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Random Fields: non-anticipating derivative and differentiation formulae

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Abstract

The non-anticipating stochastic derivative represents the integrand in the best L_2-approximation for random variables by Itô non-anticipating integrals with respect to a general stochastic measure with independent values on a space-time product. In this paper some explicit formulae for this derivative are obtained.

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1 Introduction

In many physical stochastic phenomena interesting for applications the randomness is generated by a large number of "small" independent factors which appear in their corresponding "place" and "time". Having this in mind we consider some elements of the calculus with respect to stochastic measures with indepenent values on general space-time products.

In Section 2 we introduce the integrator: the *stochastic measure* $\mu = \mu(d\theta dt)$, $(\theta, t) \in \Theta \times \mathbb{T}$, on the *space-time product* $\Theta \times \mathbb{T}$ with values in $L_2(\Omega)$. Here Θ is a general separable measurable space and $\mathbb{T}$ is a time interval. In the applications Θ can be specified ad-hoc. The measure μ considered has *independent values* (not necessarily homogeneous) and the distribution of its values follows the infinitely divisible law (3.1). Details about this law are given in Section 3. At pleasure the stochastic measure considered can be regarded as a generalization to the field case (here on space-time products) of the measures generated by additive processes (cf. e.g. [38]), thus on the time line, with values in $L_2(\Omega)$. In fact to retrieve this case it is enough to consider Θ consisting only of a single point.

In line with classical stochastic calculus we treat stochastic functions as limit of simple functions in the standard $L_2(\Theta \times \mathbb{T} \times \Omega)$. Explicit *simple approximations* are studied both for general stochastic function and for non-anticipating (i.e. adapted to

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the corresponding filtration) ones. See Theorem 2.1 and Theorem 2.2. The *partitions of* $\Theta \times \mathbb{T}$, see (2.7)-(2.10), play a central role in our approach.

In this paper our major interest is stochastic differentiation for non-anticipating calculus. The *non-anticipating derivative* $D\xi = D\xi(\theta, t)$, $(\theta, t) \in \Theta \times \mathbb{T}$ (cf. (2.15)) is well defined for all random variables $\xi \in L_2(\Omega)$ and represents the integrand in the non-anticipating integral which gives the best aproximation to ξ in $L_2(\Omega)$ by non-anticipating stochastic integrals. Namely, it is

$$\xi = \xi^0 + \iint_{\Theta\times\mathbb{T}} D\xi(\theta, t)\mu(d\theta dt),$$

where $\xi^0 \in L_2(\Omega)$ is such that $\mathfrak{D}\xi^0 \equiv 0$ and

$$\Big\|\xi - \iint_{\Theta\times\mathbb{T}} D\xi(\theta, t)\mu(d\theta dt)\Big\| = \min_{\varphi} \Big\|\xi - \iint_{\Theta\times\mathbb{T}} \varphi(\theta, t)\mu(d\theta dt)\Big\|.$$

See [11]. In this paper we provide some explicit formula for the derivative $D\xi$ with respect to stochastic measures with independent values. See Section 4, Theorem 4.1.

Some of the results here presented will be framed in a wider context in a comprehensive survey paper on integration and differentiation for random fields, cf. [14].

2 Framework and preliminary results

The stochastic measure. Let $(\Omega, \mathfrak{A}, P)$ be a complete probability space. In the line of the results and the terminology of [11], we deal with the stochastic measure with independent values $\mu = \mu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, of the type

$$E\mu(\Delta) = 0, \quad E\mu(\Delta)^2 = M(\Delta). \tag{2.1}$$

The values $\mu(\Delta)$ are real random variables in the standard (complex) space $L_2(\Omega)$ of random variables $\xi = \xi(\omega)$, $\omega \in \Omega$, with finite norm $\|\xi\| := \big(E|\xi|^2\big)^{1/2}$. The variance indicated above is represented by the σ-finite measure $M = M(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, on the separable measurable product space $\Theta \times \mathbb{T}$. Here, Θ is a general separable measurable space $\mathbb{T} \subseteq \mathbb{R}$ is a time interval.

Event σ-algebras on Ω. The randomness is represented by events related to the sets in $\Theta \times \mathbb{T}$. We write

$$\mathfrak{A}_\Delta, \quad \Delta \subseteq \Theta \times \mathbb{T}, \tag{2.2}$$

for the σ-algebras of the events generated by the stochastic measure μ over the indicated sets Δ. To be more precise, $\mathfrak{A}_\Delta$ is generated by the values $\mu(\Delta')$ for all $\Delta' \subseteq \Delta$ and all the events of P-null measure. In the sequel, the main results concern random variables $\xi \in L_2(\Omega)$ measurable with respect to the σ-algebra $\mathfrak{A}_{\Theta\times\mathbb{T}}$. For convenience of notation we assume that this σ-algebra represents *all* the events in Ω, i.e. $\mathfrak{A} = \mathfrak{A}_{\Theta\times\mathbb{T}}$. Again for convenience in notation we fix $\mathbb{T} = (0, T]$. Consistently the flow of events in the course of time is represented by the increasing σ-algebras

$$\mathfrak{A}_t := \mathfrak{A}_{\Theta \times (0,t]}, \quad 0 < t \leq T. \tag{2.3}$$

