Dept. of Math. University of Oslo
Pure Mathematics No. 32
ISSN 0806–2439 October 2004

A CONVERGENCE RATE FOR SEMI-DISCRETE SPLITTING APPROXIMATIONS FOR DEGENERATE PARABOLIC EQUATIONS WITH SOURCE TERMS

ESPEN ROBSTAD JAKOBSEN AND KENNETH H. KARLSEN

Abstract. We study a semi-discrete splitting method for computing approximate viscosity solutions of the initial value problem for a class of nonlinear degenerate parabolic equations with source terms. It is fairly standard to prove that the semi-discrete splitting approximations converge to the desired viscosity solution as the splitting step Δt tends to zero. The purpose of this paper is, however, to consider the more difficult problem of providing a precise estimate of the convergence rate. Using viscosity solution techniques we establish the L^∞ convergence rate $\mathcal{O}(\sqrt{\Delta t})$ for the approximate solutions, and this estimate is robust with respect to the regularity of the solutions. We also provide an extension of this result to weakly coupled systems of equations, and in the case of more regular solutions we recover the "classical" rate $\mathcal{O}(\Delta t)$. Finally, we analyze in an example a fully discrete splitting method.

1. Introduction

The purpose of this paper is to study the error associated with a widely used time-splitting method for computing approximate solutions of the initial value problem for a class of nonlinear degenerate parabolic equations.

A representative for the class of equations that we study is the following Hamilton-Jacobi equation perturbed by a nonlinear possibly degenerate viscous term:

$$\begin{aligned} &u_t + F(Du) - c(Du)\Delta u = G(u) \quad \text{in } \ Q_T = \mathbb{R}^N \times (0,T), \\ &u(x,0) = u_0(x) \quad \text{in } \ \mathbb{R}^N, \end{aligned} \tag{1.1}$$

Here, $u(x,t)$ is the scalar function that is sought, u_0 is the initial function, F is the Hamiltonian, $c \geq 0$ is a scalar function representing "diffusion" effects, G is the source term, D denotes the gradient with respect to $x = (x_1, \dots, x_N)$, and D^2 denotes the Hessian with respect to x. Note that the first order Hamilton-Jacobi equation is a special case of (1.1). We shall later consider more general equations than (1.1), but for the moment we restrict our attention to (1.1). It is also possible to consider weakly coupled systems of equations. We will come back to this in the final section of the paper (see also (1.6) below).

Degenerate parabolic equations arise in a variety of applications, ranging from image processing, via mathematical finance, to the description of evolving interfaces (front propagation problems), see the lecture notes [1] for an overview. Due to the

---

Date: October 7, 2004.

Key words and phrases. nonlinear degenerate parabolic equation, viscosity solution, numerical method, operator splitting, convergence rate.

This work is supported by the Norwegian Research Council via grant no. 121531/410 and the BeMatA program, and the European network HYKE, funded by the EC as contract HPRN-CT-2002-00282.

possibly degenerate diffusion operator, problems such as (1.1) do not have classical solutions and it becomes necessary to work with a certain type of generalized solutions. More precisely, it turns out that the correct mathematical framework in which to analyze partial differential equations such as (1.1) and their numerical schemes is provided by the theory of viscosity solutions. We refer to Crandall, Ishii, and Lions [13] for an overview of this theory, which applies to fully nonlinear first and second order partial differential equations.

In this paper, we are concerned with a semi-discrete numerical method for calculating approximate viscosity solutions of (1.1). Roughly speaking, the method studied herein is based on "splitting off" or isolating the effect of the source term G. This operator splitting technique has been used frequently in the literature to extend sophisticated numerical methods for homogeneous first order partial differential equations to non-homogeneous first order partial differential equations, see, e.g., [23, 24, 30, 38, 37]. The present paper represents one of the first attempts to thoroughly analyze this source splitting technique for second order, possibly degenerate, partial differential equations.

To describe the operator splitting method in our "second order" context, let $v(x,t) = S(t)v_0(x)$ denote the unique viscosity solution of the homogeneous second order viscous Hamilton-Jacobi equation

$$v_t + F(Dv) - c(Dv)\Delta v = 0, \qquad v(x,0) = v_0(x), \qquad x \in \mathbb{R}^N,\ t > 0. \tag{1.2}$$

Here $S(t)$ is the so-called solution operator associated with (1.2) at time t. Furthermore, let $E(t)$ denote the explicit Euler operator, i.e., $v(x,t) = E(t)v_0(x)$ is defined by

$$v(x,t) = v_0(x) + t\,G(v_0(x)).$$

Observe that $E(t)$ is a (fully discrete) approximate solution operator associated with the ordinary differential equation $v_t = G(v)$. Fix a splitting (or time) step $\Delta t > 0$ and an integer $n \geq 1$ such that $n\Delta t = T$. Our operator splitting method then takes the form

$$v(x,t_i) := \Big[S(\Delta t)E(\Delta t)\Big]^i u_0(x), \tag{1.3}$$

where $t_i = i\Delta t$, $i = 1,\dots,n$. It fairly easy to prove that the approximate solutions generated by (1.3) converge to the exact viscosity solution of (1.1) as $\Delta t \to 0$, thereby justifying the term "approximate solution". The main result of this paper is, however, that these approximate solutions converge with an explicit rate as $\Delta t \to 0$ (see below).

Regarding turning (1.3) into a fully discrete splitting method, we simply have to choose an appropriate numerical method for the homogeneous problem (1.2), and a variety of different methods exist for that purpose. It is not, however, the goal of this paper to study the error induced by a numerical discretization of (1.2). This is a separate and difficult task for which we refer to [4, 5, 19, 26, 27] (so far general results exist only in the context of convex Hamilton-Jacobi-Bellman equations). Nevertheless, in Section 5 we provide a fully discrete example where the convergence rate is obtained using the methods of [26, 19].

The convergence analysis (without error estimates) of numerical methods for degenerate equations has been conducted by many authors. We do not intend to give a survey here but refer only to a few papers currently known to the authors: Barles and Souganidis [7], Barles [2], Barles, Daher, and Romano [3], Camilli and

Falcone [10], Davis, Panas, and Zariphopoulou [14], Fleming and Soner [16], Krylov [26, 27], Kuo and Trudinger [28], Kushner and Dupuis [29]. Following the guidelines set forth by Barles and Perthame [6] and Barles and Souganidis [7], many authors exploit the strong comparison principle for viscosity sub- and supersolutions when proving convergence of their approximate viscosity solutions. The disadvantage with the Barles-Perthame-Souganidis approach is that it seems difficult to get an explicit estimate of the rate of convergence, i.e., an error estimate. Indeed, very few papers seem to provide such estimates, and we only know of the following ones: Krylov [26, 27], Barles and Jakobsen [4, 5], Jakobsen [18, 19], Cockburn, Gripenperg, and Londen [11], Jakobsen and Karlsen [22, 21], and Deckelnick [15]. Krylov and Barles and Jakobsen deal with the degenerate Bellman equation and prove convergence rates for finite difference schemes. Deckelnick considers a certain finite difference scheme for the mean curvature equation. Cockburn, Gripenperg, and Londen and Jakobsen and Karlsen prove continuous dependence estimates, which immediately imply convergence rates for vanishing viscosity approximations.

For smooth solutions, it is not difficult to show via a classical truncation error analysis that the approximate solutions generated by the splitting method (1.3) are first order accurate (see, e.g., [35]). We are, on the other hand, interested in the accuracy of (1.3) when the solutions of (1.1) are non-smooth. Indeed, the main result of this paper is that the L^∞ error associated with the time splitting (1.3) is of order $\sqrt{\Delta t}$. More precisely, we prove that

$$\max_{i=1,\dots,n} \left\| u(\cdot,t_i) - v(x,t_i) \right\|_{L^\infty} \leq K\sqrt{\Delta t}, \tag{1.4}$$

for some constant $K > 0$ depending on the data of the problem (and the x-Lipschitz norm of u, v) but not Δt. It is interesting to compare the convergence rate in (1.4) with the linear rate $\mathcal{O}(\Delta t)$ obtained in [23] for first order Hamilton-Jacobi equations. Roughly speaking, the loss of convergence rate of $1/2$ is due to the second order differential operator in (1.1) and the fact we are working with functions that are merely Lipschitz continuous in space. On the other hand, if the involved solutions are more regular (in x), say, uniformly bounded in $W^{2,\infty}$, then we prove that the rate convergence improves to $\mathcal{O}(\Delta t)$.

Although there are similarities, the proof of an explicit convergence rate for the time-splitting method is more involved here in the second order case than in the first order Hamilton-Jacobi case [23]. Let us also mention that the approximation theory developed in [4, 5, 19, 26, 27] for convex equations cannot be applied to quasi-linear equations like (1.1). The proof of (1.4) consists of several steps. Here we will comment only on one of them. As in [23], we introduce a conveniently chosen comparison function $q(x,t_i)$ which is "close" to the splitting solution $v(x,t_i)$ for each i (see Section 4 for details). A central idea of the proof is then to estimate (instead of $u(\cdot,t) - v(\cdot,t)$) the quantity

$$\left\| u(\cdot,t) - q(\cdot,t) \right\|_{L^\infty} \text{ for all } t \in [t_{i-1}, t_i) \text{ for each } i.$$

As it turns out, the function $q(x,t)$ satisfies (in the sense of viscosity solutions) a nonlinear degenerate parabolic equation of the form

$$\begin{aligned} & q_t + \tilde{F}(x, Dq) - \tilde{c}(x, Dq)\Delta q = \tilde{G}(x) \quad \text{in } \mathbb{R}^N \times (t_{i-1}, t_i), \\ & q(x, t_{i-1}) = q_i(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{1.5}$$

where $q_i(x)$, $\tilde{F}$, and $\tilde{c}$ are "close" to $v(x, t_{i-1})$, F, and c, respectively. Moreover, $\tilde{G}(x)$ is "close" to $G(q(x,t))$. Consequently, the proof of (1.4) is reduced to having an *explicit* continuous dependence estimate for viscosity solutions of nonlinear degenerate parabolic equations. A new aspect here is the need for a continuous dependence estimate for the coefficient c in the second order differential operator in (1.1). Estimates of this type are not a part of the standard theory of viscosity solutions [13]. In fact, continuous dependence estimates for viscosity solutions of second order equations were obtained only recently by Cockburn, Gripenberg, and Londen [11] and Jakobsen and Karlsen [22, 21]. As is the case nowadays with the comparison/uniqueness proofs for viscosity solutions of second order equations, the continuous dependence estimates in [11, 22, 21] are consequences of the maximum principle for semicontinuous functions [12, 13].

As will be explained in Section 5, our analysis applies to weakly coupled systems of equations. As an example of such a system we can take

$$u_t = \sum_{i,j=1}^{N} a_{i,j} u_{x_i x_j} + G(u,v), \qquad v_t = H(u,v), \tag{1.6}$$

where $A = (a_{i,j})_{i,j=1}^N$ is a nonnegative symmetric constant matrix, $u : Q_T \to \mathbb{R}$, $v = (v_1, \dots, v_M) : Q_T \to \mathbb{R}^N$, $M \geq 1$, and the nonlinearities $G, H = (H_1, \dots, H_M)$ are such that the initial value problem for the above system possesses a unique bounded viscosity solutions. A common semi-discrete splitting algorithm is then to alternatively solve the following two split problems:

$$u_t = \sum_{i,j=1}^{N} a_{i,j} u_{x_i x_j}$$

and

$$u_t = G(u,v), \qquad v_t = H(u,v).$$

The latter problem is herein solved with the Euler method. For this splitting method our results provide an explicit L^∞ rate of convergence of order $\mathcal{O}(\sqrt{\Delta t})$, which is robust with respect to the regularity of the solutions.

For example, mathematical models for wave processes in the cardiac tissue give raise to parabolic PDEs coupled to systems of ODEs, for which (1.6) can be viewed as a simple model example. The systems of ODEs describe the electro-chemical reactions taking place in the heart cells. In recent years there has been a lot of activity on numerically solving such coupled systems of equations, and many of the numerical approaches use operator splitting in one way or another to decouple the PDEs from the ODEs, see, for example, [33, 32, 31, 36] and the references cited therein. In [36], Sundnes, Lines, and Tveito use numerical experiments to study the error induced by operator splitting in the context of the so-called bidomain model for the electric activity in the heart. In particular, they observed reduced rates of convergence for sharp wave front solutions and "coarse grids".

For a model example like (1.6), our $\mathcal{O}(\sqrt{\Delta t})$ error estimate for source splitting is consistent with the numerical observation that the convergence rate is reduced when solutions are non-smooth or nearly so. We recall that for first-order equations, see [23, 24, 30, 38, 37], the rate of convergence is $\mathcal{O}(\Delta t)$, even in the non-smooth regime. Finally, we mention that convergence (without a rate) of a source splitting method for scalar convection-diffusion-reaction equations is proved in [25].

