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On smooth surfaces in $\mathbb{P}^4$ with a family of plane curves

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Abstract

Smooth surfaces $S \subset \mathbb{P}^4$ containing a family of plane curves not forming a fibration are studied. The result, under two extra assumptions, is that S is either the (projected) Veronese surface, the rational normal scroll or the quintic elliptic scroll or S is contained in a quadric cone of rank 3 or 4 and linked to a plane by a complete intersection.

Introduction

It is a well known result of C. Segre that the only surfaces in $\mathbb{P}^4$ containing a two dimensional family of plane curves are the Veronese surface, the rational normal scroll and the cones.

In [8] and [3] A. Lanteri and A. Aure studied scrolls in $\mathbb{P}^4$, i.e. smooth surfaces with a family of lines. There are exactly two scrolls, namely the rational normal scroll and the quintic elliptic scroll.

Afterwards, P. Ellia and G. Sacchiero showed in [5] that if S is a smooth surface in $\mathbb{P}^4$ ruled in conics, then S is either the Del Pezzo surface of degree 4, the Castelnuovo surface of degree 5 or an elliptic conic bundle of degree 8 discovered by H. Abo, W. Decker and N. Sasakura in [1]. The same result was also obtained in [4] by R. Braun and K. Ranestad in a more geometric way.

Finally, K. Ranestad proved in [9] that if S is a fibration by plane curves of degree greater than two, then S is bielliptic or abelian of degree 10 or S is contained in a quadric cone of rank 4.

In this paper we deal with smooth surfaces $S \subset \mathbb{P}^4$ having a family of plane curves not forming a fibration, i.e. such that any two curves of the family intersect. Our result, under two extra assumptions, is that S is either the quintic elliptic scroll, the (projected) Veronese surface or the rational normal scroll or S is contained in a quadric cone of rank 3 or 4 and linked to a plane by a complete intersection.

We also conjecture that if through the general point of S there pass more than two plane curves, then S is the Veronese surface or the rational normal scroll.

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1 Preliminaries

We will work over $\mathbb{C}$ the field of complex numbers.

We denote by S an irreducible nondegenerate smooth surface in $\mathbb{P}^4$ equiped with an algebraic family of plane curves. A curve of the family will be denoted by C.

By $G(k,n)$ we mean the Grassmann variety of k-planes in $\mathbb{P}^n$.

We denote by $\Sigma \subset G(2,4)$ the curve in the Grassmannian parametrising the plane curves C. We call $V_\Sigma \subset \mathbb{P}^4$ the hypersurface covered by the planes of Σ.

Two plane curves are algebraically equivalent, in particular numerically equivalent (see [6] V, Ex. 1.7) so we can define the number C^2 to express the intersection of two plane curves of the family. The case $C^2 = 0$ was studied by the authors mentioned in the introduction, so we will study the case $C^2 \geq 1$.

We will denote the elements of the Grassmannian $G(k,n)$ by small letters and use the corresponding capital letter for the linear subspace in $\mathbb{P}^n$ that they define.

Let $\check{\Sigma} \subset G(1,4)$ be the dual curve of Σ in the dual Grassmannian of lines in $\mathbb{P}^4$. Furthermore, for $W \subset G(k,n)$ we denote by $\check{W} \subset G(n-k-1,n)$ the dual variety.

$S_{\check{\Sigma}} \subset \check{\mathbb{P}}^4$ will denote the surface covered by the lines of $\check{\Sigma}$.

With $\Sigma^{(2)}$ we will denote the second symmetric product of Σ.

For $a_i \geq 0$, we denote by $S(a_0, \ldots, a_n)$ the rational normal scroll of n-planes.

We will use the following well known facts:

1. *Severi's theorem.* The only smooth surface that can be isomorphically projected to $\mathbb{P}^4$ is the Veronese surface.

2. *Zariski's Main Theorem.* Let $f : X \to Y$ be a birational projective morphism of noetherian integral schemes, and assume that Y in normal. Then for every $y \in Y$, $f^{-1}(y)$ is connected.

