THE MODULAR ISOMORPHISM PROBLEM

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Abstract. We show that the isomorphism class of a finite $p$-group $G$ is
determined by $\mathbb{F}_p G$, the group algebra of $G$ over the field of $p$ elements,
hence solving the modular isomorphism problem.

1. Introduction

The modular isomorphism problem goes back to the 1950s and asks
whether the isomorphism class of a finite $p$-group $G$ can be determined
by its modular group algebra $\mathbb{F}_p G$ where $\mathbb{F}_p$ is the field of $p$ elements. In
[4] we considered this problem in the context of deformation theory, introducing
generalized matric Massey products, see also [5], where these ideas
were applied to deformations of schemes. Using the classical Massey prod-
uct structure on $H^\ast(G, \mathbb{F}_p)$, we could in [1] prove some partial results, and
in [2] we discussed the limitations of the methods in [1].

This last work was inspired by the work done by Laudal in noncom-
mutative geometry, [6] and [7]. Going back to this framework, and more
specifically, to noncommutative deformation theory, we are now able to give
a positive answer to the modular isomorphism problem. This is done in
Section 4, where we show that the isomorphism class of a finite $p$-group $G$
is in fact determined by its modular group algebra $\mathbb{F}_p G$ (Theorem 4.3).

Quillen showed in [9] that the restricted graded Lie algebra of a finite
$p$-group $G$ is determined by $\mathbb{F}_p G$. The question is how does one get from
the graded object to the corresponding filtered object? The idea is to use
deformation theory. Both $G$ and $\mathbb{F}_p G$ have a unique simple module, $\mathbb{F}_p$.
Also, both the group and the group algebra have an obstruction theory, and
so in both cases we can work with the obstructions for deforming this simple,
trivial module.

In Theorem 2.7, we give a detailed new proof of the main result in [4],
proving that $G$ is, in a sense, the formal moduli of $\mathbb{F}_p$ in the category of

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pro-p groups. The essential new element is the use of the Brauer-Jennings-Zassenhaus series, replacing the lower central p-series used in [4].

Section 3 treats the algebra case. The Generalized Burnside Theorem, see [6], implies that $\mathbb{F}_p G$ is the formal moduli of $\mathbb{F}_p$ in the category of complete local $\mathbb{F}_p$-algebras. We choose to reprove this result as Theorem 3.7, insisting on the analogy with the proof of Theorem 2.7.

The properties of the Brauer-Jennings-Zassenhaus series now imply that the obstructions needed in the group construction can be made to coincide with the obstructions needed in the algebra construction, from which our main result, Theorem 4.3, readily follows.

2. Obstruction theory for pro-p groups

Any finite dimensional vector space over $\mathbb{F}_p$ will be called an elementary $p$-group. If $G$ is a finite $p$-group and $K$ an elementary $p$-group on which $G$ acts trivially, we shall call $K$ an elementary $G$-module.

The cohomology groups $H^n(G, K) := \text{Ext}_{\mathbb{F}_p G}^n(\mathbb{F}_p, K)$ can be computed using the bar resolution and the corresponding cocomplex $(B^*(G, K), \delta)$ defined by

\begin{equation}
B^n(G, K) = \text{Maps}(G^n, K),
\end{equation}

the set of maps from $G^n = G \times \cdots \times G$ to $K$, and with differentials

\begin{equation}
\delta^n: B^n(G, K) \longrightarrow B^{n+1}(G, K)
\end{equation}

given for $\phi \in B^n(G, K)$ by

\[
\delta^n(\phi)(g_1, \ldots, g_{n+1}) = \phi(g_2, \ldots, g_{n+1}) + \sum_{i=1}^n (-1)^i \phi(g_1, \ldots, g_ig_{i+1}, \ldots, g_n) + (-1)^{n+1} \phi(g_1, \ldots, g_n).
\]

In particular, for $\phi \in B^1(G, K)$,

\[
\delta^1(\phi)(g_1, g_2) = \phi(g_2) - \phi(g_1, g_2) + \phi(g_1),
\]

and for $\phi \in B^2(G, K)$,

\begin{equation}
\delta^2(\phi)(g_1, g_2, g_3) = \phi(g_2, g_3) - \phi(g_1, g_2, g_3) + \phi(g_1, g_2g_3) - \phi(g_1, g_2),
\end{equation}

We have $H^n(G, K) \simeq H^n(B^*(G, K), \delta)$.

Our first lemma relates to the following lifting situation. Let

\begin{equation}
D: 1 \longrightarrow K_D \longrightarrow^\iota H_1 \longrightarrow^\kappa H_2 \longrightarrow 1 \quad \phi \downarrow \quad G
\end{equation}
be a diagram where \( \kappa \) is a surjective homomorphism of groups and \( K_D = \ker \kappa \) is a central elementary subgroup of \( H_1 \), i.e. \( \kappa(K_D) \subset Z(H_1) \) and \( K_D \cong \mathbb{F}_p^m \), some \( m \), but written multiplicatively.

**Definition 2.1.** Let \( \mathcal{D} \) be the category of all such diagrams (2.3) for a fixed group \( G \) together with the obvious morphisms of such diagrams. An object \( D \) in \( \mathcal{D} \) is thus a central extension of groups with elementary kernel \( K_D \) together with a homomorphism \( \phi \) from \( G \) to the base group in the extension.

We have the following well-known result.

**Lemma 2.2.** To each diagram \( D \) in \( \mathcal{D} \) one associates a cohomology class \( \text{obs}(D) \in H^2(G, K_D) \) such that \( \text{obs}(D) = 0 \) if and only if there exists a group homomorphism \( \phi' : G \rightarrow H_1 \) with \( \phi = \kappa \circ \phi' \).

**Proof.** Let \( \sigma : H_2 \rightarrow H_1 \) be a set-theoretic section of \( \kappa \). For every pair \( g_1, g_2 \in G \), consider the element

\[
\psi(g_1, g_2) = (\phi(g_1 g_2))^{-1} \sigma(\phi(g_1))^{-1}
\]

in \( H_2 \). Since \( \kappa(\psi(g_1, g_2)) = 1 \), this defines an element \( \psi \in \text{Maps}(G \times G, K_D) \), and using (2.2), we check that \( \delta^2(\psi)(g_1, g_2, g_3) = 1 \).

Put \( \text{obs}(D) = [\psi] \in H^2(G, K_D) \), the cohomology class of the cocycle \( \psi \). We note that a different choice \( \tilde{\sigma} \) for the section \( \sigma \) in (2.4) will give a map \( \tilde{\psi} \) with \( \tilde{\psi} = \delta^1(\tilde{\sigma} \circ \phi - \sigma \circ \phi) \), so the cohomology class \( \text{obs}(D) \) is well-defined.

Assume \( \text{obs}(D) = 0 \). Then \( \psi \in \text{im} \delta^1 \), and so there exists \( \alpha \in \text{Maps}(G, K_D) \) such that \( \delta^1(\alpha) = \psi \). Define \( \phi' : G \rightarrow H_2 \) by

\[
\phi'(g) = \sigma(\phi(g))\alpha(g), \quad g \in G.
\]

Then, since we have a central extension, \( \phi' \) is a group homomorphism, and \( \phi = \kappa \circ \phi' \).

Conversely, suppose there exists a \( \phi' \) such that \( \phi = \kappa \circ \phi' \). We need to show that \( \text{obs}(D) = 0 \) as an element of \( H^2(G, K_D) \). Let \( \gamma : G \rightarrow H_2 \) be defined by \( \gamma(g) = \phi'(g)\sigma(\phi(g))^{-1} \). Then, for \( g \in G \), we have that \( \kappa(\gamma(g)) = 1 \), so \( \gamma \in \text{Maps}(G, K_D) \), and

\[
\psi(g_1, g_2) = \sigma(\phi(g_1 g_2))\sigma(\phi(g_2))^{-1} \sigma(\phi(g_1))^{-1}
\]

\[
= \sigma(\phi(g_1 g_2))\phi'(g_1 g_2)^{-1} \sigma(\phi(g_2))^{-1} \sigma(\phi(g_1))^{-1}
\]

\[
= \gamma(g_1 g_2)^{-1} \gamma(g_2) \gamma(g_1)
\]

since \( \gamma(g_2) \) commutes with \( \phi'(g_1) \). Therefore,

\[
\psi(g_1, g_2) = \gamma(g_2) \gamma(g_1)^{-1} \gamma(g_1),
\]

which proves the lemma. \( \square \)
Consider a morphism $\zeta: D \to D'$ in $\mathcal{D}$. This is the same as a commutative diagram

$$
\begin{array}{c}
D': & 1 & \longrightarrow & K_{D'} & \longrightarrow & H'_1 & \longrightarrow & H'_2 & \longrightarrow & 1 \\
\downarrow \zeta & & \downarrow \eta & & \downarrow \kappa & & \downarrow \theta & & \downarrow \phi \\
D: & 1 & \longrightarrow & K_D & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & 1
\end{array}
$$

The morphism $\zeta$ induces a map making $H^2(G, \_)$ a functor from the category $\mathcal{D}$ to the category of vector spaces over $\mathbb{F}_p$. Moreover, to every diagram $D$ in $\mathcal{D}$, we have shown that there exists a cohomology class $\text{obs}(D) \in H^2(G, K_D)$.

**Corollary 2.3.** The map $\zeta_*: H^2(G, K_D) \longrightarrow H^2(G, K_{D'})$, induced by $\zeta$, maps $\text{obs}(D)$ to $\text{obs}(D')$, i.e. the cohomology class $\text{obs}(D)$ is functorial.

