The Itô-Ventzell Formula and Forward Stochastic Differential Equations Driven by Poisson Random Measures

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Abstract

In this paper we obtain existence and uniqueness of solutions of forward stochastic differential equations driven by compensated Poisson random measures. To this end, an Itô-Ventzell formula for jump processes is proved and the flow properties of solutions of stochastic differential equations driven by compensated Poisson random measures are studied.

Key words and phrases: Itô-Ventzell formula, Lévy processes, Poisson random measures, Skorohod integrals, forward integrals, forward differential equations, Sobolev imbedding theorems.

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1 Introduction.

In recent years, there has been growing interests on jump processes, especially Lévy processes, partly due to the applications in mathematical finance. In [7] a Malliavin calculus was developed for Lévy processes. Among other things, the authors in [7] introduced a forward integral with respect to compensated Poisson random measures and showed that the forward integrals coincide with the Itô integrals when the integrands are non-anticipating. The purpose of this paper is to solve the following forward stochastic differential equation

\begin{equation}
X_t = X_0 + \int_0^t b(\omega, s, X_s) ds + \int_0^t \int_R \sigma(X_{s-}, z) \tilde{N}(d^- s, dz)
\end{equation}

with possibly anticipating coefficients and anticipating initial values, where \( \tilde{N}(d^- s, dz) \) indicates a forward integral. To this end, we adopt a same strategy as in [21] where anticipating stochastic differential equations driven by Brownian motion were studied. We first prove an Itô-Ventzell formula for jump processes and then go on to study the properties of the solution of the stochastic differential equation:

\begin{equation}
\phi_t(x) = x + \int_0^t \sigma(\phi_{s-}, z) \tilde{N}(ds, dz).
\end{equation}

Surprisingly little is known in the literature about the flow properties of \( \phi_t(x) \) (see, however, [6] for the case of multidimensional Lévy processes). We obtain bounds on
\( \phi_t(x) \), \( \phi_t'(x) \) and \( (\phi_t'(x))^{-1} \) under reasonable conditions on \( \sigma \), where \( \phi_t'(x) \) stands for the derivative of \( \phi_t(x) \) with respect to the space variable \( x \). Finally we show that the composition of \( \phi_t \) with a solution of a random differential equation gives rise to a solution to our equation (1.1). We also mention that a pathwise approach to forward stochastic differential equations driven by Poisson processes is considered in [13].

The rest of the paper is organized as follows. Section 2 is the preliminaries. In Section 3, we prove the Itô-Ventzell formula. The flow properties of solutions of stochastic differential equations driven by compensated Poisson random measures are studied in Section 4, where the main result is also presented.

## 2 Preliminaries.

In this section, we recall some of the framework and preliminary results from [7], which we will use later. Let \( \Omega = \mathcal{S}'(\mathbb{R}) \) be the Schwartz space of tempered distributions equipped with its Borel \( \sigma \)-algebra \( \mathcal{F} = \mathcal{B}(\Omega) \). The space \( \mathcal{S}'(\mathbb{R}) \) is the dual of the Schwartz space \( \mathcal{S}(\mathbb{R}) \) of rapidly decreasing smooth functions on \( \mathbb{R} \). We denote the action of \( \omega \in \Omega = \mathcal{S}'(\mathbb{R}) \) on \( f \in \mathcal{S}(\mathbb{R}) \) by \( \langle \omega, f \rangle = \omega(f) \).

Thanks to the Bochner-Milmos-Sazonov theorem, the white noise probability measure \( P \) can be defined by the relation

\[
\int_{\Omega} e^{i\langle \omega, f \rangle} dP(\omega) = e^{\int_{\mathbb{R}} \psi(f(x))dx - i\alpha \int_{\mathbb{R}} f(x)dx}, \quad f \in \mathcal{S}(\mathbb{R}),
\]

where the real constant \( \alpha \) and

\[
\psi(u) = \int_{\mathbb{R}} (e^{iuz} - 1 - iuz\mathbb{1}_{(1<|z|<1)}) \nu(dz)
\]

are the elements of the exponent in the characteristic functional of a pure jump Lévy process with the Lévy measure \( \nu(dz), z \in \mathbb{R} \), which, we recall, satisfies

\[
(2.1) \quad \int_{\mathbb{R}} 1 \wedge z^2 \nu(dz) < \infty.
\]

Assuming that

\[
(2.2) \quad M := \int_{\mathbb{R}} z^2 \nu(dz) < \infty,
\]

we can set \( \alpha = \int_{\mathbb{R}} z \mathbb{1}_{(|z|>1)} \nu(dz) \) and then we obtain that

\[
E[\langle \cdot, f \rangle] = 0 \quad \text{and} \quad E[\langle \cdot, f \rangle^2] = M \int_{\mathbb{R}} f^2(x)dx, \quad f \in \mathcal{S}(\mathbb{R}).
\]

Accordingly the pure jump Lévy process with no drift

\[
\eta = \eta(\omega, t), \quad \omega \in \Omega, t \in \mathbb{R}_+,
\]

that we do consider here and in the following, is the cadlag modification of \( \langle \omega, \chi_{[0,t]} \rangle \), \( \omega \in \Omega, t > 0 \), where

\[
(2.3) \quad \chi_{[0,t]}(x) = \begin{cases} 1, & 0 < x \leq t \\
0, & \text{otherwise}, \end{cases} \quad x \in \mathbb{R},
\]
with \( \eta(\omega, 0) := 0, \omega \in \Omega \). We remark that, for all \( t \in \mathbb{R}_+ \), the values \( \eta(t) \) belong to \( L_2(P) := L_2(\Omega, \mathcal{F}, P) \).

The Lévy process \( \eta \) can be expressed by

\[
\eta(t) = \int_0^t \int_{\mathbb{R}} z\tilde{N}(ds, dz), \quad t \in \mathbb{R}_+,
\]

where \( \tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt \) is the compensated Poisson random measure associated with \( \eta \).

Let \( \mathcal{F}_t, \ t \in \mathbb{R}_+ \), be the completed filtration generated by the Lévy process in (2.4). We fix \( \mathcal{F} = \mathcal{F}_\infty \).

Let \( L_2(\lambda) = L_2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \lambda) \) denote the space of the square integrable functions on \( \mathbb{R}_+ \) equipped with the Borel \( \sigma \)-algebra and the standard Lebesgue measure \( \lambda(dt), \ t \in \mathbb{R}_+ \). Denote by \( L_2(\nu) := L_2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu) \) the space of the square integrable functions on \( \mathbb{R} \) equipped with the Borel \( \sigma \)-algebra and the Lévy measure \( \nu \). Write \( L_2(P) := L_2(\Omega, \mathcal{F}, P) \) for the space of the square integrable random variables.

For the symmetric function \( f \in L_2((\lambda \times \nu)^m) \) \((m = 1, 2, \ldots)\), define \( I_0(f) := f \) for \( f \in \mathbb{R} \).

\[
I_m(f) := m! \int_0^\infty \int_0^\infty \cdots \int_0^t f(t_1, x_1, \ldots, t_m, x_m)\tilde{N}(dt_1, dx_1)\cdots\tilde{N}(dt_m, dx_m) \quad (m = 1, 2, \ldots)
\]

and set \( I_0(f) := f \) for \( f \in \mathbb{R} \). We have

**Theorem 2.1 (Chaos expansion).** Every \( F \in L_2(P) \) admits the (unique) representation

\[
F = \sum_{m=0}^\infty I_m(f_m)
\]

via the unique sequence of symmetric functions \( f_m \in L_2((\lambda \times \nu)^m), \ m = 0, 1, \ldots \).