3. *Castelnuovo's bound.* If $C \subset \mathbb{P}^n$, $n \geq 3$, is a smooth, nondegenerate, irreducible curve of degree d and genus g, then:

$$g \leq m(m-1)(n-1)/2 + me \text{ where } d-1 = m(n-1)+e \text{ and } 0 \leq e \leq n-2.$$

2 Examples

Example 2.1 Consider $\mathbb{P}^2$ and a curve $\Sigma \subset \check{\mathbb{P}}^2$ of degree d. Let $v_2(\mathbb{P}^2)$ be the double Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. This surface can be isomorphically projected to $\mathbb{P}^4$ from a general $P \in \mathbb{P}^5$. We have a curve in $G(2,5)$, which will be also called Σ, corresponding to planes containing a conic $C = v_2(L)$ for $l \in \Sigma$. Then $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ (or its projection to $\mathbb{P}^4$) is a surface with a family of conics corresponding to Σ, $C^2 = 1$ and through a point of $v_2(\mathbb{P}^2)$ there pass d conics of the family.

In particular, when $d = 1$ (corresponding to the case in which $\Sigma \subset \check{\mathbb{P}}^2$ is a pencil) the associated $\Sigma \subset G(2,5)$ is a twisted cubic via Plücker embedding and $V_\Sigma \subset \mathbb{P}^5$ is the rational normal scroll $S(0,1,2)$.

If $d = 2$ (corresponding to the case in which $\Sigma \subset \check{\mathbb{P}}^2$ is the dual of a conic), then note that through a general point $P \in \mathbb{P}^5$ there pass a one dimensional family of $\mathbb{P}^{4's}$ spanned by a couple of planes of $\Sigma \subset G(2,5)$. Hence, when we project $v_2(\mathbb{P}^2)$ from P we get that the general plane of $\Sigma \subset G(2,4)$ is intersected by another plane of Σ.

Example 2.2 Consider $\mathbb{P}^2$ and a curve $\Sigma \subset \check{\mathbb{P}}^2$ of degree d. Fix a point $P \in \mathbb{P}^2$ and consider the complete linear system of conics through P. Its image in $\mathbb{P}^4$ turns out to be an embedding of $Bl_p\mathbb{P}^2$ as the rational normal scroll $S(1,2)$. Lines $l \in \Sigma$ not passing through P go to conics $C = v_2(L)$, while the d lines through P go to degenerate conics, namely a pair of different lines in a plane. Then $S(1,2) \subset \mathbb{P}^4$ is a surface with a family of conics corresponding to Σ, $C^2 = 1$ and through a point of $S(1,2)$ there pass d conics of the family.

When $d = 1$ (corresponding to the case in which $\Sigma \subset \check{\mathbb{P}}^2$ is a pencil), $\Sigma \subset G(2,4)$ is a conic via Plücker embedding and $V_\Sigma \subset \mathbb{P}^4$ is a quadric cone.

If $d = 2$ (corresponding to the case in which $\Sigma \subset \check{\mathbb{P}}^2$ is the dual of a conic), $\Sigma \subset G(2,4)$ is a rational normal quartic via Plücker embedding. In this case,

$S_{\check{\Sigma}} \subset \check{\mathbb{P}}^4$ is the projected $S(2,2)$ with a singular point corresponding to the pair of lines of $\Sigma \subset \check{\mathbb{P}}^2$ through the point we are blowing up.

Example 2.3 Denote by E_5 the quintic elliptic scroll in $\mathbb{P}^4$. Two lines $L_1, L_2 \subset E_5$ span a $\mathbb{P}^3$ and its residual intersection with E_5 is an elliptic curve C of degree 3. In this way we get a one dimensional family of plane curves in E_5 and $C^2 = 1$. Moreover the curve $\Sigma \subset G(2,4)$ parametrising these plane curves is the quintic elliptic curve in $\mathbb{P}^4$ via Plücker embedding (see for instance [4]). Hence, $S_{\check{\Sigma}} \subset \check{\mathbb{P}}^4$ is again E_5.

