

The Donsker Delta function of a Lévy Process with Application to Chaos Expansion of Local Time

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Abstract

We give an explicit formula for the Donsker delta function of a certain class of Lévy processes in the Lévy-Hida distribution space. As an application we use the Donsker delta function to derive an explicit chaos expansion of local time for Lévy processes, in terms of iterated integrals with respect to the associated compensated Poisson random measure.

Key words and phrases: Lévy processes, local time, white noise, Donsker delta function, chaos expansion

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1 Introduction

Let $\eta(t)$ be a pure jump Lévy process without drift. It is known (see e.g. [B]) that under certain conditions on the characteristic exponent Ψ of $\eta(\cdot)$ the *local time* $L_t(x) = L_t(x, \omega)$ of $\eta(\cdot)$ at the point $x \in \mathbb{R}$ up to time t exists and the map

$$(x, \omega) \longmapsto L_t(x, \omega)$$

belongs to $L^2(\lambda \times P)$ for all t , where λ is the Lebesgue measure on \mathbb{R} and P is the probability law of $\eta(\cdot)$, defined on a probability space (Ω, \mathcal{F}) . Additional information about local time of Lévy processes can be found e.g. in [B], [Ba] and [Sa].

The main purpose of this paper is to give the explicit chaos expansion of $L_t(\cdot)$, in terms of iterated integrals with respect to the associated compensated Poisson random measure $\tilde{N}(dt, d\zeta)$. See Theorem 3.2.4.

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To achieve this we first prove results of independent interest about the *Donsker delta function* $\delta_x(\eta(t))$ of $\eta(t)$. For a certain class of pure jump Lévy processes we show that $\delta_x(\eta(t))$ exists as an element of the Lévy-Hida stochastic distribution space $(\mathcal{S})^*$ and we give the explicit representation

$$\begin{aligned} & \delta_x(\eta(t)) & (1.1) \\ = & \frac{1}{2\pi} \int_{\mathbb{R}} \exp^{\circ} \left(\int_0^t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1) \tilde{N}(ds, d\varsigma) + t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1 - i\lambda\varsigma) \nu(d\varsigma) - i\lambda x \right) d\lambda. \end{aligned}$$

See Theorem 3.1.4. Based on a different approach the Donsker delta function for Lévy processes was defined in [LS]. Contrary to the latter authors' definition we attain an explicit formula for the Donsker delta function in the case of a special class of Lévy processes.

Further, we show that $L_t(x)$ is related to $\delta_x(\eta(t))$ by the formula

$$L_T(x) = \int_0^T \delta_x(\eta(t)) dt, \quad (1.2)$$

just as in the Brownian motion case (Theorem 3.2.2).

Finally these results are applied to obtain the chaos expansion of $L_T(x)$.

Our approach is inspired by the method in [HØ], where the chaos expansion of local time of fractional Brownian motion is obtained. For the chaos expansion of local time of classical Brownian motion see [NV].

2 Some concepts of a white noise analysis for Lévy processes

Here we briefly elaborate a framework for the paper, using concepts and results developed in [DØP], [ØP] and [LØP].

A *Lévy process* can be considered a random walk in continuous time, that is a stochastic process $\eta(t)$ with *independent* and *stationary* increments, starting at zero, i.e. $\eta(0) = 0$ a.e. The state space may be a general topological group, but we confine ourselves to the real case \mathbb{R} . Lévy processes can be characterized in distribution by the famous Lévy-Khintchine formula, revealing the correspondence to infinitely divisible distributions. This formula gives a closed form expression for the characteristic function of Lévy processes $\eta(t)$, i.e.

$$E e^{i\lambda\eta(t)} = \exp(-t\Psi(\lambda)), \quad \lambda \in \mathbb{R}, \quad t \in \mathbb{R}_+, \quad (2.1)$$

where Ψ is the characteristic exponent, given by

$$\Psi(\lambda) = ia\lambda + \frac{1}{2}\sigma\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda\varsigma} + i\lambda\varsigma\chi_{\{|\varsigma|<1\}}) \nu(d\varsigma)$$

for constants $a \in \mathbb{R}$ and $\sigma \geq 0$. The measure ν denotes the Lévy measure, which plays the role in governing the jumps of $\eta(t)$. We recommend the books of [B] and [Sa] for general information about Lévy processes.

In this paper we will exclusively deal with pure jump Lévy processes $\eta(t)$ without drift. We assume that $\eta(t)$ is defined on the filtered probability space (Ω, \mathcal{F}, P) , $(\mathcal{F}_t)_{t \geq 0}$ with $\mathcal{F} = \mathcal{F}_\infty$, where \mathcal{F}_t is the completed filtration, generated by the Lévy process.

Next we aim at recalling the construction of various spaces of *stochastic test functions* and *stochastic distributions*, which are based on chaos expansions. In the sequel we adopt the notation in [HØUZ]. Let us define \mathcal{J} to be the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \dots)$, which have only finitely many non-zero values $\alpha_i \in \mathbb{N}_0$. Define $\text{Index}(\alpha) = \max\{i : \alpha_i \neq 0\}$ and $|\alpha| = \sum_i \alpha_i$ for $\alpha \in \mathcal{J}$. Now we choose the Laguerre functions of order $\frac{1}{2}$, denoted by $\{\xi_k\}_{k \geq 1}$, as a complete orthonormal system of $L^2(\mathbb{R}_+)$ (for its definition see e.g. [T]). Further denote by $(\pi_j)_{j \geq 1}$ any orthonormal basis of $L^2(\nu)$. Then define the bijective map

$$\kappa : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}; (i, j) \longmapsto j + (i + j - 2)(i + j - 1)/2 \quad (2.2)$$