In this paper we deal with stochastic functions $\varphi = \varphi(\theta, t)$, $(\theta, t) \in \Theta \times \mathbb{T}$, having values $\varphi(\theta, t) := \varphi(\theta, t, \omega)$, $\omega \in \Omega$, in $L_2(\Omega)$ and belonging to the standard (complex) space $L_2(\Theta \times \mathbb{T} \times \Omega)$ with norm

$$\|\varphi\|_{L_2} := \Big(\iint_{\Theta \times \mathbb{T}} \|\varphi(\theta, t)\|^2 M(d\theta dt) \Big)^{1/2}.$$

In line with the common terminology we say that a stochastic function $\varphi = \varphi(\theta, t)$, $(\theta, t) \in \theta \times \mathbb{T}$ is *non-anticipating* with respect to $\mathfrak{A}_t$, $t \in \mathbb{T}$, if for every (θ, t) the values

$$\varphi(\theta, t) := \varphi(\theta, t, \omega), \qquad \omega \in \Omega,$$

are measurable with respect to $\mathfrak{A}_t$.

To be able to grasp all the non-anticipating stochastic functions $\varphi \in L_2(\Theta \times \mathbb{T} \times \Omega)$ as integrands with respect to the stochastic measure μ, the flow of σ-algebras (2.3) is required to be *left-continuous*, i.e. for all t it is

$$\mathfrak{A}_t = \lim_{s \to t^-} \mathfrak{A}_s := \bigvee_{s<t} \mathfrak{A}_s \tag{2.4}$$

Accordingly it is required that

$$M(\Theta \times \{t\}) = 0, \quad t \in \mathbb{T}. \tag{2.5}$$

From now on it is assumed that the variance measure M satisfies the above continuity condition. Actually, the filtration (2.3) is always *right-continuous*, i.e. $\mathfrak{A}_t = \lim_{u \to t^+} \mathfrak{A}_u := \bigcap_{u>t} \mathfrak{A}_u$. See e.g. [9], [11]. With respect to filtrations generated by processes with independent increments, we refer to e.g. [24], see also e.g. [26]. However we do not exploit the right-continuity of the filtration in this paper.

In the arguments forthcoming we are also going to consider the σ-algebras

$$\mathfrak{A}_{]\Delta[}, \qquad]\Delta[:= \Theta \times \mathbb{T} \setminus \Delta \quad (\Delta \subseteq \Theta \times \mathbb{T}), \tag{2.6}$$

i.e. $\mathfrak{A}_{]\Delta[}$ is the σ-algebra generated by all the events in the complement of Δ.

Partitions of $\Theta \times \mathbb{T}$ and corresponding σ-algebras. The measurable sets $\Delta \subseteq \Theta \times \mathbb{T}$ are here treated as generated by a certain *semi-ring* which we refer to as *the partitions of* $\Theta \times \mathbb{T}$. The elements of this semi-ring have the basic form

$$\Delta = B \times (s, u], \quad B \subseteq \Theta, \ (s, u] \subseteq \mathbb{T},$$

and are arranged in series of sets. The sets

$$\Delta_{nk} = B_{nk} \times (s_{nk}, u_{nk}], \quad k = 1, ..., \kappa_n, \tag{2.7}$$

of each n^{th}-series ($n = 1, 2, ...$) are *disjoint* (always meant pairwise disjoint) and represent the partitions of some increasing sets $\Theta_n \times \mathbb{T} := \bigsqcup_{k=1}^{\kappa_n} \Delta_{nk}$, $n = 1, 2, ...$, yielding

$$\Theta \times \mathbb{T} = \lim_{n\to\infty} \Theta_n \times \mathbb{T} := \bigcup_{n=1}^{\infty} \Theta_n \times \mathbb{T}.$$

We can consider (and refer to) each n^{th}-series as the "partitions of $\Theta \times \mathbb{T}$ corresponding to the n^{th}-level of refinement". For $n = 1, 2, ...$, the partitions elements are decreasing so that any set of the n^{th}-series can be represented as finite (disjoint) union of some appropriate elements of the $(n+1)^{th}$-series and for $n \to \infty$ it is

$$\max_{k=1,...,\kappa_n} (u_{nk} - s_{nk}) \longrightarrow 0, \quad \text{for} \quad \max_{k=1,...,\kappa_n} M(\Delta_{nk}) \longrightarrow 0. \tag{2.8}$$

All the elements of *all* the n^{th}-series of partitions of $\Theta \times \mathbb{T}$ constitute a *semi-ring.* It is assumed that their finite unions constitute the *ring* which generates the σ-algebra of all the measurable sets in $\Theta \times \mathbb{T}$. Here we have applied the standard approximation of $\Delta \subseteq \Theta \times \mathbb{T}$: $M(\Delta) < \infty$, by the finite union $\Delta^{(n)} = \bigsqcup_k \Delta_{nk}$ of some elements of the n^{th}-series of partitions of $\Theta \times \mathbb{T}$, namely,

$$\Delta = \lim_{n\to\infty} \Delta^{(n)}, \text{ i.e. } \quad M\big((\Delta \setminus \Delta^{(n)}) \sqcup (\Delta^{(n)} \setminus \Delta\big) \longrightarrow 0, \quad n \to \infty. \tag{2.9}$$

Note that for any group of *disjoint* sets Δ_j, $j = 1, ..., m$ with $M(\Delta_j) < \infty$, the approximation above can be given by corresponding sequences of *disjoint* $\Delta_j^{(n)}$, $j = 1, ..., m$ ($n = 1, 2, ...$). We will refer to the finite unions of elements of the *same* series of the partitions as *simple sets* in $\Theta \times \mathbb{T}$.

