WELLPOSEDNESS OF SOLUTIONS OF A
PARABOLIC-ELLIPITIC SYSTEM

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Abstract. We show existence of a unique, regular global solution of the
parabolic-elliptic system

\[ u_t + f(t, x, u)_x + g(t, x, u) + P_x = (a(t, x)u_x)_x \]

and

\[ -P_{xx} + P = h(t, x, u, u_x) + k(t, x, u) \]

with initial data \( u|_{t=0} = u_0 \). Here \( \inf_{(t, x)} a(t, x) > 0 \). Furthermore, we show that the solution is stable with re-
spect to variation in the initial data \( u_0 \) and the functions \( f, g \) etc. Explicit
stability estimates are provided. The regularized generalized Camassa–Holm
equation is a special case of the model we discuss.

1. Introduction

In this paper we study a system that constitutes a generalized and regularized
Camassa–Holm equation. More specifically, we consider the system

\[

t + f(t, x, u)\right)_x + g(t, x, u) + P_x = \left(a(t, x)u_x\right)_x, \\
-P_{xx} + P = h(t, x, u, u_x) + k(t, x, u), \\
\]

on the domain \((t, x) \in \Pi_T := [0, T] \times \mathbb{R}\). Consider the special case, where we in
particular assume the inviscid case \( a = 0 \), given by

\[

u_t + \frac{1}{2}(u^2)_x + P_x = 0, \\
-P_{xx} + P = u^2 + \frac{1}{2}(u_x)^2 + \gamma u, \\
\]

This system formally reduces, by applying the operator \( u \mapsto u - u_{xx} \) to the first
equation in (1.2), to the Camassa–Holm equation

\[

u_t - u_{txx} + \gamma u_x + 3u u_x = 2u_x u_{xx} + u u_{xxx}. 
\]

Due to the presence of the term \( a(t, x) \) in (1.1), we see that it constitutes a viscous
regularization of a spatially and temporally varying Camassa–Holm equation. In
this paper we address the question of wellposedness of the system (1.1). In par-
ticular, we focus on stability of solutions with respect to variation not only in the
initial data, but also variation with respect to the functions \( f, a \), etc. Furthermore,
we are interested in the vanishing viscosity limit of (1.1), i.e., when \( a \to 0 \), and this
problem is discussed in a subsequent paper [4].

Formally, by applying the operator \((1 - \partial_x^3)^{-1}\) to the second equation, we see that the system (1.1) can be written as the integro-differential Cauchy problem

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\[ u_t + \left( f(t, x, u) \right)_x + g(t, x, u) + \frac{1}{2} \frac{\partial}{\partial x} \int_{\mathbb{R}} e^{-|x-y|} \left( h(t, y, u(t, y), u_x(t, y)) + k(t, y, u(t, y)) \right) dy = \left( a(t, x)u_x \right)_x, \]
\[ u|_{t=0} = u_0. \]

The Camassa–Holm equation [2] has received extensive attention the last decade. It can be used as a model of unidirectional gravitational shallow water waves on a flat bottom when \( \gamma \) is positive [8]. The velocity is given by \( u \). The equation possesses intriguing properties; it has a bi-Hamiltonian structure, it is completely integrable, and it experiences wave breaking, that is, the breakdown of smooth solutions in finite time, for a large class of initial data. The Cauchy problem has been extensively studied. We refer to [5, 7, 14, 15] and references therein for more complete information. Furthermore, Dai [6] derived the equation
\[ u_t - uu_x + 3uu_x = \gamma(2uu_x + a u_{xxx}) \]
as a model for small amplitude radial deformation waves in cylindrical compressible hyperelastic rods.

Another related example in the inviscid case is that of a simplified model for radiating gases given by the hyperbolic-elliptic system
\[ u_t + \frac{1}{2} (u^2)_x = -q_x, \quad -q_{xx} + q = -u_x, \quad u|_{t=0} = u_0, \]
see [9, 11]. However, this system is not covered by our assumptions because of the presence of the viscous term in (1.1) and the assumptions on the functions \( f, h \).

In this paper we study the Cauchy problem for the viscous regularization. More precisely, we study the system (1.1) where \( a \) is bounded away from zero. We establish short-time existence of a unique smooth solution by showing a contraction mapping principle. An energy estimate makes it possible to extend the result to a global result in time. Stability is established by a homotopy argument, see also [1, 3], where one connects by a smooth path two distinct solutions \( u \) and \( v \) with different data and coefficients. Our main result reads: Let \( u \) and \( v \) be solutions of
\[ u_t + \left( f_0(t, x, u) \right)_x + g_0(t, x, u) + P_x = \left( a_0(t, x)u_x \right)_x, \]
\[ -P_{xx} + P = h_0(t, x, u, u_x) + k_0(t, x, u), \]
\[ u|_{t=0} = u_0, \]
\[ v_t + \left( f_1(t, x, v) \right)_x + g_1(t, x, v) + Q_x = \left( a_1(t, x)v_x \right)_x, \]
\[ -Q_{xx} + Q = h_1(t, x, v, v_x) + k_1(t, x, v), \]
\[ v|_{t=0} = v_0, \]
respectively. In addition to certain regularity assumptions (essentially boundedness of various derivatives) on the functions, we assume that
\[ h_i(t, x, u, q)q - \frac{1}{2} f_{i, uu}(t, x, u)q^2 \leq C_0(u^2 + q^2), \quad i = 0, 1, \]
\[ |h_i(t, x, u, q)| \leq C_0(|u| + u^2 + q^2), \quad i = 0, 1, \]
\[ f_{0,1,1}(t, x, 0) = f_{1,1}(t, x, 0), \]
\[ g_{0,1}(t, x, 0) = g_{1,1}(t, x, 0), \]
\[ k_{0,1}(t, x, 0) = k_{1,1}(t, x, 0), \]
\[ h_{0,1,u}(t, x, 0, q) = h_{1,u}(t, x, 0, q), \]
\[ h_{0,1,q}(t, x, u, 0) = h_{1,q}(t, x, u, 0). \]
Then we show that
\[(1.5) \quad \|u(t, \cdot) - v(t, \cdot)\|_{H^1(\mathbb{R})} \leq \|u_0 - v_0\|_{H^1(\mathbb{R})} e^{K_0 t} + K_1 t \left[ \|f_1, u, - f_0, u, \|_{L^\infty(\mathcal{I})} + \|f_1, u, - f_0, u, \|_{L^\infty(\mathcal{J})} \right]
+ \|g_1, u, - g_0, u, \|_{L^\infty(\mathcal{I})}
+ \|a_1, x, - a_0, x, \|_{L^\infty(\mathcal{I})} + \|a_1, x, - a_0, x, \|_{L^\infty(\mathcal{J})}
+ \|h_0, u, - h_1, u, \|_{L^\infty(\mathcal{J})} + \|k_0, u, - k_1, u, \|_{L^\infty(\mathcal{J})}
+ \|h_0, q, - h_1, q, \|_{L^\infty(\mathcal{J})}],\]
where
\[
\mathcal{I} := \Pi_T \times \left[ - \frac{C_2}{\sqrt{2}} \frac{C_2}{\sqrt{2}} \right], \quad \mathcal{J} := \mathcal{I} \times \left[ - \frac{C_2}{\sqrt{2}} \frac{C_2}{\sqrt{2}} \right],
\]
and $K_0$, $K_1$ and $C_2$ are constants that may only depend on the time horizon $T$ and on the viscous coefficient $\alpha$.

2. Existence and uniqueness

We consider the parabolic-elliptic initial-value problem
\[
(2.1) \quad u_t + (f(t, x, u))_x + g(t, x, u) + P_x = (a(t, x)u_x)_x,
\]
\[
-P_x + P = h(t, x, u, u_x) + k(t, x, u),
\]
\[
u|_{t=0} = u_0,
\]
for $(t, x) \in \Pi_T$, where $T > 0$.

**Remark 2.1.** Since $e^{-|x|/2}$ is the Green’s function of the operator $u \mapsto -u_{xx} + u$,
(2.1) is equivalent to the following integro-differential Cauchy problem
\[
(2.2) \quad P(t, x) = \frac{1}{2} \int_\mathbb{R} e^{-|x-y|} \left( h(t, y, u(t, y), u_x(t, y)) + k(t, y, u(t, y)) \right) dy,
\]
\[
u|_{t=0} = u_0.
\]

Regarding the initial data $u_0: \mathbb{R} \to \mathbb{R}$ we assume
\[
(2.3) \quad u_0 \in H^\ell(\mathbb{R}) \quad \text{for some } \ell \geq 2.
\]

In the following sections we shall assume:

**H.1** the coefficients $a$, $f$, $g$, $h$, and $k$ are smooth;

**H.2** there exists a constant $C_0 > 0$ such that the following hold
\[
(2.4a) \quad f(\cdot, \cdot, 0) = 0,
\]
\[
(2.4b) \quad \frac{1}{C_0} \leq a(\cdot, \cdot) \leq C_0,
\]
\[
(2.4c) \quad \left\| \frac{\partial f}{\partial x}(\cdot, \cdot, u) \right\|_{L^\infty} \leq C_0 |u|, \quad 1 \leq i \leq \ell + 1,
\]
\[
(2.4d) \quad \left\| \frac{\partial^2 a}{\partial x^2} \right\|_{L^\infty} \leq C_0, \quad j \geq 1, \quad 2 \leq i + j \leq \ell + 1
\]
\[
(2.4e) \quad \left\| k_x(\cdot, \cdot, u) \right\|_{L^\infty}, \left\| \frac{\partial f}{\partial x}(\cdot, \cdot, u) \right\|_{L^\infty} \leq C_0 |u|, \quad 0 \leq i \leq \ell,
\]
\[
(2.4f) \quad \left\| \frac{\partial^{i+j} a}{\partial x^i \partial y^j} \right\|_{L^\infty} \leq C_0, \quad j \geq 1, \quad 1 \leq i + j \leq \ell + 1,\]
\[
(2.4g) \quad u \mapsto \left\| \frac{\partial^{i+j} k}{\partial x^i \partial y^j}(\cdot, \cdot, u) \right\|_{L^\infty} \text{ is in } L^\infty_{loc}(\mathbb{R}), \quad 0 \leq i + j \leq \ell + 1,
\]
(2.4h) \[ (u, q) \mapsto \left\| \frac{\partial^{i+j+p} p}{\partial x^i \partial u^j \partial q^p}(\cdot, \cdot, u, q) \right\|_{L^\infty} \text{ is in } L^\infty_{loc}(\mathbb{R}^2), \quad 0 \leq i + j + p \leq \ell + 1, \]

(2.4i) \[ h(t, x, u, q) - \frac{1}{2} f_{uu}(t, x, u)q^2 \leq C_0(u^2 + q^2), \]

(2.4j) \[ \|h(\cdot, \cdot, u, q)\|_{L^\infty} \leq C_0(|u| + u^2 + q^2), \]

for \((t, x) \in \Pi_T\) and \(u, q \in \mathbb{R}\).

