

On the Existence and Explicit Representability of Strong Solutions of Lévy Noise Driven SDE's with Irregular Coefficients

Thilo Meyer-Brandis¹ and Frank Proske¹

Abstract

We present a method to study strong solutions of stochastic differential equations driven by Lévy processes, whose coefficients are admitted to be irregular. Furthermore we give explicit representations of solutions of such SDE's.

Key words and phrases: Lévy processes, stochastic differential equations, white noise analysis

AMS 2000 classification: 60G51; 60G35; 60H15; 60H40; 60H15; 91B70

1 Introduction

In recent years stochastic differential equations (SDE's) for Lévy processes have become of much current interest for applications to mathematical finance, neuro-biology and other areas of natural sciences. In this paper we want to demonstrate how white noise concepts for Lévy processes can be effectively employed to solve a fully non-linear problem. More precisely, we present a method which enables to determine solutions of stochastic differential equations (SDE's) driven by Lévy processes, explicitly. Moreover, this approach provides criteria for the existence of strong solutions of SDE's whose coefficients are not Lipschitz continuous. These criteria also comprise

¹Centre of Mathematics for Applications (CMA), Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N-0316 Oslo, Norway.

E-mail address: meyerbr@math.uio.no, proske@math.uio.no

the case of discontinuous coefficients. To find explicit solutions is usually a challenging and difficult task of both theoretical and practical significance. We are convinced that our method grants new insights into the nature of solutions of SDE's with Lévy noise. In addition, it exhibits potential to yield dividends in other important applications.

In [LP1] the authors apply concepts from Gaussian white noise analysis to represent solutions of SDE's driven by a Brownian motion. Consider the 1-dimensional Itô-diffusion

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t, \quad Y_0 = y, \quad 0 \leq t \leq T$$

where b is the drift, σ the diffusion coefficient and B_t the Brownian motion. It is proven in [LP1] that under certain conditions on b and σ a (global) strong solution Y_t of the SDE takes the explicit form

$$Y_t = E_{\hat{\mu}} \left[u(\hat{B}_t) M_T^\diamond \right], \quad (1.1)$$

where

$$M_T^\diamond = \exp^\diamond \left\{ \int_0^T \left(W_t + \frac{b(u(\hat{B}_t))}{\sigma(u(\hat{B}_t))} - \frac{1}{2} \sigma'(u(\hat{B}_t)) \right) d\hat{B}_t - \frac{1}{2} \int_0^T \left(W_t + \frac{b(u(\hat{B}_t))}{\sigma(u(\hat{B}_t))} - \frac{1}{2} \sigma'(u(\hat{B}_t)) \right)^{\diamond 2} dt \right\}$$

and where u is a solution of the ordinary differential equation

$$u' = \sigma(u), \quad u(0) = y.$$

Here $W_s = W_s(\omega)$ is the (singular) white noise and \diamond is the Wick product with respect to the white noise probability space $(\Omega, \mathcal{F}, \mu)$. The process $\hat{B}_t = \hat{B}_t(\hat{\omega})$ is an auxiliary Brownian motion on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})$, which is a copy of the initial white noise space. The formula involves (stochastic) Bochner integrals on the Hida distribution space $(\mathcal{S})^*$. In [LP2] the result (2.2.3) is extended to SDE's on Hilbert spaces.

Using Lévy white noise theory, we adapt these ideas in this paper to Lévy-Itô diffusions, i.e. jump diffusions of the type

$$dY_t = b(Y_{t-})dt + \sigma(Y_{t-})dB_t + \gamma(Y_{t-})dL_t, \quad Y_0 = y, \quad 0 \leq t \leq T, \quad (1.2)$$

where L_t is a pure jump Lévy process, B_t a Brownian motion and where $b, \sigma, \gamma : \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions.

As in the purely Gaussian case we deduce under certain conditions on b, σ, γ a general solution formula for this SDE. Moreover, we establish an existence (and uniqueness) result for strong solutions of (1.2) with irregular coefficients, i.e. with coefficients, which are not necessarily Lipschitz continuous. In particular for $b = \sigma = 0$, when the Lévy measure ν has density a φ w.r.t. to the Lebesgue measure, we supply a condition for strong solutions, which requires L^p -integrability of the Doleans-Dade exponential

$$\mathcal{E}(M)_t,$$

where M_t is the (local) martingale

$$M_t = \int_0^t \int_{\mathbb{R}_0} \left(\frac{\varphi\left(\frac{x}{\gamma(L_{s-})}\right)}{\varphi(x)\gamma(L_{s-})} - 1 \right) \tilde{N}(ds, dx)$$

with compensated Poisson random measure $\tilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$.

The study of SDE's with less "nicely" behaved coefficients is important, since they arise in a broad range of stochastic control problems. In the current literature one finds a scarce number of results pertaining to strong solutions of Lévy-Itô diffusions with irregular coefficients. Let us mention that [Ba] derives a condition in the case of symmetric stable processes of order $\alpha > 1$, which guarantees strong solutions and pathwise uniqueness. This condition can be regarded as an analogue of the Yamada-Watanabe condition for the Brownian motion case, which presumes continuity on the coefficient γ . Other results on this topic can be e.g. found in [Wi] and [BBC]. Our approach even applies to coefficients, which are only measurable. With this paper we aim at contributing to a better understanding of Lévy-Itô diffusions.

Strong solutions of Brownian motion driven SDE's with non-Lipschitzian coefficients are treated e.g. by [V1,2,3], [Zv], [ZvK], [GK]. Recently, new ideas were developed in [FZ]. To the best of our knowledge, results about explicit strong solutions of SDE's are only in existence for continuous processes as driving noise. For example [VK], [RK] obtain explicit chaos expansions of strong solutions of SDE's driven by a Brownian motion. However, the kernels of the expansions must be determined as solutions of systems of ODE's or PDE's. Another approach is due to [D], [S], who construct strong solutions of SDE's driven by Brownian motion from solutions of ODE's,

pathwisely. However, these methods reveal the deficiency to fail, if the coefficients of the SDE are not regular enough. We underline that we even obtain explicit solutions of jump SDE's, when the coefficients of the SDE are only measurable.

The paper is organized as follows: In Section 2 we recall some concepts from white noise theory for Lévy processes, developed in [LP], [LØP] and [P]. In Section 3, for convenience of transparency we first focus on pure jump Lévy processes and derive a general solution formula for SDE's in this case. In Section 4 we turn to the general case and illustrate how the methodology of the previous Section carries over to attain similar results. Section 5 is devoted to strong solutions of Lévy-Itô SDE's with irregular coefficients. In this Section we directly verify the explicit formulas as strong solutions. Finally, Section 6 deals with explicit chaos expansions of strong solutions. In addition a different type of explicit representation for solutions is presented.

2 White noise framework

In this Section we provide a brief review of some concepts of a white noise theory for Lévy processes, developed in [LØP] and [P]. In Section 3 and 4 we will use this theory as a basic tool to establish criteria for the existence of strong solutions of Lévy-Itô diffusions with non-Lipschitzian coefficients. For general information about white noise theory the reader is referred to the excellent accounts of [HKPS], [Ku] and [O].

Let us recall that a *Lévy process* $\eta(t)$ is a stochastic process on \mathbb{R}_+ , which has independent and stationary increments starting at zero, i.e. $\eta(0) = 0$. The process $\eta(t)$ is by its nature a càdlàg semimartingale, which is uniquely determined by the characteristic triplet

$$(B_t, C_t, \hat{\mu}) = (a \cdot t, \sigma \cdot t, dt\nu(dx)), \quad (2.1)$$

where a, σ are constants and where ν is the *Lévy measure* on $\mathbb{R}_0 := \mathbb{R} - \{0\}$. For more information about Lévy processes consult e.g. [B], [Sa] or [JS].