Example 2.4 Let $V_\Sigma \subset \mathbb{P}^4$ be a quadric cone of rank 4 and $\Sigma \subset G(2,4)$ the corresponding conic parametrising one of the families of planes inside V_Σ. Let $Q \in \mathbb{P}^4$ be the vertex of the cone. Consider $S \subset \mathbb{P}^4$ the smooth surface linked to a plane of V_Σ (of the other family) by a complete intersection with a hypersurface of degree e. Then the planes of Σ contain a curve C of degree e and $C^2 = 1$.

To illustrate this, we will concentrate on the cases $e = 2$ and $e = 3$. Note that when $e = 2$ we get the rational normal scroll $S(1,2)$ of Example 2.2 with $d = 1$.

If $e = 3$ we get the Castelnuovo surface of $\mathbb{P}^4$, i.e. the blow up of $\mathbb{P}^4$ in 8 points $P_0, \ldots, P_7$ embedded by the complete linear system $|4L - 2E_0 - E_1 \ldots - E_7|$. Plane cubics through the 8 points go to cubics of the Castelnuovo surface and they intersect precisely in one point, namely the fixed point of the system of plane cubics through the 8 points.

Example 2.5 Let $V_\Sigma \subset \mathbb{P}^4$ be a quadric cone of rank 3 and $\Sigma \subset G(2,4)$ the corresponding conic parametrising the family of planes inside V_Σ. Let $L \subset \mathbb{P}^4$ be the vertex of the cone. Consider $S \subset \mathbb{P}^4$ the smooth surface linked to a plane of Σ by a complete intersection with a hypersurface of degree e. Then the planes of Σ contain a curve C of degree e and let us see that $C^2 = 1$.

First of all note that $L \subset S$ (Cf. Proposition 4.2). Then $C_i = L + D_i$ where D_i is an effective plane curve of degree $e-1$. Call H a hyperplane section of S containing L. Clearly $H^2 = 2e - 1$ and hence a straightforward computation gives $L^2 = 3 - 2e$ and $C^2 = 1$, having in mind that necessarily $D_i \cdot D_j = 0$.

We analyse the cases $e = 2$ and $e = 3$. When $e = 2$ we are just considering Example 2.2 in the case $d = 1$ and $\Sigma \subset \check{\mathbb{P}}^2$ the pencil of lines through the point we are blowing up.

If $e = 3$ we get again the Castelnuovo surface of $\mathbb{P}^4$ in the particular case when $P_1, \ldots, P_7$ lay on a conic. This conic goes to the line L which is the vertex of the cone V_Σ.

Remark 2.6 *In Example 2.5 we get two different families of plane curves in S, namely C'^s and D'^s depending on whether L is considered or not. Then $C^2 = 1$ but*

$D^2 = 0$, and hence we also get a fibration of S. We would like to remark that the quadric cone of rank 3 was not considered in [9].

3 The geometry of Σ

Let us start by analysing the curve $\Sigma \subset G(2,4)$.

Case 1. If two general planes of Σ intersect along a line, it is easy to see that either all the planes are contained in a $\mathbb{P}^3$ (in which case S would be degenerate) or all the planes contain a line $L \subset \mathbb{P}^4$ and hence $V_\Sigma \subset \mathbb{P}^4$ is a cone of vertex L. A priori, two cases should be distinguished:

1.1. L is contained in S.

1.2. L is not contained in S.

Case 2. Otherwise two general planes of Σ intersect in a point which is contained in S, so $C^2 = 1$ necessarily. There are two possibilities:

2.1. If two general planes of Σ intersect in a fixed point Q, then $V_\Sigma \subset \mathbb{P}^4$ is a cone of vertex Q.

2.2. If the intersection points move on the surface, then note that S is obtained by considering the union of the intersection points of general planes and taking the closure of this variety.

Again there are two possibilities that should be considered separately:

2.2.1. Through the general point $P \in S$ there pass exactly two planes of Σ.

2.2.2. Through the general point $P \in S$ there pass more than two planes of Σ.

Before stating our result, we would like to point the extra assumptions we need to proceed in Case 2.2:

(1) Through the general point of S there pass exactly two plane curves of S. So we are not considering Case 2.2.2 above.

(2) The general plane of Σ is not intersected by another plane of Σ along a line (neither the infinitely close one).

