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Backward Stochastic Partial Differential Equations with Jumps and Application to Optimal Control of Random Jump Fields

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Abstract

We prove an existence and uniqueness result for a general class of backward stochastic partial differential equations with jumps. This is a type of equations which appear as adjoint equations in the maximum principle approach to optimal control of systems described by stochastic partial differential equations driven by Lévy processes.

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1 Introduction

Let $B_t, t \geq 0$ and $\eta(t) = \int_0^t \int_{\mathbb{R}^n} z\widetilde{N}(ds, dz)$; $t \geq 0$ be an m-dimensional Brownian motion and a pure jump Lévy process, respectively, on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Fix $T > 0$ and let $\phi(\omega)$ be an $\mathcal{F}_T$-measurable random variable. Let

$$b : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times m} \to \mathbb{R}^n$$

be a given vector field. Consider the problem to find three $\mathcal{F}_t$-adapted processes $p(t) \in \mathbb{R}^n, q(t) \in \mathbb{R}^{n\times m}$ and $r(t, z) \in \mathbb{R}^{n\times m}$ such that

$$dp(t) = b(t, p(t), q(t))dt + q(t)dB_t + \int_{\mathbb{R}^n} r(t, z)\widetilde{N}(ds, dz), t \in (0, T)1.1 \qquad (1.1)$$

$$p(T) = \phi \quad a.s.1.2 \qquad (1.2)$$

This is a backward stochastic (ordinary) differential equation (BSDE). It is called backward because it is the terminal value $p(T) = \phi$ that is given, not the initial value $p(0)$. Still $p(t)$ is required to be $\mathcal{F}_t$-adapted. In general this is only possible if we also are free to choose $q(t)$ and $r(t, z)$ (in an $\mathcal{F}_t$-adapted way).

The theory of BSDEs, when $\eta = 0$, is now well developed. See e.g. [EPQ], [MY], [PP1], [PP2] and [YZ] and the references therein. In the jump case ($\eta \neq 0$) BSDE's have been studied. See [FØS], [NS], [S] and the references therein.

There are many applications of this theory. Examples include the following:

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(i) The problem of finding a replicating portfolio of a given contingent claim in a complete financial market can be transformed into a problem of solving a BSDE.

(ii) The maximum principle method for solving a stochastic control problem involves a BSDE for the adjoint processes $p(t), q(t), r(t,z)$.

For more information about these and other applications of BSDEs we refer to [EPQ] and [YZ] and references therein.

The purpose of this paper is to study backward stochastic *partial* differential equations (BSPDEs) with jumps. They are defined in a similar way as BSDEs, but with the basic equation being a stochastic partial differential equation rather than a stochastic ordinary differential equation. More precisely, we will study a class of BSPDEs which includes the following:

Find adapted processes $Y(t,x)$, $Z(t,x)$, $Q(t,x,z)$ such that

$$dY(t,x) = AY(t,x)dt + b(t,x,Y(t,x),Z(t,x))dt + Z(t,x)dB_t + \int_{\mathbb{R}^n} Q(t,x,z)\widetilde{N}(dt,dz), (t,x) \in (0,T)\times\mathbb{R}^n \tag{1.3}$$

$$Y(T,x) = \phi(x,\omega) \tag{1.4}$$

Here $dY(t,x)$ denotes the Itô differential with respect to t, while A is a partial differential operator with respect to x and $\widetilde{N}(dt,dz)$ is the compensated Poisson random measure associated with a Lévy process $\eta(\cdot)$.

The function $b : [0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is given and so is the terminal value function $\phi(x) = \phi(x,\omega)$. We assume that $\phi(x)$ is $\mathcal{F}_T$-measurable for all x and that

$$E[\int_{\mathbb{R}^n} \phi(x)^2 dx] < \infty, \tag{1.5}$$

where E denotes expectation with respect to P. We are seeking the processes $Y(t,x)$, $Z(t,x)$ and $Q(t,x,z)$ such that (1.3) and (1.4) hold. The processes $Y(t,x)$, $Z(t,x)$ and $Q(t,x,z)$ are required to be $\mathcal{F}_t$-adapted, i.e., $Y(t,x)$, $Z(t,x)$ and $Q(t,x,z)$ are $\mathcal{F}_t$-measurable for all $x\in\mathbb{R}^n$, $z\in\mathbb{R}$ and we also require that

$$E\left[\int_{\mathbb{R}^n}\int_0^T \{Y(t,x)^2 + Z(t,x)^2 + \int_{\mathbb{R}^n} Q(t,x,z)^2\nu(dz)\}dt\,dx\right] < \infty, \tag{1.6}$$

where $\nu(\cdot)$ is the Lévy measure of the underlying Lévy process. Equations of this type are of interest because they appear as adjoint equations in a maximum principle approach to optimal control of stochastic partial differential equations. See Section 2.

Example 1.1 *Consider the following BSPDE:*

$$dY(t,x) = -\tfrac{1}{2}\Delta Y(t,x)dt + Z(t,x)dB_t + \int_{\mathbb{R}^n} Q(t,x,z)\widetilde{N}(dt,dz), (t,x)\in(0,T)\times\mathbb{R}^n \tag{1.7}$$

$$Y(T,x) = \phi(x) \tag{1.8}$$

Here $\Delta Y(t,x) = \sum_{i=1}^n \frac{\partial^2 Y(t,x)}{\partial x_i^2}$ *is the Laplacian with respect to* x *applied to* $Y(t,x)$, *and* $\phi(x)$ *satisfies* $E[\int_{\mathbb{R}^n}\phi(x)^2dx] < \infty$.

In this simple case, we are able to find the solution explicitly:

We first use the Itô representation theorem to write, for almost all x,

$$\phi(x) = h(x) + \int_0^T g(s,x,\omega)dB_s + \int_0^T \int_{\mathbb{R}^n} k(s,x,z,\omega)\widetilde{N}(ds,dz)$$

where

$$h(x) = E[\phi(x)],$$

$g(s,x,\cdot)$ and $k(s,x,z)$ are $\mathcal{F}_s$-measurable for all s, x and

$$E[\int_{\mathbb{R}^n} \int_0^T \left\{ g^2(s,x,\cdot) + \int_{\mathbb{R}^n} k^2(s,x,z)\nu(dz) \right\} dsdx] < \infty.$$

Let

$$R_t f(x) = \int_{\mathbb{R}^n} (2\pi t)^{-\frac{n}{2}} f(y) exp(-\frac{|x-y|^2}{2t})dy, \qquad t > 0$$

be the transition operator for Brownian motion defined for all measurable $f : R^n \to R$ such that the integral converges. Then it is well known that

$$\frac{\partial}{\partial t}(R_t f(x)) = \tfrac{1}{2}\Delta(R_t f(x)) \tag{1.9}$$

Now define

$$\begin{aligned} Y(t,x) &= R_{T-t}(\int_0^t g(s,\cdot,\omega)dB_s + \int_0^t \int_{\mathbb{R}^n} k(s,\cdot,z,\omega)\widetilde{N}(ds,dz) + h(\cdot))(x) \\ &= \int_0^t (R_{T-t}g(s,\cdot,\omega))(x)dB_s + \int_0^t \int_{\mathbb{R}^n} (R_{T-t}k(s,\cdot,z,\omega))(x)\widetilde{N}(ds,dz) + (R_{T-t}h)(x) \end{aligned} \tag{1.10}$$

Then

$$\begin{aligned} &dY(t,x) \\ &= \left[\int_0^t -\tfrac{1}{2}\Delta(R_{T-t}g(s,\cdot,\omega))(x)dB_s - \tfrac{1}{2}\int_0^t \int_{\mathbb{R}^n} \Delta(R_{T-t}k(s,\cdot,z,\omega))(x)\widetilde{N}(ds,dz) - \tfrac{1}{2}\Delta R_{T-t}h(\cdot)(x) \right] dt \\ &\qquad + (R_{T-t}g(t,\cdot,\omega))(x)dB_t + \int_{\mathbb{R}^n} R_{T-t}(k(t,\cdot,z,\omega))(x)\widetilde{N}(dt,dz) \\ &= -\tfrac{1}{2}\Delta Y(t,x)dt + Z(t,x)dB_t + \int_{\mathbb{R}^n} Q(t,x,z)\widetilde{N}(dt,dz), \end{aligned}$$

where

$$Z(t,x) = (R_{T-t}g(t,\cdot,\omega))(x) \tag{1.11}$$

and

$$Q(t,x,z) = (R_{T-t}k(t,\cdot,z))(x). \tag{1.12}$$

Hence the processes $Y(t,x), Z(t,x)$ and $Q(t,x,z)$ given by (1.10)–(1.12) solve the BSPDE (1.7)–(1.8).