To avoid unnecessary technical complications we first recapitulate our white noise framework in the case of pure jump Lévy processes, that is we consider Lévy processes in (2.1) with $a = \sigma = 0$. At the end of this Section

we shortly explain the extension of the pure jump setting to the general case.

In the following we denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space on \mathbb{R}^d . The space $\mathcal{S}'(\mathbb{R}^d)$ is the dual of $\mathcal{S}(\mathbb{R}^d)$, that is the space of tempered distributions. We want to work with a white noise measure, which is constructed on the nuclear algebra $\tilde{\mathcal{S}}'(X)$, introduced in [LØP]. The space $\tilde{\mathcal{S}}(X)$ is defined as the quotient algebra

$$\tilde{\mathcal{S}}(X) = \mathcal{S}(X)/\mathcal{N}_\pi, \quad (2.2)$$

where $\mathcal{S}(X)$ is a subspace of $\mathcal{S}(X)$, given by

$$\mathcal{S}(X) := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^2) : \varphi(t, 0) = \left(\frac{\partial}{\partial x} \varphi \right)(t, 0) = 0 \right\} \quad (2.3)$$

and where the closed ideal \mathcal{N}_π in $\mathcal{S}(X)$ is defined as

$$\mathcal{N}_\pi := \{ \phi \in \mathcal{S}(X) : \|\phi\|_{L^2(\pi)} = 0 \} \quad (2.4)$$

The space $\tilde{\mathcal{S}}(X)$ is a (countably Hilbertian) nuclear algebra. We indicate by $\tilde{\mathcal{S}}'(X)$ its dual.

From the Bochner-Minlos theorem we deduce that there exists a unique probability measure μ on the Borel sets of $\tilde{\mathcal{S}}'(X)$ such that

$$\int_{\tilde{\mathcal{S}}'(X)} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = \exp \left(\int_X (e^{i\phi} - 1) d\pi \right) \quad (2.5)$$

for all $\phi \in \tilde{\mathcal{S}}(X)$, where $\langle \omega, \phi \rangle := \omega(\phi)$ denotes the action of $\omega \in \tilde{\mathcal{S}}'(X)$ on $\phi \in \tilde{\mathcal{S}}(X)$. The measure μ on $\Omega = \tilde{\mathcal{S}}'(X)$ is called (*pure jump*) *Lévy white noise probability measure*.

In the sequel we consider a compensated Poisson random measure

$$\tilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$$

associated with a Lévy process $\eta(t)$, which is defined on the *white noise probability space*

$$(\Omega, \mathcal{F}, P) = \left(\tilde{\mathcal{S}}'(X), \mathcal{B}(\tilde{\mathcal{S}}'(X)), \mu \right).$$

By using generalized Charlier polynomials $C_n(\omega) \in \left(\widetilde{\mathcal{S}}(X)^{\widehat{\otimes} n}\right)'$ (dual of the n -th completed symmetric tensor product of $\widetilde{\mathcal{S}}(X)$ with itself) it is possible to construct an orthogonal $L^2(\mu)$ -basis $\{\mathcal{K}_\alpha(\omega)\}_{\alpha \in \mathcal{J}}$ defined by

$$\mathcal{K}_\alpha(\omega) = \left\langle C_{|\alpha|}(\omega), \delta^{\widehat{\otimes} \alpha} \right\rangle, \quad (2.6)$$

where \mathcal{J} is the multiindex set of all $\alpha = (\alpha_1, \alpha_2, \dots)$ with finitely many non-zero components $\alpha_i \in \mathbb{N}_0$. The symbol $\delta^{\widehat{\otimes} \alpha}$ denotes the symmetrization of $\delta_1^{\otimes \alpha_1} \otimes \dots \otimes \delta_j^{\otimes \alpha_j}$, where $\{\delta_j\}_{j \geq 1} \subset \widetilde{\mathcal{S}}(X)$ is an orthonormal basis of $L^2(\mathbb{R} \times \mathbb{R}_0, dt\nu(dx))$.

So every $X \in L^2(\mu)$ has the unique representation

$$X = \sum_{\alpha \in \mathcal{J}} c_\alpha \mathcal{K}_\alpha$$

with Fourier coefficients $c_\alpha \in \mathbb{R}$. Moreover we have the isometry

$$\|X\|_{L^2(\mu)}^2 = \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2 \quad (2.7)$$

with $\alpha! := \alpha_1! \alpha_2! \dots$ for $\alpha \in \mathcal{J}$. The *Lévy-Hida test function space* (\mathcal{S}) consists of all $f = \sum_{\alpha \in \mathcal{J}} c_\alpha \mathcal{K}_\alpha \in L^2(\mu)$ such that

$$\|f\|_{0,k}^2 := \sum_{\gamma \in \mathcal{J}^m} \alpha! c_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty \quad (2.8)$$

holds for all $k \in \mathbb{N}_0$ with weights $(2\mathbb{N})^{k\alpha} = (2 \cdot 1)^{k\alpha_1} (2 \cdot 2)^{k\alpha_2} \dots (2 \cdot l)^{k\alpha_l}$, if $Index(\alpha) := l$. The space (\mathcal{S}) is given the projective topology, induced by the norms $(\|\cdot\|_{0,k})_{k \in \mathbb{N}_0}$ in (2.8). The *Lévy-Hida distribution space*, denoted by $(\mathcal{S})^*$ is the topological dual of (\mathcal{S}). So we obtain the following Gel'fand triple

$$(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^*. \quad (2.9)$$

We can endow $(\mathcal{S})^*$ with structure of a topological algebra by introducing the *Wick product* \diamond , defined by

$$(\mathcal{K}_\alpha \diamond \mathcal{K}_\beta)(\omega) = (\mathcal{K}_{\alpha+\beta})(\omega), \quad \alpha, \beta \in \mathcal{J} \quad (2.10)$$

The product is linearly extensible to $(\mathcal{S})^* \times (\mathcal{S})^*$. It can be proven e.g. that

$$\langle C_n(\omega), f_n \rangle \diamond \langle C_m(\omega), g_m \rangle = \langle C_{n+m}(\omega), f_n \widehat{\otimes} g_m \rangle \quad (2.11)$$

for $f_n \in \tilde{\mathcal{S}}(X)^{\hat{\otimes} n}$ and $g_m \in \tilde{\mathcal{S}}(X)^{\hat{\otimes} m}$ (see [LØP]).

A nice feature of the Lévy-Hida distribution space is that it carries the white noise $\dot{\tilde{N}}(t, x)$ of the Poisson random measure $\tilde{N}(dt, dx)$, that is the formal Radon-Nikodym derivative of $\tilde{N}(dt, dx)$ defined as

$$\dot{\tilde{N}}(t, x) = \sum_{k \geq 1} \delta_k(t, x) \mathcal{K}_\alpha(\omega)$$

is in $(\mathcal{S})^*$ $dt\nu(dx)$ -a.e. The Wick product relates to stochastic integrals w.r.t. to $\tilde{N}(dt, dx)$ in the following way: If $Y(t, x, \omega)$ is a predictable process, fulfilling the condition $E \int_0^T \int_{\mathbb{R}_0} Y^2(t, z, \omega) dt\nu(dz) < \infty$, then $Y(t, z, \omega) \diamond \dot{\tilde{N}}(t, z)$ is $\lambda \times \nu$ -Bochner integrable in $(\mathcal{S})^*$ and

$$\int_0^T \int_{\mathbb{R}_0} Y(t, z, \omega) \tilde{N}(dt, dx) = \int_0^T \int_{\mathbb{R}_0} Y(t, z, \omega) \diamond \dot{\tilde{N}}(t, z) dt\nu(dz). \quad (2.12)$$

An analogous relation is also valid for the Brownian motion. See [LØP] or [ØP] for definitions.

