Forward integrals and an Itô formula for fractional Brownian motion

Francesca Biagini\(^1\) Bernt Øksendal\(^2,3\)

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1) Department of Mathematics, University of Bologna, Piazza di Porta S. Donato, 5 I-40127 Bologna, Italy Email: biagini@dm.unibo.it

2) Center of Mathematics for Applications (CMA) Department of Mathematics, University of Oslo Box 1053 Blindern, N-0316 Oslo, Norway Email: oksendal@math.uio.no

3) Norwegian School of Economics and Business Administration, Helleveien 30, N-5045 Bergen, Norway

Abstract

We consider the forward integral with respect to fractional Brownian motion \( B^{(H)}(t) \) and relate this to the Wick-Itô-Skorohod integral by using the \( M \)-operator introduced by [10] and the Malliavin derivative \( D_t^{(H)} \). Using this connection we obtain a general Itô formula for the Wick-Itô-Skorohod integrals with respect to \( B^{(H)}(t) \), valid for \( H \in (\frac{1}{2}, 1) \).

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1 Introduction

Fractional Brownian motion $B^{(H)}(t) = B^{(H)}(t, \omega), \ t \geq 0, \omega \in \Omega$, with Hurst parameter $H \in (0, 1)$ is a real-valued Gaussian process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the property that

$$ E \left[ B^{(H)}(t) \right] = B^{(H)}(0) = 0 \text{ for all } t \geq 0 $$

and

$$ E \left[ B^{(H)}(t) B^{(H)}(s) \right] = \frac{1}{2} \left[ t^{2H} + s^{2H} - |t - s|^{2H} \right]; \ t, s \geq 0 $$

where $E$ denotes expectation with respect to $\mathbb{P}$.

Because of its properties the fractional Brownian motion has been used to model a number of phenomena, e.g. in biology, meteorology, physics and finance. See e.g. [24], [6], [7], [21] and the references therein. In that connection, it is of interest to develop a stochastic calculus based on $B^{(H)}(t)$.

In particular, one wants an integration theory, a white noise theory and a Malliavin calculus for such processes. See e.g. [6] and the references therein for an account of this.

There are several different integral concepts of independent interest, among which the pathwise integral and the Wick-Itô-Skorohod integral. For each of these integrals several versions of an Itô formula have been obtained. See for example [5], [7], [9], [15], [18], [19], [11].

The purpose of this paper is to prove a new general Itô formula for the Wick-Itô-Skorohod integral based on the $M$-operator of [10] and the Malliavin derivative $D^{(H)}_t$, valid for $H \in (\frac{1}{2}, 1)$.

2 Some preliminaries

Here we recall the approach of [10], [16],[7] to white-noise calculus for fractional Brownian motion.

We begin by recalling the standard setup for the classical white noise probability space. See e.g. [13], [17], [14] or [1] for more details.

**Definition 2.1** Let $S(\mathbb{R})$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$ and let $\Omega := S'(\mathbb{R})$ be its dual, usually called the space of tempered distributions. Let $\mathbb{P}$ be the probability measure on the Borel sets $\mathcal{B}(S'(\mathbb{R}))$ defined by the property that

$$ \int_{S'(\mathbb{R})} \exp(i < \omega, f >) d\mathbb{P}(\omega) = \exp(-\frac{1}{2} \| f \|^{2}_{L^2(\mathbb{R})}); \ f \in S(\mathbb{R}), $$

(2.1)
where \( i = \sqrt{-1} \) and \( \langle \omega, f \rangle = \omega(f) \) is the action of \( \omega \in \Omega = \mathcal{S}'(\mathbb{R}) \) on \( f \in \mathcal{S}(\mathbb{R}) \).

The measure \( \mathbb{P} \) is called the white noise probability measure. Its existence follows from the Bochner–Minlos theorem.

In the following we let
\[
h_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}); \quad n = 0, 1, 2, \ldots
\]
(2.2)
denote the Hermite polynomials and we let
\[
\xi_n(x) = \pi^{-\frac{1}{4}} ((n - 1)!)^{-\frac{1}{2}} h_{n-1}(\sqrt{2}x)e^{-\frac{x^2}{2}}; \quad n = 1, 2, \ldots
\]
(2.3)
be the Hermite functions. Then \( \xi_n \in \mathcal{S}(\mathbb{R}) \). From [25], \( \{\xi_n\}_{n=1}^{\infty} \) constitutes an orthonormal basis for \( L^2(\mathbb{R}) \). Let \( \mathcal{J} \) be the set of all multi-indices \( \alpha = (\alpha_1, \alpha_2, \ldots) \) of finite length \( l(\alpha) = \max \{i; \alpha_i \neq 0\} \), with \( \alpha_i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) for all \( i \). For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{J} \) we put \( \alpha! = \alpha_1!\alpha_2! \cdots \alpha_n! \) and \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and we define
\[
\mathcal{H}_\alpha(\omega) = h_{\alpha_1}(\langle \omega, \xi_1 \rangle) h_{\alpha_2}(\langle \omega, \xi_2 \rangle) \cdots h_{\alpha_n}(\langle \omega, \xi_n \rangle).
\]
(2.4)

In particular special cases are the unit vectors
\[
\epsilon^{(k)} = (0, 0, \ldots, 0, 1)
\]
with 1 on the \( k \)'th entry, 0 otherwise; \( k = 1, 2, \ldots \). We now use the well-known Wiener-Itô chaos expansion Theorem to define the following space (\( \mathcal{S} \)) of stochastic test functions and the dual space (\( \mathcal{S}^* \) of stochastic distributions:

**Definition 2.2 a)** We define the Hida space (\( \mathcal{S} \)) of stochastic test functions to be all \( \psi \in L^2(\mathbb{P}) \) whose expansion
\[
\psi(\omega) = \sum_{\alpha \in \mathcal{J}} a_\alpha \mathcal{H}_\alpha(\omega)
\]
satisfies
\[
\|\psi\|_k^2 := \sum_{\alpha \in \mathcal{J}} a_\alpha^2 \alpha!(2N)^{k\alpha} < \infty \quad \text{for all } k = 1, 2, \ldots
\]
(2.6)
where
\[
(2N)^\gamma = (2 \cdot 1)^{\gamma_1}(2 \cdot 2)^{\gamma_2} \cdots (2 \cdot m)^{\gamma_m} \quad \text{if } \gamma = (\gamma_1, \ldots, \gamma_m) \in \mathcal{J}.
\]
(2.7)
We define the Hida space \((S)^*\) of stochastic distributions to be the set of formal expansions
\[
G(\omega) = \sum_{\alpha \in J} b_\alpha H_\alpha(\omega)
\]
such that
\[
\|G\|_q^2 := \sum_{\alpha \in J} b_\alpha^2 \alpha!(2N)^{\alpha q} < \infty \quad \text{for some } q < \infty.
\] (2.8)

We equip \((S)\) with the projective topology and \((S)^*\) with the inductive topology. Convergence in \((S)\) means convergence in \(\| \cdot \|_k\) for every \(k = 1, 2, \cdots\), while convergence in \((S)^*\) means convergence in \(\| \cdot \|_q\) for some \(q < \infty\). Then \((S)^*\) can be identified with the dual of \((S)\) and the action of \(G \in (S)^*\) on \(\psi \in (S)\) is given by
\[
\langle G, \psi \rangle_{(S)^*, (S)} := \sum_{\alpha \in J} \alpha! a_\alpha b_\alpha
\] (2.9)

In the sequel, we will denote the action \(\langle \cdot, \cdot \rangle_{(S)^*, (S)}\) simply with the symbol \(\langle \cdot, \cdot \rangle\). We can in a natural way define \((S)^*\)-valued integrals as follows:

**Definition 2.3 (Integration in \((S)^*\))** Suppose \(Z : \mathbb{R} \to (S)^*\) has the property that
\[
\langle Z(t), \psi \rangle \in L^1(\mathbb{R}, dt) \quad \text{for all } \psi \in (S).
\]
Then the integral
\[
\int_{\mathbb{R}} Z(t) dt
\]
is defined to be the unique element of \((S)^*\) such that
\[
\left\langle \int_{\mathbb{R}} Z(t) dt, \psi \right\rangle = \int_{\mathbb{R}} \langle Z(t), \psi \rangle dt \quad \text{for all } \psi \in (S). \quad (2.10)
\]
Such functions \(Z(t)\) are called \(dt\)-integrable in \((S)^*\).