In the general case it is not possible to find explicit solutions of a BSPDE. However, in Section 3 we will prove an existence and uniqueness result for a general class of such equations. We will achieve this by regarding the BSPDE of type (1.3)–(1.4) as a special case of a backward stochastic evolution equation for Hilbert space valued processes. This, in turn, is studied by taking finite dimensional projections and then taking the limit. This is the well known Galerkin approximation method which has been used by several authors in other connections. See e.g. [B1], [B2] and [P]. We also refer readers to [PZ] for the general theory of stochastic evolution equations on Hilbert spaces.

The rest of the paper is organized as follows: In Section 2 we prove a (sufficient) maximum principle for optimal control of random jump fields, i.e. solutions of SPDE's driven by Lévy processes (Theorem 2.1). This principle involves a BSPDE of the form (1.3)-(1.4) in the associated adjoint processes. In Section 3 we give the precise framework of our general existence and uniqueness result. The existence and uniqueness result and its proof are given in Section 4.

2 The stochastic maximum principle

In this section we prove a verification theorem for optimal control of a process described by a stochastic partial differential equation (SPDE) driven by a Brownian motion $B(t)$ and a Poisson random measure $\widetilde{N}(dt,dz)$. We call such a process a (controlled) *random jump field.* The verification theorem has the form of a sufficient stochastic maximum principle and the adjoint equation for this principle turns out to be a backward SPDE driven by $B(\cdot)$ and $\widetilde{N}(\cdot,\cdot)$. This part of the paper is an extension of [Ø] to the case including jumps and an extension of [FØS] to SPDE control.

Let $\Gamma(t,x) = \Gamma^{(u)}(t,x)$; $t \in [0,T]$, $x \in \mathbb{R}^k$ be the solution of a (controlled) stochastic reaction-diffusion equation of the form

$$\begin{aligned} d\Gamma(t,x) &= [(L\Gamma)(t,x) + b(t,x,\Gamma(t,x),u(t,x))]\,dt + \sigma(t,x,\Gamma(t,x),u(t,x))dB(t) \\ &\quad + \int_{\mathbb{R}} \theta(t,x,\Gamma(t,x),u(t,x),z)\widetilde{N}(dt,dz); (t,x) \in [0,T]\times G \qquad (2.1) \\ \Gamma(0,x) &= \xi(x); x \in \overline{G} \qquad (2.2) \\ \Gamma(t,x) &= \eta(t,x); (t,x) \in (0,T)\times \partial G \qquad (2.3) \end{aligned}$$

Here $d\Gamma(t,x) = d_t\Gamma(t,x)$ is the differential with respect to t and $L = L_x$ is a given partial differential operator of order m acting on the variable $x \in \mathbb{R}^k$. We assume that $G \subset \mathbb{R}^k, \mho \subset \mathbb{R}^l$ are given open and closed sets, respectively, and that $b : [0,T]\times G\times\mathbb{R}\times\mho \to \mathbb{R}$, $\sigma : [0,T]\times G\times\mathbb{R}\times\mho \to \mathbb{R}$, $\theta : [0,T]\times G\times\mathbb{R}\times\mho\times\mathbb{R} \to \mathbb{R}$, $\xi : \overline{G} \to \mathbb{R}$ and $\eta : (0,T)\times\partial G \to \mathbb{R}$ are given measurable functions. The process

$$u : [0,T]\times G \to \mho$$

is called an *admissible* control if the equation (2.1)-(2.3) has a unique continuous solution $\Gamma(t,x) = \Gamma^{(u)}(t,x)$ which satisfies

$$E\left[\int_0^T \left(\int_G |f(t,x,\Gamma(t,x),u(t,x))|\,dx\right)dt + \int_G |g(x,\Gamma(T,x))|\,dx\right] < \infty, \qquad (2.4)$$

where $f:[0,T]\times G\times\mathbb{R}\times\mho\to\mathbb{R}$ and $g:G\times\mathbb{R}\to\mathbb{R}$ are given C^1 functions. The set of all admissible controls is denoted by $\mathcal{A}$. If $u\in\mathcal{A}$ we define the *performance* of u, $J(u)$, by

$$J(u)=E\left[\int_0^T\left(\int_G f(t,x,\Gamma(t,x),u(t,x))dx\right)dt+\int_G g(x,\Gamma(T,x))dx\right]. \tag{2.5}$$

We consider the problem to find $J^*\in\mathbb{R}$ and $u^*\in\mathcal{A}$ such that

$$J^*:=\sup_{u\in\mathcal{A}}J(u)=J(u^*). \tag{2.6}$$

J^* is called the *value* of the problem and $u^*\in\mathcal{A}$ (if it exists) is called an *optimal* control.

In the following we let L^* denote the *adjoint* of the operator L, defined by

$$(L^*\,\varphi_1,\varphi_2)=(\varphi_1,L\varphi_2)\,;\;\varphi_1,\varphi_2\in C_0^\infty(G), \tag{2.7}$$

where $(\varphi,\psi)=(\varphi,\psi)_{L^2(G)}=\int_G\varphi(x)\psi(x)dx$ and $C_0^\infty(G)$ is the set of infinitely many times differentiable functions with compact support in G.

We now formulate a (sufficient) stochastic maximum principle for the problem (2.6):

Let $\mathcal{R}$ denote the set of functions $r:[0,T]\times G\times\mathbb{R}\times\Omega\to\mathbb{R}$.

Define the *Hamiltonian*

$$H:[0,T]\times G\times\mathbb{R}\times\mho\times\mathbb{R}\times\mathbb{R}\times\mathcal{R}\to\mathbb{R}$$

by

$$\begin{aligned}&H(t,x,\gamma,u,p,q,r(t,x,\cdot))\\ =\;&f(t,x,\gamma,u)+b(t,x,\gamma,u)p+\sigma(t,x,\gamma,u)q+\int_{\mathbb{R}}\theta(t,x,\gamma,u,z)r(t,x,z)\nu(dz)2.8\end{aligned} \tag{2.3}$$

For each $u\in\mathcal{A}$ consider the following adjoint backward SPDE in the 3 unknown adapted processes $p(t,x)$, $q(t,x)$ and $r(t,x,z)$:

$$\begin{aligned}dp(t,x)=&-\{\frac{\partial H}{\partial\gamma}(t,x,\Gamma^{(u)}(t,x),u(t,x),p(t,x),q(t,x),r(t,x,\cdot))\\ &+L^*p(t,x)\}dt+q(t,x)dB(t)+\int_{\mathbb{R}}r(t,x,z)\widetilde{N}(dt,dz);t<T.\end{aligned} \tag{2.9}$$

$$p(T,x)=\frac{\partial g}{\partial\gamma}(x,\Gamma^{(u)}(t,x));\;x\in\overline{G} \tag{2.10}$$

$$p(t,x)=0;\;(t,x)\in(0,T)\times\partial G. \tag{2.11}$$

The following result may be regarded as a synthesis of Theorem 2.1 in [Ø] and Theorem 2.1 in [FØS]:

Theorem 2.1 ***(Sufficient SPDE maximum principle for optimal control of reaction-diffusion jump fields)***

Let $\widehat{u} \in \mathcal{A}$ with corresponding solution $\widehat{\Gamma}(t,x)$ of (2.1)-(2.3) and let $\widehat{p}(t,x), \widehat{q}(t,x), \widehat{r}(t,x,\cdot)$ be a solution of the associated adjoint backward SPDE (2.9)-(2.11). Suppose the following, (2.12)-(2.15), holds:

$$\begin{aligned} & \textit{The functions}2.12 \\ (\gamma,u) &\mapsto H(\gamma,u) := H(t,x,\gamma,u,\widehat{p}(t,x),\widehat{q}(t,x),\widehat{r}(t,x,\cdot));\ (\gamma,u)\in\mathbb{R}\times\mho \\ & \textit{and} \\ \gamma &\mapsto g(x,\gamma);\ \gamma\in\mathbb{R} \\ \textit{are concave for all } (t,x) &\in [0,T]\times G. \end{aligned} \tag{2.4}$$

$$\begin{aligned} & H(t,x,\widehat{\Gamma}(t,x),\widehat{u}(t,x),\widehat{p}(t,x),\widehat{q}(t,x),\widehat{r}(t,x,\cdot))2.13 \\ =\ & \sup_{v\in\mho} H(t,x,\widehat{\Gamma}(t,x),v,\widehat{p}(t,x),\widehat{q}(t,x),\widehat{r}(t,x,\cdot)) \\ \textit{for all } (t,x) \in\ & [0,T]\times G. \end{aligned} \tag{2.5}$$

For all $u \in \mathcal{A}$

$$E\left[\int_G \int_0^T \left(\Gamma(t,x)-\widehat{\Gamma}(t,x)\right)^2 \left\{\widehat{q}(t,x)^2 + \int_{\mathbb{R}} \widehat{r}(t,x,z)^2\nu(dz)\right\} dtdx\right] < \infty \tag{2.14}$$

and

$$E\left[\int_G \int_0^T \widehat{p}(t,x)^2 \left\{\sigma(t,x,\Gamma(t,x),u(t,x))^2 + \int_{\mathbb{R}} \theta(t,x,\Gamma(t,x),u(t,x),z)^2\nu(dz)\right\} dtdx\right] < \infty \tag{2.15}$$

Then $\widehat{u}(t,x)$ is an optimal control for the stochastic control problem (2.6).

Proof. Let u be an arbitrary admissible control with corresponding solution $\Gamma(t,x) = \Gamma^{(u)}(t,x)$ of (2.1)-(2.3). Then by (2.5)

$$J(\widehat{u}) - J(u) = E\left[\int_0^T \int_G \left\{\widehat{f} - f\right\} dxdt + \int_G \{\widehat{g} - g\}\, dx\right], \tag{2.16}$$

where

$$\begin{aligned} \widehat{f} &= f(t,x,\widehat{\Gamma}(t,x),\widehat{u}(t,x)),\ f = f(t,x,\Gamma(t,x),u(t,x)) \\ \widehat{g} &= g(x,\widehat{\Gamma}(T,x)) \text{ and } g = g(x,\Gamma(T,x)). \end{aligned}$$

Similarly we put

$$\begin{aligned} \widehat{b} &= b(t,x,\widehat{\Gamma}(t,x),\widehat{u}(t,x)),\ b = b(t,x,\Gamma(t,x),u(t,x)) \\ \widehat{\sigma} &= \sigma(t,x,\widehat{\Gamma}(t,x),\widehat{u}(t,x)),\ \sigma = \sigma(t,x,\Gamma(t,x),u(t,x)) \end{aligned}$$

and

$$\widehat{\theta} = \theta(t,x,\widehat{\Gamma}(t,x),\widehat{u}(t,x),z),\ \theta = \theta(t,x,\Gamma(t,x),u(t,x),z).$$

Moreover, we set

$$\begin{aligned}\widehat{H} &= H(t,x,\widehat{\Gamma}(t,x),\widehat{u}(t,x),\widehat{p}(t,x),\widehat{q}(t,x),\widehat{r}(t,x,\cdot)) \\ H &= H(t,x,\Gamma(t,x),u(t,x),\widehat{p}(t,x),\widehat{q}(t,x),\widehat{r}(t,x,\cdot)).\end{aligned}$$

Combining this notation with (2.8) and (2.16) we get

$$J(\widehat{u}) - J(u) = I_1 + I_2, \tag{2.17}$$

where

$$I_1 = E\left[\int_0^T \int_G \left\{\widehat{H} - H - (\widehat{b}-b)\widehat{p} - (\widehat{\sigma}-\sigma)\widehat{q} - \int_{\mathbb{R}} (\widehat{\theta}-\theta)\widehat{r}\nu(dz)\right\} dxdt\right] \tag{2.18}$$

and

$$I_2 = E\left[\int_G \{\widehat{g} - g\}\, dx\right]. \tag{2.19}$$

Since $\gamma \mapsto g(x,\gamma)$ is concave we have

$$g - \widehat{g} \leq \frac{\partial g}{\partial \gamma}(x,\widehat{\Gamma}(T,x)) \cdot (\Gamma(T,x) - \widehat{\Gamma}(T,x)).$$

Therefore, if we put

$$\widetilde{\Gamma}(t,x) = \Gamma(t,x) - \widehat{\Gamma}(t,x)$$

we get, by the Itô formula (or integration by parts formula) for jump diffusions ([ØS, Ex. 1.7])

$$\begin{aligned} I_2 \geq\ & -E\left[\int_G \frac{\partial g}{\partial \gamma}(x,\widehat{\Gamma}(T,x)) \cdot \widetilde{\Gamma}(T,x)dx\right] \\ =\ & -E\left[\int_G \widehat{p}(T,x)\cdot\widetilde{\Gamma}(T,x)dx\right] \\ =\ & -E\left[\int_G \Big(\widehat{p}(0,x)\cdot\widetilde{\Gamma}(0,x)\right. \\ & + \int_0^T \left\{\widetilde{\Gamma}(t,x)d\widehat{p}(t,x) + \widehat{p}(t,x)d\widetilde{\Gamma}(t,x) + (\sigma-\widehat{\sigma})\widehat{q}(t,x)\right\} dt \\ & \left. + \int_0^T \int_{\mathbb{R}} (\theta-\widehat{\theta})\widehat{r}(t,x,z)N(dt,dz)\Big)\, dx\right] \\ =\ & -E\left[\int_G \left(\int_0^T \left\{\widetilde{\Gamma}(t,x)\cdot\left[-\left(\frac{\partial H}{\partial\gamma}\right)^{\wedge} - L^*\widehat{p}(t,x)\right]\right.\right.\right. \\ & + \widehat{p}(t,x)\left[L\widetilde{\Gamma}(t,x) + (b-\widehat{b})\right] + (\sigma-\widehat{\sigma})\widehat{q}(t,x) \\ & \left.\left.\left. + \int_{\mathbb{R}}(\theta-\widehat{\theta})\widehat{r}(t,x,z)\nu(dz)\right\} dt\right) dx\right], 2.20 \end{aligned} \tag{2.6}$$

where

$$\left(\frac{\partial H}{\partial \gamma}\right)^{\wedge} = \frac{\partial H}{\partial \gamma}(t,x,\widehat{\Gamma}(t,x),\widehat{u}(t,x),\widehat{p}(t,x),\widehat{q}(t,x),\widehat{r}(t,x,\cdot)).$$