If $k = \kappa(i, j)$ for $i, j \in \mathbb{N}$, define the product

$$\delta_k(t, \varsigma) = \xi_i(t)\pi_j(\varsigma)$$

Let $\text{Index}(\alpha) = j$ and $|\alpha| = m$ for $\alpha \in \mathcal{J}$ and introduce the function $\delta^{\otimes \alpha}$ as

$$\delta^{\otimes \alpha}((t_1, \varsigma_1), \dots, (t_m, \varsigma_m)) = \quad (2.3)$$

$$\delta_1^{\otimes \alpha_1} \otimes \dots \otimes \delta_j^{\otimes \alpha_j}((t_1, \varsigma_1), \dots, (t_m, \varsigma_m)) = \delta_1(t_1, \varsigma_1) \cdot \dots \cdot \delta_1(t_{\alpha_1}, \varsigma_{\alpha_1})$$

$$\dots \cdot \delta_j(t_{\alpha_1 + \dots + \alpha_{j-1} + 1}, \varsigma_{\alpha_1 + \dots + \alpha_{j-1} + 1}) \cdot \dots \cdot \delta_j(t_m, \varsigma_m),$$

where $\delta_i^{\otimes 0} := 1$. Then we define the *symmetrized tensor product* of the δ_k 's, denoted by $\delta^{\widehat{\otimes} \alpha}$, to be the symmetrization of the function $\delta^{\otimes \alpha}$ with respect to the variables $(t_1, \varsigma_1), \dots, (t_m, \varsigma_m)$. Using the symmetrized tensor products, one constructs an orthogonal $L^2(P)$ basis $\{K_\alpha(\omega)\}_{\alpha \in \mathcal{J}}$, given by

$$K_\alpha(\omega) := I_{|\alpha|}(\delta^{\widehat{\otimes} \alpha}), \quad \alpha \in \mathcal{J} \quad (2.4)$$

with

$$I_n(f) := n! \int_0^\infty \int_{\mathbb{R}} \dots \int_0^{t_2} \int_{\mathbb{R}} f(t_1, \varsigma_1, \dots, t_n, \varsigma_n) \widetilde{N}(dt_1, d\varsigma_1) \dots \widetilde{N}(dt_n, d\varsigma_n) \quad (2.5)$$

for symmetric functions $f \in L^2((\lambda \times \nu)^n)$, where $\widetilde{N}(ds, d\varsigma) = N(ds, d\varsigma) - \nu(d\varsigma)\lambda(ds)$ denotes the compensated Poisson random measure associated with η . Here $N(ds, d\varsigma)$ is the Poisson random measure and λ stands for the Lebesgue measure on \mathbb{R} . Note that the isometry $E[I_n^2(f)] = n! \|f\|_{L^2((\lambda \times \nu)^n)}^2$ holds and that the orthogonality relation $I_n(f) \perp I_m(g)$, $n \neq m$ is valid for symmetric f, g (see [I]).

Hence every $F \in L^2(P)$ can be uniquely represented as

$$F = \sum_{\alpha \in \mathcal{J}} c_\alpha K_\alpha, \quad c_\alpha \in \mathbb{R}, \quad (2.6)$$

where

$$\|F\|_{L^2(P)}^2 = E_P[F^2] = \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2. \quad (2.7)$$

with $\alpha! := \alpha_1! \alpha_2! \dots$ for $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathcal{J}$. Let $0 \leq \rho \leq 1$. The stochastic test function space $(\mathcal{S})_\rho$ can be characterized as the space of all $f = \sum_{\alpha \in \mathcal{J}} c_\alpha K_\alpha(\omega) \in L^2(P)$ such that

$$\|f\|_{\rho, k}^2 := \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1+\rho} c_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty \quad (2.8)$$

for all $k \in \mathbb{N}$, where $(2\mathbb{N})^{q\alpha} = (2 \cdot 1)^{q\alpha_1} (2 \cdot 2)^{q\alpha_2} \dots (2 \cdot m)^{q\alpha_m}$, if $\text{Index}(\alpha) = m$.

In the same manner the space $(\mathcal{S})_{-\rho}$ of stochastic distributions can be described as the collection of all formal expansions $F = \sum_{\alpha \in \mathcal{J}} c_\alpha K_\alpha(\omega)$ such that there exists a $q \in \mathbb{N}$ with

$$\|F\|_{-\rho, -q}^2 := \sum_{\alpha \in \mathcal{J}} (\alpha!)^{1-\rho} c_\alpha^2 (2\mathbb{N})^{-q\alpha} < \infty. \quad (2.9)$$

The seminorms $\{\|\cdot\|_{\rho, q}\}$ naturally induce the projective topology on $(\mathcal{S})_\rho$ and the inductive topology on $(\mathcal{S})_{-\rho}$. The space $(\mathcal{S})_{-\rho}$ becomes the dual of $(\mathcal{S})_\rho$ by the action

$$\langle F, f \rangle = \sum_{\alpha \in \mathcal{J}} a_\alpha b_\alpha \alpha! \quad (2.10)$$

if $F = \sum_{\alpha \in \mathcal{J}} a_\alpha K_\alpha \in (\mathcal{S})^*$ and $f = \sum_{\alpha \in \mathcal{J}} b_\alpha K_\alpha \in (\mathcal{S})$. We set $(\mathcal{S}) = (\mathcal{S})_0$ and $(\mathcal{S})^* = (\mathcal{S})_{-0}$. The space (\mathcal{S}) resp. $(\mathcal{S})^*$ is a Lévy version of the *Hida test function space* resp. *Hida distribution space* (see e.g. [HKPS], [HØUZ], [Ku] and [O]). For general $0 \leq \rho \leq 1$ we have the following chain of sets

$$(\mathcal{S})_1 \subset (\mathcal{S})_\rho \subset (\mathcal{S}) \subset L^2(P) \subset (\mathcal{S})^* \subset (\mathcal{S})_{-\rho} \subset (\mathcal{S})_{-1}.$$

One of the important properties of $(\mathcal{S})^*$ is that it is rich enough to carry *Lévy white noise*. We recall its precise definition. For the convenience of a simplified notation we choose for π_j in (2.3) a certain orthonormal basis of polynomials, which we now describe. For this reason we restrict the Lévy measure ν to fulfill the following integrability condition: For every $\varepsilon > 0$ there exists a $\lambda > 0$ such that

$$\int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \exp(\lambda |\zeta|) \nu(d\zeta) < \infty. \quad (2.11)$$

It follows from (2.11) that ν integrates all monomials of degree ≥ 2 . We define π_j to be

$$\pi_j(\varsigma) = \frac{1}{\|l_{j-1}\|_{L^2(\nu_0)}} \varsigma \cdot l_{j-1}(\varsigma) \quad (2.12)$$

where $\{l_m\}_{m \geq 0}$ is the orthogonalization of $\{1, \varsigma, \varsigma^2, \dots\}$ with respect to the inner product of $L^2(\nu_0)$ for $\nu_0(d\varsigma) := \varsigma^2 \nu(d\varsigma)$. Further condition (2.11) implies that $\eta(t)$ can be expressed as

$$\eta(t) = \int_0^t \int_{\mathbb{R}} \varsigma \tilde{N}(ds, d\varsigma). \quad (2.13)$$