One of our main tools in the study of Lévy-Itô diffusions is the *Lévy Hermite transform* \mathcal{H} , which is used to give a characterization of distributions in $(\mathcal{S})^*$ (see characterization theorem 2.3.8 in [LØP]). Similar to the Gaussian case the definition of \mathcal{H} rests on the basis $\{\mathcal{K}_\alpha(\omega)\}_{\alpha \in \mathcal{J}}$ in (2.6). The *Lévy Hermite transform* of $X(\omega) = \sum_\alpha c_\alpha \mathcal{K}_\alpha(\omega) \in (\mathcal{S})^*$, denoted by $\mathcal{H}X$ or \tilde{X} , is defined by

$$\mathcal{H}X(z) = \tilde{X}(z) = \sum_\alpha c_\alpha z^\alpha \in \mathbb{C}, \quad (2.13)$$

where $z = (z_1, z_2, \dots) \in \mathbb{C}^{\mathbb{N}}$, i.e. in the space of \mathbb{C} -valued sequences, and where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots$. For example $\mathcal{H}X(z)$ in (2.13) is absolutely convergent on the infinite dimensional neighbourhood

$$\mathbb{K}_q(R) := \left\{ (z_1, z_2, \dots) \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha \neq 0} |z^\alpha|^2 (2\mathbb{N})^{q\alpha} < R^2 \right\} \quad (2.14)$$

for some $0 < q \leq R < \infty$. For example, the Hermite transform of $\dot{\tilde{N}}(t, x)$ can be evaluated as

$$\mathcal{H}(\dot{\tilde{N}}(t, x))(z) = \sum_{k \geq 1} \delta_k(t, x) z_k. \quad (2.15)$$

The Hermite transform translates the Wick product into an ordinary (complex) product, that is

$$\mathcal{H}(X \diamond Y)(z) = \mathcal{H}(X)(z) \cdot \mathcal{H}(Y)(z). \quad (2.16)$$

As a consequence of theorem 2.3.8 in [LØP] the last relation can be generalized to Wick versions of complex analytical functions g : If the function $g : \mathbb{C} \rightarrow \mathbb{C}$ can be expanded into a Taylor series around $\xi_0 = \mathcal{H}(X)(0)$ with real valued coefficients, then there exists a unique distribution $Y \in (\mathcal{S})^*$ such that

$$\mathcal{H}(Y)(z) = g(\mathcal{H}(X)(z)) \quad (2.17)$$

on $\mathbb{K}_q(R)$ for some $0 < q \leq R < \infty$. We set $g^\diamond(X) = Y$.

For example, the Wick version of the exponential function \exp can be written as

$$\exp^\diamond X = \sum_{n \geq 0} \frac{1}{n!} X^{\circ n}. \quad (2.18)$$

Finally, let us outline how the preceding concepts and results can be generalized to capture the case of Lévy processes with Brownian motion and pure jump part (see [P]). Indicate by μ_G the Gaussian white noise measure on the measurable space

$$(\Omega_G, \mathcal{F}_G) = (\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R}))).$$

Further recall the construction of the orthogonal $L^2(\mu_G)$ basis $\{H_\alpha(\omega)\}_{\alpha \in J}$, given by

$$\mathcal{H}_\alpha(\omega) = \prod_{j \geq 1} h_{\alpha_j}(\langle \omega, \xi_j \rangle),$$

where $\langle \omega, \cdot \rangle = \omega(\cdot)$ and where ξ_j resp. $h_j, j = 1, 2, \dots$ are the Hermite functions resp. Hermite polynomials. Using μ_J to denote the pure jump white noise measure on $(\Omega_J, \mathcal{F}_J) = (\tilde{\mathcal{S}}'(X), \mathcal{B}(\tilde{\mathcal{S}}'(X)))$, we can define the *Lévy white noise measure* μ as the product measure $\mu_G \times \mu_J$ on

$$(\Omega, \mathcal{F}) = (\Omega_G \times \Omega_J, \mathcal{F}_G \otimes \mathcal{F}_J). \quad (2.19)$$

Set

$$\mathcal{L}_\gamma(\omega) = \mathcal{L}_\gamma(\omega_1, \omega_2) = \mathcal{H}_\alpha(\omega_1) \mathcal{K}_\beta(\omega_2), \quad (2.20)$$

if $\gamma = (\alpha, \beta) \in \mathcal{I} := \mathcal{J}^2$. Thus $(\mathcal{L}_\gamma(\omega))_{\gamma \in \mathcal{I}}$ constitutes an $L^2(\mu)$ -basis with norm expression

$$\|\mathcal{L}_\gamma\|_{L^2(\mu)}^2 = \gamma!,$$

where $\gamma! := \alpha!\beta!$ for $\gamma = (\alpha, \beta) \in \mathcal{I}$.

As in the the pure jump setting, we employ the basis $(\mathcal{L}_\gamma(\omega))_{\gamma \in \mathcal{I}}$ to establish the concepts of Hida space, Wick product or Hermite transform to the mixture of Gaussian and pure jump Lévy noise.

3 Explicit solution of a pure jump diffusion

In this Section we study the Lévy process driven diffusion

$$\begin{aligned} dY_t &= \gamma(Y_{t-})dL_t \\ &= \int_{\mathbb{R}_0} \gamma(Y_{t-})zN(dt, dx), \quad Y_0 = y, \quad 0 \leq t \leq T. \end{aligned} \tag{3.1}$$

where L_t is a (uncompensated) pure jump Lévy process with jump measure $N(dt, dx)$ on the white noise probability space $(\Omega, \mathcal{F}, \mu)$. The Lévy process L_t is characterized by the Lévy measure $\nu(dx)$. For more information about Lévy processes consult for example the excellent textbooks [B], [Sa]. In the sequel we assume the Lévy measure to be finite and to be equivalent to the Lebesgue measure that is

$$\nu(dx) = \varphi(x)dx, \tag{3.2}$$

where $\varphi(x)$ is strictly positiv. Further, we require the diffusion coefficient $\gamma : \mathbb{R} \rightarrow \mathbb{R}_+$ to be strictly positiv and such that there exists a unique (global) strong solution Y_t of (3.1). It is well-known that this is for example the case as soon as γ is locally Lipschitz continuous and of linear growth.

First observe the following notation. Given our stochastic basis $(\Omega, \mathcal{F}, \mu)$, we use the symbol $\hat{\cdot}$ to define a copy $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})$ of our initial white noise space as well as to denote the corresponding copied objects on this new basis. For example, $\hat{L}_t = \hat{L}_t(\hat{\omega})$ denotes a Lévy process with same characteristics as L_t on the auxiliary probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})$.

Using the white noise concepts presented in Section 2, we are able to derive a general solution formula for Y_t in (3.1).

Theorem 3.1 Let ρ be a Borel measurable function from \mathbb{R} to \mathbb{R} . Assume that the Bochner integral

$$E_{\hat{\mu}}^y \left[\rho \left(\widehat{L}_t \right) J_T^\diamond \right] \quad (3.3)$$

exists, where

$$\begin{aligned} J_T^\diamond &= \exp^\diamond \int_0^T \int_{\mathbb{R}_0} \log^\diamond \left(\frac{(1 + \dot{\tilde{N}}(\omega, s, \frac{x'}{\gamma(\widehat{L}_{s-}))})\varphi(\frac{x'}{\gamma(\widehat{L}_{s-}))})}{\varphi(x')\gamma(\widehat{L}_{s-})} \right) N(\widehat{\omega}, ds, dx') \\ &\diamond \exp^\diamond \int_0^T \int_{\mathbb{R}_0} \left(1 - \frac{(1 + \dot{\tilde{N}}(\omega, s, \frac{x'}{\gamma(\widehat{L}_{s-}))})\varphi(\frac{x'}{\gamma(\widehat{L}_{s-}))})}{\varphi(x')\gamma(\widehat{L}_{s-})} \right) \nu(dx') ds. \end{aligned}$$

Here the Wick product \diamond is with respect to ω . Then $\rho(Y_t(\omega))$ is given by the expression (3.3). The integrals occurring in (3.3) are (stochastic) Bochner integrals on the Lévy-Hida space.