In particular, (2.4d) shows that \(f_{uu}\) is bounded, and hence \(f\) is at most quadratic in \(u\).

**Example 2.2.** Consider the viscosity approximation of the generalized Camassa–Holm equation

\[
\frac{\partial u}{\partial t} + \gamma u \frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2},
\]

\[
-\frac{\partial^2 P}{\partial x^2} + P = \frac{1}{2} g(u_x) + \frac{\gamma}{2} \left( \frac{\partial u}{\partial x} \right)^2 - \frac{\gamma}{2} u_x^2 + \kappa u_x,
\]

where \(u_{x,0}\) satisfies (2.3). With \(f = \gamma u^2/2, h = \gamma q^2/2, \text{ and } k = g(u) - \frac{1}{2} u_x^2 + \kappa u\),

we see that our assumptions are fulfilled, see [4]. With \(\varepsilon = 0\) the system formally reduces to, with \(u = u_{x}\),

\[ u_t - u_{txx} + \frac{1}{2} g(u)_x + \kappa u_x = \gamma(2u_xu_{xx} + uu_{xxx}), \]

which is denoted the hyperelastic-rod wave equation, see [6].

The main result of this section is the following.

**Theorem 2.3.** Let \(T > 0\). Assume (H.1), (H.2), and (2.3). Then there exists a unique smooth solution \(u \in C([0, T]; H^\ell(\mathbb{R}))\) of the Cauchy problem (2.1).

The proof of this theorem is divided into a local (in time) existence result, which is discussed first, and an extension theorem.

2.1. **Local existence and uniqueness.** We begin by proving the following result.

**Theorem 2.4 (Local existence).** Assume (H.1), (H.2), and (2.3). There exists a positive time \(T_0\) such that (2.1) has a unique, local, smooth solution defined on \([0, T_0] \times \mathbb{R}\).

Let \(G = G(t, s, x, y)\) be the Green’s function associated to the operator \(u \mapsto u_t - (a(t, x)u_x)_x\), see [10, Chapter IV, Section 11]. Let \(T_0 > 0\). Define the following quantities:

\[ U(t, x) := \int_{\mathbb{R}} G(t, 0, x, y)u_0(y)dy, \]

\[ \Lambda_1(u)(t, x) := \int_0^t \int_{\mathbb{R}} G(t, s, x, y) \left( f(s, y, u(s, y)) \right)_y + g(s, y, u(s, y)) \right) dyds, \]

\[ \Lambda_2(u)(t, x) := \frac{1}{2} \int_0^t \int_{\mathbb{R}} G(t, s, x, y) e^{-y-\xi} \left[ h(s, \xi, u(s, \xi), u_\xi(s, \xi)) \right. \]

\[ + k(s, \xi, u(s, \xi)) \right] d\xi dyds, \]

\[ \Lambda(u) := U - \Lambda_1(u) - \Lambda_2(u), \]

for each \((t, x) \in \Pi_{T_0}, u \in C([0, T_0]; H^1(\mathbb{R})). \]

The following lemmas are needed.
Lemma 2.5. Let $T_0 > 0$ and assume (2.3), (H.1), (H.2). Then

\begin{equation}
U \in C([0, T_0]; H^1(\mathbb{R})), \quad \frac{\partial U}{\partial t} \in C([0, T_0]; H^{-1}(\mathbb{R})).
\end{equation}

Moreover, $U$ is the smooth solution of

\begin{equation}
\begin{cases}
U_t = (a(t, x)U_x)_x, & t > 0, \ x \in \mathbb{R}, \\
U|_{t=0} = u_0.
\end{cases}
\end{equation}

Proof. Due to regularity of $u_0$, the fact that $U$ solves (2.6) is consequence of [10, Chapter IV, Section 14]. We have to prove (2.7). From (2.9) and (2.10), we get

\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} (U^2 + U_x^2) dx = \int_{\mathbb{R}} (UU_t + U_x U_{tx}) dx
\end{equation}

\begin{equation}
= \int_{\mathbb{R}} (U(aU_x)_x + U_x(a_x U_x + a U_{xx})_x) dx
\end{equation}

\begin{equation}
= - \int_{\mathbb{R}} (a U^2_x + a_x U_x U_{xx} + a U_{xx}^2) dx
\end{equation}

\begin{equation}
= - \int_{\mathbb{R}} (a U^2_x - \frac{a_x}{2} U_x^2 + a U_{xx}^2) dx \leq \frac{C_0}{2} \|U(t, \cdot)\|_{H^1(\mathbb{R})}^2.
\end{equation}

Clearly this implies that $U \in L^\infty([0, T_0]; H^1(\mathbb{R}))$. Writing $U$ as convolution of the Green’s function $G$ and of the initial condition $u_0$, the first part of (2.5) is consequence of [10, Chapter 13, (13.3)]. For the second part, we consider equation (2.6) and observe that $U_{xx} \in C([0, T_0]; H^{-1}(\mathbb{R})).$ \hfill \Box

Lemma 2.6. Let $T_0 > 0$ and assume (2.3), (H.1), and (H.2). Then

\begin{equation}
\Lambda_1(u) \in C([0, T_0]; H^1(\mathbb{R})), \quad \frac{\partial \Lambda_1(u)}{\partial t} \in C([0, T_0]; H^{-1}(\mathbb{R})),
\end{equation}

for each $u \in C([0, T_0]; H^1(\mathbb{R}))$, and

\begin{equation}
\|\Lambda_1(u_1) - \Lambda_1(u_2)\|_{L^\infty([0, T_0]; H^1(\mathbb{R}))} \leq C_1 e^{C_1 T_0} \|u_1 - u_2\|_{L^\infty([0, T_0]; H^1(\mathbb{R}))}^2,
\end{equation}

for each $u_1, u_2 \in C([0, T_0]; H^1(\mathbb{R}))$ and some constant $C_1 = C_1(C_0) > 0$. Moreover, $\Lambda_1(u)$ is a smooth solution of

\begin{equation}
\begin{cases}
v_t = (a(t, x)v_x)_x + (f(t, x, u))_x + g(t, x, u), & t > 0, \ x \in \mathbb{R}, \\
v|_{t=0} = 0, & x \in \mathbb{R}.
\end{cases}
\end{equation}

Proof. Due to regularity of $f, g, u$, the fact that $\Lambda_1(u)$ solves (2.9) is consequence of [10, Chapter IV, Section 14]. We have to prove (2.7). From (2.9) and (H.2) we get\footnote{Observe that, e.g., $f_x$ and $(f)_x$ are distinct as $(f)_x = (f(t, x, u(t)))_x = f_x + f_u u_x.$}

\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}} \frac{1}{2} (v^2 + v_x^2) dx = \int_{\mathbb{R}} (vv_t + v_x v_{tx}) dx
\end{equation}

\begin{equation}
= \int_{\mathbb{R}} (-av_x^2 + \frac{a_x}{2} v_x^2 - av_{xx} - v_x f + v g - v_{xx}(f)_x - v_{xx} g) dx
\end{equation}

\begin{equation}
\leq \int_{\mathbb{R}} \left( \frac{1}{C_0} - a \right) (v_x^2 + v_{xx}^2) + \frac{a_x}{2} v_x^2 + \frac{C_0}{2} v_{xx}^2 + \frac{C_0}{4} (f)_x^2 + \frac{2 + C_0^2}{4C_0} g^2 + \frac{C_0}{4} f^2 \right) dx
\end{equation}

\begin{equation}
\leq \frac{C_0}{2} \|v(t, \cdot)\|_{H^1(\mathbb{R})}^2 + c_1 \|u(t, \cdot)\|_{H^1(\mathbb{R})}^4.
\end{equation}

for some constant $c_1 = c_1(C_0) > 0$. Hence, using the Gronwall inequality

\begin{equation}
\|v\|_{L^\infty([0, T_0]; H^1(\mathbb{R}))} \leq e^{C_0 T_0/2} T_0 \|u\|_{L^\infty([0, T_0]; H^1(\mathbb{R}))}^4.
\end{equation}
Clearly this implies \( v \in L^\infty([0, T_0]; H^1(\mathbb{R})) \). Writing \( U \) as convolution of the Green’s function \( G \) and of the source, the first part of (2.5) is consequence of [10, Chapter 13, (13.3)]. For the second part, we look at the equation (2.9) and observe that \( v_{xx} \in C([0, T_0]; H^{-1}(\mathbb{R})) \).

To prove (2.8), observe that \( \Lambda_1(u_1) - \Lambda_1(u_2) \) is the smooth solution of

\[
\begin{aligned}
    w &= (a(t,x)w_x)_x + \int f(t,x,u_1) - f(t,x,u_2) \, dx + \int [g(t,x,u_1) - g(t,x,u_2)] \, dx, \\
    w|_{t=0} &= 0,
\end{aligned}
\]

since the source is in \( L^\infty([0, T_0]; H^{-1}(\mathbb{R})) \), we can argue as for (2.10).