Combining (2.17)-(2.20) we get

$$\begin{aligned} J(\widehat{u}) - J(u) &\geq E\left[\int_0^T \left(\int_G \left\{\widetilde{\Gamma} L^* \widehat{p} - \widehat{p} L \widetilde{\Gamma}\right\} dx\right) dt\right] \\ &\quad + E\left[\int_G \left(\int_0^T \left\{\widehat{H} - H + \left(\frac{\partial H}{\partial \gamma}\right)^{\wedge} \cdot \widetilde{\Gamma}(t,x)\right\} dt\right) dx\right] .2.21 \end{aligned} \tag{2.7}$$

Since $\widetilde{\Gamma}(t,x) = \widehat{p}(t,x) = 0$ for all $(t,x) \in (0,T) \times \partial G$ we get by an extension of (2.7) that

$$\int_G \left\{\widetilde{\Gamma} L^* \widehat{p} - \widehat{p} L \widetilde{\Gamma}\right\} dx = 0 \text{ for all } t \in (0,T).$$

Combining this with (2.21) we get

$$J(\widehat{u}) - J(u) \geq E\left[\int_G \left(\int_0^T \left\{\widehat{H} - H + \left(\frac{\partial H}{\partial \gamma}\right)^{\wedge} \cdot \widetilde{\Gamma}(t,x)\right\} dt\right) dx\right]. \tag{2.22}$$

From the concavity assumption in (2.12) we deduce that

$$H - \widehat{H} \leq \frac{\partial H}{\partial \gamma}(\widehat{\Gamma}, \widehat{u}) \cdot (\Gamma - \widehat{\Gamma}) + \frac{\partial H}{\partial u}(\widehat{\Gamma}, \widehat{u}) \cdot (u - \widehat{u}). \tag{2.23}$$

From the maximality assumption in (2.13) we get that

$$\frac{\partial H}{\partial u}(\widehat{\Gamma}, \widehat{u}) \cdot (u - \widehat{u}) \leq 0. \tag{2.24}$$

Combining (2.23) and (2.24) we get

$$H - \widehat{H} - \frac{\partial H}{\partial \gamma}(\widehat{\Gamma}, \widehat{u}) \cdot (\Gamma - \widehat{\Gamma}) \leq 0,$$

which substituted in (2.22) gives

$$J(\widehat{u}) - J(u) \geq 0.$$

Since $u \in \mathcal{A}$ was arbitrary this proves Theorem 2.1. ■

3 Framework

We now present the general setting in which we will prove our main existence and uniqueness result for backward SPDE's with jumps:

Let V, H be two separable Hilbert spaces such that V is continuously, densely imbedded in H. Identifying H with its dual we have

$$V \subset H \cong H^* \subset V^*, \tag{3.1}$$

where V^* stands for the topological dual of V. Let A be a bounded linear operator from V to V^* satisfying the following coercivity hypothesis: There exist constants $\alpha > 0$ and $\lambda \geq 0$ such that

$$2\langle Au, u\rangle + \lambda |u|_H^2 \geq \alpha ||u||_V^2 \qquad \text{for all } u \in V, \tag{3.2}$$

where $\langle Au, u\rangle = Au(u)$ denotes the action of $Au \in V^*$ on $u \in V$.

Remark that A is generally not bounded as an operator from H into H. Let K be another separable Hilbert space. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\{B_t, t \geq 0\}$ be a cylindrical Brownian motion with covariance space K on the probability space $(\Omega, \mathcal{F}, P)$, i.e., for any $k \in K$,$\langle B_t, k\rangle$ is a real valued-Brownian motion with $E[\langle B_t, k\rangle^2] = t|k|_K^2$. Let $(X, \mathcal{B}(X))$ be a measurable space, where X is a topological vector space. Further let $\eta(t)$ be a Lévy process on X. Denote by $\nu(dx)$ the Lévy measure of η. Denote by $L^2(\nu)$ the L^2-space of square integrable H-valued measurable functions associated with ν. Set $p(t) = \Delta\eta(t) = \eta(t) - \eta(t-)$. Then $p = (p(t), t \in D_p)$ is a stationary Poisson point process on X with characteristic measure ν. See [IW] for details on Poisson point processes. Denote by $N(dt, dx)$ the Poisson counting measure associated with the Lévy process, i.e., $N(t, A) = \sum_{s \in D_p, s \leq t} I_A(p(s))$. Let $\widetilde{N}(dt, dx) := N(dt, dx) - dt\nu(dx)$ be the compensated Poisson mesasure. Define

$$\mathcal{F}_t = \sigma(B_s, N(s, A), A \in \mathcal{B}(X), s \leq t).$$

Recall that a linear operator S from K into H is called Hilbert-Schmidt if $\sum_{i=1}^{\infty} |Sk_i|_H^2 < \infty$ for some orthonormal basis $\{k_i, i \geq 1\}$ of K. $L_2(K, H)$ will denote the Hilbert space of Hilbert-Schmidt operators from K into H equipped with the inner product $\langle S_1, S_2\rangle_{L_2(K,H)} = \sum_{i=1}^{\infty} \langle S_1 k_i, S_2 k_i\rangle_H$. Let $b(t, y, z, q, \omega)$ be a measurable mapping from $[0, T] \times H \times L_2(K, H) \times L^2(\nu) \times \Omega$ into H such that $b(t, y, z, q, \omega)$ is $\mathcal{F}_t$-adapted,i.e., $b(t, y, z, q, \cdot)$ is $\mathcal{F}_t$-measurable for all t, y, z, q. Suppose we are given an $\mathcal{F}_T$ -measurable, H-valued random variable $\phi(\omega)$. We are looking for $\mathcal{F}_t$-adapted processes Y_t, Z_t, Q_t with values in H, $L_2(K, H)$ and $L^2(\nu)$ respectively, such that the following backward stochastic evolution equation holds:

$$\begin{aligned} dY_t &= AY_t dt + b(t, Y_t, Z_t, Q_t)dt + Z_t dB_t \\ &+ \int_X Q_t(x)\widetilde{N}(dt, dx), t \in (0, T)3.3 \qquad (3.8) \\ Y_T &= \phi(\omega) \ \text{ a.s.}3.4 \qquad (3.9) \end{aligned}$$

From now on we assume that the following, (3.5) and (3.6), hold:

There exists a constant $c < \infty$ such that

$$\begin{aligned} &|b(t, y_1, z_1, q_1)(\omega) - b(t, y_2, z_2, q_2)(\omega)|_H \\ &\leq c(|y_1 - y_2|_H + |z_1 - z_2|_{L_2(K,H)} + |q_1 - q_2|_{L^2(\nu)}) \qquad (3.5) \end{aligned}$$

for all $t, y_1, z_1, q_1, y_2, z_2, q_2$.

$$E[\int_0^T |b(t, 0, 0, 0)|_H^2 dt] < \infty \qquad (3.6)$$

4 Existence and Uniqueness

We now state and prove the main existence and uniqueness result of this paper.