The rest of this paper is organized as follows: In Section 2, we state existence, uniqueness, comparison, and regularity results for viscosity solutions of the problem under consideration. Then we recall a continuous dependence estimate from [22] and use it to derive some a priori regularity estimates for exact viscosity solutions. In Section 3, we state the operator splitting algorithm precisely as well as the main convergence results. In Section 4, we give detailed proof of the result stated in Section 3. In Section 5 we give various extensions our our main result: (i). An extension to weakly coupled systems of equations. (ii). For more regular solutions we obtain the classical rate $\mathcal{O}(\Delta t)$. Finally, in Chapter 5, we also provide results for a fully discrete scheme using finite differences.

2. Definitions and Preliminary Results

In this section we first recall the notion of viscosity solutions, and give existence, uniqueness, and comparison results for the class of equations we shall study. We then recall a stability (continuous dependence) result from [22] (see also [11]), and derive from it some a priori estimates for exact viscosity solutions. Finally, we state regularity results for our solutions.

We need to introduce some notation. First let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^N$ and also the Frobenius matrix norm $|A| = \mathrm{tr}[A^T A]$ for any matrix A, where A^T denotes the transpose of A and tr denotes the trace. If X is a domain, and $f : X \to \mathbb{R}$ is a bounded measurable function on X, then $\|f\| := \text{ess sup}_{x\in X}|f(x)|$. For any continuous function $f : \mathbb{R}^N \times I \to \mathbb{R}$, where $I \subset [0, \infty)$ is a time interval, $Df(x, t)$ is the spatial gradient of $f(x, t)$ in the sense of distributions. In particular $\|Df\| < \infty$ means that $|f(x, t) - f(y, t)| \le \|Df\||x - y|$ for all $t \in I$ and $x, y \in \mathbb{R}^N$, that is Lipschitz continuity in x (uniformly in t). For functions $f : \mathbb{R}^N \to \mathbb{R}$, the same holds, just remove any mention of time t. We let $C(X; Y)$, $C_b(X; Y)$, and $W^{1,\infty}(X; Y)$ denote the spaces of continuous functions, bounded continuous functions, and bounded Lipschitz functions from X to Y (for some domains X, Y) respectively. Let $S(N)$ denote the spaces of $N \times N$ symmetric matrices. In this space we use the partial ordering $\le$, which is defined as follows: $X \le Y$ whenever $eXe \le eYe$ for every $e \in \mathbb{R}^N$. Finally, let $Q_T = \mathbb{R}^N \times (0, T)$.

In the rest of this section we shall consider the following initial value problem:

$$u_t + f(t, x, u, Du) - \mathrm{tr}[A(t, Du)D^2u] = 0 \quad \text{in } Q_T, \tag{2.1}$$

$$u(0, x) = u_0(x) \quad \text{in } \mathbb{R}^N. \tag{2.2}$$

We do not display the source term in this equation (think of it as hidden in the f term) because we want to give general definitions and results. In particular, (1.1) is special case of (2.1) with $f(t, x, u, Du) = F(Du) - G(u)$ and $A(t, Du) = c(Du)I$.

There are several equivalent ways to define viscosity solutions [13]. We will need only one of these definitions in this paper.

Definition 2.1 (Viscosity Solution). *Suppose $f \in C(\bar{Q}_T, \mathbb{R}, \mathbb{R}^N)$ and $0 \le A \in C([0, T] \times \mathbb{R}^N)$.*

(1) *A function $u \in C(Q_T)$ is a viscosity subsolution (supersolution) of (2.1) if for every $\phi \in C^2(Q_T)$, if $u - \phi$ attains a local maximum (minimum) at*

$(x_0, t_0) \in Q_T$, then

$$\phi_t(x_0,t_0)+f(t_0,x_0,u(x_0,t_0),D\phi(x_0,t_0)) + \mathrm{tr}[A(t,D\phi(x_0,t_0))D^2\phi(x_0,t_0)] \le 0\,(\ge 0).$$

(2) *A function $u \in C(Q_T)$ is a viscosity solution of (2.1) if it is both a viscosity sub- and supersolution of (2.1).*

(3) *A function $u \in C(\bar{Q}_T)$ is viscosity solution of the initial value problem (2.1) and (2.2) if u is a viscosity solution of (2.1) and $u(x,0) = u_0(x)$ in $\mathbb{R}^N$.*

We will require that (2.1) satisfies the following conditions:

(C1) $f \in C(\bar{Q}_T \times \mathbb{R} \times \mathbb{R}^N)$ is uniformly continuous on $\bar{Q}_T \times [-R,R] \times B_N(0,R)$ for each $R > 0$, where $B_m(0,R) = \{x \in \mathbb{R}^m : |x| \le R\}$.

(C2) $C^f := \sup_{\bar{Q}_T} |f(t,x,0,0)| < \infty$.

(C3) For each $R > 0$ there is a constant $C_R^f > 0$ such that $|f(t,x,r,p) - f(t,y,r,p)| \le C_R^f(1+|p|)|x-y|$, for $t \in [0,T]$, $|r| \le R$, $x,y,p \in \mathbb{R}^N$.

(C4) For every t,x,p,X and for $R > 0$, there is $\gamma_R \in \mathbb{R}$ such that for $-R \le s \le r \le R$ $f(t,x,r,p) - f(t,x,s,p) \ge \gamma_R(r-s)$.

(C5) For every t,p, $A(t,p) = a(t,p)a(t,p)^T$ for some matrix $a \in C([0,T] \times \mathbb{R}^N; \mathbb{R}^{N\times P})$.

Remark 2.2. *It is sufficient to consider $\gamma_R \le 0$ in (C4), because if $\gamma_R > 0$ the inequality still holds if you set the right-hand side to zero. It is also sufficient to consider only symmetric matrices A in (C5). This is a consequence of the fact that the trace of a matrix equals the trace of the symmetric part of the same matrix.*

We have the following result concerning existence, uniqueness, and comparison of viscosity solutions of (2.1):

Theorem 2.3 (Existence, uniqueness, and comparison)**.** *Assume that (C1)–(C5) hold, that γ_R in (C4) is independent of R, and that $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. Then there exists a unique bounded viscosity solution u of the initial value problem* (2.1) *and* (2.2)*. Moreover the following comparison result holds: Let u and v be viscosity solutions of* (2.1) *with initial data u_0 and v_0 respectively, where $u_0, v_0 \in W^{1,\infty}(\mathbb{R}^N)$, then*

$$\sup_{\mathbb{R}^N}(u(\cdot,t) - v(\cdot,t)) \le e^{-\gamma t}\|(u_0 - v_0)^+\|.$$

We give the outline of a proof inspired by Zhan [39].

Outline of proof. 1. Conditions (C1), (C3) – (C5) imply that a strong comparison result holds for bounded viscosity solutions. It is by now quite standard to prove this result, and we omit this proof. This result implies uniqueness.

2. The comparison result stated in the theorem follows from the strong comparison result in the following way: Check that $w(t,x) = u(t,x) - e^{-\gamma t}\|(u_0 - v_0)^+\|$ is a subsolution of (2.1) and note that $w(0,x) \le v_0(x)$. Strong comparison then yields $w(t,x) \le v(t,x)$ in $\bar{Q}_T$ which is the desired result.

3. Take u_ε to be the solution of (2.1) with smooth initial data $u_{0,\varepsilon} := u_0 * \rho_\varepsilon$, where ρ_ε is a mollifier (a smooth function with unit mass and support in $B(0, \varepsilon)$).

4. Since $u_{0,\varepsilon} \in W^{2,\infty}(\mathbb{R}^N)$ and (C2) holds, it is easy to check that for K_ε big enough, $\pm K_\varepsilon t + u_{0,\varepsilon}(x)$ are classical sub and supersolutions of (2.1).

5. Perron's method then yields the existence of a bounded continuous function u_ε solving (2.1) in the viscosity sense, satisfying $-K_\varepsilon t + u_{0,\varepsilon}(x) \leq u_\varepsilon(t,x) \leq K_\varepsilon t + u_{0,\varepsilon}(x)$. This also means that u_ε takes the initial values $u_{0,\varepsilon}$.

6. The sequence $\{u_\varepsilon\}_\varepsilon$ is Cauchy in $C_b(\bar{Q}_T)$. This follows from an easy application of the comparison result: $|u_\varepsilon(t,\cdot) - u_{\varepsilon'}(t,\cdot)|_0 \leq e^{\gamma t}|u_{0,\varepsilon} - u_{0,\varepsilon'}|_0 \leq C(\varepsilon + \varepsilon')$.

7. Since $C_b(\bar{Q}_T)$ is complete (under the supremum norm), the existence of $\lim_{\varepsilon\to 0} u_\varepsilon =: u \in C_b(\bar{Q}_T)$ follows. Moreover by the stability result for viscosity solutions (Lemma 6.1 in [13]) u is the viscosity solution of (2.1), so the proof is complete. □

Now we state a key result, namely an estimate for continuous dependence on the nonlinearities. Consider the two equations

$$(\mathrm{EQ}_i) \qquad u_t^i + f_i(t,x,u^i,Du^i) - \mathrm{tr}[A_i(t,Du^i)D^2u^i] = 0, \quad i = 1,2.$$

Then the following theorem, which is proved in [22] (Theorem 3.2 b)), gives an estimate of $u^1 - u^2$:

Theorem 2.4 (Continuous Dependence Estimate). *Assume (C1), (C3) – (C5) hold for f_i and A_i with constants γ_R^i for $i = 1,2$. Furthermore assume that there are functions $u^i \in C(\bar{Q}_T)$ with $\|u^i\|, \|Du^i\| \leq \infty$ for $i = 1,2$, such that u^1 and u^2 are respectively a viscosity subsolution of (EQ_1), and a viscosity supersolution of (EQ_2). Let $R_0 = \max(\|u^1\|, \|u^2\|)$, $\gamma = \min\left(\gamma_{R_0}^1, \gamma_{R_0}^2\right)$, and $D_{s,t}$ be the following set*

$$D_{s,t} := \Big\{ (\tau,x,r,p) : \tau \in [s,t], x \in \mathbb{R}^N, |r| \leq e^{-\gamma(t-s)} \min\left(\|u^1\|, \|u^2\|\right), \\ |p| \leq e^{-\gamma(t-s)} \min(\|Du^1\|, \|Du^2\|) \Big\}.$$

Then for $0 \leq s \leq t \leq T$ there exists a constant $\tilde{M}$ depending only on $T, \gamma, C_R^{f_i}$, and $\|Du^i\|$ for $i = 1,2$, such that

$$e^{\gamma(t-s)}\|\left(u^1(t,\cdot) - u^2(t,\cdot)\right)^+\| \leq \|u^1(s,\cdot) - u^2(s,\cdot)\| \\ + \sup_{D_{s,t}} \Big\{ (t-s)e^{\gamma(\tau-s)}|f_1(\tau,x,r,p) - f_2(\tau,x,r,p)| \\ + \tilde{M}(t-s)^{1/2}\,|a_1(\tau,p) - a_2(\tau,p)| \Big\}.$$

Note that if u^1 and u^2 are solutions (not only sub- and supersolutions), then by interchanging the roles of u^1 and u^2, the above result yields an estimate of $\|u^1 - u^2\|$. From Theorem 2.4 we can derive the following a priori estimates:

Corollary 2.5 (A priori estimates). *Assume (C1)–(C5) hold with $\gamma_R \leq 0$, and let $u \in C(\bar{Q}_T)$ be a viscosity solution of* (2.1) *with initial data u_0. Moreover assume that $R := \|u\| < \infty$ and define $L := \|Du\|$ $(\leq \infty)$, $\gamma := \gamma_R$. Then the following statements are true for every $t,s \in [0,T]$:*

(a) If $\gamma = \gamma_R$ is independent of R, then $\|u(\cdot,t)\| \leq e^{-\gamma t}\left(\|u_0\| + tC^f\right)$.

(b) $\|Du(\cdot,t)\| \leq e^{-\gamma t}(\|Du_0\| + tC^f_R(1+L))$, *where*

$$L \leq e^{T(2C^f_R e^{-\gamma T}-\gamma)}(\|Du_0\| + TC^f_R).$$

(c) If $L < \infty$*, then there is a finite constant* $K_0 > 0$ *such that*

$$\|u(\cdot,t) - u(\cdot,s)\| \leq K_0|t-s|^{1/2},$$

where

$$K_0 = e^{\gamma(t-s)}\big\{\tilde{M} \sup_{\substack{[s,t]\times\\ \{|p|\leq e^{-\gamma t}L\}}} |a(t,p)| + \sqrt{|t-s|}\big(C^f + \omega_f(1)(1+R+L)\big)\big\},$$

$\tilde{M}$ *is defined in Theorem 2.4, and* ω_f *is the modulus of continuity of* $f(t,x,r,p)$ *provided by (C1) when* $|r| \leq R$ *and* $|p| \leq L$.

Proof. (a) Note that 0 is a viscosity solution of $u_t - \mathrm{tr}[A(t,Du)D^2u] = 0$. The result now follows by applying Theorem 2.4 to u and 0 and also using (C2).

(b) Let $v(x,t) = u(x+h,t)$, then v is the viscosity solution to the following initial value problem,

$$v_t + f(t,x+h,v,Dv) - \mathrm{tr}[A(t,Dv)D^2v] = 0, \quad v(x,0) = u_0(x+h).$$

By Theorem 2.4 and (C3) we get

$$e^{\gamma t}\|u(t,\cdot) - v(t,\cdot)\| \leq \|u(0,\cdot) - v(0,\cdot)\| + tC^f_R(1+L)h.$$

This is exactly the first inequality in (b).