\( \square \)

**Lemma 2.7.** Let \( T_0 > 0 \) and assume (2.3), (H.1), (H.2). Then

(2.11) \[ \Lambda_2(u) \in C([0, T_0]; H^1(\mathbb{R})), \quad \frac{\partial \Lambda_2(u)}{\partial t} \in C([0, T_0]; H^{-1}(\mathbb{R})), \]

for each \( u \in C([0, T_0]; H^1(\mathbb{R})) \), and

(2.12) \[ \| \Lambda_2(u_1) - \Lambda_2(u_2) \|_{L^\infty([0, T_0]; H^1(\mathbb{R}))} \leq C_2 \sqrt{T_0} \| u_1 - u_2 \|_{L^\infty([0, T_0]; H^1(\mathbb{R}))} + \| u_1 - u_2 \|_{L^\infty([0, T_0]; H^1(\mathbb{R}))}^2, \]

for each \( u_1, u_2 \in C([0, T_0]; H^1(\mathbb{R})) \) and some constant \( C_2 = C_2(C_0) > 0 \). Moreover, \( w = \Lambda_2(u) \) is the smooth solution of

(2.13) \[ \begin{cases}
    w_t = (a(t,x)w_x)_x + P_x, & t > 0, \ x \in \mathbb{R}, \\
    -P_{xx} + P = h(t,x,u,x) + k(t,x,u), & t > 0, \ x \in \mathbb{R}, \\
    w|_{t=0} = 0.
\end{cases} \]

**Proof.** Due to regularity of \( h, k, u \), the fact that \( \Lambda_2(u) \) solves (2.13) is consequence of [10, Chapter IV, Section 14] and Remark 2.1. We have to prove (2.11). From (2.13) and (H.2), we get

\[
\frac{d}{dt} \int_\mathbb{R} \frac{1}{2} (w^2 + w_x^2) \, dx = \int_\mathbb{R} (w_{xx} + w_x w_{xx}) \, dx = \int_\mathbb{R} \left( -aw^2_x + \frac{a_{xx}}{2} w_x^2 - aw^2 - wP_x + w_x P - w_x h - w_x k \right) \, dx \\
\leq \int_\mathbb{R} \left( \frac{1}{C_0} - a \right) w_x^2 - aw^2 + \frac{a_{xx}}{2} w_x^2 + \frac{C_0}{2} h^2 + \frac{C_0}{2} k^2 \, dx \\
\leq \frac{C_0}{2} \| w(t, \cdot) \|^2_{H^1(\mathbb{R})} + c_2 \left( \| u(t, \cdot) \|^2_{H^1(\mathbb{R})} + \| u(t, \cdot) \|^2_{H^1(\mathbb{R})} \right),
\]

for some constant \( c_2 = c_2(C_0) > 0 \). Clearly this implies \( w \in L^\infty([0, T_0]; H^1(\mathbb{R})) \). Writing \( U \) as convolution of the Green’s function \( G \) and of the source, the first part of (2.5) is consequence of [10, Chapter 13, (13.3)]. For the second part, we have simply to consider equation (2.13) and note that \( w_{xx} \in C([0, T_0]; H^{-1}(\mathbb{R})) \).

To prove (2.12), observe that \( \Lambda_1(u_1) - \Lambda_1(u_2) \) is the smooth solution of

\[
\begin{aligned}
    w_t &= (a(t,x)w_x)_x + P_x, \\
    -P_{xx} + P &= h(t,x,u_1,u_{1,x}) - h(t,x,u_2,u_{2,x}) + [k(t,x,u_1) - k(t,x,u_2)], \\
    w|_{t=0} &= 0.
\end{aligned}
\]

and use the previous argument.

\( \square \)

**Proof of Theorem 2.4.** Choose a time

\[
0 \leq T_0 \leq \min \{1, c_3, c_4\},
\]
where
\[
  c_3 := \frac{1}{16C_1^2 (\|U\|_{L^{\infty}([0,1];H^1(\mathbb{R}))} + 3)^2 e^{2C_1}},
\]
\[
  c_4 := \frac{1}{8C_2^2 (\|U\|_{L^{\infty}([0,1];H^1(\mathbb{R}))} + 2\|U\|_{L^{\infty}([0,1];H^1(\mathbb{R}))} + 3)^2},
\]
and consider the ball
\[
  \mathcal{B} := \left\{ u \in C([0,T_0];H^1(\mathbb{R})) \mid \|u - U\|_{L^{\infty}([0,T_0];H^1(\mathbb{R}))} \leq 1 \right\}.
\]

Observe that
\[
  u \in \mathcal{B} \implies \Lambda(u) \in \mathcal{B},
\]
\[
  u, v \in \mathcal{B} \implies \|\Lambda(u) - \Lambda(v)\|_{L^{\infty}([0,T_0];H^1(\mathbb{R}))} \leq \frac{1}{\sqrt{2}} \|u - v\|_{L^{\infty}([0,T_0];H^1(\mathbb{R}))},
\]

indeed, by (2.8) and (2.12),
\[
  \|\Lambda(u) - U\|_{L^{\infty}([0,T_0];H^1(\mathbb{R}))} \leq \|\Lambda_1(u)\|_{L^{\infty}([0,T_0];H^1(\mathbb{R}))} + \|\Lambda_2(u)\|_{L^{\infty}([0,T_0];H^1(\mathbb{R}))}
\]
\[
  \leq C_1 \sqrt{T_0} e^{C_1 T_0} \|u\|_{L^{\infty}([0,T_0];H^1(\mathbb{R}))}^2 + \|u\|_{L^{\infty}([0,T_0];H^1(\mathbb{R}))}^2 + 2\|u - U\|_{L^{\infty}([0,T_0];H^1(\mathbb{R}))}^2
\]
\[
  \leq 2C_1 \sqrt{T_0} (1 + \|u\|_{L^{\infty}([0,T_0];H^1(\mathbb{R}))}^2) + 2\|u\|_{L^{\infty}([0,T_0];H^1(\mathbb{R}))}^2
\]
\[
  \leq \sqrt{T_0} \left( \frac{1}{2\sqrt{2c_3}} + \frac{1}{2\sqrt{2c_4}} \right) \leq \frac{1}{\sqrt{2}},
\]

for each \( u, v \in \mathcal{B} \). Hence, the operator \( \Lambda \) is a contraction on \( \mathcal{B} \). Due to the contraction mapping principle, there exists a unique \( u \in \mathcal{B} \) such that
\[
  u = \Lambda(u).
\]

In particular,
\[
  u \in C([0,T_0];H^1(\mathbb{R})),
\]
and from (2.5), (2.7), and (2.11), we get
\[ \frac{\partial u}{\partial t} \in C([0,T_0]; H^{-1}(\mathbb{R})), \]
namely \( u \) is the unique weak solution of (2.1) defined on \([0,T_0] \times \mathbb{R} \). The proof of the smoothness of this solutions follows by the classical regularity theory for parabolic problems (see, e.g., [13, Section 2.2]). Here, we simply sketch the steps of the argument. We begin by observing that \( u_x \in L^2_{\text{loc}}([0,T_0] \times \mathbb{R}) \), then we prove that
\[ u \in L^2([0,T_0]; H^{\ell+1}(\mathbb{R})) \cap C([0,T_0]; H^\ell(\mathbb{R})), \quad \frac{\partial^k u}{\partial t^k} \in L^2([0,T_0]; H^{\ell_k}(\mathbb{R})), \]
for \( k = 0, \ldots, \lfloor (\ell + 1)/2 \rfloor \), where \( \lfloor x \rfloor \) denotes the largest integer not exceeding \( x \), and where
\[ l_k := \begin{cases} \ell + 1 - 2k, & \text{if } k = 0, \ldots, \lfloor \ell/2 \rfloor, \\ 2, & \text{if } k = \lfloor (\ell + 1)/2 \rfloor. \end{cases} \]
To conclude we use Sobolev embedding. \( \square \)

A trivial consequence of the previous theorem is the following corollary.

**Corollary 2.8 (Uniqueness).** Assume that (H.1), (H.2), and (2.3) hold. The initial value problem (2.1) has at most one smooth solution defined in \([0,T] \times \mathbb{R}\).

2.2. Energy estimate. We prove the following a priori estimates.

**Theorem 2.9 (H^1-estimate).** Assume that (H.1), (H.2), and (2.3) hold. Let \( u \) be the unique, local, smooth solution to (2.1). Then
\[ \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \|u(t, \cdot)\|_{H^1(\mathbb{R})} \leq \frac{1}{\sqrt{2}} e^{11C_0 t^{1/2}} \|u_0\|_{H^1(\mathbb{R})} \]
for each \( 0 \leq t \leq T_0 \). In particular,
\[ u \in C([0,T_0]; H^1(\mathbb{R})). \]

**Proof.** Fix \( 0 < t \leq T_0 \). Multiplying the first equation in (2.1) by \( u \) and integrating over \( \mathbb{R} \), we get
\[ \int_\mathbb{R} u_t u \, dx = \int_\mathbb{R} (a(t,x)u_x)_x u \, dx - \int_\mathbb{R} (f(t,x,u))_x u + g(t,x,u)u + P_x u \, dx. \]
Integrating by parts and using (H.1), (H.2),
\[ \int_\mathbb{R} u_t u \, dx = \frac{d}{dt} \int_\mathbb{R} u^2 \, dx = \frac{d}{dt} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2, \]
\[ \int_\mathbb{R} (a(t,x)u_x)_x u \, dx = - \int_\mathbb{R} a(t,x)u_x^2 \, dx \leq 0, \]
\[ \int_\mathbb{R} g(t,x,u)u \, dx \leq C_0 \int_\mathbb{R} u^2 \, dx, \]
\[ \int_\mathbb{R} (f(t,x,u))_x u \, dx = - \int_\mathbb{R} f(t,x,u)u_x \, dx \]
\[ = - \int_\mathbb{R} F_u(t,x,u)u_x \, dx \]
\[ = - \int_\mathbb{R} (F(t,x,u))_x \, dx + \int_\mathbb{R} F_x(t,x,u) \, dx \]
\[ \leq \frac{C_0}{2} \int_\mathbb{R} u^2 \, dx, \]
Differentiating the first equation in (2.1) with respect to \( t \), we get

\[
\frac{1}{2} \frac{d}{dt} \| u(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq C_0 \frac{3}{2} \int_{\mathbb{R}} u^2 \, dx - \int_{\mathbb{R}} P_x u \, dx.
\]

