REDUCIBLE LOCALLY COHEN-MACAULAY SURFACES

ALESSANDRA DRAGOTTO

1. Introduction

In this paper we study degenerations of surfaces whose general fibre is a smooth projective algebraic surface and whose central fiber is a reduced, connected surface $T$ in $\mathbb{P}^4$. Furthermore $T$ is assumed to be a locally Cohen-Macaulay surfaces whose irreducible components are a finite number of planes and a smooth surface. Here we present a first set of results on the classification by degree of $T$. In particular we want to prove

**Theorem 1.1.** If $T$ a reducible locally Cohen-Macaulay surface in $\mathbb{P}^4$ of degree $d \leq 10$ whose irreducible components are a plane and a smooth rational surface then $T$ is a degeneration of a smooth surface $T'$. In particular

(i) for $d \leq 4$, $T'$ is a rational surface;
(ii) for $5 \leq d \leq 10$, $T'$ is a non special rational surface or a K3 surface.

In this paper we present a proof of the result up to degree 7 and a partial results in degree 8 and 9. The complete proof is still in progress and will appear later. The starting point is the following

**Definition 1.1.** Let $R$ be a Noetherian ring, then $R$ is Cohen-Macaulay if and only if for every ideal $I \subseteq R$ we have $\text{depth}(I, R) = \text{codim} I$. In particular a variety $X$ is arithmetically Cohen-Macaulay if and only if its coordinate ring is a Cohen-Macaulay ring. If $\dim X \geq 1$ this is equivalent to

$$H^i(I_X (n)) = 0, \quad n \in \mathbb{Z}, \quad 0 < i \leq \dim X$$

where $I_X$ is the ideal sheaf of $X$.

If $(R, P)$ is a regular local ring, then any minimal set of generators for $P$ is a regular sequence. Thus the natural inequality $\text{depth}(I, R) \leq \text{codim} I$ becomes an equality also in this case. This implies that if $X$ is a smooth variety, then at every point the local ring $\mathcal{O}_{X, p}$ is a Cohen-Macaulay ring. Clearly the opposite is not true, but we can still say the the Cohen-Macaulay property is local in the following sense

**Definition 1.2.** A variety $X$ is locally Cohen-Macaulay at a point $p$ if and only if the local ring $\mathcal{O}_{X, p}$ is Cohen-Macaulay.

First of all observe that if $X$ is locally Cohen-Macaulay at a point $p$, then $p$ cannot lie on two components of different dimension. Furthermore a result of Hartshorne says that, at a Cohen-Macaulay point, a variety must be locally "connected in codimension 1" i.e. removing a subvariety of codimension 2 or more cannot disconnect it. Algebraically the result reads as follows:
Theorem 1.2 (Hartshorne's Connectedness Theorem). Let $R$ be a local ring and let $I$ and $J$ be proper ideals of $R$ whose radicals are incomparable. If $I \cap J$ is nilpotent, then $\text{depth}(I + J) \geq 1$. In particular, if $R$ is a Cohen-Macaulay ring, then $\text{codim}(I + J) \geq 1$.

For a proof see [[Ei94], Theorem 18.12].

If now $X$ is a connected reducible surface in $\mathbb{P}^4$, the points where two components meet are singular points of $X$. Hartshorne's results shows that if $X$ is locally Cohen-Macaulay then the components must meet in codimension 1 in $X$. In the case we are actually interested we have the following

Corollary 1.3. Let $X = S_1 \cup S_2$ an equidimensional connected reducible surface in $\mathbb{P}^4$ where $S_1$ is a smooth surface and $S_2$ is locally Cohen-Macaulay. If $X$ is locally Cohen-Macaulay, then $S_1 \cap S_2$ is an effective divisor on $S_1$.

Among all the families of smooth surfaces in $\mathbb{P}^4$ those of smooth rational surfaces have been classified up to degree 10. Furthermore all known rational surfaces in $\mathbb{P}^4$ can be represented as blow-ups of the projective plane $\mathbb{P}^2$ and the classification of these surfaces consists of giving explicit open and closed conditions on the configuration of the points in such a way that the linear system $|H|$ embedding the surface is very ample. Given a smooth rational surface $S$ in $\mathbb{P}^4$, we are interested to find all the possible plane curves that satisfy Corollary 1.3. Since any plane curve $C$ on $S$ is given in terms of the linear system $|H|$ it is furthermore possible to give a numerical condition on $C$ for $C$ to satisfy the locally Cohen-Macaulay property. For instance

Lemma 1.4. If $S_1$ and $S_2$ are locally Cohen-Macaulay surfaces in $\mathbb{P}^4$ and $S_1 \cap S_2$ is of codimension 1 in $S_1$ and $S_2$, then $S_1 \cup S_2$ is locally Cohen-Macaulay if and only if $S_1 \cap S_2$ is locally Cohen-Macaulay.

Proof. Let $X := S_1 \cup S_2$ and let $C := S_1 \cap S_2$. Let furthermore $p$ a closed point of $S_1 \cup S_2$. Since $S_1$ and $S_2$ are locally Cohen-Macaulay, the only possible not Cohen-Macaulay points must lie on $C$. Hence we can suppose that $p \in S_1 \cap S_2$. Let $P$ the maximal ideal defining $p$. Then $\text{codim}(P, \mathcal{O}_{X,p}) = 2$ and $\text{codim}(P, \mathcal{O}_{C,p}) = 1$. Then proving the lemma is equivalent to prove that depth$(P, \mathcal{O}_{X,p}) = 2$ if and only if depth$(P, \mathcal{O}_{C,p}) = 1$. On the other hand the depth of a module may be computed from the vanishing behaviour of some Ext -modules. In fact, since the depth of a module is the same if calculated over $A = \mathcal{O}_{\mathbb{P}^4,p}$ and using the Auslander-Buchsbaum formula, we have depth$(P, \mathcal{O}_{X,p}) = 2$ (resp. depth$(P, \mathcal{O}_{C,p}) = 1$) if and only if $\text{pd}_A \mathcal{O}_{X,p} \leq 2$ (resp. $\text{pd}_A \mathcal{O}_{C,p} \leq 3$). On the other hand this is equivalent to $\text{Ext}^i_A(\mathcal{O}_{X,p}, A) = 0$ for all $i \geq 3$ (resp. $\text{Ext}^i_A(\mathcal{O}_{C,p}, A) = 0$ for all $i \geq 4$).

Considering finally the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

the associated long exact sequence of Ext gives

$$\text{Ext}_A^{3+j}(\mathcal{O}_{X,p}, A) \cong \text{Ext}_A^{3+j}(\mathcal{O}_{C,p}, A),$$

for all $j \geq 0$. Here we use the fact that since $p$ is Cohen-Macaulay point for $S_1$ and $S_2$, then $\text{Ext}_A^{3+j}(\mathcal{O}_{S_1,p}, A) = \text{Ext}_A^{3+j}(\mathcal{O}_{S_2,p}, A) = 0$ for all $j \geq 0$. 

$\square$
Proposition 1.5. Let $T$ be a reducible surface in $\mathbb{P}^4$ whose irreducible components are a plane $\Pi$ and a smooth surface $S$. Suppose furthermore that $C$ is the divisorial part of $\Pi \cap S$ and denote by $H - C$ the residual intersection of $S$ with an hyperplane containing $\Pi$. Then $T$ is locally Cohen-Macaulay if and only if $(H - C)^2 = 0$.

Proof. One direction is clear. In fact if $(H - C)^2 = 0$ then any two curves in $|H - C|$ does not meet on $\Pi$. In particular $|H - C|$ is base point free. This implies that $\Pi$ intersects $S$ in codimension one and furthermore $C$ is locally Cohen-Macaulay. For the other direction suppose that $(H - C)^2 > 0$. Then the linear system $|H - C|$ has base points contained in $\Pi$. By assumption $T$ is locally Cohen-Macaulay so, by Hartshorne Connectedness theorem, $S$ and $\Pi$ must intersect in codimension one on $S$ and $\Pi$. As a consequence we have that, if $p$ is a base point for $|H - C|$, then $p \in C$. Let $Z := S \cap \Pi$.

Claim 1. $p$ is an embedded point for $Z$.

If we prove Claim 1 then the proposition follows because if $p$ is an embedded point of $Z$ then $Z$ is not Cohen-Macaulay in $p$, contradicting Lemma 1.4.

Proof of Claim 1. After a change of coordinates we can assume that $p = (0, 0, 0, 0)$ in the open affine subset $k^4 = \text{Spec } k[x, y, z, t]$ and that $\Pi$ has equations $(z = t = 0)$. If $H_{\mu, \lambda} : (\mu z + \lambda t = 0)$ is the generic hyperplane containing $\Pi$, let $S \cap H_{\mu, \lambda} = C \cup D_{\mu, \lambda}$. Then $p \in C$ and $p \notin D_{\mu, \lambda}$ for all $\mu, \lambda$. Working in the local ring $O_{x^4, p}$ and since $p$ is a smooth point for $S$, we can assume that $S$ has local equations $(f_1 = f_2 = 0)$.

Claim 2. The tangent plane $T_p S$ to $S$ at $p$ coincide with $\Pi$.

Proof of Claim 2. Since $p$ is a singular point for $C \cup D_{\mu, \lambda}$ for all $\mu, \lambda$, then the tangent plane $T_p (C \cup D_{\mu, \lambda}) = T_p S$ for all $\mu, \lambda$. On the other hand $T_p (C \cup D_{\mu, \lambda}) \subset H_{\mu, \lambda}$ for all $\mu, \lambda$ and the intersection of all the $H_{\mu, \lambda}$'s is exactly $\Pi$. Then $T_p S = \Pi$. \hfill \Box

From Claim 2 it follows that the local equations of $S$ at $p$ are of the form

\[ z + F(x, y, z, t) = t + G(x, y, z, t) = 0 \]

Furthermore we can write $F$ and $G$ as

\[ F(x, y, z, t) = z^a F_1(x, y, z, t) + t^b F_2(x, y, t) + F_3(x, y) \]
\[ G(x, y, z, t) = z^c G_1(x, y, z, t) + t^d G_2(x, y, t) + G_3(x, y) \]

where $z \notin F_1$, $z \notin G_1$, $t \notin F_2$ and $t \notin G_2$ Restricting the local equations of $S$ to $\Pi$ we get

\[ I(Z) = (z + F, t + G, z, t) = (F_3, G_3) \]

By assumption $C$ is a divisor on $\Pi$ then we can assume that $C$ has local equation $(R(x, y) = 0)$ on $\Pi$.

Since $I(C) \supseteq I(Z)$ we have $F_3 = \bar{F}_3 R$ and $G_3 = \bar{G}_3 R$. Then it is enough to prove that $\bar{F}_3$ and $\bar{G}_3$ are in the maximal ideal $m_p$ of $p$. In order to prove it consider the two hyperplanes $H_z : \{z = 0\}$ and $H_t : \{t = 0\}$ and let $H_z \cap S = C \cup D_z$, $H_t \cap S = C \cup D_t$. Then

\[ I(C \cup D_z) = (z + F, t + G, z) = (t^b F_2 + \bar{F}_3 R, t + t^d G_2 + \bar{G}_3 R) \]
In particular \( A : (t + t^4G_2 + \tilde{G}_3R = 0) \) (resp. \( B : (z + z^4F_1 + \tilde{F}_3R = 0) \)) define a surface in \( H_z \) (resp. \( H_z \)) smooth in \( p \). Modulo \( A \) (resp. \( B \)) the equations \( t^bF_2 + \tilde{F}_3R = 0 \) (resp. \( z^cG_1 + \tilde{G}_3R = 0 \)) must define the union of \( C \) and \( D_z \) (resp. \( C \) and \( D_z \)). Hence the two equations must factorize in the following way

\[
t^bF_2 + \tilde{F}_3R = R(t^b\tilde{F}_2 + \tilde{F}_3)
\]

\[
z^cG_1 + \tilde{G}_3R = R(z^c\tilde{G}_1 + \tilde{G}_3)
\]

where \( t^b\tilde{F}_2 + \tilde{F}_3 \) (resp \( z^c\tilde{G}_1 + \tilde{G}_3 \)) belongs to \( m_p \). Since \( t^b\tilde{F}_2 \) (resp. \( z^c\tilde{G}_1 \)) are in \( m_p \), it follows that \( \tilde{F}_3 \) and \( \tilde{G}_3 \) are in \( m_p \) concluding the proof.

\[\Box\]

The existence of plane curves on a rational surface often imposes extra conditions on the configuration of points in the blow-up so that in some cases proving that a linear system is very ample i.e. (the surface is still embedded in \( \mathbb{P}^4 \) as a smooth surface) turns out to be very subtle. This problem was widely studied by Alexander and we would like to recall some of the very ampleness criteria he introduced and that we will use later on.