**Proof.** Let $\sigma'$ be a set-theoretic section of $\kappa'$. Referring to the proof of Lemma 2.2, $\text{obs}(D') = [\psi']$, where $\psi' \in \text{Maps}(G \times G, K_{D'})$ is defined by

$$
\psi'(g_1, g_2) = \sigma' (\theta (\phi (g_1 g_2))) \sigma' (\theta (\phi (g_1)))^{-1} \sigma' (\theta (\phi (g_1)))^{-1}.
$$

On the other hand, $\zeta_* (\psi) = \zeta \circ \psi$. We may pick the section $\sigma'$ such that $\eta \circ \sigma = \sigma' \circ \theta$. Then

$$
\eta (\psi (g_1, g_2)) = \eta (\sigma (\phi (g_1 g_2))) \eta (\sigma (\phi (g_2))^{-1}) \eta (\sigma (\phi (g_1))^{-1})
\quad = \eta (\sigma (\phi (g_1 g_2))) \eta (\sigma (\phi (g_2)))^{-1} \eta (\sigma (\phi (g_1)))^{-1}
\quad = \sigma' (\theta (\phi (g_1 g_2))) \sigma' (\theta (\phi (g_2)))^{-1} \sigma' (\theta (\phi (g_1)))^{-1}
\quad = \psi'(g_1, g_2),
$$

hence $[\zeta \circ \psi] = [\psi']$ and the result follows. $\square$

Generalising the above, we make the following definition.

**Definition 2.4.** A map $\mathbf{o}$ defined on the set of objects of $\mathcal{D}$ mapping $D \in \text{ob} (\mathcal{D})$ to an element $\mathbf{o}_D \in H^2 (G, K_D)$ will be called an obstruction if it satisfies the following two conditions:

**OG1:** $\mathbf{o}_D$ is functorial, i.e. if $D \longrightarrow D'$ is a morphism in $\mathcal{D}$, then $H^2 (G, K_D) \longrightarrow H^2 (G, K_{D'})$ maps $\mathbf{o}_D$ to $\mathbf{o}_{D'}$;

**OG2:** $\mathbf{o}_D = 0$ if and only if $\phi: G \longrightarrow H_2$ lifts to $H_1$.

In Lemma 2.2, we have constructed such an obstruction.

Now observe that functorially in $\mathcal{D}$ we have

$$
H^2 (G, K_D) \simeq H^2 (G, \mathbb{F}_p) \otimes_{\mathbb{F}_p} K_D \simeq \text{Hom}_{\mathbb{F}_p} (H^2 (G, \mathbb{F}_p)^*, K_D).
$$

Moreover, we have the following easy lemma.
Lemma 2.5. Let \( \mathbf{o} \) and \( \mathbf{o}' \) be two obstructions. For every \( D \in \text{ob}(D) \), consider \( o_D, o'_D \in H^2(G, K_D) \cong \text{Hom}_{\mathbb{F}_p}(H^2(G, \mathbb{F}_p)^*, K_D) \). Then \( \text{im} o_D = \text{im} o'_D \) as vector subspaces in \( K_D \).

Proof. Consider a diagram \( D \) in \( D \) with \( K_D, H_1 \) and \( H_2 \) making up the central extension. Since \( \mathbf{o} \) is an obstruction, we can lift \( \phi \) to \( H_1/\text{im} o_D \):

\[
\begin{array}{cccc}
1 & \to & K_D/\text{im} o_D & \xrightarrow{\iota} & H_1/\text{im} o_D & \xrightarrow{\kappa} & H_2 & \to & 1 \\
1 & \to & K_D & \xrightarrow{\iota} & H_1 & \xrightarrow{\kappa} & H_2 & \to & 1 \\
& & & & & & & & \\
& & & \phi & & & \phi' & & \\
& & & \phi' & & & G & & \end{array}
\]

Now, \( o'_D \) is also an obstruction for lifting \( \phi \), and \( \text{im} o'_D \) is therefore, by functoriality, sent to 0 in \( K_D/\text{im} o_D \). Hence \( \text{im} o'_D \subseteq \text{im} o_D \), and by symmetry we get equality. \( \square \)

From this result we see that the image \( \text{im} o_D \subset K_D \) of the obstruction is the interesting thing. This image will from now on be called the obstruction.

We introduce the notation \( \langle - \rangle \) and \( \overline{(-)} \) to denote the normal subgroup generated by \(-\) and the closed normal subgroup generated by \(-\) respectively.

Let \( T^i \) be the free pro-\( p \) group on a dual basis for \( H^i(G, \mathbb{F}_p) \), \( i = 1, 2 \). Since \( H^1(G, \mathbb{F}_p) \cong (G/\langle [G, G], G^p \rangle)^* \), there is a surjective homomorphism \( \pi: T^1 \to G \). In the next theorem we construct a morphism \( o_G \) from \( T^2 \) to \( T^1 \) with \( T^1/\langle \text{im} o_G \rangle \cong G \).

It is well-known that there exists such a morphism \( \phi: T^2 \to T^1 \) with \( T^1/\langle \text{im} o \rangle \cong G \), see [10]. The interest of our result lies in the particular construction of \( o_G \).

We construct the morphism \( o_G \) as the inverse limit of a certain coherent sequence of morphisms from \( T^2 \) to quotients of \( T^1 \) using the above obstruction calculus at each level. The quotients involve a filtration \( \{ F_i(T^1) \}_{i \geq 1} \) of \( T^1 \). This construction is written down in some detail in [4], but then using the lower central \( p \)-series as the filtration. However, for our purposes, we shall use the \( \mathcal{M} \)-series, also known as the Brauer-Jennings-Zassenhaus series.

**Definition 2.6.** The \( \mathcal{M} \)-series of a pro-\( p \) group \( G \) is defined inductively by

\[
\begin{align*}
\mathcal{M}_1(G) &= G, \\
\mathcal{M}_i(G) &= \frac{[\mathcal{M}_{i-1}(G), G]}{\mathcal{M}_{i-1}(G)}(p),
\end{align*}
\]

where \( (i/p) \) is the least integer \( \geq i/p \) and \( \mathcal{M}_i(G)(p) \) denotes the set of \( p \)-th powers in \( \mathcal{M}_i(G) \).
We note that $T^i$ is the projective limit of the quotients $T_i^i / \mathcal{M}_n(T^i)$.

The successive quotients $\mathcal{M}_i(G)/\mathcal{M}_{i+1}(G)$ are vector spaces over $\mathbb{F}_p$. The series $\{\mathcal{M}_i(G)\}_{i \geq 1}$ is not the fastest descending series with this property, however, it is the fastest descending series satisfying

\begin{equation}
[\mathcal{M}_n(G), \mathcal{M}_m(G)] \leq \mathcal{M}_{n+m}(G),
\end{equation}

\begin{equation}
\mathcal{M}_n(G) \leq \mathcal{M}_{np}(G).
\end{equation}

Hence the $\mathcal{M}$-series is the “restricted mod $p$ lower central series”, see [3] and [8, chapter 11].

**Theorem 2.7.** Let $G$ be a finite $p$-group and let $T^i$ be the free pro-$p$ group on a dual basis for $H^i(G, \mathbb{F}_p)$, $i = 1, 2$. Then the above obstruction calculus induces an obstruction morphism $\alpha_G : T^2 \longrightarrow T^1$ with $T^1 / (\text{im} \alpha_G) \cong G$.

**Proof.** We introduce the notation $T'_n := T^1 / \mathcal{M}_n(T^1)$ and $G_n := G / \mathcal{M}_n(G)$. Let $\sigma_n : G \longrightarrow G_n$ denote the canonical homomorphism.

We know that $\dim \mathbb{F}_p H^1(G, \mathbb{F}_p)$ is the minimal number of generators for $G$. It is also known that $\dim \mathbb{F}_p H^2(G, \mathbb{F}_p)$ is the minimal number of pro-$p$ relations for $G$, but we are not going to use this fact.

Fix a surjective homomorphism $\pi : T^1 \longrightarrow G$. For the first step, we note that $\pi$ induces an isomorphism

$$
\theta_2 : T'_2 \xrightarrow{\cong} G_2
$$

and consider the following commutative diagram

$$
D_2 : 1 \longrightarrow \mathcal{M}_2(T^1) / \mathcal{M}_3(T^1) \xrightarrow{i} T'_3 \xrightarrow{\pi^2_3} T'_2 \longrightarrow 1
$$

which is an object in the category $D$. By Lemma 2.2 we have a unique element $\text{obs}(D_2) \in H^2(G, \mathcal{M}_2(T^1) / \mathcal{M}_3(T^1))$ such that $\text{obs}(D_2) = 0$ if and only if there exists a lifting $\phi_2$ of $\phi_1$ to $T'_3$.

Now,

$$
H^2(G, \mathcal{M}_2(T^1) / \mathcal{M}_3(T^1)) \cong H^2(G, \mathbb{F}_p \otimes \mathcal{M}_2(T^1) / \mathcal{M}_3(T^1))
$$

$$
\cong \text{Hom}(H^2(G, \mathbb{F}_p), \mathcal{M}_2(T^1) / \mathcal{M}_3(T^1))
$$

$$
\cong \text{Mor}(T^2, \mathcal{M}_2(T^1) / \mathcal{M}_3(T^1))
$$

and

$$
\text{Mor}(T^2, \mathcal{M}_2(T^1) / \mathcal{M}_3(T^1)) \subseteq \text{Mor}(T^2, T'_3),
$$
so let $o_2$ be obs$(D_2)$ as an element in Mor$(T^2, T^1_3)$. We shall refer to
imobs$(D_2)$ as the obstruction on the first level, which lies in

$$K_2 := \mathcal{M}_2(T^1)/\mathcal{M}_3(T^1) = \mathcal{M}_2(T^1_3).$$

**Remark:** By Lemma 2.2, $\langle \text{im} o_2 \rangle$ is the smallest normal subgroup of $T^1_3$
containing the elements that need to be reduced to 1 in order to get a lifting
of $\bar{\phi}_1$. In other words, the universal functorial property of these obstructions
implies that $T^1_3/\langle \text{im} o_2 \rangle$ is the largest quotient of $T^1_3$ making a lifting $\phi_2$ of
$\bar{\phi}_1$ possible. So for all other quotients $T^1_3/N$ and a map $\tilde{\phi}_2 : G \rightarrow T^1_3/N$
lifting $\bar{\phi}_1$ there exists a homomorphism $\rho : T^1_3/\langle \text{im} o_2 \rangle \rightarrow T^1_3/N$ such that
the following diagram commutes

$$
\begin{array}{ccc}
T^1_3/\langle \text{im} o_2 \rangle & \xrightarrow{\rho} & T^1_3/N \\
\downarrow{\phi_2} & & \downarrow{\tilde{\phi}_2} \\
G & & G.
\end{array}
$$