Let us explain more precisely the above sentence. We say that a plane $\pi \in \Sigma$ is *intersected by the infinitely close one along a line* when the embedded tangent line to Σ at π (via Plücker embedding) is contained in the Grassmannian.

Theorem 3.1 *Let $S \subset \mathbb{P}^4$ be a smooth surface covered by an algebraic family of plane curves C. Suppose that $C^2 \geq 1$ and that condition (1) and (2) hold. Then:*

(i) S is contained in a quadric cone of rank 3 in Case 1, and Case 1.2 is not possible.

(ii) S is contained in a quadric cone of rank 4 (unless S is the Veronese surface) in Case 2.1.

(iii) S is the rational normal scroll $S(1,2)$ or the quintic elliptic scroll E_5 in Case 2.2.

Furthermore, we always get $C^2 = 1$.

Up to now, we are not able to remove this conditions. Of course, the (projected) Veronese surface of Example 2.1 with $d = 2$ satisfies that the general plane of Σ is intersected by some plane along a line, so (2) is not satisfied.

On the other hand, we have Examples 2.1 and 2.2 which satisfy (1) when considering $d \geq 3$. In these cases we actually have a two dimensional family of conics covering the surface. Hence it is reasonable to make the following conjecture.

Conjecture 3.2 *If $S \subset \mathbb{P}^4$ is a smooth surface covered by a one dimensional family of plane curves and through the general point of S there pass more than two plane curves of the family, then S is either the (projected) Veronese surface or the rational normal scroll.*

The proof of Theorem 3.1 will be developed throughout the rest of the paper analysing every case separately.

4 V_Σ is a cone

This section is devoted to Case 1 and Case 2.1 of the previous section in which $V_\Sigma \subset \mathbb{P}^4$ is a cone of vertex a line or a point respectively.

Let us see that something else can be said about the curve $\Sigma \subset G(2,4)$. Indeed, we can prove the following lemma.

Lemma 4.1 *If V_Σ is a cone, then Σ is a rational curve.*

Proof. Let $\Sigma \subset G(2,4)$ be the curve parametrising the planes of the family and $V_\Sigma \subset \mathbb{P}^4$ the hypersurface which is the union of the planes. In the natural incidence variety in $\mathbb{P}^4 \times G(2,4)$ between points and planes, there is a $\mathbb{P}^2$-fibration U_Σ over Σ whose projection into $\mathbb{P}^4$ is V_Σ. Then, we have the following diagram

$$
\begin{array}{ccc}
 & \widetilde{S} \subset U_\Sigma & \\
{}^{p}\swarrow & \swarrow & \searrow \\
S \subset V_\Sigma & & \Sigma
\end{array}
$$

where $U_\Sigma := \{(P, \pi) \mid P \in \pi\} \subset \mathbb{P}^4 \times \Sigma$ and $\widetilde{S}$ is the strict transform of S in U_Σ.

We observe that $\widetilde{S}$ is smooth, since the map $p : \widetilde{S} \longrightarrow S$ is a birational morphism and S is smooth. We proceed then with a case-by-case analysis.

Case 2.1. Since all the planes of V_Σ meet in one fixed point Q, the strict transform of Q in $\widetilde{S}$ is a connected curve γ (Zariski's Main Theorem) such that one of its components meets all the planes of U_Σ. Since γ is mapped onto the point Q by the map p, we deduce that the above component of γ is a (-1)-curve which dominates Σ. Then Σ is rational.

Case 1.1. Consider the strict transform by p of $L \subset S$. This is a curve γ such that there exists, at least, one irreducible component of γ meeting all the planes of U_Σ. Since the map p is birational, we have that this component is rational and since it dominates Σ, we conclude again that Σ is rational.

Case 1.2. Since $L \not\subset S$, we deduce that $L \cap S$ consists of a finite number of points. Moreover, we have that all the plane curves of the family pass through, at least, a point $Q \subset L$. As in Case 2.1, if we consider the strict transform of Q we obtain a curve γ such that one of its components meets all the planes of U_Σ. Hence, by the same reason, Σ is rational. $\square$

Finally, as a consequence of the above result, we can give a more precise description of the surface $S \subset \mathbb{P}^4$.