Body Math **Theorem 4.1** *Assume that* $E[|\phi|_H^2] < \infty$. *Then there exists a unique* $H \times L_2(K, H) \times L^2(\nu)$*-valued progressively measurable process* (Y_t, Z_t, Q_t) *such that*

(i) $E[\int_0^T |Y_t|_H^2 dt] < \infty, E[\int_0^T |Z_t|_{L_2(K,H)}^2 dt] < \infty, E[\int_0^T |Q_t|_{L^2(\nu)}^2 dt] < \infty.$

(ii) $\phi = Y_t + \int_t^T AY_s ds + \int_t^T b(s, Y_s, Z_s, Q_s)ds + \int_t^T Z_s dB_s + \int_t^T \int_X Q_s(x)\widetilde{N}(ds, dx); \quad 0 \leq t \leq T.$

As the proof is long, we split it into lemmas.

Body Math **Lemma 4.2** *Assume that* $E[|\phi|_H^2] < \infty$, *and that* $b(t, y, z, q, \omega) = b(t, \omega)$ *is independent of* y, z *and* q, *and* $E[\int_0^T |b(t)|_H^2 dt] < \infty$. *Then there exists a unique* $H \times L_2(K, H) \times L^2(\nu)$*-valued progressively measurable process* (Y_t, Z_t, Q_t) *such that*

(i) $E[\int_0^T |Y_t|_H^2 dt] < \infty, E[\int_0^T |Z_t|_{L_2(K,H)}^2 dt] < \infty, E[\int_0^T |Q_t|_{L^2(\nu)}^2 dt] < \infty.$

(ii) $\phi = Y_t + \int_t^T AY_s ds + \int_t^T b(s)ds + \int_t^T Z_s dB_s + \int_t^T \int_X Q_s(x)\widetilde{N}(ds, dx); \quad 0 \le t \le T.$

Proof.

Existence of solution.

Set $D(A) = \{v; v \in V, Av \in H\}$. Then $D(A)$ is a dense subspace of H. Thus we can choose and fix an orthonormal basis $\{e_1, ...e_n, ...\}$ of H such that $e_i \in D(A)$. Set $V_n = span(e_1, e_2, ..., e_n)$. Denote by P_n the projection operator from H into V_n. Put $A_n = P_n A$. Then A_n is a bounded linear operator from V_n to V_n. For the cylindrical Brownian motion B_t, it is well known that the following decomposition holds:

$$B_t = \sum_{i=1}^{\infty} \beta_t^i k_i \tag{4.1}$$

where $\{k_1, k_2, ..., k_i, ...\}$ is an orthonormal basis of K, and $\beta_t^i, i = 1, 2, 3, ...$ are independent standard Brownian motions. Set $B_t^n = (\beta_t^1, ..., \beta_t^n)$. Define $\mathcal{F}_t^n = \sigma(B_s^n, N(s, A), A \in \mathcal{B}(X), s \le t)$ completed by the probability measure P, and put $\phi_n = E[P_n\phi|\mathcal{F}_T^n]$ and $b_n(t) = E[P_n b(t)|\mathcal{F}_t^n]$. Consider the following backward stochastic differential equation on the finite dimensional space V_n:

$$\begin{aligned} dY_t^n &= A_n Y_t^n dt + b_n(t)dt + Z_t^n dB_t^n \\ &+ \int_X Q_t^n(x)\widetilde{N}(dt, dx)\,; \qquad t < T4.2 \qquad (4.10) \\ Y_T^n &= \phi_n(\omega) \text{ a.s.}4.3 \qquad (4.11) \end{aligned}$$

As A_n is a bounded linear operator from V_n to V_n, it follows from the results of Situ [S] that (4.2)–(4.3) admits a unique $\mathcal{F}_t^n$- adapted solution (Y_t^n, Z_t^n, Q_t^n), where $Y_t^n \in V_n$, $Z_t^n \in L_2(K_n, V_n)$, $K_n = \text{span}(k_1, k_2, ..., k_n)$ and $Q_t^n \in L^2(\nu)$. We are going to show that the sequence (Y_t^n, Z_t^n, Q_t^n) admits a convergent subsequence. Using Itô's formula, we find that

$$E[|Y_t^n|_H^2] = E[|\phi_n|_H^2] - 2E[\int_t^T \langle Y_s^n, P_n A Y_s^n\rangle ds]$$

$$-2E[\int_t^T < Y_s^n, b_n(s) > ds] - E[\int_t^T |Z_s^n|_{L_2(K_n,V_n)}^2 ds] - E[\int_t^T ds \int_X |Q_s^n(x)|_H^2 \nu(dx)], \tag{4.4}$$

where $|Z_s^n|_{L_2(K_n,V_n)}^2 = \sum_{i,j=1}^n (Z_s^n(i,j))^2$ stands for the Hilbert-Schmidt norm. It follows from (3.2) that

$$E[|Y_t^n|_H^2] \le E[|\phi|_H^2] - \alpha E[\int_t^T ||Y_s^n||_V^2 ds] + \lambda E[\int_t^T |Y_s^n|_H^2 ds]$$

$$+E[\int_t^T |Y_s^n|_H^2 ds] + E[\int_t^T |b(s)|_H^2 ds] - E[\int_t^T |Z_s^n|^2_{L_2(K_n,V_n)} ds] - E[\int_t^T ds \int_X |Q_s^n(x)|_H^2 \nu(dx)].$$

Hence,

$$E[|Y_t^n|_H^2] + \alpha E[\int_t^T ||Y_s^n||_V^2 ds] + E[\int_t^T |\bar{Z}_s^n|^2_{L_2(K,H)} ds] + E[\int_t^T ds \int_X |Q_s^n(x)|_H^2 \nu(dx)]$$
$$\leq E[|\phi|_H^2] + (\lambda+1)E[\int_t^T |Y_s^n|_H^2 ds] + E[\int_t^T |b(s)|_H^2 ds]$$

where $\bar{Z}_s^n = Z_s^n \bar{P}_n$, and $\bar{P}_n$ is the projection from K into $K_n = span(k_1, ..., k_n)$. In particular,

$$E[|Y_t^n|_H^2] \leq E[|\phi|_H^2] + (\lambda+1)E[\int_t^T |Y_s^n|_H^2 ds] + E[\int_t^T |b(s)|_H^2 ds] \tag{4.5}$$

Set $\bar{Y}_t^n = \int_t^T |Y_s^n|_H^2 ds$. Then (4.5) implies that

$$-\frac{d(e^{(\lambda+1)t}\bar{Y}_t^n)}{dt} \leq e^{(\lambda+1)t}(E[|\phi|_H^2] + E[\int_t^T |b(s)|_H^2 ds])$$

Hence,

$$\int_0^T E[|Y_s^n|_H^2] ds \leq C(E[|\phi|_H^2] + E[\int_0^T |b(s)|_H^2 ds]),$$

where C is an appropriate constant. This further implies that

$$\sup_n \{\int_0^T E[|Y_s^n|_H^2] ds\} < \infty$$
$$\sup_n \{\int_0^T E[||Y_s^n||_V^2] ds\} < \infty \tag{4.6}$$
$$\sup_n \{\int_0^T E[|\bar{Z}_s^n|^2_{L_2(K,H)}] ds\} < \infty \tag{4.7}$$
$$\sup_n \{\int_0^T E[|Q_s^n|^2_{L^2(\nu)}] ds\} < \infty \tag{4.8}$$

