi}\right)\right\}_{\varepsilon = 0} \right]
$$

$$
= \mathbb{E}\left[ F(\omega) G(\omega) \int_\mathbb{R} \gamma dB^{(H)} \right] - \mathbb{E}[F(\omega) D_\gamma G(\omega)].
$$

\[\Box\]

We now apply the above to the fractional gradient

$$
D^\phi_t F = \int_\mathbb{R} D_s F \cdot \phi(s, t) ds = \sum_{k=1}^m \int_\mathbb{R} D_{k,s} F \cdot \phi_k(s, t) ds = D^\phi_0 F(\omega)
$$

(2.41)

Theorem 2.10 (Fractional integration by parts II)
Suppose $F, G \in L^2(\mu)$ are differentiable, with fractional gradients $D^\phi_t F$, $D^\phi_t G$. Then for each $t \in \mathbb{R}$, $k \in \{1, \ldots, m\}$ we have

$$
\mathbb{E}[D^\phi_{k,t} F \cdot G] = \mathbb{E}[F \cdot G \cdot B^{(H)}_k(t)] - \mathbb{E}[F \cdot D^\phi_{k,t} G].
$$

(2.42)

**Proof.** Choose a sequence $\tilde{\gamma}^{(j)}_k \in L^2_{\phi_k}$, $j = 1, 2, \ldots$, such that $\lim_{j \to \infty} \tilde{\gamma}^{(j)}_k = \delta_t(\cdot)$ (the point mass at $t$), in the sense that if we define

$$
\tilde{\phi}^{(j)}_k(s) = \int_\mathbb{R} \tilde{\gamma}^{(j)}_k \phi_k(s, r) dr
$$

then $\tilde{\phi}^{(j)}_k(\cdot) \to \phi_k(\cdot, t)$ in $L^2(\mathbb{R})$. Then by Theorem 2.9

$$
\mathbb{E}[D^\phi_{k,t} F \cdot G] = \mathbb{E}\left[ \lim_{j \to \infty} D_{\tilde{\phi}^{(j)}_k} F \cdot G \right]
$$

$$
= \lim_{j \to \infty} \mathbb{E}\left[ F \cdot G \cdot \int_\mathbb{R} \tilde{\gamma}^{(j)} \cdot dB^{(H)} \right] - \mathbb{E}[F \cdot D_{\tilde{\phi}^{(j)}_k} G]
$$

$$
= \mathbb{E}[F \cdot G \cdot B^{(H)}_k(t)] - \mathbb{E}[F \cdot D_{k,t} G].
$$

\[\Box\]
3 Application to minimal variance hedging

Consider the multidimensional version of the fractional mathematical market model introduced by [9] and by [7], consisting of $n+1$ independent fractional Brownian motions $B_1^{(H)}(t), \ldots, B_m^{(H)}(t)$ with Hurst coefficients $H_1, \ldots, H_m$ respectively ($\frac{1}{2} < H_i < 1$), as follows:

(bond price) \[ dS_0(t) = r(t, \omega)dt ; \quad S_0(0) = s_0, \quad 0 \leq t \leq T \] (3.1)

(stock prices) \[ dS_i(t) = \mu_i(t, \omega)dt + \sum_{j=1}^m \sigma_{ij}(t, \omega)dB_j^{(H)}(t) ; \quad S_i(0) = s_i, \] \[ i = 1, \ldots, n, \quad 0 \leq t \leq T. \] (3.2)

Here $r(t, \omega), \mu_i(t, \omega)$ and $\sigma_{ij}(t, \omega)$ are $\mathcal{F}_t^{(H)}$-adapted processes satisfying reasonable growth conditions. We refer to [7], [9], [14] and [21] for a general discussion of such markets.