Multiplying this equation by \( u_x \) and integrating on \( \mathbb{R} \), we get

\[
\int_{\mathbb{R}} u_{tx} u_x \, dx = - \int_{\mathbb{R}} (f_x(t, x, u) + f_u(t, x, u) u_x) u_x \, dx
\]

\[
+ \int_{\mathbb{R}} (a_x(t, x) u_x + a(t, x) u_{xx}) u_x \, dx
\]

\[
- \int_{\mathbb{R}} P u_x \, dx + \int_{\mathbb{R}} h(t, x, u, u_x) u_x \, dx + \int_{\mathbb{R}} k(t, x, u) u_x \, dx.
\]

Again, integrating by parts and using (H.1) and (H.2), we find

\[
\int_{\mathbb{R}} u_{tx} u_x \, dx = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u_x^2 \, dx = \frac{1}{2} \frac{d}{dt} \| u_x(t, \cdot) \|_{L^2(\mathbb{R})}^2,
\]

\[
\int_{\mathbb{R}} (a_x(t, x) u_x + a(t, x) u_{xx}) u_x \, dx
\]

\[
= - \frac{1}{2} \int_{\mathbb{R}} a_x(t, x) (u_x^2)_x \, dx - \int_{\mathbb{R}} a(t, x) u_{xx}^2 \, dx
\]

\[
\leq \frac{1}{2} \int_{\mathbb{R}} a_{xx}(t, x) u_x^2 \, dx \leq \frac{C_0}{2} \int_{\mathbb{R}} u_x^2 \, dx,
\]

\[
- \int_{\mathbb{R}} (f_x(t, x, u) u_x + (f_u(t, x, u) u_x^2 + \frac{1}{2} f_{uu}(t, x, u) u_x^3) u_x \, dx
\]

\[
\leq C_0 \frac{5}{2} \int_{\mathbb{R}} u_x^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} f_{uu}(t, x, u) u_x^3 \, dx,
\]

\[
- \int_{\mathbb{R}} g_x(t, x, u) u_x \, dx \leq C_0 \int_{\mathbb{R}} |u u_x| \, dx + C_0 \int_{\mathbb{R}} u_x^2 \, dx
\]

\[
\leq C_0 \frac{1}{2} \int_{\mathbb{R}} u_x^2 \, dx + C_0 \frac{3}{2} \int_{\mathbb{R}} u_x^2 \, dx,
\]

\[
\int_{\mathbb{R}} k(t, x, u) u_x \, dx = \int_{\mathbb{R}} (K(t, x, u))_x \, dx - \int_{\mathbb{R}} K_x(t, x, u) \, dx
\]

\[
= - \int_{\mathbb{R}} K_x(t, x, u) \, dx \leq \frac{C_0}{2} \int_{\mathbb{R}} u^2 \, dx,
\]

where

\[
K(t, x, z) := \int_0^z k(t, x, \zeta) \, d\zeta, \quad 0 \leq t \leq T, \, x, \, z \in \mathbb{R}.
\]
Therefore, from (2.18) and (H.2), we get
\[
\frac{1}{2} \frac{d}{dt} \|u_x(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq C_0 \int_\mathbb{R} u^2 \, dx + C_0 \frac{9}{2} \int_\mathbb{R} u_x^2 \, dx + \int_\mathbb{R} (hu_x - \frac{1}{2} f_{uu} u_x^2) \, dx + \int_\mathbb{R} P_x u \, dx
\]
\leq 2C_0 \int_\mathbb{R} u^2 \, dx + C_0 \frac{11}{2} \int_\mathbb{R} u_x^2 \, dx + \int_\mathbb{R} P_x u \, dx, \tag{2.19}
\]
using (2.4i). Summing (2.16) and (2.19),
\[
\frac{d}{dt} \|u(t, \cdot)\|^2_{H^1(\mathbb{R})} \leq C_0 \frac{7}{2} \int_\mathbb{R} u^2 \, dx + C_0 \frac{11}{2} \int_\mathbb{R} u_x^2 \, dx \leq C_0 \frac{11}{2} \|u(t, \cdot)\|^2_{H^1(\mathbb{R})}. \tag{2.20}
\]
The Gronwall inequality implies
\[
\|u(t, \cdot)\|^2_{H^1(\mathbb{R})} \leq \|u_0\|^2_{H^1(\mathbb{R})} e^{11C_0 t}, \quad 0 \leq t \leq T_0,
\]
that proves the second inequality in (2.14). Finally, since (see [12, Theorem 8.5])
\[
\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \|u(t, \cdot)\|_{H^1(\mathbb{R})}, \quad 0 \leq t \leq T_0,
\]
also the first one is proved. \(\square\)

**Remark 2.10.** Estimate (2.19) requires the assumption (2.4i). In the case of the Camassa–Holm equation [4] we can improve (2.19) and obtain the energy conservation. Observe that the system for radiating gases, see equation (1.3), does not satisfy (2.4i), and indeed in that model there is no energy conservation.

### 2.3. Global existence
In this section we prove Theorem 2.3. The following lemma is needed.

**Lemma 2.11 (H²-estimate).** Let \(T_0 > 0\). Assume that \((H.1), (H.2),\) and (2.3) hold. Let \(u\) be the unique, local, smooth solution to (2.1). Then, there exists a positive constants such that
\[
\|u_{xx}(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq \|u_{0,xx}\|^2_{L^2(\mathbb{R})} e^{At} + \frac{B}{A} (e^{At} - 1), \tag{2.22}
\]
for each \(0 \leq t \leq T_0\), where
\[
A := 2\left[27C_0^3 \|u_0\|^2_{H^1(\mathbb{R})} e^{11C_0 T} + 6C_0^3 + C_0\right],
\]
\[
B := 2\left[39C_0^3 \|u_0\|^2_{H^1(\mathbb{R})} e^{11C_0 T} + \left(\frac{1}{2} + 6C_0\right) \|u_0\|^2_{H^1(\mathbb{R})} e^{22C_0 T}\right].
\]
In particular
\[
u \in C([0, T_0]; H^2(\mathbb{R})).
\]

**Proof.** Fix \(0 < t \leq T_0\). Differentiating (2.17) with respect to \(x\),
\[
u_{xx} = -(f_{xx} + 2f_{ux} u_x + f_{uu} u_x^2 + f_u u_{xx})_x
\]
\[- (g_x + g_u u_x)_x + (a_{xx} u_x + 2a_x u_{xx} + a_{xxx})_x
\]
\[- P_x + (h(t, x, u, u_x))_x + (k(t, x, u))_x.
\]

Multiplying (2.23) by \(u_{xx}\) and integrating on \(\mathbb{R}\), we get
\[
\int_\mathbb{R} u_{txx} u_{xx} \, dx = - \int_\mathbb{R} (f_{xx} + 2f_{ux} u_x + f_{uu} u_x^2 + f_u u_{xx})_x u_{xx} \, dx
\]
\[= - \int_\mathbb{R} (g_x + g_u u_x)_x u_{xx} \, dx + \int_\mathbb{R} (a_{xx} u_x + 2a_x u_{xx} + a_{xxx})_x u_{xx} \, dx
\]
\[= - \int_\mathbb{R} P_x u_{xx} \, dx + \int_\mathbb{R} (h(t, x, u, u_x))_x u_{xx} \, dx + \int_\mathbb{R} (k(t, x, u))_x u_{xx} \, dx.
\]

\[\square\]
Observe that, by (2.14), (H.1), and (H.2), we find
\[
\int_R u_{xx} u_{xx} \, dx = \frac{d}{dt} \left\| u_{xx}(t, \cdot) \right\|^2_{L^2(R)} ,
\]
\[
- \int_R \left( f_{xx} + 2 f_{ux} u_x + f_{uu} u_x^2 + f_{uxx} u_{xx} \right) u_{xx} \, dx
\]
\[
= \int_R \left( f_{xx} + 2 f_{ux} u_x + f_{uu} u_x^2 + f_{uxx} u_{xx} \right) u_{xx} \, dx
\]
\[
\leq \frac{3C_0}{2} \int_R \left( f_{xx} + 2 f_{ux} u_x + f_{uu} u_x^2 + f_{uxx} u_{xx} \right)^2 \, dx + \frac{1}{6C_0} \int_R u^2_{xx} \, dx
\]
\[
\leq 3C_0 \int_R \left( C_0 u_x + 2C_0 u_x + C_0 u_x^2 + C_0 u_{xx} \right)^2 \, dx + \frac{1}{6C_0} \int_R u^2_{xx} \, dx
\]
\[
\leq 6C_0 \int_R \left( C_0 u_x + 4C_0 u_x^2 + C_0 u_{xx} \right)^2 \, dx + \frac{1}{6C_0} \int_R u^2_{xx} \, dx
\]
\[
\leq 24C_0^3 \| u(t, \cdot) \|^2_{H^1(R)} + 6C_0^3 \| u_x(t, \cdot) \|^2_{L^2(R)}
\]
\[
+ 6C_0^3 \| u_x(t, \cdot) \|^2_{L^4(R)} + \frac{1}{6C_0} \int_R u^2_{xx} \, dx
\]
\[
\leq 24C_0^3 \| u(t, \cdot) \|^2_{H^1(R)} + 18C_0^3 \| u(t, \cdot) \|^2_{H^1(R)} \| u_x(t, \cdot) \|^2_{L^2(R)}
\]
\[
+ 6C_0^3 \| u_x(t, \cdot) \|^2_{L^2(R)} + \frac{1}{6C_0} \int_R u^2_{xx} \, dx
\]
\[
\leq 24C_0^3 e^{11C_0 t} \| u_0 \|^2_{H^1(R)}
\]
\[
+ 6C_0^3 (3e^{11C_0 t} \| u_0 \|^2_{H^1(R)} + 1) \| u_x(t, \cdot) \|^2_{L^2} + \frac{1}{6C_0} \int_R u^2_{xx} \, dx ,
\]
\[
- \int_R \left( g_x + g_{ux} \right) u_x \, dx = \int_R \left( g_x + g_{ux} \right) u_{xx} \, dx
\]
\[
\leq \frac{3C_0}{2} \int_R \left( g_x + g_{ux} \right)^2 \, dx + \frac{1}{6C_0} \int_R u^2_{xx} \, dx
\]
\[
\leq 3C_0 \int_R \left( \frac{u_x}{2} + \frac{u_x}{2} \right)^2 \, dx + \frac{1}{6C_0} \int_R u^2_{xx} \, dx
\]
\[
\leq 3C_0 \int_R \left( u_x + u_x \right) \, dx + \frac{1}{6C_0} \int_R u^2_{xx} \, dx
\]
\[
= 3C_0 \| u_x(t, \cdot) \|^2_{L^2(R)} + \frac{1}{6C_0} \int_R u^2_{xx} \, dx
\]
\[
\leq 3C_0 e^{11C_0 t} \| u_0 \|^2_{L^2(R)} + \frac{1}{6C_0} \int_R u^2_{xx} \, dx ,
\]
\[
\int_R \left( a_{xx} u_x + 2a_x u_{xx} + a u_{xx} \right) u_{xx} \, dx
\]
\[
= - \int_R a_{xx} u_x u_{xx} \, dx - 2 \int_R a_x u_{xx} u_{xx} \, dx - \int_R a u_x^2 \, dx
\]
\[
\leq \frac{3C_0}{2} \int_R a_{xx} u_x^2 \, dx - \frac{5}{6C_0} \int_R u^2_{xx} \, dx - \int_R a_x u_x^2 \, dx
\]
\[
\leq \frac{3C_0}{2} \| u_x(t, \cdot) \|^2_{L^2(R)} + \int_R a_{xx} u_x^2 \, dx - \frac{5}{6C_0} \int_R u^2_{xx} \, dx
\]
\[
\leq \frac{3C_0}{2} e^{11C_0 t} \| u_0 \|^2_{H^1(R)} + C_0 \| u_x(t, \cdot) \|^2_{L^2(R)} - \frac{5}{6C_0} \int_R u^2_{xx} \, dx ,
\]
\[
\int_R \left( h(t, x, u, u_x) \right) u_{xx} \, dx = - \int_R h u_{xx} \, dx
\]
where using the generalized Hölder inequality.