Also note that $\pi$ induces a commutative diagram of surjective homomorphisms,

$$
\begin{array}{ccc}
T^1_3 & \xrightarrow{\alpha_3} & G_3 \\
\downarrow{\sigma_2} & & \downarrow{\sigma_3} \\
T^1_2 & \xrightarrow{\bar{\phi}_1} & G, \\
\end{array}
$$

and therefore a surjective homomorphism $\theta_3$ in the commutative diagram

$$
\begin{array}{ccc}
T^1_3/\langle \text{im} o_2 \rangle & \xrightarrow{\theta_3} & G_3 \\
\omega \downarrow{\theta_3} & & \sigma_3 \downarrow{\sigma_3} \\
T^1 & \xrightarrow{\pi} & G.
\end{array}
$$
To show that $\theta_3$ is an isomorphism, consider the diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & M_2(T_3^1) & \longrightarrow & T_3^1 & \longrightarrow & T_2^1 & \longrightarrow & 1 \\
\uparrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_2(T_3^1)/\langle \text{im} o_2 \rangle & \longrightarrow & T_3^1/\langle \text{im} o_2 \rangle & \longrightarrow & T_2^1 & \longrightarrow & 1 \\
\quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \quad & \quad \\
1 & \longrightarrow & G_3 & \quad & \quad & \quad & \quad & \quad & \quad & \quad
\end{array}
\]

and observe that (any) $\phi_2$ lifting $\tilde{\phi}_1$ factorizes via $G_3$ since $\phi_2$ maps $M_3(G)$ to 1 in $T_3^1/\langle \text{im} o_2 \rangle$. Let $\psi_3: G_3 \longrightarrow T_3^1/\langle \text{im} o_2 \rangle$ be the induced homomorphism. Since $\phi_2$ must be surjective, $\psi_3$ is surjective. Therefore $G_3$ and $T_3^1/\langle \text{im} o_2 \rangle$ have the same order, thus $\theta_3$ (as well as $\psi_3$) is an isomorphism induced by $\pi$.

We have now seen most of the first non-trivial step in the construction of $o_G$: So far we have

1) a surjective homomorphism $\pi: T^1 \longrightarrow G$ and
2) a homomorphism $o_2: T^2 \longrightarrow T_3^1$ inducing the isomorphism $\theta_3$ in 

the commutative diagram

\[
\begin{array}{ccccc}
T^1 & \longrightarrow & G \\
\downarrow & & \downarrow & \sigma_3 \\
T_3^1/\langle \text{im} o_2 \rangle & \longrightarrow & G_3.
\end{array}
\]

Our aim is to find a coherent sequence of homomorphisms

\[
\{o_n: T^2 \longrightarrow T_{n+1}^1\}_{n \geq 1},
\]
o_1 being trivial, inducing an isomorphism $\theta_{n+1}$ in the commutative diagram

\[
\begin{array}{ccccc}
T^1 & \longrightarrow & G \\
\downarrow & & \downarrow & \sigma_{n+1} \\
T_{n+1}^1/\langle \text{im} o_n \rangle & \longrightarrow & G_{n+1}.
\end{array}
\]
Let \( \tilde{\phi}_n : G \rightarrow T_{n+1}^1 / \langle \text{im} \sigma_n \rangle \) be the obvious composition of \( \sigma_{n+1} \) and \( (\theta_{n+1})^{-1} \) for \( n \geq 2 \).

To finish the first step of the construction, we need to see the coherence between the first and second step. So, before we do the general step, let us see what happens when we try to lift \( \tilde{\phi}_2 \).

We start with the diagram

\[
1 \longrightarrow \langle \mathcal{M}_3(T^1), L_2 \rangle / \mathcal{M}_4(T^1) \longrightarrow T_4^1 \longrightarrow T_3^1 / \langle \text{im} \sigma_2 \rangle \longrightarrow 1
\]

where \( L_2 \) is the preimage of \( \langle \text{im} \sigma_2 \rangle \) in \( T^1 \) (so \( L_2 \triangleleft T^1 \)).

However, this is not a diagram to which we can apply Lemma 2.2. We want the kernel to be both central and elementary, hence we must consider the quotient diagram

\[
D_3 : 1 \longrightarrow K_3 \longrightarrow T^1 / N_4 \longrightarrow T_3^1 / \langle \text{im} \sigma_2 \rangle \longrightarrow 1
\]

where \( N_4 := \langle \mathcal{M}_4(T^1), [T^1, L_2], (L_2)^p \rangle \) and \( K_3 := \langle \mathcal{M}_3(T^1), L_2 \rangle / N_4 \). From Lemma 2.2 we get a unique cohomology class

\[
\text{obs}(D_3) \in H^2(G, K_3) \simeq \text{Mor}(T^2, K_3) \subseteq \text{Mor}(T^2, T^1 / N_4),
\]

which has to be 0 in order to get a lifting of \( \tilde{\phi}_2 \). Hence the obstruction \( \text{imobs}(D_3) \) on the second level lies in \( K_3 \).
Let $o(\tilde{\phi}_2)$ be $\text{obs}(D_3)$ as an element in $\text{Mor}(T^2, T^1/N_4)$ and consider the following diagram where the four arrows $\sim\sim\sim$ denote two short-exact sequences

\begin{equation}
(2.7)
\end{equation}

Recall that $\langle \text{im} o_2 \rangle$ lies in $\mathcal{M}_2(T^1)/\mathcal{M}_3(T^1)$ and that $L_2$ lies in $\mathcal{M}_2(T^1)$, so $\langle [T^1, L_2], (L_2)^p \rangle \subseteq \mathcal{M}_3(T^1)$. Therefore there exists a homomorphism $\mu$ which factors $\pi_3^4$. It also factors $\pi_4^4$ and induces a homomorphism

$$\mu: K_3 \rightarrow K_2,$$

which again induces a linear map

$$\mu_*: H^2(G, K_3) \rightarrow H^2(G, K_2).$$

At the same time we see that $L_2 \subset T^1$ is mapped to 1 in $G_3$, therefore into $\mathcal{M}_3(G)$ in $G$. But then $\pi$ maps $\langle [T^1, L_2], (L_2)^p \rangle$ into $\mathcal{M}_3p(G) \subseteq \mathcal{M}_4(G)$. Therefore the homomorphism $T^1_4 \rightarrow G_4$ induced by $\pi$ factorizes through a surjection $T^1/N_4 \rightarrow G_4$. By functoriality of the obstructions (see Corollary 2.3), it also factorizes via the surjection

$$\theta_4: (T^1/N_4)/\text{imobs}(D_3) \rightarrow G_4.$$

As above we prove that (any) lifting $\phi_3$ of $\tilde{\phi}_2$ must be surjective and must map $\mathcal{M}_4(G)$ to 1, inducing a surjection

$$\psi_4: G_4 \rightarrow (T^1/N_4)/\text{imobs}(D_3).$$
Therefore $\theta_4$ is an isomorphism induced by $\pi$.

Pick an $o_3$ such that $\mu \circ o_3 = o(\tilde{\phi}_2)$. By construction,
\[
\pi_3^4 \circ o_3 = \mu \circ \rho_4 \circ o_3 = \mu \circ o(\tilde{\phi}_2) = \overline{\mu}_\ast(\text{obs}(D_3)),
\]
which as an element of $\text{Mor}(T^2, K_2) \subset \text{Mor}(T^2, T^1_3)$ is equal to $o_2$, since by functoriality the map $\overline{\mu}_\ast$ sends $\text{obs}(D_3)$ to $\text{obs}(D_2)$. Let $L_3$ be the preimage of $\langle \text{im}o_3 \rangle$ in $T^1$. Then, by the properties (2.5) and (2.6) of the $\mathcal{M}$-series, we find that
\[
\langle [T^1, L_2], (L_2)^p \rangle \subseteq L_3.
\]
Hence there is a canonical surjection $T^1/N_4 \xrightarrow{\pi_3^4 T^1_4/(\text{im}o_3)}$. Moreover, $\langle \text{im}o_3 \rangle$ maps onto $\text{imobs}(D_3)$ in $K_3$, inducing the canonical isomorphism $B_4$ which identifies $T^1_4/(\text{im}o_3)$ and $(T^1/N_4)/\text{imobs}(D_3)$. Put
\[
\tilde{\phi}_3 = (\theta_4)^{-1} \circ \sigma_4 : G \rightarrow T^1_4/(\text{im}o_3).
\]

We have now seen the first non-trivial step of the construction of $o_G$ in detail. The construction continues: Assume, by induction, that we have constructed $o_{n-1}$ such that there exists an isomorphism
\[
\theta_n : T^1_n/(\text{im}o_{n-1}) \xrightarrow{\cong} G_n
\]
induced by $\pi$. Put $\tilde{\phi}_{n-1} = (\theta_n)^{-1} \circ \sigma_n$. Then, for lifting $\tilde{\phi}_{n-1}$, we have the diagram
\[
\begin{array}{c}
1 \rightarrow \langle \mathcal{M}_n(T^1), L_{n-1} \rangle/\mathcal{M}_{n+1}(T^1) \rightarrow T^1_{n+1} \rightarrow T^1_n/(\text{im}o_{n-1}) \rightarrow 1 \\
\downarrow \tilde{\phi}_{n-1} \uparrow G
\end{array}
\]
where $L_{n-1}$ is the preimage of $\langle \text{im}o_{n-1} \rangle$ in $T^1$.