Proof By applying the Hermite transform to the solution Y_t we obtain

$$\begin{aligned} \mathcal{H}(\rho(Y_t))(z) &= E_{\hat{\mu}}^y \left[\rho(Y_t) \mathcal{E} \left(\int_0^T \int_{\mathbb{R}_0} \phi_z(s, x) \tilde{N}(\omega, ds, dx) \right) \right] \\ &= E_{\hat{\mu}}^y \left[\rho(\widehat{Y}_t) \mathcal{E} \left(\int_0^T \int_{\mathbb{R}_0} \phi_z(s, x) \tilde{N}(\widehat{\omega}, ds, dx) \right) \right], \end{aligned}$$

where $\phi_z(s, x) = \sum_k z_k \delta_k(s, x)$, $z \in \mathbb{C}_c^{\mathbb{N}}$ and $\tilde{N}(ds, dx) = N(ds, dx) - \nu(dx)ds$. Note that

$$\begin{aligned} &\mathcal{E} \left(\int_0^T \int_{\mathbb{R}_0} \phi_z(s, x) \tilde{N}(ds, dx) \right) \\ &= \exp \left\{ \int_0^T \int_{\mathbb{R}_0} \log(1 + \phi_z(s, x)) N(ds, dx) - \int_0^T \int_{\mathbb{R}_0} \phi_z(s, x) \nu(dx) ds \right\}. \end{aligned}$$

In virtue of the Girsanov theorem for random measures (see [JS]), it follows that

$$\mathcal{H}(\rho(Y_t))(z) = E_Q^y \left[\rho(\widehat{Y}_t) \right]$$

where Q is the probability measure, given by

$$dQ = \mathcal{E}\left(\int_0^T \int_{\mathbb{R}_0} \phi_z(s, x) \tilde{N}(ds, dx)\right) d\hat{\mu}$$

and where \hat{Y} under Q is a jump process whose jump measure has the predictable compensation

$$\nu^*(ds, dx) = (1 + \phi_z(s, x))\nu(dx)ds.$$

The process Y_t can be rewritten in terms of its jump measure denoted by $N'(dt, dx)$:

$$dY_t = \int_{\mathbb{R}_0} x N'(dt, dx),$$

where the Q -compensation $\nu'(\omega, ds, dx)$ of $N'(ds, dx)$ is given by the relation

$$\int_0^T \int_{\mathbb{R}_0} f(x) \nu'(\omega, ds, dx) = \int_0^T \int_{\mathbb{R}_0} f(\gamma(Y_{s-})x) (1 + \phi_z(s, x)) \nu(dx) ds \quad (3.4)$$

for all ν' -integrable f . The substitution $x' = \gamma(Y_{s-})x$ on the right hand side of the latter relation yields

$$\int_0^T \int_{\mathbb{R}_0} f(x) \nu'(\omega, ds, dx) = \int_0^T \int_{\mathbb{R}_0} f(x') \pi_z(\omega, s, x') dx' ds,$$

where

$$\pi_z(\omega, s, x') = (1 + \phi(s, \frac{x'}{\gamma(Y_{s-})}, z)) \cdot \varphi(\frac{x'}{\gamma(Y_{s-})}) \cdot \frac{1}{\gamma(Y_{s-})}.$$

By invoking the Girsanov theorem again, we get that

$$\begin{aligned} & E_Q^y \left[\rho(\hat{Y}_t) \right] \\ &= E_Q^y \left[\hat{Y}_t \mathcal{E} \left(\int_0^T \int_{\mathbb{R}_0} \theta_z(\hat{\omega}, s, x') (N' - \nu')(\hat{\omega}, ds, dx') \right) \right. \\ & \quad \left. \mathcal{E}^{-1} \left(\int_0^T \int_{\mathbb{R}_0} \theta_z(\hat{\omega}, s, x') (N' - \nu')(\hat{\omega}, ds, dx') \right) \right] \end{aligned}$$

$$\begin{aligned}
&= E_{Q'}^y \left[\widehat{Y}_t \mathcal{E}^{-1} \left(\int_0^T \int_{\mathbb{R}_0} \theta_z(\widehat{\omega}, s, x') (N' - \nu')(\widehat{\omega}, ds, dx') \right) \right] \\
&= E_{Q'}^y \left[\rho \left(\widehat{Y}_t \right) \exp \left\{ \int_0^T \int_{\mathbb{R}_0} \log \left(\frac{1}{\theta_z(\widehat{\omega}, s, x') + 1} N'(\widehat{\omega}, ds, dx') \right) \right. \right. \\
&\quad \cdot \exp \left\{ \int_0^T \int_{\mathbb{R}_0} \left(1 - \frac{1}{\theta_z(\widehat{\omega}, s, x') + 1} \right) \nu(dx') ds \right\} \left. \right] \tag{3.5}
\end{aligned}$$

$$\begin{aligned}
&= E_{\mu}^y \left[\rho \left(\widehat{L}_t \right) \exp \left\{ \int_0^T \int_{\mathbb{R}_0} \log \left(\frac{(1 + \phi_z(s, \frac{x'}{\widehat{L}_{s-}}), z)) \varphi(\frac{x'}{\widehat{L}_{s-}})}{\varphi(x') \gamma(\widehat{L}_{s-})} \right) N(\widehat{\omega}, ds, dx') \right. \right. \\
&\quad \cdot \exp \left\{ \int_0^T \int_{\mathbb{R}_0} \left(1 - \frac{(1 + \phi_z(s, \frac{x'}{\widehat{L}_{s-}})) \varphi(\frac{x'}{\widehat{L}_{s-}})}{\varphi(x') \gamma(\widehat{L}_{s-})} \right) \nu(dx') ds \right\} \left. \right] \tag{3.6}
\end{aligned}$$

where

$$\theta_z(\widehat{\omega}, s, x') = \frac{\varphi(x')}{\pi_z(\widehat{\omega}, s, x')} - 1.$$

and where

$$dQ' = \mathcal{E} \left(\int_0^T \int_{\mathbb{R}_0} \theta_z(\widehat{\omega}, s, x') (N' - \nu')(\widehat{\omega}, ds, dx') \right) dQ.$$

Here we have employed the relation (3.4) to derive identity (3.5), and identity (3.6) is due to the fact that Y_t under Q' has the same law as L_t under μ .

Therefore we obtain the relation

$$\begin{aligned}
&\mathcal{H}(\rho(Y_t))(z) \\
&= E_{\mu}^y \left[\rho \left(\widehat{L}_t \right) \exp \left\{ \int_0^T \int_{\mathbb{R}_0} \log \left(\frac{(1 + \phi_z(s, \frac{x'}{\widehat{L}_{s-}})) \varphi(\frac{x'}{\widehat{L}_{s-}})}{\varphi(x') \gamma(\widehat{L}_{s-})} \right) N(\widehat{\omega}, ds, dx') \right. \right. \\
&\quad \cdot \exp \left\{ \int_0^T \int_{\mathbb{R}_0} \left(1 - \frac{(1 + \phi_z(s, \frac{x'}{\widehat{L}_{s-}})) \varphi(\frac{x'}{\widehat{L}_{s-}})}{\varphi(x') \gamma(\widehat{L}_{s-})} \right) \nu(dx') ds \right\} \left. \right] \tag{3.7}
\end{aligned}$$

for $z \in \mathbb{K}_q(R)$ with some $0 < q, R < \infty$.