To prove the second part of (b), we use an inductive argument by Souganidis [34]. First choose an m such that

$$0 < \frac{TC^f_R}{m}e^{-\gamma T} \leq \frac{1}{2}.$$

Define $Q_i := \mathbb{R}^N \times (\frac{i-1}{m}T, \frac{i}{m}T]$, $\bar{Q}_i := \mathbb{R}^N \times [\frac{i-1}{m}T, \frac{i}{m}T]$, $u_i := u\,|_{\bar{Q}_i}$, and $L_i := \sup_{\bar{Q}_i}|Du(x,t)|$. Then u_i is the viscosity solution of (2.1) in Q_i with initial value $u_i(x, \frac{i-1}{m}T) = u(x, \frac{i-1}{m}T)$. By part one, we get

$$L_i \leq e^{-\gamma\frac{T}{m}}\Big(L_{i-1} + C^f_R\frac{T}{m}(1+L_i)\Big).$$

Solving this inequality for L_i, we get

$$L_i \leq \frac{e^{-\gamma\frac{T}{m}}}{1 - C^f_R\frac{T}{m}e^{-\gamma\frac{T}{m}}}\Big(L_{i-1} + C^f_R\frac{T}{m}\Big) \leq e^{2C^f_R\frac{T}{m}e^{-\gamma\frac{T}{m}}-\gamma\frac{T}{m}}\Big(L_{i-1} + C^f_R\frac{T}{m}\Big).$$

The last inequality follows from the fact that for $0 \leq x \leq \frac{1}{2}$, $\frac{1}{1-x} \leq e^{2x}$. By iterating this formula we get the second part of (b).

(c) Let $v(t,x) \equiv u(x,s)$ for all $t \in [s,T]$, i.e., v is the viscosity solution of the initial value problem $v_t = 0$, $v(x,s) = u(x,s)$. As in (a) we use Theorem 2.4 to get

$$\begin{aligned}&e^{\gamma(t-s)}\|u(\cdot,t) - u(\cdot,s)\| = e^{\gamma(t-s)}\|u(\cdot,t) - v(\cdot,t)\|\\ &\leq 0 + (t-s)\sup_{D_{s,t}}\big|f(\tau,x,r,p)\big| + (t-s)^{1/2}\tilde{M}\sup_{D_{s,t}}|a(\tau,p)|,\end{aligned}$$

The term $\sup_{D_{s,t}} |a(\tau,p)|$ is bounded by (C5), and by (C1) and (C2) we get

$$\sup_{D_{s,t}} \big|f(\tau,x,r,p)\big| \leq \sup_{D_{s,t}} \Big|f(\tau,x,0,0)+f(\tau,x,r,p)-f(\tau,x,0,0)\Big|$$
$$\leq C^f + \omega_f\Big(\sup_{[0,T]} \|u(\cdot,\tau)\| + \sup_{[0,T]} \|Du(\cdot,\tau)\|\Big) \leq C^f + \omega_f(1)(1+R+L).$$

□

As a direct consequence of part (b) and (c) in the previous theorem we get the following regularity result:

Proposition 2.6 (Regularity). *Assume (C1)–(C5) hold with $\gamma_R \leq 0$, $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, and u is the viscosity solution of the initial value problem* (2.1) *and* (2.2)*. Then there is a constant $K > 0$ such that*

$$|u(t,x)-u(s,y)| \leq K(|x-y|+|t-s|^{1/2}) \tag{2.3}$$

for all $t,s \in [0,T]$ and $x,y \in \mathbb{R}^N$.

3. Statement of the Main Result

In this section we state the main results concerning the convergence of the semi-discrete splitting method for the scalar initial value problem

$$\begin{aligned} &u_t + F(t,x,u,Du) - \mathrm{tr}[A(t,Du)D^2u] = G(t,x,u) \quad \text{in } \ Q_T, \\ &u(x,0) = u_0(x) \quad \text{in } \ \mathbb{R}^N. \end{aligned} \tag{3.1}$$

Observe that (3.1) is more general than (1.1). In applications, the F-term would normally not depend on u. However this u dependence is irrelevant for the analysis, so we keep it for the sake of generality.

We start by giving conditions on the data of the problem (3.1).

Conditions on F.

(F1) $F \in C(\bar{Q}_T \times \mathbb{R} \times \mathbb{R}^N)$ is uniformly continuous on $\bar{Q}_T \times [-R,R] \times B_N(0,R)$ for each $R > 0$.

(F2) $C^F := \sup_{\bar{Q}_T \times S(N)} |F(t,x,0,0)| < \infty$.

(F3) For each $R > 0$ there is a constant $C^F_R > 0$ such that
$$|F(t,x,r,p) - F(s,y,r,p)| \leq C^F_R(1+|p|)(|x-y|+|t-s|^{1/2})$$
for $t,s \in [0,T]$, $|r| \leq R$, $x,y,p \in \mathbb{R}^N$.

(F4) There is a constant $L^F > 0$ such that
$$|F(t,x,r,p) - F(t,x,s,p)| \leq L^F|r-s|$$
for $t \in [0,T]$, $r,s \in \mathbb{R}$, $x,p \in \mathbb{R}^N$.

(F5) For each $R > 0$ there is a constant $M^F_R > 0$ such that
$$|F(t,x,r,p) - F(t,x,r,q)| \leq M^F_R|p-q|$$
for $t \in [0,T]$, $|r| \leq R$, $x,p,q \in \mathbb{R}^N$ and $|p|,|q| \leq R$.

Conditions on G.

(G1) $G \in C(\bar{Q}_T \times \mathbb{R})$ is uniformly continuous on $\bar{Q}_T \times [-R, R]$ for each $R > 0$.

(G2) $C^G := \sup_{\bar{Q}_T} |G(t, x, 0)| < \infty$.

(G3) For each $R > 0$ there is a constant $C_R^G > 0$ such that $|G(t, x, r) - G(s, y, r)| \leq C_R^G(|x - y| + |t - s|^{1/2})$ for $s, t \in [0, T]$, $|r| \leq R$, $x, y \in \mathbb{R}^N$.

(G4) There is a constant $L^G > 0$ such that $|G(t, x, r) - G(t, x, s)| \leq L^G|r - s|$ for $t \in [0, T]$, $r, s \in \mathbb{R}$, $x \in \mathbb{R}^N$.

Conditions on A.

(A1) For every t, p, $A(t, p) = a(t, p)^T a(t, p), a \in C([0, T] \times \mathbb{R}^N; \mathbb{R}^{P \times N})$.

(A2) For each $R > 0$ there is a constant $M_R^a > 0$ such that $|a(t, p) - a(t, q)| \leq M_R^a|p - q|$ for $t \in [0, T]$, $p \in \mathbb{R}^N$, and $|p| \leq R$.

We note that under these assumptions and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, the conditions of Theorems 2.3 and 2.4, Proposition 2.6, and Corollary 2.5 are all satisfied for the initial value problem (3.1). In particular we have existence and uniqueness of bounded Hölder continuous viscosity solutions:

Theorem 3.1. *If (F1)–(A2) hold and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, then there exists a unique viscosity solution $u \in C_b(\bar{Q}_T)$, to the initial value problem* (3.1)*. Moreover, there is a $K > 0$ such that for all $t \in [0, T]$ and $x, y \in \mathbb{R}^N$*

$$|u(x, t) - u(y, s)| \leq K(|x - y| + |t - s|^{1/2}).$$

To define the operator splitting for (3.1), let $E(t, s) : W^{1,\infty}(\mathbb{R}^N) \to W^{1,\infty}(\mathbb{R}^N)$ denote the Euler operator defined by

$$E(t, s)v_0(x) = v_0(x) + (t - s)G(s, x, v_0(x)) \tag{3.2}$$

for $0 \leq s \leq t \leq T$ and $v_0 \in W^{1,\infty}(\mathbb{R}^N)$. Furthermore, let $S(t, s) : W^{1,\infty}(\mathbb{R}^N) \to W^{1,\infty}(\mathbb{R}^N)$ be the solution operator of the homogeneous parabolic equation

$$\begin{aligned} &v_t + F(t, x, v, Dv, D^2v) - \mathrm{tr}[A(t, Dv)D^2v] = 0 \quad \text{in } \mathbb{R}^N \times (s, T), \\ &v(x, s) = v_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{3.3}$$

where $v_0 \in W^{1,\infty}(\mathbb{R}^N)$. Note that S is well-defined on the time interval $[s, T]$ by Theorem 3.1, since (3.3) is a special case of (3.1).

The operator splitting solution $\{v(x, t_i)\}_{i=1}^n$, where $t_i = i\Delta t$ and $t_n \leq T$, is defined by

$$\begin{aligned} &v(x, t_i) = S(t_i, t_{i-1})E(t_i, t_{i-1})v(\cdot, t_{i-1})(x), \\ &v(x, 0) = v_0(x). \end{aligned} \tag{3.4}$$

Note that this approximate solution is defined only at discrete t-values. The main result in this paper states that the operator splitting solution, when (3.3) is solved exactly, converges with rate $\frac{1}{2}$ in Δt to the viscosity solution of (3.1).

Theorem 3.2. *Assume that conditions (F1)–(A2) hold. If $u(x, t) \in C_b(\bar{Q}_T)$ is the viscosity solution of (3.1) and $v(x, t_i)$ is the operator splitting solution (3.4), then* *there exists a constant $\bar{K} > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, F, a, and G, such that for $i = 1, \dots, n$*

$$\|u(\cdot, t_i) - v(\cdot, t_i)\| \leq \bar{K}(\|u_0 - v_0\| + \sqrt{\Delta t}).$$

We will prove this theorem in the next section.

Before we give the proof, we mention that two extensions of the above result are given in Chapter 5: (i) An extension to weakly coupled systems of equations. (ii) For more ($W^{2,\infty}$) regular solutions we establish the classical rate $\mathcal{O}(\Delta t)$.

Finally in Chapter 5, we consider a particular equation for which we can provide an error estimate for a fully discrete scheme where the S operator is approximated using finite differences.

4. Proof of the Main Result

In this section we provide a detailed proof of Theorem 3.2. We proceed by several steps. A key step is to introduce a suitable comparison function.

a) The comparison function.
The main step in the proof of Theorem 3.2 is to estimate the error between u and v for one single time interval of length Δt. Hence we are interested in estimating

$$\|u(\cdot, t_i) - S(t_i, t_{i-1})E(t_i, t_{i-1})v(\cdot, t_{i-1})\|, \qquad i = 1, \dots, n.$$

Now fix i, $i = 1, \dots, n$, and define the function $\zeta : \mathbb{R}^N \times [t_{i-1}, t_i] \to \mathbb{R}$ as follows

$$\zeta(x, t) := S(t, t_{i-1})E(t_i, t_{i-1})v(\cdot, t_{i-1})(x).$$

Note that ζ solves the homogeneous equation (3.3) on $[t_{i-1}, t_i]$, and that $\zeta(x, t_i) = v(x, t_i)$. To estimate the difference between $u(\cdot, t_i)$ and $v(\cdot, t_i)$, we introduce the comparison function $q^\delta : \mathbb{R}^N \times [t_{i-1}, t_i] \to \mathbb{R}$ defined by

$$q^\delta(x, t) = \zeta(x, t) + \psi^\delta(x, t), \tag{4.1}$$

where $\psi^\delta : \mathbb{R}^N \times [t_{i-1}, t_i] \to \mathbb{R}$ is defined by

$$\psi^\delta(x, t) = -(t_i - t) \int_{\mathbb{R}^N} \eta_\delta(z) G(t_{i-1}, x - z, v(x - z, t_{i-1}))\, dz. \tag{4.2}$$

Here $\eta_\delta(x) := \frac{1}{\delta^N}\eta(\frac{x}{\delta})$, where η is the standard mollifier satisfying

$$\eta \in C_0^\infty(\mathbb{R}^N), \quad \|D\eta\| \leq 2, \quad \eta(x) = 0 \text{ when } |x| > 1, \quad \int_{\mathbb{R}^N} \eta(x)\, dx = 1. \tag{4.3}$$

For each $x \in \mathbb{R}^N$ we see that $q^\delta(x, t_i) = v(x, t_i)$ and we will later show that

$$q^\delta(x, t_{i-1}) \to v(x, t_{i-1}) \ \text{ as } \ \delta \to 0.$$

The difference

$$u(\cdot, t_i) - v(\cdot, t_i) = u(\cdot, t_i) - q^\delta(\cdot, t_i)$$

will be estimated by deriving a bound on the difference

$$u(\cdot, t) - q^\delta(\cdot, t) \text{ for all } t \in [t_{i-1}, t_i].$$

To this end, observe that if ζ was a classical C^2 solution of the homogeneous equation (3.3), then q^δ would be a classical C^2 solution of

$$\begin{aligned} &q^\delta_t + F(t, x, q^\delta - \psi^\delta, Dq^\delta - D\psi^\delta) \\ &\qquad - \operatorname{tr}\left[A(t, Dq^\delta - D\psi^\delta)(D^2 q^\delta - D^2\psi^\delta)\right] = \psi^\delta_t \quad \text{in } \mathbb{R}^N \times (t_{i-1}, t_i), \\ &q^\delta(x, t_{i-1}) = \zeta(x, t_{i-1}) + \psi^\delta(x, t_{i-1}) \qquad \text{in } \ \mathbb{R}^N. \end{aligned} \tag{4.4}$$

It is easy to extend this result to the viscosity solution setting (see [23]), so we have that q^δ is a viscosity solution of (4.4).