Define $N_{n+1}$ by
\[
N_{n+1} := \langle \mathcal{M}_{n+1}(T^1), [T^1, L_{n-1}], (L_{n-1})^p \rangle
\]
and consider the central elementary extension, i.e. the diagram
\[
\begin{array}{c}
1 \rightarrow K_n \rightarrow T^1/N_{n+1} \rightarrow T^1_n/(\text{im}o_{n-1}) \rightarrow 1 \\
\downarrow \tilde{\phi}_{n-1} \uparrow G
\end{array}
\]
where
\[
K_n := \langle \mathcal{M}_n(T^1), L_{n-1} \rangle/N_{n+1}.
\]

As before, we get the obstruction $\text{imobs}(D_n) \subseteq K_n$ and a lifting
\[
\phi_n : G \rightarrow (T^1/N_{n+1})/\text{imobs}(D_n),
\]
inducing an isomorphism

\[ \theta_{n+1} : (T^1 / N_{n+1})/\text{imobs}(D_n) \xrightarrow{\cong} G_{n+1} \]

commuting with \( \pi \).

Now we need to check that there exists an \( o_n \) and a commutative diagram

\[ \begin{array}{ccc}
T^2 & \xrightarrow{\phi_{n-1}} & T^1_{n+1} \\
\downarrow \rho_{n+1} & & \downarrow \rho_n \\
T^1 / N_{n+1} & \xrightarrow{\gamma} & T^1 / N_n \\
\end{array} \]

with the properties we want, in particular, the canonical isomorphism

\[ B_{n+1} : T^1_{n+1}/(\text{im} o_n) \xrightarrow{\cong} (T^1 / N_{n+1})/\text{imobs}(D_n), \]

identifying the two groups.

We denote by \( L_n \) the preimage of \( (\text{im} o_n) \) in \( T^1 \) under the map \( T^1 \longrightarrow T^1_{n+1} \). Observe that, by induction, \( L_{n-1} \subseteq L_{n-2} \), implying

\[ (\langle T^1, L_{n-1} \rangle, (L_{n-1})^p) \subseteq (\langle T^1, L_{n-2} \rangle, (L_{n-2})^p). \]

Note that (2.11) is an equality modulo \( M_{n+1}(T^1) \).

Recall that we defined \( N_n = \langle M_n(T^1), [T^1, L_{n-2}], (L_{n-2})^p \rangle \) and that \( N_{n+1} = \langle M_{n+1}(T^1), [T^1, L_{n-1}], (L_{n-1})^p \rangle \). From (2.11) it now follows that we have a surjection

\[ M_n(T^1)/M_{n+1}(T^1) \longrightarrow N_n/N_{n+1}. \]

Consider the diagram

\[ \begin{array}{ccc}
T^2 & \xrightarrow{o_{n-1}} & T^1_{n+1} \\
\downarrow \rho_{n+1} & & \downarrow \rho_n \\
T^1 / N_{n+1} & \xrightarrow{\gamma} & T^1 / N_n \\
\end{array} \]

We need to show that there exists an \( o_n \) such that \( \pi_{n+1}^n \circ o_n = o_{n-1} \) and \( \rho_{n+1} \circ o_n = o(\phi_{n-1}) \). By definition, \( T^2 \) is the free pro-\( p \) group on a dual basis for \( H^2(G, \mathbb{F}_p) \). Fix generators \( y_1, \ldots, y_r \) for \( T^2 \) corresponding to a basis for
\(H^2(G, \mathbb{F}_p)^*\). Let \(y := y_i \in T^2\) for an \(i \in \{1, \ldots, r\}\) and pick \(y' \in T^1_{n+1}\) such that \(\pi_n^{n+1}(y') = o_{n-1}(y)\). We have
\[
\gamma((\rho_{n+1}(y'))^{-1} \cdot o(\phi_{n-1})(y)) = (\rho_n(\pi_n^{n+1}(y'))^{-1} \cdot o(\phi_{n-2})(y)
= (\rho_n(o_{n-1}(y)))^{-1} \cdot o(\phi_{n-2})(y)
= 1,
\]
hence \((\rho_{n+1}(y'))^{-1} \cdot o(\phi_{n-1})(y) \in \ker \gamma = N_1/N_{n+1}\). By (2.12), there exists an element \(a \in \ker \pi_n^{n+1} = \mathcal{M}_n(T^1)/\mathcal{M}_{n+1}(T^1)\) such that \(\rho_{n+1}(a) = (\rho_{n+1}(y'))^{-1} \cdot o(\phi_{n-1})(y)\). Hence if we let \(o_n(y) = y' \cdot a\), we get \(\pi_n^{n+1}(o_n(y)) = o_{n-1}(y)\) and \(\rho_{n+1}(o_n(y)) = o(\phi_{n-1})(y)\), which is what we needed to show.

By construction of \(o_n\), there is a canonical surjection
\[
B_{n+1} : T^1_{n+1}/(\text{im} \ o_n) \longrightarrow (T^1/N_{n+1})/\text{im} \text{obs}(D_n).
\]
By definition, \(L_n\) is the preimage in \(T^1\) of \((\text{im} \ o_n)\), and \(T^1/L_n = T^1_{n+1}/(\text{im} \ o_n)\), therefore \(\mathcal{M}_{n+1}(T^1) \subseteq L_n\).

Now, \(N_{n+1} = \langle \mathcal{M}_{n+1}(T^1), [T^1, L_{n-1}], (L_{n-1})^p \rangle\) is contained in \(L_n\), since \(\langle [T^1, L_{n-1}], (L_{n-1})^p \rangle \subseteq L_n\). Therefore we have an obvious commutative diagram
\[
\begin{array}{ccc}
T^1_{n+1} & \xrightarrow{\rho_{n+1}} & T^1/N_{n+1} \\
\downarrow & & \downarrow \\
T^1_{n+1}/(\text{im} \ o_n) & \xrightarrow{B_{n+1}} & (T^1/N_{n+1})/\text{im} \text{obs}(D_n)
\end{array}
\]
By construction, \(\rho_{n+1}((\text{im} \ o_n)) = \text{im} \text{bs}(D_n)\), hence \(B_{n+1}\) is an equality.

As before, we let \(\phi_n = (\theta_{n+1})^{-1} \circ \sigma_{n+1} : G \longrightarrow T^1_{n+1}/(\text{im} \ o_n)\), completing the induction process.

To finish, the map \(o_G : T^2 \longrightarrow T^1\) is now the inverse limit of the system \(\{o_n : T^2 \longrightarrow T^1_{n+1}\}_{n \geq 1}\). Moreover, the liftings \(\{\phi_n\}_{n \geq 1}\) induce isomorphisms \(T^1_{n+1}/(\text{im} \ o_n) \simeq G_{n+1}\) for all \(n \geq 1\), and so \(T^1/(\text{im} o_G) \simeq G\).

\[\boxed{3. \text{Obstruction theory for complete local } \mathbb{F}_p\text{-algebras}}\]

There exists a noncommutative deformation theory for families of modules over \(k\)-algebras (\(k\) a field), see [6]. In particular, the theory will apply to the case where the family is reduced to \(\mathbb{F}_p\), the only simple \(\mathbb{F}_p\) \(G\)-module for a finite \(p\)-group \(G\).

Recall that in the previous section, \(T^i\) denoted the free pro-\(p\) group on a dual basis for \(H^i(G, \mathbb{F}_p)\), \(i = 1, 2\). Now let \(T^i_{\mathbb{F}_p G}\) denote the completed tensor algebra of the dual vector space of \(\text{Ext}^i_{\mathbb{F}_p G}(\mathbb{F}_p, \mathbb{F}_p)\) over \(\mathbb{F}_p\), \(i = 1, 2\). Therefore, if we put \(E_i = \text{Ext}^i_{\mathbb{F}_p G}(\mathbb{F}_p, \mathbb{F}_p)\), we have
\[
T^i_{\mathbb{F}_p G} \simeq \mathbb{F}_p \times E^*_i \times (E^*_i \otimes E^*_i) \times (E^*_i \otimes E^*_i \otimes E^*_i) \times \cdots
\]
Let \((-\)) denote the two-sided ideal generated by \(-\), and let \(T\) denote the closure of the two-sided ideal \(I\) in the complete algebra. The closure notation will sometimes be omitted.

General theory provides us with the existence of an obstruction morphism

\[
\sigma_{F_p G} : T^2_{F_p G} \longrightarrow T^1_{F_p G}
\]

with \(T^1_{F_p G}/(\text{im} \sigma_{F_p G}) \simeq F_p G\).

In the language of deformation theory, this says that \(F_p G\) is “the formal moduli” of the trivial module \(F_p\). Similarly, we have seen in Theorem 2.7 that \(G\) is “the formal moduli” of \(F_p\) when we are in the category of pro-\(p\) groups. Whereas the construction of the obstruction morphism in the previous section used a filtration of \(T^1\) by its \(M\)-series, we will now filter \(T^1_{F_p G}\) by powers of its maximal ideal.

To make it clear that the construction of \(\sigma_{F_p G}\) just depends on the \(F_p\)-algebra structure of \(F_p G\) and not on the group structure, we shall construct the obstruction morphism for any complete associative local \(F_p\)-algebra \(A\) with residue field \(F_p\).

Since \(\text{Ext}^1_{F_p}(F_p, F_p) \simeq (m/m^2)^*\) where \(m\) is the maximal ideal of \(A\), there is a surjective homomorphism \(\pi_A : T^1 \longrightarrow A\).

We shall need an algebra version of Lemma 2.2. In order to formulate this, we have to give some easy (and well-known) results on the Hochschild cohomology of \(A\).