Here we have used the inequality
\[
\int_R \left( k(t, x, u) \right)_x u_{x} dx = - \int_R k u_{xxx} dx
\]
\[
\leq \frac{3C_0}{2} \int_R k^2 dx + \frac{1}{6C_0} \int_R u_{xx}^2 dx
\]
\[
= \frac{3C_0}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{6C_0} \int_R u_{xx}^2 dx
\]
\[
\leq \frac{3C_0}{2} \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{6C_0} \int_R u_{xx}^2 dx.
\]

which follows from
\[
\int_R u^3_x(t, x) dx = \int_R (u u^3_x - 3 u u^2_x u_{xx}) dx
\]
\[
= -3 \int_R u u^2_x u_{xx} dx
\]
\[
\leq 3 \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \|u_x(t, \cdot)\|_{L^2(\mathbb{R})} \|u_{xx}(t, \cdot)\|_{L^2(\mathbb{R})}
\]

using the generalized Hölder inequality.

Finally, we have to estimate the nonlocal term \(P\). Observe that
\[
- \int_R P_x u_{xx} dx = \int_R P u_{xxx} dx \leq \frac{3C_0}{2} \int_R P^2 dx + \frac{1}{6C_0} \int_R u_{xx}^2 dx,
\]
moreover, using Remark 2.1 and (H.2),
\[
P(t, x) = \frac{1}{2} \int_R e^{-|x-y|} \left( k(t, y, u(t, y), u_x(t, y)) + k(t, y, u(t, y)) \right) dy
\]
\[
\leq C_0 (P_1 + P_2),
\]
where
\[
P_1(t, x) := \int_R e^{-|x-y|} |u(t, y)| dy,
\]
\[
P_2(t, x) := \int_R e^{-|x-y|} (u^2(t, y) + u_x^2(t, y)) dy.
\]
Therefore, from (2.26), (2.27), (2.28), (2.29), and Theorem 2.9, we get
\[(2.29)\]
\[\text{hence}\]
\[(2.28)\]
\[\text{C in the large. Due to Theorem 2.9 and Lemma 2.11, there exists a unique (global)}\]
\[\text{posed in time (see Lemma 2.4); it remains to prove that the solutions are defined}\]
\[(2.31)\]
\[\text{Proof of Theorem}\]
\[\text{we are done.}\]
\[\text{□}\]
\[\text{using the Tonelli theorem and the Hölder inequality, we find}\]
\[\int_{\mathbb{R}} |P_2(t,x)|^2 \, dx = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-|x-y|} |u(t,y)| \, dy \right)^2 \, dx\]
\[\leq \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} e^{-|x-y|} |u(t,y)| \right)^2 \, dx \, dy\]
\[= 2 \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-|x-y|} \, dy \right) |u(t,y)|^2 \, dx\]
\[= 4 \|u(t, \cdot)\|_{L^4(\mathbb{R})}^2 \leq 4 \|u(t, \cdot)\|_{H^1(\mathbb{R})}^2,\]
\[|P_2(t,x)| = \int_{\mathbb{R}} e^{-|x-y|}(u^2(t,y) + u_x^2(t,y)) \, dy\]
\[\leq \int_{\mathbb{R}} (u^2(t,y) + u_x^2(t,y)) \, dx = \|u(t, \cdot)\|_{H^1(\mathbb{R})}^2,\]
\[\int_{\mathbb{R}} |P_2(t,x)| \, dx = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} e^{-|x-y|} \, dy \right) (u^2(t,y) + u_x^2(t,y)) \, dx \, dy\]
\[= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-|x-y|} \, dy \right) (u^2(t,y) + u_x^2(t,y)) \, dy\]
\[\text{hence}\]
\[\int_{\mathbb{R}} |P_2(t,x)|^2 \, dx \leq \|P_2(t, \cdot)\|_{L^\infty(\mathbb{R})} \|P_2(t, \cdot)\|_{L^6(\mathbb{R})} \leq 2 \|u(t, \cdot)\|_{H^1(\mathbb{R})}^4.\]
\[\text{Therefore, from (2.26), (2.27), (2.28), (2.29), and Theorem 2.9, we get}\]
\[- \int_{\mathbb{R}} P_x u_{xx} \, dx \leq 6 C_0^3 \left( \|u(t, \cdot)\|_{H^1(\mathbb{R})}^2 + \|u(t, \cdot)\|_{H^1(\mathbb{R})}^4 \right) + \frac{1}{6 C_0} \int_{\mathbb{R}} u_{xx}^2 \, dx\]
\[\leq 6 C_0^3 e^{11 C_0 t} \|u_0\|_{H^1(\mathbb{R})}^2 \left(1 + e^{11 C_0 t} \|u_0\|_{H^1(\mathbb{R})}^2 \right) + \frac{1}{6 C_0} \int_{\mathbb{R}} u_{xx}^2 \, dx.\]
\[\text{Then, from (2.24),}\]
\[\frac{d}{dt} \|u_{xx}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq A \|u_{xx}(t, \cdot)\|_{H^1(\mathbb{R})}^2 + B,\]
\[\text{where A and B are defined in the statement. Then, using the Gronwall inequality}\]
\[\text{we are done.}\]
\[\text{□}\]
\[\text{Proof of Theorem 2.3. We know that the initial value problem (2.1) is locally well-}\]
\[\text{posed in time (see Lemma 2.4); it remains to prove that the solutions are defined in the large. Due to Theorem 2.9 and Lemma 2.11, there exists a unique (global)}\]
\[\text{solution that belongs to C([0,T]; H^2(\mathbb{R})), which is a subset of C([0,T]; W^{1,\infty}(\mathbb{R}))}\]
\[\text{(see [12, Theorem 8.5]). We have to prove that}\]
\[\|u(t, \cdot)\|_{H^1(\mathbb{R})} < +\infty, \quad 0 \leq t \leq T,\]
\[\text{and}\]
\[\|u(t, \cdot)\|_{H^1(\mathbb{R})} \in C^\infty((0,T) \times \mathbb{R}).\]
\[\text{We begin by proving (2.31). Since the argument is similar to the one of Lemma 2.11,}\]
\[\text{we only sketch it. Let i \in \{0, \ldots, \ell\}. Differentiating (2.17) i times with respect to}\]
\[\text{x we find}\]
\[\frac{\partial^{i+1} u}{\partial t^{i+1}} + \frac{\partial^{i+1}}{\partial x^{i+1}} (f(t,x,u)) + \frac{\partial^i}{\partial x^i} (g(t,x,u)) + \frac{\partial^{i+1} p}{\partial x^{i+1}} = \frac{\partial^{i+1}}{\partial x^{i+1}} (a(t,x) u_x).\]
Multiplying (2.33) by \( \frac{\partial^{i+1}u}{\partial x^i} \), integrating over \( \mathbb{R} \) and summing on \( i \),

\[
\sum_{i=0}^{\ell} \int_{\mathbb{R}} \frac{\partial^{i+1}u}{\partial x^i} \frac{\partial^i u}{\partial x^i} dx + \sum_{i=0}^{\ell} \int_{\mathbb{R}} \frac{\partial^{i+1}u}{\partial x^{i+1}} (f(t,x,u)) \frac{\partial^i u}{\partial x^i} dx \\
+ \sum_{i=0}^{\ell} \int_{\mathbb{R}} \frac{\partial^i}{\partial x^i} (g(t,x,u)) \frac{\partial^i u}{\partial x^i} dx + \sum_{i=0}^{\ell} \int_{\mathbb{R}} \frac{\partial^{i+1}P}{\partial x^i} \frac{\partial^i u}{\partial x^i} dx
\]

\[
= \sum_{i=0}^{\ell} \int_{\mathbb{R}} \frac{\partial^{i+1}}{\partial x^{i+1}} (a(t,x)u_x) \frac{\partial^i u}{\partial x^i} dx.
\]