Let \(B(t)\) a standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\). If we consider \(B(t)\) as a map \(B(\cdot) : \mathbb{R} \to (S)^*\), then \(B(t)\) is differentiable with respect to \(t\) and its derivative \(W(t) := \frac{d}{dt} B(t)\) exists in \((S)^*\) and is called white noise.

A fundamental property of the Wick product is the following relation to (Itô-)Skorohod integration. We recall the definition of Skorohod integral.

Let \(u(t, \omega), \omega \in \Omega, t \in [0, T]\) be a stochastic process (always assumed to be \((t, \omega)\)-measurable), such that
\[
u(t, \cdot) \quad \text{is } \mathcal{F}\text{-measurable for all } t \in [0, T]
\] (2.11)
and
\[ E[u^2(t,\omega)] < \infty \quad \text{for all } t \in [0,T]. \quad (2.12) \]

**Definition 2.4** Suppose \( u(t,\omega) \) is a stochastic process satisfying (2.11), (2.12) and with Wiener-Itô chaos expansion
\[ u(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)). \quad (2.13) \]

Then we define the Skorohod integral of \( u \) by
\[ \delta(u) : = \int_{\mathbb{R}} u(t,\omega) \delta B(t) := \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n) \quad \text{(when convergent)} \quad (2.14) \]
where \( \tilde{f}_n \) is the symmetrization of \( f_n(t_1,\ldots,t_n,t) \) as a function of \( n+1 \) variables \( t_1,\ldots,t_n,t \).

We say \( u \) is Skorohod-integrable and write \( u \in \text{dom}(\delta) \) if the series in (2.14) converges in \( L^2(\mathbb{P}) \). This occurs iff
\[ E[\delta(u)^2] = \sum_{n=0}^{\infty} (n+1)! \|\tilde{f}_n\|_{L^2(\mathbb{R}^{n+1})}^2 < \infty. \quad (2.15) \]

**Theorem 2.5** Suppose \( f(t,\omega) : \mathbb{R} \times \Omega \to \mathbb{R} \) is Skorohod integrable. Then \( f(t,\cdot) \diamond W(t) \) is dt-integrable in \((\mathbb{S})^*\) and
\[ \int_{\mathbb{R}} f(t,\omega) \delta B(t) = \int_{\mathbb{R}} f(t,\omega) \diamond W(t) dt, \quad (2.16) \]
where the integral on the left is the Skorohod integral (which coincides with the Itô integral if \( f \) is adapted) and \( f(t,\omega) \diamond W(t) \) denotes the Wick product in \((\mathbb{S})^*\).

### 2.1 Integration

We now review briefly how the classical white noise theory can be used in order to construct a stochastic integral with respect to a fractional Brownian motion \( B^{(H)}(t) \) for any \( H \in (0,1) \) as in the approach of [10]. The main idea is to relate the fractional Brownian motion \( B^{(H)}(t) \) with Hurst parameter \( H \in (0,1) \) to classical Brownian motion \( B(t) \) via the following operator \( M \):
Definition 2.6 The operator $M = M^{(H)}$ is defined on functions $f \in S(\mathbb{R})$ by

$$
\hat{M}f(y) = |y|^{\frac{1}{2}-H} \hat{f}(y); \quad y \in \mathbb{R}
$$

(2.17)

where

$$
\hat{g}(y) := \int_\mathbb{R} e^{-ixy}g(x)dx
$$

(2.18)

denotes the Fourier transform.

For further details on the operator $M$, we refer to [10] and to [6]. The operator $M$ extends in a natural way from $S(\mathbb{R})$ to the space

$$
L^2_H(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ (deterministic); } |y|^{\frac{1}{2}-H} \hat{f}(y) \in L^2(\mathbb{R}) \right\}
$$

where $\|f\|_{L^2_H(\mathbb{R})} = \|Mf\|_{L^2(\mathbb{R})}$.

The inner product on this space is

$$
\langle f, g \rangle_{L^2_H(\mathbb{R})} = \langle Mf, Mg \rangle_{L^2(\mathbb{R})}.
$$

(2.19)

If $(\xi_n)_{n \in \mathbb{N}}$ is the orthonormal basis of $L^2(\mathbb{R})$ introduced in (2.3), then

$$
e_n := M^{-1}\xi_n, \quad \forall n \in \mathbb{N}
$$

(2.20)

is an orthonormal basis for $L^2_H(\mathbb{R})$. In particular, the indicator function $\chi_{[0,t]}(\cdot)$ is easily seen to belong to this space, for fixed $t \in \mathbb{R}$, and we write

$$
M\chi_{[0,t]}(x) = M[0,t](x).
$$

We now define, for $t \in \mathbb{R}$

$$
\tilde{B}^{(H)}(t) := \tilde{B}^{(H)}(t, \omega) := \langle \omega, M[0,t](\cdot) \rangle
$$

(2.21)

Then $\tilde{B}^{(H)}(t)$ is Gaussian, $\tilde{B}^{(H)}(0) = E[\tilde{B}^{(H)}(t)] = 0$ for all $t \in \mathbb{R}$ and

$$
E \left[ \tilde{B}^{(H)}(s) \tilde{B}^{(H)}(t) \right] = \frac{1}{2} \|t|^{2H} + |s|^{2H} - |s-t|^{2H}
$$

as follows by [10], (A.10). Therefore the continuous version of $B^{(H)}(t)$ of $\tilde{B}^{(H)}(t)$ is a fractional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $f \in L^2_H(\mathbb{R})$ and define

$$
\int_\mathbb{R} f(t)dB^{(H)}(t) := \int_\mathbb{R} Mf(t)dB(t); \quad f \in L^2_H(\mathbb{R}).
$$

(2.22)
Now define the fractional white noise \( W^H(t) \) as the derivative with respect to \( t \) of \( B^H(t) \)

\[
\frac{dB^H(t)}{dt} = W^H(t) \text{ in } (S)^*.
\]

In particular, by [7] we obtain that the relation between fractional and classical white noise is given by

\[
W^H(t) = MW(t).
\]

In view of Theorem 2.5 the following definition is natural:

**Definition 2.7 (The fractional Wick-Itô-Skorohod (WIS) integral)**

Let \( Y : \mathbb{R} \rightarrow (S)^* \) be such that \( Y(t) \circ W^H(t) \) is \( dt \)-integrable in \((S)^*\). Then we say that \( Y \) is \( dB^H \)-integrable and we define the Wick-Itô-Skorohod (WIS) integral of \( Y(t) = Y(t, \omega) \) with respect to \( B^H(t) \) by

\[
\int_{\mathbb{R}} Y(t, \omega) dB^H(t) := \int_{\mathbb{R}} Y(t) \circ W^H(t) dt.
\]

Note that this definition coincides with (2.22) if \( Y = f \in L^2_H(\mathbb{R}) \).