**Definition 3.1.** Let \(A\) be an associative \(k\)-algebra, \(k\) a field, and let \(Q\) be an \(A\)-bimodule. The Hochschild cocomplex \((C^*(A, Q), \partial)\) is given by

\[
\cdots \longrightarrow \text{Hom}_k(A \otimes^n Q) \xrightarrow{\partial^n} \text{Hom}_k(A \otimes^{n+1} Q) \longrightarrow \cdots
\]

where the differential \(\partial\) is given by the formula

\[
\partial^n(\phi)(a_1 \otimes \cdots \otimes a_{n+1}) = \begin{align*}
&= a_1 \phi(a_2 \otimes \cdots \otimes a_{n+1}) + \\
&\quad + \sum_{i=1}^n (-1)^i \phi(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + \\
&\quad + (-1)^{n+1} \phi(a_1 \otimes \cdots \otimes a_n) a_{n+1}
\end{align*}
\]

for \(\phi \in C^n(A, Q), a_1, \ldots, a_{n+1} \in A\). The cohomology groups of this cocomplex are the Hochschild cohomology groups, denoted \(HH^n(A, Q)\).

We note that if \(M\) and \(N\) are right \(A\)-modules, \(\text{Hom}_k(M, N)\) is an \(A\)-bimodule, and we have the isomorphism

\[
(3.1) \quad \text{Ext}^n_A(M, N) \simeq HH^n(A, \text{Hom}_k(M, N))
\]

for \(n \geq 0\).
Consider the diagram

\[
E : 0 \to V_E \xrightarrow{\kappa} B_1 \xrightarrow{\phi} B_2 \to 0
\]

where \( \kappa \) is a small surjective homomorphism of finite dimensional local algebras over \( \mathbb{F}_p \) and \( V_E = \ker \kappa \), therefore \( m_1 V_E = V_E m_1 = 0 \) where \( m_1 \) is the maximal ideal of \( B_1 \). Thus \( V_E \) is a vector space over \( \mathbb{F}_p \) as a \( B_1 \)-module.

**Definition 3.2.** Let \( \mathcal{E} \) denote the category of all such diagrams \( (3.2) \) for a fixed associative algebra \( A \) together with the obvious morphisms of such diagrams. An object \( E \) in \( \mathcal{E} \) is thus a small extension of finite dimensional \( \mathbb{F}_p \)-algebras with kernel \( V_E \) and a homomorphism \( \phi \) from \( A \) to the base algebra in the extension.

**Lemma 3.3.** To each diagram \( E \) in \( \mathcal{E} \) one associates a cohomology class \( \text{obs}(E) \in \text{Ext}^2_A(\mathbb{F}_p, V_E) \) such that \( \text{obs}(E) = 0 \) if and only if there exists an algebra homomorphism \( \phi' : A \to B_1 \) with \( \phi = \kappa \circ \phi' \).

**Proof.** Let \( \sigma \) be a section from \( B_2 \) to \( B_1 \) as vector spaces over \( \mathbb{F}_p \). For every pair \( a_1, a_2 \in A \), consider the element

\[
\psi(a_1 \otimes a_2) = \sigma(\phi(a_1a_2)) - \sigma(\phi(a_1))\sigma(\phi(a_2))
\]

in \( B_2 \). Then \( \psi \in \text{Hom}_{\mathbb{F}_p}(A \otimes A, V_E) \) since \( \phi \) is a homomorphism. We check that \( (\partial^2 \psi)(a_1 \otimes a_2 \otimes a_3) = 0 \) and let \( \text{obs}(E) = [\psi] \in HH^2(A, V_E) \). Note that this cohomology class is independent of the choice of \( \sigma \).

Assume \( \text{obs}(E) = 0 \). Then \( \psi \in \text{im} \partial^1 \), and so there exists \( \alpha \in \text{Hom}_{\mathbb{F}_p}(A, V_E) \) such that \( \partial^1 \alpha = \psi \). Define \( \phi' : A \to B_2 \) by

\[
\phi'(a) = \sigma(\phi(a)) + \alpha(a), \quad a \in A.
\]

Then, since we have a small extension, \( \phi' \) is an algebra homomorphism, and \( \phi = \kappa \circ \phi' \).

Conversely, suppose there exists a \( \phi' \) such that \( \phi = \kappa \circ \phi' \). We need to show that \( \text{obs}(E) = 0 \) as an element in \( \text{Ext}^2_A(\mathbb{F}_p, V_E) \). Let \( \gamma : A \to B_2 \) be defined by \( \gamma(a) = \phi'(a) - \sigma(\phi(a)) \). Then we have \( \kappa(\gamma(a)) = 0 \), hence \( \gamma \in \text{Hom}_{\mathbb{F}_p}(A, V_E) \). Furthermore, \( \psi(a_1 \otimes a_2) = \gamma(a_1) - \gamma(a_1a_2) + \gamma(a_2) \), which proves the lemma.

We make a similar definition of an obstruction as in the previous section. The functoriality and universal property will also be similar.
Consider a morphism $\tilde{\zeta}: E \longrightarrow E'$ in $\mathcal{E}$, i.e. a commutative diagram

$$
\begin{array}{c}
E': 0 \longrightarrow V_{E'} \xrightarrow{\iota} B_1' \xrightarrow{\kappa'} B_2' \longrightarrow 0 \\
E: 0 \longrightarrow V_E \xrightarrow{\iota} B_1 \xrightarrow{\kappa} B_2 \longrightarrow 0
\end{array}
$$

The morphism $\tilde{\zeta}$ induces a map making $\text{Ext}^2_A(\mathbb{F}_p, -)$ a functor from the category $\mathcal{E}$ to the category of vector spaces over $\mathbb{F}_p$. Moreover, to every diagram $E$ in $\mathcal{E}$, we have shown that there exists a cohomology class $\text{obs}(E) \in \text{Ext}^2_A(\mathbb{F}_p, V_E)$.

**Corollary 3.4.** The map $\zeta_*: \text{Ext}^2_A(\mathbb{F}_p, V_E) \longrightarrow \text{Ext}^2_A(\mathbb{F}_p, V_{E'})$, induced by $\tilde{\zeta}$, maps $\text{obs}(E)$ to $\text{obs}(E')$, i.e. the cohomology class $\text{obs}(E)$ is functorial.

**Proof.** Similar to the proof of Corollary 2.3. \qed

As in the previous section, we generalize the above and make the following definition.

**Definition 3.5.** A map $\circ$ defined on the set of objects of $\mathcal{E}$ mapping $E \in \text{ob}(\mathcal{E})$ to an element $o_E \in \text{Ext}^2_A(\mathbb{F}_p, V_E)$ will be called an obstruction if it satisfies the following two conditions:

**OA1:** $o_E$ is functorial, i.e. if $E \longrightarrow E'$ is a morphism in $\mathcal{E}$, then $\text{Ext}^2_A(\mathbb{F}_p, V_E) \longrightarrow \text{Ext}^2_A(\mathbb{F}_p, V_{E'})$ maps $o_E$ to $o_{E'}$;

**OA2:** $o_E = 0$ if and only if $\phi: A \longrightarrow B_2$ lifts to $B_1$.

In Lemma 3.3, we have constructed such an obstruction. Now observe that functorially in $\mathcal{E}$ we have

$$
\text{Ext}^2_A(\mathbb{F}_p, V_E) \cong \text{Ext}^2_A(\mathbb{F}_p, \mathbb{F}_p) \otimes_{\mathbb{F}_p} V_E \cong \text{Hom}_{\mathbb{F}_p}(\text{Ext}^2_A(\mathbb{F}_p, \mathbb{F}_p)^*, V_E).
$$

**Lemma 3.6.** Let $\circ$ and $\circ'$ be two obstructions. For every $E \in \text{ob}(\mathcal{E})$ consider $o_E, o'_E \in \text{Ext}^2_A(\mathbb{F}_p, V_E) \cong \text{Hom}_{\mathbb{F}_p}(\text{Ext}^2_A(\mathbb{F}_p, \mathbb{F}_p)^*, V_E)$. Then $\text{im} o_E = \text{im} o'_E$ as vector subspaces in $V_E$.

**Proof.** Similar to the proof of Lemma 2.5. \qed

Again, we see that the image of the obstruction is the interesting thing. As in the group situation, this image will from now on be called the obstruction.

We are now ready to state and prove the algebra version of Theorem 2.7.

**Theorem 3.7.** Let $A$ be a complete associative local algebra over $\mathbb{F}_p$ with residue field $\mathbb{F}_p$. Let $T_A^1$ be the completed tensor algebra of the dual vector
space of $\text{Ext}^i_A(\mathbb{F}_p, \mathbb{F}_p)$, $i = 1, 2$, over $\mathbb{F}_p$. Then the above obstruction calculus induces an obstruction morphism $o_A : T^2_A \rightarrow T^1_A$ with $T^1_A/(\text{Im}o_A) \cong A$.

Proof. In the proof we will use $T^i$ for $T^i \mathcal{G}$, $m$ and $m_A$ will denote the maximal ideals of $T^1$ and $A$ respectively, and $T^1_n$ and $A_n$ will be short for $T^1/m^n$ and $A/m^n$ respectively.

We note that $\text{Ext}^1_A(\mathbb{F}_p, \mathbb{F}_p) \cong (m_A/m^2_A)^*$ and that $A/m^2_A \cong T^1/m^2$. Moreover, $T^1$ is the completed tensor algebra, and so $T^1_n$ is always the universal extension of $T^1_{n-1}$ with $(\text{Ext}^1_A(\mathbb{F}_p, \mathbb{F}_p)^*)^{\otimes (n-1)}$, in particular, $T^1/m^2 \cong \mathbb{F}_p \oplus \text{Ext}^1_A(\mathbb{F}_p, \mathbb{F}_p)^*$. Fix a surjective homomorphism $\pi_A : T^1 \rightarrow A$. Let $\sigma_n : A \rightarrow A_n$ be the canonical homomorphism.

We want to lift algebra homomorphisms from $A$ to quotients of $T^1$ involving powers of $m$. For the first step of the construction we note that $\pi_A$ induces an isomorphism $\theta_2 : T^1_2 \cong A_2$ and consider the following commutative diagram

$$
E_2 : 0 \rightarrow m^2/m^3 \rightarrow T^1_3 \rightarrow T^1_2 \rightarrow 0
$$

For the analogy with the proof of Theorem 2.7, it helps to keep the following “correspondence” in mind: Let $T^1_3$ be $T^1$ in the proof of Theorem 2.7. Then $\mathcal{M}_n(T^1_3)$ corresponds to $m^n$ for $n \geq 2$.