Because of a Lévy version of Theorem 2.6.12 in [HØUZ] and the finiteness of the Lévy measure we can extract the Hermite transform on both sides of (3.7), which concludes the proof. \square

Remark 3.2. (i) A set of sufficient conditions for the existence of the Bochner integral (3.3) is that γ is bounded and that

$$M \geq \frac{\varphi\left(\frac{x}{\gamma(y)}\right)}{\varphi(x)\gamma(y)} > 0$$

holds for all $x, y \in \mathbb{R}$, where M is a constants. See Corollary 5.3 in Section 5.

(ii) The formula (3.3) necessitates stochastic integration of the form

$$\int_0^t \int_{\mathbb{R}_0} \Phi(\omega, s, x) N(\omega, ds, dx),$$

where Φ is a process on the conuclear space $(\mathcal{S})^*$ and N the Poisson random measure. For general information about stochastic integration on conuclear spaces with respect to Poisson random measures we refer to [KX].

In particular, if $\rho = id$ we get

Corollary 3.3 The solution $Y_t \in L^2(\mu)$ in (3.1) has the form

$$\begin{aligned} & Y_t \\ &= E_{\hat{\mu}}^y \left[\hat{L}_t \exp^{\diamond} \left\{ \int_0^T \int_{\mathbb{R}_0} \log^{\diamond} \left(\frac{(1 + \overset{\bullet}{\tilde{N}}(\omega, s, \frac{x'}{\hat{L}_{s-}})) \varphi(\frac{x'}{\hat{L}_{s-}})}{\varphi(x') \gamma(\hat{L}_{s-})} \right) N(\hat{\omega}, ds, dx') \right. \right. \\ & \quad \left. \left. \diamond \exp^{\diamond} \left\{ \int_0^T \int_{\mathbb{R}_0} \left(1 - \frac{(1 + \overset{\bullet}{\tilde{N}}(\omega, s, \frac{x'}{\hat{L}_{s-}})) \varphi(\frac{x'}{\hat{L}_{s-}})}{\varphi(x') \gamma(\hat{L}_{s-})} \right) \nu(dx') ds \right\} \right]. \quad (3.8) \end{aligned}$$

4 Explicit solution of a mixed Lévy-Itô diffusion

In the sequel we operate on the white noise space (2.19) and denote again by $\overset{\circ}{\omega}$ the copy mechanism to an auxiliary white noise space. This Section is devoted to the investigation of strong solutions of Lévy-Itô diffusion of the type

$$\begin{aligned} dY_t &= b(Y_{t-})dt + \sigma(Y_{t-})dB_t + \gamma(Y_{t-})dL_t \\ &= b(Y_{t-})dt + \sigma(Y_{t-})dB_t + \int_{\mathbb{R}_0} \gamma(Y_{t-})xN(dt, dx), \quad Y_0 = y, \quad 0 \leq t \leq T, \end{aligned} \quad (4.1)$$

where we impose the same conditions as in Section 3 on the coefficient γ and the pure jump Lévy process L_t , and where the coefficients b and σ are Lipschitz continuous of linear growth. In addition we restrict σ to be strictly positive. It is known that under these assumptions there exists a unique (global) strong solution Y_t of (4.1) and we can now state the analogue to Theorem 3.1:

Theorem 4.1 Suppose that the Lévy process L_t has only positive jumps. Let ρ be a Borel measurable function from \mathbb{R} to \mathbb{R} . Consider the strictly increasing function $\Lambda : \mathbb{R} \rightarrow \mathbb{R}$, given by

$$\Lambda(y) := \begin{cases} \int_x^y \frac{1}{\sigma(u)} du, & y > x \\ -\int_y^x \frac{1}{\sigma(u)} du, & y \leq x \end{cases},$$

and define the function $\gamma^*(y, x)$ as

$$\gamma^*(y, x) = \Lambda(\Lambda^{-1}(y) + \gamma(\Lambda^{-1}(y))x) - y,$$

which is invertible in $x > 0$ for all y .

Assume that the Bochner integral

$$E_{\hat{\mu}}^y \left[\rho \left(\Lambda^{-1}(\hat{B}_t + \hat{L}_t) \right) M_T^\diamond \diamond J_T^\diamond \right] \quad (4.2)$$

exists, where

$$\begin{aligned} M_T^\diamond &= \exp^\diamond \left\{ \int_0^T \left(W_s(\omega) + \left(\frac{b}{\sigma} \right) (\Lambda^{-1}(\hat{B}_t)) - \frac{1}{2} \sigma'(\Lambda^{-1}(\hat{B}_t))) \right) d\hat{B}_t - \right. \\ &\quad \left. \frac{1}{2} \int_0^T \left(W_s(\omega) + \left(\frac{b}{\sigma} \right) (\Lambda^{-1}(\hat{B}_t)) - \frac{1}{2} \sigma'(\Lambda^{-1}(\hat{B}_t))) \right)^{\diamond 2} ds \right\} \end{aligned}$$

and

$$\begin{aligned} J_T^\diamond &= \exp^\diamond \left\{ \int_0^T \int_{\mathbb{R}_0} \log^\diamond \left((1 + \dot{\tilde{N}}(\omega, s, \gamma^{*-1}(\hat{L}_{s-}, x'))) \right. \right. \\ &\quad \left. \left. \frac{\varphi(\gamma^{*-1}(\hat{L}_{s-}, x'))}{\varphi(x')} \frac{\partial \gamma^{*-1}}{\partial x'}(\hat{L}_{s-}, x') \right) N(\hat{\omega}, ds, dx') \right\} \\ &\diamond \exp^\diamond \left\{ \int_0^T \int_{\mathbb{R}_0} \left(1 - (1 + \dot{\tilde{N}}(\omega, s, \gamma^{*-1}(\hat{L}_{s-}, x'))) \right. \right. \\ &\quad \left. \left. \frac{\varphi(\gamma^{*-1}(\hat{L}_{s-}, x'))}{\varphi(x')} \frac{\partial \gamma^{*-1}}{\partial x'}(\hat{L}_{s-}, x') \right) \nu(dx') ds \right\}. \end{aligned}$$

Proof W.l.o.g. we set $\rho = id$. For $Z_t := \Lambda(Y_t)$ Itô's Lemma implies that

$$\begin{aligned} Z_t &= \left(\left(\frac{b}{\sigma} \right) (\Lambda^{-1}(Z_t)) - \frac{1}{2} \sigma'(\Lambda^{-1}(Z_t)) \right) dt + dB_t + \\ &\quad \int_{\mathbb{R}_0} (\Lambda(\Lambda^{-1}(Z_t) + \gamma(\Lambda^{-1}(Z_t))x) - \Lambda(\Lambda^{-1}(Z_t))) N(dt, dx) \\ &= b^*(Z_t)dt + dB_t + \int_{\mathbb{R}_0} \gamma^*(Z_t, x)N(dt, dx), \end{aligned}$$

where

$$b^*(y) = \left(\frac{b}{\sigma} \right) (\Lambda^{-1}(y)) - \frac{1}{2} \sigma'(\Lambda^{-1}(y))$$

and

$$\gamma^*(y, x) = \Lambda(\Lambda^{-1}(y) + \gamma(\Lambda^{-1}(y))x) - y.$$

Observe that $\gamma^*(y, x) \neq 0$ for all x, y and that for y given $\gamma^*(y, x)$ is invertible in $x > 0$.

To complete the proof it is sufficient to give the explicit representation for Z_t .