**Definition 2.8**

A process \( Y(t) = \sum_{\alpha \in J} c_\alpha(t) \mathcal{H}_\alpha(\omega) \in (S)^* \) belongs to the space \( \mathcal{M} \) if \( c_\alpha(\cdot) \in L^2_H(\mathbb{R}) \) and \( \sum_{\alpha \in J} M c_\alpha(t) \mathcal{H}_\alpha(\omega) \) converges in \((S)^*\) for all \( t \).

Then the following fundamental relation holds.

**Proposition 2.9 (Integration)** [BØSW, (5.2)], [Ø, (3.16)] Suppose \( Y : \mathbb{R} \rightarrow (S)^* \) is \( dB^H \)-integrable (Definition 2.7) and \( Y \in \mathcal{M} \). Then

\[
\int_{\mathbb{R}} Y(t) dB^H(t) = \int_{\mathbb{R}} MY(t) \delta B(t).
\]

### 2.2 Differentiation

We now recall the approach in [16] to differentiation, as modified and extended by [10]:

**Definition 2.10** Let \( F : \Omega \rightarrow \mathbb{R} \) and choose \( \gamma \in \Omega \). Then we say \( F \) has a directional \( M \)-derivative in the direction \( \gamma \) if

\[
D^{(H)}_\gamma F(\omega) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [F(\omega + \varepsilon M\gamma) - F(\omega)]
\]

exists almost surely in \((S)^*\). In that case we call \( D^{(H)}_\gamma F \) the directional \( M \)-derivative of \( F \) in the direction \( \gamma \).
Definition 2.11 We say that $F: \Omega \to \mathbb{R}$ is differentiable if there exists a function 
\[ \Psi : \mathbb{R} \to (\mathbb{S})^* \]
in $\mathcal{M}$ such that 
\[ D_{\gamma}^{(H)} F(\omega) = \int_{\mathbb{R}} M\Psi(t)M\gamma(t)dt \quad \text{for all} \quad \gamma \in L^2_H(\mathbb{R}). \] (2.28)

Then we write 
\[ D_t^{(H)} F := \frac{\partial^{(H)}}{\partial \omega} F(t, \omega) = \Psi(t) \] (2.29)
and we call $D_t^{(H)} F$ the Malliavin derivative or the stochastic gradient of $F$ at $t$.

In the classical case ($H = \frac{1}{2}$) we use the notation $D_t$ for the corresponding Malliavin derivative.

Proposition 2.12 [BØSW, (5.1)] Let $F \in (\mathbb{S})^*$. Then 
\[ D_t F = M D_t^{(H)} F \quad \text{for a.a.} \quad t \in \mathbb{R}. \] (2.30)

Proposition 2.13 [BØSW, Theorem 5.3] Suppose $Y: \mathbb{R} \to (\mathbb{S})^*$ is $dB^{(H)}$-integrable. If $D_t Y(\cdot): \mathbb{R} \to (\mathbb{S})^*$ is $dB^{(H)}$-integrable for every $t$, then 
\[ D_t^{(H)} \left( \int_{\mathbb{R}} Y(s)dB^{(H)}(s) \right) = \int_{\mathbb{R}} D_t^{(H)} Y(s)dB^{(H)}(s) + Y(t). \] (2.31)

Definition 2.14 Let $\mathbb{D}_{1,2}^{(H)}$ be the set of all $F \in L^2(\mathbb{P})$ such that the Malliavin derivative $D_t^{(H)} F$ exists and 
\[ E \left[ \int_{\mathbb{R}} [D_t^{(H)} F]^2 dt \right] < \infty \] (2.32)

The following result has been obtained with a different proof in Lemma 2 of [18].

Lemma 2.15 Suppose $g \in L^2_H(\mathbb{R})$ and let $F \in \mathbb{D}_{1,2}^{(H)}$. Then 
\[ F \circ \int_{\mathbb{R}} g(t)dB^{(H)}(t) = F \cdot \int_{\mathbb{R}} g(t)dB^{(H)}(t) - \langle g, D_t^{(H)} F \rangle_{L^2_H(\mathbb{R})} \] (2.33)
3 The forward integral

By following the approach of [23], we now define the forward integral with respect to the fractional Brownian motion as follows:

**Definition 3.1**

a) The (classical) forward integral of a real valued measurable process $Y$ with integrable trajectories is defined by

$$
\int_0^T Y(t) d^- B^{(H)}(t) = \lim_{\epsilon \to 0} \int_0^T Y(t) \frac{B^{(H)}(t + \epsilon) - B^{(H)}(t)}{\epsilon} dt,
$$

provided that the limit exists in probability under $\mathbb{P}$.

b) The (generalized) forward integral of a real valued measurable process $Y$ with integrable trajectories is defined by

$$
\int_0^T Y(t) d^- B^{(H)}(t) = \lim_{\epsilon \to 0} \int_0^T Y(t) \frac{B^{(H)}(t + \epsilon) - B^{(H)}(t)}{\epsilon} dt,
$$

provided that the limit exists in $(\mathcal{S})^*$.

Note that in the generalized definition of forward integral, the limit is required to exist in the Hida space of stochastic distributions $(\mathcal{S})^*$ introduced in Definition 2.2. Convergence in $(\mathcal{S})^*$ is also explained in Section 2.

**Corollary 3.2** Let $\psi(t) = \psi(t, \omega)$ be a measurable forward integrable process and assume that $\psi$ is càglàd. The forward integral of $\psi$ with respect to the fractional Brownian motion $B^{(H)}$ coincides with

$$
\int_0^T \psi(t) d^- B^{(H)}(t) = \lim_{|\Delta| \to 0} \sum_{j=1}^N \psi(t_j) \Delta B^{(H)}_{t_j}
$$

whenever the left-hand limit exists in probability, where $\pi : 0 = t_0 < t_1 < \cdots < t_N = T$ is a partition of $[0, T]$ with mesh size $|\Delta| = \sup_{j=0, \ldots, N-1} |t_{j+1} - t_j|$ and $\Delta B^{(H)}_{t_j} = B^{(H)}_{t_{j+1}} - B^{(H)}_{t_j}$.

**Proof.** Let $\psi$ be a càglàd forward integrable process and

$$
\psi^{(\Delta)}(t) = \sum_k \psi(t_k) \chi_{(t_k, t_{k+1}]}(t)
$$

(3.2)
be a càglàd step function approximation to \( \psi \). Then \( \psi(\Delta)(t) \) converges bound-
edly almost surely to \( \psi(t) \) as \( |\Delta| \to 0 \). The forward integral of \( \psi(\Delta)(t) \) is then given by

\[
\int_0^T \psi(\Delta)(t) d\mathcal{H}(t) = \lim_{\epsilon \to 0} \int_0^T \psi(\Delta)(s) \frac{B(\mathcal{H})(s + \epsilon) - B(\mathcal{H})(s)}{\epsilon} ds
\]

\[
= \lim_{\epsilon \to 0} \sum_k \psi(t_k) \int_{t_k}^{t_{k+1}} \frac{1}{\epsilon} \int_s^{s+\epsilon} dB(\mathcal{H})(u) ds
\]

\[
= \lim_{\epsilon \to 0} \sum_k \psi(t_k) \Delta B_{t_k}(\mathcal{H}), \tag{3.3}
\]

where \( \Delta B_{t_k}(\mathcal{H}) = B_{t_k+1}(\mathcal{H}) - B_{t_k}(\mathcal{H}) \). Hence (3.1) follows by the dominated con-
vergence theorem and by (3.3).