Now, $E_2$ is an object in the category $\mathcal{E}$. By Lemma 3.3 we have a unique cohomology class $\text{obs}(E_2) \in \text{Ext}^2_A(\mathbb{F}_p, m^2/m^3)$ such that $\text{obs}(E_2) = 0$ if and only if there exists a lifting $\phi_2$ of $\phi_1$ to $T^1_3$.

Furthermore,

$$
\text{Ext}^2_A(\mathbb{F}_p, m^2/m^3) \cong \text{Ext}^2_A(\mathbb{F}_p, \mathbb{F}_p) \otimes_{\mathbb{F}_p} m^2/m^3
$$

$$
\cong \text{Hom}_{\mathbb{F}_p}(\text{Ext}^2_A(\mathbb{F}_p, \mathbb{F}_p)^*, m^2/m^3)
$$

$$
\cong \text{Mor}(T^2, m^2/m^3),
$$

and

$$
\text{Mor}(T^2, m^2/m^3) \subseteq \text{Mor}(T^2, T^1_3).
$$

Let $o_2$ be $\text{obs}(E_2)$ as an element in $\text{Mor}(T^2, T^1_3)$. Again, we shall refer to $\text{imobs}(E_2)$ as the obstruction on the first level, which lies in $V_2 := m^2/m^3 = m^2(T^1_3)$, the square of the maximal ideal of $T^1_3$.

In analogy with the group situation we shall write $(\text{im}o_n)$ for the two-sided ideal generated by $o_n(m_T^2)$. 


By Lemma 3.3, \((\text{im} \phi_2)\) is the smallest ideal of \(T^1_3\) containing the elements that need to be reduced to 0 in order to get a lifting of \(\tilde{\phi}_1\). In other words, the universal property of these obstructions implies that \(T^1_3/(\text{im} \phi_2)\) is the largest quotient of \(T^1_3\) making a lifting \(\phi_2\) of \(\tilde{\phi}_1\) possible. So for all other quotients \(T^1_3/I\) and a map \(\tilde{\phi}_2 : A \rightarrow T^1_3/I\) lifting \(\tilde{\phi}_1\), there exists a homomorphism \(\rho : T^1_3/(\text{im} \phi_2) \rightarrow T^1_3/I\) such that the following diagram commutes

\[
\begin{array}{ccc}
T^1_3/(\text{im} \phi_2) & \xrightarrow{\rho} & T^1_3/I \\
\downarrow{\phi_2} & & \downarrow{\phi_2} \\
A & & A.
\end{array}
\]

Also note that \(\pi_A\) induces a commutative diagram of surjective homomorphisms,

\[
\begin{array}{ccc}
T^1_3 & \xrightarrow{\sigma_3} & A_3 \\
\downarrow{\sigma_1} & & \downarrow{\sigma_3} \\
T^1_2 & \xrightarrow{\phi_2} & A,
\end{array}
\]

and therefore a surjective homomorphism \(\theta_3\) in the commutative diagram

\[
\begin{array}{ccc}
T^1_3/(\text{im} \phi_2) & \xrightarrow{\theta_2} & A_3 \\
\downarrow{\sigma_1} & & \downarrow{\sigma_3} \\
T^1 & \xrightarrow{\pi_A} & A.
\end{array}
\]

To show that \(\theta_3\) is an isomorphism, consider the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & m^2(T^1_3) & \xrightarrow{m^2(T^1_3)/(\text{im} \phi_2)} & T^1_3 & \xrightarrow{\phi_2} & T^1_2 & \rightarrow & 0 \\
\downarrow{\tilde{\phi}_1} & & \uparrow{\phi_2} & & \uparrow{\phi_2} & & \downarrow{\phi_2} & & \downarrow{\phi_2} \\
\rightarrow & & T^1_3/(\text{im} \phi_2) & \xrightarrow{\theta_1} & A_3 & \xrightarrow{\phi_2} & A.
\end{array}
\]

Observe that (any) \(\phi_2\) lifting \(\tilde{\phi}_1\) factorizes via \(A_3\) since \(\phi_2\) maps \(m^3_1\) to 0 in \(T^1_3/(\text{im} \phi_2)\). Let \(\psi_3 : A_3 \rightarrow T^1_3/(\text{im} \phi_2)\) be the induced homomorphism.
Since $\phi_2$ is surjective, $\psi_3$ is surjective. Therefore $A_3$ and $T^1_3/(\text{im}\,\phi_2)$ have the same dimension as $\mathbb{F}_p$-vector spaces, thus $\theta_3$ (and $\psi_3$) is an isomorphism induced by $\pi_A$.

We have now seen most of the first non-trivial step in the construction of $o_A$: So far we have

1): a surjective homomorphism $\pi_A: T^1 \rightarrow A$ and

2): a homomorphism $\phi_2: T^2 \rightarrow T^1_3$ inducing the isomorphism $\theta_3$ in the commutative diagram

\[
\begin{array}{ccc}
T^1 & \xrightarrow{\pi_A} & A \\
\downarrow & & \downarrow \theta_3 \\
T^1_3/(\text{im}\,\phi_2) & \cong & A_3.
\end{array}
\]

Our aim is to find a coherent sequence of homomorphisms $\{o_n: T^2 \rightarrow T^n_{n+1}\}_{n \geq 1}$, $o_1$ being trivial, inducing an isomorphism $\theta_{n+1}$ in the commutative diagram

\[
\begin{array}{ccc}
T^1 & \xrightarrow{\pi_A} & A \\
\downarrow & & \downarrow \theta_{n+1} \\
T^1_{n+1}/(\text{im}\,o_n) & \cong & A_{n+1}.
\end{array}
\]

Let $\tilde{\phi}_n: A \rightarrow T^1_{n+1}/(\text{im}\,o_n)$ be the obvious composition of $\sigma_{n+1}$ and $(\theta_{n+1})^{-1}$ for $n \geq 2$.

To see the coherence between the first and second step, let us see what happens when we try to lift $\tilde{\phi}_2$.

We start with the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & (m^2 + I_2)/m^4 \\
\downarrow & & \downarrow \phi_2 \\
T^1_4 & \xrightarrow{\kappa'} & T^1_3/(\text{im}\,\phi_2) & \rightarrow & 0 \\
\downarrow & & \downarrow \phi_2 & & \\
A & & & & \\
\end{array}
\]

where $I_2$ is the preimage of $(\text{im}\,\phi_2)$ in $T^1$. However, this is not a diagram to which we can apply Lemma 3.3. We want $\kappa'$ to be small such that the kernel becomes elementary. Hence we must consider the diagram

\[
E_3: \quad \begin{array}{c}
0 \rightarrow V_3 \rightarrow T^1/J_4 \rightarrow T^1_3/(\text{im}\,\phi_2) \rightarrow 0 \\
\downarrow \phi_2 & \quad \phi_2 & \\
A & & \\
\end{array}
\]
where \( J_4 := (m^4 + m \cdot I_2) \) and \( V_3 := (m^3 + I_2)/(m^4 + m \cdot I_2) \). From Lemma 3.3 we get a unique cohomology class \( \text{obs}(E_3) \in \text{Ext}^2_A(\mathbb{F}_p, V_3) \cong \text{Mor}(T^2, V_3) \subset \text{Mor}(T^2, T^1/J_4) \) which has to be 0 in order to get a lifting of \( \bar{\phi}_2 \). Hence the obstruction \( \text{im obs}(E_3) \) on the second level lies in \( V_3 \).

Let \( o(\bar{\phi}_2) \) be \( \text{obs}(E_3) \) as an element in \( \text{Mor}(T^2, T^1/J_4) \) and consider the following diagram where the four arrows \( \sim \sim \) denote two short-exact sequences

\[
\begin{array}{cccccc}
T^2 & \sim o_2 & V_2 & \sim o_3 & T_4^1 & \sim \pi_2 \& T_2^1 \\
\sim \pi_3 & T_4^1 & \sim \pi_4 & \sim \pi_5 & \sim \pi_6 & \sim \pi_7 & \sim \pi_8 \\
V_3 & \sim T^1/J_4 & \sim (T^1/J_4)/\text{im obs}(E_3) & \sim \phi_3 \\
\sim \phi_4 & T_4^1/(\text{im} \phi_3) & \sim \phi_5 & \sim \phi_6 & \sim \phi_7 & \sim \phi_8 \\
A & \sim B_4 & A & \sim A & \sim A & \sim A \\
\end{array}
\]

Recall that \( \text{im} \phi_2 \) lies in \( m^2/m^3 \) and \( I_2 \) lies in \( m^2 \), so \( m \cdot I_2 \subseteq m^3 \). Therefore there exists a homomorphism \( \mu \) which factors \( \pi_3^4 \). It also factors \( \pi_3^4 \) and induces a homomorphism \( \mu : V_3 \rightarrow V_2 \), which again induces a linear map \( \mu : \text{Ext}^2_A(\mathbb{F}_p, V_3) \rightarrow \text{Ext}^2_A(\mathbb{F}_p, V_2) \).

At the same time we see that \( I_2 \subset T^1 \) is mapped to 0 in \( A_3 \), therefore into \( m^3 \) in \( A \). But then \( \pi_4 \) maps \( m \cdot I_2 \) into \( m^4 \). Therefore the homomorphism \( T^1_4 \rightarrow A_4 \) induced by \( \pi_4 \) factorizes through a surjection \( T^1/J_4 \rightarrow A_4 \).

By functoriality of the obstructions (see Corollary 3.4), it also factorizes via the surjection

\( \theta_4 : (T^1/J_4)/\text{im obs}(E_3) \rightarrow A_4 \).