Proposition 4.2 *If S is not the Veronese surface of Example 2.1 with $d = 1$, then V_Σ is a quadric cone. Furthermore, S is linked to a plane by the complete intersection of V_Σ and a hypersurface of $\mathbb{P}^4$ and Case 1.2 is not possible.*

Proof. Since S is either linearly normal or the Veronese surface by Severi's Theorem, we deduce from Lemma 4.1 that, except in the Veronese case, $V_\Sigma \subset \mathbb{P}^4$ is a quadric cone of rank 3 or 4. Otherwise, V_Σ is a projected rational scroll of planes and hence S would be projected from a bigger projective space. It is a well known result (see for instance [3], Proposition 1.3.1) that in this case we have the following two possibilities:

(a) S is a complete intersection of V_Σ and another hypersurface.

(b) S is linked to a plane by the quadric cone V_Σ and another hypersurface.

We observe that the case (a) does not occur, since we would have that S should be singular in the points of the vertex of V_Σ. In fact, every embedded tangent space to S is the intersection of the embedded tangent spaces to both hypersurfaces defining S.

Finally, we remark that in Case 1 the vertex L is contained in S, i.e. Case 1.2 is not possible. In fact, attending to (b), S turns out to have odd degree. Then, if $L \not\subset S$, the intersection of S with a hyperplane containing L would be a reducible curve of even degree, which is a contradiction. Moreover we proved in Example 2.5 that $C^2 = 1$ also in this case. $\square$

5 V_Σ is not a cone

We start this section by considering the general case in which the intersection point of two general planes of Σ moves on S. Recall the two extra assumptions we use:

(1) Through the general point of S there pass exactly two plane curves of S.

(2) The general plane of Σ is not intersected by another plane of Σ along a line (neither the infinitely close one).

Consider the map $\alpha : \Sigma^{(2)} \dashrightarrow S \subset \mathbb{P}^4$ which associates to each pair of general planes (π_1, π_2) its intersection point $P = \Pi_1 \cap \Pi_2$.

Lemma 5.1 *The map α is birational and through every point of S there pass at most two plane curves.*

Proof. The map α is clearly birational because of assumption (1). Call $\Gamma(\Sigma) \subset \Sigma^{(2)} \times S$ the closure of the graph of the map α and p_1, p_2 its projections to $\Sigma^{(2)}$ and S respectively. If $P \in S$ is contained in three or more planes, we apply Zariski's Main Theorem to $p_2 : \Gamma(\Sigma) \longrightarrow S$ and we get that all the planes pass through P, which is a contradiction. $\square$

Lemma 5.2 *Σ is smooth.*

Proof. Suppose by contradiction that Σ is singular at x and let $\widetilde{\Sigma} \to \Sigma$ be its normalization. Then $\Sigma^{(2)}$ is singular along the curve $\sigma_x = \{(x,y) \,|\, y \in \Sigma\} \subset \Sigma^{(2)}$ and $\widetilde{\Sigma}^{(2)} \to \Sigma^{(2)}$ is a birational morphism. Now take a point $(x,y) \in \sigma_x$ in which α is actually defined. We claim that $\alpha(x,y)$ is a singular point of S. Indeed, its preimage in $\widetilde{\Sigma}^{(2)}$ is the finite set of points $\{(\tilde{x}_1, y) \ldots (\tilde{x}_n, y)\}$ where $\tilde{x}_i \in \widetilde{\Sigma}$ is the set of points over x.

Therefore S must be singular by Zariski's Main Theorem. But S is smooth, so we have arrived to a contradiction. $\square$

In what follows we will work with the dual curve $\check{\Sigma} \subset G(1,4)$. Two general lines of $\check{\Sigma}$ do not intersect in this case. Moreover $\check{\Sigma}$ is nondegenerate, in the sense that is not contained in a $G(1,3)$. Hence we can consider the rational map $\varphi : \check{\Sigma}^{(2)} \dashrightarrow \check{\mathbb{P}}^4$ which associates to each pair of skew lines (l, l') its linear span $\Lambda = \langle L, L' \rangle$. We define the secant variety $S(\check{\Sigma}) \subset \check{\mathbb{P}}^4$ of $\check{\Sigma}$ to be the closure of the image of this map. Note that $S(\check{\Sigma}) = \check{S}$ and that no $\Lambda \in S(\check{\Sigma})$ contains more than two lines of $\check{\Sigma}$ (properly counted) because of Lemma 5.1.