Next define $\epsilon^l \in \mathcal{J}$ by

$$\epsilon^l(j) = \begin{cases} 1 & \text{for } j = l \\ 0 & \text{else} \end{cases}, \quad l \geq 1 \quad (2.14)$$

Since $\delta^{\otimes \epsilon^l} = \delta_l(t, \varsigma) = \xi_i(t) \pi_j(\varsigma)$, if $l = \kappa(i, j)$, $\eta(t)$ in (2.13) can be rewritten as

$$\eta(t) = \sum_{k \geq 1} m \int_0^t \xi_k(s) ds \cdot K_{\epsilon^{\kappa(k, 1)}}$$

where $m = \|\varsigma\|_{L^2(\nu)}$. Then differentiation of $\eta(t)$ with respect to time, denoted by $\dot{\eta}(t)$, in the $(\mathcal{S})^*$ -topology yields

$$\dot{\eta}(t) = m \sum_{k \geq 1} \xi_k(t) \cdot K_{\epsilon^{\kappa(k, 1)}} \quad (2.15)$$

for all t , where $\xi_k(t)$ are the Laguerre functions and $\kappa(i, j)$ the map in (2.2). We call $\dot{\eta}(t)$ *Lévy white noise*. A more general definition, which comprises $\dot{\eta}(t)$, is the *white noise* of the Poisson random measure $\tilde{N}(t, \varsigma) \in (\mathcal{S})^*$ a.e., given by

$$\tilde{N}(t, \varsigma) = \sum_{k, m \geq 1} \xi_k(t) \pi_m(\varsigma) \cdot K_{\epsilon^{\kappa(k, m)}} \quad (2.16)$$

The *Lévy Wick product* $F \diamond G$ of $F(\omega) = \sum_{\alpha \in \mathcal{J}} a_\alpha K_\alpha$ and $G(\omega) = \sum_{\beta \in \mathcal{J}} b_\beta K_\beta \in (\mathcal{S})_{-1}$ is defined by

$$(F \diamond G)(\omega) = \sum_{\alpha, \beta \in \mathcal{J}} a_\alpha b_\beta K_{\alpha+\beta}(\omega) \quad (2.17)$$

This Wick product makes the spaces $(\mathcal{S})_1$, (\mathcal{S}) , $(\mathcal{S})^*$, $(\mathcal{S})_{-1}$ topological algebras (see [DØP]). Later on we will utilize the following interesting property of the

Wick product, exposing its relation to Itô-Skorohod integration (see [ØP]). The relation is described by the representation

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} Y(t, \varsigma) \tilde{N}(\delta t, d\varsigma) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} Y(t, \varsigma) \diamond \dot{\tilde{N}}(t, \varsigma) \nu(d\varsigma) dt. \quad (2.18)$$

where the left side defines the Skorohod integral of a random field $Y : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ with respect to the compensated Poisson random measure (which coincides with the corresponding Itô integral, if $Y(t, \varsigma)$ is predictable). The right hand side is in terms of an $(\mathcal{S})^*$ -valued Bochner-integral with respect to the measure $\nu \times \lambda$ (see Definition 3.11 in [ØP]). In the sequel we need the important tool of the *Lévy Hermite transform* (see [LØP]). This transform enables the application of methods of complex analysis, since it maps the algebra $(\mathcal{S})_{-1}$ homomorphically into the algebra of power series in infinitely many complex variables. The definition of this transform is analogous to that in the Gaussian case, which was first introduced in [LØU]. It is based on the expansion along the basis $\{K_\alpha\}_{\alpha \in \mathcal{J}}$. Let $F(\omega) = \sum_\alpha a_\alpha K_\alpha(\omega) \in (\mathcal{S})_{-1}$. Then the *Lévy Hermite transform* of F , denoted by $\mathcal{H}F$, is defined as

$$(\mathcal{H}F)(z) = \sum_\alpha a_\alpha z^\alpha \in \mathbb{C}, \quad (2.19)$$

provided convergence holds, where $z := (z_1, z_2, \dots) \in \mathbb{C}^{\mathbb{N}}$ (the space of all sequences of complex numbers) and where $z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n} \dots$ for $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathcal{J}$ with $z_j^0 = 1$. It can be shown that the sum in (2.19) converges for some $0 < q, R < \infty$ in the infinite-dimensional neighborhood $U_q(R)$ in $\mathbb{C}^{\mathbb{N}}$, defined by

$$U_q(R) = \{(\xi_1, \xi_2, \dots) \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha \neq 0} |\xi^\alpha|^2 (2\mathbb{N})^{q\alpha} < R^2\}. \quad (2.20)$$

Furthermore, it can be verified that any element in $(\mathcal{S})_{-1}$ is uniquely characterized through its \mathcal{H} -transform (Theorem 2.3.8 in [LØP]). For example, the Hermite transform of $K_{\epsilon^{\kappa(k,j)}}$ is

$$(\mathcal{H}K_{\epsilon^{\kappa(k,j)}})(z) = z_{\kappa(k,j)}; \quad z = (z_1, z_2, \dots)$$

and the Hermite transform of $\dot{\eta}(t)$ can be determined as

$$(\mathcal{H}\dot{\eta}(t))(z) = \sum_{k \geq 1} \xi_k(t) z_{\kappa(k,1)}$$

Since \mathcal{H} is an algebra-homomorphism, we have

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}(F)(z) \cdot \mathcal{H}(G)(z) \quad (2.21)$$

Relation (2.21) can be extended to Wick versions of complex analytical functions g , which have a Taylor expansion around $\xi_0 = \mathcal{H}(F)(0)$ with real valued

coefficients. It is an immediate consequence of the characterization theorem (Theorem 2.3.8 in [LØP]) that there exists a unique $Y \in (\mathcal{S})_{-1}$ such that

$$(\mathcal{H}Y)(z) = (g \circ \mathcal{H}(F))(z) \quad (2.22)$$

We set $g^\diamond(F) = Y$ to indicate the Wick version of g applied to F . As an example the Wick version of the exponential function \exp is given by

$$\exp^\diamond(F) = \sum_{n \geq 0} \frac{1}{n!} F^{\diamond n}$$

with $F^{\diamond n} = F \diamond F \diamond \dots \diamond F$ (n -times). In conclusion we state the *chain rule* in $(\mathcal{S})^*$: Assume that $X : \mathbb{R} \rightarrow (\mathcal{S})^*$ is continuously differentiable. Further let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function with $f(\mathbb{R}) \subset \mathbb{R}$ and $f^\diamond(X(t)) \in (\mathcal{S})^*$ for all t . Then we have

$$\frac{d}{dt} f^\diamond(X(t)) = (f')^\diamond(X(t)) \diamond \frac{d}{dt} X(t) \text{ in } (\mathcal{S})^*. \quad (2.23)$$