As above we prove that (any) lifting \( \phi_3 \) of \( \bar{\phi}_2 \) must be surjective and must map \( m^4 \) to 0, inducing a surjection

\( \psi_4 : A_4 \rightarrow (T^1/J_4)/\text{im obs}(E_3) \).
Therefore $\theta_4$ is an isomorphism induced by $\pi_A$.  
Pick an $o_3$ such that $\rho_4 \circ o_3 = o(\tilde{\phi}_2)$. By construction, 
\[
\pi_3^4 \circ o_3 = \mu \circ \rho_4 \circ o_3 = \mu \circ o(\tilde{\phi}_2) = \overline{\mu}(\text{obs}(E_3)),
\]
which as an element of $\text{Mor}(T^2, V_2) \subset \text{Mor}(T^2, T_3)$ is equal to $o_2$, since by functoriality the map $\overline{\mu}$ sends $\text{obs}(E_3)$ to $\text{obs}(E_2)$. Let $I_3$ be the preimage of $(\text{im}o_3)$ in $T^1$. Then $\overline{m} \cdot I_2 \subseteq I_3$, so there is a canonical surjection $T^1/J_4 \twoheadrightarrow T^1_4/(\text{im}o_3)$. Moreover, $(\text{im}o_3)$ maps onto imobs($E_3$) in $V_3$, inducing the canonical isomorphism $B_4$ which identifies $T^1_4/(\text{im}o_3)$ and 
\[
(T^1/J_4)/\text{imobs}(E_3).
\]
Put 
\[
\tilde{\phi}_3 = (\theta_4)^{-1} \circ \sigma_4 : A \longrightarrow T^1_4/(\text{im}o_3).
\]
We have now seen the first non-trivial step of the construction of $o_A$ in detail. The construction continues: Assume, by induction, that we have constructed $o_{n-1}$ such that there exists an isomorphism 
\[
\theta_n : T^1_n/(\text{im}o_{n-1}) \cong A_n
\]
induced by $\pi_A$. Put $\tilde{\phi}_{n-1} = (\theta_n)^{-1} \circ \sigma_n$. Then, for lifting $\tilde{\phi}_{n-1}$, we have the diagram 
\[
\begin{array}{ccc}
0 & \longrightarrow & \overline{m}^n + I_{n-1} / \overline{m}^{n+1} \\
& & \longrightarrow \ T^1_{n+1} \longrightarrow T^1_n/(\text{im}o_{n-1}) \longrightarrow 0 \\
& & \uparrow \tilde{\phi}_{n-1} \\
& & A,
\end{array}
\]
where $I_{n-1}$ is the preimage of $(\text{im}o_{n-1})$ in $T^1$.

Put $J_{n+1} := (\overline{m}^{n+1} + \overline{m} \cdot I_{n-1})$ and consider the small extension, i.e. the diagram 
\[
\begin{array}{ccc}
E_n : 
0 & \longrightarrow & V_n \\
& & \longrightarrow \ T^1/J_{n+1} \longrightarrow T^n_1/(\text{im}o_{n-1}) \longrightarrow 0 \\
& & \uparrow \tilde{\phi}_{n-1} \\
& & A,
\end{array}
\]
where $V_n := (\overline{m}^n + I_{n-1})/J_{n+1}$.

As before, we get the obstruction $\text{imobs}(E_n) \subseteq V_n$ and a lifting 
\[
\phi_n : A \longrightarrow (T^1/J_{n+1})/\text{imobs}(E_n),
\]
inducing an isomorphism $\theta_{n+1} : (T^1/J_{n+1})/\text{imobs}(E_n) \cong A_{n+1}$ commuting with $\pi_A$. 

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Now we need to check that there exists an $o_n$ and a commutative diagram

\[
\begin{array}{c}
T^2 \xrightarrow{o_n} T_{n+1}^1 \xrightarrow{\rho_{n+1}} T_n^1 \\
\downarrow o(\tilde{\phi}_{n-1}) \quad \downarrow \rho_{n+1} \\
T^1/J_{n+1} \xrightarrow{\gamma} T^1/J_n.
\end{array}
\]

with the properties we want, in particular, the canonical isomorphism

\[
B_{n+1} : T_{n+1}^1/(\text{im} o_n) \xrightarrow{\sim} (T^1/J_{n+1})/\text{im obs}(E_n),
\]

identifying the two algebras.

We denote by $I_n$ the preimage of $(\text{im} o_n)$ in $T^1$ under the map $T^1 \longrightarrow T_{n+1}^1$. Observe that, by induction, $I_{n-1} \subset I_{n-2}$, which implies

(3.4) \hspace{1cm} m \cdot I_{n-1} \subset m \cdot I_{n-2}.

Note that (3.4) is an equality modulo $m^{n+1}$.

Recall that we defined $J_n = (m^n + m \cdot I_{n-2})$. From (3.4) it now follows that we have a surjection

(3.5) \hspace{1cm} m^n/m^{n+1} \longrightarrow J_n/J_{n+1}.

Consider the diagram

\[
\begin{array}{c}
T^2 \xrightarrow{o_n} T_{n+1}^1 \xrightarrow{\pi_{n+1}} T_n^1 \\
\downarrow o(\tilde{\phi}_{n-1}) \quad \downarrow \rho_{n+1} \\
T^1/J_{n+1} \xrightarrow{\gamma} T^1/J_n.
\end{array}
\]

We need to show that there exists an $o_n$ such that $\pi_{n+1}^n \circ o_n = o_{n-1}$ and $\rho_{n+1} \circ o_n = o(\tilde{\phi}_{n-1})$. By definition, $T^2$ is the completed tensor algebra of the dual vector space of $\text{Ext}_{\mathbb{P}_G}^2(\mathbb{P}_p, \mathbb{P}_p)$. Fix generators $y_1, \ldots, y_r$ for $T^2$ corresponding to a basis for $\text{Ext}_{\mathbb{P}_G}^2(\mathbb{P}_p, \mathbb{P}_p)^*$. Let $y := y_i \in T^2$ for an $i \in \{1, \ldots, r\}$ and pick $y' \in T_{n+1}^1$ such that $\pi_{n+1}^n(y') = o_{n-1}(y)$. We have

\[
\gamma(-\rho_{n+1}(y') + o(\tilde{\phi}_{n-1})(y)) = -\rho_n(\pi_{n+1}^n(y')) + o(\tilde{\phi}_{n-2})(y) = -\rho_n(o_{n-1}(y)) + o(\tilde{\phi}_{n-2})(y) = 0,
\]

hence $-\rho_{n+1}(y') + o(\tilde{\phi}_{n-1})(y) \in \ker \gamma = J_n/J_{n+1}$. By (3.5), there exists an element $a \in \ker \pi_{n+1}^n = m^n/m^{n+1}$ such that $\rho_{n+1}(a) = -\rho_{n+1}(y') +$
\( o_1(\tilde{\phi}_{n-1})(y) \). Hence if we let \( o_n(y) = y' + a \), we get \( \pi_{n+1}^n(o_n(y)) = o_{n-1}(y) \) and \( \rho_{n+1}(o_n(y)) = o(\tilde{\phi}_{n-1})(y) \), which is what we needed to show.

By construction of \( o_n \), there is a canonical surjection

\[
B_{n+1} : T^1_{n+1}/(\text{im} o_n) \longrightarrow (T^1/J_{n+1})/\text{imobs}(E_n).
\]

By definition, \( I_n \) is the preimage in \( T^1 \) of \( (\text{im} o_n) \), and \( T^1/I_n = T^1_{n+1}/(\text{im} o_n) \), therefore \( \overline{m}^{n+1} \subseteq I_n \). Now, \( J_{n+1} = (\overline{m}^{n+1} + \overline{m} \cdot I_{n-1}) \) is contained in \( I_n \), since \( \overline{m} \cdot I_{n-1} \subseteq I_n \). Therefore we have an obvious commutative diagram

\[
\begin{array}{ccc}
T^1_{n+1} & \xrightarrow{\rho_{n+1}} & T^1/J_{n+1} \\
\downarrow & & \downarrow \\
T^1_{n+1}/(\text{im} o_n) & \xrightarrow{B_{n+1}} & (T^1/J_{n+1})/\text{imobs}(E_n).
\end{array}
\]

By construction, \( \rho_{n+1}((\text{im} o_n)) = \text{imobs}(E_n) \), hence \( B_{n+1} \) is an equality.

As before, we let \( \tilde{\phi}_n = (\theta_{n+1})^{-1} \circ \sigma_{n+1} : A \longrightarrow T^1_{n+1}/(\text{im} o_n) \), completing the induction process.

To finish, the map \( o_A : T^2 \longrightarrow T^1 \) will be the inverse limit of the system \( \{o_n : T^2 \longrightarrow T^1_{n+1} \}_{n \geq 1} \). Moreover, the liftings \( \{\tilde{\phi}_n\}_{n \geq 1} \) induce isomorphisms \( T^1_{n+1}/(\text{im} o_n) \simeq A_{n+1} \) for all \( n \geq 1 \), and so for a complete local \( \mathbb{F}_p \)-algebra \( A \), \( T^1/(\text{im} o_A) \simeq A \).

**4. The main theorem**

In [3], Jennings proves that the \( M \)-series is the same as the modular dimension subgroup-series, i.e.

\[
g \in M_i(G) \text{ if and only if } (g - 1) \in (IG)^i,
\]

where \( IG \) is the augmentation ideal of \( \mathbb{F}_p G \). We know that \( \mathbb{F}_p G \) is local with \( IG \) being the maximal ideal, so we get the following lemma.

**Lemma 4.1.** Let \( G \) be a finite \( p \)-group and let \( \overline{m} \) be the maximal ideal of \( \mathbb{F}_p G \). Then there is an injection

\[
\iota : G_i := G/M_i(G) \longrightarrow \mathbb{F}_p G/\overline{m}^i
\]

inducing an isomorphism \( \beta_i : \mathbb{F}_p[G_i]/\overline{m}^i \longrightarrow \mathbb{F}_p G/\overline{m}^i \). Moreover, the restriction of \( \iota \), \( M_i(G)/M_{i-1}(G) \longrightarrow M_{i+1}/M_i \) is a linear map for all \( i \geq 2 \).