Now we proceed by deriving a priori estimates for u, v, ψ^δ, and q^δ that are independent of Δt.

b) A priori estimates.
We start by analyzing S and E. Let $w, \tilde{w} \in W^{1,\infty}(\mathbb{R}^N)$ and assume that

$$R_1 := \max\Big\{\sup_{0\le s\le t\le T} \|E(t,s)w\|, \sup_{0\le s\le t\le T} \|S(t,s)w\|\Big\} < \infty. \tag{4.5}$$

For $0 \le s \le t \le T$, let $\bar{w}(x, t-s) = S(t,s)w(x)$. This function is a viscosity solution of equation (3.3) on $[0, T-s]$ when $F(t,x,r,p)$, $A(t,p)$ is replaced by $F(t+s,x,r,p)$, $A(t+s,p)$ respectively. The initial condition is $\bar{w}(x,0) = w(x)$. Applying Corollary 2.5 (a), (b), (c), and the comparison principle from Theorem 2.3 to $\bar{w}$ and then using $S(\tau+s,s)w(x) = \bar{w}(x,\tau)$, we get the following estimates

$$\|S(t,s)w\| \le e^{L^F(t-s)}\Big\{\|w\| + (t-s)C^F\Big\}, \tag{4.6}$$

$$\|D\{S(t,s)w\}\| \le e^{(L^F+K_1(R_1))(t-s)}\Big\{\|Dw\| + (t-s)C^F_{R_1}(1+TK_1(R_1))\Big\}, \tag{4.7}$$

$$\|S(t,s)w - S(t,s)\tilde{w}\| \le e^{L^F(t-s)}\|w-\tilde{w}\|, \tag{4.8}$$

$$\|S(t,s)w - w\| \le K_0\sqrt{t-s}, \tag{4.9}$$

where

$$K_1(R) = C^F_R e^{T(2C^F_R e^{L^F T} + L^F)} \tag{4.10}$$

and K_0 is as defined in Corollary 2.5 by replacing u by w, and depends on F, a, w in such a way that $\|w\|, \|Dw\| < \infty$ implies $K_0 < \infty$. Note that $\gamma = -L^F$, and that in the expression (4.7), the constant L in Corollary 2.5 (b) is replaced by its bound.

Let us turn to E. The following estimates follow from the definition of E, $E(t,s)w(x) = w(x) + (t-s)G(s,x,w(x))$, and the properties of G and w:

$$\|E(t,s)w\| \le (1 + L^G(t-s))\|w\| + (t-s)C^G \tag{4.11}$$

$$\|D\{E(t,s)w\}\| \le (1 + L^G(t-s))\|Dw\| + (t-s)C^G_{R_1} \tag{4.12}$$

$$\|E(t,s)w - w\| \le (t-s)(C^G + L^G\|w\|) \tag{4.13}$$

Now we see that assumption (4.5) holds. Just replace $t-s$ by T in expressions (4.6) and (4.11).

Let us define the following constants,

$$\begin{aligned}
&\bar{L} := 2\max(L^F, L^G),\\
&C := C^F + C^G,\\
&C_R := C^F_R + C^G_R \quad \text{for every} \quad R > 0,\\
&M_R := \max\{M^F_R, M^a_R\} \quad \text{for every} \quad R > 0.
\end{aligned} \tag{4.14}$$

Now we give the a priori estimates.

Lemma 4.1. *There exists a constant R_2 independent of Δt such that $\max_{1\le i\le n}\|v(\cdot,t_i)\| < R_2$. Moreover with $K_1(R)$ defined in (4.10), for every $1 \le i \le n$ the following statements hold:*

(a) $\|v(\cdot,t_i)\| \le e^{\bar{L}t_i}\Big\{\|v_0\| + t_iC\Big\}$,

(b) $\|Dv(\cdot,t_i)\| \le e^{(\bar{L}+K_1(R_2))t_i}\Big\{\|Dv_0\| + t_iC_{R_2}(1+TK_1(R_2))\Big\}$.

Proof. By the definition of v (3.4), $v(x,t_i) = S(t_i,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})(x)$ and $v(x,0) = v_0(x)$. Assume there is a constant R_2 independent of Δt such that

$$\max_{1\le i\le n}\|v(\cdot,t_i)\| < R_2. \tag{4.15}$$

In expressions (4.6) – (4.13) replace R_1 by R_2, t by t_i, s by t_{i-1}, and w by $v(\cdot,t_{i-1})$. Successive use of expressions (4.6) and (4.11) yield (a), and similarly (b) follows from (4.7) and (4.12). In expression (a), replace t_i by T and we see that the assumption (4.15) holds. □

Lemma 4.2. *Let V_N denote the volume of the unit ball in $\mathbb{R}^N$. Then for every $1 \le i \le n$ and $t \in [t_{i-1},t_i]$,*

(a) $\|\psi^\delta(\cdot,t)\| \le (t_i - t)\Big\{C^G + L^G\|v(\cdot,t_{i-1})\|\Big\}$,

(b) $\|D\psi^\delta(\cdot,t)\| \le (t_i - t)\Big\{C^G_{R_2} + L^G\|Dv(\cdot,t_{i-1})\|\Big\}$.

(c) $\|D^2\psi^\delta(\cdot,t)\| \le \frac{t_i-t}{\delta}2NV_N\Big\{C^G_{R_2} + L^G\|Dv(\cdot,t_{i-1})\|\Big\}$.

Proof. From the definition (4.2) of ψ^δ it is easy to see that (a) and (b) hold. We will only prove (c). Let e_j be the j-th basis vector in $\mathbb{R}^N$, and $h \in \mathbb{R}$. We then calculate

$$\begin{aligned}
&|\psi^\delta_{x_ix_j}(x,t)| = (t-t_i)\Big|\{G(t_{i-1},\cdot,v(\cdot,t_{i-1})) * \eta_{\delta\,x_ix_j}\}(x)\Big|\\
&= (t_i-t)\lim_{h\to 0}\Big|\Big\{G(t_{i-1},\cdot,v(\cdot,t_{i-1})) * \frac{1}{h}(\eta_{\delta\,x_i}(\cdot+he_j) - \eta_{\delta\,x_i}(\cdot))\Big\}(x)\Big|\\
&= (t_i-t)\lim_{h\to 0}\Big|\Big\{\frac{1}{h}\big(G(t_{i-1},\cdot-he_j,v(\cdot-he_j,t_{i-1})) - G(t_{i-1},\cdot,v(\cdot,t_{i-1})\big)*\eta_{\delta\,x_i}\Big\}(x)\Big|\\
&\le (t_i-t)\Big\{\big(C^G_{R_2} + L^G\|Dv(\cdot,t_{i-1})\|\big)\frac{2}{\delta^{N+1}}\delta^N V_N\Big\},
\end{aligned}$$

where the first equality is a property of convolutions, the second equality follows from the definition of the (partial) derivative and Lebesgue dominated convergence theorem, and the third equality is a change of variables. Finally, the inequality follows from (G3) and (G4) which imply that

$$|\eta_{\delta\,x_i}(x)| = \Big|\frac{1}{\delta^{N+1}}\eta_{x_i}\Big(\frac{x}{\delta}\Big)\Big| \le \frac{2}{\delta^{N+1}}$$

and

$$\begin{aligned}
&\big|G(t_{i-1},x-he_j,v(x-he_j,t_{i-1})) - G(t_{i-1},x,v(x,t_{i-1}))\big|\\
&\le C^G_{R_2}|h| + \bar{L}\|Dv(\cdot,t_{i-1})\||h|.
\end{aligned}$$

□

Now we are in a position to prove the following estimates:

Lemma 4.3. *Let $K_1(R)$ be defined in (4.10). For every $1 \le i \le n$ and $t \in [t_{i-1},t_i]$,* *(a)* $\|q^\delta(\cdot,t)\| \leq e^{2\bar{L}\Delta t}\Big\{\|v(\cdot,t_{i-1})\| + 2\Delta t C\Big\}$,

(b) $\|Dq^\delta(\cdot,t)\| \leq e^{(2\bar{L}+K_1(R_2))\Delta t}\Big\{\|v(\cdot,t_{i-1})\| + \Delta t\, C_{R_2}(2+TK_1(R_2))\Big\}$,

(c) There exists at constant M independent of t, i, and Δt such that

$$\|q^\delta(\cdot,t) - v(\cdot,t_{i-1})\| \leq M\sqrt{\Delta t}.$$

Proof. We only give the proof of (c). The other statements are easy consequences of expressions (4.6), (4.7), (4.11), (4.12), and Lemma 4.2 a) and b).

By Lemma 4.1 and estimates (4.6), (4.7), (4.11), and (4.12) there are finite constants R', L' (independent of i and Δt) such that

$$\sup_{[t_{i-1},t_i]} \|S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \leq R',$$

$$\sup_{[t_{i-1},t_i]} \|D\{S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})\}\| \leq L'.$$

Because of these bounds, estimate (4.9) gives the existence of a finite constant K_0' (also independent of i and Δt – see the the remarks below (4.9)) – such that

$$\|S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \leq K_0'\sqrt{\Delta t}.$$

By using expression (4.13) and Lemma 4.1 we can show that

$$\|E(t_i,t_{i-1})v(\cdot,t_{i-1}) - v(\cdot,t_{i-1})\| \leq \text{Const } \Delta t,$$

where the constant is independent of i and Δt. By Lemmas 4.2 and 4.1 we can find a constant independent of t, i, and Δt such that

$$\|\psi^\delta\| \leq \text{Const } \Delta t.$$

We conclude the proof by noting that $\Delta t \leq \sqrt{T}\sqrt{\Delta t}$ and that by the definition of q^δ, expression (4.1),

$$\begin{aligned}\|q^\delta(\cdot,t) - v(\cdot,t_{i-1})\| \leq &\|S(t,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})\| \\ &+ \|E(t_i,t_{i-1})v(\cdot,t_{i-1}) - v(\cdot,t_{i-1})\| + \|\psi^\delta\|.\end{aligned}$$

□

Finally we come to u. Using Corollary 2.5 with $f(t,x,r,p) = F(t,x,r,p) - G(t,x,r)$ we get the following estimates (see also the derivation of (4.6) and (4.7)):

Lemma 4.4. *There exists a constant R_3 such that $\max_{[0,T]}\|u(\cdot,t)\| < R_3$. Moreover with $K_2(R) = C_R \exp\big\{T(2C_R e^{\bar{L}T} + \bar{L})\big\}$, for $t \in [0,T]$ the following statements hold:*

(a) $\|u(\cdot,t)\| \leq e^{\bar{L}t}\Big\{\|u_0\| + tC\Big\}$,

(b) $\|Du(\cdot,t)\| \leq e^{(\bar{L}+K_2(R_3))t}\Big\{\|Du_0\| + tC_R(1+TK_2(R_3))\Big\}$.

There is a constant R_4 independent of t, i, and Δt such that $\|q^\delta(\cdot,t)\| \leq R_4$. This follows from Lemma 4.3 a) by replacing $\|v(\cdot,t_{i-1})\|$ by R_2 and Δt by T. Similarly there is a constant R_5 independent of t, i, and Δt such that $\|\psi^\delta(\cdot,t)\| \leq R_5$. Define

$$R := \max(R_2, R_3, R_4, R_5). \tag{4.16}$$

By a similar argument there is an L independent of t, i, and Δt such that

$$\max_{1\leq i\leq n}\|Dv(\cdot,t_i)\|,\ \sup_{[t_{i-1},t_i]}\|D\psi^\delta(\cdot,t)\|,\ \sup_{[t_{i-1},t_i]}\|Dq^\delta(\cdot,t)\|, \sup_{[0,T]}\|Du(\cdot,t)\| \leq L. \tag{4.17}$$

Furthermore, in view of equation (4.4), we set

$$\bar{M} = M_{2\max\{L,R\}}. \tag{4.18}$$

We are now in a position to prove Theorem 3.2.

c) The proof of Theorem 3.2
We prove Theorem 3.2 by applying Theorem 2.4 to u and q^δ. To do this we will prove that q^δ is a subsolution of a certain equation and a supersolution of another (closely related) equation. Actually we will find a function $\bar{A}$ and a constant $k(\Delta t, \delta)$ such that q^δ solves $\left|v_t + F[v] - \mathrm{tr}[\bar{A}[v]D^2 v]\right| \le k(\Delta t, \delta)$ in the viscosity sense.