We say that $g = (g_1, \ldots, g_n)$ is an admissible portfolio if $g(t)$ is $\mathcal{F}_t^{(H)}$-adapted, $g \sigma \in \mathcal{L}_o^{1,2}(m)$ and $\mathbb{E} \left[ \int_0^T \sum_{i=1}^n |g_i(t)\mu_i(t)|dt \right] < \infty$. Here we denote by $\sigma$ the volatility matrix $[\sigma]_{i,j} (\cdot) = \sigma_{ij}(\cdot)$. Suppose we are only allowed to trade in some, say $k$, of the securities $S_0, \ldots, S_n$. Let $K$ be the set of $i \in \{1, \ldots, n\}$ such that trading in $S_i$ is allowed. Then, according to our model, the wealth hedged by an initial value $z \in \mathbb{R}$ and an admissible portfolio $g(t) = (g_i(t, \omega))_{i \in K} \in \mathbb{R}^k$ up to time $t$ is

\[ V(t) = V_z^g(t) = z + \sum_{i \in K} \int_0^t g_i(u) dS_i(u) ; \quad 0 \leq t \leq T. \] (3.3)

Now let $F(\omega)$ be a $T$-claim, i.e. an $\mathcal{F}_T^{(H)}$-measurable random variable in $L^2(\mu)$.

The minimal variance hedging problem is to find a $z^* \in \mathbb{R}$ and an admissible portfolio $g^*$ such that

\[ \mathbb{E}[(F - V_z^{g^*}(T))^2] = \inf_{z \in \mathbb{R}} \mathbb{E}[(F - V_z^g(T))^2]. \] (3.4)

This is a difficult problem even in the classical Brownian motion setting. See e.g. [8], [17] and the references therein. For a recent general martingale approach see [5]. For fractional Brownian motion markets a special case is solved in [1] by using optimal control theory.

Here we will discuss the two-dimensional case only, and we will simply assume that

\[ dS_0(t) = 0, \quad dS_1(t) = dB_1^{(H)}(t) \quad \text{and} \quad dS_2(t) = dB_2^{(H)}(t). \]

Assume that only trading in $S_0$ and $S_1$ is allowed. Then the problem is to minimize

\[ J(z, g_1) = \mathbb{E} \left[ \left( F - \left( z + \int_0^T g_1 dS_1 \right) \right)^2 \right]. \] (3.5)

over all $z \in \mathbb{R}$ and all admissible portfolios $g_1$.

By the fractional Clark-Haussmann-Ocone formula ([9, Theorem 4.15]) we can write

\[ F(\omega) = \mathbb{E}[F] + \int_0^T f_1(t) dB_1^{(H)}(t) + \int_0^T f_2(t) dB_2^{(H)}(t) \] (3.6)
where
\[ f_i(t) = \mathbb{E}[D_{i,t} F \mid \mathcal{F}_t^{(H)}] ; \quad i = 1, 2. \]
Substituting this into (3.5) we get, by (1.8),
\[
J(z, g_1) = \mathbb{E} \left[ (\mathbb{E}[F] - z + \int_0^T (f_1 - g_1) dB_1^{(H)} + \int_0^T f_2 dB_2^{(H)})^2 \right] \\
= (\mathbb{E}[F] - z)^2 + \mathbb{E} \left[ \left( \int_0^T (f_1 - g_1) dB_1^{(H)} + \int_0^T f_2 dB_2^{(H)} \right)^2 \right].
\]
Hence it is optimal to choose \( z = z^* := \mathbb{E}[F]. \) The remaining problem is therefore to minimize
\[
J_0(g_1) = \mathbb{E} \left[ \left( \int_0^T (f_1 - g_1) dB_1^{(H)} + \int_0^T f_2 dB_2^{(H)} \right)^2 \right].
\]
From now on we assume that \( f_1 \in L_{\phi_1}^{1,2} \) for \( i = 1, 2. \) By a Hilbert space argument on \( L^2(\mu) \) we see that \( g_1^* \) minimizes (3.8) if and only if
\[
\mathbb{E} \left[ \left( \int_0^T (f_1 - g_1) dB_1^{(H)} + \int_0^T f_2 dB_2^{(H)} \right) \cdot \left( \int_0^T \gamma dB_1^{(H)} \right) \right] = 0 \\
\text{for all adapted } \gamma \in L_{\phi_1}^{1,2}.
\]
By Theorem 2.1 (3.9) is equivalent to
\[
\mathbb{E} \left[ \int_0^T \int_0^T (f_1(t) - g_1(t)) \gamma(s) \phi_1(s, t) ds dt + \left( \int_0^T D_{1,t}^\phi(f_1(t) - g_1(t)) dt \right) \left( \int_0^T D_{1,t}^\phi \gamma(t) dt \right) \\
+ \left( \int_0^T D_{2,t}^\phi f_2(t) dt \right) \cdot \left( \int_0^T D_{2,t}^\phi \gamma(t) dt \right) \right] \\
= 0 \quad \text{for all adapted } \gamma \in L_{\phi}^{1,2}.
\]
From this we immediately deduce