**Definition 1.3.** If \( X \) is a projective scheme over an algebraically closed field and \( \mathcal{L} \) is an invertible \( \mathcal{O}_X \)-module, then \( \mathcal{L} \) is very ample on \( X \) if and only if for every affine closed subscheme \( Z_0 \) on \( X \) of co-length 2, the canonical map

\[
(1) \quad H^0(X, \mathcal{L}) \to H^0(Z_0, \mathcal{L}_{|Z_0})
\]

is surjective.

In the case \( X \) is a surface it is possible to reduce the problem of very ampleness of \( \mathcal{L} \) to the study of linear systems on curves. In fact, the following holds

**Lemma 1.6.** Let \( \mathcal{L} \) be an invertible \( \mathcal{O}_X \)-module and \( G \) a linear system on \( X \) of dimension \( \geq 3 \) satisfying \( h^1(\mathcal{L}(-G)) = 0 \). Then \( \mathcal{L} \) is very ample on \( X \) if and only if for every curve \( D \) in \( G \), \( \mathcal{L}_D \) is very ample.

**Proof.** Let \( Z_0 \) a finite closed subscheme of co-length 2. Since \( \dim G \geq 3 \), there exists a curve \( D \) in \( G \) containing \( Z_0 \). From the exact sequence

\[
0 \to \mathcal{L}(-D) \to \mathcal{L} \to \mathcal{L}_D \to 0
\]

and since \( h^1(\mathcal{L}(-D)) = 0 \), it follows that the map

\[
H^0(X, \mathcal{L}) \xrightarrow{j} H^0(D, \mathcal{L}_D)
\]

is surjective. Consider finally the map

\[
H^0(D, \mathcal{L}_D) \xrightarrow{g} H^0(Z_0, \mathcal{L}_{Z_0})
\]

Then \( \mathcal{L} \) is very ample if and only if \( g \circ f \) is surjective i.e., if and only if \( f \) is surjective.

\[\Box\]
Lemma 1.7. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and $D = D_1 + D_2$ a curve on $X$ with irreducible components $D_1$ and $D_2$. Suppose that for every $i > 0$ the following conditions are satisfied

(i) $h^i(\mathcal{L}) = 0$
(ii) $h^i(\mathcal{L}(-D)) = 0$
(iii) $h^j(\mathcal{L}(-D_j)) = 0$ for $j = 1, 2$.

Then the following conditions are sufficient for $\mathcal{L}_D$ to be very ample on $D$:

(a) $\mathcal{L}_{D_j}$ is very ample on $D_j$ for $j = 1, 2$
(b) either $\mathcal{L}_{D_1}(-D_2)$ or $\mathcal{L}_{D_2}(-D_1)$ is base point free

For a proof see [AI92] Lemma 2.7.

If in particular $X$ is a rational surface embedded with the complete linear system $|H|$, and $C$ is a plane curve on $X$, we can use the hyperplanes through $C$ to construct a residual linear system $|D|$ such that $H \equiv C + D$ and $\dim |D| \geq 1$. Since the divisor $H$ correspond to the $\mathcal{O}_X$-module $\mathcal{O}_X(H)$ and $|H|_C$ corresponds to $\mathcal{O}_C(H)$ then the following applies

Theorem 1.8 (Alexander-Bauer). Let $S$ be a smooth projective variety and let $C, D$ be effective divisors with $\dim |D| \geq 1$. Let $H$ be the divisor $H \equiv C + D$. If $|H|_C$ is very ample and for all $D'$ in a 1-dimensional subsystem of $|D|$, $|H|_{D'}$ is very ample, then $|H|$ is very ample.

For a proof see [AI92] Theorem 2.

Once we established a criteria for very ampleness, it remains the problem to determine whether a linear system is very ample on a curve.

Theorem 1.9. A divisor $H$ is very ample on $C$ if for every subcurve $Y$ of $C$ of arithmetic genus $g(Y)$

(i) $H \cdot Y \geq 2p(Y) + 1$ or
(ii) $H \cdot Y \geq 2p(Y)$ and there is no 2-cycle $Z$ of $Y$ such that $I_{Z,Y} \equiv \omega_Y(-H)$.

For a proof see [CaFrHuRe96] Theorem 1.1.

In many cases we will use these criteria to see whether a surface $S$ realized as the blow-up of $\mathbb{P}^2$ in a finite number of points is still embedded in $\mathbb{P}^4$ as a smooth surface when the points are chosen in special position. On the other hand there will be other situations in which we will construct explicidy the "special" surface using linkage. For this purpose we recall the following

Definition 1.4. Given two surfaces $S_1$ and $S_2$ in $\mathbb{P}^4$ with no embedded components, we will say that $S_1$, $S_2$ are linked in a complete intersection $(a, b)$ if there exist hypersurfaces $X_a$, $X_b$ of degree $a$ and $b$ respectively such that $X_a \cap X_b = S_1 \cup S_2$.

The invariants of $S_1$ and $S_2$ are related via the standard sequence of linkage

$$0 \rightarrow \mathcal{O}_{S_1}(K_{S_1}) \rightarrow \mathcal{O}_X(K_X) \rightarrow \mathcal{O}_{S_2}(K_X) \rightarrow 0.$$ 

where $X = X_a \cap X_b$ and $K_X \equiv (a + b - 5)H$ is the canonical divisor of $X$.

In particular we have

$$\chi(\mathcal{O}_{S_j}) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_{S_i}(a + b - 5)).$$
Restricting the sequence to $S_i$ we get a new exact sequence

$$0 \rightarrow \mathcal{O}_{S_i}(K_{S_i}) \rightarrow \mathcal{O}_{S_i}(K_X) \rightarrow \mathcal{O}_{C}(K_X) \rightarrow 0.$$  

where $C = S_1 \cup S_2$. From the standard sequence

$$0 \rightarrow \mathcal{O}_{S_i}(-C) \rightarrow \mathcal{O}_{S_i} \rightarrow \mathcal{O}_{C} \rightarrow 0$$

it follows that $C \equiv (a + b - 5)H - K_{S_i}$ on $S_i$.

The corresponding sequence for linkage of curves in $\mathbb{P}^3$ yields the following relation between the sectional genera

$$(2) \quad \pi(S_i) - \pi(S_j) = \frac{1}{2}(a + b - 5)(d_i - d_j)$$

where $d_i$ and $d_j$ are the degrees of $S_i$ and $S_j$ and they are related by the relation $d_1 + d_2 = ab$.

In order to determine the geometric properties of the surfaces we link, we will use the following

**Proposition 1.10.** If $S_1$ and $S_2$ are linked in a complete intersection, then $S_1$ is locally Cohen-Macaulay if and only if $S_2$ is locally Cohen-Macaulay.

For a proof see [PS74] Proposition 1.3]

**Proposition 1.11.** If $S_i$ is a local complete intersection surface in $\mathbb{P}^4$, which scheme-theoretically is cut out by hypersurfaces of degree $d$, then $S_i$ is linked to a smooth surface $S_2$ in the complete intersection of two hypersurfaces of degree $d$.

For a proof see [PS74] Proposition 1.3]

**Remark 1.12.** In the case in which $S_1$ is locally Cohen-Macaulay the proposition can be modified assuming that at any point where $S_1$ is not locally a complete intersection, the tangent cone at that point is linked to a plane in a complete intersection. For a proof see

For a reference see [[AuRa92] Remark 1.12].

Once we construct $T$ we will see in the next section how to compute its invariants. If $T$ has the same invariants, in particular the same Hilbert polynomial $p_i$ of a smooth surface $S$ then $S$ and $T$ must belong to the same component $U$ of the Hilbert scheme $Hilb^p(\mathbb{P}^4)_{CM}$. Here the subscript denotes the restriction of $Hilb^p(\mathbb{P}^4)$ to equidimensional Cohen-Macaulay subschemes of $\mathbb{P}^4$. If $U$ is irreducible then Theorem 1.1 is finally proven. For a certain class of surfaces the irreducibility problem has been solved. In fact among all the smooth surfaces $S$ in $\mathbb{P}^4$ there are those which are also arithmetically Cohen-Macaulay which can be fully described in terms of the minimal free resolution of the ideal of $S$. In fact this is one of the cases in which the Hilbert-Burch theorems applies and it can be stated in the following way

**Theorem 1.13.** If $S$ is an arithmetically Cohen-Macaulay surface in $\mathbb{P}^4$ and $\mathcal{I}_S$ is the ideal sheaf of $S$ then there exists a short exact sequence
\[
0 \longrightarrow \bigoplus_{j=1}^{2} \mathcal{O}_{\mathbb{P}^4}(-n_{2j}) \xrightarrow{\varphi} \bigoplus_{j=1}^{3} \mathcal{O}_{\mathbb{P}^4}(-n_{1j}) \longrightarrow \mathcal{I}_S \longrightarrow 0.
\]

Furthermore the $2 \times 2$ minors of the matrix $M$ for $\varphi$ define a minimal set of generators for the ideal of $S$.

For a proof see [[Ek94] Theorem 20.15].

The Hilbert scheme of Cohen-Macaulay schemes has been studied by G. Ellingsrud and in the case of surfaces we have

**Theorem 1.14.** Let the $n_{ij}$'s as before and let $U_{(n_{ij})} \subseteq \text{Hilb}^{d}(\mathbb{P}^4)$ the set of points in the Hilbert scheme corresponding to arithmetically Cohen-Macaulay surfaces with fixed Betti numbers $\{n_{ij}\}$, $i = 1, 2$. Then $U_{(n_{ij})}$ is open, smooth and connected. In particular $U_{(n_{ij})}$ is irreducible.

For a proof see [[Ek75] Theorem 2].

As a consequence of Theorem 1.14, we have that in order to prove a surface $T$ is smootheable to an arithmeticaly Cohen-Macaulay surface, it is enough to give explicitly a matrix $M$ as in Theorem 1.13. Furthermore all the smooth Cohen-Macaulay surfaces in $\mathbb{P}^4$ have been classified. In fact

**Theorem 1.15.** Let $S$ be a smooth non general type surface in $\mathbb{P}^4$ of degree $d$, sectional genus $\pi$ and self-intersection of the canonical divisor $K^2$. If $S$ is arithmetically Cohen-Macaulay then $S$ has the following invariants

- a) $d = 8$, $\pi = 7$, $K^2 = 0$ and $S$ is an elliptic surface;
- b) $d = 7$, $\pi = 5$, $K^2 = -1$ and $S$ is a K3 surface;
- c) $d = 7$, $\pi = 6$, $K^2 = 0$ and $S$ is an elliptic surface;
- d) $d = 6$, $\pi = 3$, $K^2 = -1$ and $S$ is a Bordiga surface
- e) $d = 5$, $\pi = 2$, $K^2 = 1$ and $S$ is a Castelnuovo surface;
- f) $d = 4$, $\pi = 1$, $K^2 = 4$ and $S$ is a del Pezzo surface;
- g) $d = 3$, $\pi = 0$, $K^2 = 8$ and $S$ is a rational normal scroll.

For the general case, if $T$ can be constructed by linkage, we can use the theory of linkage of families which has been studied in particular by J. Kleppe. For instance if $U := \{S_t \subseteq \mathbb{P}^4\}$ is a family of surfaces of $\mathbb{P}^4$, $U \subseteq \text{Hilb}^{d}(\mathbb{P}^4)_{CM}$, and each member $S_t$ is contained in a complete intersection $X_t$ of type $(f_1, f_2)$ then the type does not vary with $t \in U$. Let $U' := \{S'_t \subseteq \mathbb{P}^4\} \subseteq \text{Hilb}^{d'}(\mathbb{P}^4)_{CM}$ the family of subschemes obtained by linkage. Then we have the following

**Proposition 1.16.** Let $U$ and $U'$ as above and let $a_1, a_2$ two integers. Suppose that $U$ is locally closed and irreducible and that any element $X$ in $U$ satisfies $h^0(\mathcal{I}_X(f_i)) = a_i$ for $i = 1, 2$. Then the linked family $U'$ is irreducible.