Let ϕ be a C^2 function, and assume that $q^\delta - \phi$ has a local maximum point in (x,t). Then by the definition of viscosity subsolution and equation (4.4) we get

$$\begin{aligned}&\phi_t(x,t) - \psi_t^\delta(x,t)\\ &\quad + F(t,x,q^\delta(x,t) - \psi^\delta(x,t), D\phi(x,t) - D\psi^\delta(x,t))\\ &\qquad \le \mathrm{tr}\left[A\big(t, D\phi(x,t) - D\psi^\delta(x,t)\big)\big(D^2\phi(x,t) - D^2\psi^\delta(x,t)\big)\right].\end{aligned} \tag{4.19}$$

Now we estimate $\psi_t^\delta(x,t)$ and $F(t,x,q^\delta(x,t) - \psi^\delta(x,t), D\phi(x,t) - D\psi^\delta(x,t))$. First note that

$$\begin{aligned}&|\psi_t^\delta(x,t) - G(t_{i-1},x,q^\delta(x,t))|\\ &\quad = \Big|\int_{\mathbb{R}^N}\eta_\delta(z)G(t_{i-1},x-z,v(x-z,t_{i-1}))dz - G(t_{i-1},x,q^\delta(x,t))\Big|\\ &\quad \le \int_{\mathbb{R}^N}\eta_\delta(z)\Big|G(t_{i-1},x-z,v(x-z,t_{i-1})) - G(t_{i-1},x-z,q^\delta(x-z,t))\Big|dz\\ &\qquad + \int_{\mathbb{R}^N}\eta_\delta(z)\Big|G(t_{i-1},x-z,q^\delta(x-z,t)) - G(t_{i-1},x,q^\delta(x-z,t))\Big|dz\\ &\qquad + \int_{\mathbb{R}^N}\eta_\delta(z)\Big|G(t_{i-1},x,q^\delta(x-z,t)) - G(t_{i-1},x,q^\delta(x,t))\Big|dz\\ &\quad \le \bar{L}M\sqrt{\Delta t} + C_R\delta + \bar{L}L\delta,\end{aligned}$$

where M is given by Lemma 4.3 (c), and we have also used (G3) and (G4). Using this computation and (G3) again, we see that

$$\begin{aligned}\psi_t^\delta(x,t) &\le G(t,x,q^\delta(x,t)) + |G(t_{i-1},x,q^\delta(x,t)) - G(t,x,q^\delta(x,t))|\\ &\quad + |{\psi^\delta}_t(x,t) - G(t_{i-1},x,q^\delta(x,t))|\\ &\le G(t,x,q^\delta(x,t)) + \sqrt{\Delta t}\,\{\bar{L}M + C_R\} + \delta\{C_R + \bar{L}L\}.\end{aligned} \tag{4.20}$$

Regarding F, we have

$$\begin{aligned}&F(t,x,q^\delta(x,t) - \psi^\delta(x,t), D\phi(x,t) - D\psi^\delta(x,t))\\ &\ge F(t,x,q^\delta(x,t), D\phi(x,t)) - \bar{L}|\psi^\delta(x,t)| - \bar{M}|D\psi^\delta(x,t)|\\ &\ge F(t,x,q^\delta(x,t), D\phi(x,t)) - \Delta t\{\bar{L}(C + \bar{L}R) + \bar{M}(C_R + \bar{L}L)\}.\end{aligned} \tag{4.21}$$

Here we have used (F4), (F5), and Lemma 4.2. We turn to the trace term. Using the fact that (x,t) is a maximum point, we can get $|D\phi(x,t)| \le L$. We will use this fact to bound $|a(t, D\phi(x,t) - D\psi^\delta(t,x))|$. By (A2) and (4.17) we get

$$\begin{aligned}
&|a(t, D\phi(x,t) - D\psi^\delta(t,x))| \\
&\le |a(t,0)| + \bar{M}|D\phi(x,t) - D\psi^\delta(t,x)| \\
&\le \sup_{[0,T]} |a(t,0)| + 2\bar{M}L.
\end{aligned}$$

Now we note that $|\operatorname{tr} X| \le N|X|$ for any $N \times N$ matrix X. Using Lemma 4.2 enables us to get the following estimate,

$$\begin{aligned}
&\operatorname{tr}\left[A(t, D\phi(x,t) - D\psi^\delta(t,x))(D^2\phi(x,t) - D^2\psi^\delta(t,x))\right] \\
&\le \operatorname{tr}\left[A(t, D\phi(x,t) - D\psi^\delta(t,x))D^2\phi(x,t)\right] \\
&\qquad + \Big(\sup_{[0,T]} |a(t,0)| + 2\bar{M}L\Big)^2 \frac{\Delta t}{\delta} 2N^2 V_N (C_R + \bar{L}L).
\end{aligned} \tag{4.22}$$

Define the constants M_0, M_1 by

$$\begin{aligned}
M_0 &:= \sqrt{T}\bar{L}\{C + \bar{L}R\} + \sqrt{T}\bar{M}\{C_R + \bar{L}L\} + \bar{L}M + C_R, \\
M_1 &:= 2N^2 V_N (C_R + \bar{L}L)\big(\sup_{[0,T]} |a(t,0)| + 2\bar{M}L\big)^2.
\end{aligned} \tag{4.23}$$

Substituting (4.20), (4.21), and (4.22) into (4.19), we get

$$\begin{aligned}
&\phi_t(x,t) + F\big(t, x, q^\delta(x,t), D\phi(x,t)\big) - G\big(t, x, q^\delta(x,t)\big) \\
&\qquad - \operatorname{tr}\left[A(t, D\phi(x,t) - D\psi^\delta(t,x))D^2\phi(x,t)\right] \le k(\Delta t, \delta),
\end{aligned}$$

where

$$k(\Delta t, \delta) := \sqrt{\Delta t}\, M_0 + \delta\{C_R + \bar{L}L\} + \frac{\Delta t}{\delta} M_1. \tag{4.24}$$

In a similar way we can show that if $\bar{\phi}$ is C^2 and $q^\delta - \bar{\phi}$ has a local minimum in (x,t), then

$$\begin{aligned}
&\bar{\phi}_t(x,t) + F\big(t, x, q^\delta(x,t), D\bar{\phi}(x,t)\big) - G\big(t, x, q^\delta(x,t)\big) \\
&\qquad - \operatorname{tr}\left[A(t, D\phi(x,t) - D\psi^\delta(t,x))D^2\phi(x,t)\right] \ge -k(\Delta t, \delta).
\end{aligned}$$

Two applications of Theorem 2.4 to u and q^δ on the time interval $[t_{i-1}, t_i]$ then yields

$$\begin{aligned}
e^{-\bar{L}\Delta t}\|u(\cdot, t_i) - q^\delta(\cdot, t_i)\| &\le \|u(\cdot, t_{i-1}) - q^\delta(\cdot, t_{i-1})\| + \Delta t\, k(\Delta t, \delta) \\
&\qquad + \sqrt{\Delta t}\, K \sup_{D_{t_{i-1}, t_i}} \big|a(t,p) - a(t, p + D\psi^\delta(x,t))\big|.
\end{aligned} \tag{4.25}$$

The quantities D_{t_{i-1}, t_i} and K are defined in Theorem 2.4, and from the definition of K we see that it is independent of Δt and i.

Remember that $q^\delta(x, t_i) = v(x, t_i)$. To finish the proof we must estimate $\|u(\cdot, t_{i-1}) - q^\delta(\cdot, t_{i-1})\|$ and the a-term and choose δ in an appropriate way. First note that

$$
\begin{aligned}
&\left|v(x,t_{i-1}) - q^\delta(x,t_{i-1})\right| \\
&= \left|v(x,t_{i-1}) - E(t_i,t_{i-1})v(\cdot,t_{i-1})(x) - \psi^\delta(x,t_{i-1})\right| \\
&= \left|\Delta t\, G(t_{i-1},x,v(x,t_{i-1})) + \psi^\delta(x,t_{i-1})\right| \\
&\leq \Delta t \int_{\mathbb{R}^N} \eta_\delta(z) \Big| G(t_{i-1},x,v(x,t_{i-1})) \\
&\qquad\qquad - G(t_{i-1},x-z,v(x-z,t_{i-1}))\Big| dz \\
&\leq \Delta t\, \delta \bar{L} \|Dv(\cdot,t_{i-1})\| + \Delta t\, \delta C_R,
\end{aligned}
\tag{4.26}
$$

where the last estimate follows from the triangle inequality, (G4), and (G3). Furthermore using (A2) and Lemma 4.2 we get

$$
\sup_{D_{t_{i-1},t_i}} \left|a(t,p) - a(t,p+D\psi^\delta(x,t))\right| \leq \bar{M} \sup_{D_{t_{i-1},t_i}} |D\psi^\delta(x,t)| \leq \Delta t \bar{M}(C_R + \bar{L}L).
\tag{4.27}
$$

Combining (4.24), (4.25), (4.26), and (4.27), we get

$$
\begin{aligned}
&e^{-\bar{L}\Delta t}\|u(\cdot,t_i) - v(\cdot,t_i)\| = e^{-\bar{L}\Delta t}\|u(\cdot,t_i) - q^\delta(\cdot,t_i)\| \\
&\leq \|u(x,t_{i-1}) - v(x,t_{i-1})\| + \delta\Delta t\,\{C_R + \bar{L}L\} \\
&\quad + \Big(\Delta t^{3/2} M_0 + \Delta t\, \delta\{C_R + \bar{L}L\} + \frac{\Delta t^2}{\delta} M_1\Big) + \Delta t^{3/2} K \bar{M}(C_R + \bar{L}L).
\end{aligned}
$$

We choose $\delta = \sqrt{\Delta t}$, and with this choice we see that there is a constant K' such that

$$
\|u(\cdot,t_i) - v(\cdot,t_i)\| \leq \; e^{\bar{L}\Delta t}\|u(x,t_{i-1}) - v(x,t_{i-1})\| + \Delta t\sqrt{\Delta t}K',
$$

and K' does only depend on $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, F, G, a, and T, but not on Δt. This follows from the definition of $\bar{L}$, M_0, M_1, and Lemmas 4.1 – 4.4.

Since the fixed number i, $i = 1,\dots,n$, was arbitrary, successive use of the previous formula gives us

$$
\begin{aligned}
\|u(\cdot,t_j) - v(\cdot,t_j)\| &\leq e^{\bar{L}t_j}\|u_0 - v_0\| + \Delta t\sqrt{\Delta t}K' \sum_{i=1}^{j} e^{\bar{L}t_i} \\
&\leq e^{\bar{L}t_j}\|u_0 - v_0\| + \sqrt{\Delta t}K'Te^{\bar{L}T} \quad \text{for } j = 1,\dots,n.
\end{aligned}
$$

Let $\bar{K} := (1 + K'T)e^{\bar{L}T}$, and our theorem is proved.

5. Extensions and a fully discrete example

In this section we will give some extensions of the main result. Moreover, as an example, we show how to obtain the rate of convergence for a fully discrete splitting method for a particular degenerate parabolic equation.

5.1. **Weakly coupled systems.** In this section we extend our main result (see Theorem 3.2) to weakly coupled systems of equations. For first order equations such results were obtained in [24]. The results in this section follow easily from the estimates in the previous section and the arguments in [24].

We consider the weakly coupled problem

$$(5.1)\qquad \begin{aligned} &\frac{\partial u_i}{\partial t} + H_i(t,x,u_i,Du_i) - \operatorname{tr}[A_i(t,Du_i)D^2u_i] \\ &\qquad = G_i(t,x,u) \quad \text{in } \; Q_T = \mathbb{R}^N \times (0,T), \quad i = 1,\ldots,m, \\ &u(x,0) = u_0(x) \quad \text{in } \; \mathbb{R}^N, \end{aligned}$$

where $u = (u_1,\ldots,u_m)$ is vector of unknowns.

The phrase "weakly coupled" refers to the fact that the equations in (5.1) are coupled only through the source term $G = (G_1,\ldots,G_m)$.

We assume the following conditions:

(H1) – (H5) For each i, H_i satisfies conditions (F1) – (F5).

(G1) $G \in C\left(\bar{Q}_T \times \mathbb{R}^m; \mathbb{R}^m\right)$ is uniformly continuous on $\bar{Q}_T \times B_m(0,R)$ for each $R > 0$.

(G2) There is a constant $C^G > 0$ such that $C^G = \sup_{\bar{Q}_T} |G(t,x,0)| < \infty$.

(G3) For each $R > 0$ there is a constant $C_R^G > 0$ such that
$$|G(t,x,r) - G(s,y,r)| \leq C_R^G(|x-y| + |t-s|^{1/2})$$
for $t,s \in [0,T]$, $|r| \leq R$, and x, $y \in \mathbb{R}^N$.

(G4) There is a constant $L^G > 0$ such that
$$|G(t,x,r) - G(t,x,s)| \leq L^G|r-s|$$
for $(t,x) \in \bar{Q}_T$ and $r,s \in \mathbb{R}^m$.

(B1) – (B2) For each i, A_i satisfies conditions (A1) – (A2).

Let $u_0 \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^m)$ and assume that there exists a unique bounded viscosity solution u to the initial value problem (5.1) with the additional regularity condition (2.3). We refer to [17] for existence results for systems of equations.