Thus, we have changed the problem to study smooth curves in $G(1,4)$ without trisecant Λ'^s and, by duality, assumption (2) means that the general line of $\check{\Sigma} \subset G(1,4)$ is not intersected by another line of $\check{\Sigma}$ (neither the infinitely close one). We remark that, in this situation, a plane curve $C \subset \Pi$ of the family is precisely the dual curve of the curve of Λ'^s containing the line $L = \check{\Pi}$.

Proposition 5.3 *Let $\check{\Sigma} \subset G(1,4)$ be a smooth nondegenerate curve as above. Then $\check{\Sigma}$ is either rational of degree 3 or 4 or is elliptic of degree 5.*

Proof. Let $l \in \check{\Sigma}$ be a general line of $\check{\Sigma}$. Consider the projection $\pi_l : \check{\Sigma} \dashrightarrow G(1,2)$ and let $\check{\Sigma}'$ be the image of $\check{\Sigma}$ under this projection. Note that $\check{\Sigma}'$ is smooth because we have no trisecant Λ'^s as was discussed before. By duality, this claim is equivalent to say that a general plane curve C is smooth. Denote d and d' the degrees of $\check{\Sigma}$ and $\check{\Sigma}'$ respectively (d' turns out to be the degree of C). We are assuming (2), so the general $l \in \check{\Sigma}$ is not intersected by another line of $\check{\Sigma}$. Then we prove that $d' = d - 2$.

In fact, d' is the number of lines of $\check{\Sigma}'$ through a point $P \in \mathbb{P}^2$, which corresponds to lines of $\check{\Sigma}$ meeting the plane $\Pi = \langle P, L \rangle \subset \mathbb{P}^4$. This is the Schubert variety $\Omega(\Pi, 4)$, which is singular along the lines of Π with multiplicity 2. To prove that the intersection multiplicity of $\check{\Sigma}$ and $\Omega(\Pi, 4)$ along l is exactly 2, we have to check that the embedded tangent line to $\check{\Sigma}$ at l is not contained in the tangent cone to $\Omega(\Pi, 4)$ (see [7], Corollary 12.4). This is true because by assumption (2) the embedded tangent line to $\check{\Sigma}$ at l is not contained in the Grassmannian.

Then we have $g(\check{\Sigma}) = \frac{(d-3)(d-4)}{2}$. Compairing $g(\check{\Sigma})$ with Castelnuovo bounds for curves in $\mathbb{P}^4$ (if the span of $\check{\Sigma}$ is a $\mathbb{P}^3$ then is necessarily a twisted cubic or is contained in a quadric, in which case $\check{\Sigma}$ is contained in a $G(1,3)$), we get that $d = 3, 4$ or 5. $\square$

To finish the proof of Theorem 3.1, we just have to check that the case $d = 3$ in Proposition 5.3 does not produce a new example. In fact, $S_{\check{\Sigma}} \subset \check{\mathbb{P}}^4$ is necessarily

$S(1,2)$ and hence $\check{S} \subset \check{\mathbb{P}}^4$ is nothing but the set of $\mathbb{P}^{3's}$ containig the exceptional line of the scroll, so $S = \mathbb{P}^2$.

On the other hand, if $\check{\Sigma} \subset G(1,4)$ is either rational of degree 4 or elliptic of degree 5, then $S_{\check{\Sigma}} \subset \check{\mathbb{P}}^4$ is either a projected $S(2,2)$ or E_5 respectively, and in these cases S is either $S(1,2)$ or E_5 as in Examples 2.2 and 2.3.

Remark 5.4 *The projected Veronese surface in Case 2.2.1 corresponds to Example 2.1 with $d = 2$, so condition (2) does not hold.*

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