Let \( X(t, z), \ t \in \mathbb{R}_+, \ z \in \mathbb{R} \), be a random field taking values in \( L_2(P) \). Then, for all \( t \in \mathbb{R}_+ \) and \( z \in \mathbb{R} \), Theorem 2.1 provides the chaos expansion via symmetric functions

\[
X(t, z) = \sum_{m=0}^\infty I_m(f_m(t_1, z_1, \ldots, t_m, z_m; t, z)).
\]

Let \( \hat{f}_m = \hat{f}_m(t_1, z_1, \ldots, t_{m+1}, z_{m+1}) \) be the symmetrization of \( f_m(t_1, z_1, \ldots, t_m, z_m; t, z) \) as a function of the \( m + 1 \) variables \((t_1, z_1), \ldots, (t_{m+1}, z_{m+1})\) with \( t_{m+1} = t \) and \( z_{m+1} = z \).

**Definition 2.1** [11], [12] The random field \( X(t, z), \ t \in \mathbb{R}_+, \ z \in \mathbb{R} \), is Skorohod integrable if

\[
\sum_{m=0}^\infty (m + 1)!\|\hat{f}_m\|^2_{L_2((\lambda \times \nu)^{m+1})} < \infty.
\]

Then its Skorohod integral with respect to \( \tilde{N} \), i.e.

\[
\int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t, z)\tilde{N}(dt, dz),
\]

is defined by

\[
\int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t, z)\tilde{N}(dt, dz) := \sum_{m=0}^\infty I_{m+1}(\hat{f}_m).
\]
The Skorohod integral is an element of $L^2(P)$ and
\begin{equation}
\left\| \int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t, z) \tilde{N}(\delta t, dz) \right\|_{L^2(P)}^2 = \sum_{m=0}^{\infty} (m+1)! \left\| \hat{f}_m \right\|_{L^2((\lambda \times \nu)^{m+1})}^2.
\end{equation}
Moreover,
\begin{equation}
E \int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t, z) \tilde{N}(\delta t, dz) = 0.
\end{equation}

The Skorohod integral can be regarded as an extension of the Itô integral to anticipating integrands. In fact, the following result can be proved. Cf. [11], [12], [5], [7], [18] and [21].

**Proposition 2.2** Let $X(t, z), t \in \mathbb{R}_+, z \in \mathbb{R}$, be a non-anticipating (adapted) integrand. Then the Skorohod integral and the Itô integral coincide in $L^2(P)$, i.e.
\begin{equation}
\int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t, z) \tilde{N}(\delta t, dz) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} X(t, z) \tilde{N}(dt, dz).
\end{equation}

**Definition 2.2** The space $\mathcal{D}_{1,2}$ is the set of all the elements $F \in L^2(P)$ whose chaos expansion: $F = E[F] + \sum_{m=1}^{\infty} I_m(f_m)$, satisfies
\begin{equation}
\|F\|_{\mathcal{D}_{1,2}} := \sum_{m=1}^{\infty} m \cdot m! \|f_m\|_{L^2((\lambda \times \nu)^m)} < \infty.
\end{equation}
The Malliavin derivative $D_{t,z}$ is an operator defined on $\mathcal{D}_{1,2}$ with values in the standard $L^2$-space $L^2(P \times \lambda \times \nu)$ given by
\begin{equation}
D_{t,z} F := \sum_{m=1}^{\infty} m I_{m-1}(f_m(\cdot, t, z)),
\end{equation}
where $f_m(\cdot, t, z) = f_m(t_1, z_1, ..., t_{m-1}, z_{m-1}; t, z)$.

Note that the operator $D_{t,z}$ is proved to be closed and to coincide with a certain difference operator defined in [22].

We recall the forward integral with respect to the Poisson random measure $\tilde{N}$ defined in [7].

**Definition 2.3** The forward integral
\begin{equation}
J(\theta) := \int_0^T \int_{\mathbb{R}} \theta(t, z) \tilde{N}(dt, dz)
\end{equation}
with respect to the Poisson random measure $\tilde{N}$, of a caglad stochastic function $\theta(t, z), t \in \mathbb{R}_+, z \in \mathbb{R}$, with
\begin{equation}
\theta(t, z) := \theta(t, z, \omega), \quad \omega \in \Omega,
\end{equation}
is defined as
\begin{equation}
\int_0^T \int_{\mathbb{R}} \theta(t, z) \tilde{N}(dt, dz) := \lim_{m \to \infty} \int_0^T \int_{\mathbb{R}} \theta(t, z) I_{U_m} \tilde{N}(dt, dz)
\end{equation}
if the limit exists in $L^2(P)$. Here $U_m, m = 1, 2, ..., \text{ is an increasing sequence of compact sets } U_m \subset \mathbb{R} \setminus \{0\}$ with $\nu(U_m) < \infty$ such that $\lim_{m \to \infty} U_m = \mathbb{R} \setminus \{0\}$.
The relation between the forward integral and the Skorohod integral is the following.

**Lemma 2.1** [7] If $\theta(t,z)+D_{t+},\theta(t,z)$ is Skorohod integrable and $D_{t+},\theta(t,z) := \lim_{s\to t^+} D_s,\theta(t,z)$ exists in $L^2(P \times \lambda \times \nu)$, then the forward integral exists in $L^2(P)$ and
\[
\int_0^T \int_\mathbb{R} \theta(t,z)\tilde{N}(dt, dz) = \int_0^T \int_\mathbb{R} D_{t+},\theta(t,z)\nu(dz)dt + \int_0^T \int_\mathbb{R} (\theta(t,z)+D_{t+},\theta(t,z))\tilde{N}(dt, dz).
\]

3 The Itô-Ventzell formula.

Consider the following two forward processes depending on a parameter $x \in \mathbb{R}$:
\[
F_t(x) = F_0(x) + \int_0^t G_s(x)ds + \int_0^t \int_\mathbb{R} H_s(z,x)\tilde{N}(ds, dz),
\]
\[
Y_t(x) = Y_0(x) + \int_0^t K_s(x)ds + \int_0^t \int_\mathbb{R} J_s(z,x)\tilde{N}(ds, dz),
\]
where the integrands are such that the above integrals belong to $L^2(\Omega \times \mathbb{R}, P \times dx)$. Let $<,>$ denote the inner product in the space $L^2(\mathbb{R}, dx)$.