3 The Donsker delta function of a pure jump Lévy process

In this section we investigate the local time $L_T(x)$ of a certain class of pure jump Lévy processes $\eta(t)$, which can be heuristically described by

$$L_T(x) = \int_0^T \delta(\eta(t) - x) dt, \quad (3.1)$$

where $\delta(u)$ is the *Dirac delta function*, which is approximated by

$$P_\varepsilon(u) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{|u|^2}{2\varepsilon}} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iyu - \frac{1}{2}\varepsilon y^2} dy, \quad u \in \mathbb{R}, \quad (3.2)$$

with $i = \sqrt{-1}$. Formally, this implies

$$\delta(u) = \lim_{\varepsilon \rightarrow 0} P_\varepsilon(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iyu} dy \quad (3.3)$$

The justification for this heuristical line of reasoning is similar to that of the Gaussian case (see [AHZ] and [H]). In the following subsections we will make the above considerations rigorous, by showing that the *Donsker delta function* $\delta(\eta(t) - x)$ of $\eta(t)$ may be realized as a generalized Lévy functional. Furthermore we provide an explicit formula for $\delta(\eta(t) - x)$. We also prove that the identity (3.1) makes sense in terms of a Bochner integral. To demonstrate an application we use the Donsker delta function to derive a chaos expansion of $L_T(x)$ with explicit kernels.

3.1 An explicit formula for the Donsker delta function

We proceed as in the Gaussian case (see [AØU]) to define the Donsker delta function.

Definition 3.1.1 Suppose that $X : \Omega \rightarrow \mathbb{R}$ is a random variable and that $X \in (\mathcal{S})_{-1}$. The Donsker delta function of X is a continuous function $\delta.(X) : \mathbb{R} \rightarrow (\mathcal{S})_{-1}$ such that

$$\int_{\mathbb{R}} h(y) \delta_y(X) dy = h(X) \quad (3.1.1)$$

for all measurable functions $h : \mathbb{R} \rightarrow \mathbb{R}$ under the assumption that the integral converges in $(\mathcal{S})_{-1}$.

As the main result of this subsection we will determine an explicit formula for the Donsker delta function in the case of a certain class of pure jump Lévy processes. From now on we limit ourselves to consider Lévy measures, whose characteristic exponent Ψ satisfies the following condition: There exists a $\varepsilon \in (0, 1)$ such that

$$\lim_{|\lambda| \rightarrow \infty} |\lambda|^{-(1+\varepsilon)} \operatorname{Re} \Psi(\lambda) = \infty, \quad (3.1.2)$$

where $\operatorname{Re} \Psi$ is the real part of Ψ .

Remark 3.1.2 Condition (3.1.2) entails the strong Feller property of the semigroup of our Lévy process $\eta(t)$, implying that the probability law of $\eta(t)$ is absolutely continuous with respect to the Lebesgue measure. The assumption covers e.g. the following Lévy process $\eta(t)$ of unbounded variation with Lévy measure ν , given by

$$\nu(d\zeta) = \chi_{(0,1)}(\zeta) \zeta^{-(2+\alpha)} d\zeta,$$

where $0 < \alpha < 1$. We shall also emphasize that various other conditions of the type (3.1.2) are conceivable, as the proof of Theorem 3.1.4 unveils.

The main theorem is preceded by the following Lemma.

Lemma 3.1.3 Let $\lambda \in \mathbb{C}$, $t \geq 0$, then

$$\exp(\lambda \eta(t)) = \exp^{\diamond} \left(\int_0^t \int_{\mathbb{R}} (e^{\lambda \zeta} - 1) \tilde{N}(ds, d\zeta) + t \int_{\mathbb{R}} (e^{\lambda \zeta} - 1 - \lambda \zeta) \nu(d\zeta) \right). \quad (3.1.3)$$

Proof Define

$$Y(t) = \exp(\lambda \eta(t)) - t \int_{\mathbb{R}} (e^{\lambda \zeta} - 1 - \lambda \zeta) \nu(d\zeta). \quad (3.1.4)$$

Then Itô's formula shows that Y satisfies the stochastic differential equation

$$dY(t) = Y(t_-) \int_{\mathbb{R}} (e^{\lambda\varsigma} - 1) \tilde{N}(dt, d\varsigma); \quad Y(0) = 1.$$

By relation (2.18) the last equation can be rewritten as

$$\frac{d}{dt}Y(t) = Y(t_-) \diamond \int_{\mathbb{R}} (e^{\lambda\varsigma} - 1) \overset{\bullet}{\tilde{N}}(t, \varsigma) \nu(d\varsigma); \quad Y(0) = 1.$$

With the help of the chain rule (2.23) one checks that the solution of the last equation is given by

$$\begin{aligned} Y(t) &= \exp^{\diamond} \left(\int_0^t \int_{\mathbb{R}} (e^{\lambda\varsigma} - 1) \overset{\bullet}{\tilde{N}}(t, \varsigma) \nu(d\varsigma) dt \right) & (3.1.5) \\ &= \exp^{\diamond} \left(\int_0^t \int_{\mathbb{R}} (e^{\lambda\varsigma} - 1) \tilde{N}(ds, d\varsigma) \right). \end{aligned}$$

This solution is unique and if we compare (3.14) and (3.1.5) we receive the desired formula. \square

Theorem 3.1.4 Assume that condition (3.1.2) holds. Then $\delta_y(\eta(t))$, i.e. the Donsker delta function of $\eta(t)$, exists uniquely. Moreover $\delta_y(\eta(t))$ takes the explicit form

$$\begin{aligned} &\delta_y(\eta(t)) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \exp^{\diamond} \left(\int_0^t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1) \tilde{N}(ds, d\varsigma) + t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1 - i\lambda\varsigma) \nu(d\varsigma) - i\lambda y \right) d\lambda \end{aligned}$$

Proof The proof is essentially based on the application of the Hermite transform \mathcal{H} in (2.19) and the use of the Fourier inversion formula.

To ease the notation we define

$$X_t = X_t(\lambda) = \int_0^t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1) \tilde{N}(ds, d\varsigma)$$

and

$$f(\lambda) = \exp(X_t + t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1 - i\lambda\varsigma) \nu(d\varsigma)).$$

Further we set

$$g(\lambda, z) = (\mathcal{H}f^{\diamond}(\lambda))(z),$$

where f^{\diamond} is the Wick version of f . Thus according to relation (2.22) we can write

$$\begin{aligned} g(\lambda, z) &= \exp((\mathcal{H}X_t)(z) + t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1 - i\lambda\varsigma) \nu(d\varsigma)) \\ &= \exp((\mathcal{H}X_t)(z) - t\Psi(\lambda)) \end{aligned}$$

We subdivide the proof into several steps.