**Proposition 3.1** The portfolio
\[
g_1(t) = g_1^*(t) := f_1(t)
\]
minimizes (3.8) if and only if
\[
\int_0^T D_{1,t}^\phi f_2(t) dt = 0 \quad \text{a.s.}
\]
This result is surprising in view of the corresponding situation for classical Brownian motion, when it is always optimal to choose \( g_1(t) = g_1^*(t) = f_1(t). \)

We also get
Proposition 3.2 Suppose $g_1(t)$ minimizes (3.8). Then

$$
\mathbb{E} \left[ \int_0^T (f_1(t) - g_1(t)) dt \right] = 0. \tag{3.12}
$$

Proof. This follows by choosing $\gamma(t)$ deterministic in (3.10). \qed

Now assume that $D_{1,t}^\phi(f_1(t))$ and $D_{1,t}^\phi(g_1(t))$ are differentiable with respect to $D_{1,s}^\phi$ and
that $D_{1,t}^\phi f_2(t)$ is differentiable with respect to $D_{2,s}^\phi$ for all $s \in [0,T]$. Then we can use integration by parts (Theorem 2.10) to rewrite equation (3.10) as follows:

$$
\mathbb{E} \left[ \int_0^T \int_0^T \{ (f_1(t) - g_1(t)) \gamma(s) \phi_1(s,t) + D_{1,t}^\phi(f_1(t) - g_1(t)) \cdot D_{1,s}^\phi \gamma(s) \\
+ D_{1,t}^\phi f_2(t) \cdot D_{2,s}^\phi \gamma(s) \} ds \ dt \right]
= \int_0^T \int_0^T \mathbb{E} \left[ (f_1(t) - g_1(t)) \gamma(s) \phi_1(s,t) \gamma(s) + D_{1,t}^\phi(f_1(t) - g_1(t)) \gamma(s) B_1^{(H)}(s) \\
- D_{1,s}^\phi D_{1,t}^\phi(f_1(t) - g_1(t)) \gamma(s) + D_{1,t}^\phi f_2(t) \gamma(s) B_2^{(H)}(s) \\
- D_{2,s}^\phi D_{1,t}^\phi f_2(t) \gamma(s) \right] ds \ dt
= \mathbb{E} \left[ \int_0^T K(s) \gamma(s) ds \right] = 0, \tag{3.13}
$$

where

$$
K(s) = \int_0^T G(s,t) dt, \tag{3.14}
$$

with

$$
G(s,t) = (f_1(t) - g_1(t)) \phi_1(s,t) + D_{1,t}^\phi(f_1(t) - g_1(t)) B_1^{(H)}(s) \\
- D_{1,s}^\phi D_{1,t}^\phi(f_1(t) - g_1(t)) + D_{1,t}^\phi f_2(t) B_2^{(H)}(s) - D_{2,s}^\phi D_{1,t}^\phi f_2(t). \tag{3.15}
$$

Since $\gamma(s)$ is $\mathcal{F}_s^{(H)}$-measurable we get from (3.13) that

$$
0 = \int_0^T \mathbb{E}[K(s) \gamma(s)] ds = \int_0^T \mathbb{E} \left[ \mathbb{E}[K(s) \gamma(s) | \mathcal{F}_s^{(H)}] \right] ds
= \int_0^T \mathbb{E} \left[ \gamma(s) \mathbb{E}[K(s) | \mathcal{F}_s^{(H)}] \right] ds = \mathbb{E} \left[ \int_0^T \mathbb{E}[K(s) | \mathcal{F}_s^{(H)}] \gamma(s) ds \right]. \tag{3.16}
$$