**Lemma 3.1** It holds that
\[
<F_t,Y_t>=<Y_0,F_0> + \int_0^t <F_s,K_s>ds + \int_0^t <Y_s,G_s>ds + \int_0^t \int_\mathbb{R} <H_s(z,\cdot),J_s(z,\cdot)>\nu(dz)ds
\]
\[
+ \int_0^t \int_\mathbb{R} [<F_{s-},J_s(z,\cdot)> + <H_s(z,\cdot),Y_{s-}> + <H_s(z,\cdot),J_s(z,\cdot)>|\tilde{N}(d^-s, dz).
\]

**Proof.** Let $e_i, i \geq 1$ be an orthonormal basis of $L^2(\mathbb{R}, dx)$. For each $i \geq 1$, we have
\[
<F_t,e_i>=<F_0,e_i> + \int_0^t <G_s,e_i>ds + \int_0^t \int_\mathbb{R} <H_s(z,\cdot),e_i>\tilde{N}(ds, dz),
\]
\[
<Y_t,e_i>=<Y_0,e_i> + \int_0^t <K_s,e_i>ds + \int_0^t \int_\mathbb{R} J_s(z,\cdot),e_i>\tilde{N}(ds, dz).
\]
By the Itô’s formula for forward processes in [7],
\[
<F_t,e_i><Y_t,e_i>=<F_0,e_i><Y_0,e_i> + \int_0^t <F_s,e_i><Y_s,e_i>ds + \int_0^t <Y_s,e_i><G_s,e_i>ds
\]
\[
+ \int_0^t \int_\mathbb{R} [<F_{s-},e_i><J_s(z,\cdot),e_i> + <H_s(z,\cdot),e_i><Y_{s-},e_i>
\]
\[
+ <H_s(z,\cdot),e_i><J_s(z,\cdot),e_i>\tilde{N}(d^-s, dz) + \int_0^t \int_\mathbb{R} <H_s(z,\cdot),e_i><J_s(z,\cdot),e_i>\nu(dz)ds.
\]
Taking the summation over $i$, we get (3.11).

We now state and prove an Itô-Ventzell formula for forward processes. Let $X_t$ be a forward process given by
\[
X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \int_\mathbb{R} \gamma(s,z)\tilde{N}(d^-s, dz).
\]
Theorem 3.1 Assume that $F_t(x)$ is $C^1$ w. r. t. the space variable $x \in \mathbb{R}$. Then

$$F_t(X_t) = F_0(X_0) + \int_0^t F'_s(X_s) \alpha_s \, ds + \int_0^t \int \left[ F_s(X_s + \gamma(s,z)) - F_s(X_s) - F'_s(X_s) \gamma(s,z) \right] \nu(dz) \, ds$$

$$+ \int_0^t G_s(X_s) \, ds + \int_0^t \int \left[ H_s(z, X_s + \gamma(s,z)) - H_s(z, X_s) \right] \nu(dz) \, ds$$

(3.14) $$+ \int_0^t \int \left[ F_s(X_s - \gamma(s,z)) - F_s(X_s) + H_s(z, X_s - \gamma(s,z)) \right] \tilde{N}(d^- s, dz).$$

Here, and in the following, $F'_s(x)$ denotes the derivative of $F_s(x)$ with respect to $x$.

**Proof.** We are using the same method as in [21]. Let $\phi \in C_0^\infty(\mathbb{R}, \mathbb{R}^+)$ with $\int_{\mathbb{R}} \phi(x) \, dx = 1$. Define for $\varepsilon > 0$, $\phi_{\varepsilon}(x) = \varepsilon^{-1} \phi(\frac{x}{\varepsilon})$. It follows from Theorem 4.6 in [7] that

$$\phi_{\varepsilon}(X_t - x) = \phi_{\varepsilon}(X_0 - x) + \int_0^t \phi_{\varepsilon}'(X_s - x) \alpha_s \, ds$$

$$+ \int_0^t \int \phi_{\varepsilon}(X_s + \gamma(s,z) - x) - \phi_{\varepsilon}(X_s - x) - \phi_{\varepsilon}'(X_s - x) \gamma(s,z) \nu(dz) \, ds$$

(3.15) $$+ \int_0^t \int \phi_{\varepsilon}(X_s - \gamma(s,z) - x) - \phi_{\varepsilon}(X_s - x) \tilde{N}(d^- s, dz).$$

Using Lemma 3.1 we get that

$$\int_{\mathbb{R}} F_t(x) \phi_{\varepsilon}(X_t - x) \, dx = \int_{\mathbb{R}} F_0(x) \phi_{\varepsilon}(X_0 - x) \, dx + \int_0^t \int F_s(x) \alpha_s \phi_{\varepsilon}'(X_s - x) \, dx$$

$$+ \int_0^t \int F_s(x) \, ds \int \phi_{\varepsilon}(X_s + \gamma(s,z) - x) - \phi_{\varepsilon}(X_s - x) - \phi_{\varepsilon}'(X_s - x) \gamma(s,z) \nu(dz) \, ds$$

$$+ \int_0^t \int G_s(x) \phi_{\varepsilon}(X_s - x) \, dx + \int_0^t \int \nu(dz) \int H_s(z, X_s) \phi_{\varepsilon}(X_s + \gamma(s,z) - x) - \phi_{\varepsilon}(X_s - x) \, dx$$

$$+ \int_0^t \int H_s(z, X_s) \tilde{N}(d^- s, dz).$$

Integrating by parts,

$$\int_{\mathbb{R}} F_t(x) \phi_{\varepsilon}(X_t - x) \, dx = \int_{\mathbb{R}} F_0(x) \phi_{\varepsilon}(X_0 - x) \, dx + \int_0^t \int F'_s(x) \alpha_s \phi_{\varepsilon}(X_s - x) \, dx$$

$$+ \int_0^t \int F'_s(x) \, ds \int \phi_{\varepsilon}(X_s + \gamma(s,z) - x) - \phi_{\varepsilon}(X_s - x) \nu(dz) - \int_0^t \int F'_s(x) \, ds \int \phi_{\varepsilon}(X_s - x) \gamma(s,z) \nu(dz) \, ds$$

$$+ \int_0^t \int G_s(x) \phi_{\varepsilon}(X_s - x) \, dx + \int_0^t \int \nu(dz) \int H_s(z, X_s) \phi_{\varepsilon}(X_s + \gamma(s,z) - x) - \phi_{\varepsilon}(X_s - x) \, dx$$

$$+ \int_0^t \int H_s(z, X_s) \tilde{N}(d^- s, dz).$$
\[ + \int_0^t \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} F_s(x) \phi(x, x+\gamma(s, z) - x) - \phi(x, x+\gamma(s, z) - x) \right\} dx \]

(3.17) \[ + \int_{\mathbb{R}} H_s(z, x) \phi(x, x+\gamma(s, z) - x) dx \tilde{N}(d^-s, dz). \]

Since \( \phi_\varepsilon \) approximates to identity as \( \varepsilon \to 0 \), letting \( \varepsilon \to 0 \) we obtain that

\[
F_t(X_t) = F_0(X_0) + \int_0^t F'_s(X_s) \alpha_s ds + \int_0^t \int_{\mathbb{R}} [F_s(X_s + \gamma(s, z)) - F_s(X_s) - F'_s(X_s) \gamma(s, z)] \nu(dz) ds
\]

(3.18) \[ + \int_0^t \int_{\mathbb{R}} [F_s(X_s + \gamma(s, z)) - F_s(X_s) + H_s(z, X_s + \gamma(s, z))] \tilde{N}(d^-s, dz). \]

Next we are going to deduce an Itô-Ventzell formula for Skorohod integrals using the relation between the forward integral and the Skorohod integral. Consider

(3.19) \[ X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \int_{\mathbb{R}} \gamma(s, z) \tilde{N}(ds, dz), \]

\[ F_t(x) = F_0(x) + \int_0^t G_s(x) ds + \int_0^t \int_{\mathbb{R}} H_s(z, x) \tilde{N}(ds, dz). \]

The stochastic integrals here are understood as Skorohod integrals. Let \( \hat{H}_s(z, x) = S_{s,z} H_s(z, x), \), \( S_{s,z} = S_{s,z} \gamma(s, z) \), where \( S_{s,z} \) is an operator satisfying

\[
S_{s,z}G + D_{t+,z} \left( S_{s,z}G \right) = G
\]