Taking the Hermite transform we get

$$\begin{aligned} &\mathcal{H}(Z_t)(z) \\ &= E_{\tilde{\mu}}^y \left[\tilde{Z}_t \mathcal{E} \left(\int_0^T \phi'_z(t) d\tilde{B}_t \right) \cdot \mathcal{E} \left(\int_0^T \int_{\mathbb{R}_0} \phi''_z(t, x) \tilde{N}(\tilde{\omega}, dt, dx) \right) \right], \end{aligned}$$

where $\phi'_z(s) = \sum_j z_k \xi_j(s, x)$ and $\phi''_z(s, x) = \sum_k z_k \delta_k(s, x)$, $z \in \mathbb{C}_c^{\mathbb{N}}$.

The rest of the proof goes the line of reasoning of Theorem 3.1. \square

5 Existence of strong solutions of Lévy-Itô diffusions with non-Lipschitz coefficients

In this Section we want to apply the explicit formula in Theorem 4.1 to examine the existence of strong solutions of Lévy -Itô diffusions, whose coefficients are not necessarily Lipschitzian and of linear growth. Under these conditions it is well-known that uniqueness of solutions is not ensured.

We present a novel approach for the study of strong solutions of Lévy -Itô diffusions. This method is constructive in the sense that we consider a

concrete distributional object in the Lévy-Hida space, which we verify to be a strong solution of a diffusion.

For convenience we confine ourselves to the study of the pure jump diffusion (3.1)

$$\begin{aligned} dY_t &= \gamma(Y_{t-})dL_t \\ &= \int_{\mathbb{R}_0} \gamma(Y_{t-})xN(dt, dx), \quad Y_0 = y, \quad 0 \leq t \leq T. \end{aligned}$$

Before we come to the main result of the Section we give the following Lemma:

Lemma 5.1 Consider the Doleans-Dade exponential $\mathcal{E}(M_t)$, where the (local) martingale M_t is given by

$$M_t(\omega) = \int_0^t \int_{\mathbb{R}_0} \left\{ \frac{\varphi\left(\frac{x}{\gamma(L_{s-})}\right)}{\varphi(x)\gamma(L_{s-})} - 1 \right\} \tilde{N}(\omega, ds, dx).$$

Assume that $\mathcal{E}(M_t)$ is a martingale, which belongs to $L^p(\mu)$ for some $p > 1$. In addition suppose that

$$\int_0^T \|\gamma(L_s)\|_{L^{2p/(p-1)}(\mu)}^2 ds < \infty.$$

Then the process Y_t defined by

$$Y_t(\omega) = E_{\hat{\mu}}^y \left[\widehat{L}_t J_T^\circ \right]$$

is contained in $L^2(\mu)$ for all t , where J_T° is as in (4.2). Moreover, for a given measurable function $\rho: \mathbb{R} \rightarrow \mathbb{R}$ with $\rho(Y_t) \in L^2(\mu)$ the following property holds:

$$\rho(Y_t) = E_{\hat{\mu}}^y \left[\rho(\widehat{L}_t) J_T^\circ \right], \quad (5.1)$$

provided the Bochner expectation on the right hand side of the last relation exists.

Proof For convenience we subdivide the proof into two steps.

(i) We want to show that $Y_t \in L^2(\mu)$ for all t : First we observe that given the integrability assumption on γ , the Lévy process L_t itself is a weak

solution of (3.1) by Girsanov's theorem. More precisely, L_t is a solution of the equation

$$dL_t = \gamma(L_{t-})dL_t^*$$

on the probability space

$$(\Omega, \mathcal{F}^*, \mu^*)$$

where the measure μ^* is given by $d\mu^* = \mathcal{E}(M_t) d\mu$ and where L_t^* is a Lévy process under μ^* with same characteristics as the original Lévy process. Using the integrability assumption on $\mathcal{E}(M_t)$ and Hölder's inequality we see that L_t is μ^* -square integrable. By applying Corollary 3.3 (on a different white noise space) we obtain that L_t can be represented by

$$L_t = E_{\hat{\mu}^*}^y \left[\widehat{L}_t^* J_T^{\diamond*} \right], \quad (5.2)$$

where \widehat{L}_t^* is a Lévy process with same characteristics on a copy $(\widehat{\Omega}, \widehat{\mathcal{F}}^*, \widehat{\mu}^*)$ of the white noise space w.r.t. μ^* and where $J_T^{\diamond*}$ is as J_T^{\diamond} with the μ^* -Wick product \diamond^* . To ensure that the right hand side of (5.2) is well defined, we have to verify that $\widehat{L}_t^* J_T^{\diamond*}$ is Bochner integrable. For this purpose it is sufficient to show that

$$E_{\hat{\mu}^*}^y \left[\sup_{z \in \mathbb{K}_q(R)} \left| \mathcal{H}(\widehat{L}_t^* J_T^{\diamond*})(z) \right| \right] < \infty$$

for some q, R . We find the estimate

$$\sup_{z \in \mathbb{K}_q(R)} |\phi_z(s, x)| \leq R \left\| \overset{\bullet}{\widetilde{N}}(s, x) \right\|_{0, -2} ds \times \nu - \text{a.e.},$$

where $\phi_z(s, x)$ denotes the Hermite transform of $\overset{\bullet}{\widetilde{N}}(s, x)$ (the white noise of the jump measure of \widehat{L}_t^*) and where $\|\cdot\|_{0, -2}$ is the norm for distributions, which corresponds to the norm $\|\cdot\|_{0, 2}$ (see Section 2). So we get in connection with substitution that

$$\begin{aligned} & E_{\hat{\mu}^*}^y \left[\sup_{z \in \mathbb{K}_q(R)} \left| \mathcal{H}(\widehat{L}_t^* J_T^{\diamond*})(z) \right| \right] \\ & \leq \text{const.} E_{\hat{\mu}^*}^y \left[\left| \widehat{L}_t^* \right| \mathcal{E} \left(\int_0^T \int_{\mathbb{R}_0} \left\{ \left(1 + R \left\| \overset{\bullet}{\widetilde{N}} \left(s, \frac{x}{\gamma(\widehat{L}_{s-}^*)} \right) \right\|_{0, -2} \right) \right\} \right) \right] \end{aligned}$$

$$\begin{aligned} & \left. \frac{\varphi\left(\frac{x}{\gamma(\widehat{L}_{s-}^*)}\right)}{\varphi(x)\gamma(\widehat{L}_{s-}^*)} - 1 \right\} \widetilde{N}(\widehat{\omega}, ds, dx) \Bigg) \\ & \exp \left\{ \int_0^T \int_{\mathbb{R}_0} \left\{ \left(1 + R \left\| \overset{\bullet}{\widetilde{N}}\left(s, \frac{x}{\gamma(\widehat{L}_{s-}^*)}\right) \right\|_{0,-2} \right) \right. \right. \\ & \left. \left. \frac{\varphi\left(\frac{x}{\gamma(\widehat{L}_{s-}^*)}\right)}{\varphi(x)\gamma(\widehat{L}_{s-}^*)} - 1 \right\} \nu(dx) ds \right\}. \end{aligned}$$

This implies

$$\begin{aligned} & E_{\widehat{\mu}^*}^y \left[\sup_{z \in \mathbb{K}_q(R)} \left| \mathcal{H}(\widehat{L}_t^* J_T^{\circ*})(z) \right| \right] \\ & \leq \text{const.} E_{\widehat{\mu}^*}^y \left[\left| \widehat{L}_{s-}^* \right| \mathcal{E} \left(\int_0^T \int_{\mathbb{R}_0} \left\{ \left(1 + R \left\| \overset{\bullet}{\widetilde{N}}\left(s, \frac{x}{\gamma(\widehat{L}_{s-}^*)}\right) \right\|_{0,-2} \right) \right. \right. \right. \\ & \left. \left. \frac{\varphi\left(\frac{x}{\gamma(\widehat{L}_{s-}^*)}\right)}{\varphi(x)\gamma(\widehat{L}_{s-}^*)} - 1 \right\} \widetilde{N}(\widehat{\omega}, ds, dx) \right) \right]. \end{aligned}$$