We would like to point the attention to the following fact which will be used in the sequel. Let (θ, t) be fixed and let us consider the elements $\Delta_{nk} = B_{nk} \times (s_{nk}, u_{nk}]$ belonging to the partitions of $\Theta \times \mathbb{T}$ which contain (θ, t). Then it is

$$\lim_{n\to\infty} \mathfrak{A}_{]\Delta_{nk}[} = \mathfrak{A}_{\Theta\times\mathbb{T}} \quad \text{and} \quad \lim_{n\to\infty} \mathfrak{A}_{s_{nk}} = \mathfrak{A}_t. \tag{2.10}$$

Simple approximations. In this paper we are dealing with the stochastic functions $\varphi = \varphi(\theta, t)$, $(\theta, t) \in \Theta \times \mathbb{T}$, having values $\varphi(\theta, t) \in L_2(\Omega)$ and belonging to the standard (complex) space $L_2(\Theta \times \mathbb{T} \times \Omega)$. Our approach will be to consider the stochastic functions as limits of simple functions of the form

$$\varphi = \sum_k \varphi_k 1_{\Delta_k} \text{ with } \varphi_k := \frac{1}{M(\Delta_k)} \iint_{\Delta_k} \varphi(\theta, t) M(d\theta dt), \tag{2.11}$$

where the sum is finite and the sets Δ_k involved are *disjoint.* A fundamental role is played by the simple functions of the form

$$\varphi = \sum_k \varphi_k 1_{\Delta_k} \text{ with } \varphi_k = \frac{1}{M(\Delta_k)} E\Big(\iint_{\Delta_k} \varphi(\theta, t) M(d\theta dt) \big| \mathfrak{A}_{]\Delta_k[} \Big), \tag{2.12}$$

and of the form

$$\varphi = \sum_k \varphi_k 1_{\Delta_k} \quad \text{with} \quad \varphi_k = \frac{1}{M(\Delta_k)} E\Big(\iint_{\Delta_k} \varphi(\theta,t) M(d\theta dt) \big| \mathfrak{A}_{s_k} \Big), \tag{2.13}$$

for $\Delta_k = B_k \times (s_k, u_k]$. These last ones are non-anticipating stochastic functions.

We denote the subspace of all the *non-anticipating* stochastic functions in $L_2(\Theta \times \mathbb{T} \times \Omega)$ by

$$L_2^I(\Theta \times \mathbb{T} \times \Omega) \subseteq L_2(\Theta \times \mathbb{T} \times \Omega).$$

Theorem 2.1 *Any stochastic function $\varphi \in L_2^I(\Theta \times \mathbb{T} \times \Omega)$ can be represented as the limit*

$$\varphi = \lim_{n\to\infty} \varphi^{(n)}, \quad i.e. \quad \|\varphi - \varphi^{(n)}\|_{L_2} \longrightarrow 0, \quad n \to \infty,$$

of simple functions of the type (2.13)*:*

$$\varphi^{(n)} = \sum_{k=1}^{\kappa_n} \varphi_{nk} 1_{\Delta_{nk}} \quad \textit{with} \quad \varphi_{nk} = \frac{1}{M(\Delta_{nk})} E\Big(\iint_{\Delta_{nk}} \varphi(\theta,t) M(d\theta dt) \big| \mathfrak{A}_{s_{nk}} \Big),$$

Here the sets Δ_{nk} are the elements of the n^{th}-series of the partitions of $\Theta \times \mathbb{T}$.

Proof. First let us show that *any* function $\varphi \in L_2(\Theta \times \mathbb{T} \times \Omega)$ is the limit of simple functions of the form (2.11):

$$\varphi^{(n)} = \sum_{k=1}^{\kappa_n} \varphi_{nk} 1_{\Delta_{nk}} \quad \text{with} \quad \varphi_{nk} := \frac{1}{M(\Delta_{nk})} \iint_{\Delta_{nk}} \varphi(\theta,t) M(d\theta dt),$$

where the sets Δ_{nk} are the elements of the n^{th}-series of the partitions of $\Theta \times \mathbb{T}$. Being $\varphi \in L_2(\Theta \times \mathbb{T} \times \Omega)$, it can be represented as the limit $\varphi = \lim_{n\to\infty} \psi^{(n)}$ in $L_2(\Theta \times \mathbb{T} \times \Omega)$ of some simple functions

$$\psi^{(n)}(\theta,t) = \sum_{k=1}^{\kappa_n} \psi_{nk} 1_{\Delta_{nk}}(\theta,t), \quad (\theta,t) \in \Theta \times \mathbb{T},$$

where, for every k, the element $\psi_{nk} \in L_2(\Omega)$ is the value taken on the element Δ_{nk} of the n^{th}-series of the given partitions of $\Theta \times \mathbb{T}$. Note that

$$\begin{aligned}\|\varphi_{nk} - \psi_{nk}\|^2 &= \Big\| \frac{1}{M(\Delta_{nk})} \iint_{\Delta_{nk}} \big[\varphi(\theta,t) - \psi^{(n)}(\theta,t)\big] M(d\theta dt) \Big\|^2 \\ &\le \frac{1}{M(\Delta_{nk})} \iint_{\Delta_{nk}} \big\|\varphi(\theta,t) - \psi^{(n)}(\theta,t)\big\|^2 M(d\theta dt).\end{aligned}$$