**Theorem 3.2** Assume that \( F_t(x) \) is \( C^1 \) w. r. t. the space variable \( x \in \mathbb{R} \). Then

\[
F_t(X_t) = F(X_0) + \int_0^t F'_s(X_s) \alpha_s ds - \int_0^t D_{s+,z} \hat{\gamma}(s, z) \nu(dz) ds + \int_0^t G_s(X_s) ds
\]

(3.20) \[ + \int_0^t ds \int_{\mathbb{R}} [F_s(X_s + \hat{\gamma}(s, z)) - F_s(X_s) - F'_s(X_s) \hat{\gamma}(s, z)] \nu(dz)
\]

\[ + \int_0^t ds \int_{\mathbb{R}} [\hat{H}_s(z, X_s + \hat{\gamma}(s, z)) - \hat{H}_s(z, X_s)] \nu(dz)
\]

\[ + \int_0^t ds \int_{\mathbb{R}} D_{s+,z} [F_{s-}(X_{s-} + \hat{\gamma}(s, z)) - F_{s-}(X_{s-}) - \hat{H}_s(z, X_s + \hat{\gamma}(s, z))] \nu(dz) ds
\]

\[ + \int_0^t ds \int_{\mathbb{R}} \{[F_{s-}(X_{s-} + \hat{\gamma}(s, z)) - F_{s-}(X_{s-}) + \hat{H}_s(z, X_s + \hat{\gamma}(s, z))]\} \tilde{N}(ds, dz). \]
Proof. Using the relation between forward integrals and Skorohod integrals, we rewrite \( X_t \) and \( F_t(x) \) as

\[
X_t = X_0 + \int_0^t [\alpha_s - \int_{\mathbb{R}} D_{s+,z} \tilde{\gamma}(s,z)\nu(dz)]ds + \int_0^t \int_{\mathbb{R}} \tilde{\gamma}(s,z)\tilde{N}(d^-s,dz),
\]

\[
F_t(x) = F_0(x) + \int_0^t [G_s(x) - \int_{\mathbb{R}} D_{s+,z} \tilde{H}_s(z,x)\nu(dz)]ds + \int_0^t \int_{\mathbb{R}} \tilde{H}_s(z,x)\tilde{N}(d^-s,dz).
\]

It follows from Theorem 3.1 that

\[
F_t(X_t) = F_0(X_0) + \int_0^t F'_s(X_s)[\alpha_s - \int_{\mathbb{R}} D_{s+,z} \tilde{\gamma}(s,z)\nu(dz)]ds
\]

\[+
\int_0^t ds \int_{\mathbb{R}} [F_s(X_s + \tilde{\gamma}(s,z)) - F_s(X_s) - F'_s(X_s)\tilde{\gamma}(s,z)]\nu(dz) + \int_0^t G_s(X_s)ds
\]

\[+
\int_0^t ds \int_{\mathbb{R}} [\tilde{H}_s(z,X_s + \tilde{\gamma}(s,z)) - \tilde{H}_s(z,X_s)]\nu(dz)
\]

\[= F(X_0) + \int_0^t F'_s(X_s)[\alpha_s - \int_{\mathbb{R}} D_{s+,z} \tilde{\gamma}(s,z)\nu(dz)]ds + \int_0^t G_s(X_s)ds
\]

\[+
\int_0^t ds \int_{\mathbb{R}} [F_s(X_s + \tilde{\gamma}(s,z)) - F_s(X_s) - F'_s(X_s)\tilde{\gamma}(s,z)]\nu(dz)
\]

\[+
\int_0^t ds \int_{\mathbb{R}} [\tilde{H}_s(z,X_s + \tilde{\gamma}(s,z)) - \tilde{H}_s(z,X_s)]\nu(dz)
\]

\[+
\int_0^t ds \int_{\mathbb{R}} D_{s+,z}[F_s(X_s + \tilde{\gamma}(s,z)) - F_s(X_s)]\nu(dz)ds.
\]

\[+
\int_0^t ds \int_{\mathbb{R}} [\{F_s(X_s + \tilde{\gamma}(s,z)) - F_s(X_s) + \tilde{H}_s(z,X_s + \tilde{\gamma}(s,z))\}]
\]

\[+
D_{s+,z}[F_s(X_s + \tilde{\gamma}(s,z)) - F_s(X_s)] + \tilde{H}_s(z,X_s + \tilde{\gamma}(s,z))\tilde{N}(d^-s,dz).
\]

Example 3.1 (Stock price influenced by a large investor with inside information)

Suppose the price \( S_t = S_t(x) \) at time \( t \) of a stock is modelled by a geometric Lévy process of the form

\[
(3.21) \quad dS_t(x) = S_{t-}(x)[\mu(t,x)dt + \int_{\mathbb{R}} \theta(t,z)\tilde{N}(dt,dz)], \quad S_0 > 0 \quad (\text{constant}).
\]

(See e. g. [2] for more information about the use of this type of process in financial modelling.) Here \( x \in \mathbb{R} \) is a parameter and for each \( x \) and \( z \) the processes \( \mu(t) = \mu(t,x,\omega) \) and \( \theta(t,z) = \theta(t,z,\omega) \) are \( \mathcal{F}_t \)-adapted, where \( \mathcal{F}_t \) is the filtration generated by the driving Lévy process

\[
\eta(t) = \int_0^t \int_{\mathbb{R}} z\tilde{N}(ds,dz).
\]
Suppose the value of this “hidden parameter” \( x \) is influenced by a large investor with inside information, so that \( x \) can be represented by a stochastic process \( X_t \) of the form

\[
x = X_t = X_0 + \int_0^t \alpha(s) ds + \int_0^t \int_\mathbb{R} \gamma(s, z) \tilde{N}(d^- s, dz); \quad X_0 \in \mathbb{R}
\]

where \( \alpha(t) \) and \( \gamma(t, z) \) are processes adapted to a larger insider filtration \( \mathcal{G}_t \), satisfying \( \mathcal{F}_t \subseteq \mathcal{G}_t \) for all \( t \geq 0 \). (For a justification of the use of forward integrals in the modelling of insider trading, see e.g. [7]).

Combining (3.21) and (3.22) and using Theorem 3.1 we see that the dynamics of the corresponding stock price \( S_t(X_t) \) is, with \( S'_t(x) = \frac{\partial}{\partial x} S_t(x) \),

\[
d(S_t(X_t)) = S'_t(X_t)\alpha(t) dt
\]

\[
+ \int_\mathbb{R} \{ S_t(X_t + \gamma(t, z)) - S_t(X_t) - \gamma(t, z) S'_t(X_t) \} \nu(dz) dt
\]

\[
+ S_t(X_t) \mu(t, X_t) dt
\]

\[
+ \int_\mathbb{R} \{ S_t(X_t + \gamma(t, z)) - S_t(X_t) \} \theta(t, z) \nu(dz) dt
\]

\[
(3.23)
\]

By the Itô formula

\[
S_t(x) = S_0 \exp \left( \int_0^t \mu(s, x) ds + \int_0^t \int_\mathbb{R} \left( \ln \left( 1 + \theta(s, z) \right) - \theta(s, z) \right) \nu(dz) ds \right)
\]

\[
+ \int_0^t \int_\mathbb{R} \ln \left( 1 + \theta(s, z) \right) \tilde{N}(ds, dz),
\]

and hence

\[
S'_t(x) = S_t(x) \int_0^t \mu'(s, x) ds,
\]

where

\[
\mu'(s, x) = \frac{\partial}{\partial x} \mu(s, x).
\]