Observe that by integrating by parts,

\[
\sum_{i=0}^{\ell} \int_{\mathbb{R}} \frac{\partial^{i+1}u}{\partial x^i} \frac{\partial^i u}{\partial x^i} dx = \frac{1}{2} \frac{d}{dt} \|u(t, \cdot)\|^2_{H^\ell(\mathbb{R})}.
\]

From (2.4b) and (2.4d),

\[
\sum_{i=0}^{\ell} \int_{\mathbb{R}} \frac{\partial^{i+1}}{\partial x^{i+1}} (a(t,x)u_x) \frac{\partial^i u}{\partial x^i} dx \leq c_1 \|u(t, \cdot)\|^2_{H^\ell(\mathbb{R})} - \frac{1}{C_0} \left\| \frac{\partial u}{\partial x} (t, \cdot) \right\|^2_{H^\ell(\mathbb{R})},
\]

where \( c_1 = c_1(\ell, C_0) > 0 \). From (2.4c) and (2.4d),

\[
\sum_{i=0}^{\ell} \int_{\mathbb{R}} \frac{\partial^{i+1}}{\partial x^{i+1}} (f(t,x,u)) \frac{\partial^i u}{\partial x^i} dx \\
\leq C_0 \sum_{i=0}^{\ell} \int_{\mathbb{R}} \left( \frac{\partial^i}{\partial x^i} (f(t,x,u)) \right)^2 dx + \frac{1}{2C_0} \sum_{i=0}^{\ell} \int_{\mathbb{R}} \left( \frac{\partial^{i+1}u}{\partial x^{i+1}} \right)^2 dx \\
\leq c_2 \|u(t, \cdot)\|^2_{H^\ell(\mathbb{R})} + \frac{1}{2C_0} \left\| \frac{\partial u}{\partial x} (t, \cdot) \right\|^2_{H^\ell(\mathbb{R})},
\]

where \( c_2 = c_2(\ell, C_0, \|u_0\|_{H^2(\mathbb{R})}, T) > 0 \). From (2.4e) and (2.4f),

\[
\sum_{i=0}^{\ell} \int_{\mathbb{R}} \frac{\partial^i}{\partial x^i} (g(t,x,u)) \frac{\partial^i u}{\partial x^i} dx.
\]
\[ \leq \frac{1}{2} \sum_{i=0}^{\ell} \int_{\mathbb{R}} \left( \frac{\partial^i}{\partial x^i} f(t, x, u) \right)^2 \, dx + \frac{1}{2} \sum_{i=0}^{\ell} \int_{\mathbb{R}} \left( \frac{\partial^i u}{\partial x^i} \right)^2 \, dx \leq c_3 \|u(t, \cdot)\|_{H^2(\mathbb{R})}, \]

where \( c_3 = c_3(\ell, C_0, \|u_0\|_{H^2(\mathbb{R})}, T) > 0 \). From (2.4e), (2.4g), and (2.4h), since \( u, u_x \) are bounded,

\begin{equation}
\sum_{i=0}^{\ell} \int_{\mathbb{R}} \frac{\partial^{i+1} P}{\partial x^{i+1}} \, dx \\
= - \sum_{i=0}^{\ell} \int_{\mathbb{R}} \frac{\partial^i P}{\partial x^i} \frac{\partial^{i+1} u}{\partial x^{i+1}} \, dx \\
\leq \frac{C_0}{2} \sum_{i=0}^{\ell} \int_{\mathbb{R}} \left( \frac{\partial^i P}{\partial x^i} \right)^2 \, dx + \frac{1}{2C_0} \sum_{i=0}^{\ell} \int_{\mathbb{R}} \left( \frac{\partial^{i+1} u}{\partial x^{i+1}} \right)^2 \, dx \\
\leq c_4 \|u(t, \cdot)\|_{H^2(\mathbb{R})}^2 + \frac{1}{2C_0} \|\frac{\partial u}{\partial x}(t, \cdot)\|_{H^2(\mathbb{R})}^2 + c_4,
\end{equation}

where \( c_4 = c_4(\ell, C_0, \|u_0\|_{H^2(\mathbb{R})}, T) > 0 \). Then, by (2.34), (2.35), (2.36), (2.37), (2.38) and (2.39),

\[ \frac{d}{dt}\|u(t, \cdot)\|_{H^2(\mathbb{R})}^2 \leq c_5 \|u(t, \cdot)\|_{H^2(\mathbb{R})}^2 + c_5, \]

where \( c_5 = c_5(\ell, C_0, \|u_0\|_{H^2(\mathbb{R})}, T) > 0 \), clearly this implies (2.31).

We conclude by proving (2.32). Let \( 0 < \tau < T \). Consider the new Cauchy problem

\[
\begin{aligned}
\omega_t + (f(t, x, \omega))_x + g(t, x, \omega) + \Omega_x &= (a(t, x) \omega_x)_x, \\
-\Omega_{xx} + \Omega &= h_0(t, x, \omega, \omega_x) + k_0(t, x, \omega), \\
\omega|_{t=0} &= u(\tau, \cdot).
\end{aligned}
\]

Since \( u(\tau, \cdot) \in H^2(\mathbb{R}) \) (see (2.31)), by Theorem 2.4, there exists a unique smooth solution \( \omega = \omega(t, x) \) defined on the strip \([\tau, \tau + \varepsilon) \times \mathbb{R}\), for some \( \varepsilon > 0 \). By Corollary 2.8, we have

\[ u(t, x) = \omega(t - \tau, x), \quad \tau \leq t \leq \tau + \varepsilon, \quad x \in \mathbb{R}, \]

and, due to the smoothness of \( \omega \), this implies (2.32).

\[ \square \]

3. Stability

Let \( u = u(t, x) \) be the solution of the Cauchy problem (see Theorem 2.3)

\[
\begin{aligned}
&u_t + (f_0(t, x, u))_x + g_0(t, x, u) + P_x = (a_0(t, x) u_x)_x, \\
&-P_{xx} + P = h_0(t, x, u, u_x) + k_0(t, x, u), \\
&u|_{t=0} = u_0,
\end{aligned}
\]

for \((t, x) \in \Pi_T\), and \( v = v(t, x) \) be the solution of

\[
\begin{aligned}
&v_t + (f_1(t, x, v))_x + g_1(t, x, v) + Q_x = (a_1(t, x) v_x)_x, \\
&-Q_{xx} + Q = h_1(t, x, v, v_x) + k_1(t, x, v), \\
v|_{t=0} = v_0,
\end{aligned}
\]

for \((t, x) \in \Pi_T\). Regarding the initial data \( u_0, v_0 : \mathbb{R} \to \mathbb{R} \), we assume

\[ u_0, v_0 \in H^\ell(\mathbb{R}) \quad \text{for some } \ell \geq 2. \]

In the following sections we shall assume:

(H.3) the coefficients \( f_i, g_i, a_i, h_i \) and \( k_i \) are smooth;
(H.4) there exists a constant $C_0 > 0$ such that the following hold
\[ f_i(\cdot, \cdot, 0) = 0, \]
\[ \frac{1}{C_0} \leq a_i(\cdot, \cdot) \leq C_0, \]
\[ \|\partial^l f_i(\cdot, \cdot, u)\|_{L^\infty} \leq C_0 |u|, \quad 1 \leq l \leq \ell + 1, \]
\[ \|a_i\|_{C^\infty}, \quad \|\partial^{j+i} f_i\|_{L^\infty} \leq C_0, \quad j \geq 1, \quad 2 \leq l + j \leq \ell + 1, \]
\[ \|k_{i,x}(\cdot, \cdot, u)\|_{L^\infty}, \quad \|\partial^{j+i} g_i(\cdot, \cdot, u)\|_{L^\infty} \leq C_0 |u|, \quad 0 \leq l \leq \ell, \]
\[ u \mapsto -\partial^i k_i(\cdot, \cdot, u), \quad u \in L^\infty_{\text{loc}}(\mathbb{R}), \quad 0 \leq l + j \leq \ell + 1, \]
\[ (u, q) \mapsto \|\partial^{j+i+p} h_i(\cdot, \cdot, u, q)\|_{L^\infty} \text{ is in } L^\infty_{\text{loc}}(\mathbb{R}^2), \quad 0 \leq l + j + p \leq \ell + 1, \]
\[ h_i(t, x, u, q) - \frac{1}{2} f_i(u, u)(t, x, u)q^3 \leq C_0(u^2 + q^2), \]
\[ \|h_i(\cdot, \cdot, u, q)\|_{L^\infty} \leq C_0(|u| + u^2 + q^2), \]
for every for $i = 0, 1, (t, x) \in \Pi_T$, and $u, q \in \mathbb{R}$;

(H.5) the following identities hold
\[ f_{0,x}(t, x, 0) = f_{1,x}(t, x, 0), \]
\[ g_0(t, x, 0) = g_1(t, x, 0), \]
\[ k_0(t, x, 0) = k_1(t, x, 0), \]
\[ h_{0,u}(t, x, 0, q) = h_{1,u}(t, x, 0, q), \]
\[ h_{0,q}(t, x, u, 0) = h_{1,q}(t, x, u, 0), \]
for every $(t, x) \in \Pi_T$ and $u, q \in \mathbb{R}$.

Clearly, these assumptions are satisfied in the case of the viscosity approximation of the generalized Camassa–Holm equation (see Example 2.2).

From Theorem 2.9 and Lemma 2.11 we know that
\[ \|u(t, \cdot)\|_{H^2(\mathbb{R})}, \|v(t, \cdot)\|_{H^2(\mathbb{R})} \leq C_2, \quad 0 \leq t \leq T, \]
where
\[ C_2 := \left( \|u_0\|_{H^1(\mathbb{R})} + \|v_0\|_{H^1(\mathbb{R})} \right) e^{11C_0T} + \left( \|u_{0,xx}\|_{H^1(\mathbb{R})} + \|v_{0,xx}\|_{H^1(\mathbb{R})} \right) e^{11C_0T} e^{A_1T} + \frac{B_1}{A_1} (e^{A_1T} - 1), \]
\[ A_1 := 2 \left[ 9c_1C_0^3 \left( \|u_0\|_{H^2(\mathbb{R})}^2 + \|v_0\|_{H^2(\mathbb{R})}^2 \right) e^{11C_0T} + 6C_0^3 + C_0 \right], \]
\[ B_1 := 2 \left[ 39C_0^3 \left( \|u_0\|_{H^2(\mathbb{R})}^2 + \|v_0\|_{H^2(\mathbb{R})}^2 \right) e^{11C_0T} + \left( \frac{1}{2} + 6C_0 \right) \left( \|u_0\|_{H^1(\mathbb{R})}^4 + \|v_0\|_{H^1(\mathbb{R})}^4 \right) 4e^{22C_0T} \right]. \]

The main result of this section is the following.