For a separable Hilbert space L, we denote by $M^2([0,T], L)$ the Hilbert space of progressively measurable, square integrable, L-valued processes equipped with the inner product $< a, b >_M = E[\int_0^T < a_t, b_t >_L dt]$. By the weak compactness of a Hilbert space, it follows from (4.6),(4.7)and (4.8) that a subsequence $\{n_k, k \geq 1\}$ can be selected so that $Y_\cdot^{n_k}, k \geq 1$ converges weakly to some limit Y in $M^2([0,T], V)$, $\bar{Z}_\cdot^{n_k}, k \geq 1$ converges weakly to some limit Z in $M^2([0,T], L_2(K,H))$ and $Q_\cdot^{n_k}, k \geq 1$ converges weakly to some limit Q in $M^2([0,T], L^2(\nu))$. Let us prove that (a version of)(Y, Z, Q) is a solution to the backward stochastic evolution equation (3.3) and (3.4). For $n \geq i \geq 1$, we have that

$$d\langle Y_t^n, e_i\rangle = \langle P_n A Y_t^n, e_i\rangle dt + \langle b_n(t), e_i\rangle dt + \langle \bar{Z}_t^n dB_t, e_i\rangle + \int_X \langle Q_t^n(x), e_i\rangle \widetilde{N}(dt, dx)$$
$$= \langle A Y_t^n, e_i\rangle dt + \langle b_n(t), e_i\rangle dt + \langle \bar{Z}_t^n dB_t, e_i\rangle + \int_X \langle Q_t^n(x), e_i\rangle \widetilde{N}(dt, dx) \tag{4.9}$$

Let $h(t)$ be an absolutely continuous function from $[0,T]$ to $\mathbb{R}$ with $h'(\cdot) \in L^2([0,T])$ and $h(0)=0$. By the Itô formula,

$$\langle Y_T^n, e_i\rangle h(T) = \int_0^T h(t)\langle AY_t^n, e_i\rangle dt + \int_0^T h(t)\langle b_n(t), e_i\rangle dt$$

$$+ \int_0^T h(t) d\langle \int_0^t \bar{Z}_s^n dB_s, e_i\rangle + \int_0^T \int_X h(t)\langle Q_t^n(x), e_i\rangle \widetilde{N}(dt,dx) + \int_0^T \langle Y_t^n, e_i\rangle h'(t)dt. \quad (4.10)$$

Replacing n by n_k in (4.10) and letting $k \to \infty$ to obtain

$$\begin{aligned}&\langle \phi, e_i\rangle h(T)\\ &= \int_0^T h(t)\langle AY_t, e_i\rangle dt + \int_0^T h(t)\langle b(t), e_i\rangle dt + \int_0^T \int_X h(t)\langle Q_t(x), e_i\rangle \widetilde{N}(dt,dx)\\ &\quad + \int_0^T h(t) d\langle \int_0^t Z_s dB_s, e_i\rangle + \int_0^T \langle Y_t, e_i\rangle h'(t)dt.4.11 \qquad (4.12)\end{aligned}$$

From (4.10) to (4.11), we have used the fact that the linear mapping G from $M^2([0,T], L_2(K,H))$ into $L^2(\Omega)$ defined by

$$G(Z) = \int_0^T h(t) d\langle \int_0^t Z_s dB_s, e_i\rangle = \sum_{j=1}^{\infty} \int_0^T h(t)\langle Z_t(k_j), e_i\rangle d\beta_t^j$$

is continuous and also the linear mapping F from $M^2([0,T], L^2(\nu))$ into $L^2(\Omega)$ defined by

$$F(Q) = \int_0^T \int_X h(t)\langle Q_t(x), e_i\rangle \widetilde{N}(dt,dx)$$

is continuous. So, the convergence of (4.10) to (4.11) takes place weakly in $L^2(\Omega)$. Fix $t \in (0,T)$ and choose , for $n \geq 1$,

$$h_n(s) = \begin{cases} 1, & s \geq t + \frac{1}{2n},\\ 1 - \frac{1}{n}(t + \frac{1}{2n} - s), & t - \frac{1}{2n} \leq s \leq t + \frac{1}{2n},\\ 0, & s \leq t - \frac{1}{2n}\end{cases}$$

With $h(\cdot)$ replaced by $h_n(\cdot)$ in (4.11), it follows that

$$\begin{aligned}\langle \phi, e_i\rangle &= \int_0^T h_n(s)\langle AY_s, e_i\rangle ds + \int_0^T h_n(s)\langle b(s), e_i\rangle ds + \int_0^T \int_X h(t)\langle Q_t(x), e_i\rangle \widetilde{N}(dt,dx)\\ &\quad + \int_0^T h_n(s) d\langle \int_0^s Z_u dB_u, e_i\rangle + \frac{1}{n}\int_{t-\frac{1}{2n}}^{t+\frac{1}{2n}} \langle Y_s, e_i\rangle ds. \qquad (4.12)\end{aligned}$$

Sending n to infinity in (4.12) we get that

$$\begin{aligned}\langle \phi, e_i\rangle &= \int_t^T \langle AY_s, e_i\rangle ds + \int_t^T \langle b(s), e_i\rangle ds\\ &\quad + \int_t^T \int_X \langle Q_s(x), e_i\rangle \widetilde{N}(ds,dx) + \int_t^T d\langle \int_0^s Z_u dB_u, e_i\rangle + \langle Y_t, e_i\rangle.\end{aligned}$$

for almost all $t \in [0,T]$ (with respect to Lebesgue measure).

As i is arbitrary, this implies that

$$\phi = \int_t^T AY_s ds + \int_t^T b(s)ds + \int_t^T Z_s dB_s + \int_t^T \int_X Q_s(x)\widetilde{N}(ds,dx) + Y_t. \tag{4.13}$$

for almost all $t \in [0,T]$ (with respect to Lebesgue measure).

For $t \in [0,T]$, define

$$\hat{Y}_t = \phi - \int_t^T AY_s ds - \int_t^T b(s)ds - \int_t^T Z_s dB_s - \int_t^T \int_X Q_s(x)\widetilde{N}(ds,dx).$$

Then we see that $(\hat{Y}_t, Z_t, Q_t)$ also satisfies (ii) in the Theorem 4.1 with Y replaced by $\hat{Y}$ for all $t \in [0,T]$. Hence, $(\hat{Y}_t, Z_t, Q_t)$ is a solution to the equations (3.3) and (3.4).

Uniqueness:

Let (Y_t, Z_t, Q_t) and $(\bar{Y}_t, \bar{Z}_t, \bar{Q}_t)$ be two solutions of the equation (3.3). Then

$$\int_t^T A(Y_s - \bar{Y}_s)ds + \int_t^T (Z_s - \bar{Z}_s)dB_s + (Y_t - \bar{Y}_t) + \int_t^T \int_X (Q_s(x) - \bar{Q}_s(x))\widetilde{N}(ds,dx) = 0$$

Applying Itô's formula, we get

$$\begin{aligned} 0 = {} & |Y_t - \bar{Y}_t|_H^2 + 2\int_t^T \langle Y_s - \bar{Y}_s, dM_s\rangle \\ & + \int_t^T \int_X [|Y_{s-} - \bar{Y}_{s-} + Q_s(x) - \bar{Q}_s(x)|_H^2 - |Y_{s-} - \bar{Y}_{s-}|_H^2]\widetilde{N}(ds,dx) \\ & + 2\int_t^T \langle A(Y_s - \bar{Y}_s), Y_s - \bar{Y}_s \langle ds + \int_t^T |Z_s - \bar{Z}_s|_{L_2(K,H)}^2 ds \\ & + \int_t^T \int_X |Q_s(x) - \bar{Q}_s(x)|_H^2 ds\nu(dx) \end{aligned} \tag{4.14}$$

where $M_t = \int_0^t (Z_s - \bar{Z}_s)dB_s$. By (3.2),we get that

$$\begin{aligned} E[|Y_t - \bar{Y}_t|_H^2] = {} & -2\int_t^T E[\langle A(Y_s - \bar{Y}_s), Y_s - \bar{Y}_s\rangle]ds - E[\int_t^T |Z_s - \bar{Z}_s|_{L_2(K,H)}^2 ds] \\ & - E[\int_t^T \int_X |Q_s(x) - \bar{Q}_s(x)|_H^2 ds\nu(dx)] \\ & \le -\alpha \int_t^T E[||Y_s - \bar{Y}_s||_V^2]ds + \lambda \int_t^T E[|Y_s - \bar{Y}_s|_H^2]ds \\ & \le \lambda \int_t^T E[|Y_s - \bar{Y}_s|_H^2]ds. \end{aligned}$$

By a Gronwall type inequality, it follows that $E[|Y_t - \bar{Y}_t|_H^2] = 0$, which proves the uniqueness.