Hence we have

$$\begin{aligned}\|\varphi^{(n)} - \psi^{(n)}\|_{L_2}^2 &\le \sum_{k=1}^{\kappa_n} \iint_{\Delta_{nk}} \|\varphi(\theta,t) - \psi^{(n)}(\theta,t)\|^2 M(d\theta dt) \\ &= \|\varphi - \psi^{(n)}\|_{L_2}^2.\end{aligned}$$

Thus it is

$$\|\varphi - \varphi^{(n)}\|_{L_2} \le 2\|\varphi - \psi^{(n)}\|_{L_2} \longrightarrow 0, \quad n \to \infty.$$

Now let us consider the *non-anticipating* functions $\varphi \in L_2^I(\Theta \times \mathbb{T} \times \Omega)$ such that for *some* level of refinement n of the partitions of $\Theta \times \mathbb{T}$ we have that the values

$$\varphi(\theta, t) \in L_2(\Omega), \qquad (\theta, t) \in \Delta_{nk} = B_{nk} \times (s_{nk}, u_{nk}],$$

are $\mathfrak{A}_{s_{nk}}$-measurable. For these kind of stochastic functions we remark that the simple approximations of type (2.11) and (2.13) considered on the partitions of $\Theta \times \mathbb{T}$ are identical, for n big enough.

Finally we show that *any non-anticipating* stochastic function $\varphi \in L_2^I(\Theta \times \mathbb{T} \times \Omega)$ admits approximations via non-anticipating functions of the type described above. To this aim we recall that, for any (θ, t) and all the sets $\Delta_{nk} = B_{nk} \times (s_{nk}, u_{nk}]$ of the partitions of $\Theta \times \mathbb{T}$ such that $\Delta_{nk} \ni (\theta, t)$, we have that $\lim_{n\to\infty} \mathfrak{A}_{s_{nk}} = \mathfrak{A}_t$ - cf. (2.10). Then we also have

$$\varphi(\theta, t) = E\Big(\varphi(\theta, t)\big|\mathfrak{A}_t\Big) = \lim_{n\to\infty} E\Big(\varphi(\theta, t)\big|\mathfrak{A}_{s_{nk}}\Big)$$

in $L_2(\Omega)$ and we can also see that $\varphi = \lim_{n\to\infty} \phi^{(n)}$ in $L_2(\Theta \times \mathbb{T} \times \Omega)$ with

$$\phi^{(n)}(\theta, t) := \sum_{k=1}^{\kappa_n} E\Big(\varphi(\theta, t)\big|\mathfrak{A}_{s_{nk}}\Big)\, 1_{\Delta_{nk}}(\theta, t), \quad (\theta, t) \in \Theta \times \mathbb{T}.$$

The proof can be finished by observing that, for any n, the simple approximations of type (2.11) and (2.13) of the function $\phi^{(n)}$ are identical and it is

$$\begin{aligned}\iint_{\Delta_{nk}} \phi^{(n)}(\theta, t) M(d\theta, t) &= \iint_{\Delta_{nk}} E\big[\varphi(\theta, t)|\mathfrak{A}_{s_{nk}}\big] M(d\theta dt)\\ &= E\Big[\iint_{\Delta_{nk}} \varphi(\theta, t) M(d\theta, t)|\mathfrak{A}_{s_{nk}}\Big].\end{aligned}$$

By this the proof is complete. ■

Proposition 2.2 *Any stochastic function $\varphi \in L_2(\Theta \times \mathbb{T} \times \Omega)$ can be represented as the limit*

$$\varphi = \lim_{n\to\infty} \varphi^{(n)}$$

in $L_2(\Theta \times \mathbb{T} \times \Omega)$ of simple functions of the type (2.12)*:*

$$\varphi^{(n)} = \sum_{k=1}^{\kappa_n} \varphi_{nk} 1_{\Delta_{nk}} \quad \textit{with } \varphi_{nk} = \frac{1}{M(\Delta_{nk})} E\Big(\iint_{\Delta_{nk}} \varphi(\theta, t) M(d\theta dt)\big|\mathfrak{A}_{]\Delta_{nk}[}\Big) 1_{\Delta_{nk}}.$$

Here the sets Δ_{nk} are the elements of the n^{th}-series of the partitions of $\Theta \times \mathbb{T}$.