For a proof see [[?] Proposition 3.4].

In the case of families of specializations the results reads as follows

**Proposition 1.17.** Let $T$ and $S$ be two points of $\text{Hilb}^{d}(\mathbb{P}^4)_{CM}$, $T$ a specialization of $S$. Suppose furthermore that there is a complete intersection $X$ of type $(f_1, f_2)$ containing $T$ such that
\[ h^0(I_X(f_1)) = h^0(I_S(f_i)) \quad \text{and} \quad h^0(O_X(f_i - 5)) = h^0(O_S(f_i - 5)). \]

Then there is a complete intersection \( X_1 \) of type \((f_1, f_2)\) containing \( S \) such that if \( S' \), resp. \( T' \), is the subscheme of \( \mathbb{P}^4 \) linked to \( S \) by \( X_1 \), resp. to \( T \) by \( X \), then \( T' \) is a specialization of \( S' \).

For a proof see [??] Proposition 3.7.

2. THE INVARIANTS OF A REDUCIBLE SURFACE

From now on \( S \) will be a smooth rational surface of degree \( d \), sectional genus \( \pi_S \) and self-intersection of the canonical divisor \( K_S^2 \). If furthermore \([H]\) is the linear system embedding \( S \) in \( \mathbb{P}^4 \), we will denote by \( C \) a plane curve on \( S \) and by \( D \) the generic element in the pencil \( H - C \) corresponding to the residual intersection of \( S \) with a generic hyperplane \( H \) containing the plane \( \Pi \) spanned by \( C \).

Let \( T = S \cup \Pi \) then, if \( T \) is a degeneration of a smooth surface \( T' \) of degree \( d + 1 \), \( T \) and \( T' \) have the same invariants.

The sectional genus \( \pi_T \) of \( T \) is by adjunction,

\[(3) \quad \pi_T = \pi_S + \deg(C) - 1\]

If in particular \((H - C)^2 = 0\), then \( T = S \cup \Pi \) is locally Cohen-Macaulay by Proposition 1.5. In this case \( T \) has a dualizing sheaf \( \omega_T^2 \) [cf. [Ha71]] and for which the following holds

**Proposition 2.1.** Let \( T = S_1 \cup S_2 \) be a locally Cohen-Macaulay surface whose irreducible components are two smooth surfaces and let \( C = S_1 \cap S_2 \) a reduced curve. Then

\[(4) \quad \omega_{S_i} \cong \omega_T^2 \otimes O_{S_i}(-C)\]

for \( i = 1, 2 \).

**Proof.** Let \( \bar{\mathbb{P}}^4 \to \mathbb{P}^4 \) the blow-up of \( \mathbb{P}^4 \) along \( C \) and let \( E \) the exceptional divisor. Then

\[ K_{\bar{\mathbb{P}}^4} \cong \pi^* K_{\mathbb{P}^4} + 2E. \]

In fact if \( \omega \) is a meromorphic 4-form on \( \mathbb{P}^4 \), with \( C \) not contained in the zero or polar divisor of \( \omega \), then the divisor of the pullback form \( \pi^* \omega \) on \( \bar{\mathbb{P}}^4 \) is away from \( E \), just the pull back of the divisor of \( (\omega) \). To see how \( \pi^* \omega \) behaves around \( E \), let \( p \) a generic point of \( C \) and \( z_1, \ldots, z_4 \) local coordinates in a neighborhood \( U \) of \( p \) with

\[ C \cap U = (z_2, z_3, z_4 = 0) \]

Let \( \omega = g(z)dz_1 \wedge \cdots \wedge dz_4 \) with \( g \) non zero and holomorphic around \( p \). The inverse image of \( U \) in \( \bar{\mathbb{P}}^4 \) is covered by the complements \( U_2 \), \( U_3 \) and \( U_4 \) of the proper transforms of the coordinates hyperplanes \((z_2 = 0), (z_3 = 0)\) and \((z_4 = 0)\). So for example in \( U_2 \) we have coordinates

\[ z'_1 = z_1, \quad z'_2 = z_2, \quad z'_3 = \frac{z_3}{z_2}, \quad \frac{z_4}{z_2} \]
and
\[ dz'_1 = dz_1, \quad dz'_2 = dz_2, \quad dz'_3 = z'_3 dz_2 + z_2 dz'_3, \quad dz'_4 = z'_4 dz_2 + z_2 dz'_4. \]

It follows that
\[ \pi^* \omega = \pi^* g(z) z_2^2 dz_1 \wedge dz_2 \wedge dz'_3 \wedge dz'_4 \]
vanishes to order 2 along \( E = (z_2) \) and the formula is verified.

Let now \( \tilde{T} \) (resp. \( \tilde{S}_i \)) the strict transforms of \( T \) (resp. \( S_i \)) in \( \mathbb{P}^4 \). Then
\[ \tilde{T} \sim \pi^* T - 2E, \quad \tilde{S}_i \sim \pi^* S_i - E. \]

Let now \( X \) be any smooth hypersurface containing \( T \) and let \( \widetilde{X} \) the proper transform of \( X \) in the blow-up. Then as in the case of \( \mathbb{P}^4 \) we can see that
\[ K_{\widetilde{X}} \cong \pi^* K_X + E. \]

Using the adjunction formula we get
\[ K_{\tilde{T}} = K_{\tilde{X}} + \tilde{T} |_{\tilde{T}} = \pi^*(K_X + T) - E \]
i.e., \( \omega_{\tilde{T}} \equiv \omega_T^0 \otimes \mathcal{O}_T(-C) \). On the other hand we have
\[ K_{\tilde{S}_i} = K_{\tilde{X}} + \tilde{S}_i |_{\tilde{S}_i} = \pi^*(K_X + S_i) \]
i.e., \( \omega_{\tilde{S}_i} \equiv \omega_{\tilde{X}} \otimes \omega_{\tilde{S}_i} \). Finally, since \( \omega_{\tilde{S}_i} \equiv \omega_{\tilde{T}} \otimes \omega_{\tilde{S}_i} \), we get
\[ \omega_{\tilde{S}_i} \equiv \omega_{\tilde{S}_i} \equiv \omega_{\tilde{T}} \otimes \omega_{\tilde{S}_i} \equiv \omega_T^0 \otimes \mathcal{O}_T(-C) \equiv \omega_{\tilde{S}_i} \equiv \omega_T^0 \otimes \mathcal{O}_{S_i}(-C) \]

\[ \square \]

3. Rational surfaces of degree \( d \leq 4 \)

In order to prove Theorem 1.1, we need to investigate all the possible plane curves on the classified smooth rational surfaces. For this purpose we recall the following

**Proposition 3.1.** If \( S \) is a non degenerate smooth surface in \( \mathbb{P}^4 \) and \( S \) is ruled in lines, then \( S \) is either the rational cubic scroll or the quintic elliptic scroll.

For a proof see [Al92].

**Proposition 3.2.** If \( S \) is a non degenerate smooth surface in \( \mathbb{P}^4 \) and \( S \) is ruled in conics then \( S \) has the following invariants

(a) \( d = 4, \pi = 1, K^2 = 4 \) and \( S \) is a Del Pezzo surface;

(b) \( d = 5, \pi = 2, K^2 = 1 \) and \( S \) is a Castelnuovo surface;

(c) \( d = 8, \pi = 5, q = 1 \).

For a proof see [AbDeSa98] and [ElSa98].

**Corollary 3.3.** If \( S \) is a smooth surface in \( \mathbb{P}^4 \) of degree \( d \) and \( S \) is not a scroll, then \( S \) cannot contain plane curves of degree \( d - 1 \). Furthermore if \( S \) is not as in Proposition 3.2 then \( S \) cannot contain plane curves of degree \( d - 2 \).
Proof. If $C$ is a plane curve of degree $d - 1$ (resp. $d - 2$) on $S$ then the residual intersection of $S$ with any hyperplane containing $C$ is a line (resp. a conic). Then the pencil of hyperplanes containing $C$ determines a pencil of lines (resp. conics) on $S$. Hence $S$ is a scroll (resp. as in Proposition 3.2)

\[ \square \]

Remark 3.4. If $s$ is the maximal degree in a minimal set of generators of $I_S$, then $S$ cannot contain plane curves of degree $> s$.

Remark 3.5. If $C$ is a effective divisor of degree $d$ on $S$ then $C$ is a plane curve if and only if its aritmetic genus given by adjunction is equal to the aritmetic genus of a plane curve of degree $d$, i.e.

\[ (d - 1)(d - 2) = C^2 + C.K_S + 2 \]

where $K_S$ is the canonical divisor of $S$.

Let now $S$ be a smooth rational surface of degree $d \leq 4$. From the classification of smooth surfaces in $\mathbb{P}^3$ we have

If $d < 3$, then $S$ is degenerate.

If $d = 3$, then $\pi = 0$ and $S$ is a cubic scroll, cut out by a net of quadrics.

If $d = 4$, then $\pi = 0$ and $S$ is a Veronese Surface projected from $\mathbb{P}^5$ or $\pi = 1$ and $S$ is a Del Pezzo surface, a complete intersection $(2, 2)$.

Now if $S$ is a plane then it is clear that the union of $S$ with any other plane intersecting $S$ along a line is a degeneration of a smooth quadric. Hence the first result reads as follows

Proposition 3.6. If $S$ is a degenerate locally Cohen-Macaulay quadric surface in $\mathbb{P}^3$ then $S$ is the union of two planes meeting along a line.

In the case of the cubic surfaces we have

Proposition 3.7. If $R$ is a reduced degenerate locally Cohen-Macaulay cubic scroll then $R$ is either the union of a smooth quadric and a plane intersecting along a line or $R$ is the union of 3 planes $\Pi_1 \cup \Pi_2 \cup \Pi_3$ where $\Pi_2 \cap \Pi_3 = p_{23}$ is a point and $\Pi_1 \cap \Pi_j = l_{1j}$ is a line.

Proof. First of all a simple computation on the invariants shows that if $R$ is a reduced specialization of a cubic scroll then $S$ is neccesserly one of the described. Since a cubic scroll is aritmetically Cohen-Macaulay whose ideal is genearted by the $2 \times 2$ minors of a $3 \times 3$ matrix $M$ of linear forms, by Theorem 1.13 it is enough to specialize $M$ in a suitable way. In the first case, let $R$ be the union of a smooth quadric and a plane intersecting along a line. After a change of coordinate we can assume that the smooth quadric is given by $(x_0 x_1 - x_2 x_3 = 0)$. Then its very easy to check that the $2 \times 2$ minors of the matrix

\[
\begin{pmatrix}
  x_0 & x_2 & x_3 \\
  x_3 & x_1 & 0
\end{pmatrix}
\]

define the union of the quadric and the plane $(x_1 = x_3 = 0)$.

In the second case let $R$ the union of 3 planes $\Pi_1 \cup \Pi_2 \cup \Pi_3$ where $\Pi_2 \cap \Pi_3 = p_{23}$ is a point and $\Pi_1 \cap \Pi_j = l_{1j}$ is a line. Then it is enogh to degenerate the quadric in two planes meeting along a line to get $R$. The matrix in this case will be
\[
\begin{pmatrix}
    x_4 & x_0 & 0 \\
    0 & x_4 + x_3 & x_1
\end{pmatrix}
\]

\[\square\]

**Proposition 3.8.** If \( T \) is a cubic reducible locally Cohen-Macaulay surface whose irreducible components are a smooth quadric and a plane then \( T \) is a degeneration of a smooth cubic scroll or a degeneration of a smooth cubic in \( \mathbb{P}^3 \).

**Proof.** It is very well known that the only plane curves on a smooth quadric are curves of the type \((1,0), (0,1) (1,1)\) which are lines of the two rulings on the quadric or conics. If \( \Pi \) is a plane intersecting the quadric in a line then we already know, by Proposition 3.7, that \( T \) is a degeneration of a smooth cubic scroll. If \( \Pi \) is a plane intersecting the quadric in a conic then \( \Pi \) must be contained in the \( \mathbb{P}^3 \) of the quadric. Hence \( T \) is degenerate and the equation of \( T \) is simply the product of the equation of \( \Pi \) and the equation of the quadric. Then \( T \) is a degeneration of a cubic surface in \( \mathbb{P}^3 \).

\[\square\]

In degree 4 we have the following result

**Proposition 3.9.** If \( T \) is a reduced locally Cohen-Macaulay degeneration of a Veronese surface in \( \mathbb{P}^4 \) containing a plane, then \( T \) is

- (a) the union of a smooth quadric \( Q \) and two planes \( \Pi_1, \Pi_2 \) such that \( \Pi_1 \cap \Pi_2 \) is a line of the quadric or
- (b) the union of four planes \( \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup \Pi_4 \) where \( \Pi_i \cap \Pi_j = p_{ij} \) is a point for \( i \neq j \neq 4 \) and \( \Pi \) is the plane generated by the \( p_{ij} \)'s.