For the sequel we will use the same notation as in Section 2.

**Definition 3.3** The space \( L_{1,2}^{\mathcal{H}} \) consists of all càglàd processes

\[
\psi(t) = \sum_{\alpha \in J} c_{\alpha}(t) \mathcal{H}_{\alpha}(\omega) \in (S)^* 
\]

for every \( t \in [0, T] \) and such that

\[
\|\psi\|_{L_{1,2}^{\mathcal{H}}}^2 := \sum_{\alpha \in J} \sum_{i=1}^{\infty} \alpha_i \alpha_i! \|c_{\alpha}\|_{L^2([0, T])}^2 < \infty. \tag{3.4}
\]

Note that if \( \psi(t) \in (S)^* \) for every \( t \in [0, T] \), then \( D_s \psi(t) \) exists in \((S)^* \) (see Lemma 3.10 of [1]). We recall a preliminary lemma needed in the following.

**Lemma 3.4** Let \((\Gamma, \mathcal{G}, m)\) be a measure space. Let \( f_\epsilon : \Gamma \to B, \epsilon \in \mathbb{R}, \) be measurable functions with values in a Banach space \((B, \| \cdot \|_B)\). If \( f_\epsilon(\gamma) \to f_0(\gamma) \) as \( \epsilon \to 0 \) for almost every \( \gamma \in \Gamma \) and there exists \( K < \infty \) such that

\[
\int_\Gamma \|f_\epsilon(\gamma)\|_B dm(\gamma) < K \tag{3.5}
\]

for all \( \epsilon \in \mathbb{R}, \) then

\[
\int_\Gamma f_\epsilon(\gamma) dm(\gamma) \to \int_\Gamma f_0(\gamma) dm(\gamma) \tag{3.6}
\]

in \( \| \cdot \|_B \).
Proof. The proof is analogous to the one of Theorem II.21.2 of [22]. □

Lemma 3.5 Suppose that $\psi \in \mathbb{L}_{1/2}^{(H)}$. Then

$$M_{t^+}D_{t^+}\psi(t) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} M_s D_s \psi(t) ds \quad (3.7)$$

exists in $L^2(\mathbb{P})$ for all $t$. Moreover

$$\int_0^T M_{t^+}D_{t^+}\psi(t) dt = \lim_{\epsilon \to 0} \int_0^T \left( \frac{1}{\epsilon} \int_t^{t+\epsilon} M_s D_s \psi(t) ds \right) dt \quad (3.8)$$

in $L^2(\mathbb{P})$ and

$$E \left[ \left( \int_0^T M_{s^+}D_{s^+}\psi(s) ds \right)^2 \right] < \infty. \quad (3.9)$$

Proof. Suppose that $\psi(t)$ has the expansion

$$\psi(t) = \sum_{\alpha \in J} c_{\alpha}(t) \mathcal{H}_\alpha(\omega).$$

In the sequel we drop $\omega$ in $\mathcal{H}_\alpha(\omega)$ for the sake of simplicity. Then we have

$$D_s \psi(t) = \sum_{\alpha \in J} \sum_{i=1}^{\infty} c_{\alpha}(t) \alpha_i \mathcal{H}_{\alpha - e_i(s)} \xi_i(s)$$

and

$$M_s D_s \psi(t) = \sum_{\alpha \in J} \sum_{i=1}^{\infty} c_{\alpha}(t) \alpha_i \mathcal{H}_{\alpha - e_i(s)} \eta_i(s),$$

where $\eta_i(s) = M\xi_i(s)$. Hence

$$\frac{1}{\epsilon} \int_t^{t+\epsilon} M_s D_s \psi(t) ds = \sum_{\alpha \in J} \sum_{i=1}^{\infty} (c_{\alpha}(t) \frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_i(s) ds) \alpha_i \mathcal{H}_{\alpha - e_i(s)}.$$

Since $\eta_i(s) = M\xi(s)$ is a continuous function, we have that

$$\frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_i(s) ds \to \eta_i(t)$$

as $\epsilon \to 0$.

We apply now Lemma 3.4 with $\gamma = (\alpha, i)$, $dm(\gamma) = \sum_{\alpha \in J} \sum_{i=1}^{\infty} \delta(\alpha, i)$, where
\(\delta_x\) denotes the point mass at \(x\), \(B = L^2(\mathbb{P})\) and \(f_\epsilon = (c_\alpha(t) \frac{1}{\epsilon} \int_{t}^{t+\epsilon} \eta_i(s) ds) \alpha_i \mathcal{H}_{\alpha - \epsilon(i)}\).

We obtain

\[
\int_G \|f_\epsilon(\gamma)\|_B^2 dm(\gamma) = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} \|f_\epsilon(\gamma)\|_B^2 \mathcal{H}_\alpha = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} (c_\alpha(t) \frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_i(s) ds)^2 \alpha_i \alpha!
\]

since

\[
\frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_i(s) ds = \langle M\xi_i, \frac{1}{\epsilon} \chi_{[t,t+\epsilon]} \rangle = \frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_i(s) ds \leq \frac{(t + \epsilon)^{2H} - t^{2H}}{\epsilon},
\]

where we have used that the fact that \(\|e_i\|_{L^2_\mathbb{R}(\mathbb{R})} = 1\) and the equality

\[
\int_{\mathbb{R}} |M[a,b](x)|^2 dx = (b - a)^{2H}.
\]

Since we have \(\sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_\alpha(t) \alpha_i \alpha! < \infty\) for almost every \(t\), by Lemma 3.4 it follows that \(\sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} (c_\alpha(t) \frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_i(s) ds) \alpha_i \mathcal{H}_{\alpha - \epsilon(i)}\) converges to

\[
\sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} c_\alpha(t) \eta_i(t) \alpha_i \mathcal{H}_{\alpha - \epsilon(i)}
\]

in \(L^2(\mathbb{P})\).

We now prove (3.8). Consider

\[
\int_0^T \frac{1}{\epsilon} \int_t^{t+\epsilon} M_x D_s \psi(t) ds dt = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} \int_0^T \left( c_\alpha(t) \frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_i(s) ds \right) dt \alpha_i \mathcal{H}_{\alpha - \epsilon(i)}.
\]

Now assuming \(f_\epsilon = \int_0^T \left( c_\alpha(t) \frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_i(s) ds \right) dt \alpha_i \mathcal{H}_{\alpha - \epsilon(i)}\) and as before \(\gamma = (\alpha, i), B = L^2(\mathbb{P}), dm(\gamma) = \sum_{\alpha \in \mathcal{J}} \sum_{i=1}^{\infty} \delta_{\alpha, i}\), where \(\delta_x\) denotes the point mass.
at $x$, we use again Lemma 3.4. We obtain

$$\int_{\Gamma} \| f_\epsilon(\gamma) \|^2_2 dm(\gamma) = \sum_{\alpha \in J} \sum_{i=1}^{\infty} \| f_\epsilon(\gamma) \|^2_{L^2(\mathbb{P})} = \sum_{\alpha \in J} \sum_{i=1}^{\infty} \left( \int_{0}^{T} c_\alpha(t) \frac{1}{\epsilon} \int_{t}^{t+\epsilon} \eta_i(s) ds \frac{1}{\epsilon} \right)^2 \alpha_i \alpha!$$

$$\leq \sum_{\alpha \in J} \sum_{i=1}^{\infty} \left( \int_{0}^{T} c_\alpha(t) \left[ \frac{(t+\epsilon)^{2H} - t^{2H}}{\epsilon} \right] dt \right)^2 \alpha_i \alpha!$$

$$\leq \sum_{\alpha \in J} \sum_{i=1}^{\infty} \left( \int_{0}^{T} c_\alpha(t)^2 dt \right) \left( \int_{0}^{T} \left[ \frac{(t+\epsilon)^{2H} - t^{2H}}{\epsilon} \right]^2 dt \right) \alpha_i \alpha! \tag{3.10}$$

Since $\psi \in L^{(H)}_{1,2}$ by Lemma 3.4 we can conclude that the limit 3.8 exists in $L^2(\mathbb{P})$ and also that (3.9) holds. 