Proof. We proceed with arguments similar to the ones in the proof of Theorem 2.1. Let us recall that for any (θ, t) and all the sets Δ_{nk} of the partitions of $\Theta \times \mathbb{T}$ such that $\Delta_{nk} \ni (\theta, t)$, we have that $\lim_{n\to\infty} \mathfrak{A}_{]\Delta_{nk}[} = \mathfrak{A}_{\Theta\times\mathbb{T}}$ - cf. (2.10). Thus it is

$$\varphi(\theta,t) = E\Big(\varphi(\theta,t)|\mathfrak{A}_{\Theta\times\mathbb{T}}\Big) = \lim_{n\to\infty} E\Big(\varphi(\theta,t)|\mathfrak{A}_{]\Delta_{nk}[}\Big)$$

in $L_2(\Omega)$. Moreover we can see that $\varphi = \lim_{n\to\infty} \phi^{(n)}$ in $L_2(\Theta \times \mathbb{T} \times \Omega)$ with

$$\phi^{(n)}(\theta,t) := \sum_{k=1}^{\kappa_n} E\Big(\varphi(\theta,t)\big|\mathfrak{A}_{]\Delta_{nk}[}\Big)\, 1_{\Delta_{nk}}(\theta,t), \quad (\theta,t) \in \Theta \times \mathbb{T}.$$

For all the functions $\phi^{(n)}$ we can see that their simple approximations of type (2.11) and (2.12), for n big enough, are identical. We conclude the proof with arguments similar to the ones for Theorem 2.1. ∎

The Itô non-anticipating integral. Following the classical work [23], we can apply stochastic integration on the space-time product $\Theta \times \mathbb{T}$ - cf. e.g. [11]. The *Itô non-anticipating integral*

$$I\varphi := \int\int_{\Theta\times\mathbb{T}} \varphi(\theta,t)\mu(d\theta dt)$$

is well-defined for all $\varphi \in L_2^I(\Theta \times \mathbb{T} \times \Omega)$, i.e. for all the non-anticipating functions in $L_2(\Theta \times \mathbb{T} \times \Omega)$. This stochastic integral is represented by the *isometric linear operator* I:

$$L_2^I(\Theta \times \mathbb{T} \times \Omega) \ni \varphi \Longrightarrow I\varphi \in L_2(\Omega),$$

and integration can be carried out via the limit (2.13) as

$$I\varphi := \lim_{n\to\infty} \sum_{k=1}^{\kappa_n} \varphi_{nk}\mu(\Delta_{nk}) \tag{2.14}$$

with

$$\varphi_{nk} = \frac{1}{M(\Delta_{nk})} E\Big(\iint_{\Delta_{nk}} \varphi(\theta,t)M(d\theta dt)\big|\mathfrak{A}_{s_{nk}}\Big),$$

by means of the simple approximations of type (2.13) related to the partitions of $\Theta \times \mathbb{T}$.

The non-anticipating derivative. We refer to the adjoint linear operator $D = I^*$:

$$L_2(\Omega) \ni \xi \Longrightarrow D\xi \in L_2^I(\Theta \times \mathbb{T} \times \Omega)$$

as the *non-anticipating derivative* $D\xi = D\xi(\theta, t)$, $(\theta, t) \in \Theta \times \mathbb{T}$. According to [11] (see also [10] and [13]), the *non-anticipating differentiation* can be done through the limit

$$D\xi = \lim_{n\to\infty} \sum_{k=1}^{\kappa_n} \frac{1}{M(\Delta_{nk})} E\big(\xi\mu(\Delta_{nk})|\mathfrak{A}_{s_{nk}}\big) 1_{\Delta_{nk}}, \tag{2.15}$$

in $L_2(\Theta \times \mathbb{T} \times \Omega)$. Here the sets Δ_{nk} are the elements of the n^{th}-series of the partitions of $\Theta \times \mathbb{T}$.

For *all* the elements $\xi \in L_2(\Omega)$ the non-anticipating derivative provides the integrand in the *best approximation* $\hat{\xi}$ in $L_2(\Omega)$ to ξ via Itô integrals. Namely,

$$\hat{\xi} = \iint_{\Theta \times \mathbb{T}} D\xi(\theta, t) \mu(d\theta dt) \tag{2.16}$$

and it is

$$\|\xi - \hat{\xi}\| = \min_{\varphi \in L_2^I(\Theta \times \mathbb{T} \times \Omega)} \|\xi - I\varphi\|.$$

Equivalently we can regard the non-anticipating derivative as an explicit way of characterizing the integrand in the (unique) integral representation of ξ:

$$\xi = \xi^0 + \hat{\xi}$$

where $\xi^0 \in L_2(\Omega)$ is orthogonal to all stochastic integrals. Note that ξ^0 can be characterized by $D\xi^0 \equiv 0$. See [10], [11]. With respect to integral representations, a part from the fundamental work [24], we can refer for example to the seminal papers [4], [5], [8], [19], [29], [34].

The problem to determine the integrand in the best stochastic integral approximation $\hat{\xi}$ to ξ is of general interest in applications. In mathematical finance, for example, it is related to problems of quadratic optimal hedging which are widely studied in literature. See e.g. [16], [17], [37], [39] and references therein. With respect to applications to finance, the non-anticipating derivative represents the minimal variance hedging strategy for general market models considered under the risk-neutral probability measure, see [2], [10] for more details.