**Proof.** If \( T \) is a reduced locally Cohen-Macaulay degeneration of a Veronese surface in \( \mathbb{P}^4 \) then it must satisfy \( \pi_T = 0 \). This implies immediately that \( T \) cannot be the union of a plane \( \Pi \) and a smooth cubic surface \( S \). In fact if \( S \) is a degenerate cubic surface, then \( \pi_T = 0 \) iff \( \Pi \) intersects \( S \) in codimension 2 on \( S \), contradicting Hartshorne's theorem. If instead \( S \) is a smooth cubic scroll, then \( \Pi \) must intersect \( S \) along a line.

Now in geometric terms \( S \) is \( \mathbb{P}^2 \) blown-up in 1 point and embedded in \( \mathbb{P}^4 \) with a linear system of the form

\[ |H| = |2L - E_0| \]

Then it is straightforward that any plane curve \( C \) on \( S \) is the strict transform of

- (1) \( C \equiv L \) and \( C \) is a conic or
- (2) \( C \equiv L - E_0 \) and \( C \) is a line or
- (3) \( C \equiv E_0 \) and \( C \) is the exceptional line.

In particular, if \( C \) is a line on \( S \), \((H - C)^2 \neq 0 \) contradicting Proposition 1.5. Then \( S \) must be a locally Cohen-Macaulay degenerate cubic scroll. By Proposition 3.7 \( S \) is completely described. If \( S \) is the union of a smooth quadric \( Q \) and a plane \( \Pi \) intersecting \( Q \) along a line \( L \), then \( \Pi \) must intersect \( S \) in \( L \) by Hartshorne's theorem.

If \( S \) is the union of three planes \( \Pi_1 \cup \Pi_2 \cup \Pi_3 \) where \( \Pi_2 \cap \Pi_3 = p_{23} \) is a point and \( \Pi_1 \cap \Pi_j = l_{ij} \) is a line, then \( \Pi \) must intersect \( \Pi_2 \) (resp. \( \Pi_3 \)) in a point \( p \) (resp. \( q \)) and \( \Pi_1 \) is precisely the plane spanned by \( p_{23}, p, q \). \[\square\]
Proposition 3.10. If $T$ is a locally Cohen-Macaulay reducible surface of degree 4 whose irreducible components are a plane and a smooth cubic surface then $T$ is

(a) a degeneration of a smooth quartic in $\mathbb{P}^3$ or
(b) a degeneration of a smooth Del Pezzo surface which is a complete intersection $(2, 2)$

Proof. Then we can divide the proof in two different cases.

Case 1: If $S$ is a smooth rational normal scroll, we know by Proposition 3.9 that the only plane curves $C$ on $S$ satisfying $(H - C)^2 = 0$ are the conics, strict transforms of lines in $\mathbb{P}^2$. Then $T$ must satisfy $\pi_T = 1$ by formula (3) in Section 1. Furthermore we know that the cubic scroll is cut out by a net of quadrics and since $\Pi$ contains a conic, containing $\Pi$ is one condition on the net of quadrics. Hence $h^0(\mathcal{I}_T(2)) \geq 2$. Finally if $T$ is smoothable to a surface $T'$, then by formula (4) in Section 1, we can compute the self-intersection of the canonical divisor of $T'$ in the following way

$$(K'_T s)^2 = (K_S + C)^2 = (-3L + E_0 + L)^2(-2L + E_0)^2 = 3$$

Hence $K'_T s^2 = 4$ and $T'$ has the same invariants of a Del Pezzo surface which is a complete intersection $(2, 2)$. In order to construct $T$, let $Q_1, Q_2$ two quadrics containing $S$, then $S$ is linked to a plane in the complete intersection $Q_1 \cap Q_2$. Since the sectional genus of a complete intersection $(2, 2)$ is 1, then $\Pi$ intersects $S$ in a conic. Hence $T$ is a complete intersection $(2, 2)$ by construction.

Case 2: If $S$ is a cubic surface in $\mathbb{P}^3$ then $\Pi$ intersects $S$ in a line or $\Pi$ is contained in the $\mathbb{P}^3$ of $S$. In the first case $T$ has $\pi_T = 1$ and as in Case 1 it is a complete intersection of two quadrics hypersurfaces containing $S$. In the second case $T$ is trivially a degeneration of a smooth quartic surface in $\mathbb{P}^3$.

\[\square\]

Proposition 3.11. If $T$ is a locally Cohen-Macaulay reducible surface of degree 5 whose irreducible components are a plane and a smooth rational surface of degree 4 then $T$ is a degeneration of a smooth Castelnuovo surface surface realized as $\mathbb{P}^2$ blown-up in 8 points.

Proof. From the classification of smooth surfaces in $\mathbb{P}^4$ it follows that if $S$ is a smooth quartic surface, then $S$ is a projected Veronese surface or $S$ is a Del Pezzo Surface. A simple geometric argument shows that, if $S$ is a Veronese surface, then $S$ cannot be an irreducible component of $T$. In fact, since the linear system $|H|$ embedding the Veronese surface in $\mathbb{P}^4$ is contained in the linear system $|2L|$, then $S$ contain only conics as plane curves. But if $C$ is conic on the Veronese surface and $\Pi$ is the plane spanned by $C$, then any hyperplane containing $C$ will cut the surface in an extra conic $C'$. Hence the planes of $C$ and $C'$ must intersect in a line $L$. Since $C.C' = 1$ then $\Pi$ will intersect the surface outside the $C$, contradicting Hartshorne’s theorem.

Let then $S$ be a Del Pezzo surface of degree 4, sectional genus $\pi = 1$ and self intersection of the canonical divisor $K^2 = 4$. Then $S$ is a complete intersection
(2, 2) and geometric terms it is $\mathbb{P}^2$ blown-up in 5 points and embedded by the linear system

$$|H| = |3L - \sum_{i=1}^{5} E_i|.$$ 

By remark 3.4 $S$ can contains only lines and conics, and by Proposition 3.2 it is ruled in conics. In terms of linear sistem, if $C$ is a plane curve on $S$ then $C$ is a strict transform of

$$C \equiv nL - \sum_{i=1}^{5} b_i E_i.$$ 

with $n < 3$. Since $0 < H.C \leq 2$, then $(H - C)^2 = 0$ if and only if $C^2 = 0$ or $-2$. Then it is very easy to check that the only plane curves satisfying these conditions are conics and they are

(a) $C \equiv L - E_i$
(b) $C \equiv 2L - \sum_{i \in \Delta} E_i$, for $|\Delta| = 4$.

Let $C$ one of these conics and let $\Pi$ the plane of $C$. Then $T = S \cup \Pi$ is a degree 5 surface with sectional genus $\pi_T = 2$, by formula (3) in Section 1. Furthermore if $T$ is smoothable to $T'$, by formula (4) we have for example

$$(K_T^l|S)^2 = (K_S + C)^2 = (-3L + \sum_{i=1}^{5} E_i + L - E_i)^2 = 0$$

$$(K_T^l|\Pi)^2 = (K_\Pi + C)^2 = (-3L + 2L)^2 = (-L)^2 = 1.$$ 

Hence $K_T^2 = 1$ and $T'$ has the same invariants of a Castelnuovo surface wich is aritmetically Cohen-Macaulay. In particular its ideal is generated by the $2 \times 2$ minors of a $2 \times 3$ matrix of the form

$$\begin{pmatrix}
L_1 & L_2 & Q_1 \\
L_3 & L_4 & Q_2
\end{pmatrix}$$

where $L_i$ are linear forms and $Q_i$ are quadric forms. As in the proof of Proposition 3.7 it is enough to specialize this matrix to prove that $T$ is a degeneration of a smooth Castelnuovo surface. Let $R_1, R_2$ the quadrics in the ideal of $S$ then it is enough to choose the $L_i$’s and the $Q_i$’s such that $L_1L_4 - L_2L_3 = R_1, Q_2 = R_2$ and $Q_1 = 0$.

\[\square\]

4. The non special rational surface of degree 5 in $\mathbb{P}^4$

Let $S$ be a smooth rational surface of degree 5, sectional genus $\pi = 2$, self-intersection of the canonical divisor $K^2 = 1$ and speciality $h = h^1(O_S(1)) = 0$. In geometric terms it is $\mathbb{P}^2$ blow-up in 8 points embedded by a linear system of the form

$$|H| = |4L - 2E_0 - \sum_{i=1}^{7} E_i|$$

and for a general position of the points $p_i$, $|H|$ embeds $S$ into $\mathbb{P}^4$. More precisely
Theorem 4.1. The linear system $|H| = |4L - 2E_0 \sum_{i=1}^{7} E_i|$ embeds the surface $S = \widetilde{\mathbb{P}^2(p_0, \ldots, p_7)}$ into $\mathbb{P}^4$ if and only if

1. no $p_i$ is infinitely near,
2. $|L - E_0 - \sum_{i \in \Delta} E_i| = 0$ for $|\Delta| \geq 2$,
3. $|L - \sum_{i \in \Delta} E_i| = 0$ for $|\Delta| = 4$,
4. $|2L - E_0 - \sum_{i \in \Delta} E_i| = 0$ for $|\Delta| \geq 6$.

Proof. We shall first show that the conditions stated are necessary.

Since $H.(E_i - E_j) = 0$ and $H.A > 0$ for every divisor $A$ then conditions (1) to (4) are satisfied.

Now assume that conditions (1) to (5) holds. Let $C \equiv 2L - E_0 - \sum_{i \in \Delta} E_i$ for $|\Delta| = 4$ and $D \equiv H - C = 2L - E_0 - \sum_{i \in \Delta'} E_i$ where $|\Delta'| = 2$. To prove that $|H|$ is very ample we apply theorem 1.3 to the pencil with fixed component $C + |D| \subset |H|$.

**step 1:** $h^0(O_S(H)) = 5$ and for all $D \subset |D|$ the canonical maps $H^0(O_S(H)) \to H^0(O_C(H))$ and $H^0(O_S(H)) \to H^0(O_D(H))$ are surjective.

Let $K \equiv -3L + \sum_{i=0}^{7} E_i$ be the canonical divisor of $S$. Since $D$ moves in a pencil, we have $h^0(O_S(D)) = 2$ and $h^2(O_S(D)) = h^0(O_S(K - D)) = 0$. Hence by Riemann-Roch $h^1(O_S(D)) = 0$ since $\chi(O_S(D)) = 2$.

Consider now the exact sequence

$$0 \to O_S(C - E_k - E_h) \to O_S(C) \to O_{E_k + E_h}(C) = O_{E_k + E_h} \to 0.$$  

By condition (4) $h^0(O_S(C - E_k - E_h)) = 0$ and clearly $h^2(O_S(C - E_k - E_h)) = h^0(O_S(K - C + E_k + E_h)) = 0$. Hence by Riemann-Roch $h^1(O_S(C - E_k - E_h)) = 0$.

This shows that $h^1(O_S(C)) = 0$. In particular $h^0(O_S(C)) = 1$, i.e. $C$ is uniquely determined.

Finally consider the exact sequence

$$0 \to O_S(D) \to O_S(H) \to O_C(H) \to 0.$$  

Since $C$ is a conic, by Riemann-Roch we have $\chi(O_C(H)) = 3$. Furthermore $h^1(O_C(H)) = h^0(O_C(K - H)) = 0$.

This shows that $|H|$ maps $S$ to $\mathbb{P}^4$ and $H$ restricts to complete linear systems on $C$ and $D$.

**step 2:** For every subcurve $D' \subset |D|$ we have

$$H.D' \geq 2p(D') + 1.$$  

Suppose $D'$ is not on of the exceptional divisors for which (10) is fulfiled, then let $D' \equiv aL - \sum_{i=0}^{7} b_i E_i$. For $1 \leq a \leq 2$ then $b_i$ and $c_i$ can be just 0 and 1 and $p(D') = 0$. Hence (10) holds if $H.D' > 0$, i.e conditions from(2) to (5) are satisfied.