Lemma 3.6 Suppose that $\psi \in L^{(H)}_{1,2}$ and let

$$\psi^{(\Delta)}(s) = \sum_{k} \psi(t_k) \chi(t_k, t_{k+1})(s) \tag{3.11}$$

be a càglàd step function approximation to $\psi$, where $\Delta = \max_i |\Delta t_i|$ is the maximal length of the subinterval in the partition $0 = t_0 < \cdots < t_n = T$ of $[0, T]$. Then $\psi^{(\Delta)} \in L^{(H)}_{1,2}$ for all $\Delta$ and

$$\int_{0}^{T} M_{s+D_s+} \psi^{(\Delta)}(s) ds \rightarrow \int_{0}^{T} M_{s+D_s+} \psi(s) ds \quad \text{in } L^2(\mathbb{P}) \tag{3.12}$$

as $|\Delta| \rightarrow 0$. 

Proof. Since $\psi^{(\Delta)}(s) = \sum_{\alpha \in J} c_\alpha^{(\Delta)}(s) \mathcal{H}_\alpha(\omega)$ with

$$c_\alpha^{(\Delta)}(s) = \sum_{k} c_\alpha(t_k) \chi(t_k, t_{k+1})(s)$$

and

$$\| c_\alpha^{(\Delta)} \|_{L^2([0,T])} \leq \text{const.} \| c_\alpha \|_{L^2([0,T])} \quad \forall \alpha, \tag{3.13}$$
it follows that \( \psi^{(\Delta)} \in \mathbb{L}^{(H)}_{1,2} \). We have
\[
\frac{1}{\epsilon} \int_{t}^{t+\epsilon} M_s D_s \psi^{(\Delta)}(t) ds = \sum_{\alpha \in \mathbb{J}} \sum_{i=1}^{\infty} \left( \int_{0}^{T} (c^{(\Delta)}_\alpha(t) - \sum_{j=1}^{\infty} \psi(t_j) \chi_{(t_j, t_{j+1})}(t) \right) \quad \alpha;_{-\epsilon(\cdot)}.
\]

If we assume \( \gamma = (\alpha, i), B = L^{2}(\mathbb{P}), m(d\gamma) = \sum_{\alpha \in \mathbb{J}} \sum_{i=1}^{\infty} \delta_{(\alpha, i)}, \) where \( \delta_{x} \) denotes the point mass at \( x \), and \( f_{\Delta} = \left( \int_{0}^{T} c^{(\Delta)}_\alpha(t) \sum_{j=1}^{\infty} \psi(t_j) \right) \alpha;_{-\epsilon(\cdot)}, \)

with the same argument as in (3.10) by Lemma 3.4 we obtain that
\[
\int_{0}^{T} \left( \frac{1}{\epsilon} \int_{t}^{t+\epsilon} M_s D_s \psi(t) ds \right) dt = \lim_{|\Delta| \rightarrow 0} \int_{0}^{T} \left( \frac{1}{\epsilon} \int_{t}^{t+\epsilon} M_s D_s \psi^{(\Delta)}(t) ds \right) dt
\]
in \( L^{2}(\mathbb{P}) \) for almost every \( s \), since \( c^{(\Delta)}_\alpha \) converges by dominated convergence to \( c^{(\Delta)}_\alpha \) in \( L^{2}(\mathbb{P}) \) and \( \psi^{(\Delta)} \in \mathbb{L}^{(H)}_{1,2} \). Using (3.14) and Lemma 3.5 we conclude that (3.12) holds.

We now investigate the relation among forward integrals and WIS-integrals for \( H > \frac{1}{2} \).

In [4] and [19] a similar relation is established between the symmetric integral and the divergence, in [9] between the forward integral and the fractional Wick-Itô-Skorohod integral. For the case \( H < \frac{1}{2} \), we refer to [2].

**Theorem 3.7** Let \( H \in (0, 1) \). Suppose \( \psi \in \mathbb{L}^{(H)}_{1,2} \) and that one of the following conditions holds:

i) \( \psi \) is Wick-Itô-Skorohod integrable (Definition 2.7);

ii) \( \psi \) is forward integrable in \((S)^*\) (Definition 3.1).

Then
\[
\int_{0}^{T} \psi(t) dB^{(H)}(t) = \int_{0}^{T} \psi(t) dB^{(H)}(t) + \int_{0}^{T} M_{t+} D_{t+} \psi(t) dt, \quad (3.15)
\]
holds as an identity in \((S)^*\), where here \( \int_{0}^{T} \psi(t) dB^{(H)}(t) \) is the WIS-integral of Definition 2.7.

**Proof.** We prove (3.15) assuming that hypothesis i) is in force. The argument works symmetrically under hypothesis ii). Let \( \psi \in \mathbb{L}^{(H)}_{1,2} \). Since \( \psi \) is càglàd, we can approximate it as
\[
\psi(t) = \lim_{|\Delta| \rightarrow 0} \sum_{j} \psi(t_j) \chi_{(t_j, t_{j+1})}(t) \quad \text{a.e.}
\]
where for any partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0, T]$, with $\Delta t_j = t_{j+1} - t_j$, we have put $|\Delta t| = \sup_{j=0, \ldots, N-1} \Delta t_j$.

As before we put $\psi^{(\Delta)}(t) = \sum_{j=0}^{N-1} \psi(t_k) \chi_{(t_k, t_{k+1})}(t)$ and evaluate

\[
\int_0^T \psi^{(\Delta)}(t) d^- B^{(H)}(t) = \lim_{\epsilon \to 0} \int_0^T \psi^{(\Delta)}(t, \omega) \frac{B^{(H)}(t + \epsilon) - B^{(H)}(t)}{\epsilon} dt =
\]

\[
\lim_{\epsilon \to 0} \int_0^T \left( \sum_j \psi(t_j) \chi_{(t_j, t_{j+1})}(t) \right) \frac{1}{\epsilon} \int_t^{t+\epsilon} d B^{(H)}(u) dt =
\]

\[
\lim_{\epsilon \to 0} \int_0^T \left( \sum_j \psi(t_j) \chi_{(t_j, t_{j+1})}(t) \right) \frac{1}{\epsilon} \int_t^{t+\epsilon} W^{(H)}(u) du dt +
\]

\[
\lim_{\epsilon \to 0} \sum_j \int_0^T \chi_{(t_j, t_{j+1})}(t) \frac{1}{\epsilon} \int_{\mathbb{R}} \chi_{[t,t+\epsilon]}(u) M^2_u D^{(H)}_u \psi(t_j) du dt.
\]