So by Girsanov's theorem and the assumption on $\mathcal{E}(M_t)$ we obtain that

$$\begin{aligned} & E_{\widehat{\mu}^*}^y \left[\sup_{z \in \mathbb{K}_q(R)} \left| \mathcal{H}(\widehat{L}_t^* J_T^{\circ*})(z) \right| \right] \\ & \leq \text{const.} E_{\widehat{\mu}^*}^y \left[\left| \widehat{L}_t \right| \mathcal{E} \left(\int_0^T \int_{\mathbb{R}_0} \left(R \left\| \overset{\bullet}{\widetilde{N}}(s, x) \right\|_{0,-2} \right) \widetilde{N}(\widehat{\omega}, ds, dx) \right) \right] \\ & < \infty. \end{aligned}$$

We recall that \widehat{L}_t is a solution of

$$d\widehat{L}_t = \gamma(\widehat{L}_{t-}) d\widehat{L}_t^*$$

on the space $(\widehat{\Omega}, \widehat{\mathcal{F}}^*, \widehat{\mu}^*)$. Thus the Bochner integrability follows.

Since $\widehat{L}_t^* J_T^{\circ*}$ is Bochner integrable and has the same Fourier coefficients as $\widehat{L}_t J_T^{\circ}$, we conclude from relation (3.3) that $Y_t \in L^2(\mu)$ for all t .

(ii) We give the proof of the last statement of the Lemma:

Next let us write J_T^\diamond as

$$J_T^\diamond = G_T^\diamond \mathcal{E}(M_t),$$

where G_T^\diamond is the remaining factor. Thus it follows that

$$\mathcal{H}(\rho(Y_t))(z) = \mathcal{H}(\rho(E_\mu^y \left[\widehat{L}_t G_T^\diamond \mathcal{E}(M_t) \right]))(z)$$

The Girsanov transform implies

$$\mathcal{H}(\rho(Y_t))(z) = \mathcal{H}(\rho(E_\mu^y \left[\widehat{Y}_t R_T^\diamond \right]))(z),$$

where

$$\begin{aligned} & R_T^\diamond \\ = & \exp^\diamond \left\{ \int_0^T \int_{\mathbb{R}_0} \left\{ \log^\diamond \left(1 + \dot{\tilde{N}}(\omega, s, x) \right) \right\} N(\widehat{\omega}, ds, dx) \right. \\ & \left. - \int_0^T \int_{\mathbb{R}_0} \dot{\tilde{N}}(\omega, s, x) \nu(dx) ds \right\}. \end{aligned}$$

and where the law \widehat{Y}_t is induced by the Girsanov transformation $\mathcal{E}(M_t)$. Actually, this process is a weak solution of (3.1). Therefore we get that

$$\begin{aligned} \mathcal{H}(\rho(Y_t))(z) &= E_\mu^y \left[\rho(\widehat{Y}_t) \mathcal{H}(R_T^\diamond)(z) \mathcal{E}(M_t) \mathcal{E}^{-1}(M_t) \right] \\ &= E_\mu^y \left[\rho(L_t) \mathcal{H}(J_T^\diamond)(z) \right]. \end{aligned}$$

Extracting the Hermite transform gives (ii) and completes the proof. \square

We state the main result of this Section.

Theorem 5.2 Assume again that the Doleans-Dade exponential $\mathcal{E}(M_t)$ in Lemma 5.1 is a L^p -integrable martingale for some $p > 1$ and that

$$\int_0^T \|\gamma(L_s)\|_{L^{2p/(p-1)}(\mu)}^2 ds < \infty.$$

Then Y_t in Lemma 5.1 is a square integrable càdlàg-adapted process, which solves the SDE (3.1) in the strong sense.

Proof We know that

$$\mathcal{H}(Y_t)(z) = E_{\mu^*}^y \left[\widehat{L}_t \mathcal{E} \left(\int_0^T \int_{\mathbb{R}_0} \phi_z(s, x) \tilde{N}(\widehat{\omega}, ds, dx) \right) \right],$$

Since \widehat{L}_t is a (strong) solution of (3.1) under the measure $\widehat{\mu}^*$, we can write

$$\begin{aligned} & \mathcal{H}(Y_t)(z) \\ &= E_{\widehat{\mu}^*}^y \left[\int_0^t \int_{\mathbb{R}_0} \gamma(\widehat{L}_{s-}) N(\widehat{\omega}, ds, dx) \mathcal{E} \left(\int_0^T \int_{\mathbb{R}_0} \phi_z(s, x) \widetilde{N}(\widehat{\omega}, ds, dx) \right) \right]. \end{aligned}$$

Using relation (2.12) yields

$$\begin{aligned} & \mathcal{H}(Y_t)(z) \\ &= E_{\widehat{\mu}^*}^y \left[\int_0^t \int_{\mathbb{R}_0} \gamma(\widehat{L}_{s-}) \diamond \left(1 + \overset{\bullet}{\widetilde{N}}(\widehat{\omega}, s, x) \right) \nu(dx) ds \right. \\ & \quad \left. \mathcal{E} \left(\int_0^T \int_{\mathbb{R}_0} \phi_z(s, x) \widetilde{N}(\widehat{\omega}, ds, dx) \right) \right] \\ &= \int_0^t \int_{\mathbb{R}_0} \mathcal{H}^*(\gamma(\widehat{L}_{s-}))(z) (1 + \phi_z(s, x)) \nu(dx) ds, \end{aligned}$$

where \mathcal{H}^* denotes the Hermite transform w.r.t. to the measure $\widehat{\mu}^*$. So it follows from the square integrability of Y_t and with $\rho = \gamma$ in Lemma 5.1 that $\mathcal{H}^*(\gamma(\widehat{L}_t))(z) = \mathcal{H}(\gamma(Y_t))(z)$ and we get

$$\mathcal{H}(Y_t)(z) = \int_0^t \int_{\mathbb{R}_0} \mathcal{H}(\gamma(Y_{s-}))(z) (1 + \phi_z(s, x)) \nu(dx) ds.$$

Taking relation (2.12) again leads to

$$\begin{aligned} & \mathcal{H}(Y_t)(z) \\ &= \mathcal{H} \left(\int_0^t \int_{\mathbb{R}_0} \gamma(Y_{s-}) \diamond \left(1 + \overset{\bullet}{\widetilde{N}}(\omega, s, x) \right) \nu(dx) ds \right) \\ &= \mathcal{H} \left(\int_0^t \int_{\mathbb{R}_0} \gamma(Y_{s-}) N(\omega, ds, dx) \right). \end{aligned}$$

Using the inverse Hermite transform gives

$$Y_t = \int_0^t \int_{\mathbb{R}_0} \gamma(Y_{s-}) N(\omega, ds, dx).$$

Note that Y_t possesses a càdlàg version in virtue of the last statement of Lemma 5.1 and Kolmogorov's continuity theorem. Altogether the proof follows. \square

Corollary 5.3 If the condition

$$M \geq \frac{\varphi\left(\frac{x}{\gamma(L_{s-}(\omega))}\right)}{\varphi(x)\gamma(L_{s-}(\omega))} > 0 \quad \mu \times \nu \times ds - \text{a.e.}$$

holds and if γ is bounded, then there exists a strong $L^2(\mu)$ -solution of (3.1), which has the explicit form (3.3).

A similar result to Theorem 5.2 holds in the mixed diffusion case.