Since this holds for all adapted $\gamma \in \mathcal{L}_{\phi}^{1,2}$ we conclude that

$$
\mathbb{E}[K(s) | \mathcal{F}_s^{(H)}] = 0 \quad \text{for a.a. } (s,\omega). \tag{3.17}
$$
or, using (3.14),

\[
\int_0^T \{ \mathbb{E}_s[f_1(t) - g_1(t)]\phi_1(s, t) + \mathbb{E}_s[D_{1,t}^\phi(f_1(t) - g_1(t))]B_1^{(H)}(s) \\
- \mathbb{E}_s[D_{1,s}^\phi D_{1,t}^\phi(f_1(t) - g_1(t))] + \mathbb{E}_s[D_{1,s}^\phi f_2(t)]B_2^{(H)}(s) - \mathbb{E}_s[D_{2,s}^\phi D_{1,t}^\phi f_2(t)] \} \, dt = 0 ,
\]

(3.18)

where we have used the shorthand notation

\[
\mathbb{E}_s[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_s^{(H)}].
\]

We have proved:

**Theorem 3.3**  Suppose the claim \( F \) represented by (3.6) is such that \( D_{1,s}^\phi D_{1,t}^\phi f_1(t) \) and \( D_{2,s}^\phi D_{1,t}^\phi f_2(t) \) exist for all \( s, t \in [0, T] \). Suppose \( \hat{g}_1(t) \) is an adapted process in \( \mathcal{L}_\phi^{1,2} \) such that \( D_{1,t}^\phi \hat{g}_1(t) \) and \( D_{1,s}^\phi D_{1,t}^\phi \hat{g}_1(t) \) exist for all \( s, t \in [0, T] \). Then the following are equivalent:

(i) \( \hat{g}_1(t) \) is a minimal variance hedging portfolio for \( F \), i.e. \( \hat{g}_1(t) \) minimizes (3.8) over all adapted \( g_1(t) \in \mathcal{L}_\phi^{1,2} \).

(ii) \( g_1(t) = \hat{g}_1(t) \) satisfies equation (3.18).

Note that the same method also applies if we assume a fractional exponential dynamics for the asset prices, which represents a more realistic financial model. To illustrate this result we consider the following special case:

**Example 3.4**  Suppose \( f_1(t) = 0 \) and

\[
D_{1,t}^\phi f_2(t) = h(t) , \quad \text{a deterministic function}.
\]

We seek a minimal variance hedging portfolio \( g_1^*(t) \) for the claim

\[
F(\omega) = \int_0^T f_2(t) dB_2^{(H)}(t).
\]

(3.20)

In this case (3.18) gets the form

\[
\int_0^T \{ -\mathbb{E}_s[g_1(t)]\phi_1(s, t) - \mathbb{E}_s[D_{1,t}^\phi g_1(t)]B_1^{(H)}(s) + \mathbb{E}_s[D_{1,s}^\phi D_{1,t}^\phi g_1(t)] \\
+ h(t)B_2^{(H)}(s) \} \, dt = 0 \quad \text{for a.a. } (s, \omega).
\]

(3.21)

Let us try to choose \( g_1(t) \) such that

\[
D_{1,t}^\phi g_1(t) = 0.
\]

(3.22)

Then (3.19) reduces to

\[
\int_0^T \mathbb{E}_s[g_1(t)]\phi_1(s, t) \, dt = B_2^{(H)}(s) \int_0^T h(t) \, dt
\]

(3.23)
or, since $g_1$ is adapted,
\[
\int_0^s g_1(t)\phi_1(s,t)dt + \int_s^T E_s[g_1(t)]\phi_1(s,t)dt = B_2^{(H)}(s) \int_0^T h(t)dt, \quad s \in [0,T].
\] (3.24)
In particular, if we choose $s = T$ we get the equation
\[
\int_0^T g_1(t)\phi_1(T,t)dt = B_2^{(H)}(T) \int_0^T h(t)dt,
\] (3.25)
which clearly has no adapted solution $g_1(t)$. (However, it obviously has a non-adapted solution.) Therefore an optimal portfolio $g_1(t) = g_{1*}(t)$ for the claim (3.20), if it exists, cannot satisfy (3.22).

References