**Theorem 3.1.** Assume (H.3), (H.4), (H.5), and (3.3). Then
\[ \|u(t, \cdot) - v(t, \cdot)\|_{H^1(\mathbb{R})} \leq \|u_0 - v_0\|_{H^1(\mathbb{R})} e^{K_0T} \]
\[ + K_1 T \left( \|f_{1,ux} - f_{0,ux}\|_{L^\infty(\mathbb{R})} + \|f_{1,u} - f_{0,u}\|_{L^\infty(\mathbb{R})} \right) \]
for some positive constant \( \kappa \leq \) homotopy argument. Let 0 for each 3.1. problem (see Theorem 2.3)

\[
\frac{\partial}{\partial t} \Omega_{\theta,xx} + \Omega_{\theta} = h_\theta(t,x,\omega_{\theta,x}) + k_\theta(t,x,\omega_{\theta}),
\]

\[
\Omega_{\theta} = h_\theta(t,x,\omega_{\theta,x}) + k_\theta(t,x,\omega_{\theta}),
\]

\[
\|u(t, \cdot) - v(t, \cdot)\|_{H^1(\mathbb{R})} \leq \text{dist}_{H^1(\mathbb{R})}(u(t, \cdot), v(t, \cdot)) \leq \text{length}_{H^1(\mathbb{R})}(\omega_{\theta}(t, \cdot)),
\]

for each \( t \geq 0 \).

**Lemma 3.2** (Smoothness of \( \theta \mapsto \omega_{\theta} \)). Assume (H.3), (H.4), and (3.3). The curve

\[
\theta \mapsto \omega_{\theta}(t, x)
\]

is of class \( C^1 \). In particular, we infer

\[
\text{length}_{H^1(\mathbb{R})}(\omega_{\theta}(t, \cdot)) = \int_0^1 \left\| \frac{\partial \omega_{\theta}}{\partial \theta}(t, \cdot) \right\|_{H^1(\mathbb{R})} d\theta,
\]

for each \( 0 \leq t \leq T \).
Proof. Consider the map
\[ F = (F_1, F_2): \mathcal{D} \rightarrow C(\mathbb{R}, H^{l-2}(\mathbb{R})) \times H^{l-2}(\mathbb{R}), \]
\[ F_1(\theta, \omega, \Omega) := w_1 + (f_0(t, x, \omega))_x + g_0(t, x, \omega) + \Omega_x - (a_0(t, x)\omega_x)_x, \]
\[ F_2(\theta, \omega, \Omega) := -\Omega_{xx} + \Omega - h_0(t, x, \omega, \omega_x) - k_0(t, x, \omega), \]
where
\[ \mathcal{D} := \{(\theta, \omega, \Omega) \in [0,1] \times C([0,T]; H^l(\mathbb{R})) \mid \omega|_{t=0} = \theta v_0 + (1-\theta)u_0\}. \]

From the definition of \( \omega_\theta \) and \( \Omega_\theta \),
\[ F(\theta, \omega_\theta, \Omega_\theta) \equiv 0, \quad 0 \leq \theta \leq 1. \]
Observe that \( F \) is of class \( C^1 \), and
\[ \frac{\partial F_1}{\partial \theta}(\theta, \omega, \Omega) \equiv 0, \quad \frac{\partial F_2}{\partial \theta}(\theta, \omega, \Omega) \equiv 0, \]
\[ \frac{\partial F_1}{\partial \omega}(\theta, \omega, \Omega)[(\theta', z, Z)] = Z_x, \]
\[ \frac{\partial F_2}{\partial \omega}(\theta, \omega, \Omega)[(\theta', z, Z)] = -Z_{xx} + Z. \]

Using
\[ \frac{\partial F_i}{\partial \omega}(\theta, \omega, \Omega)[(\theta', z, Z)] = \frac{\partial F_i}{\partial \epsilon}(\theta, \omega + \epsilon z, \Omega) \bigg|_{\epsilon=0}, \quad i = 1, 2, \]
we find
\[ \frac{\partial F_1}{\partial \omega}(\theta, \omega, \Omega)[(\theta', z, Z)] = z_t + (f_\theta, u(t, x, \omega)z)_x + g_\theta, u(t, x, \omega)z - (a_\theta(t, x)z_x)_x, \]
\[ \frac{\partial F_2}{\partial \omega}(\theta, \omega, \Omega)[(\theta', z, Z)] = -h_\theta, u(t, x, \omega)z - h_\theta, q(t, x, \omega, \omega_x)z_x - k_\theta, u(t, x, \omega)z. \]

Observe that \((\theta', z, Z) \in \mathcal{D} \) satisfies the equation
\[ \frac{\partial F}{\partial (\omega, \Omega)}(\theta, \omega, \Omega)[(\theta', z, Z)] = (\zeta, \xi) \]
if and only if \((\theta', z, Z) \) is solution of the linear initial value problem
\[ z_t + (f_\theta, u(t, x, \omega)z)_x + g_\theta, u(t, x, \omega)z + Z_x = (a_\theta(t, x)z_x)_x + \zeta(t, x), \]
\[ -Z_{xx} + Z = h_\theta, u(t, x, \omega_x)z + k_\theta, u(t, x, \omega)z + h_\theta, q(t, x, \omega, \omega_x)z_x + \xi(t, x), \]
\[ z|_{t=0} = \theta' v_0 + (1-\theta')u_0. \]

Since this problem admits a unique solution (see Theorem 2.3), \( \frac{\partial F}{\partial (\omega, \Omega)}(\theta, \omega, \Omega) \) is invertible. By the implicit function theorem, the curve \( \theta \rightarrow \omega_\theta \) is of class \( C^1 \) and clearly (3.8) holds. This concludes the proof. \( \square \)

Differentiating the equations in (3.6) with respect to \( \theta \), we have
\[ \frac{\partial^2 \omega_\theta}{\partial \theta^2} + \frac{\partial}{\partial x} \left( \frac{\partial f_\theta}{\partial u}(t, x, \omega_\theta) \frac{\partial \omega_\theta}{\partial \theta} \right) + \frac{\partial}{\partial x} \left( f_1(t, x, \omega_\theta) - f_0(t, x, \omega_\theta) \right) \]
\[ + \frac{\partial g_\theta}{\partial u}(t, x, \omega_\theta) \frac{\partial \omega_\theta}{\partial \theta} + (g_1(t, x, \omega_\theta) - g_0(t, x, \omega_\theta)) + \frac{\partial^2 \Omega_\theta}{\partial x \partial \theta} \]
Denoting $z_{\theta} := \frac{\partial \omega_{\theta}}{\partial \theta}$,

$Z_{\theta} := \frac{\partial \Omega_{\theta}}{\partial \theta}$,

$\alpha(\theta, t, x) := \frac{\partial f_{\theta}}{\partial u}(t, x, \omega_{\theta})$,

$\beta(\theta, t, x) := \frac{\partial g_{\theta}}{\partial u}(t, x, \omega_{\theta})$,

$\gamma(\theta, t, x) := \frac{\partial}{\partial x} \left( f_{1}(t, x, \omega_{\theta}) - f_{0}(t, x, \omega_{\theta}) + (g_{1}(t, x, \omega_{\theta}) - g_{0}(t, x, \omega_{\theta})) \right)$

$\delta(\theta, t, x) := a_{\theta}(t, x)$,

$\eta(\theta, t, x) := (h_{0}(t, x, \omega_{\theta}, \omega_{\theta,x}) - h_{1}(t, x, \omega_{\theta}, \omega_{\theta,x})) + (k_{0}(t, x, \omega_{\theta}) - k_{1}(t, x, \omega_{\theta}))$,

$\mu(\theta, t, x) := \frac{\partial h_{\theta}}{\partial u}(t, x, \omega_{\theta}, \omega_{\theta,x}) + \frac{\partial k_{\theta}}{\partial u}(t, x, \omega_{\theta})$,

$\nu(\theta, t, x) := \frac{\partial k_{\theta}}{\partial q}(t, x, \omega_{\theta}, \omega_{\theta,x})$,

for each $0 \leq \theta \leq 1$, we find

\[ z_{\theta,t} + (\alpha z_{\theta})_{x} + \beta z_{\theta} + \gamma + Z_{\theta,x} = \left( \delta z_{\theta,x} \right)_{x}, \]

\[ -Z_{\theta,xx} + Z_{\theta} = \mu z_{\theta} + \nu z_{\theta, x} + \eta, \]

for $(t, x) \in I_{T}$. Moreover, observe that

\[ z_{\theta} \big|_{t=0} = v_{0} - u_{0}, \quad 0 \leq \theta \leq 1. \]

### 3.2. Energy estimate

In this section we prove Theorem 3.1.