Body Math **Lemma 4.3** *Assume that* $E[|\phi|_H^2] < \infty$, *and that* $b(t,y,z,q,\omega) = b(t,z,q,\omega)$ *is independent of* y. *Then there exists a unique* $H \times L_2(K,H) \times L^2(\nu)$*-valued progressively measurable process* (Y_t, Z_t, Q_t) *such that*

(i) $E[\int_0^T |Y_t|_H^2 dt] < \infty, E[\int_0^T |Z_t|_{L_2(K,H)}^2 dt] < \infty, E[\int_0^T |Q_t|_{L^2(\nu)}^2 dt] < \infty.$

(ii) $\phi = Y_t + \int_t^T AY_s ds + \int_t^T b(s, Z_s, Q_s) ds + \int_t^T Z_s dB_s + \int_t^T \int_X Q_s(x) \widetilde{N}(ds, dx); \quad 0 \leq t \leq T.$

Proof.

Set $Z_t^0 = 0, Q_t^0 = 0$. Denote by (Y_t^n, Z_t^n, Q_t^n) the unique solution of the backward stochastic evolution equation:

$$dY_t^n = AY_t^n dt + b(t, Z_t^{n-1}, Q_t^{n-1})dt + Z_t^n dB_t + \int_X Q_t^n(x)\widetilde{N}(dt, dx)4.15 \quad (4.13)$$

$$Y_T^n = \phi(\omega).4.16 \quad (4.14)$$

The existence of such a solution (Y_t^n, Z_t^n, Q_t^n) has been proved in Lemma 4.2. Putting $M_t^n = \int_0^t Z_s^n dB_s$, and by Itô's formula we get that

$$\begin{aligned}
0 &= |Y_T^{n+1} - Y_T^n|_H^2 \\
&= |Y_t^{n+1} - Y_t^n|_H^2 + 2\int_t^T \langle A(Y_s^{n+1} - Y_s^n), Y_s^{n+1} - Y_s^n \rangle ds \\
&+ 2\int_t^T \langle b(t, Z_s^n, Q_s^n) - b(t, Z_s^{n-1}, Q_s^{n-1}), Y_s^{n+1} - Y_s^n \rangle ds \\
&+ \int_t^T \int_X [|Y_{s-}^{n+1} - Y_{s-}^n + Q_s^{n+1} - Q_s^n|_H^2 - |Y_{s-}^{n+1} - Y_{s-}^n|_H^2]\widetilde{N}(ds, dx) \\
&+ \int_t^T \int_X |Q_s^{n+1} - Q_s^n|_H^2] ds \nu(dx) \\
&+ 2\int_t^T \langle Y_s^{n+1} - Y_s^n, d(M_s^{n+1} - M_s^n) \rangle + \int_t^T |Z_s^{n+1} - Z_s^n|_{L_2(K,H)}^2 ds
\end{aligned}$$

In virtue of (3.2), for $\varepsilon > 0$,

$$\begin{aligned}
&E[|Y_t^{n+1} - Y_t^n|_H^2] + E[\int_t^T |Z_s^{n+1} - Z_s^n|_{L_2(K,H)}^2 ds] + E[\int_t^T \int_X |Q_s^{n+1} - Q_s^n|_H^2] ds \nu(dx)] \\
&= -2E[\int_t^T \langle A(Y_s^{n+1} - Y_s^n), Y_s^{n+1} - Y_s^n \rangle ds] \\
&- 2E[\int_t^T \langle b(t, Z_s^n, Q_s^n) - b(t, Z_s^{n-1}, Q_s^{n-1}), Y_s^{n+1} - Y_s^n \rangle ds] \\
&\leq \lambda E[\int_t^T |Y_s^{n+1} - Y_s^n|_H^2 ds] - \alpha E[\int_t^T ||Y_s^{n+1} - Y_s^n||_V^2 ds] \\
&+ \varepsilon E[\int_t^T |b(t, Z_s^n, Q_s^n) - b(t, Z_s^{n-1}, Q_s^{n-1})|_H^2 ds] + \frac{1}{\varepsilon} E[\int_t^T |Y_s^{n+1} - Y_s^n|_H^2 ds] \quad (4.17)
\end{aligned}$$

Choose $\varepsilon < \frac{1}{4c}$, where c is the Lipschitz constant in (3.5). It follows from (4.17) that

$$\begin{aligned}
& E[|Y_t^{n+1} - Y_t^n|_H^2] + E[\int_t^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds] \\
& +\alpha E[\int_t^T ||Y_s^{n+1} - Y_s^n||_V^2 ds] + E[\int_t^T \int_X |Q_s^{n+1} - Q_s^n|_H^2] ds\nu(dx)] \\
\leq\ & (\lambda + \frac{1}{\varepsilon}) E[\int_t^T |Y_s^{n+1} - Y_s^n|_H^2 ds] + \tfrac{1}{2} E[\int_t^T |Z_s^n - Z_s^{n-1}|^2_{L_2(K,H)} ds] \\
& +\frac{1}{2} E[\int_t^T \int_X |Q_s^n - Q_s^{n-1}|_H^2] ds\nu(dx)] 4.18 \qquad (4.15)
\end{aligned}$$

Let $\beta = \lambda + \frac{1}{\varepsilon}$. Then we have from (4.18) that

$$\begin{aligned}
& -\frac{d}{dt}(e^{\beta t} E[\int_t^T |Y_s^{n+1} - Y_s^n|_H^2 ds] \\
& + e^{\beta t} E[\int_t^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds] \\
& + e^{\beta t} E[\int_t^T \int_X |Q_s^{n+1} - Q_s^n|_H^2] ds\nu(dx)] \\
& + \alpha e^{\beta t} E[\int_t^T ||Y_s^{n+1} - Y_s^n||_V^2 ds] \\
\leq & \frac{1}{2} e^{\beta t} E[\int_t^T |Z_s^n - Z_s^{n-1}|^2_{L_2(K,H)} ds] \\
& + \frac{1}{2} e^{\beta t} E[\int_t^T \int_X |Q_s^n - Q_s^{n-1}|_H^2] ds\nu(dx)] \qquad (4.19)
\end{aligned}$$

From here, following a similar proof as in [PP1] we will show that (Y^n, Z^n, Q^n) converges to some limit (Y, Z, Q) in the product space of $M^2([0,T], V)$,$M^2([0,T], L_2(K,H))$ and $M^2([0,T], L^2(\nu))$.