The operator splitting algorithm can now be defined as follows. Let

$$E(t,s) : W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^m) \to W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^m)$$

denote the Euler operator defined by

$$(5.2)\qquad E(t,s)w(x) = w(x) + (t-s)G(s,x,w(x))$$

for $0 \leq s \leq t \leq T$. Furthermore, let

$$S_H(t,s) : W^{1,\infty}(\mathbb{R}^N) \to W^{1,\infty}(\mathbb{R}^N)$$

be the solution operator of the scalar equation without source term

$$(5.3)\qquad u_t + H(t,x,u,Du) - \operatorname{tr}[A(t,Du)D^2u] = 0, \qquad u(x,s) = \bar{w}(x),$$

i.e., we write the viscosity solution of (5.3) as $S_H(t,s)\bar{w}(x)$. Then let S denote the operator defined by by

$$S(t,s)w = (S_{H_1}(t,s)w_1,\ldots,S_{H_m}(t,s)w_m)$$

for any $w = (w_1, \dots, w_m) \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^m)$. We can now define the operator splitting solution. For $\Delta t > 0$ and $j = 1, 2, \dots$, we set $t_j = j\Delta t$ and define

$$\begin{aligned} v(x, t_j) &= S(t_j, t_{j-1})E(t_j, t_{j-1})v(\cdot, t_{j-1})(x), \\ v(x, 0) &= v_0(x) \end{aligned} \tag{5.4}$$

Under these assumptions it is possible to obtain the rate of convergence by using the method of [24] and the estimates in the previous section (thus we state the result without a proof).

Theorem 5.1. *Assume (H1)–(B2) hold. Suppose there exists a unique bounded viscosity solution $u(x,t)$ of* (5.1) *satisfying* (2.3)*, and let $v(x, t_j)$ be the operator splitting solution defined in* (5.4)*. Then there exists a constant $K > 0$, depending only on T, $\|u_0\|$, $\|Du_0\|$, $\|v_0\|$, $\|Dv_0\|$, H_i, A_i, and G, such that for $j = 1, \dots, n$*

$$\|u(\cdot, t_j) - v(\cdot, t_j)\| \le K(\|u_0 - v_0\| + \sqrt{\Delta t}).$$

5.2. **More regularity implies better rate.** In this section we show that if the solutions are more regular, then we can obtain an improved convergence rate. In particular, we show that when the relevant solutions belong to $W^{1,2,\infty}$ (see below), the convergence rate of our operator splitting procedure becomes $\mathcal{O}(\Delta t)$. For the purpose of comparison, we recall that classical truncation analysis requires fours times continuously x-differentiable functions to achieve a linear rate of convergence.

Before we continue, we introduce the following Banach spaces

$$\begin{aligned} W^{2,\infty}(\mathbb{R}^N) &= \{f : \mathbb{R}^N \to \mathbb{R} \mid \|f\| + \|Df\| + \|D^2 f\| < \infty\}, \\ W^{1,2,\infty}(\bar{Q}_T) &= \{f : \bar{Q}_T \to \mathbb{R} \mid \|f\| + \|f_t\| + \|Df\| + \|D^2 f\| < \infty\}. \end{aligned}$$

Introduce the following conditions on a function f:

(C1) For every $R > 0$, $f \in C(\bar{Q}_T \times \mathbb{R} \times \mathbb{R}^N \times S(N))$ is uniformly continuous on $\bar{Q}_T \times [-R, R] \times B_N(0, R) \times B_{N\times N}(0, R)$.

(C2) There is $\gamma \le 0$ such that for every t, x, s, r, p, X, Y,

$$X \le Y \text{ and } s \le r \quad \Rightarrow \quad f(t, x, r, p, X) - f(t, x, s, p, Y) \ge \gamma(r - s).$$

Consider the following initial value problem

$$\begin{aligned} u_t + F(t, x, u, Du, D^2 u) &= G(t, x, u) \quad \text{in } Q_T, \\ u(x, 0) &= u_0(x) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{5.5}$$

where we assume that (C1) and (C2) hold for $f = F$ and $f = G$. It is not difficult to see that (C2) implies that the comparison principle holds for smooth classical solutions of (5.5). Furthermore, this result can be extended to strong $W^{1,2,\infty}$ solutions of (5.5) (i.e., solutions satisfying (5.5) a.e.) by Bony's maximum principle [8] and continuity of the equation (C1).

Remark 5.2. *It is well-known that viscosity solutions satisfy the equation pointwise at any point where it is differentiable once in t and twice in x, see* [13]*. Since $W^{1,2,\infty}$ functions are a.e. differentiable (once in t, twice in x), it follows that all viscosity solutions of* (5.5) *belonging $W^{1,2,\infty}$ are strong solutions. Furthermore, by the comparison principle for strong $W^{1,2,\infty}$ solutions of* (5.5)*, such solutions are unique. Hence we may conclude that the unique strong $W^{1,2,\infty}$ solution of* (5.5) *is a $W^{1,2,\infty}$ viscosity solution of* (5.5) *whenever such a viscosity solution exists. Since* *we will assume the existence of $W^{1,2,\infty}$ viscosity solutions in this section, there is no need to distinguish between viscosity and strong solutions here.*

Let S, S_F, S_G denote the solution semigroups of (5.5),

$$u_t + F(t,x,u,Du,D^2u) = 0, \qquad \text{in} \quad Q_T,$$
$$u_t = G(t,x,u) \qquad \text{in} \quad Q_T,$$

respectively. Assume that S, S_F, S_G maps $W^{2,\infty}$ into $W^{2,\infty}$. By the comparison principle and (C2) we have for $R = S, S_F, S_G$:

(D1) The semi-group $R : W^{2,\infty}(\mathbb{R}^N) \to W^{2,\infty}(\mathbb{R}^N)$ satisfies

$$\|R(t,s)\phi - R(t,s)\psi\| \le e^{-2\gamma(t-s)}\|\phi - \psi\|,$$

for every $\phi, \psi \in W^{2,\infty}(\mathbb{R}^N), 0 \le s \le t$.

To obtain rigorous error estimates in the $W^{2,\infty}$ case we need to produce uniform a priori bounds in the $W^{2,\infty}$ norm of the (operator splitting) solutions. Such bounds can be difficult to obtain, and in general they do not exist. We refer to Caffarelli and Cabré [9] (and the references therein) for the regularity theory of non-linear uniformly elliptic and parabolic equations and to [20] for $W^{2,\infty}$ estimates for some non-linear degenerate parabolic equations. In this section we will simply assume that such bounds exist, and hence the merit of Theorem 5.3 below is simply to show that with the techniques used in this paper we can recover the classical error estimate $\mathcal{O}(\Delta t)$ when the relevant functions are sufficiently smooth.

To be precise, for $R = S, S_F, S_G$ we will assume:

(D2) There are functions K_1, K_2, K_3 such that the semi-group $R : W^{2,\infty}(\mathbb{R}^N) \to W^{2,\infty}(\mathbb{R}^N)$ satisfies

$$\|R(t,s)\phi\| \le e^{-2\gamma(t-s)}(\|\phi\| + (t-s)K_1),$$
$$\|D(R(t,s)\phi)\| \le e^{-2\gamma(t-s)}\left(\|D\phi\| + (t-s)K_2(\|\phi\|)\right),$$
$$\|D^2(R(t,s)\phi)\| \le e^{-2\gamma(t-s)}\left(\|D^2\phi\| + (t-s)K_3(\|\phi\|, \|D\phi\|)\right),$$

for every $\phi \in W^{2,\infty}(\mathbb{R}^N), 0 \le s \le t \le T$.

From these bounds and the equations we can obtain estimates on the time derivative of the semigroup solutions. If we assume that F and G are bounded when $x \in \mathbb{R}^N$ and t, r, p, X are bounded, we immediately have for $R = S, S_F, S_G$:

(D3) There is a function K_4 such that the semi-group $R : W^{2,\infty}(\mathbb{R}^N) \to W^{2,\infty}(\mathbb{R}^N)$ satisfies

$$\|R(t,s)\phi - \phi\| \le (t-s)K_4(\|\phi\|, \|D\phi\|, \|D^2\phi\|)$$

for every $\phi \in W^{2,\infty}(\mathbb{R}^N), 0 \le s \le t \le T$.

The final assumption we need is a smoothness assumption on G (in order to have a result like Lemma 4.2), we take the following:

(C3) There is a function K_5 such that

$$|D_x^n\left(G(t,x,\phi(x))\right)| \le K_5(\|\phi\|, \|D\phi\|, \|D^2\phi\|), \quad n = 0,1,2,$$

for every x and $t(< T)$, and every $\phi \in W^{2,\infty}(\mathbb{R}^N)$.

This assumption together with (C1) and (C2) implies that (D2) holds for $R = S_G$ and $R = E$, where E is the Euler operator defined in (3.2). Similar to what we did in (3.4), we now define the operator splitting solution $\{v(x,t_i)\}_{i=1}^n$ by

$$\begin{aligned} v(x,t_i) &= S_F(t_i,t_{i-1})E(t_i,t_{i-1})v(\cdot,t_{i-1})(x), \\ v(x,0) &= v_0(x). \end{aligned} \tag{5.6}$$

If we repeat the argument leading to Theorem 3.2 we see that, due to the assumption of additional regularity of the involved functions, the estimates become independent of the mollification parameter δ (see Lemma 4.2) and that wherever $\sqrt{\Delta t}$ appeared before, now Δt appears. Therefore these arguments lead to the following result.

Theorem 5.3 ($W^{2,\infty}$ case)**.** *Assume (C1)–(C3) hold, and S, S_F, E (defined above) satisfy (D1) – (D3). If $u(x,t) \in W^{1,2,\infty}(\bar{Q}_T)$ is the viscosity solution of (5.5) and $v(x,t_i)$ is the operator splitting solution (5.6), then there exists a constant $\bar{K} > 0$, depending only on T, $\|D^n u_0\|$ and $\|D^n v_0\|$ for $n = 0,1,2$, F, and G, such that for $i = 1,\dots,n$*

$$\|u(\cdot,t_i) - v(\cdot,t_i)\| \le \bar{K}(\|u_0 - v_0\| + \Delta t).$$

This result can be extended to weakly coupled systems in the same way we indicated it in the previous section.

5.3. **A fully discrete example.** In this section we provide an example of a fully discrete splitting method based on a finite difference scheme for the PDE part. We then show how to derive an error estimate for this operator splitting method. In general, however, finite difference schemes are harder to analyze than operator splitting methods, and error bounds are not available in most cases, including quasilinear equations. We refer to [5] for the best and most general results available up to now, see also [4, 19, 26, 27]. Here we will consider a "simple" problem that falls within the scope of the results in this paper and for which the finite difference part can be analyzed using available machinery.

The problem we have in mind reads

$$\begin{aligned} u_t + H(u_x,u_y) - \lambda u_{yy} &= G(x,y,u) \quad \text{in } Q_T := (0,T)\times\mathbb{R}^2, \\ u(0,x,y) &= u_0(x,y) \qquad \text{in } \mathbb{R}^2, \end{aligned} \tag{5.7}$$

where H is bounded, convex, and Lipschitz continuous, $\lambda > 0$, and $G \in W^{2,\infty}(\mathbb{R}^3)$. The assumption on G is used to avoid unnecessary technicalities. Indeed, all results below hold under the weaker assumptions (G1) – (G4) in Section 3. Note that this equation degenerates in the x-direction. We assume that $u_0 \in W^{1,\infty}(\mathbb{R}^2)$. These assumptions then imply the existence and uniqueness of a bounded viscosity solution u satisfying (2.3) (see Theorem 2.3 and Proposition 2.6).

We will analyze a fully discrete splitting method, so in view of the previous sections it remains to discretize the homogeneous equation

$$u_t + H(u_x,u_y) - \lambda u_{yy} = 0 \quad \text{in } Q_T. \tag{5.8}$$

We do this using an explicit finite difference scheme based on a central difference approximation of the second order term and the Engquist-Osher flux approximation of the Hamiltonian, but any monotone, consistent, and stable finite difference scheme for (5.8) will do. Let $U = U(t,x,y)$ denote the numerical solution, and note that sometimes we suppress the x, y dependence and write $U(t)$ instead of $U(x, y, t)$. Let $\Delta t, \Delta x, \Delta y > 0$, and define

$$\begin{aligned} U(t+\Delta t) &= U(t) - \Delta t\, F\Big(D_{x,+}U(t), D_{x,-}U(t), D_{y,+}U(t), D_{y,-}U(t)\Big) \\ &\quad + \Delta t \lambda D_{yy}^2 U(t) \qquad \text{in} [0, T-\Delta t] \times \mathbb{R}^2, \\ U(t) &= \left(1 - \frac{t}{\Delta t}\right) u_0 + \frac{t}{\Delta t} U(\Delta t) \qquad \text{in } [0, \Delta t) \times \mathbb{R}^2, \end{aligned} \tag{5.9}$$

where the Engquist-Osher flux F is defined as

$$\begin{aligned} &F(p_1, p_2, q_1, q_2) \\ &= H(p_1, q_1) + \int_{p_1}^{p_2} \min\left(\frac{\partial H}{\partial p}(p, q_1), 0\right) dp + \int_{q_1}^{q_2} \min\left(\frac{\partial H}{\partial q}(p_1, q), 0\right) dq, \end{aligned}$$

and $D_{x,\pm}$, D_{yy}^2 denote the difference operators defined by

$$\begin{aligned} D_{x,\pm}\phi(x,y) &= \pm\frac{1}{\Delta x}\Big(\phi(x \pm \Delta x, y) - \phi(x,y)\Big), \\ D_{yy}^2\phi(x,y) &= \frac{1}{(\Delta y)^2}\Big(\phi(x, y+\Delta y) - 2\phi(x,y) + \phi(x, y-\Delta y)\Big). \end{aligned}$$

The y-directional difference operators $D_{y,\pm}$ are defined similarly. Note that for technical reasons, the scheme is defined for every point (x, y, t) (and not just on some grid). Consequently, we need initial values on the entire time-strip $[0, \Delta t)$, and our particular choice of initial values makes the function U continuous in t. Also note that F is convex and Lipschitz continuous since H has these properties.