**Remark 3.3.** By (H.4), (H.5), and (3.4), we infer

\[ \|a\|_{L^{\infty}} \leq \|f_{0,u}\|_{L^{\infty}(I)} + \|f_{1,u}\|_{L^{\infty}(I)} \leq C_{3}, \]

\[ \|\alpha_{x}\|_{L^{\infty}} \leq \|f_{0,ux}\|_{L^{\infty}(I)} + \|f_{1,ux}\|_{L^{\infty}(I)} \]

\[ + \left( \|f_{0,uu}\|_{L^{\infty}(I)} + \|f_{1,uu}\|_{L^{\infty}(I)} \right) \sqrt{2C_{2}} \leq C_{3}, \]

\[ \|\beta\|_{L^{\infty}} \leq \|g_{0,u}\|_{L^{\infty}(I)} + \|g_{1,u}\|_{L^{\infty}(I)} \leq C_{3}, \]

\[ \|\delta_{xx}\|_{L^{\infty}} \leq \|a_{0,xx}\|_{L^{\infty}(\Pi_{T})} + \|a_{1,xx}\|_{L^{\infty}(\Pi_{T})} \leq C_{3}, \]

\[ \|\mu\|_{L^{\infty}} \leq \|h_{0,uu}\|_{L^{\infty}(I)} + \|h_{1,uu}\|_{L^{\infty}(I)} + \|k_{0,u}\|_{L^{\infty}(I)} + \|k_{1,u}\|_{L^{\infty}(I)} \leq C_{3}, \]

\[ \|\nu\|_{L^{\infty}} \leq \|h_{0,q}\|_{L^{\infty}(I)} + \|h_{1,q}\|_{L^{\infty}(I)} \leq C_{3}, \]

where

\[ C_{3} := c_{2}c_{0}C_{2}, \]
for some positive constant $c_2$ independent on $C_0$, $u_0$, and $v_0$. Moreover,

$$\frac{1}{C_0} \leq \delta(\cdot, \cdot, \cdot) \leq C_0.$$

(3.13) \hspace{1cm} \|\gamma(\theta, t, \cdot)\|_{L^2(\mathbb{R})} \leq \|f_{1,ux} - f_{0,ux}\|_{L^\infty(\mathcal{T})}\|\omega_0(t, \cdot)\|_{L^2(\mathbb{R})} + \|f_{1,u} - f_{0,u}\|_{L^\infty(\mathcal{T})}\|\omega_0(t, \cdot)\|_{L^2(\mathbb{R})} + \|g_{1,u} - g_{0,u}\|_{L^\infty(\mathcal{T})}\|\omega_0(t, \cdot)\|_{L^2(\mathbb{R})} + \|a_{1,x} - a_{0,x}\|_{L^\infty(\Pi_T)}\|\omega_0(t, \cdot)\|_{L^2(\mathbb{R})} + \|a_1 - a_0\|_{L^\infty(\Pi_T)}\|\omega_0(t, \cdot)\|_{L^2(\mathbb{R})} \leq C_2 \left[ \|f_{1,ux} - f_{0,ux}\|_{L^\infty(\mathcal{T})} + \|f_{1,u} - f_{0,u}\|_{L^\infty(\mathcal{T})} + \|g_{1,u} - g_{0,u}\|_{L^\infty(\mathcal{T})} + \|a_{1,x} - a_{0,x}\|_{L^\infty(\Pi_T)} + \|a_1 - a_0\|_{L^\infty(\Pi_T)} \right],

for each $0 \leq \theta \leq 1$ and $0 \leq t \leq T$.

The following lemma is needed.

**Lemma 3.4.** Assume (H.3), (H.4), (H.5), and (3.3). Then

(3.14) \hspace{1cm} \|\eta(\theta, t, \cdot)\|_{L^2(\mathbb{R})} \leq \|h_{0,u} - h_{1,u}\|_{L^\infty(\mathcal{T})}\|\omega_0(t, \cdot)\|_{L^2(\mathbb{R})} + \|k_{0,u} - k_{1,u}\|_{L^\infty(\mathcal{T})}\|\omega_0(t, \cdot)\|_{L^2(\mathbb{R})} + \|h_{0,q} - h_{1,q}\|_{L^\infty(\mathcal{T})}\|\omega_0(t, \cdot)\|_{L^2(\mathbb{R})} \leq C_2 \left[ \|h_{0,u} - h_{1,u}\|_{L^\infty(\mathcal{T})} + \|k_{0,u} - k_{1,u}\|_{L^\infty(\mathcal{T})} + \|h_{0,q} - h_{1,q}\|_{L^\infty(\mathcal{T})} \right],

for each $0 \leq \theta \leq 1$ and $0 \leq t \leq T$, where

$$K_2 := 5C_3 + 3C_0C_3^2 + C_2 + 2.$$

**Proof.** Let $0 < \theta < 1$ and $t > 0$. Multiplying the first equation of (3.10) by $\omega_0$ and integrating on $\mathbb{R}$,

$$\int_{\mathbb{R}} \omega_0 \omega_0 \omega_0 dx = -\int_{\mathbb{R}} (\alpha \omega_0) x \omega_0 dx - \int_{\mathbb{R}} \beta \omega_0^2 dx - \int_{\mathbb{R}} \gamma \omega_0 dx = \int_{\mathbb{R}} Z_{\theta,x} \omega_0 dx + \int_{\mathbb{R}} (\delta \omega_0, x) \omega_0 dx.$$

Integrating by parts and using (H.3), (H.4), (H.5), (3.12),

(3.17) \hspace{1cm} -\int_{\mathbb{R}} \beta \omega_0^2 dx \leq \|\beta\|_{L^\infty} \int_{\mathbb{R}} \omega_0^2 dx \leq C_4 \|\omega_0(t, \cdot)\|^2_{H^1(\mathbb{R})},

$$\int_{\mathbb{R}} (\delta \omega_0, x) \omega_0 dx = -\int_{\mathbb{R}} \delta \omega_0^2 dx \leq 0,$n

$$\int_{\mathbb{R}} (\alpha \omega_0) x \omega_0 dx = \int_{\mathbb{R}} \omega_0 \omega_0 \omega_0 dx \leq \|\alpha\|_{L^\infty} \int_{\mathbb{R}} \omega_0^2 dx \leq \frac{1}{2} \int_{\mathbb{R}} \omega_0^2 dx + \frac{1}{2} \int_{\mathbb{R}} \omega_0^2 dx \leq \frac{C_3^2}{2} \omega_0(t, \cdot) \omega_0(t, \cdot) \|\omega_0(t, \cdot)\|^2_{H^1(\mathbb{R})},$$
where

Multiplying this equation by $z$

Differentiating the first equation in (3.10) with respect to $x$,

Then from (3.16) and (3.17),

(3.18) \[ \frac{1}{2} \int \frac{d}{dt} \|z_0(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_4 \|z_0(t, \cdot)\|^2_{H^1(\mathbb{R})} - \int \theta_{t,x}z_0 dx + \frac{1}{2} \|\gamma(\theta, t, \cdot)\|^2_{L^2(\mathbb{R})}, \]

where

\[ C_4 := C_3 + \frac{C^2}{2} + \frac{1}{2}. \]

Differentiating the first equation in (3.10) with respect to $x$, we have

(3.19) \[ z_{t,x} + (\alpha_x z_0 + \alpha z_{\theta,x})\cdot x + (\beta z_0)\cdot x + \gamma_x = -z_{\theta,xx} + (\delta_x z_0 + \delta z_{\theta,xx})\cdot x = -z_\theta + \mu z_\theta + \nu z_{\theta,x} + \eta + (\delta_x z_\theta + \delta z_{\theta,xx})\cdot x. \]

Multiplying this equation by $z_{\theta,x}$ and integrating on $\mathbb{R}$, we get

(3.20) \[ \int \theta_{t,x}z_{\theta,x} dx = -\int \left( \alpha_x z_\theta + \alpha z_{\theta,x} \right) z_{\theta,x} dx \]

Again, integrating by parts and using (H.3), (H.4), (H.5), (3.12),

(3.21) \[ \int \theta_{t,x}z_{\theta,x} dx = \frac{d}{dt} \int \theta_{\theta,x} dx = \frac{1}{2} \frac{d}{dt} \|z_\theta(x, t, \cdot)\|_{L^2(\mathbb{R})}^2, \]

Finally, integrating by parts and using (H.3), (H.4), (H.5), (3.12),

(3.22) \[ \int \left( \alpha_x z_\theta + \alpha z_{\theta,x} \right) z_{\theta,x} dx = \int \left( \alpha_x z_\theta + \alpha z_{\theta,x} \right) z_{\theta,x} dx \]

where

\[ C_4 := C_3 + \frac{C^2}{2} + \frac{1}{2}. \]
\[
\begin{align*}
&= -\frac{1}{2} \int_{\mathbb{R}} \delta_x(z^2_{\theta,x})dx - \int_{\mathbb{R}} \delta_x^2 dx \\
&= \frac{1}{2} \int_{\mathbb{R}} \delta_{xx} z^2_{\theta,x} dx - \int_{\mathbb{R}} \frac{1}{C_0} \int_{\mathbb{R}} z^2_{\theta,x} dx \\
&\leq \frac{1}{2} \|\delta_{xx}\|_{L^\infty} \int_{\mathbb{R}} z^2_{\theta,x} dx - \frac{1}{C_0} \int_{\mathbb{R}} z^2_{\theta,x} dx \\
&\leq \frac{C_3}{2} \|\theta(t, \cdot)\|_{H^1(\mathbb{R})}^2 - \frac{1}{C_0} \int_{\mathbb{R}} z^2_{x,x,x} dx,
\end{align*}
\]

Therefore, from (3.20) and (3.21),

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|z_{\theta,x}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_5 \|z_{\theta}(t, \cdot)\|_{H^1(\mathbb{R})}^2 + \int_{\mathbb{R}} z_{x,x,x} z_{\theta} dx \\
+ \frac{3C_0}{4} \|\gamma(\theta, t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \|\eta(\theta, t, \cdot)\|_{L^2(\mathbb{R})}^2,
\end{align*}
\]

where

\[
C_5 := \frac{3C_0C_3^2}{4} + C_2 + \frac{C_3}{2} + \frac{1}{2}.
\]

Summing (3.18) and (3.22),

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|z_{\theta}(t, \cdot)\|_{H^1(\mathbb{R})}^2 \leq (C_4 + C_5) \|z_{\theta,x}(t, \cdot)\|_{H^1(\mathbb{R})}^2 \\
+ \frac{5C_0}{4} \|\gamma(\theta, t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \|\eta(\theta, t, \cdot)\|_{L^2(\mathbb{R})}^2.
\end{align*}
\]

The claim now follows from (3.23) and the Gronwall inequality. \(\square\)

**Proof of Theorem 3.1.** The claim is direct consequence of (3.8), (3.11), (3.15), (3.13), and (3.14). \(\square\)

**References**


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