Wick product and semigroup

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May 5, 2003

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Abstract

The purpose of this paper is to establish a relation between Wick version of analytic functions with respect to the brownian motion and its associated semigroup.

Key words and phrases: semigroup, Wick product, Hermite transform, Ito formula.

1 Introduction

The motivation for this paper comes from the study of Wick version of analytic functions in the white noise analysis setting; the starting point is to understand the behaviour of the operator:

$$\Diamond : f \rightarrow f^\diamond$$

where $f$ is an analytic function; in [5] a connection between the brownian motion case and the backward heat equation is proved.

Here a more general result is proved: the Wick version of analytic functions works in opposite direction with respect to the action of the semigroup associated to the Brownian motion. This characterization will give a very useful and intuitive representation of the Hermite transform, which is a fundamental tool in the theory of stochastic differential equations (see [3]).

In Section 2 we briefly recall some basic white noise theory. Then in Section 3 the main result is proved and in section 4 we give a representation formula for the Wick version of analytic functions which can be used for the extension to more general functions. Finally in Section 5 we give a very short prove of the Ito formula based on the previous results.
2 Framework

Here we briefly recall some of the main concepts and results from white noise theory. For more information we refer the reader to [2] and [3]. Our notation will follow that from [3]. From now on we will assume that our Brownian motion is constructed on a white noise probability space $(\Omega, \mathcal{F}, P)$ and we let $(S)$ and $(S)^*$ denote the space /nn/sarpanitu/gjestern_2/albertol/new.ten of stochastic test functions and the space of stochastic distribution functions (Hida distribution), respectively. Every $X \in L^2(P)$ has a unique representation

$$X(\omega) = \sum_{\alpha} c_{\alpha} H_{\alpha}(\omega), \quad c_{\alpha} \in \mathbb{R},$$

where

$$\| X \|_{L^2(P)}^2 = E_P[X^2] = \sum_{\alpha} |c_{\alpha}|^2$$

and where $\alpha = \alpha_1 \alpha_2 \cdots$ when $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathcal{I}$, $\mathcal{I}$ denotes the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots)$ of arbitrary but finite length, where $\alpha_1, \alpha_2, \ldots$ are nonnegative integers, and

$$\{H_{\alpha}(\omega)\}_{\alpha \in \mathcal{I}},$$

is a orthogonal $L^2(P)$ basis constructed using the Hermite functions $e_1(x), e_2(x), \ldots$ (which form an orthonormal basis for $L^2(\mathbb{R})$) and the Hermite polynomials. The space $(S)$ of stochastic test functions is defined to be the set of all $X(\omega) = \sum_{\alpha} c_{\alpha} H_{\alpha}(\omega) \in L^2(P)$ such that

$$\| X \|_{L^2(P)}^2 := \sum_{\alpha} |c_{\alpha}|^2 (2N)_{\alpha} < \infty \text{ for all } q \in \mathbb{R},$$

where

$$(2N)_{\beta} = 2^{\beta_1} \cdots (2k)^{\beta_k} \cdots \text{ if } \beta = (\beta_1, \beta_2, \ldots) \in \mathcal{I}.$$  

Similarly, the space $(S)^*$ of Hida distributions can be described as the set of formal series

$$X(\omega) = \sum_{\alpha} c_{\alpha} H_{\alpha}(\omega)$$

such that there exist $q \in \mathbb{R}$ such that

$$\| X \|_{L^2(P)}^2 := \sum_{\alpha} |c_{\alpha}|^2 (2N)^{-\alpha q} < \infty.$$

Thus we have

$$(S) \subset L^2(P) \subset (S)^*.$$  

The family of seminorms $\| \cdot \|_{0,k}$ $k \in \mathbb{R}$ gives a natural projective topology on $(S)$ and an inductive topology on $(S)^*$. With this topologies $(S)^*$ becomes the dual of $(S)$. The action of $F(\omega) = \sum_{\alpha} a_{\alpha} H_{\alpha}(\omega) \in (S)$ on $f(\omega) = \sum_{\alpha} b_{\alpha} H_{\alpha}(\omega) \in (S)$ is given by

$$\langle F, f \rangle = \sum_{\alpha} a_{\alpha} b_{\alpha}.$$  

One of the important features about the Hida space $(S)^*$ is that it contains the singular white noise $W_t(\omega)$ for all $t \in \mathbb{R}$.  