Before we continue, let us define S_{num} to be the solution operator of (5.9), so that

$$U(t+\Delta t) = S_{\text{num}}(\Delta t)U(t) \qquad \text{in } [0, T-\Delta t] \times \mathbb{R}^2.$$

The scheme (5.9) is monotone provided an appropriate CFL condition holds. Recall that monotonicity of the scheme means that for any functions $\phi, \psi : \mathbb{R}^2 \to \mathbb{R}$,

$$\phi \le \psi \quad \Rightarrow \quad S_{\text{num}}(t)\phi \le S_{\text{num}}(t)\psi, \quad t > 0.$$

It is standard to prove that for any $t > 0$,

$$\|S_{\text{num}}(t)\phi\| \le \|\phi\| \quad \text{and} \quad \|DS_{\text{num}}(t)\phi\| \le \|D\phi\|.$$

(We refer to Section 3 in [19] for similar but more difficult estimates).

The splitting solution can now be defined for any $t \in [0, T]$ using the following iterative scheme:

$$\begin{aligned} v(t+\Delta t) &= S_{\text{num}}(\Delta t)E(\Delta t)v(t) \qquad &&\text{in } [0, T-\Delta t] \times \mathbb{R}^2, \\ v(t) &= \left(1 - \frac{t}{\Delta t}\right) u_0 + \frac{t}{\Delta t} v(\Delta t) \qquad &&\text{in } [0, \Delta t) \times \mathbb{R}^2, \end{aligned} \tag{5.10}$$

where the E is the Euler solution operator defined in (3.2) (since G is independent of t, E only depend on Δt). Note that the choice of initial values makes v continuous in t. In fact, using the properties of S_{num} and E (see above and Section 4) and the $W^{1,\infty}(\mathbb{R}^2)$ regularity of u_0, one can show that v is bounded and satisfies the regularity condition (2.3) with bounds independent of $\Delta t, \Delta x, \Delta y$. Regularity in x

follows directly from the previous estimates, while regularity in t needs in addition a barrier argument. We refer to [19] for the details.

The convergence rate for the fully discrete operator splitting method is stated in the following theorem:

Theorem 5.4. *Under the assumptions stated above,*

$$\|v(t_k) - u(t_k)\| \leq C((\Delta t)^{1/4} + (\Delta x)^{1/2} + (\Delta y)^{1/2}),$$

for every $t_k := k\Delta t \in [0, T]$ where $k \in \mathbb{N}$ and C is independent of $\Delta t, \Delta x, \Delta y, k$.

Remark 5.5. *The rate obtained here is the same as the rate obtained in* [19, 4] *for a pure finite difference method. This rate is lower than the rate $\mathcal{O}(\Delta t^{1/2})$ obtained for the semi-discrete operator splitting scheme. In other words, the dominating contribution to the total error comes from the finite differencing.*

Proof of Theorem 5.4. We will use a variant of the procedure of Krylov [26], see [19] for the time dependent case. This procedure consists of proving separately an upper and a lower bound for $v(t_k) - u(t_k)$. Let us start with the upper bound.

First we mollify the solution u of (5.7): For every $\varepsilon > 0$, define

$$u_\varepsilon(t, x, y) := (u * \rho_\varepsilon)(t, x, y) = \int_{\bar{Q}_T} u(t - \tau, x - r, y - s)\rho_\varepsilon(\tau, r, s)d\tau\, dr\, ds,$$

where ρ_ε is a mollifier defined by

$$\rho_\varepsilon(t, x, y) = \frac{1}{\varepsilon^4}\rho\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$$

for some smooth function ρ with unit mass and support in $(0, 1) \times [-1, 1]^2$. Note that for u_ε to be defined for all positive t, we must extend the solution u to times $t \in [-\varepsilon^2, 0]$. We assume that this has been done and refer to [19] for the details.

The key step in obtaining the upper bound is the following lemma:

Lemma 5.6. *For $t \in [\Delta t, T - \Delta t)$,*

$$u_\varepsilon(t + \Delta t) - S_{\text{num}}(\Delta t)E(\Delta t)u_\varepsilon(t) \leq \Delta t\, K(\varepsilon, \Delta t, \Delta x, \Delta y),$$

where

$$K(\varepsilon, \Delta t, \Delta x, \Delta y) := C\left(\varepsilon + \varepsilon^{-1}(\Delta x + \Delta y) + \varepsilon^{-3}(\Delta t + (\Delta y)^2) + (1 + \varepsilon^{-1})\Delta t\right).$$

Proof. Insert u_ε into the splitting scheme:

$$\begin{aligned}
&u_\varepsilon(t + \Delta t) - S_{\text{num}}(\Delta t)E(\Delta t)u_\varepsilon(t) \\
&= u_\varepsilon(t + \Delta t) - S_{\text{num}}(\Delta t)\left[u_\varepsilon(t) + \Delta t\, G(x, y, u_\varepsilon(t))\right] \\
&= u_\varepsilon(t + \Delta t) - \left[u_\varepsilon(t) + \Delta t\, G(x, y, u_\varepsilon(t))\right] \\
&\quad + \Delta t\, F\left(D_{x,+}[u_\varepsilon(t) + \Delta t G(x, y, u_\varepsilon(t))], \dots\right) \\
&\quad - \Delta t \lambda D_{yy}^2[u_\varepsilon(t) + \Delta t G(x, y, u_\varepsilon(t))].
\end{aligned}$$

Taylor expand F to see that

$$\begin{aligned}
&F\left(D_{x,+}[u_\varepsilon(t) + \Delta t G(x, y, u_\varepsilon(t))], \dots\right) \\
&\leq F\left(D_{x,+}u_\varepsilon(t), \dots, D_{y,-}u_\varepsilon(t)\right) + \Delta t\|DF(\cdots) \cdot (D_{x,+}G(\cdots), \dots, D_{y,-}G(\cdots)\|.
\end{aligned}$$

Then Taylor expand u_ε and use consistency of F to get

$$F\left(D_{x,+}u_\varepsilon(t), \dots, D_{y,-}u_\varepsilon(t)\right) \leq H(u_{\varepsilon\, x}, u_{\varepsilon\, y}) + C\|DH\|(\|u_{\varepsilon\, xx}\|\Delta x + \|u_{\varepsilon\, yy}\|\Delta y).$$

Now we see that a Taylor expansion of u_ε (and F) leads to

$$\begin{aligned}
&u_\varepsilon(t+\Delta t) - S_{\text{num}}(\Delta t)E(\Delta t)u_\varepsilon(t) \\
&\leq -\Delta t\, G(x,y,u_\varepsilon(t)) + \Delta t\,(u_{\varepsilon\, t} + H(u_{\varepsilon\, x}, u_{\varepsilon\, y}) - \lambda u_{\varepsilon\, yy}) \\
&\quad + \Delta t\, C(\|u_{\varepsilon\, xx}\|\Delta x + \|u_{\varepsilon\, yy}\|\Delta y + \|u_{\varepsilon\, yyyy}\|(\Delta y)^2). \\
&\quad + (\Delta t)^2 \|DF(\cdots)\cdot(D_{x,+}G(\cdots),\dots,D_{y,-}G(\cdots)\| \\
&\quad + \lambda(\Delta t)^2\, \|D^2_{yy}G(\cdot,\cdot,u_\varepsilon)\|.
\end{aligned}$$

By mollifying equation (5.7) and using convexity of H and Jensen's inequality, we see that u_ε satisfies (see the appendix in [4] for the details)

$$u_{\varepsilon\, t} + H(u_{\varepsilon\, x}, u_{\varepsilon\, y}) - \lambda u_{\varepsilon\, yy} \leq \rho_\varepsilon * G(\cdot,\cdot,u(\cdot,\cdot,\cdot))(t,x,y) \qquad \text{in } [\Delta t, T)\times\mathbb{R}^N.$$

Furthermore, our assumptions imply that

$$\begin{aligned}
&\|u_\varepsilon\| + \|Du_\varepsilon\| + \varepsilon\|D^2u_\varepsilon\| + \varepsilon^3(\|\partial_t^2 u_\varepsilon\| + \|D^4u_\varepsilon\|) \leq C, \\
&\|D_{x,\pm}G(\cdot,\cdot,u_\varepsilon)\| + \|D_{y,\pm}G(\cdot,\cdot,u_\varepsilon)\| \leq \|DG\|(1+\|u_\varepsilon\| + \|Du_\varepsilon\|) \leq C, \\
&\|D^2_{yy}G(\cdot,\cdot,u_\varepsilon)\| \leq C(\|DG\| + \|D^2G\|)(1+\|u_\varepsilon\| + \|u_{\varepsilon yy}\|) \leq C(1+\varepsilon^{-1}), \\
&\|DF(\cdots)\| \leq C\|DH(\cdots)\| \leq C.
\end{aligned}$$

The desired result follows from the above calculations and the fact that

$$\rho_\varepsilon * G(\cdot,\cdot,u) - G(\cdot,\cdot,u_\varepsilon) \leq C\varepsilon.$$

□

By iterations, an immediate consequence of this lemma is

$$u_\varepsilon(t_k) - [S_{\text{num}}(\Delta t)E(\Delta t)]^{k-1}u_\varepsilon(\Delta t) \leq t_{k-1}K(\varepsilon,\Delta t,\Delta x,\Delta y).$$

Now we write $u(t_k) - v(t_k)$ as

$$\begin{aligned}
u(t_k) - u_\varepsilon(t_k) + u_\varepsilon(t_k) &- [S_{\text{num}}(\Delta t)E(\Delta t)]^{k-1}u_\varepsilon(\Delta t) \\
&+ [S_{\text{num}}(\Delta t)E(\Delta t)]^{k-1}u_\varepsilon(\Delta t) - v(t_k).
\end{aligned}$$

The first difference is bounded by $C\varepsilon$, and as we have just seen, the second difference is upper bounded by $t_{k-1}K(\varepsilon,\Delta t,\Delta x,\Delta y)$. Since $v(t_k) = [S_{\text{num}}(\Delta t)E(\Delta t)]^k u_0$, contraction properties of S_{num} and E implies that the third difference is bounded by

$$e^{t_{k-1}\|DG\|}\|u_\varepsilon(\Delta t) - v(\Delta t)\|.$$

Writing $u_\varepsilon(\Delta t) - v(\Delta t) = u_\varepsilon(\Delta t) - u_\varepsilon(0) + u_\varepsilon(0) - u_0 + u_0 - v(\Delta t)$ and using the regularity of u and v and the properties of mollifiers we have

$$\|u_\varepsilon(\Delta t) - v(\Delta t)\| \leq C\Delta t^{1/2} + C\varepsilon.$$

It follows that

$$u(t_k) - v(t_k) \leq C(\varepsilon + \Delta t^{1/2} + K(\varepsilon,\Delta t,\Delta x,\Delta y)),$$

where C is independent of $k,\varepsilon,\Delta t,\Delta x,\Delta y$. If we now minimize w.r.t. ε, we get the following result:

Lemma 5.7. *For $t_k \in (0,T]$,*

$$u(t_k) - v(t_k) \leq C((\Delta t)^{1/4} + (\Delta x)^{1/2} + (\Delta y)^{1/2}).$$

To get the lower bound, we reverse the roles of u and v, extend v to times $t \in [-\varepsilon^2, 0)$ (see [19] for the details), and consider $v_\varepsilon = \rho_\varepsilon * v$. The key step is the next lemma.