However the non-anticipating derivative is not easy to be computed and more explicit fomulae for its computation are searched. In this paper we provide some explicit formulae for the non-anticipating derivative (2.15) in the framework we have introduced. Our results are in the same line of interest as a series of results which are mostly related to Malliavin calculus for Lévy processes, see e.g. [2], [12], [15], [30]. For the specific case of Brownian motion we can refer to e.g. [1], [6], [32], [33], [35], [41] and to [36] for some results on a space-time Brownian field.

We would like to stress that we are following a rather different approach than what was taken in the existing literature which allows us to obtain more general results.

3 The deFinetti-Kolmogorov law

B. deFinetti and A.N. Kolmogorov were the first pioneers in the study of stochastic processes with independent increments and infinitely divisible distributions. See [7] and [28]. The focus was on processes with random variables in $L_2(\Omega)$. Particular relevance for our discussion is their explicit expression of the characteristic function of the increments of such processes: *the deFinetti-Kolmogorov formula*. The generalization of this formula to all stochastic processes with stationary independent increments is first due to the works of P. Lévy and A.N. Khintchine: *the Lévy-Khintchine formula.* See [31] and [27]. We can refer to the classical works of [18], [22] and [40], and also to the recent monographies of [3] and [38] for detailed reading and references. In the line of the results above we have to mention a version for random measures on some topological space studied in [25]. Here below we detail our version of the deFinetti-Kolmogorov law in the context of the general random measures with independent values (2.1) we are dealing with. In order to be coherent in our exposition we present the result for random measures on the space-time product $\Theta \times \mathbb{T}$ equipped with a measure M satisfying (2.5). This formula will be thoroughly exploited in the sequel.

Theorem 3.1 *Let $\mu = \mu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, be a stochastic measure with independent values of the type* (2.1)-(2.5)*, then it is*

$$\log Ee^{iu\mu(\Delta)} = \iint_\Delta \Big[-\frac{u^2}{2}\sigma^2(\theta,t) + \int_{\mathbb{R}_0} \big(e^{iux} - 1 - iux\big) L(dx,\theta,t)\Big] M(d\theta dt), \ u \in \mathbb{R}, \tag{3.1}$$

where $\sigma^2 = \sigma^2(\theta,t)$, $(\theta,t) \in \Theta \times \mathbb{T}$, is a non-negative function and $L = L(dx,\theta,t)$, $x \in \mathbb{R}_0$, is, for every $(\theta,t) \in \Theta \times \mathbb{T}$, a Borel σ-finite measure on $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ such that

$$\sigma^2(\theta,t) + \int_{\mathbb{R}_0} x^2 L(dx,\theta,t) \equiv 1.$$

Proof. Recall that the product σ-algebra on $\Theta \times \mathbb{T}$ is separable and that the partitions of $\Theta \times \mathbb{T}$ generate this σ-algebra. Then it is immediate to see that for any simple set:

$$\Delta = \bigsqcup_k \Delta_{nk}, \quad \text{where} \qquad \max_k M(\Delta_{nk}) \longrightarrow 0, \quad n \to \infty,$$

it is

$$\mu(\Delta) = \sum_k \mu(\Delta_{nk})$$

which shows that $\mu(\Delta)$ is a sum of independent and uniformly infinitely small random variables $\mu(\Delta_{nk})$; namely,

$$\max_k E\Big[\mu(\Delta_{nk})^2\Big] \longrightarrow 0, \quad n \to \infty.$$

Accordingly for any $\Delta = \Theta \times \mathbb{T}$ the corresponding value can be represented as sum

$$\mu(\Delta) = \sum_{k=1}^{\kappa_n} \mu(\Delta \cap \Delta_{nk}) + \mu\big(\Delta \setminus \bigsqcup_{k=1}^{\kappa_n} \Delta_{nk}\big)$$

of independent and uniformly infinitely small random variables - cf. (2.8)-(2.9). Hence for any fixed $\Delta \subseteq \Theta \times \mathbb{T}$, the random variable $\mu(\Delta) \in L_2(\Omega)$ obeys the deFinetti-Kolmogorov law

$$\log Ee^{iu\mu(\Delta)} = -\frac{u^2}{2}\sigma^2_\Delta + \int_{\mathbb{R}_0} \big(e^{iux} - 1 - iux\big) L_\Delta(dx), \quad u \in \mathbb{R},$$

where the non-negative constant σ^2_Δ and the integrator $L_\Delta = L_\Delta(dx)$, $x \in \mathbb{R}_0$, satisfy the condition

$$\sigma^2_\Delta + \int_{\mathbb{R}_0} x^2 L_\Delta(dx) = M(\Delta).$$

Moreover σ^2_Δ and L_Δ depend on $\Delta \subseteq \Theta \times \mathbb{T}$ as additive set-functions, being the values $\mu(\Delta)$ corresponding to disjoint $\Delta \subseteq \Theta\times\mathbb{T}$ independent random variables. The condition above shows that $\sigma^2_\Delta \leq M(\Delta)$, thus the additive set-function σ^2_Δ, $\Delta \subseteq \Theta\times\mathbb{T}$, is actually a measure with representation