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Definition 2.1 The Wick product $X \diamond Y$ of $X(\omega) = \sum a_a H_a(\omega) \in (S)^*$ and $Y(\omega) = \sum b_\beta H_\beta(\omega) \in (S)^*$ is defined by

$$(X \diamond Y)(\omega) = \sum_{a, \beta} a_a b_\beta H_{a+\beta}(\omega) = \sum_{\gamma} \left( \sum_{a+\beta=\gamma} a_a b_\beta \right) H_\gamma(\omega).$$

This Wick product satisfies the associative, commutative and distributive law. Using the associative law we can define Wick powers

$$X^{\diamond n} = X \diamond X \cdots \diamond X, \quad (n \text{ times}).$$

More generally, if

$$f(z) = \sum_{k=0}^\infty a_k z^k$$

is entire, i.e., an analytic function of the complex variable $z$ in the complex plane $\mathbb{C}$, we can, for some $X \in (S)^*$, define the Wick version

$$f^{\diamond} (X) = \sum_{k=0}^\infty a_k X^{\diamond k} \in (S)^*.$$

For example, if $\phi \in L^2(\mathbb{R})$ is deterministic, then

$$\exp^{\diamond} \left[ \int \phi(s) dB_s \right] = \exp \left[ \int \phi(s) dB_s - \frac{1}{2} \int \phi^2(s) ds \right].$$

We recall the following important connection between Ito integration and the Wick product: let $u(t, \omega)$ be an $\mathcal{F}_t$-adapted process such that $E[\int_a^b u^2(t, \omega) dt] < \infty$. Then $u(t, \omega) \diamond W_t$ is integrable in $(S)^*$ and

$$\int_a^b u(t, \omega) dB_t(\omega) = \int_a^b u(t, \omega) \diamond W_t(\omega) dt.$$

A very important tool in the white noise analysis is:

Definition 2.2 Let $X(\omega) = \sum a_a H_a(\omega) \in (S)^*$, then the Hermite transform of $X$ (with respect to the basis $\{e_k\}_k$), denoted by $\mathcal{H}X$ or $\hat{X}$, is defined by

$$\mathcal{H}X(z) = \hat{X}(z) = \sum a_a z^a \in \mathbb{C}, \quad \text{(when convergent)}$$

where $z = (z_1, z_2, \ldots) \in \mathbb{C}^N$, and

$$z^a = z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}.$$ 

if $\alpha = (a_1, a_2, \ldots) \in \mathcal{I}$, where $z^0_j = 1$.

One can verify that the previous sum converges for all $z \in \mathbb{C}_{\mathcal{I}}^N$ (the set of all finite length sequences of complex numbers), and that any element in $(S)^*$ is uniquely characterized through its $\mathcal{H}$-transform. We recall the important relations

$$\mathcal{H}[X \diamond Y](z) = \mathcal{H}[X](z) \cdot \mathcal{H}[Y](z).$$
and
\[ \mathcal{H}[f^\diamond(X)](z) = f(\mathcal{H}[X](z)), \quad \text{(when convergent)} \]
if \( f: \mathbb{C} \to \mathbb{C} \) is entire, \( f(\mathbb{R}) \subseteq \mathbb{R} \) and \( f^\diamond(X) \in (S)^* \).

We also mention the chain rule in \((S)^*\): suppose \( X: \mathbb{R} \to (S)^* \) is continuously differentiable and let \( f: \mathbb{C} \to \mathbb{C} \) be entire function such that \( f(\mathbb{R}) \subseteq \mathbb{R} \) and \( f^\diamond(X_t) \in (S)^* \) for all \( t \), then
\[
\frac{d}{dt} f^\diamond(X_t) = f^\diamond(X_t) \frac{d}{dt} X_t, \quad \text{in } (S)^*.
\]

3 Main result

Let \( \varphi: \mathbb{R} \to \mathbb{R} \) be an analytic function of the form
\[
\varphi(x) = \sum_{n=0}^{+\infty} a_n x^n
\]
where \( a_n \in \mathbb{R} \) for all \( n \in \mathbb{N} \cup \{0\} \) and consider its Wick version with respect to the brownian motion, i.e.
\[
\varphi^\diamond(B_t) = \sum_{n=0}^{\infty} a_n B_t^\diamond n.
\]
We introduce the notation
\[
(T_t \varphi)(x) := \varphi^\diamond(B_t)|_{B_t=x}.
\]
Then we have:

**Theorem 3.1** If \( \varphi \) is an analytic function and \( \{P_t\}_{t \geq 0} \) is the semigroup associated to the brownian motion, i.e.
\[
(P_t \varphi)(x) = E^x[\varphi(B_t)] = \int_\mathbb{R} \varphi(x+y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy
\]
then
\[
(T_s (P_t \varphi))(x) = (P_{t-s} \varphi)(x) \quad \text{for all } 0 \leq s \leq t, x \in \mathbb{R}.
\]