Lemma 5.8. *For $t \in [0, T - 2\Delta t)$,*

$$v_\varepsilon(t + \Delta t) - S(\Delta t)v_\varepsilon(t) \leq \Delta t\, K(\varepsilon, \Delta t, \Delta x, \Delta y),$$

where S is the solution operator of the full equation (5.7) *(not* (5.8)*!) and K is defined in Lemma 5.6.*

Outline of proof. Similar to what we did in the proof of Lemma 5.6 we insert v_ε into equation (5.7) and use Taylor expansion to see that

$$\begin{aligned} &v_{\varepsilon,t}(t) + H(v_{\varepsilon\, x}(t), v_{\varepsilon\, y}(t)) - \lambda v_{\varepsilon\, xx}(t) - G(x, y, v_\varepsilon(t)) \\ &\leq \frac{1}{\Delta t}\Big(v_\varepsilon(t + \Delta t) - S_{\mathrm{num}}(\Delta t)E(\Delta t)v_\varepsilon(t)\Big) + K(\varepsilon, \Delta t, \Delta x, \Delta y) \end{aligned}$$

for $t \in (0, T - \Delta t]$. By the definition of S_{num}, convexity of F, and Jensen's inequality we get

$$\rho_\varepsilon * [S_{\mathrm{num}}(\Delta t)E(\Delta t)v](t) \leq S_{\mathrm{num}}(\Delta t)\ [\rho_\varepsilon * [E(\Delta t)v]]\,(t),$$

so mollifying scheme (5.10) and using the definition of E leads to

$$v_\varepsilon(t + \Delta t) \leq S_{\mathrm{num}}(\Delta t)\Big[v_\varepsilon(t) - \Delta t\, \rho_\varepsilon * G(\cdot, \cdot, v)(t)\Big], \quad t \in (0, T - \Delta t].$$

Monotonicity of S_{num} implies that

$$\begin{aligned} &S_{\mathrm{num}}(\Delta t)\Big[v_\varepsilon(t) - \Delta t(\rho_\varepsilon * G(\cdot, \cdot, v)(t)\Big] \\ &\leq S_{\mathrm{num}}(\Delta t)\Big[v_\varepsilon(t) - \Delta t\, G(x, y, v_\varepsilon(t)) + \Delta t\|G(\cdot, \cdot, v_\varepsilon(t)) - \rho_\varepsilon * G(\cdot, \cdot, v)(t)\|\Big] \\ &\leq S_{\mathrm{num}}(\Delta t)\Big[v_\varepsilon(t) - \Delta t G(x, y, v_\varepsilon(t))\Big] + \Delta t\|G(\cdot, \cdot, v_\varepsilon(t)) - \rho_\varepsilon * G(\cdot, \cdot, v)(t)\| \\ &= S_{\mathrm{num}}(\Delta t)E(\Delta t)v_\varepsilon(t) + \Delta t\|G(\cdot, \cdot, v_\varepsilon(t)) - \rho_\varepsilon * G(\cdot, \cdot, v)(t)\|. \end{aligned}$$

The second inequality follows since S_{num} satisfies

$$S_{\mathrm{num}}(\Delta t)(\phi + k) = S_{\mathrm{num}}(\Delta t)\phi + k$$

for any constant k. Combining the above computations yields

$$v_{\varepsilon,t}(t) + H(v_{\varepsilon,x}(t), v_{\varepsilon,y}(t)) - \lambda v_{\varepsilon,xx}(t) - G(x, y, v_\varepsilon(t)) \leq C\varepsilon + K(\varepsilon, \Delta t, \Delta x, \Delta y)$$

for $t \in (0, T - \Delta t]$. It follows that for $t_0 \in (0, T - 2\Delta t]$ and $t \in [0, \Delta t]$,

$$v_\varepsilon(t_0 + t) + te^{\|DG\|t}\,[C\varepsilon + K(\varepsilon, \Delta t, \Delta x, \Delta y)]$$

is subsolution of equation (5.7) with initial data $v_\varepsilon(t_0)$. Since $S(\Delta t)v_\varepsilon(t)$ is the solution of (5.7) at $t = \Delta t$ with $v_\varepsilon(t)$ as initial data, the lemma holds by the comparison principle. □

Consider the case $t_k \in (0, T - \Delta t]$. We write $v(t_k) - u(t_k)$ as

$$v(t_k) - v_\varepsilon(t_k) + v_\varepsilon(t_k) - [S(\Delta t)]^k v_\varepsilon(0) + [S(\Delta t)]^k v_\varepsilon(0) - u(t_k).$$

By properties of mollifiers the first term is bounded by $C\varepsilon$. The second term is bounded by $K(\varepsilon, \Delta t, \Delta x, \Delta y)$, as can be seen by Lemma 5.8 and iteration. The last term is bounded by $C\varepsilon$ by the contraction properties of S, properties of mollifiers, and the fact that $v(0) = u_0$. Minimizing again w.r.t. ε, we get the following result:

Lemma 5.9. *For $t_k \in (0, T-\Delta t]$,*

$$v(t_k) - u(t_k) \leq C((\Delta t)^{1/4} + (\Delta x)^{1/2} + (\Delta y)^{1/2}).$$

Finally, we consider the case $t_k \in [T-\Delta t, T]$. By Lemma 5.8 and regularity of u and v we have

$$\begin{aligned} v(t_k) - u(t_k) &= v(t_k) - v(t_{k-1}) + v(t_{k-1}) - u(t_{k-1}) - u(t_{k-1}) - u(t_k) \\ &\leq C\Delta t^{1/2} + C((\Delta t)^{1/4} + (\Delta x)^{1/2} + (\Delta y)^{1/2}) + C\Delta t^{1/2}, \end{aligned}$$

for $t_k \in [T-\Delta t, T]$. Hence the conclusion of Lemma 5.9 holds for all $t_k \in (0, T]$ and this concludes the proof the lower bound.

Combining this lower bound with the upper bound of Lemma 5.7 concludes the proof of Theorem 5.4. □

References

[1] M. Bardi, M. G. Crandall, L. C. Evans, H. M. Soner, and P. E. Souganidis. *Viscosity solutions and applications.* Springer-Verlag, Berlin, 1997. Lectures given at the 2nd C.I.M.E. Session held in Montecatini Terme, June 12–20, 1995, Edited by I. Capuzzo Dolcetta and P. L. Lions, Fondazione C.I.M.E.. [C.I.M.E. Foundation].

[2] G. Barles. Convergence of numerical schemes for degenerate parabolic equations arising in finance theory. In *Numerical methods in finance*, pages 1–21. Cambridge Univ. Press, Cambridge, 1997.

[3] G. Barles, C. Daher, and M. Romano. Convergence of numerical schemes for parabolic equations arising in finance theory. *Math. Models Methods Appl. Sci.*, 5(1):125–143, 1995.

[4] G. Barles and E. R. Jakobsen. On the convergence rate of approximation schemes for Hamilton-Jacobi-Bellman equations. *M2AN Math. Model. Numer. Anal.*, 36(1):33–54, 2002.

[5] G. Barles and E. R. Jakobsen. Error bounds for monotone approximation schemes for Hamilton-Jacobi-Bellman equations. Preprint.

[6] G. Barles and B. Perthame. Comparison principle for Dirichlet-type Hamilton-Jacobi equations and singular perturbations of degenerated elliptic equations. *Appl. Math. Optim.*, 21(1):21–44, 1990.

[7] G. Barles and P. E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Anal.*, 4(3):271–283, 1991.

[8] J.-M. Bony. Principe du maximum dans les espaces de Sobolev. *C. R. Acad. Sci. Paris Sér. A-B*, 265:333–336, 1967.

[9] L. A. Caffarelli and X. Cabré. *Fully Nonlinear Elliptic Equations.* AMS Colloquium Publications, 43. Amer. Math. Soc., Providence, RI, 1995.

[10] F. Camilli and M. Falcone. An approximation scheme for the optimal control of diffusion processes. *RAIRO Modél. Math. Anal. Numér.*, 29(1):97–122, 1995.

[11] B. Cockburn, G. Gripenberg, and S.-O. Londen. Continuous dependence on the nonlinearity of viscosity solutions of parabolic equations. *J. Differential Equations*, 170(1):180–187, 2001.

[12] M. G. Crandall and H. Ishii. The maximum principle for semicontinuous functions. *Differential Integral Equations*, 3(6):1001–1014, 1990.

[13] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[14] M. H. A. Davis, V. G. Panas, and T. Zariphopoulou. European option pricing with transaction costs. *SIAM J. Control Optim.*, 31(2):470–493, 1993.

[15] K. Deckelnick and G. Dziuk. Error estimates for a semi-implicit fully discrete finite element scheme for the mean curvature flow of graphs. *Interfaces Free Bound.*, 2(4):341–359, 2000.

[16] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions.* Springer-Verlag, New York, 1993.

[17] H Ishii and S Koike. Viscosity solutions for monotone systems of second-order elliptic PDEs. *Comm. PDE.*, 16(6&7):1095–1128, 1991.

[18] E. R. Jakobsen. On error bounds for approximation schemes for non-convex degenerate elliptic equations. To appear in *BIT*.

[19] E. R. Jakobsen. On the rate of convergence of approximation schemes for Bellman equations associated with optimal stopping time problems. *Math. Models Methods Appl. Sci. (M3AS)* 13(5):613-644, 2003.

[20] E. R. Jakobsen. $W^{2,\infty}$ regularizing effect in a nonlinear, degenerate parabolic equation in one space dimension. To appear in *Proc. Amer. Math. Soc.*

[21] E. R. Jakobsen and K. H. Karlsen. Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate elliptic equations. *Electron. J. Differential Equations*, pages No. 39, 10 pp. (electronic), 2002.

[22] E. R. Jakobsen and K. H. Karlsen. Continuous dependence estimates for viscosity solutions of fully nonlinear degenerate parabolic equations. *J. Differential Equations*, 183(2):497–525, 2002.

[23] E. R. Jakobsen, K. H. Karlsen, and N. H. Risebro. On the convergence rate of operator splitting for Hamilton-Jacobi equations with source terms. *SIAM J. Numer. Anal.*, 39(2):499–518 (electronic), 2001.

[24] E. R. Jakobsen, K. H. Karlsen, and N. H. Risebro. On the convergence rate of operator splitting for weakly coupled systems of Hamilton-Jacobi equations. In *Hyperbolic problems: theory, numerics, applications, Vol. I, II (Magdeburg, 2000)*, volume 141 of *Internat. Ser. Numer. Math., 140*, pages 553–562. Birkhäuser, Basel, 2001.

[25] K. H. Karlsen and K.-A. Lie. An unconditionally stable splitting scheme for a class of nonlinear parabolic equations. *IMA J. Numer. Anal.*, 19(4):609–635, 1999.

[26] N. V. Krylov. On the rate of convergence of finite-difference approximations for Bellman's equations. *Algebra i Analiz*, 9(3):245–256, 1997.

[27] N. V. Krylov. On the rate of convergence of finite-difference approximations for bellmans equations with variable coefficients. *Probab. Theory Related Fields*, 117(1):1–16, 2000.

[28] H. J. Kuo and N. S. Trudinger. Discrete methods for fully nonlinear elliptic equations. *SIAM J. Numer. Anal.*, 29(1):123–135, 1992.

[29] H. J. Kushner and P. G. Dupuis. *Numerical methods for stochastic control problems in continuous time.* Springer-Verlag, New York, 1992.

[30] J. O. Langseth, A. Tveito, and R. Winther. On the convergence of operator splitting applied to conservation laws with source terms. *SIAM J. Numer. Anal.*, 33(3):843–863, 1996.

[31] G. T. Lines, M. L. Buist, P. Grøttum, A. J. Pullan, J. Sundnes, and A. Tveito. Mathematical models and numerical methods for the forward problem in cardiac electrophysiology. *Comput. Vis. Sci.*, 5:215–239, 2003.

[32] G. T. Lines, P. Grøttum, and A. Tveito. Modeling the electrical activity of the heart: a bidomain model of the ventricles embedded in a torso. *Comput. Vis. Sci.*, 5:195–213, 2003.

[33] Z. Qu and A. Garfinkel. An advanced algorithm for solving partial differential equation in cardiac conduction. *IEEE Transactions of Biomedical Engineering*, 46(9):1166–1168, 1999.

[34] P. E. Souganidis. Existence of viscosity solutions of Hamilton-Jacobi equations. *J. Differential Equations*, 56(3):345–390, 1985.

[35] G. Strang. On the construction and comparison of difference schemes. *SIAM J. Numer. Anal.*, 5:506–517, 1968.

[36] J. Sundnes, G. T. Lines, and A. Tveito. A second order scheme for solving the coupled bidomain and forward problem. Technical report, SIMULA Research Laboratory, 2002.

[37] T. Tang. Convergence analysis for operator-splitting methods applied to conservation laws with stiff source terms. *SIAM J. Numer. Anal.*, 35(5):1939–1968 (electronic), 1998.

[38] T. Tang and Z. H. Teng. Error bounds for fractional step methods for conservation laws with source terms. *SIAM J. Numer. Anal.*, 32(1):110–127, 1995.

[39] Y. Zhan. Viscosity solution theory of a class of nonlinear degenerate parabolic equations. I. Uniqueness and existence of viscosity solutions. *Acta Math. Appl. Sinica (English Ser.)*, 13(2):136–144, 1997.

(Espen Robstad Jakobsen)
Department of Mathematical Sciences
Norwegian University of Science and Technology
N–7491 Trondheim, Norway
E-mail address: erj@math.ntnu.no
URL: http://www.math.ntnu.no/~erj/

(Kenneth Hvistendahl Karlsen)
Centre of Mathematics for Applications (CMA)
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo, Norway
and
Department of Scientific Computing
Simula Research Laboratory
P.O.Box 134
N–1325 Lysaker, Norway
E-mail address: kennethk@math.uio.no
URL: http://www.math.uio.no/~kennethk/