**step 3:** For every subcurve $C' \subset C$ we have $H.C' \geq 2p(C') + 1$. In fact the argument in step 2 applies also in this case.

**Theorem 4.2.** If $T$ is a reducible locally Cohen-Macaulay surface in $\mathbb{P}^4$ of degree 6 whose irreducible components are a plane and a smooth rational surface of degree 5 then $T$ is
(i) a degeneration of a smooth rational surface with $\pi_T = 3$, $K_T^2 = -1$ and realized as $\mathbb{P}^2$ blown-up in 10 points or

(ii) a degeneration of a smooth K3 surface which is a complete intersection

(2, 3)

Proof. We shall prove the theorem in different steps.

step 1: Let $C$ be a curve of degree $d$ and arithetic genus $p_a(C)$ contained in $S$. Then $C$ is the strict transform of a plane curve of the form

$$C \equiv nL - \sum_{i=0}^{7} b_i E_i$$

with $n \leq 3$. Since the ideal of $S$ is generated by 1 quadric and 7 cubics then, by Remark 3.4, $S$ cannot contain plain curves of degree $d \geq 4$. This observations together with Remark 3.5 says that $C$ is a plane curve if the following conditions are satisfied

$$\begin{cases} 4n - 2b_0 - \sum_{i=1}^{7} b_i = d \\ n^2 - \sum_{i=0}^{7} b_i^2 - 3n + \sum_{i=0}^{7} b_i + 2 = (d-1)(d-2), \quad d \leq 3, \; n \leq 3 \end{cases}$$

The idea is to consider systematically all the possible values of $d$ and $n$ and solve the correspondent systems of equations in terms of $b_1, \ldots, b_{10}$. So for example for $d = 1, \; n = 3$ we have

$$\begin{cases} 2b_0 + \sum_{i=1}^{7} b_i = 11 \\ \sum_{i=0}^{7} b_i^2 = 13 - b_0 \end{cases}$$

Since an irreducible plane cubic has at most one double point, then the only irreducible plane cubics which are mapped into lines are of the form $3L - 2E_0 - \sum_{i=1}^{7} E_i$. Proceeding in this way we get the following classification:

If $C$ is a line then $C$ is the strict transform of

1. $L - \sum_{i \in \Delta} E_i$, $|\Delta| = 3$
2. $L - E_0 - E_i$
3. $2L - \sum_{i=1}^{7} E_i$
4. $2L - E_0 - \sum_{i \in \Delta} E_i$, $|\Delta| = 5$
5. $3L - 2E_0 - \sum_{i=1}^{7} E_i$

If $C$ is a conic then $C$ is the strict transform of

1. $L - E_i - E_j$
2. $L - E_0$
3. $2L - \sum_{j \in \Delta} E_i$, $|\Delta| = 6$
4. $2L - E_0 - \sum_{i \in \Delta} E_i$, $|\Delta| = 4$
5. $3L - 2E_0 - \sum_{i \in \Delta, j \neq i} E_j$, $|\Delta| = 6$

If $C$ is a cubic then $C$ is the strict transform of

1. $3L - \sum_{i=0}^{7} E_i$
Excluding the cases for which \((H - C)^2 > 0\), here are finally described the admissible linear systems:

(I) \(C \equiv L - \sum_{i \in \Delta} E_i\), \(|\Delta| = 2\)

(II) \(C \equiv 2L - E_0 - \sum_{i \in \Delta} E_i\), \(|\Delta| = 6\)

(III) \(C \equiv 3L - 2E_0 - \sum_{i \in \Delta} E_i\), \(|\Delta| = 6\)

(IV) \(C \equiv 3L - \sum_{i = 0}^{7} E_i\).

(V) \(C \equiv 2L - \sum_{i = 1}^{7} E_i\).

**Step 2:** If the linear system \(2L - \sum_{i = 1}^{7} E_i\) is not empty then by Theorem 4.1, we know that \(H\) is very ample. Furthermore any plane containing \(C\) contains also a conic of the ruling \(\mathfrak{g} S\). In fact if \(C \equiv 2L - \sum_{i = 1}^{7} E_i\), then the residual curve is \(H - C \equiv 2L - 2E_0\) which is the union of two conics of the ruling. Since \(C\) must intersect both conics, the planes of the conics intersect exactly along the line \(C\). Hence there is a 3-dimensional family of pairs of conics which described the only quadric hypersurface containing \(S\). In this way we proved that \(S\) contains only conics or plane cubics.

If \(C\) is a conic and \(\Pi\) is the plane of \(C\). Then \(T = S \cup \Pi\) is a degree 6 surface with sectional genus \(\pi_T = 3\), by formula (3). Furthermore if \(T\) is smoothable to \(T'\), by formula (4) we have for example

\[
(K_T^2|s)^2 = (K_S + C)^2 = (-3L + \sum_{i=0}^{7} E_i + L - E_i - E_j)^2 = -2
\]

\[
(K_T^2|n)^2 = (K_{\Pi} + C)^2 = (-3L + 2L)^2 = (-L)^2 = 1.
\]

Hence \(K_T^2 = -1\) and \(T'\) has the same invariants of a Bordiga surface which is arithmetically Cohen-Macaulay. In particular its ideal is generated by 4 cubics that correspond to the \(3 \times 3\) minors of a \(3 \times 4\) matrix of linear forms. Furthermore any Bordiga surface is linked to a cubic scroll in a complete intersection (3,3). On the other hand we now that the ideal of \(S\) is generated by one quadric \(Q\) and two cubics \(X_1, X_2\). Then taking any hyperplane \(H\) containing \(\Pi\), we can link \(T\) to a cubic surface \(R\) in a complete intersection (3,3). Here one of the cubics is \(X_i\) and the other one is \(H \cup Q\). In particular \(R\) is reducible and is the union of a quadric surface and a plane intersecting the quadric along a line. By Proposition 3.7 \(R\) is a degeneration of a smooth cubic scroll, hence \(T\) is a degeneration of a Bordiga surface by Proposition 1.16.

Finally if \(C\) is a plane cubic curve on \(S\) and \(\Pi\) is the plane of \(C\), then \(T = S \cup \Pi\) is a degree 6 surface with sectional genus \(\pi_T = 4\), by (3). Furthermore if \(T\) is smoothable to \(T'\), by (4) we have for example

\[
(K_T^2|s)^2 = (K_S + C)^2 = (-3L + \sum_{i=0}^{7} E_i + 3L - \sum_{i=0}^{7} E_i)^2 =
\]

\[
(K_T^2|n)^2 = (K_{\Pi} + C)^2 = (-3L + 3L)^2 = (-L)^2 = 0.
\]

Hence \(K_T^2 = 0\) and \(T'\) has the same invariants of a complete intersection of type (2,3) \(K^3\) surface. On the other hand any Castelnuovo surface \(S\) is linked to a plane in a complete intersection (3,3) and the plane must intersects \(S\) precisely along a plane cubic. This concludes the proof.

\[\square\]
5. The non special rational surface of degree 6 in $\mathbb{P}^4$

In this section we’ll classified families of reducible locally Cohen-Macaulay surfaces of degree 7 whose irreducible components are a plane and a Bordiga surface of degree 6. In particular we will prove the following

**Theorem 5.1.** If $T$ is a reducible locally Cohen-Macaulay surface in $\mathbb{P}^4$ of degree 7 whose irreducible components are a plane and a smooth rational surface of degree 6 then $T$ is

(i) a degeneration of a smooth rational surface with $\pi_T = 4$, $K_T^2 = -2$ and realized as $\mathbb{P}^2$ blown-up in 11 points or

(ii) a degeneration of a smooth $K3$ surface with $\pi_T = 5$, $K_T^2 = -1$ and realized as the blow-up of its minimal model in one point.

From the classification of smooth rational surfaces there is only one non special smooth rational surface of degree 6. In geometric terms it is $\mathbb{P}^2$ blown-up in 10 points embedded by the linear system

$$|H| = |4L - \sum_{i=1}^{10} E_i|$$

For a general position of the $p_i$’s the linear system $|H|$ clearly embeds $S = \tilde{\mathbb{P}}^2(p_1, \ldots, p_{10})$ into $\mathbb{P}^4$. Here we want to apply the decomposition method to this surface and give explicit open conditions for the position of the points for $|H|$ to be very ample.

**Theorem 5.2.** The linear system $|H| = |4L - \sum_{i=1}^{10} E_i|$ embeds the surface $S = \tilde{\mathbb{P}}^2(p_1, \ldots, p_{10})$ into $\mathbb{P}^4$ if and only if

1. no $p_i$ is infinitely near,
2. $|L - \sum_{i \in \Delta} E_i| = \emptyset$ for $|\Delta| \geq 4$,
3. $|2L - \sum_{i \in \Delta} E_i| = \emptyset$ for $|\Delta| \geq 8$,
4. $|3L - \sum_{i=1}^{10} E_i| = \emptyset$,
5. $|3L - 2E_i - \sum_{j \neq i} E_i| = \emptyset$.

**Proof.** We shall first show that the conditions stated are necessary.

Since $H.(E_i - E_j) = 0$ then (1) follows. The very ampleness of $|H|$ implies also (2) and (3) because $|H|$ has only effective divisors. Finally if the linear system $|3L - \sum_{i=1}^{10} E_i|$ contains some element $A$, then $H.A = 2$ and $p(A) = 1$ which contradicts the very ampleness of $H$.

Now assume that conditions (1) to (4) holds. Let $C \equiv 3L - \sum_{i \in \Delta} E_i$ for $|\Delta| = 9$ and $D \equiv H - C = L - E_j$ where $j \notin \Delta$. To prove that $|H|$ is very ample we apply Theorem 1.3 to the pencil with fixed component $C + |D| \subseteq |H|$.

**step 1:** $h^0(O_S(H)) = 5$ and for all $D \subseteq |D|$ the canonical maps $H^0(O_S(H)) \to H^0(O_C(H))$ and $H^0(O_S(H)) \to H^0(O_D(H))$ are surjective.

In fact let $K \equiv -3L + \sum_{i=1}^{10} E_i$ be the canonical divisor of $S$. Since $D$ moves in a pencil, we have $h^0(O_S(D)) = 2$ and $h^2(O_S(D)) = h^0(O_S(K - D)) = 0$. Hence by Riemann-Roch $h^1(O_S(C)) = 0$ since $\chi(O_S(D)) = 2$.

Consider now $-K \equiv 3L - \sum_{i=1}^{10} E_i \equiv C - E_j$. By condition (4) $h^0(O_S(-K)) = 0$ and clearly $h^2(O_S(-K)) = h^0(O_S(K)) = 0$. Hence by Riemann-Roch $h^1(O_S(-K)) = 0$. 
Consider now the exact sequence

\[ 0 \rightarrow \mathcal{O}_S(-K) \rightarrow \mathcal{O}_S(C) \rightarrow \mathcal{O}_{E_i}(C) = \mathcal{O}_{E_i} \rightarrow 0. \]

This shows that \( h^1(\mathcal{O}_S(C)) = 0 \). In particular \( h^0(\mathcal{O}_S(C)) = 1 \), i.e. \( C \) is uniquely determined.

Finally consider the exact sequence

\[ 0 \rightarrow \mathcal{O}_S(D) \rightarrow \mathcal{O}_S(H) \rightarrow \mathcal{O}_C(H) \rightarrow 0. \]

Since \( C \) is a plane cubic curve, by Riemann-Roch we have \( \chi(\mathcal{O}_C(H)) = 3 \). Furthermore \( K_C \) is trivial so \( h^1(\mathcal{O}_C(H)) = h^0(\mathcal{O}_C(-H)) \). By condition (4) and (5) \( C \) cannot contain exceptional divisors. As a plane curve it can decompose in a conic and a line or in three lines. Conditions (2) and (3) guaranty that \( H \) has positive degree on every component therefore \( h^0(\mathcal{O}_C(H)) = 0 \). It follows that \( h^0(\mathcal{O}_C(H)) = 3 \) and \( h^0(\mathcal{O}_S(H)) = 5 \).

This shows that \( |H| \) maps \( S \) to \( \mathbb{P}^4 \) and \( H \) restricts to complete linear systems on \( C \) and \( D' \) for all \( D' \in |D| \).