**Proof.** Using Jennings’ result (4.1), let \( g \in G \). Then

\[
\iota(gM_i(G)) = (g - 1) + \overline{m}^i.
\]

Since \( gh - 1 = (g - 1) + (h - 1) + (g - 1)(h - 1) \), the lemma follows. 

\( \Box \)
Our goal is to prove that the obstruction calculus for G (Theorem 2.7) is determined by the obstruction calculus for $\mathbb{F}_p G$ (Theorem 3.7).

Let us start with the following lemma.

**Lemma 4.2.** Let G be a finite p-group and let $A = \mathbb{F}_p G$. Choose an embedding $\psi: G \hookrightarrow A^*$. Then a diagram in $\mathcal{E}$ induces a diagram in $\mathcal{D}$.

**Proof.** Let

\[
    E : \quad 0 \longrightarrow V \longrightarrow B_1 \xrightarrow{\pi} B_2 \longrightarrow 0
\]

be a diagram in $\mathcal{E}$. Then $\psi$ induces a diagram

\[
    D : \quad 1 \longrightarrow V' \longrightarrow \pi^{-1}(\psi'(G)) \xrightarrow{\pi} \psi'(G) \longrightarrow 1
\]

in $\mathcal{D}$, where $V' := V + 1$ and $\psi' = \phi \circ \psi$. \hfill $\square$

**Theorem 4.3.** The isomorphism class of a finite p-group G is determined by its modular group algebra $\mathbb{F}_p G$.

**Proof.** We let $A$ denote $\mathbb{F}_p G$. As before, we use $A_n$, $G_n$ and $T^1_n$ for $A/m^n$, $G/M_n(G)$ and $T^1/M_n(T^1)$ respectively, where $m$ is the maximal ideal of $A$ and $T^1$ is the free pro-p group on a dual basis for $H^1(G, \mathbb{F}_p)$.

We know that $G_2 = T^1_2$ depends only on $H^1(G, \mathbb{F}_p) = \text{Ext}^1_A(\mathbb{F}_p, \mathbb{F}_p)$. Moreover, there is a surjective homomorphism $A \twoheadrightarrow \mathbb{F}_p[G_2]/m^2 = A_2$. By induction, we will show that $G_n$, $n \geq 3$, can be constructed using obstructions depending only on $A$.

Assume that $G_{n-1}$ has been constructed as in the proof of Theorem 2.7. Then the following diagram of exact sequences is given (see diagrams (2.7) and (2.8))

\[
    1 \longrightarrow K'_{n-1} \longrightarrow T^1_{n-1} \longrightarrow G_{n-1} \longrightarrow 1
\]

with $N_n$ and $K_{n-1}$ defined as in (2.9) and (2.10) respectively.
We put \( G'_n := T^1 / N_n \). Then we have \( K_{n-1} \subset \mathcal{M}_{n-1}(G'_n) \) and \( \mathcal{M}_{n-1}(G'_n) \supset \mathcal{M}_n(G'_n) = \{1\} \), \( \mathcal{M}_2(K_{n-1}) = \{1\} \).

Note that \( G_{n-1} = G'_n / \mathcal{M}_{n-1}(G'_n) \). Consider the following diagram which, by induction, only depends on the structure of \( A \)

\[
\begin{array}{ccc}
D_{n-1} : & 1 & \longrightarrow K_{n-1} \longrightarrow G'_n \longrightarrow G_{n-1} \longrightarrow 1 \\
& & \downarrow \phi_{n-1} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{E} : & 0 & \longrightarrow V_{n-1} \longrightarrow \mathbb{F}_p[G'_n] / m^n \longrightarrow \mathbb{F}_p[G_{n-1}] / m^{n-1} \longrightarrow 0 \\
& & \downarrow \kappa_A \\
0 & \longrightarrow V_{n-1} \longrightarrow B'_n \longrightarrow B_{n-1} \longrightarrow 0. \\
\end{array}
\]

The injectivity of \( \phi \) follows from Lemma 4.1. Here \( D_{n-1} \) is an exact sequence of groups, and \( \mathcal{E} \) an exact sequence corresponding to a surjection of \( \mathbb{F}_p \)-algebras. Since \( K_{n-1} \subset \mathcal{M}_{n-1}(G'_n) \) and \( K_{n-1} \subset Z(G'_n) \), and since \( V_{n-1} \) is the ideal generated by \( \{g - 1 | g \in K_{n-1}\} \subset m^n \) in \( \mathbb{F}_p[G'_n] \), we have that \( m \cdot V_{n-1} = 0 \) in \( B'_n \), i.e. \( \kappa_A \) is small.

Clearly, \( \phi \) induces an injective linear map \( \tau : K_{n-1} \longrightarrow V_{n-1} \) where \( \tau(g) = g - 1 \).

Consider now the following object in the category \( \mathcal{D} \)

\[
\begin{array}{ccc}
D_{n-1} : & 1 & \longrightarrow K_{n-1} \longrightarrow G'_n \longrightarrow G_{n-1} \longrightarrow 1 \\
& & \downarrow \phi_{n-1} \\
& & G, \\
\end{array}
\]

where \( \phi_{n-1} = \sigma_{n-1} : G \longrightarrow G_{n-1} \) is the canonical homomorphism. Pick any embedding \( \psi : G' \longrightarrow A^* \) inducing an isomorphism from \( \mathbb{F}_p G \) to \( A \). By Lemma 4.1, \( \phi_{n-1} \) therefore induces an isomorphism

\[
\beta_{n-1} : B_{n-1} \xrightarrow{\cong} A_{n-1}
\]
and the following diagram

$$
E : 0 \longrightarrow V_{n-1} \longrightarrow B'_n \longrightarrow A_{n-1} \longrightarrow 0 \\
E' : 0 \longrightarrow V_{n-1} \longrightarrow B'_n \underset{\kappa_A}{\longrightarrow} B_{n-1} \longrightarrow 0 \\
\mu : \phi_{n-1} \longrightarrow A_{n-1} \underset{\beta_{n-1}}{\longrightarrow} A,
$$

where $\sigma_{n-1}^A = \beta_{n-1} \circ \phi_{n-1}^A : A \longrightarrow A_{n-1}$ is the canonical homomorphism
and $\phi_{n-1}^A$ is the algebra homomorphism induced by $\phi_{n-1}$. By functoriality, the obstructions $\text{im} \sigma_E$ and $\text{im} \sigma_{E'}$ are equal as vector subspaces of $V_{n-1}$.

Now, by Lemma 4.2, $E'$ induces a diagram $D'$ in the category $\mathcal{D}$

$$
D' : 1 \longrightarrow V'_{n-1} \longrightarrow \kappa_A^{-1}(G_{n-1}) \longrightarrow G_{n-1} \longrightarrow 1 \\
\mu : \phi_{n-1} \longrightarrow G,
$$

where $V'_{n-1} := V_{n-1} + 1$, since $(\phi_{n-1}^A \circ \psi)(G) = G_{n-1}$.

Let $\tau' : K_{n-1} \longrightarrow V'_{n-1}$ and $\mu : V_{n-1} \longrightarrow V'_{n-1}$ be the obvious injections.
The obstruction $\text{im} \sigma_{D_{n-1}}$ is now mapped to $\text{im} \sigma_{D'}$, hence

$$
\tau'(\text{im} \sigma_{D_{n-1}}) = \text{im} \sigma_{D'}.
$$

The morphism $\phi_{n-1}^A : A \longrightarrow B_{n-1}$ can be (minimally) lifted to

$$
A \longrightarrow B'_n / \text{im} \sigma_{E'},
$$

therefore the morphism $\phi_{n-1} : G \longrightarrow G_{n-1}$ can be lifted to

$$
G \longrightarrow \kappa_A^{-1}(G_{n-1})/\langle \text{im} \sigma_{E'} + 1 \rangle,
$$

which implies

$$
\text{im} \sigma_{D'} \subseteq \mu(\text{im} \sigma_{E'}).
$$

Now, the morphism $\phi_{n-1} : G \longrightarrow G_{n-1}$ can be lifted to

$$
G \longrightarrow G'_n / \text{im} \sigma_{D_{n-1}},
$$

therefore the morphism $\bar{\phi}_{n-1}^A : A \longrightarrow B_{n-1}$ can be lifted to

$$
A \longrightarrow B'_n / (\tau(\text{im} \sigma_{D_{n-1}})) ,
$$

hence

$$
\text{im} \sigma_{E'} \subseteq \tau(\text{im} \sigma_{D_{n-1}}).
$$
From (4.5), (4.3) and (4.4) it follows that
\[
\mu(\text{im}o_{E'}) \subseteq \mu(\text{im}o_{D'_{n-1}}) = \text{im}o_{D'} \subseteq \mu(\text{im}o_E).
\]
Therefore, \(\tau(\text{im}o_{D'_{n-1}}) = \text{im}o_{E'}\), so \(\text{im}o_{D'_{n-1}}\) is (entirely) determined by the obstruction \(\text{im}o_{E'} = \text{im}o_E\). Since, from Section 2, we know that
\[
G_n \simeq G'_n/\text{im}o_{D'_{n-1}},
\]
the theorem follows.

We note that \(\lim_{n \to \infty} B_n \simeq A\).

As a final remark we note that the last proof also gives us a criterion for when a local (complete) \(\mathbb{F}_p\)-algebra \(A\) with \(\mathbb{F}_p\) as the only simple module is the group algebra \(\mathbb{F}_pG\) for a \(p\)-group \(G\): The necessary and sufficient conditions are that, inductively, the map \(\tau: K_n \rightarrow V_n\) in the diagram (4.2) is injective, and that \(\text{im}o_E \subset \tau(K_n) \subset V_n\).

Further consequences will be treated in a forthcoming paper.

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