Substituted into (3.23) this gives

\[
d(S_t(X_t)) = S_t(X_t) \{ \alpha(t) + \mu(t, X_t) + \int_0^t \mu'(s, X_t) ds \} dt
\]

\[
+ \int_\mathbb{R} \{ S_t(X_t + \gamma(t, z)) - S_t(X_t) \} (1 + \theta(t, z)) \tilde{N}(d^- t, dz)
\]

\[
+ \int_\mathbb{R} \{ S_t(X_t + \gamma(t, z)) - S_t(X_t) \} \nu(dz) dt
\]

\[
(3.25)
\]
4 Forward SDEs Driven by Poisson Random Measures

Let $b(\omega, s, x) : \Omega \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$, $\sigma(x, z) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be measurable mappings (possibly anticipating). Let $X_0$ be a random variable. In this section, we are going to solve the following forward sde:

$$X_t = X_0 + \int_0^t b(\omega, s, X_s) \, ds + \int_0^t \int_{\mathbb{R}} \sigma(X_{s-}, z) \tilde{N}(d^{-} s, dz).$$

Let $\phi_t(x), t \geq 0$ be the stochastic flow determined by the following non-anticipating SDE:

$$\phi_t(x) = x + \int_0^t \int_{\mathbb{R}} \sigma(\phi_{s-}(x), z) \tilde{N}(ds, dz).$$

Define

$$\hat{b}(\omega, s, x) = (\phi'_s)^{-1}(x)b(\omega, s, \phi_s(x)).$$

Consider the differential equation:

$$\frac{dY_t}{dt} = \hat{b}(\omega, t, Y_t), \quad Y_0 = X_0.$$

Theorem 4.1 If $Y_t, t \geq 0$ is the unique solution to equation (4.28), then $X_t = \phi_t(Y_t), t \geq 0$ is the unique solution to equation (4.26).

Proof. It follows from Theorem 3.1 that

$$X_t = \phi_t(Y_t) = X_0 + \int_0^t \phi'_s(Y_s) \hat{b}(\omega, s, Y_s) \, ds + \int_0^t \int_{\mathbb{R}} \sigma(\phi_{s-}(Y_{s-}), z) \tilde{N}(d^{-} s, dz)$$

$$= X_0 + \int_0^t b(\omega, s, X_s) \, ds + \int_0^t \int_{\mathbb{R}} \sigma(X_{s-}, z) \tilde{N}(d^{-} s, dz).$$

Next we are going to provide appropriate conditions under which (4.28) has a unique solution. To this end, we need to study the flow generated by the solution of the following equation:

$$X_t(x) = x + \int_0^t \int_{\mathbb{R}} \sigma(X_{s-}(x), z) \tilde{N}(ds, dz).$$

Let $(p, D_p)$ denote the point process generating the Poisson random measure $N(dt, dz)$, where $D_p$, called the domain of the point process $p$, is a countable subset of $[0, \infty)$ depending on the random parameter $\omega$.

Proposition 4.1 Let $k \geq 1$. Assume that for $l = 1, 2, \ldots, 2k$,

$$\int_{\mathbb{R}} |\sigma(y, z)|^l \nu(dz) \leq C(1 + |y|^l).$$

Let $X_t(x), t \geq 0$ be the unique solution to equation (4.29). Then, we have

$$E[\sup_{0 \leq t \leq T} |X_t(x)|^{2k}] \leq C_{T, k}(1 + |x|^{2k}).$$
Proof. It follows from Itô’s formula that
\[(X_t(x))^{2k} = x^{2k} + \int_0^t \int_R [(X_{s-}(x) + \sigma(X_{s-}(x), z))^{2k} - (X_{s-}(x))^{2k}] \tilde{N}(ds,dz)\]
(4.32) \[+ \int_0^t \int_R [(X_s(x) + \sigma(X_s(x), z))^{2k} - (X_s(x))^{2k}] \tilde{N}(ds,dz)\]
\[\leq 2k(X_s(x) + \sigma(X_s(x), z))^2 - 2k(X_s(x))^{2k-1} \sigma(X_s(x), z)] \nu(dz)ds.\]

Denote by $M_t$ the martingale part in the above equation. We have
\[\mathbb{E} \left[ \sum_{0 \leq s \leq t} (\Delta M_s)^2 \right] \leq \sum_{0 \leq s \leq t, s \in D_p} |(X_{s-}(x) + \sigma(X_{s-}(x), p(s)))^{2k} - (X_{s-}(x))^{2k}|^2.\]
(4.33) \[\leq \sum_{0 \leq s \leq t, s \in D_p} |(X_{s-}(x) + \sigma(X_{s-}(x), p(s)))^{2k} - (X_{s-}(x))^{2k}|.\]

By Burkholder’s inequality,
\[E \left[ \sup_{0 \leq s \leq t} |M_s| \right] \leq C \mathbb{E} \left[ \sum_{0 \leq s \leq t} (\Delta M_s)^2 \right]^{\frac{1}{2}} \]
\[\leq E \left[ \sum_{0 \leq s \leq t, s \in D_p} |(X_{s-}(x) + \sigma(X_{s-}(x), p(s)))^{2k} - (X_{s-}(x))^{2k}| \right] \]
\[= E \left[ \int_0^t \int_R [(X_{s-}(x) + \sigma(X_{s-}(x), z))^{2k} - (X_{s-}(x))^{2k}] \tilde{N}(ds,dz) \right] \]
\[= E \left[ \int_0^t \int_R [(X_s(x) + \sigma(X_s(x), z))^{2k} - (X_s(x))^{2k}] dB_s \right].\]

By the Mean-Value Theorem, there exists $\theta(s, z, \omega) \in [0, 1]$ such that
\[(X_s(x) + \sigma(X_s(x), z))^{2k} - (X_s(x))^{2k} = 2k(X_s(x) + \theta(s, z, \omega)\sigma(X_s(x), z))^{2k-1} \sigma(X_s(x), z).\]

Therefore,
\[E \left[ \sup_{0 \leq s \leq t} |M_s| \right] \leq C_k E \left[ \int_0^t ds |X_s(x)|^{2k-1} \int_R |\sigma(X_s(x), z)| \nu(dz) \right] \]
\[+ C_k E \left[ \int_0^t ds \int_R |\sigma(X_s(x), z)|^{2k} \nu(dz) \right] \]
\[\leq C_k + C_k \int_0^t E \left[ |X_s(x)|^{2k} \right] ds.\]
By Taylor expansion, there exists $\eta(s, z, \omega) \in [0, 1]$ such that
\[
E\left[ \int_0^t \int_{\mathbb{R}} \left| (X_s(x) + \sigma(X_s(x), z))^2 - (X_s(x))^2 - 2k(X_s(x))^{2k-1}\sigma(X_s(x), z)|\nu(dz)ds \right| \right]
\]
\[
= 2k(2k-1)E\left[ \int_0^t \int_{\mathbb{R}} \left| (X_s(x) + \eta(s, z, \omega)\sigma(X_s(x), z))^{2k-2}\right|\sigma(X_s(x), z)\nu(dz) \right]
\]
\[
\leq C_k E\left[ \int_0^t ds |X_s(x)|^{2k-2} \int_{\mathbb{R}} \left| \sigma(X_s(x), z) \right|^2 \nu(dz) \right]
\]
\[
+ C_k E\left[ \int_0^t ds \int_{\mathbb{R}} \left| \sigma(X_s(x), z) \right|^{2k} \nu(dz) \right]
\]
(4.35)
\[
\leq C_k + C_k \int_0^t E[\sup_{0 \leq s \leq t} |X_s(x)|^{2k}] ds.
\]
(4.32), (4.34) and (4.35) imply that
\[
E[\sup_{0 \leq s \leq t} |X_s(x)|^{2k}] \leq C_k + C_k \int_0^t E[|X_s(x)|^{2k}] ds.
\]
Applying Gronwall’s lemma we get
\[
E[\sup_{0 \leq t \leq T} |X_t(x)|^{2k}] \leq C_{T, k}(1 + |x|^{2k}).
\]