The first limit is equal to

\[
\lim_{\epsilon \to 0} \int_0^T \left( \sum_j \psi(t_j) \chi_{(t_j, t_{j+1})}(t) \right) \frac{1}{\epsilon} \int_t^{t+\epsilon} d B^{(H)}(u) dt =
\]

\[
\lim_{\epsilon \to 0} \int_0^T \left( \sum_j \psi(t_j) \chi_{(t_j, t_{j+1})}(t) \right) \frac{1}{\epsilon} \int_t^{t+\epsilon} W^{(H)}(u) du dt =
\]

\[
\lim_{\epsilon \to 0} \int_0^T \frac{1}{\epsilon} \left( \int_{u-\epsilon}^{u} \sum_j \psi(t_j) \chi_{(t_j, t_{j+1})}(t) \right) \frac{1}{\epsilon} \int_t^{t+\epsilon} W^{(H)}(u) du =
\]

\[
\int_0^T \psi^{(\Delta)}(u) \frac{1}{\epsilon} \int_t^{t+\epsilon} W^{(H)}(u) du,
\]

that converges in $(8)^*$ to $\int_0^T \psi(u) \frac{1}{\epsilon} \int_t^{t+\epsilon} W^{(H)}(u) du = \int_0^T \psi(u) d B^{(H)}(u)$. For the second limit we get

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \sum_j \int_0^T \chi_{(t_j, t_{j+1})}(t) \int_t^{t+\epsilon} M^2_u D^{(H)}_u \psi(t_j) du dt =
\]

\[
\lim_{\epsilon \to 0} \int_0^T \frac{1}{\epsilon} \int_t^{t+\epsilon} M^2_u D^{(H)}_u \psi^{(\Delta)}(t) du dt =
\]

\[
\lim_{\epsilon \to 0} \int_0^T \frac{1}{\epsilon} \int_t^{t+\epsilon} M_u D_u \psi^{(\Delta)}(t) du dt.
\]

By Lemmas 3.5 and 3.6 the last limit converges to

\[
\int_0^T M_u D_u \psi(u) du
\]

(3.16)
in $L^2(\mathbb{P})$. \hfill \Box

An analogous relation to the one of Theorem 3.7 between Stratonovich integrals and Wick-Itô-Skorohod integrals for fractional Brownian motion is proved under different conditions in [18].

An Itô formula for forward integrals with respect to classical Brownian motion was obtained by [23] and then extended to the fractional Brownian motion case in [12]. Here we prove the following Itô formula for forward integrals with respect to fractional Brownian motion as a consequence of Lemma 3.8.

**Lemma 3.8** Let $G \in (S)^*$ and suppose that $\psi$ is forward integrable. Then

$$G(\omega) \int_0^T \psi(t) d^- B^{(H)}(t) = \int_0^T G(\omega) \psi(t) d^- B^{(H)}(t) \quad (3.17)$$

**Proof.** This is immediate by Definition 3.1. \hfill \Box

**Definition 3.9** Let $\psi$ be a forward integrable process and let $\alpha(s)$ be a measurable process such that $\int_0^t |\alpha(s)| ds < \infty$ a.s. for all $t \geq 0$. Then the process

$$X(t) := x + \int_0^t \alpha(s) ds + \int_0^t \psi(s) d^- B^{(H)}(s); \quad t \geq 0 \quad (3.18)$$

is called a fractional forward process. As a shorthand notation for (3.18) we write

$$d^- X(t) := \alpha(t) dt + \psi(t) d^- B^{(H)}(t); \quad X(0) = x. \quad (3.19)$$

**Theorem 3.10** Let

$$d^- X(t) = \alpha(t) dt + \psi(t) d^- B^{(H)}(t); \quad X(0) = x$$

be a fractional forward process. Suppose $f \in C^2(\mathbb{R}^2)$ and put $Y(t) = f(t, X(t))$.

Then if $\frac{1}{2} < H < 1$, we have

$$d^- Y(t) = \frac{\partial f}{\partial t}(t, X(t)) dt + \frac{\partial f}{\partial x}(t, X(t)) d^- X(t)$$
Proof. Let 0 = t_0 < t_1 < \cdots < t_N = t be a partition of [0, t]. By using Taylor expansion, we get by equation (3.17)

\[ Y(t) - Y(0) = \sum_j Y(t_{j+1}) - Y(t_j) \]

\[ = \sum_j f(t_{j+1}, X(t_{j+1})) - f(t_j, X(t_j)) \]

\[ = \sum_j \frac{\partial f}{\partial t}(t_j, X(t_j)) \Delta t_j + \sum_j \frac{\partial f}{\partial x}(t_j, X(t_j)) \Delta X(t_j) \]

\[ + \frac{1}{2} \sum_j \frac{\partial^2 f}{\partial x^2}(t_j, X(t_j))(\Delta X(t_j))^2 + \sum_j o((\Delta t_j)^2) + o((\Delta X(t_j))^2) \]

\[ = \sum_j \frac{\partial f}{\partial t}(t_j, X(t_j)) \Delta t_j + \sum_j \int_{t_j}^{t_{j+1}} \frac{\partial f}{\partial x}(t_j, X(t_j)) d^-X_t \]

\[ + \frac{1}{2} \sum_j \frac{\partial^2 f}{\partial x^2}(t_j, X(t_j))(\Delta X(t_j))^2 + \sum_j o((\Delta t_j)^2) + o((\Delta X(t_j))^2) \]

where \( \Delta X(t_j) = X(t_{j+1}) - X(t_j) \). Since \( \frac{1}{2} < H < 1 \), the quadratic variation of the fractional Brownian motion is zero and we are left with the first terms of the sum above, which converges to \( \int_0^t \frac{\partial f}{\partial s}(s, X(s)) ds + \int_0^t \frac{\partial f}{\partial x}(s, X(s)) d^-X(s) \).

Using the results of Theorem 3.7 and 3.10, we obtain a general Itô formula for functionals of Wick-Itô-Skorohod integrals with respect to the fractional Brownian motion when \( \frac{1}{2} < H < 1 \). An Itô formula for \( \frac{1}{2} < H < 1 \) has been already proved in [9] and in [4], but under more restrictive hypotheses. Here we provide a different proof under weaker assumptions. If \( \frac{1}{2} < H < 1 \) this theorem extends Theorem 3.8 in [7]. A related result, obtained independently and by a different method, can be found in [11]. Moreover our results hold in a different setting.