**step 2:** For every subcurve \( C' \subset C \) we have

\[ (8) \quad H.C' \geq 2p(C') + 1 \]

First of all we can assume \( C' \) is not one of the exceptional divisors since \( H.E_i = 1 \) and (8) holds. Furthermore it is enough to prove (8) for all curves such that \( p(C') \geq 0 \). In fact if \( p(C') < 0 \) then \( C' \) is reducible. Then for every irreducible component of \( C' \) (8) holds, so \( H.C' > 0 \) and (8) holds also for \( C' \). Let \( C' \equiv aL - \sum_{i=1}^{10} b_i E_i \) for \( 1 \leq a \leq 3 \). Then \( b_i \in \{0, 1\} \) for all \( i \) and condition (8) says immediately that \( a < 3 \).

For \( a = 1 \) or 2 conditions (2) and (3) give the following possibilities:
- \( C' \equiv 2L - \sum_{i \in \Delta} E_i \) for \( |\Delta| = 6 \)
- \( C' \equiv 2L - \sum_{i \in \Delta} E_i \) for \( |\Delta| = 7 \)
- \( C' \equiv L - \sum_{i \in \Delta} E_i \) for \( |\Delta| = 3 \)

In each of these cases property (8) is fulfilled.

**step 3:** For every subcurve \( D' \in |D| \) we have \( H.D' \geq 2p(D') + 1 \). In fact the only possible subcurves of \( D \) are either exceptional divisors or lines passing through at most 2 extra points. Therefore (8) is always fulfilled. This conclude the proof of theorem 2.2

**Proof of theorem 5.1.** Let \( S \) a Bordiga surface of degree 6, sectional genus \( \pi = 3 \), self-intersection of the canonical divisor \( K^2 = -1 \) and embedded by the linear system \( H \equiv 4L - \sum_{i=1}^{10} E_i \). Let furthermore \( K_S = -3L + \sum_{i=1}^{10} E_i \) be the canonical divisor of \( S \). We shall prove the theorem in different steps.

**step 1:** Let \( C \) be a curve of degree \( d \) and aritmetic genus \( p_a(C) \) contained in \( S \). Then \( C \) is the strict transform of a plane curve of the form

\[ C \equiv nL - \sum_{i=1}^{10} b_i E_i \]
In particular, if \( C \) is a plane curve it must be a component of a pencil of hyperplane sections of \( S \) cut out by any hyperplane containing the plane of \( C \). Hence we can assume \( n \leq 3 \). Since the ideal of \( S \) is generated by four cubics then, by Remark 3.4, \( S \) cannot contain plain curves of degree \( d \geq 4 \). This observation together with Remark 3.5 says that \( C \) is a plane curve if the following conditions are satisfied

\[
\begin{align*}
4n - \sum_{i=1}^{10} b_i &= d \\
n^2 - \sum_{i=1}^{10} b_i^2 &= 3n + \sum_{i=1}^{10} b_i + 2 = (d - 1)(d - 2), & d \leq 3, \ n \leq 3
\end{align*}
\]

The idea is to consider systematically all the possible values of \( d \) and \( n \) and solve the correspondent systems of equations in terms of \( b_1, \ldots, b_{10} \). So for example for \( d = 1, \ n = 3 \) we have

\[
\begin{align*}
\sum_{i=1}^{10} b_i &= 11 \\
\sum_{i=1}^{10} b_i^2 &= 13
\end{align*}
\]

Since an irreducible plane cubic has at most one double point, then the only irreducible plane cubics which are mapped into lines are of the form \( 3L - 2E_i - \sum_{j \neq i} E_j \).

Proceeding in this way we get the following classification:

If \( C \) is a line then \( C \) is the strict transform of

\[
\begin{align*}
(1) & \quad L - \sum_{i \in \Delta} E_i, & |\Delta| &= 3 \\
(2) & \quad 2L - \sum_{i \in \Delta} E_i, & |\Delta| &= 7 \\
(3) & \quad 3L - 2E_i - \sum_{j \neq i} E_j
\end{align*}
\]

If \( C \) is a conic then \( C \) is the strict transform of

\[
\begin{align*}
(1) & \quad L - E_i - E_j, & |\Delta| &= 6 \\
(2) & \quad 2L - \sum_{i \in \Delta} E_i, & |\Delta| &= 9 \\
(3) & \quad 3L - 2E_i - \sum_{j \in \Delta, j \neq i} E_j, & |\Delta| &= 9
\end{align*}
\]

If \( C \) is a cubic then \( C \) is the strict transform of

\[
\begin{align*}
(1) & \quad 3L - \sum_{i \in \Delta} E_i, & |\Delta| &= 9
\end{align*}
\]

If \( H \) is any hyperplane passing through \( C \) and \( D \equiv H - C \) is the residual intersection curve, then by Proposition 1.5 we can exclude all the linear systems for which \( D^2 \neq 0 \). Here are finally described the admissible linear systems:

\[
\begin{align*}
(\text{I}) & \quad C \equiv 2L - \sum_{i \in \Delta} E_i, & |\Delta| &= 6 & \text{and} & \quad D \equiv 2L - \sum_{i \in \Delta} E_i, & |\Delta'| &= 4 \\
(\text{II}) & \quad C \equiv 3L - \sum_{i \in \Delta} E_i, & |\Delta| &= 9 & \text{and} & \quad D \equiv L - \sum_{i \in \Delta} E_i, & |\Delta'| &= 1
\end{align*}
\]

**step 2:** Since there always exists a plane cubic passing through 9 points, we only need to ensure that when 6 points are chosen to be on a conic, then the general Borroga surface \( S \) will be smooth. But if the linear system \( |2L - \sum_{i \in \Delta} E_i| \) for \( |\Delta| = 6 \) is non empty, then \( H \) is very ample by Theorem 5.2. Hence \( S \) is smooth.

**step 3:** Let now \( \Pi \) be the plane of \( C \) on \( S \) and let \( T = S \cup \Pi \).

**Case 1:** If \( C \) is a conic, then by formula (3) in Section 1 we have \( \pi_T = 4 \). Furthermore we know that the ideal of \( S \) is generated by 4 cubics. Let \( X \supset S \) a cubic hypersurface, then \( \Pi \) must intersect \( X \) along a cubic curve. Since \( \Pi \) contains
already a conic, then $\Pi \subset X$ iff $\Pi$ contains a line and an extra point of $X$. This imposes 3 conditions on the number of cubics so we expect that $T$ is contained in only one cubic. Furthermore if $T$ is smoothable to a surface $T'$ then by formula (4) in Section 1

\[(K'_T|S)^2 = (K_S + C)^2 = (-L + \sum_{i \in \Delta} E_i)^2 = -3 |\Delta| = 4\]

\[(K'_T|\Pi)^2 = (K_{\Pi} + C)^2 = (-3L + 2L)^2 = 1.\]

It follows that $K'_T = -3 + 1 = -2$. Then $T$ has the same invariants of a smooth rational surface of degree 7 with $\pi = 4$ and $K^2 = -2$ realized as the blow-up of $\mathbb{P}^2$ in 11 points.

**Case 2** If $C$ is a cubic on $S$ then $\pi_T = 5$. Furthermore containing a plane imposes 1 condition on the number of cubic hypersurfaces containing $S$ so we expect that $T$ is contained in 3 cubics. If finally $T$ is a degeneration of a smooth surface $T'$ then

\[(K'_T|S)^2 = (E_3)^2 = -1\]

\[(K'_T|\Pi)^2 = (-3L + 3L)^2 = 0.\]

It follows that $K'_T = -1 + 0 = -1$. Thus we can conclude that $T'$ has the same invariants a smooth $K3$ surface of degree 7 with $\pi = 5, K^2 = -1$ and realized as the blow-up of its minimal model in one point.

**step 4:** In both cases it is possible to give a geometric construction of $T$ via linkage and by Proposition 1.16 it will follow that $T$ is a degeneration of a smooth surface. First of all observe that any smooth Bordiga surface of degree 6 and sectional genus $\pi = 3$ can be linked $(3, 3)$ to a smooth cubic scroll. By Proposition 3.7 there are two possible reduced locally Cohen-Macaulay degenerations of a cubic scroll and if $R$ one of them, by Remark 1.12 any surface of degree 6 and $\pi = 3$ linked $(3, 3)$ to $R$ is smooth.

(1) Let $R = R_0 \cup \Pi_0$ where $R_0$ is a smooth quadric and $\Pi_0$ is a plane intersecting $R_0$ along a line $L_0$. Let $S$ the smooth Bordiga surface linked $(3, 3)$ to $R$ and let $T = S \cup \Pi_0$. Consider the exact sequence

\[0 \longrightarrow \mathcal{O}_{R_0}(K_{R_0}) \longrightarrow \mathcal{O}_X(H) \longrightarrow \mathcal{O}_T(H) \longrightarrow 0.\]

where $X$ is the complete intersection $(3, 3)$. Restricted to $R_0$ the sequence becomes

\[0 \longrightarrow \mathcal{O}_{R_0}(K_{R_0}) \longrightarrow \mathcal{O}_{R_0}(H) \longrightarrow \mathcal{O}_C(H) \longrightarrow 0.

where $C = (R_0 \cap S) \cup L_0$.

Comparing with the exact sequence on $R_0$

\[0 \longrightarrow \mathcal{O}_{R_0}(-C + H) \longrightarrow \mathcal{O}_{R_0}(H) \longrightarrow \mathcal{O}_C(H) \longrightarrow 0
\]

it follows that $C \equiv H - K_{R_0}$ on $R_0$. Since $K_{R_0} \equiv -2H$ it follows that $C \equiv 3H$.

Since $R$ is locally Cohen-Macaulay we can assume $L_0$ to be of type $(0, 1)$ on $R$. Hence $R_0 \cap S4L = E_i - 2E_j$ is curve of degree 5 and genus 2.
A similar argument shows that on $\Pi_0$ we have $L_0 \cup (S \cap \Pi_0) \equiv 4L$. Since $L_0 \equiv L$ on $\Pi_0$, it follows that $\Pi_0$ intersects $S$ along a plane cubic.

(2) Let $R = \Pi_1 \cup \Pi_2 \cup \Pi_3$ where $\Pi_2 \cap \Pi_3 = p_{23}$ is a point and $\Pi_1 \cap \Pi_j = l_j$ is a line. Let $S$ the smooth linked Bordiga surface in a complete intersection $(3, 3)$. As before consider the sequence of linkage restricted to each component i.e.

$$0 \longrightarrow \mathcal{O}_{\Pi_j}(K_{\Pi_j}) \longrightarrow \mathcal{O}_{\Pi_j}(H) \longrightarrow \mathcal{O}_{D_j}(H) \longrightarrow 0,$$

where $D_j$ is the intersection of $\Pi_j$ with the other components for $j = 1, \ldots, 3$. Then $D_j \equiv 4L$ for every $j$. This implies that $\Pi_1$ intersects $S$ along a conic $C$ while $\Pi_j$ intersects $S$ along a cubic for $i = 1, 2$. Unfortunately the configuration of the $3$ planes suggests immediately that $C$ must be a $-1$ conic since $\Pi_1$ intersect $S$ along $C$ and $p_{23}$.

On the other hand there are on $S$ natural $-1$ conics which are the strict transforms of curves $D_{ij} \equiv L - E_i - E_j$. Let for simplicity $D_{12} \equiv L - E_1 - E_2$ one of these conics and let $l_1$ the plane of $D_{12}$. If $X$ is a cubic hypersurface containing $S$ then $X$ contains $\Pi_1$ iff $\Pi_1$ contains two extra points of $X$. Therefore $\Pi_1 \cup S$ is contained in at least 2 cubics $X$, $X'$ and it can be linked to a quadric surface $R_0$. Since $\pi_{\Pi_1} = 4$, it follows that $\pi_{R_0} = -1$. Hence $R_0$ is reducible and $R_0 = \Pi_2 \cup \Pi_3$ where $\Pi_2 \cap \Pi_3 = P_{23}$ and $R = \Pi_1 \cup \Pi_2 \cup \Pi_3$ is a degenerate cubic scroll. In particular

$$\Pi_2 \cap S \equiv 3L - \sum_{i \neq 1} E_i,$$

$$\Pi_3 \cap S \equiv 3L - \sum_{j \neq 2} E_j.$$

Let now $\Pi_0$ be another plane such that $\Pi_0 \cup \Pi_1 \cup \Pi_2 \cup \Pi_3$ such that $\Pi_2$ and $\Pi_3$ meet pairwise in a point and $\Pi_1$ is the plane spanned by those 3 points. Then by Proposition 3.9 the union of the $\Pi_i$'s is a locally Cohen-Macaulay degenerate Veronese surface. Let $H_0$ an hyperplane containing $\Pi_0$ and let $Y = X' \cup H_0$ a quartic hypersurface containing $P = \Pi_0 \cup R$. Then $P$ is linked $(3, 4)$ to the union of a smooth Bordiga surface $S$ and a new smooth quadric $R_0$ contained in $H_0$.