\section*{Proposition 4.2}

Assume that $\frac{\partial \sigma(y, z)}{\partial y}$ exists and
\[
(4.36) \quad \sup_y \int_{\mathbb{R}} \left| \frac{\partial \sigma(y, z)}{\partial y} \right| |\nu(dz)| < \infty,
\]
for $l = 1, 2, \ldots, 2k$. Let $X'_i(x)$ denote the derivative of $X_i(x)$ w.r.t. $x$. Then there exists a constant $C_{T, k}$ such that
\[
(4.37) \quad E[\sup_{0 \leq t \leq T} |X'_i(x)|^{2k}] \leq C_{T, k}.
\]

\section*{Proof.}

Differentiating both sides of the equation (4.29) we get
\[
(4.38) \quad X'_i(x) = 1 + \int_0^t \int_{\mathbb{R}} \frac{\partial \sigma(X_{s-}(x), z)}{\partial y} X'_i(x) \tilde{N}(ds, dz).
\]

Put
\[
h(s, z) = \frac{\partial \sigma(X_{s-}(x), z)}{\partial y} X'_i(x).
\]

By Itô’s formula,
\[
(4.39) \quad (X'_i(x))^{2k} = 1 + \int_0^t \int_{\mathbb{R}} [(X'_i(x) + h(s, z))^{2k} - (X'_i(x))^{2k}] \tilde{N}(ds, dz)
\]
\[
+ \int_0^t \int_{\mathbb{R}} [(X'_i(x) + h(s, z))^{2k} - (X'_i(x))^{2k} - 2k(X'_i(x))^{2k-1}h(s, z)] \nu(dz) ds.
\]

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Denote the martingale part of the above equation by \( M \). Reasoning as in the proof of Proposition 4.1 we have that

\[
E[ \sup_{0 \leq s \leq t} |M_s|] \leq CE([M]_t^\frac{1}{2})
\]

\[
\leq CE[ \int_0^t \int_{\mathbb{R}} |(X'_s(x) + h(s,z))^{2k} - (X'_s(x))^{2k}|N(ds,dz)]
= E[ \int_0^t \int_{\mathbb{R}} |(X'_s(x) + h(s,z))^{2k} - (X'_s(x))^{2k}|ds\nu(dz)] 
\leq C_k E[\int_0^t ds|X'_s(x)|^{2k-1} \int_{\mathbb{R}} |h(s,z)|\nu(dz)]
+ C_k E[\int_0^t ds \int_{\mathbb{R}} |h(s,z)|^{2k}\nu(dz)] 
\leq C_k E[\int_0^t ds|X'_s(x)|^{2k} \int_{\mathbb{R}} |\frac{\partial \sigma(X'_s(x),z)}{\partial y}|\nu(dz)]
+ C_k E[\int_0^t ds|X'_s(x)|^{2k} \int_{\mathbb{R}} |\frac{\partial \sigma(X'_s(x),z)}{\partial y}|^{2k}\nu(dz)] 
\]

(4.40)

\[
\leq \hat{C}_k + \hat{C}_k \int_0^t E[|X'_s(x)|^{2k}]ds,
\]

where\[
\hat{C}_k = C_k (\sup_y \int_{\mathbb{R}} |\frac{\partial \sigma(y,z)}{\partial y}|\nu(dz) + \sup_y \int_{\mathbb{R}} |\frac{\partial \sigma(y,z)}{\partial y}|^{2k}\nu(dz)).
\]

A similar treatment applied to the second term in (4.39) yields \[
E[\int_0^t \int_{\mathbb{R}} |(X'_s(x) + h(s,z))^{2k} - (X'_s(x))^{2k} - 2k(X'_s(x))^{2k-1}h(s,z)|\nu(dz)ds]
\]

(4.41)

\[
\leq C_k + C_k \int_0^t E[|X'_s(x)|^{2k}]ds.
\]

Combining (4.39), (4.40) and (4.41) we get

\[
E[\sup_{0 \leq s \leq t} |X'_s(x)|^{2k}] \leq C_k(1 + \int_0^t E[|X'_s(x)|^{2k}]ds).
\]

An application of the Gronwall’s inequality completes the proof. 

Our next step is to give estimates for \((X'_t(x))^{-1}\). Define

\[
Z_t = \int_0^t \int_{\mathbb{R}} \frac{\partial \sigma(X'_s(x),z)}{\partial y} \tilde{N}(ds,dz).
\]

Then we see that \[
X'_t(x) = 1 + \int_0^t X'_{s-}(x) dZ_s.
\]

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Define

$$W_t = -Z_t + \int_0^t \int_\mathbb{R} \frac{\partial \sigma(X_s\cdot(x),z)}{\partial y}^2 N(ds,dz).$$

Let $Y_t(x), t \geq 0$ be the solution to the equation:

$$Y_t(x) = 1 + \int_0^t Y_{s-}(x)dW_s.$$  (4.42)

An application of Itô's formula shows that $Y_t(x) = (X'_t(x))^{-1}$.

**Proposition 4.3** Assume

$$\sup_y \int_{\mathbb{R}} \left| \frac{\partial \sigma(y,z)}{\partial y} \right|^2 l \nu(dz) < \infty,$$  (4.43)

for $l = 1, \ldots, 2k$. Then there exists a constant $C_{T,k}$ such that

$$E[\sup_{0 \leq t \leq T} |Y_t(x)|^{2k}] \leq C_{T,k}.$$  (4.44)

**Proof.** Note that

$$Y_t(x) = 1 - \int_0^t Y_{s-}(x) \int_{\mathbb{R}} \frac{\partial \sigma(X_s\cdot(x),z)}{\partial y} \tilde{N}(ds,dz)$$

$$+ \int_0^t Y_{s-}(x) \int_{\mathbb{R}} \frac{\partial \sigma(X_s\cdot(x),z)}{\partial y}^2 N(ds,dz).$$  (4.45)

Set

$$f(s,z) = Y_{s-}(x) \frac{\partial \sigma(X_s\cdot(x),z)}{1 + \partial \sigma(X_s\cdot(x),z)},$$

$$h(s,z) = -Y_{s-}(x) \frac{\partial \sigma(X_s\cdot(x),z)}{\partial y}.$$

By Itô's formula,

$$(Y_t(x))^{2k} = 1 + \int_0^t \int_{\mathbb{R}} [(Y_{s-}(x) + h(s,z))^{2k} - (Y_{s-}(x))^{2k}] \tilde{N}(ds,dz)$$

$$+ \int_0^t \int_{\mathbb{R}} [(Y_{s-}(x) + f(s,z))^{2k} - (Y_{s-}(x))^{2k}] N(ds,dz)$$

$$+ \int_0^t \int_{\mathbb{R}} [(Y_{s-}(x) + h(s,z))^{2k} - (Y_{s-}(x))^{2k} - 2k(Y_{s-}(x))^{2k-1} h(s,z)] \nu(dz)ds.$$  (4.46)