Integrating both sides in (4.19) we get that

$$\begin{aligned}
& E[\int_0^T |Y_s^{n+1} - Y_s^n|_H^2 ds] + \int_0^T E[\int_t^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds] e^{\beta t} dt \\
& + \int_0^T e^{\beta t} E[\int_t^T \int_X |Q_s^{n+1} - Q_s^n|_H^2] ds\nu(dx)] dt + \alpha \int_0^T E[\int_t^T ||Y_s^{n+1} - Y_s^n||_V^2 ds] e^{\beta t} dt \\
& \leq \tfrac{1}{2} \int_0^T E[\int_t^T |Z_s^n - Z_s^{n-1}|^2_{L_2(K,H)} ds] e^{\beta t} dt + \frac{1}{2} \int_0^T e^{\beta t} E[\int_t^T \int_X |Q_s^n - Q_s^{n-1}|_H^2] ds\nu(dx)] dt
\end{aligned} \qquad (4.20)$$

In particular,

$$\begin{aligned}
& \int_0^T E[\int_t^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds] e^{\beta t} dt + \int_0^T e^{\beta t} E[\int_t^T \int_X |Q_s^{n+1} - Q_s^n|_H^2] ds\nu(dx)] dt \\
& \leq \tfrac{1}{2} \left[\int_0^T E[\int_t^T |Z_s^n - Z_s^{n-1}|^2_{L_2(K,H)} ds] e^{\beta t} dt + \int_0^T e^{\beta t} E[\int_t^T \int_X |Q_s^n - Q_s^{n-1}|_H^2] ds\nu(dx)] dt \right]
\end{aligned} \qquad (4.21)$$

This implies that

$$\int_0^T E[\int_t^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds] e^{\beta t} dt + \int_0^T e^{\beta t} E[\int_t^T \int_X |Q_s^{n+1} - Q_s^n|_H^2] ds \nu(dx)] dt$$
$$\leq (\tfrac{1}{2})^n C$$

for some constant C. Thus, it follows from (4.20) that

$$E[\int_0^T |Y_s^{n+1} - Y_s^n|_H^2 ds] \leq (\frac{1}{2})^n C \tag{4.22}$$

Hence, we conclude from (4.18) that

$$E[\int_0^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds] + E[\int_0^T \int_X |Q_s^{n+1} - Q_s^n|_H^2] ds \nu(dx)] dt$$
$$\leq (\tfrac{1}{2})^n C\beta + \tfrac{1}{2}\{E[\int_0^T |Z_s^n - Z_s^{n-1}|^2_{L_2(K,H)} ds] + E[\int_0^T \int_X |Q_s^n - Q_s^{n-1}|_H^2] ds \nu(dx)]\} \tag{4.23}$$

Using the above inequality repeatedly gives

$$E[\int_0^T |Z_s^{n+1} - Z_s^n|^2_{L_2(K,H)} ds] + E[\int_0^T \int_X |Q_s^{n+1} - Q_s^n|_H^2] ds \nu(dx)]$$
$$\leq (\tfrac{1}{2})^n n C\beta \tag{4.24}$$

Combining (4.18) and (4.23) we have that

$$E[\int_0^T ||Y_s^{n+1} - Y_s^n||_V^2 ds] \leq (\frac{1}{2})^n (n+1) C\beta \tag{4.25}$$

It follows now from (4.24) and (4.25) that the the sequence (Y_t^n, Z_t^n, Q_t^n), $n \geq 1$ converges in $M^2([0,T], V) \times M^2([0,T], L_2(K,H)) \times M^2([0,T], L^2(\nu))$ to some limit (Y_t, Z_t, Q_t). Letting $n \to \infty$ in (4.14), we see that (Y_t, Z_t, Q_t) satisfies

$$Y_t + \int_t^T AY_s ds + \int_t^T b(s, Z_s) ds + \int_t^T Z_s dB_s + \int_t^T \int_X Q_s(x) \widetilde{N}(ds, dx) = \phi \tag{4.26}$$

i.e., (Y_t, Z_t, Q_t) is a solution to equation (3.3).

Uniqueness

Let (Y_t, Z_t, Q_t) , $(\bar{Y}_t, \bar{Z}_t, \bar{Q}_t)$ be two solutions. By Itô's formula , as in (4.14) we have

$$
\begin{aligned}
& E[|Y_t - \bar{Y}_t|_H^2] + E[\int_t^T |Z_s - \bar{Z}_s|^2_{L_2(K,H)} ds] \\
& + E[\int_t^T \int_X |Q_s(x) - \bar{Q}_s(x)|_H^2 ds\nu(dx)] \\
& = -2E[\int_t^T < A(Y_s - \bar{Y}_s), Y_s - \bar{Y}_s > ds] \\
& - 2E[\int_t^T \langle b(t, Z_s, Q_s) - b(t, \bar{Z}_s, \bar{Q}_s), Y_s - \bar{Y}_s \rangle ds] \\
& \leq \lambda E[\int_t^T |Y_s - \bar{Y}_s|_H^2 ds] - \alpha E[\int_t^T ||Y_s - \bar{Y}_s||_V^2 ds] \\
& + \tfrac{1}{2} E[\int_t^T |Z_s - \bar{Z}_s|^2_{L_2(K,H)} ds] + cE[\int_t^T |Y_s - \bar{Y}_s|_H^2 ds] \\
& + \frac{1}{2} E[\int_t^T \int_X |Q_s(x) - \bar{Q}_s(x)|_H^2 ds\nu(dx)] \qquad (4.27)
\end{aligned}
$$

Consequently,

$$
E[|Y_t - \bar{Y}_t|_H^2] \leq (\lambda + c) E[\int_t^T |Y_s - \bar{Y}_s|_H^2 ds] \qquad (4.28)
$$

By Gronwall's inequality ,

$$
Y_t = \bar{Y}_t
$$

which further implies $Z_t = \bar{Z}_t$ and $Q_t = \bar{Q}_t$ by (4.27).

Proof of Theorem 4.1.

Let $Y_t^0 = 0$. Define , for $n \geq 1$, $(Y_t^{n+1}, Z_t^{n+1}, Q_t^{n+1})$ to be the solution of the equation:

$$
\begin{aligned}
dY_t^{n+1} &= AY_t^{n+1} dt + b(t, Y_t^n, Z_t^{n+1}, Q_t^{n+1}) dt \\
&+ Z_t^{n+1} dB_t + \int_X Q_t^{n+1}(x) \widetilde{N}(dt, dx) 4.29 \qquad (4.16) \\
Y_T^{n+1} &= \phi 4.30 \qquad (4.17)
\end{aligned}
$$

The existence of $(Y_t^{n+1}, Z_t^{n+1}, Q_t^{n+1})$ is contained in Lemma 4.3.

Using the similar arguments as in the proof of Lemma 4.3 it can be shown that $(Y_t^{n+1}, Z_t^{n+1}, Q_t^{n+1})$ converges to some limit (Y_t, Z_t, Q_t), and moreover (Y_t, Z_t, Q_t) is the unique solution to equation (3.3). We omit the details avoiding the repeating.

Body Math **Example 4.2** *Let* $H = L^2(R^d)$, *and set*

$$
V = H_2^1(\mathbb{R}^d) = \{u \in L^2(R^d); \nabla u \in L^2(\mathbb{R}^d \to \mathbb{R}^d)\}
$$

Denote by $a(x) = (a_{ij}(x))$ *a matrix-valued function on* R^d *satisfing the uniform ellipticity condition:*

$$
\frac{1}{c} I_d \leq a(x) \leq c I_d \qquad \text{for some constant } c \in (0, \infty).
$$

Let $f(x)$ be a vector field on R^d with $f \in L^p(R^d)$ for some $p > d$. Define

$$Au = -div(a(x)\nabla u(x)) + f(x) \cdot \nabla u(x)$$

Then (3.2) is fulfilled for (H, V, A). Thus, for any choice of cylindrical Brownian motion B, any drift coefficient $b(t, y, z, \omega)$ satisfying (3.5) and (3.6) and terminal random variable ϕ, the main result in Section 4 applies.

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