Finally the usual exact sequence

$$0 \longrightarrow \mathcal{O}_{\Pi_0}(K_{\Pi_0}) \longrightarrow \mathcal{O}_{\Pi_0}(2H) \longrightarrow \mathcal{O}_{D_0}(2H) \longrightarrow 0,$$

gives $D_0 = (\Pi_0 \cap R) \cup (\Pi_0 \cap R_0) \cup (\Pi_0 \cap S) \equiv 5L$. Since $\Pi_0 \cap R \equiv L$ and $\Pi_0 \cap R_0 \equiv 2L$ then $\Pi_0 \cap S$ is a plane conic $C$. The same argument shows that $\Pi_1$ and $\Pi_2$ intersect $R_0$ along a line while $\Pi_0$ intersects $R_0$ in two points, one for each line. The existence of this two points corresponds exactly to the fact that $C$ is a $-2$ conic as we want.

This concludes the proof of the theorem. \hfill \Box

**Remark 5.3.** In the case in which the points are chosen in special position, it is actually possible to construct the linear system $[H]$. In fact, using Theorem 5.2 it is enough to choose 10 points in the plane such that 9 points lie on a cubic and 6 of them on a conic. Fixed 5 general points $p_1, \ldots, p_5$ in $\mathbb{P}^2$, let $A$ the unique conic through them and let $C$ any cubic passing through $p_1, \ldots, p_5$ transversal to $A$. Then $A$ and $C$ will intersect in an extra point $p_6$. Finally we can choose 3 more points $p_7, \ldots, p_9$ lying on $C$ but not on $A$ and a last point $p_{10}$ outside the two curves.
Notice that Theorem 5.2 implies also that if \( A \) is a reducible curve, \( A = A' + A'' \) where \( A' \) and \( A'' \) are lines, then the following two cases are the only possible:

(i) \( A'^2 = -2, A''^2 = -2 \)

(ii) \( A'^2 = -3, A''^2 = -1 \)

In case (ii) \( A'' \) is one of the exceptional divisors not intersecting \( A \).

6. The Non Special Rational Surface of Degree 7 in \( \mathbb{P}^4 \)

In this section we shall prove analogous of theorems ?? and 5.1 in degree 7. Let \( S \) be a smooth rational surface of degree 7, sectional genus \( \pi = 4 \), self-intersection of the canonical divisor \( K^2 = -2 \) and speciality \( h = h^1(\mathcal{O}_S(1)) = 0 \). In geometric terms it is \( \mathbb{P}^2 \) blow-up in 11 points embedded by a linear system of the form

\[ |H| = |6L - \sum_{i=1}^{6} 2E_i - \sum_{i=7}^{11} E_i| \]

and for a general position of the points \( p_i \), \( |H| \) embeds \( S \) into \( \mathbb{P}^4 \). Using the decomposition method we can give also in this case explicit open conditions for the position of the \( p_i \)'s for \( |H| \) to be an embedding.

**Notation:** To avoid possible confusion from now on we will write \( H \equiv 6L - \sum_{i=1}^{6} 2E_i - \sum_{j=1}^{5} F_j \)

**Theorem 6.1.** The linear system \( |H| = |6L - \sum_{i=1}^{6} 2E_i - \sum_{j=1}^{5} F_j| \) embeds the surface \( S = \tilde{\mathbb{P}}^2(p_1, \ldots, p_{11}) \) into \( \mathbb{P}^4 \) if and only if

1. no \( p_i \) is infinitely near,

2. \( |L - \sum_{i \in \Delta} E_i| = 0 \) for \( |\Delta| \geq 3 \),

3. \( |L - \sum_{i \in \Delta} E_i - \sum_{j \in \Delta'} F_j| = 0 \) for \( |\Delta'| \geq 2 \) or \( |\Delta'| \geq 1, |\Delta'| \geq 4 \),

4. \( |2L - \sum_{i=1}^{6} E_i| = 0 \)

5. \( |2L - \sum_{i \in \Delta} E_i - \sum_{j \in \Delta'} F_j| = 0 \) for \( |\Delta'| \geq 5, |\Delta'| \geq 2 \) or \( |\Delta'| = 4, |\Delta'| \geq 4 \),

6. \( |3L - \sum_{i=1}^{6} E_i - \sum_{j \in \Delta'} F_j| = 0 \) for \( |\Delta'| \geq 4 \),

7. \( |3L - 2E_k - \sum_{i \neq k} E_i - \sum_{j \in \Delta'} F_j| = 0 \) for \( |\Delta'| \geq 4 \),

8. \( |3L - \sum_{i=1}^{6} E_i - 2F_k - \sum_{j \neq k} F_j| = 0 \).

**Proof.** We shall first show that the conditions stated are necessary.

Since \( H.(E_i - E_j) = 0 \) then (1) follows. The very ampleness of \( |H| \) implies also from (2) to (5) and (7) and (8) because \( |H| \) cannot have non effective divisors. For (6), if the linear system \( |3L - \sum_{i=1}^{6} E_i - \sum_{j \in \Delta'} F_j| \) contains some element \( A \) then \( 0 < H.A \leq 2 \) and \( p(A) = 1 \) which contradicts very ampleness of \( H \).

Now assume that conditions (1) to (8) holds. Let \( C \equiv 2L - \sum_{i \in \Delta} E_i \) for \( |\Delta'| = 5 \) and \( D \equiv H - C = 4L - 2E_k - \sum_{i \in \Delta} E_i - \sum_{j=1}^{5} F_j \) where \( k \notin \Delta \). To prove that \( |H| \) is very ample we apply theorem 1.3 to the pencil with fixed component \( C + |D| \subset |H| \).

**step 1:** \( h^0(\mathcal{O}_S(H)) = 5 \) and for all \( D \subset |D| \) the canonical maps \( H^0(\mathcal{O}_S(H)) \to H^0(\mathcal{O}_C(H)) \) and \( H^0(\mathcal{O}_S(H)) \to H^0(\mathcal{O}_D(H)) \) are surjective.

Let \( K \equiv -3L + \sum_{i=1}^{6} E_i \sum_{j=1}^{5} F_j \) be the canonical divisor of \( S \). Since \( D \) moves in a pencil, we have \( h^0(\mathcal{O}_S(D)) = 2 \) and \( h^2(\mathcal{O}_S(D)) = h^0(\mathcal{O}_S(K - D)) = 0 \). Hence by Riemann-Roch \( h^1(\mathcal{O}_S(D)) = 0 \) since \( \chi(\mathcal{O}_S(D)) = 2 \).

Consider now the exact sequence
0 \to \mathcal{O}_S(C - E_k) \to \mathcal{O}_S(C) \to \mathcal{O}_{E_k}(C) = \mathcal{O}_{E_k} \to 0.

By condition (4) \( h^0(\mathcal{O}_S(C - E_k)) = 0 \) and clearly \( h^2(\mathcal{O}_S(C - E_k)) = h^0(\mathcal{O}_S(K - C + E_k)) = 0. \) Hence by Riemann-Roch \( h^1(\mathcal{O}_S(C - E_k)) = 0. \)

This shows that \( h^1(\mathcal{O}_S(C)) = 0. \) In particular \( h^0(\mathcal{O}_S(C)) = 1, \) i.e. \( C \) is uniquely determined.

Finally consider the exact sequence

\[
0 \to \mathcal{O}_S(D) \to \mathcal{O}_S(H) \to \mathcal{O}_C(H) \to 0.
\]

Since \( C \) is a conic, by Riemann-Roch we have \( \chi(\mathcal{O}_C(H)) = 3. \) Furthermore \( h^0(\mathcal{O}_C(H)) = h^0(\mathcal{O}_C(K - H)) = 0. \)

This show that \( |H| \) maps \( S \) to \( \mathbb{P}^3 \) and \( H \) restricts to complete linear systems on \( C \) and \( D. \)

**step 2:** For every subcurve \( D' \in |D| \) we have

\[
H.D' \geq 2p(D') + 1.
\]

As in the proof of Theorem 5.2 we can assume \( D' \) is not one of the exceptional divisor for which (10) is fulfilled and \( p(D') \geq 0. \) Let \( D' \equiv aL - \sum_{i=1}^5 b_iE - i \sum_{j=1}^5 c_jF_j \)

for \( 1 \leq a \leq 4. \) Then

\[
H.D' = 6a - \sum_{i=1}^6 2b_i \sum_{j=1}^5 c_j,
\]

\[
2p(D') = (a - 1)(a - 2) - \sum_{i=1}^6 b_i(b_i - 1) - \sum_{j=1}^5 c_j(c_j - 1)
\]

For \( 1 \leq a \leq 2 \) then \( b_i \) and \( c_i \) can be just 0 and 1 and \( p(D') = 0. \) Hence (10) holds
eff \( H.D' > 0, \) i.e conditions from (2) to (5) are satisfied.

For \( a = 3 \) then at most one of the \( b_i \)'s or \( c_i \)'s can be 2 and \( p(D') = 0 \) or 1. If \( p(D') = 0 \) then conditions (7) and (8) garanty at \( H.D' > 0. \) If \( p(D') = 1 \) then condition (6) implies \( H.D' \geq 3. \)

Finally if \( a = 4 \) then at most 3 of the \( b_i \)'s or \( c_i \)'s can be equal 2 or there is a triple point in \( E_k. \) If there is a triple point or 3 doble points then \( p(D') = 0 \) and (10) is always satisfied. If there are two double points then \( p(D') = 1, \)

\[
\sum_{i=1}^6 2b_i + \sum_{j=1}^5 c_j \leq 19 \text{ and (10) is fulfilled. If the only double point is } E_k \text{ then } p(D') = 2, H.D' = 5 \text{ and (10) holds.}
\]

**step 3:** For every subcurve \( C' \subset C \) we have \( H.C' \geq 2p(C') + 1. \) In fact conditions (1) (2) (4) and (5) say that \( C \) is irreducible, so the degree formula is necessarily satisfied. This conclude the proof of the theorem

\[
\square
\]

Once we studied conditions for \( |H| \) to be very ample, we can look for special plane curves on \( S. \) Than we can prove the following

**Theorem 6.2.** If \( T \) is a reducible locally Cohen-Macaulay surface in \( \mathbb{P}^4 \) of degree 8 whose irreducible components are a plane and a smooth rational surface of degree 7 then \( T \) is

(i) a degeneration of a non special smooth rational surface with \( \pi = 5, K_T^2 = -2 \) and realized as \( \mathbb{P}^2 \) blown-up in 11 points or

(ii) a degeneration of a smooth \( K^3 \) surface with \( \pi = 6, K_T^2 = -1 \) and realized as the blow-up of its minimal model in one point.
Proof. Let $S$ be a rational surface of degree 7 embedded by the linear system $H \equiv 6L - \sum_{i=1}^{6} 2E_i - \sum_{j=1}^{11} E_j$ and let
\[ K_S = -3L + \sum_{i=1}^{6} E_i - \sum_{j=1}^{5} F_j \]
be the canonical divisor of $S$.