Theorem 5.4 Retain assumptions and notation of Theorem 4.1. Define the (local) martingale C_t by

$$C_t = \int_0^t \left(\left(\frac{b}{\sigma} \right) (\Lambda^{-1}(B_t)) - \frac{1}{2} \sigma'(\Lambda^{-1}(B_t)) \right) dB_t.$$

and the purely discontinuous (local) martingale M_t as

$$M_t = \int_0^t \int_{\mathbb{R}_0} \left(\frac{\varphi(\gamma^{*-1}(L_{s-}, x))}{\varphi(x)} \frac{\partial \gamma^{*-1}}{\partial x}(L_{s-}, x) - 1 \right) \tilde{N}(\omega, ds, dx).$$

Assume that $\mathcal{E}(C_t + M_t) = \mathcal{E}(C_t)\mathcal{E}(M_t)$ is a Doleans-Dade martingale, which is $L^p(\mu)$ -integrable for some $p > 1$.

Further suppose that

$$\begin{aligned} & \int_0^T \left(\|b(\Lambda^{-1}(B_s + L_s))\|_{L^{2p/(p-1)}(\mu)}^2 + \|\sigma(\Lambda^{-1}(B_s + L_s))\|_{L^{2p/(p-1)}(\mu)}^2 \right. \\ & \quad \left. + \|\gamma(\Lambda^{-1}(B_s + L_s))\|_{L^{2p/(p-1)}(\mu)}^2 \right) ds \\ & < \infty. \end{aligned}$$

Then Y_t in (4.2) is a strong $L^2(\mu)$ -solution of the Lévy-Itô diffusion (4.1).

Proof The proof follows the same arguments as in Theorem 5.2. \square

Corollary 5.5 Consider in Theorem 5.4 the special case $\sigma = 1$ and $\gamma = 0$, that is the SDE

$$dY_t = b(Y_t)dt + dB_t; \quad Y_0 = y. \quad (5.3)$$

Suppose that

$$E^y \left[\exp \left\{ 4p \int_0^{T+1} b^2(B_s) ds \right\} \right] < \infty. \quad (5.4)$$

Then there exists a strong $L^p(\mu)$ -solution ($p \geq 2$) of the form (4.2).

Proof The proof is based on Theorem 5.4 and the following inequality for continuous local martingales X_t :

$$E[\exp\{|X_t|\}] \leq KE \left[\exp \left\{ 2 \langle X \rangle_t^{\frac{1}{2}} \right\} \right],$$

where K is a constant. Actually, this inequality is a consequence of the Burkholder inequality. \square

Remark 5.6 It is possible to guarantee condition (5.4) by assumptions only in terms of integrability conditions on the coefficient b . More precisely it is enough to require $b \in L^2(\mathbb{R})$ to guarantee the strong solution in Corollary 5.5.

6 Some additional results

In this Section we discuss some additional results, concerning the explicit formula of the pure Brownian motion diffusion. For the pure jump or mixed case analogous results are more heavily to deduce and don't reveal the same insight as they do in the Gaussian case.

First we demonstrate how the formula (1.1) can be used to derive an explicit chaos expansion of the diffusion Y_t . To ease calculations we choose from now on an orthonormal basis $\{\eta_i\}_{i \geq 1}$ of $L^2(0, T)$ instead of the Hermite functions $\{\xi_i\}_{i \geq 1}$ in the definition of \mathcal{H}_α (see Section 2). We are allowed to do this, since the Wick product is invariant under the change of such basis elements (see [HØUZ]).

Proposition 5.1 Let the solution Y_t be in $L^2(\mu)$ and given by formula (1.1). Then Y_t can be represented as

$$Y_t = \sum_{\gamma \in \mathcal{J}} c_\gamma(t) \mathcal{H}_\gamma$$

with explicit Fourier coefficients

$$c_\gamma(t) = E_{\hat{\mu}}^y \left[\Lambda^{-1}(\hat{B}_t) \mathcal{E} \left(\int_0^T \vartheta_s(\hat{\omega}) d\hat{B}_s \right) \sum_{\alpha+\beta=\gamma} \frac{1}{\alpha! \beta!} (-1)^{|\beta|} (\vartheta \cdot (\hat{\omega}), \eta)^\beta \mathcal{H}_\alpha(\hat{\omega}) \right],$$

where

$$\vartheta_s(\widehat{\omega}) := \frac{b(\Lambda^{-1}(\widehat{B}_t))}{\sigma(\Lambda^{-1}(\widehat{B}_t))} - \frac{1}{2}\sigma'(\Lambda^{-1}(\widehat{B}_t))$$

and where

$$(\vartheta \cdot (\widehat{\omega}), \eta)^\beta = \prod_{i \geq 1} (\vartheta \cdot (\widehat{\omega}), \eta_i)_{L^2}^{\beta_i}.$$

Proof By using the Hermite transform it can be checked that

$$\begin{aligned} & \exp^\diamond \left\{ \int_0^T W_s(\omega) d\widehat{B}_s(\widehat{\omega}) - \frac{1}{2} \int_0^T W_s^{\diamond 2}(\omega) ds \right\} \\ &= \sum_{\alpha} \frac{1}{\alpha!} \mathcal{H}_\alpha(\widehat{\omega}) \mathcal{H}_\alpha(\omega). \end{aligned}$$

In addition we find

$$\begin{aligned} \exp^\diamond \left\{ \int_0^T \vartheta_s(\widehat{\omega}) W_s(\omega) ds \right\} &= \exp^\diamond \left\{ \int_0^T \vartheta_s(\widehat{\omega}) dB_s(\omega) \right\} \\ &= \sum_{\alpha+\beta=\gamma} \frac{1}{\beta!} (\vartheta \cdot (\widehat{\omega}), \eta)^\beta \mathcal{H}_\beta(\omega). \end{aligned}$$

Thus the statement of the theorem follows from (1.1). \square

Finally, we state a representation of (1.1), which gives a different insight into the structure of the solution, reflecting an interesting symmetry in the interplay between the driving process $B_t(\omega)$ and its copy $\widehat{B}_s(\widehat{\omega})$. Also, the role of the Wick product as the essential operation becomes evident.

Proposition 5.2 In addition to the conditions on the coefficients in (1.1) require that the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, defined by

$$\varphi(x) = \frac{b(\Lambda^{-1}(x))}{\sigma(\Lambda^{-1}(x))} - \frac{1}{2}\sigma'(\Lambda^{-1}(x))$$

as well as Λ^{-1} are real analytic.

Then the solution Y_t in (1.1) can be rewritten as

$$\begin{aligned} & Y_t \\ &= E_{\widehat{\mu}}^y \left[\Lambda^{\diamond -1}(\widehat{B}_s + B_s) \exp^\diamond \left\{ \int_0^T \varphi^\diamond(\widehat{B}_s + B_s) d\widehat{B}_s - \frac{1}{2} \int_0^T \varphi^{\diamond 2}(\widehat{B}_s + B_s) ds \right\} \right]. \end{aligned}$$

Proof We have that

$$\begin{aligned} & \mathcal{H}(Y_t)(z) \\ = & E_{\hat{\mu}}^y \left[\Lambda^{-1}(\widehat{B}_t) \exp \left\{ \int_0^T \varphi(\widehat{B}_s) d\widehat{B}_s \right. \right. \\ & \left. \left. - \frac{1}{2} \int_0^T \varphi^2(\widehat{B}_s) ds - \int_0^T \varphi(\widehat{B}_s) \phi_z(s) ds \right\} \mathcal{E} \left(\int_0^T \phi_z(s) d\widehat{B}_s \right) \right], \end{aligned}$$

where ϕ_z is the Hermite transform of the white noise. So the application of the Girsanov transform to the process $\phi_z(s)$ and the use of the inverse Hermite transform provide the proof. \square

Acknowledgements The authors thank Professor A. Cherny for suggestions and valuable comments.

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