Denote the three terms on the right hand side of (4.46) by $I_t$, $II_t$, $III_t$ respectively. Similar arguments as in the proof of Proposition 4.2 show that there exists a constant $C_1$ such that

$$E[\sup_{0 \leq s \leq t} |I_s|] \leq C_1(1 + \int_0^t E[|Y_s(x)|^{2k}]ds).$$  (4.47)
By the Mean Value Theorem, we have

\[
E[\sup_{0 \leq s \leq t} |II_s|] \leq E[\int_0^t |(Y_{s-}(x) + f(s, z))^{2k} - (Y_{s-}(x))^{2k}| N(ds, dz)]
\]

\[
= E[\int_0^t \int_R |(Y_{s-}(x) + f(s, z))^{2k} - (Y_{s-}(x))^{2k}| \nu(dz)]
\]

\[
\leq CE[\int_0^t ds|Y_{s-}(x)|^{2k} \int_R \left| \frac{\partial \sigma(X_{s-}(x), z)}{\partial y} \right|^2 \nu(dz)
\]

\[
+ CE[\int_0^t ds|Y_{s-}(x)|^{2k} \int_R \left| \frac{\partial \sigma(X_{s-}(x), z)}{\partial y} \right|^{2k} \nu(dz)
\]

\[
\leq CE[\int_0^t ds|Y_s(x)|^{2k}],
\]

where we have used the fact that

\[
\sup_y \int_R \left| \frac{\partial \sigma(y, z)}{\partial y} \right|^l \nu(dz) < \infty,
\]

for \(l = 1, ..., 2k\). It follows from (4.46), (4.47), (4.48) and (4.49) that

\[
E[\sup_{0 \leq s \leq t} |Y_s(x)|^{2k}] \leq C_k(1 + \int_0^t E[|Y_s(x)|^{2k}] ds).
\]

The desired result follows from the Gronwall’s lemma. ■

Finally, we need some estimates for the derivative of \(Y_t(x)\). Define

\[
K(s, z) =: -Y_{s-}'(x) \frac{\partial \sigma(X_{s-}(x), z)}{\partial y} - Y_{s-}'(x) X_{s-}'(x) \frac{\partial^2 \sigma(X_{s-}(x), z)}{\partial y^2},
\]

\[
J(y, z) =: \left( \frac{\partial \sigma(y, z)}{\partial y} \right)^2 1 + \frac{\partial \sigma(y, z)}{\partial y},
\]

\[
L(y, z) =: 2 \frac{\partial \sigma(y, z)}{\partial y} \left( 1 + \frac{\partial \sigma(y, z)}{\partial y} \right) \frac{\partial^2 \sigma(y, z)}{\partial y^2} - \left( 1 + \frac{\partial \sigma(y, z)}{\partial y} \right)^2
\]

\[
m(s, z) =: Y_{s-}'(x) J(X_{s-}(x), z) + Y_{s-}(x) X_{s-}'(x) L(X_{s-}(x), z).
\]
Proposition 4.4 Assume

\begin{equation}
\sup_y \int_R \left| \frac{\partial^2 \sigma(y, z)}{\partial y^2} \right|^l \nu(dz) < \infty,
\end{equation}

and

\begin{equation}
\sup_y \int_R |L(y, z)|^l \nu(dz) < \infty, \quad \sup_y \int_R |J(y, z)|^l \nu(dz) < \infty,
\end{equation}

for \( l = 1, \ldots, 2k \). Then there exists a constant \( C_k \) such that \( E[\sup_{0 \leq s \leq t} |Y_s'(x)|^{2k}] \leq C_k \).

**Proof.** The proof is in the same nature as the proofs of previous propositions. We only sketch it. Differentiating (4.45) we see that

\begin{equation}
Y_t' = \int_0^t K(s, z)N(ds, dz) + \int_0^t m(s, z)N(ds, dz).
\end{equation}

By Itô’s formula,

\begin{equation}
\begin{split}
(Y_t')^{2k} & = \int_0^t \int_R (Y_s')^2 K(s, z)N(ds, dz) + \int_0^t \int_R (Y_s')^2 m(s, z)N(ds, dz) \\
& \quad + \int_0^t \int_R (Y_s')^2 m(s, z)N(ds, dz) - (Y_t')^{2k}.
\end{split}
\end{equation}

Let us denote the three terms on the right side by \( I_t, II_t \) and \( III_t \). Reasoning in the same way as in the proof of Proposition 4.2, we have

\begin{equation}
E[\sup_{0 \leq s \leq t} |I_t|] \leq E[\int_0^t \int_R |Y_s' + K(s, z)|^{2k} - (Y_s')^{2k} |ds\nu(dz)|]
\end{equation}

\begin{equation}
\leq CE[\int_0^t ds |Y_s'|^{2k} \int_R \left| \frac{\partial \sigma(X_s, z)}{\partial y} \right|^2 |y|^2 \nu(dz) + CE[\int_0^t ds |Y_s'|^{2k-1} |Y_s' X_s' |] \int \left| \frac{\partial^2 \sigma(X_s, z)}{\partial y^2} \right|^2 |\nu(dz)|]
\end{equation}

(4.54) is less than

(4.55)

\begin{equation}
CE[\int_0^t ds |Y_s'|^{2k}] + CE[\int_0^t ds |Y_s'|^{2k-1} |Y_s' X_s' |] + CE[\int_0^t ds |Y_s' X_s' |^{2k}].
\end{equation}

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Note that
\[ E[|Y_{s-}^\prime(x)|^{2k-1}|Y_s(x)X_s^\prime(x)|] \leq C_k(E[|Y_{s-}^\prime(x)|^{2k}] + E[|Y_s(x)X_s^\prime(x)|^{2k}]), \]
and from Proposition 4.3,
\[ E[\sup_{0 \leq s \leq T} |Y_s(x)X_s^\prime(x)|^\alpha] < \infty, \quad \text{for} \quad \alpha \leq 2k. \]

It follows from (4.55) that
\[ E[\sup_{0 \leq s \leq t} |I_s|] \leq C(1 + E[\int_0^t |Y_{s-}^\prime(x)|^{2k} ds]]. \]

By a similar argument, we can show that
\[ E[\sup_{0 \leq s \leq t} |III_s|] \leq C(1 + E[\int_0^t |Y_{s-}^\prime(x)|^{2k} ds]]. \]

For the second term, we have
\[ E[\sup_{0 \leq s \leq t} |II_s|] \leq E[\int_0^t \int_\mathbb{R} |(Y_{s-}^\prime(x) + m(s,z))^{2k} - (Y_{s-}^\prime(x))^{2k}| ds \nu(dz)] \]
\[ \leq C_k E[\int_0^t \int_\mathbb{R} |(Y_{s-}^\prime(x))^{2k-1}|m(s,z)| + |m(s,z)|^{2k}] ds \nu(dz)] \]
\[ \leq C_k E[\int_0^t \int_\mathbb{R} |Y_{s-}^\prime(x)|^{2k}(|J(X_{s-}(x),z)| + |J(X_{s-}(x),z)|^{2k}) ds \nu(dz)] \]
\[ + C_k E[\int_0^t \int_\mathbb{R} |(Y_{s-}^\prime(x))^{2k-1}|Y_{s-}^\prime(x)X_{s-}^\prime(x)||L(X_{s-}(x),z)||L(X_{s-}(x),z)|^{2k} ds \nu(dz)] \]
\[ + C_k E[\int_0^t \int_\mathbb{R} |Y_{s-}^\prime(x)X_{s-}^\prime(x)|^{2k} ds] + C_k E[\int_0^t |Y_{s-}^\prime(x)X_{s-}^\prime(x)|^{2k} ds]] \]
\[ \leq C_k E[\int_0^t |Y_{s-}^\prime(x)|^{2k} ds] + C_k E[\int_0^t |Y_{s-}^\prime(x)X_{s-}^\prime(x)|^{2k} ds], \]
\[ \leq C(1 + E[\int_0^t |Y_{s-}^\prime(x)|^{2k} ds]) \]
where we have used the assumptions (4.51) and the fact that
\[ E[\sup_{0 \leq s \leq T} |Y_{s-}^\prime(x)X_{s-}^\prime(x)|^{2k}] < \infty. \]