(i) We want to show that $g(\cdot, z)$ is an element of the Schwartz space $\mathcal{S}(\mathbb{R})$. To this end we provide some reasonable upper bound for $g(\cdot, z)$. Let $0 < t \leq T$. Since

$$\begin{aligned} X_t &= \int_0^t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1) \tilde{N}(d\varsigma, d\varsigma) \stackrel{(2.18)}{=} \int_0^t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1) \diamond \tilde{N}(s, \varsigma) \nu(d\varsigma) ds \\ &\stackrel{(2.16)}{=} \int_0^t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1) \diamond \sum_{k,j} \xi_k(s) \pi_j(\varsigma) K_{\epsilon^{\kappa(k,j)}} \nu(d\varsigma) ds \\ &= \sum_{k,j} \int_0^t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1) \xi_k(s) \pi_j(\varsigma) \nu(d\varsigma) ds \cdot K_{\epsilon^{\kappa(k,j)}} \end{aligned}$$

with basis elements ξ_k and π_j as in Section 2, we can find the estimate

$$\begin{aligned} &|(\mathcal{H}X_t)(z)| \tag{3.1.6} \\ &\leq |\lambda| \left(\sum_{k,j} \left(\int_0^t \int_{\mathbb{R}} |\varsigma| |\xi_k(s) \pi_j(\varsigma)| \nu(d\varsigma) ds \right)^2 (2\mathbb{N})^{-2\kappa(k,j)} \right)^{\frac{1}{2}} \cdot \left(\sum_{\alpha} |z^\alpha|^2 (2\mathbb{N})^{2\alpha} \right)^{\frac{1}{2}} \\ &\leq |\lambda| \left(\sum_{k,j} \left(\int_0^t \int_{\mathbb{R}} |\varsigma|^2 \nu(d\varsigma) ds \right) (2\mathbb{N})^{-2\kappa(k,j)} \right)^{\frac{1}{2}} \cdot \left(\sum_{\alpha} |z^\alpha|^2 (2\mathbb{N})^{2\alpha} \right)^{\frac{1}{2}} \\ &\leq \text{const.} \cdot |\lambda| \end{aligned}$$

for all $z \in U_2(R)$ for some $R < \infty$ (see (2.20)), where we used that

$\sum_{k,j} (2\mathbb{N})^{-2\kappa(k,j)} < \infty$ (see Proposition 2.3.3 in [HØUZ]). Therefore using the definition of $g(\lambda, z)$ we get

$$|g(\lambda, z)| \leq e^{\text{const.}|\lambda|} e^{-t \operatorname{Re} \Psi(\lambda)}$$

for all $z \in U_2(R)$. By condition (3.1.2) let us require $|\lambda|^{-(1+\varepsilon)} \operatorname{Re} \Psi(\lambda) \geq 1$ for $|\lambda| \geq L \geq 1$. This implies

$$\begin{aligned} |g(\lambda, z)| &\leq e^{\text{const.}|\lambda|} e^{-t|\lambda|^{(1+\varepsilon)} |\lambda|^{-(1+\varepsilon)} \operatorname{Re} \Psi(\lambda)} \leq e^{\text{const.}|\lambda| - t|\lambda|^{(1+\varepsilon)}} \\ &\leq e^{-tC|\lambda|^{(1+\varepsilon)}} \end{aligned}$$

for $z \in U_2(R)$ and $|\lambda| \geq M$ with positive constants M, C . Next we cast a glance at the derivatives $\frac{\partial^n}{\partial \lambda^n} g(\lambda, z)$. Since a similar estimate to (3.1.6) yields

$$\begin{aligned} &\left| \int_0^t \int_{\mathbb{R}} (i\varsigma)^n e^{i\lambda\varsigma} \xi_k(s) \pi_j(\varsigma) \nu(d\varsigma) ds \cdot z_{\kappa(k,j)} \right| \\ &\leq \int_0^t \int_{\mathbb{R}} |\varsigma|^n |\xi_k(s) \pi_j(\varsigma)| \nu(d\varsigma) ds \cdot |z_{\kappa(k,j)}| \in L^1(\mathbb{N} \times \mathbb{N}) \end{aligned}$$

for all fixed $z \in U_2(R)$ and $n \in \mathbb{N}$, we obtain that for all $n \in \mathbb{N}$ there exists a constant C_n such that

$$\left| \frac{\partial^n}{\partial \lambda^n} (\mathcal{H}X_t)(z, \lambda) \right| \leq C_n$$

for all $\lambda \in \mathbb{R}$, $z \in U_2(R)$. Further we observe that

$$\begin{aligned} & \left| \frac{d}{d\lambda} \left(t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1 - i\lambda\varsigma) \nu(d\varsigma) \right) \right| = \left| t \int_{\mathbb{R}} i\varsigma (e^{i\lambda\varsigma} - 1) \nu(d\varsigma) \right| \\ & \leq |\lambda| t \int_{\mathbb{R}} |\varsigma|^2 \nu(d\varsigma) \end{aligned}$$

or more generally that

$$\left| \frac{d^n}{d\lambda^n} \left(t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1 - i\lambda\varsigma) \nu(d\varsigma) \right) \right| \leq C_n |\lambda|.$$

Altogether we conclude that for all $k, n \in \mathbb{N}_0$ there exists a polynomial $p_{k,n}(\lambda)$ with positive coefficients such that

$$\sup_{z \in U_2(R)} \left| (1 + |\lambda|^k) \frac{\partial^n}{\partial \lambda^n} g(\lambda, z) \right| \leq p_{k,n}(|\lambda|) e^{-tC|\lambda|^{1+\varepsilon}} \quad (3.1.7)$$

for all $|\lambda| \geq M$. This implies that $g(\cdot, z) \in \mathcal{S}(\mathbb{R})$ for all $z \in U_2(R)$.