**Proof.** Consider the problem
\[
\begin{cases}
\partial_t u(t, x) + \frac{1}{2} \partial_{xx} u(t, x) = 0 & \quad (t, x) \in [0, t_0] \times \mathbb{R} \\
\quad u(t_0, x) = \varphi(x) & \quad x \in \mathbb{R}
\end{cases}
\]
(3.1)
The unique solution to this problem (see, for instance, [6]) is
\[
u(t, x) = E^x[\varphi(B_{t_0-t})] = (P_{t_0-t} \varphi)(x).
\]
Now consider this new problem
\[
\begin{cases}
\partial_t v(t, x) + \frac{1}{2} \partial_{xx} v(t, x) = 0 & \quad (t, x) \in [0, t_0] \times \mathbb{R} \\
\quad v(0, x) = (P_{t_0} \varphi)(x) & \quad x \in \mathbb{R}
\end{cases}
\]
(3.2)
A solution (see [5]) is 
\[ v(t, x) = (T_t(P_{t_0}\varphi))(x). \]

On the other hand, if 
\[ v(t_0, x) = (T_{t_0}(P_{t_0}\varphi))(x) = \varphi(x) \text{ for all } x \in \mathbb{R} \]
then by uniqueness of problem 1 we must have 
\[ v(t, x) = u(t, x) \text{ for all } (t, x) \in [0, t_0] \times \mathbb{R} \]
i.e. 
\[ (T_t(P_{t_0}\varphi))(x) = (P_{t_0-t}\varphi)(x). \]

So, in order to prove the theorem, we have only to prove that 
\[ (T_t(P_t\varphi))(x) = \varphi(x) \text{ for all } x \in \mathbb{R} \text{ and } t \geq 0. \]

By linearity it is sufficient to prove the previous relation for polynomials.
Let \( \varphi(x) = x^n, n \in \mathbb{N}; \) hence 
\[ (P_t\varphi)(x) = E_t^x[\varphi(B_t)] = \int_{\mathbb{R}} \varphi(x + y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} c_k(t) \]
where 
\[ c_k(t) = E_t^x[B_{t}^k]. \]

But 
\[ E_t^x[B_{t}^{2k}] = (2k - 1)!! t^k \text{ and } E_t^x[B_{t}^{2k+1}] = 0 \text{ for all } k \in \mathbb{N}, t \geq 0. \]

So 
\[ (P_t\varphi)(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} x^{n-2k} c_{2k}(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} x^{n-2k} (2k - 1)!! t^k. \]

Hence 
\[ (T_t(P_t\varphi))(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (2k - 1)!! t^k h_{n-2k,t}(x) \]
where \( h_{n,t} \) is the \( n \)-th Hermite polynomial with parameter \( t \). These polynomials are defined by the relation 
\[ e^{xy - \frac{1}{2} y^2} = \sum_{n=0}^{+\infty} \frac{y^n}{n!} h_{n,t}(x). \]

Moreover we have 
\[ x^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (2k - 1)!! t^k h_{n-2k,t}(x). \]

Therefore 
\[ (T_t(P_t\varphi))(x) = x^n = \varphi(x). \]
Corollary 3.2  
1. $P_sT_t = T_{t-s}$ for all $0 \leq s \leq t$;
2. $T_{s+t} = T_sT_t$ for all $s, t \geq 0$;
3. $\mathcal{H}(\varphi(B_t))(z) = E^{\mathcal{H}(B_t)(z)}[\varphi(B_t)]$.

**Proof.**

1. It follows from the semigroup property of $(P_t)_{t \geq 0}$.
2. It follows from the semigroup property of $(P_t)_{t \geq 0}$.
3. 
   \[
   \mathcal{H}(\varphi(B_t))(z) = \mathcal{H}(T_t(P_t\varphi)(B_t))(z) = \mathcal{H}((P_t\varphi)\circ(B_t))(z) = \\
   = (P_t\varphi)(\mathcal{H}(B_t)(z)) = E^{\mathcal{H}(B_t)(z)}[\varphi(B_t)].
   \]

\[
\square
\]

4 Representation formula and extension to not analytic functions

Consider the function

\[
\phi(x) = \int_{-\infty}^{+\infty} \varphi(x + iy)N_{0,t}(y)dy, \quad x \in \mathbb{R}, t \geq 0
\]

where $i$ is the imaginary constant, $N_{0,t}(\cdot)$ is the density of a normal random variable with 0 mean and variance $t$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a function such that the integral is finite for almost all $x \in \mathbb{R}$. Choose $\varphi(x) = x^n$; then

\[
\phi(x) = \int_{-\infty}^{+\infty} (x + iy)^n N_{0,t}(y)dy = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} i^k c_k(t) = \sum_{k=0}^{[\frac{n}{2}]} \left( \frac{n}{2k} \right) x^{n-2k} i^{2k} c_{2k}(t) = \\
= \sum_{k=0}^{[\frac{n}{2}]} \left( \frac{n}{2k} \right) x^{n-2k} (-1)^k (2k-1)! i^{2k} = h_{n,t}(x)
\]