**Step 1:** If $C$ is a plane curve of degree $d$ and arithmetical genus $p_a(C)$ contained in $S$, then $C$ is the strict transform of a plane curve of the form
\[ nL - \sum_{i=1}^{6} b_i E_i - \sum_{j=1}^{5} c_j F_j, \quad n \leq 6 \]
Since the ideal of $S$ is generated by one cubic and six quartics then $S$ cannot contain plane curves of degree $d \geq 5$. Using remark 3.4 and 3.5 $C$ is a plane curve if the following conditions are satisfied
\[
(12) \begin{cases}
6n - \sum_{i=1}^{6} 2b_i - \sum_{j=1}^{5} c_j = d & d \leq 4, \quad n \leq 6 \\
n^2 - \sum_{i=1}^{6} b_i^2 - \sum_{j=1}^{5} c_j^2 - 3n + \sum_{i=1}^{6} b_i + \sum_{j=1}^{5} c_j + 2 = (d - 1)(d - 2)
\end{cases}
\]
Proceeding exactly in the same way as in the proof of theorem ?? we get the following classification:

If $C$ is a line then $C$ is the strict transform of
\[
(1) \quad L - \sum_{i=1}^{5} E_i \\
(2) \quad L - \sum_{\Delta \in \delta} E_i - F_j, \quad |\Delta| = 2 \\
(3) \quad L - E_i - \sum_{\Delta \in \delta} F_j, \quad |\Delta| = 3 \\
(4) \quad 2L - \sum_{i \in \Delta} E_i - \sum_{j=1}^{5} F_j, \quad |\Delta| = 3 \\
(5) \quad 2L - \sum_{\Delta \in \delta} E_i - \sum_{\Delta \in \delta} F_j, \quad |\Delta| = 4, |\Delta'| = 3, \\
(6) \quad 2L - \sum_{\Delta \in \delta} E_i - F_j, \quad |\Delta| = 5 \\
(7) \quad 3L - 2E_k - \sum_{i \notin \Delta} E_i - \sum_{j=1}^{5} F_j, \quad |\Delta| = 4 \\
(8) \quad 3L - 2E_k - \sum_{i \notin \Delta} E_i - \sum_{j \notin \Delta} F_j, \quad |\Delta| = 5, |\Delta'| = 3, \\
(9) \quad 3L - \sum_{i=1}^{5} E_i - 2F_h - \sum_{i,h \notin \Delta} F_j, \quad |\Delta| = 3. \\
(10) \quad 5L - \sum_{i=1}^{5} 2E_i - \sum_{j=1}^{5} F_j
\]
If $C$ is a conic then $C$ is the strict transform of
\[
(1) \quad L - \sum_{\Delta \in \delta} F_j, \quad |\Delta'| = 4, \\
(2) \quad L - \sum_{i \notin \Delta} E_i, \quad |\Delta| = 2 \\
(3) \quad L - E_i - \sum_{\Delta \in \delta} F_j, \quad |\Delta| = 2, \\
(4) \quad 2L - \sum_{\Delta \in \delta} E_i, \quad |\Delta| = 5 \\
(5) \quad 2L - \sum_{\Delta \in \delta} E_i - \sum_{\Delta \in \delta} F_j, \quad |\Delta'| = 4, |\Delta'| = 2, \\
(6) \quad 2L - \sum_{\Delta \in \delta} E_i - \sum_{\Delta \in \delta} F_j, \quad |\Delta'| = 3, |\Delta'| = 4, \\
(7) \quad 3L - 2E_k - \sum_{i \notin \Delta} E_i - \sum_{j \notin \Delta} F_j, \quad |\Delta| = 5, |\Delta'| = 2, \\
(8) \quad 3L - 2E_k - \sum_{i \notin \Delta} E_i - \sum_{j \notin \Delta} F_j, \quad |\Delta| = 4, |\Delta'| = 4, \\
(9) \quad 3L - \sum_{i=1}^{5} E_i - 2F_h - \sum_{i,h \notin \Delta} F_j, \quad |\Delta| = 2, \\
(10) \quad 3L - \sum_{\Delta \in \delta} E_i - 2E_k - \sum_{j \notin \Delta} F_j, \quad |\Delta| = 5, |\Delta'| = 4, \\
If $C$ is a cubic then $C$ is the strict transform of
(1) $3L - \sum_{i=1}^{6} E_i - \sum_{j \in \Delta'} F_j$, \hspace{1cm} |\Delta'| = 3,
(2) $3L - \sum_{j \in \Delta} E_i - \sum_{i=1}^{5} F_j$ \hspace{1cm} |\Delta| = 5

Among all these possible linear system we are interested just in those which satisfy $(H - C)^2 = 0$. Her is finally the list of the admissible linear systems:

(I) $L - \sum_{j \in \Delta'} F_j$, \hspace{1cm} |\Delta'| = 4,
(II) $2L - \sum_{j \in \Delta} E_i - \sum_{j \in \Delta'} F_j$, \hspace{1cm} |\Delta'| = 3, |\Delta'| = 4,
(III) $3L - 2E_k - \sum_{i \in \Delta} E_i - \sum_{j \in \Delta'} F_j$, \hspace{1cm} |\Delta| = 4, |\Delta'| = 4,
(IV) $3L - \sum_{j \in \Delta} E_i - \sum_{j=1}^{5} F_j$ \hspace{1cm} |\Delta| = 5
(V) $4L - \sum_{k \in \Delta} 2E_k - \sum_{i \in \Delta'} E_i - \sum_{j=1}^{5} F_j$, \hspace{1cm} |\Delta| = 2, |\Delta'| = 4.

**Step 2:** There exist Cremona’s transformations that map any curve (I) in a curve in (II) and (III) and any curve in (IV) in a curve in (V).

In fact we can suppose for simplicity that

\[
C_1 \equiv L - \sum_{j=1}^{4} F_j \\
C_2 \equiv 2L - \sum_{i=1}^{3} E_i - \sum_{j=1}^{4} F_j \\
C_3 \equiv 3L - 2E_1 - \sum_{i=2}^{5} E_i - \sum_{j=1}^{4} F_j \\
C_4 \equiv 3L - \sum_{i=2}^{6} E_i - \sum_{j=1}^{5} F_j \\
C_5 \equiv 4L - \sum_{i=1}^{4} E_i - 2E_5 - 2E_6 - \sum_{j=1}^{5} F_j.
\]

Consider the birational map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ induced by the linear system

\[
|L'| = |2L - \sum_{i=1}^{3} E_i|.
\]

Let $G_k$ the strict transform of $L - E_i - F_j$ then

\[
C_1 \equiv L - E_1 - E_2 + E_1 + E_2 - \sum_{j=1}^{4} F_j \\
\equiv G_3 + L' - G_2 - G_3 + L' - G_1 - G_3 - \sum_{j=1}^{4} F_j \\
\equiv C_2
\]

In the same way

\[
C_3 \equiv L' + L - E_1 - E_2 + E_2 - E_4 - E_5 - \sum_{j=1}^{4} F_j \\
\equiv L' + G_3 + L' - G_1 - G_3 - E_4 - E_5 + \sum_{j=1}^{4} F_j \\
\equiv C_2
\]

This proves the first part of the claim.

For the second part consider the Cremona’s transformation induced by the linear system

\[
|L'| = |2L - \sum_{i=4}^{6} E_i|.
\]

Then

\[
C_5 \equiv L' + 2L - E_5 - E_6 + E_4 - E_4 - \sum_{i=1}^{3} E_i - \sum_{j=1}^{4} F_j \\
\equiv L' + L' + L' - G_5 - G_6 - \sum_{i=1}^{3} E_i - \sum_{j=1}^{4} F_j \\
\equiv C_3
\]

**Step 3:** By Step 2 it is enough to study one linear system for each degree. Then for example if the linear system $|L - \sum_{j=1}^{4} F_j|$ is non empty, $|H|$ is very ample by
Theorem 2.4. Hence $S$ is embedded in $\mathbb{P}^4$ as a smooth surface. Same conclusion if the linear system $3L - \sum_{i=1}^5 E_i - \sum_{j=1}^5 F_j$ is non empty.

Step 4: Let $\Pi$ be the plane of $C$ on $S$ and let $T = S \cup \Pi$.

Case 1: If $C$ is a conic, by formula (3) in Section 1 we have $\pi_T = 5$. Furthermore we know that the ideal of $S$ is generated by 1 cubic and 6 quartics. Let $X \supset S$ a quartic hypersurface then $\Pi$ must intersect $X$ along a quartic curve. Since $\Pi$ contains already a conic, then $\Pi \subset X$ if $\Pi$ contains another conic and an extra point of $X$. This imposes 6 conditions on the number of cubics so we expect that $T$ is contained in 5 quartics. Furthermore if $T$ is smoothable to $T'$ then

$$(K_T'|s)^2 = (K_S + C)^2 = (-2L + \sum_{i=1}^6 E_i + F_j)^2 = -3$$

$$(K_T|\Pi)^2 = (K_{\Pi} + C)^2 = (-3L + 2L)^2 = 1.$$ It follows that $K_T = -3 + 1 = -2$. Thus we can conclude that $T'$ is in the Hilbert scheme whose general point is a non special smooth rational surface of degree 8, $\pi = 5$, $K^2 = -2$ and realized as the blow-up of $\mathbb{P}^2$ in 11 points and embedded with the linear system $|7L - \sum_{i=1}^{10} 2E_i - E11$.

Case 2: If instead $C$ is cubic then $\pi_T = 6$. Furthermore containing a plane imposes 3 conditions on the number of quartics hypersurfaces containing $S$ so we expect that $T$ is contained in 8 quartics. Furthermore we have

$$(K_T'|s)^2 = (E_j)^2 = -1$$

$$(K_T'|\Pi)^2 = (-3L + 3L)^2 = 0.$$ where $T'$ is the smooth model of $T$. It follows that $K_{T'} = -1 + 0 = -1$ and $T'$ is a smooth $K3$ surface of degree 8, $\pi = 6$, $K^2 = -1$ realized as the blow-up of its minimal model in one point.

Step 5: What is left now is to prove that $T$ is a degeneration of a smooth surface, constructing explicitly $T$ via linkage and applying Proposition 1.17. First of all notice that any smooth rational surface of degree 7 is linked to a quintic elliptic scroll $E$ in a complete intersection (3,4). Furthermore $E$ can be linked to a Veronese surface $V$ in a complete intersection (3,3). Then $T$ can be linked (4,4) to the union of a smooth quintic scroll and a degenerate cubic surface starting from the complete intersection (3,3) and adding to the cubic hypersurface an hyperplane containing $\Pi$. Unfortunately it is still not clear how to choose this two quartics.

\[Q.E.D\]

Remark 6.3. Using theorem 6.1 it is possible to construct explicitly the linear system $|H|$ assuming that the points are chosen in special positions.

Case 1: If the linear system $|L - \sum_{j=1}^4 F_j|$ is non empty it is enough to choose 11 points $p_1, \ldots, p_6$ and $q_1, \ldots, q_5$ such that there exists the decomposition $C + |D| \subset |H|$ where $C \equiv 2L - \sum_{i=1}^5 E_i$ and $D \equiv 4L - 2E_1 - \sum_{i=1}^{10} E_i - \sum_{j=1}^5 F_j$. Fixed 8 general points $p_1, \ldots, p_5$ and $q_1, q_2$ in $\mathbb{P}^2$, let $C$ the unique conic passing through $p_1, \ldots, p_5$ and let $D$ any quartic passing doubly through $p_6$ and simply through $p_1, \ldots, p_5, q_1, q_2$. Let $A$ the unique line through $q_1, q_2$. By Bezout’s theorem $L$ and
$D$ will intersect in two extra distinct points $q_3$ and $q_4$. Finally choose $q_5$ on $D$ not lying on $A$ and $C$. Notice that theorem 6.1 implies also that if $A$ is a reducible curve, $A = A' + A''$ where $A'$ and $A''$ are lines, then $A''$ is the exceptional divisor $F_5$.

**Case 2:** If the linear system $|3L - \sum_{i=1}^{5} E_i - \sum_{j=7}^{11} F_j|$ is non empty then $|H|$ is very ample. In fact let $C$ and $D$ as in Case 1 Let $B$ any cubic passing through $p_1, \ldots, p_5, q_1, q_2$ transversal to $D$. By Bezout’s theorem $D$ and $B$ intersect in 4 extra points. Let $q_3, q_4, q_5$ three of them. By theorem to 6.1 if $B = B' + B''$ where $B'$ is a line and $B''$ is a conic then the following cases are the only possible:

(i) $B'^2 = -1$, $B''^2 = -4$
(ii) $B'^2 = -4$, $B''^2 = -1$
(iii) $B'^2 = -2$, $B''^2 = -3$
(iv) $B'^2 = -3$, $B''^2 = -2$

In case (i) $B'$ is one of the exceptional divisors $E_1, \ldots, E_5$ and $B''$ is smooth. In case (ii) $B''$ is one of the exceptional divisors $F_1, \ldots, F_5$ and $B'$ is smooth. Finally in the other cases $B''$ is always a smooth conic.

**References**