Since the (inverse) Fourier transform maps $\mathcal{S}(\mathbb{R})$ onto itself, we get that $\widehat{g}(y, z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iy\lambda} g(\lambda, z) d\lambda$ is L^1 -integrable and that

$$\int_{\mathbb{R}} e^{iy\lambda} \widehat{g}(y, z) d\lambda = g(\lambda, z) \quad (3.1.8)$$

for all $z \in U_2(R)$. In view of identity (2.22), Lemma 3.1.3 and condition (3.1.1) relation (3.1.8) gives rise to defining the Donsker delta function of $\eta(t)$ as

$$\delta_y(\eta(t)) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iy\lambda} f^\circ(\lambda) d\lambda, \quad (3.1.9)$$

i.e.

$$\mathcal{H}(\delta_y(\eta(t))) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iy\lambda} g(\lambda, z) d\lambda = \widehat{g}(y, z)$$

Now we proceed as follows to complete the proof: We check that the Hermite transform $\mathcal{H}(\delta_y(\eta(t))) = \widehat{g}(y, z)$ in the integrand in (3.1.8) can be extracted outside the integral and that all occurring expressions are well-defined. Then we can apply the inverse Hermite transform and Lemma 3.1.3 to show that $\delta_y(\eta(t))$ in (3.1.9) fulfills the property (3.1.1) for $h(x) = e^{ix}$. Finally, the proof follows from a well-known density argument, using trigonometric polynomials.

(ii) Let us verify that $\delta_y(\eta(t))$ exists in $(\mathcal{S})_{-1}$ for all y . By a Lévy version of Lemma 2.8.5 in [HØUZ] (analogous proof) we know that

$$Y_n(y, z) := \frac{1}{2\pi} \int_{-n}^n e^{-iy\lambda} f^\circ(\lambda) d\lambda$$

exists in $(\mathcal{S})_{-1}$ and that

$$\mathcal{H}Y_n(y, z) = \frac{1}{2\pi} \int_{-n}^n e^{-iy\lambda} g(\lambda, z) d\lambda$$

for all $n \in \mathbb{N}$. Further, the bound (3.1.7) gives

$$\sup_{n \in \mathbb{N}, z \in U_2(R)} |\mathcal{H}Y_n(y, z)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} \sup_{z \in U_2(R)} |g(\lambda, z)| d\lambda < \infty \quad (3.1.10)$$

So it follows with the help of an analogue of Theorem 2.8.1 c) in [HØUZ] that $\delta_y(\eta(t)) \in (\mathcal{S})_{-1}$ for all y .

(iii) We check that the integral

$$\int_{\mathbb{R}} e^{iy\lambda} \delta_y(\eta(t)) dy \quad (3.1.11)$$

converges in $(\mathcal{S})_{-1}$. Because of the estimate (3.1.10) we also get that

$$X_n(\lambda, z) := \int_{-n}^n e^{iy\lambda} \delta_y(\eta(t)) dy$$

exists in $(\mathcal{S})_{-1}$. By (3.1.7) and integration by parts we deduce

$$\begin{aligned} & \sup_{n \in \mathbb{N}, z \in U_2(R)} |\mathcal{H}X_n(y, z)| \\ & \leq \int_{\mathbb{R}} \frac{1}{1+y^2} \cdot (1+y^2) |\mathcal{H}(\delta_y(\eta(t)))(z)| dy \\ & \leq \text{const.} \cdot \sup_{\lambda \in \mathbb{R}, z \in U_2(R)} \left(\left| (1+\lambda^2) \frac{\partial^2}{\partial \lambda^2} g(\lambda, z) \right| + |(1+\lambda^2)g(\lambda, z)| \right) \\ & \leq M < \infty, \end{aligned}$$

where we have used that

$$y^2 \mathcal{H}(\delta_y(\eta(t)))(z) = y^2 \widehat{g}(y, z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iy\lambda} \frac{\partial^2}{\partial \lambda^2} g(\lambda, z) d\lambda.$$

Again with the help of a Lévy version of Theorem 2.8.1 c) in [HØUZ] we see that the integral (3.1.11) is well-defined in $(\mathcal{S})_{-1}$.

Finally, by using the inverse Hermite transform and Lemma 3.1.3 we obtain

$$\int_{\mathbb{R}} e^{iy\lambda} \delta_y(\eta(t)) dy = f^\circ(\lambda) = e^{i\lambda\eta(t)} \quad (3.1.12)$$

Thus we have proved relation (3.1.1) for $h(y) = e^{i\lambda y}$. Since (3.1.1) still holds for linear combinations of such functions, the general case is attained by a well-known density argument. The continuity of $y \mapsto \delta_y(\eta(t))$ is also a direct consequence of a Lévy version of Theorem 2.8.1 in [HØUZ]. \square

3.2 Chaos expansion of local time for Lévy processes

Let us recall the definition of a particular version of the density of a occupation measure, which is referred to as the *local time*.

Definition 3.2.1 Let $t \geq 0$, $x \in \mathbb{R}$. The *local time* of $\eta(t)$ at level x and time t , denoted by $L_t(x)$, is defined by

$$\begin{aligned} L_t(x) &: = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon} \int_0^t \chi_{\{|\eta(s)-x| < \varepsilon\}} ds \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon} \lambda(\{s \in [0, t] : \eta(s) \in (x - \varepsilon, x + \varepsilon)\}), \end{aligned}$$

where λ is the Lebesgue measure.

We mention that the local time of $\eta(t)$ exists, if the integrability condition

$$\int_{\mathbb{R}} \operatorname{Re} \left(\frac{1}{1 + \Psi(\lambda)} \right) d\lambda < \infty \quad (3.2.1)$$

holds, where Ψ denotes the characteristic exponent of $\eta(t)$ (see e.g. [B]). Since we have the inequality

$$\operatorname{Re} \left(\frac{1}{1 + \Psi(\lambda)} \right) \leq \frac{1}{1 + \operatorname{Re} \Psi(\lambda)},$$

condition (3.1.2) entails (3.2.1), giving the existence of $L_t(x)$. We point out that $(x, \omega) \mapsto L_t(x)(\omega)$ belongs to $L^2(\lambda \times P)$ for all $t \geq 0$ and that

$$\int_0^t f(\eta(s)) ds = \int_{\mathbb{R}} f(x) L_t(x) dx \quad (3.2.2)$$

for all measurable bounded function $f \geq 0$ a.s. Relation (3.2.2) is called the *occupation density formula*. Note that $t \mapsto L_t(x)$ is an increasing process and that $L_t(\cdot)$ has compact support for all $t > 0$ a.e. Furthermore (3.1.2) implies that $x \mapsto L_t(x)$ is Hölder-continuous a.e. for every $t > 0$ (see p. 151 in [B]).

Next we give a rigorous proof for relation (3.1).

Proposition 3.2.2 Fix $T > 0$. Then

$$L_T(x) = \int_0^T \delta_x(\eta(s)) ds$$

for all $x \in \mathbb{R}$ a.e.