$$\sigma^2_\Delta = \int_\Delta \sigma^2(\theta, t) M(d\theta dt)$$

via the non-negative function $\sigma^2 = \sigma^2(\theta, t)$, $(\theta, t) \in \Theta \times \mathbb{T}$, integrable with respect to M. Following similar arguments we also obtain the representation

$$L_\Delta(B) = \int_\Delta \tilde{L}(B, \theta, t) M(d\theta dt), \quad B \subseteq \mathbb{R}_0,$$

via the non-negative stochastic function $\tilde{L}(B, \theta, t)$, $(\theta, t) \in \Theta\times\mathbb{T}$, integrable with respect to M for every $B \subseteq \mathbb{R}_0$. Note that, for every (θ, t), $\tilde{L}(B, \theta, t)$ are additive set-functions on the Borel sets $B \subseteq \mathbb{R}_0$. For any $B \subseteq \mathbb{R}_0$, the integrand $\tilde{L}(B, \theta, t)$, $(\theta, t) \in \Theta \times \mathbb{T}$, can be modified on a set of M-null measure in a way which gives the lifting upto a new *equivalent* integrand $L(B, \theta, t)$, $(\theta, t) \in \Theta \times \mathbb{T}$ which is, for every (θ, t), a *measure* on the Borel sets $B \subseteq \mathbb{R}_0$. This is similar to the lifting of conditional probabilities - see e.g. [20], [21]. By this the proof is complete. ■

4 An explicit differentiation formula for the non-anticipating derivative.

Let $\mu = \mu(\Delta)$, $\Delta \subseteq \Theta \times \mathbb{T}$, be a general stochastic measure with independent values characterized by the probability law given by (3.1). We consider the random variables $\xi \in L_2(\Omega)$ generated by the values of μ; namely, the random variables we are dealing with are measurable with respect to $\mathfrak{A}_T$ - cf. (2.3).

In particular in this section we focus on the random variables ξ which can be treated as functions of a *finite* number of values of the stochastic measure involved. Any such random variable admits representation as a function

$$\xi = F(\xi_1, ..., \xi_m) \tag{4.1}$$

of the values $\xi_j = \mu(\Lambda_j)$, $j = 1, ..., m$, on some appropriately chosen *disjoint* sets Λ_j, $j = 1, ..., m$, in $\Theta \times \mathbb{T}$. Of course, the representation (4.1) is not unique. So, for *any* finite number of *any* particular group of *disjoint* sets

$$\Lambda_j \subseteq \Theta \times \mathbb{T} \quad \text{with} \quad M(\Lambda_j) < \infty, \qquad j = 1, ..., m,$$

we consider $\xi = F$ of type (4.1) for the functions

$$F = F(\xi_1, ..., \xi_m), \quad (\xi_1, ..., \xi_m) \in \mathbb{R}^m,$$

which we assume to be $C^1(\mathbb{R}^m)$.

In the line with the first continuous derivatives $\frac{\partial F}{\partial \xi_j}$, $j = 1, ..., m$, we write

$$\partial_j^x F := \begin{cases} \frac{\partial}{\partial \xi_j} F(..., \xi_j, ...), & x \neq 0 \\ \frac{1}{x}\big[F(..., \xi_j + x, ...) - F(..., \xi_j, ...)\big], & x = 0. \end{cases}$$

According to the characterization of μ by the deFinetti-Kolmogorov law with the parameters σ^2 and L, see (3.1), we define

$$\mathfrak{D}\xi(\theta, t) := \sum_{j=1}^{m} \big[\partial_j^0 F\, \sigma^2(\theta, t) + \int_{\mathbb{R}_0} \partial_j^x F\, x^2 \cdot L(dx, \theta, t)\big] 1_{\Lambda_j}(\theta, t), \quad (\theta, t) \in \Theta \times \mathbb{T}, \tag{4.2}$$

for $\xi = F$ of the above type. We assume that the above stochastic functions

$$\mathfrak{D}\xi = \mathfrak{D}\xi(\theta, t), \quad (\theta, t) \in \Theta \times \mathbb{T},$$

satisfy the condition

$$|||\mathfrak{D}\xi|||_{L_2}^2 := \sum_{j=1}^{m} \iint_{\Lambda_j} \Big[\|\partial_j^0 F\|^2\, \sigma^2(\theta, t) + \int_{\mathbb{R}_0} \|\partial_j^x F\|^2\, x^2\, L(dx, \theta, t)\Big] M(d\theta dt) < \infty. \tag{4.3}$$

Then we have that $\mathfrak{D}\xi \in L_2(\Theta \times \mathbb{T} \times \Omega)$. In fact

$$\|\mathfrak{D}\xi\|_{L_2} \leq |||\mathfrak{D}\xi|||_{L_2}. \tag{4.4}$$

In the scheme (4.1)-(4.4) we obtain the following result.