Theorem 3.11 (Itô formula for the WIS-integral) Suppose \( \frac{1}{2} < H < 1 \). Let \( \gamma(s) \) be a measurable process such that \( \int_0^t |\gamma(s)| ds < \infty \) a.s. for all \( t \geq 0 \), let \( \psi(t) = \sum_{\alpha \in \mathbb{J}} c_{\alpha}(t) \xi_{\alpha}(\omega) \) be càglàd, WIS-integrable and such that

\[ \sum_{\alpha \in \mathbb{J}} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \|c_{\alpha}\|_{L^2([0,T])} \alpha_i(\alpha_k + 1)\alpha_i < \infty. \]
Suppose that $M_t D_t \psi(s)$ is also WIS-integrable for almost all $t \in [0, T]$. Consider

$$X(t) = x + \int_0^t \gamma(s) ds + \int_0^t \psi(s) dB^{(H)}(s), \quad t \in [0, T],$$

or, in short-hand notation,

$$dX(t) = \gamma(t) dt + \psi(t) dB^{(H)}(t), \quad X(0) = x.$$

Suppose that $M_t$ has a càdlàg version (Remark 3.12). Let $f \in C^2(\mathbb{R}^2)$ and put $Y(t) = f(t, X(t))$. Then on $[0, T]$

$$dY(t) = \frac{\partial f}{\partial t}(t, X(t)) dt + \frac{\partial f}{\partial x}(t, X(t)) dX(t) + \frac{\partial^2 f}{\partial x^2}(t, X(t)) \psi(t) M_{t+} D_{t+} X(t) dt,$$

and equivalently

$$dY(t) = \frac{\partial f}{\partial t}(t, X(t)) dt + \frac{\partial f}{\partial x}(t, X(t)) dX(t) + \frac{\partial^2 f}{\partial x^2}(t, X(t)) \psi(t) M^2(\psi_{[0, t]}), dt$$

$$+ \left[ \frac{\partial^2 f}{\partial x^2}(t, X(t)) \psi(t) \int_0^t M^2_t D_t^{(H)} \psi(u) dB^{(H)}(u) \right] dt,$$

where $M^2(\psi_{[0, t]}), t = M^2(\psi_{[0, t]})(t)$.

**PROOF.** For simplicity we put $\alpha = 0$. By Theorem 3.7 we have

$$X(t) = \int_0^t \psi(s) dB^{(H)}(s) - \int_0^t M^2_s D_s^{(H)} \psi(s) ds$$

We note that

$$\frac{1}{\epsilon} \int_t^{t+\epsilon} M^2_s D^{(H)}_s (f'(X(t)) \psi(t)) ds = f'(X(t)) \frac{1}{\epsilon} \int_t^{t+\epsilon} M^2_s D^{(H)}_s \psi(t) ds$$

$$+ \psi(t) f''(X(t)) \frac{1}{\epsilon} \int_t^{t+\epsilon} M^2_s D^{(H)}_s X(t) dt$$

$$= \int_0^{t+\epsilon} M^2_s D^{(H)}_s (\psi_{[0, t]}(s)) ds$$

Since $\psi \in L^{(H)}_{1,2}$, the first term converges to $f'(X(t)) M^2_t D^{(H)}_t \psi(t)$ as $\epsilon \to 0$. For the second term we restrict our attention to

$$\frac{1}{\epsilon} \int_t^{t+\epsilon} M^2_s D^{(H)}_s X(t) ds = \frac{1}{\epsilon} \int_t^{t+\epsilon} \int_0^t M^2_s D^{(H)}_s \psi(u) dB^{(H)}(u) ds$$

$$+ \frac{1}{\epsilon} \int_t^{t+\epsilon} M^2_s (\psi_{[0, t]}) ds.$$
a) To study the convergence of the term a), we proceed as in Lemma 3.5. By using the chaos expansion we obtain

\[
\frac{1}{\epsilon} \int_t^{t+\epsilon} \int_0^t M^2_s D^{(H)}(u) \psi(u) dB^{(H)}(u) ds = \sum_{\alpha \in J} \sum_{i=1}^\infty \sum_{k=1}^\infty (c_{\alpha}, \xi_k)_t \frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_h(s) ds \alpha_i \mathcal{H}_{\alpha,-\epsilon(t),+\epsilon(t)}.
\]

Put \( \psi_{i,k,\alpha,\epsilon} := (c_{\alpha}, \xi_k)_t \frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_h(s) ds \alpha_i \mathcal{H}_{\alpha,-\epsilon(t),+\epsilon(t)} \). Then

\[
\sum_{\alpha \in J} \sum_{i=1}^\infty \sum_{k=1}^\infty \| \psi_{i,k,\alpha,\epsilon} \|^2_{L^2(\mathbb{P})} = \sum_{\alpha \in J} \sum_{i=1}^\infty \sum_{k=1}^\infty (c_{\alpha}, \xi_k)_t^2 \left( \frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_h(s) ds \right)^2 \alpha_i (\alpha_k + 1) ! \leq \left[ \frac{(t + \epsilon)^{2H} - \epsilon^{2H}}{\epsilon} \right]^2 \sum_{\alpha \in J} \sum_{i=1}^\infty \sum_{k=1}^\infty \| c_{\alpha} \|^2_{L^2(0,T)} \| \xi_k \|^2_{L^2(0,T)} \alpha_i (\alpha_k + 1) ! \leq \left[ \frac{(t + \epsilon)^{2H} - \epsilon^{2H}}{\epsilon} \right]^2 \sum_{\alpha \in J} \sum_{i=1}^\infty \sum_{k=1}^\infty \| c_{\alpha} \|^2_{L^2(0,T)} \alpha_i (\alpha_k + 1) !, \tag{3.23}
\]

where we have used that \( \| \xi_k \|^2_{L^2(0,T)} \leq \| \xi_k \|^2_{L^2(0,T)} = 1, \forall k = 1, 2, \ldots \). Since

\[
\frac{1}{\epsilon} \int_t^{t+\epsilon} \eta_h(s) ds \rightarrow \eta_h(t) \tag{3.24}
\]

and (3.23) holds, by Lemma 3.4 we conclude that

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} \int_0^t M^2_s D^{(H)}(u) \psi(u) dB^{(H)}(u) ds = \int_0^t M^2_t D^{(H)}(u) \psi(u) dB^{(H)}(u)
\]

in \( L^2(\mathbb{P}) \).

b) Since \( \psi \in \mathbb{H}^{1,2}_{1,2} \), we have

\[
\frac{1}{\epsilon} \int_t^{t+\epsilon} M^2_s (\psi \chi_{[0,t]}) ds \rightarrow M^2(\psi \chi_{[0,t]}), \quad \text{a.e. and in } L^2(\mathbb{P}), \tag{3.26}
\]

where for the sake of simplcity we have put \( M^2(\psi \chi_{[0,t]}))_{t} = M^2(\psi \chi_{[0,t]})(t) \).

Let \( A_t = -\int_0^t M^2_{s+} D^{(H)}_s(\psi(s) ds \). Then by the Itô formula for forward integrals
we obtain
\[ dY(t) = f'(X(t))dA_t + f'(X(t))d^-B^2(t) \]
\[ = -f'(X(t))M_t + D_t + \psi(t)dt \]
\[ + \left[ f'(X(t))M_t + D_t + \psi(t)f''(X(t))M_t + D_t + X(t) \right]dt \]
\[ = f'(X(t))dX(t) + f''(X(t))\psi(t)M_t + D_t + X(t)dt \]
and by (3.25) and (3.26) we can conclude that
\[ dY(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))dX(t) + \frac{\partial^2 f}{\partial x^2}(t, X(t))\psi(t)d\int_0^t f''(X(t))M_t + D_t + X(t)dt \]
\[ = f'(X(t))dX(t) + f''(X(t))\psi(t)M_t + D_t + X(t)dt \]
Note that all the integrands appearing in (3.27) are well-defined because \( X_t \)
is càdlàg.

\[ \square \]

**Remark 3.12** Conditions under which the integral process admits a continuous modification are proved in [3] and [4].

**Corollary 3.13** Assume that \( \psi \in L^2_H(\mathbb{R}) \), \( \alpha = 0 \) and otherwise let \( H, X, f, Y \) be as in Theorem 3.11. Then
\[ dY(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))dX(t) + \frac{\partial^2 f}{\partial x^2}(t, X(t))\psi(t)dM^2(t, X(t))dt \]
(3.27)

**Remark 3.14** In the case when \( \psi(s) \) is deterministic, a (different) Itô formula, valid for all \( H \in (0, 1) \) and for all \( x \)-entire functions \( f(t, x) \) of order 2, has been obtained in Theorem 11.1 of [15].