where $h_{n,t}(x)$ is the n-th Hermite polynomial of degree $n$ and parameter $t$. So by linearity we see that

\[
\phi(x) = (T_t\varphi)(x) \quad x \in \mathbb{R}, t \geq 0
\]

if $\varphi$ is a polynomial; moreover by a limit argument we can say that the above relation holds also for those analytic functions such that the integral

\[
(T_t\varphi)(x) = \int_{-\infty}^{+\infty} \varphi(x + iy)N_{0,t}(y)dy, \quad x \in \mathbb{R}, t \geq 0
\]

converges. Using this equation we try to define the operator $T_t$ for more general functions.
Proposition 4.1 Let $\varphi \in L^2(\mathbb{R})$ such that
\[ \xi \mapsto \hat{\varphi}(\xi) e^{\frac{i \xi^2}{2}} \in L^2(\mathbb{R}), \forall t \geq 0 \]
where
\[ \hat{\varphi}(\xi) = \int_{-\infty}^{+\infty} e^{ix\xi} \varphi(x) dx. \]
Then $T_t \varphi(\cdot) \in L^2(\mathbb{R})$ and is given by
\[ (T_t \varphi)(x) = \int_{-\infty}^{+\infty} e^{-i\xi x} e^{\frac{i \xi^2 t}{2}} \varphi(\xi) d\xi. \]

Proof.
\[
\int_{-\infty}^{+\infty} \varphi(x + iy)N_{0,t}(y)dy = \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} e^{-i\xi(x+iy)} \hat{\varphi}(\xi) d\xi \right)N_{0,t}(y)dy =
\]
\[
= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} e^{-i\xi x} e^{iy\xi} \hat{\varphi}(\xi) d\xi \right)N_{0,t}(y)dy = \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} e^{y\xi} N_{0,t}(y)dy \right) e^{-i\xi x} \hat{\varphi}(\xi) d\xi =
\]
\[
= \int_{-\infty}^{+\infty} e^{-i\xi x} e^{\frac{i \xi^2 t}{2}} \hat{\varphi}(\xi) d\xi
\]
and the last term belongs to $L^2(\mathbb{R})$ because of our hypothesis. \qed

5 Ito formula

Theorem 5.1 Let $\varphi \in C^2(\mathbb{R})$ be a function such that $(P_t \varphi) : \mathbb{R} \rightarrow \mathbb{R}$ ($(P_t)_{t \geq 0}$ is the brownian motion semigroup) is analytic for all $t \geq 0$ and let $\{B_t\}_{t \geq 0}$ be a 1-dimensional brownian motion. Then
\[ \varphi(B_b) - \varphi(B_a) = \int_a^b \varphi'(B_t) dB_t + \frac{1}{2} \int_a^b \varphi''(B_t) dt \quad \text{for all } a, b \in \mathbb{R}, a \leq b. \]

Proof.
\[
\mathcal{H}(\int_a^b \varphi'(B_t) dB_t(z)) = \mathcal{H}(\int_a^b \varphi'(B_t) dB_t(z)) = \int_a^b \mathcal{H}(\varphi'(B_t))(z) \frac{d}{dt} \mathcal{H}(B_t(z)) dt =
\]
\[
= \int_a^b (P_t \varphi)(\mathcal{H}(B_t(z))) \frac{d}{dt} \mathcal{H}(B_t(z)) dt = \int_a^b \frac{d}{dt} [(P_t \varphi)(\mathcal{H}(B_t(z)))] - \partial_t [(P_t \varphi)(\mathcal{H}(B_t(z)))] dt =
\]
\[
= (P_b \varphi)(\mathcal{H}(B_b(z))) - (P_a \varphi)(\mathcal{H}(B_a(z))) - \int_a^b \frac{1}{2} (P_t \varphi')(\mathcal{H}(B_t(z))) dt =
\]
\[
= \mathcal{H}(\varphi(B_b) - \varphi(B_a)) - \frac{1}{2} \int_a^b \varphi''(B_t) dt(z).
\]
By uniqueness of the Hermite transform, the theorem is proved. \qed
**Remark 5.2** Next important steps will be the extensions to diffusion and Levy processes and give explicit solution in terms of semigroup to stochastic differential equations of the form:

\[ dX_t = b(t, X_t)dB_t, \quad t \geq 0 \]

with initial condition

\[ X_0 = x. \]

**Acknowledgments**

I want to thank Bernt Øksendal and Frank Proske for their encouragement and interest and the Department of Mathematics, University of Oslo, for its warm hospitality.

**References**