Proof Let f be a continuous function with compact support in $[-r, r] \subset \mathbb{R}$. Define the function $Z : [0, T] \times \mathbb{R} \times [-r, r] \rightarrow (\mathcal{S})^*$ by

$$Z(t, \lambda, y) = e^{i\lambda(\eta(t)-y)}.$$

First we want to show that Z is Bochner-integrable in $(\mathcal{S})^*$ with respect to the measure $\lambda^3 = \lambda \times \lambda \times \lambda$. One observes that $(t, \lambda, y) \mapsto \langle Z(t, \lambda, y), l \rangle$ is $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}([-r, r])$, $\mathcal{B}(\mathbb{R})$ -measurable for all $l \in (\mathcal{S})$. Further we consider the exponential process

$$Y(t) := \exp\left(\int_0^t \varphi(s) d\eta(s) - \int_0^t \int_{\mathbb{R}} (e^{\varphi(s)\varsigma} - 1 - \varphi(s)\varsigma) \nu(d\varsigma) ds\right).$$

for deterministic functions φ such that $e^{\varphi(s)\varsigma} - 1 \in L^2([0, T] \times \mathbb{R}, \lambda \times \nu)$. Then by the Lévy-Itô formula we have

$$dY(t) = Y(t_-) \int_{\mathbb{R}} (e^{\varphi(t)\varsigma} - 1) \tilde{N}(dt, d\varsigma),$$

so that

$$Y(t) = 1 + \int_0^t Y(s_{1-}) \int_{\mathbb{R}} (e^{\varphi(s_1)\varsigma_1} - 1) \tilde{N}(ds_1, d\varsigma_1).$$

If we iterate this process, we get the following chaos expansion for $Y(t)$:

$$Y(t) = \sum_{n \geq 0} I_n(g_n) \text{ in } L^2(P)$$

with $g_n(s_1, \varsigma_1, \dots, s_n, \varsigma_n) = \frac{1}{n!} \prod_{j=1}^n (e^{\varphi(s_j)\varsigma_j} - 1) \chi_{t > \max(s_i)}$, where I_n denotes the iterated integral in (2.5). Thus we obtain for $\varphi(s) \equiv i\lambda$

$$Z(t, \lambda, y) = \sum_{n \geq 0} I_n(g_n \cdot h) \text{ in } L^2(P) \quad (3.2.3)$$

with the function $h(t, \lambda, y) = \exp(\int_0^t \int_{\mathbb{R}} (e^{i\lambda\varsigma} - 1 - i\lambda\varsigma) \nu(d\varsigma) ds - i\lambda y)$. We also get the isometry

$$E(Z(t, \lambda, y))^2 = \sum_{n \geq 0} n! \|g_n \cdot h\|_{L^2((\lambda \times \nu)^n)}^2$$

Now, let us have a look at the weighted sum

$$\|Z(t, \lambda, y)\|_{-q}^2 := \sum_{n \geq 0} n! \|g_n \cdot h\|_{L^2((\lambda \times \nu)^n)}^2 e^{-qn} < \infty.$$

for $q \geq 0$. Then

$$\begin{aligned} & \|Z(t, \lambda, y)\|_{-q}^2 \\ & \leq \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{R}^n} \int_{[0, T]^n} \\ & \quad \left(\prod_{j=1}^n |e^{i\lambda\varsigma_j} - 1| e^{-t \operatorname{Re} \Psi(\lambda)} \right)^2 \chi_{t > \max(s_i)} ds_1 \dots ds_n \nu(d\varsigma_1) \dots \nu(d\varsigma_n) \cdot e^{-qn} \\ & \leq \sum_{n \geq 0} \frac{1}{n!} t^n 2^n (\operatorname{Re} \Psi(\lambda))^n e^{-2t \operatorname{Re} \Psi(\lambda)} \cdot e^{-qn}, \end{aligned}$$

where we used the fact that

$$\int_{\mathbb{R}} |e^{i\lambda\zeta} - 1|^2 \nu(d\zeta) = 2 \operatorname{Re} \Psi(\lambda).$$

So it follows that

$$\|Z(t, \lambda, y)\|_{-q} \leq \sum_{n \geq 0} \frac{1}{(n!)^{\frac{1}{2}}} t^{\frac{n}{2}} 2^{\frac{n}{2}} (\operatorname{Re} \Psi(\lambda))^{\frac{n}{2}} e^{-t \operatorname{Re} \Psi(\lambda)} \cdot e^{-\frac{q}{2}n}. \quad (3.2.4)$$

Next suppose that $|\lambda|^{-(1+\varepsilon)} \operatorname{Re} \Psi(\lambda) \geq 1$ for $|\lambda| \geq K \geq 0$. Then

$$\begin{aligned} & \int_{-r}^r \int_{|\lambda| \geq K} \int_0^T \|Z(t, \lambda, y)\|_{-q} dt d\lambda dy \\ & \leq 2r \sum_{n \geq 0} \frac{1}{(n!)^{\frac{1}{2}}} 2^{\frac{n}{2}} e^{-\frac{q}{2}n} \int_{|\lambda| \geq K} (\operatorname{Re} \Psi(\lambda))^{\frac{n}{2}} \int_0^T t^{\frac{n}{2}} e^{-t \operatorname{Re} \Psi(\lambda)} dt d\lambda \\ & \leq 2r \sum_{n \geq 0} \frac{1}{(n!)^{\frac{1}{2}}} 2^{\frac{n}{2}} e^{-\frac{q}{2}n} \int_{|\lambda| \geq K} \frac{1}{\operatorname{Re} \Psi(\lambda)} d\lambda \cdot \Gamma\left(\frac{n}{2} + 1\right), \end{aligned}$$

where Γ denotes the Gamma function. Together with (3.1.2) we receive

$$\begin{aligned} & \int_{-r}^r \int_{|\lambda| \geq K} \int_0^T \|Z(t, \lambda, y)\|_{-q} dt d\lambda dy \\ & \leq \operatorname{const.} \sum_{n \geq 0} \frac{(\frac{n}{2})!}{(n!)^{\frac{1}{2}}} 2^{\frac{n}{2}} e^{-\frac{q}{2}n} \int_{|\lambda| \geq K} \frac{1}{\operatorname{Re} \Psi(\lambda)} d\lambda \\ & \leq \operatorname{const.} \sum_{n \geq 0} \frac{(\frac{n}{2})!}{(n!)^{\frac{1}{2}}} 2^{\frac{n}{2}} e^{-\frac{q}{2}n}. \end{aligned}$$