4 Examples

4.1 A special case

In [5] and [7] an Itô formula for the case when \( Y(t) = f(B^H(t)) \) is provided, valid for all \( H \in (0, 1) \). We recall here that formula
\[ dY(t) = f'(X(t))dX(t) + Ht^{2H-1}f''(X(t))\psi(t)dt \]
(4.1)
We now show that if $H > \frac{1}{2}$ then (3.20) and (4.1) coincide in this case.

**Proposition 4.1** For every $H \in (0, 1)$ we have

$$M_t + D_t + B^{(H)}(t) = H t^{2H-1}, \quad t \geq 0.$$ 

**Proof.** Let $t \geq 0$. We recall that $D_t^{(H)} B^{(H)}(u) = \chi_{[0,u)}(t)$. Hence we need to prove that

$$M_t + D_t + B^{(H)}(t) = \lim_{s \to t^-} \frac{1}{\epsilon} \int_t^{t+\epsilon} M_s^2 D_s^{(H)} B^{(H)}(t) ds$$

$$= [M_t^2 \chi_{[0,u]}(t)]_{t=\xi} = H t^{2H-1}$$

We consider $\psi(u) = \int_{\mathbb{R}} (M_t \chi_{[0,u]}(t))^2 \, dt$. Since, by [10], we have that $\psi(u) = u^{2H}$, we only need to show that $\psi'(u) = 2[M_t^2 \chi_{[0,u]}(t)]_{t=u}$. We rewrite $\psi(u)$ as follows

$$\psi(u) = \int_{\mathbb{R}} (M_t \chi_{[0,u]}(t))^2 \, dt$$

$$= \int_{\mathbb{R}} \chi_{[0,u]}(t) M_t^2 \chi_{[0,u]}(t) \, dt$$

$$= \int_0^u M_t^2 \chi_{[0,u]}(t) \, dt$$

by using the properties of the operator $M$. We compute

$$\psi(u + \epsilon) - \psi(u)$$

$$= \frac{1}{\epsilon} \left( \int_0^{u+\epsilon} M_t^2 \chi_{[0,u+a]}(t) \, dt - \int_0^u M_t^2 \chi_{[0,u]}(t) \, dt \right)$$

$$= \frac{1}{\epsilon} \left( \int_0^{u+\epsilon} M_t^2 \chi_{[0,u+a]}(t) \, dt + \int_0^u [M_t^2 \chi_{[0,u]}(t) - M_t^2 \chi_{[0,u]}(t)] \, dt \right)$$

by adding and subtracting $\int_0^u M_t^2 \chi_{[0,u]}(t) \, dt$. Since the operator $M$ transforms $\chi_{[0,u]}(t)$ into a continuous function, we obtain

1. $\int_u^{u+\epsilon} M_t^2 \chi_{[0,u+\epsilon]}(t) \, dt = [M_t^2 \chi_{[0,u+\epsilon]}(t)]_{t=\xi} \epsilon$, where $u < \xi < u + \epsilon$. By writing

$$[M_t^2 \chi_{[0,u+\epsilon]}(t)]_{t=\xi} = [M_t^2 (\chi_{[0,u+\epsilon]} - \chi_{[0,u]})](t)_{t=\xi} + [M_t^2 \chi_{[0,u]}(t)]_{t=\xi}$$

we obtain that, when taking the limit as $\epsilon \to 0$, the first term goes to zero, while the second term converges to $[M_t^2 \chi_{[0,u]}(t)]_{t=\xi}$ since $\xi_\epsilon \to u$ when $\epsilon \to 0$. 

21
2. We have that
\[
\frac{1}{\epsilon} \int_0^u [M^2_\epsilon \chi_{[0,u+c]}(t) dt - M^2_\epsilon \chi_{[0,u]}(t)] dt =
\frac{1}{\epsilon} \int_0^u M^2_\epsilon [\chi_{(u,u+c)}(t)] dt =
\frac{1}{\epsilon} \int_0^T \chi_{[0,u]}(t)(M^2_\epsilon [\chi_{(u,u+c)}(t)] dt =
\frac{1}{\epsilon} \int_{u+\epsilon}^{u} M^2_\epsilon [\chi_{[0,u]}(t)] dt
\]
converges to \([M^2_\epsilon \chi_{[0,u]}(t)]_{t=u}\) as \(\epsilon \to 0\).

Hence
\[
\psi'(u) = \lim_{\epsilon \to 0} \frac{\psi(u+\epsilon) - \psi(u)}{\epsilon} = 2[M^2_\epsilon \chi_{[0,u]}(t)]_{t=u}
\]
i.e. the equality \([M^2_\epsilon \chi_{[0,u]}(t)]_{t=u} = Hu^{2H-1}\) holds for every \(H \in (0,1)\). \(\square\)

4.2 An integration by parts formula

Let \(\psi(s) = \psi(s, \omega) \in \mathbb{L}_{1,2}^{(H)}\) be \(dB^{(H)}\)-integrable and define
\[
X(t) = \int_0^t \psi(s) dB^{(H)}(s)
\]
and
\[
Y(t) = X^2(t).
\]
By (3.25) and (3.26) we have
\[
M_t + D_t X(t) = \int_0^t M_t D_t \psi(s) dB^{(H)}(s) + M^2(\psi \chi_{[0,t]}),
\]
where \(M^2(\psi \chi_{[0,t]})) = M^2(\psi \chi_{[0,t]})(t)\). Then by Theorem 3.11 and by Proposition 2.12 we have
\[
dY(t) = 2X(t) dX(t) + 2\psi(t) \left( \int_0^t M_t D_t \psi(s) dB^{(H)}(s) + M^2(\psi \chi_{[0,t]})) dt \right) dt
\]
In particular, if \(\psi \in L^2_H(\mathbb{R})\), we get
\[
dY(t) = 2X(t) dX(t) + 2\psi(t) M^2(\psi \chi_{[0,t]})) dt
\]
By using that \(X_1X_2 = \frac{1}{2}[(X_1 + X_2)^2 - X_1^2 - X_2^2]\) this gives the following product rule:
Proposition 4.2 (Product rule) Suppose $\psi_1, \psi_2 \in L^2_H(\mathbb{R})$ and define

$$X_i(t) = \int_0^t \psi_i(s) dB^{(H)}(s); \quad i = 1, 2$$

and

$$Y(t) = X_1(t)X_2(t).$$

Then

$$dY(t) = X_1(t)dX_2(t) + X_2(t)dX_1(t)$$

$$+ \left\{ \psi_1(t)M^2(\psi_2\chi_{[0,t]})(t) + \psi_2(t)M^2(\psi_1\chi_{[0,t]})(t) \right\} dt \quad (4.5)$$

Corollary 4.3 (Integration by parts) Let $X_i(t), i = 1, 2,$ be as in Proposition 4.2. Then

$$\int_0^t X_1(s)dX_2(s) = X_1(t)X_2(t) - \int_0^t X_2(s)dX_1(s)$$

$$- \int_0^t \left\{ \psi_1(s)M^2(\psi_2\chi_{[0,s]})(s) + \psi_2(s)M^2(\psi_1\chi_{[0,s]})(s) \right\} ds. \quad (4.6)$$

References


[19] D. Nualart: Stochastic integration with respect to fractional Brownian motion