Now (4.53), (4.56), (4.57) imply
\[ E[\sup_{0 \leq s \leq t} |Y_s^\prime(x)|^{2k}] \leq C_k(1 + \int_0^t E[|Y_s^\prime(x)|^{2k}] ds), \]
which yields the desired result by Gronwall’s inequality. ■

Let \( J(y,z) \), \( L(y,z) \) be defined as in Proposition 4.4.
Proposition 4.5 Assume

\[(4.59) \sup_y \int \left| \frac{\partial \sigma(y, z)}{\partial y^l} \right| \nu(dz) < \infty, \]

\[(4.60) \sup_y \int |L(y, z)|^l \nu(dz) < \infty, \quad \sup_y \int |J(y, z)|^l \nu(dz) < \infty, \]

and

\[(4.61) \sup_y \int \left| \frac{\partial L(y, z)}{\partial y} \right| \nu(dz) < \infty, \quad \sup_y \int \left| \frac{\partial J(y, z)}{\partial y} \right| \nu(dz) < \infty, \]

for \( l = 1, \ldots, 2k, j = 1, 2, 3. \) Then there exists a constant \( C_k \) such that

\[ E[\sup_{0 \leq s \leq t} |Y''_s(x)|^{2k}] \leq C_k. \]

The proof of this proposition is entirely similar to that of Proposition 4.4. It is omitted.

Theorem 4.2 Assume that \( b(\omega, s, x) \) is locally Lipschitz in \( x \) uniformly with respect to \((\omega, s)\) and

\[(4.62) |b(\omega, s, x)| \leq C(1 + |x|^\delta), \]

for some constants \( C > 0 \) and \( \delta < 1. \) Moreover assume that \((4.30),(4.36),(4.43), (4.59), (4.60)\) and \((4.61)\) hold for some \( k > \frac{1+\delta}{1-\delta} \). Then the equation \((4.28)\) admits a unique solution. So does the equation \((4.26).\)

Proof. Recall the Sobolev imbedding theorem: if \( p > 1, \) then

\[(4.63) \sup_{x \in \mathbb{R}} |h(x)| \leq c_p \|h\|_{1,p}, \]

where \( \|h\|_{1,p} = \int_{\mathbb{R}} \left( |h(x)|^p + |h'(x)|^p \right) dx. \) Let \( \beta > 0, \alpha > 0 \) and \( p > 1 \) be any parameters with \( 2\alpha p > 1 \) and \( (2\beta - 1)p > 1. \) Set

\[ f_s(x) = (1 + x^2)^{-\beta} X_s(x), \quad g_s(x) = (1 + x^2)^{-\alpha} Y_s(x), \]

where \( Y_s(x) = (X'_s(x))^{-1}. \) For any \( T > 0, \) using Proposition 4.2,

\[ E[\sup_{0 \leq s \leq T} \|f_s\|_{1,p}^p] \]

\[ \leq C_{\beta,p} \int_{\mathbb{R}} E[\sup_{0 \leq s \leq T} |X_s(x)|^p] \left[ (1 + x^2)^{-\beta p} + |x|^p(1 + x^2)^{-(\beta+1)p} \right] dx\]

\[ + C_{\beta,p} \int_{\mathbb{R}} E[\sup_{0 \leq s \leq T} |X'_s(x)|^p(1 + x^2)^{-\beta p} dx \]

\[(4.64) \leq \int_{\mathbb{R}} \{ |x|^p \left( (1 + x^2)^{-\beta p} + |x|^p(1 + x^2)^{-(\beta+1)p} \right) + (1 + x^2)^{-\beta p} \} dx < \infty. \]

Similarly, by Proposition 4.4,

\[ E[\sup_{0 \leq s \leq T} \|g_s\|_{1,p}^p] \]

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\[
\begin{align*}
\leq C_{\alpha,p} \int_{\mathbb{R}} E\left[ \sup_{0 \leq s \leq T} |Y_s(x)|^p \right] \left[ (1 + x^2)^{-\alpha p} + |x|^p (1 + x^2)^{-(\alpha + 1)p} \right] dx \\
+ C_{\alpha,p} \int_{\mathbb{R}} E\left[ \sup_{0 \leq s \leq T} |Y'_s(x)|^p (1 + x^2)^{-\alpha p} \right] dx \\
\leq \int_{\mathbb{R}} \left\{ \left( (1 + x^2)^{-\alpha p} + |x|^p (1 + x^2)^{-(\alpha + 1)p} \right) + (1 + x^2)^{-\alpha p} \right\} dx < \infty.
\end{align*}
\] (4.65)

By the Sobolev imbedding theorem there exist random constants \( C_{\beta,T}(\omega) \) and \( C_{\alpha,T}(\omega) \) such that
\[
\sup_{0 \leq s \leq T} |X_s(x)| \leq C_{\beta,T}(\omega) (1 + x^2)\beta,
\]
and
\[
\sup_{0 \leq s \leq T} |Y_s(x)| \leq C_{\alpha,T}(\omega) (1 + x^2)^\alpha.
\]
The assumption (4.62) together with the above two inequalities gives
\[
\begin{align*}
\sup_{0 \leq s \leq T} |\hat{b}(\omega, s, x)| & = \sup_{0 \leq s \leq T} \left\{ |Y_s(x)| |b(\omega, s, X_s(x))| \right\} \\
& \leq C(\omega) (1 + x^2)^\alpha \left( 1 + |X_s(x)|^\delta \right) \\
& \leq M_{\alpha,\beta,T}(\omega) (1 + x^2)^{\alpha + \beta \delta}.
\end{align*}
\] (4.66)

If \( p > \frac{1 + \delta}{1 - \frac{\delta}{3}} \), it is possible to choose \( \beta > 0 \) and \( \alpha > 0 \) such that \( 2\alpha p > 1 \), \((2\beta - 1)p > 1 \) and \( 2\alpha + 2\beta \delta \leq 1 \). Therefore, there exists a random constant \( C_T(\omega) \) such that
\[
\sup_{0 \leq s \leq T} |\hat{b}(\omega, s, x)| \leq C_T(\omega) (1 + |x|).
\] (4.67)

On the other hand, by the Sobolev imbedding Theorem and Proposition 4.4 we see that \( (\phi'_s)^{-1}(x) \) is \( C^1 \) in \( x \) and the derivative is bounded on compact sets. Combining this fact with the assumption on \( b \), it is easily seen that for a fixed \( \omega \), \( \hat{b}(\omega, s, x) \) is locally Lipschitz in \( x \) uniformly with respect to \( s \) on any compact sets. It follows from the general theory of ordinary differential equations that (4.28) admits a unique global solution. ✷

References


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