Since

$$\frac{((\frac{n}{2})!)^2}{n!} \leq \frac{2^{\lfloor \frac{n}{2} \rfloor}}{2^{2\lfloor \frac{n+1}{2} \rfloor}} \leq 1$$

for all $n \in \mathbb{N}$, we conclude that

$$\int_{-r}^r \int_{|\lambda| \geq K} \int_0^T \|Z(t, \lambda, y)\|_{-q} dt d\lambda dy < \infty.$$

for all $q \geq 2$. It also follows directly from (3.2.4) that

$$\int_{-r}^r \int_{|\lambda| \leq K} \int_0^T \|Z(t, \lambda, y)\|_{-q} dt d\lambda dy < \infty$$

for all $q \geq 2$. Therefore we have

$$\int_{-r}^r \int_{\mathbb{R}} \int_0^T \|Z(t, \lambda, y)\|_{-q} dt d\lambda dy < \infty \quad (3.2.5)$$

for all $q \geq 2$. Let $l \in (\mathcal{S})$. Then it is not difficult to verify that

$$|\langle Z(t, \lambda, y), l \rangle| \leq \text{const.} \|Z(t, \lambda, y)\|_{-2} \quad (t, \lambda, y) - \text{a.e.}$$

The latter gives

$$\int_{-r}^r \int_{\mathbb{R}} \int_0^T |\langle Z(t, \lambda, y), l \rangle| dt d\lambda dy < \infty$$

for all $l \in (\mathcal{S})$. Note that $\|Z(t, \lambda, y)\|_{-q}$ is measurable. Thus we proved the Bochner-integrability of Z in $(\mathcal{S})^*$. By using Fubini and Lemma 3.1.3 we get

$$\int_{\mathbb{R}} f(y) \int_0^T \delta_y(\eta(t)) dt dy = \int_0^T f(\eta(t)) dt.$$

Then we can deduce from relation (3.2.2) that

$$\int_{\mathbb{R}} f(y) \int_0^T \delta_y(\eta(t)) dt dy = \int_{\mathbb{R}} f(y) L_T(y) dy \quad (3.2.6)$$

for all continuous $f : \mathbb{R} \rightarrow \mathbb{R}$ with compact support. Using a density argument we find

$$L_T(x) = \int_0^T \delta_x(\eta(t)) dt \text{ for a.a. } x \in \mathbb{R} \quad P - \text{a.e.} \quad (3.2.7)$$

Let $l \in (\mathcal{S})$. The map $x \mapsto \delta_x(\eta(t))$ is continuous by Theorem 3.1.4. Thus $\langle \delta_x(\eta(t)), l \rangle = \frac{1}{2\pi} \int_0^T \int_{\mathbb{R}} \langle Z(t, \lambda, x), l \rangle d\lambda dt$ is continuous, too. Then, based on (3.2.4), one shows that $x \mapsto \int_0^T \delta_x(\eta(t)) dt$ is continuous in $(\mathcal{S})^*$. Since $x \mapsto L_T(x)$ is Hölder-continuous, relation (3.2.7) is valid for *all* $x \in \mathbb{R}$ P -a.e. \square

Remark 3.2.3 Relation (3.2.5) in the proof of Proposition 3.2.2 shows that the Donsker delta function $\delta_x(\eta(t))$ even takes values in the Lévy-Hida distribution space $(\mathcal{S})^* \subset (\mathcal{S})_{-1}$.

We are coming to the main result of our paper.

Theorem 3.2.4 The chaos expansion of the local time $L_T(x)$ of $\eta(t)$ is given by

$$\begin{aligned} L_T(x) &= \sum_{n \geq 0} I_n(f_n) \\ &= \sum_{n \geq 0} n! \int_0^\infty \int_{\mathbb{R}} \dots \int_0^{t_2} \int_{\mathbb{R}} f_n(s_1, \varsigma_1, \dots, s_n, \varsigma_n) \tilde{N}(ds_1, d\varsigma_1) \dots \tilde{N}(ds_n, d\varsigma_n) \end{aligned}$$

in $L^2(P)$ with the symmetric functions

$$f_n(s_1, \varsigma_1, \dots, s_n, \varsigma_n) = \frac{1}{2\pi} \frac{1}{n!} \int_0^T \int_{\mathbb{R}} \left(\prod_{j=1}^n (e^{i\lambda \varsigma_j} - 1) \right) h(t, \lambda, x) \chi_{t > \max(s_i)} d\lambda dt$$

for

$$h(t, \lambda, x) = \exp\left(t \int_{\mathbb{R}} (e^{i\lambda\zeta} - 1 - i\lambda\zeta)\nu(d\zeta) - i\lambda x\right).$$

Proof By Proposition 3.2.2 we can write $L_T(x) = \int_0^T \delta_x(\eta(t))dt$. Using the definition of the function $Z(t, \lambda, y) = e^{i\lambda(\eta(t)-y)}$ in the proof of Proposition 3.2.2 and its chaos representation (3.2.3), we get

$$L_T(x) = \frac{1}{2\pi} \int_0^T \int_{\mathbb{R}} \sum_{n \geq 0} I_n(g_n \cdot h) d\lambda dt \quad (3.2.8)$$

with $g_n(s_1, \varsigma_1, \dots, s_n, \varsigma_n) = \frac{1}{n!} \prod_{j=1}^n (e^{i\lambda\varsigma_j} - 1) \chi_{t > \max(s_i)}$ and h as in the statement of this theorem. Because of the inequality (3.2.4) and similar estimates directly after this relation in the proof of Proposition 3.2.2 we can take the sum sign outside the double integral in (3.2.8). Thus we obtain

$$\frac{1}{2\pi} \int_0^T \int_{\mathbb{R}} \sum_{n \geq 0} I_n(g_n \cdot h) d\lambda dt = \frac{1}{2\pi} \sum_{n \geq 0} \int_0^T \int_{\mathbb{R}} I_n(g_n \cdot h) d\lambda dt \text{ in } (\mathcal{S})^*. \quad (3.2.9)$$

Further we can interchange the integrals in (3.2.9), so that we obtain

$$L_T(x) = \frac{1}{2\pi} \sum_{n \geq 0} I_n\left(\int_0^T \int_{\mathbb{R}} g_n \cdot h d\lambda dt\right) \text{ in } (\mathcal{S})^*.$$

Note that this is a consequence of the integrability condition (3.2.5). Since $L_T(x)$ is in $L^2(P)$, the result follows. \square

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