Theorem 4.1 *For the random variables ξ of type* (4.1) *for which* (4.3) *holds, the non-anticipating derivative $D\xi = D\xi(\theta, t)$, $(\theta, t) \in \Theta \times \mathbb{T}$, defined via the limit* (2.15), *can be obtained by the formula*

$$D\xi(\theta, t) = E\big(\mathfrak{D}\xi(\theta, t)|\mathfrak{A}_t\big), \quad (\theta, t) \in \theta \times \mathbb{T}. \tag{4.5}$$

Proof. At first let us consider random variables ξ of type (4.1) given by the functions F of form

$$F(\xi_1, ..., \xi_m) = e^{i \sum_{j=1}^m u_j \xi_j} \qquad (u_j \in \mathbb{R},\ j = 1, ..., m). \tag{4.6}$$

In this case the formula (4.2) gives

$$\mathfrak{D}\xi(\theta, t) = \xi \cdot \sum_{j=1}^{m} \big[iu_j \sigma^2(\theta, t) + \int_{\mathbb{R}_0} \big(e^{iu_j x} - 1\big) x\, L(dx, \theta, t)\big] 1_{\Lambda_j}(\theta, t), \quad (\theta, t) \in \Theta \times \mathbb{T}.$$

Having in mind (2.15) it is convenient to start considering $\xi_j = \mu(\Lambda_j)$, $j = 1, ..., m$ (see (4.1)) on *disjoint simple sets* Λ_j; we recall that each Λ_j as a *finite* union of elements of the partitions of $\Theta \times \mathbb{T}$ - cf. (2.7). For n big enough ($n \to \infty$, see (2.8)) *any* element Δ of the n^{th}-series of partitions either belongs to some Λ_j or it is disjoint with all of them. Then we note that if Δ is disjoint from all Λ_j, $j = 1, ..., m$, we have

$$E\big(\xi\,\mu(\Delta)|\mathfrak{A}_{]\Delta[}\big) = \xi\, E\mu(\Delta) = 0,$$

- cf. (2.6). Otherwise, if $\Delta \subseteq \Lambda_j$ for some j, it is

$$E\big(\xi\,\mu(\Delta)|\mathfrak{A}_{]\Delta[}\big) = e^{-iu_j\mu(\Delta)}\,\xi\, E\Big(\mu(\Delta)\, e^{iu_j\mu(\Delta)}\Big)$$

with

$$E\Big(\mu(\Delta)\, e^{iu_j\mu(\Delta)}\Big) = Ee^{iu_j\mu(\Delta)} \cdot \iint_\Delta \Big[iu_j\sigma^2(\theta,t) + \int_{\mathbb{R}_0} \big(e^{iu_jx} - 1\big)xL(dx,\theta,t)\Big] M(d\theta dt),$$

- cf. (3.1). So, for $\Delta \subseteq \Lambda_j$, we can see that

$$\begin{aligned} E\big(\xi\mu(\Delta)|\mathfrak{A}_{]\Delta[}\big) =& e^{-iu_j\mu(\Delta)} Ee^{iu_j\mu(\Delta)} \iint_\Delta \mathfrak{D}\xi(\theta,t)M(d\theta dt) \\ =& E\Big(\iint_\Delta \mathfrak{D}\xi(\theta,t)M(d\theta dt)|\mathfrak{A}_{]\Delta[}\Big). \end{aligned}$$

Then according to Proposition 2.2 the stochastic function $\mathfrak{D}\xi$ admits representation as the limit

$$\mathfrak{D}\xi = \lim_{n\to\infty} \sum_\Delta \frac{1}{M(\Delta)} E\big(\xi\mu(\Delta)|\mathfrak{A}_{]\Delta[}\big)1_\Delta, \tag{4.7}$$

in $L_2(\Theta \times \mathbb{T} \times \Omega)$ (the sum $\sum_\Delta$ refers to all the elements of the same n^{th}-series of partitions of $\Theta \times \mathbb{T}$: we have neglected writing the indexes not playing crucial role in the argument). From (4.7) by considering the sub-sequence of $n = 1, 2, ...$ for which the limits (4.7) and (2.15) converge in $L_2(\Omega)$ for almost all $(\theta, t) \in \Theta \times \mathbb{T}$, we have

$$\mathfrak{D}\xi(\theta,t) = \lim_{n\to\infty} \frac{1}{M(\Delta)} E\big(\xi\mu(\Delta)|\mathfrak{A}_{]\Delta[}\big), \quad (\theta,t) \in \Delta,$$

and

$$D\xi(\theta,t) = \lim_{n\to\infty} \frac{1}{M(\Delta)} E\big(\xi\mu(\Delta)|\mathfrak{A}_s\big), \quad (\theta,t) \in \Delta,$$

(for $\Delta = B \times (s, u]$, $\Delta \ni (\theta, t)$, elements of the n^{th}-series of partitions of $\Theta \times \mathbb{T}$). Accordingly, for (θ, t) fixed and via all the corresponding sets Δ such that $\Delta \ni (\theta, t)$, we obtain

$$E\big(D\xi(\theta,t)|\mathfrak{A}_{t-}\big) = \lim_{n\to\infty} \frac{1}{M(\Delta)} E\big(\xi\mu(\Delta)|\mathfrak{A}_{t-}\big) = E\big(\mathfrak{D}\xi(\theta,t)|\mathfrak{A}_{t-}\big)$$

for $t^- < t$. Let $t^- \to t$, then the above relationship between $D\xi$ and $\mathfrak{D}\xi$ implies that

$$D\xi(\theta,t) = \lim_{t^-\to t} E\big(D\xi(\theta,t)|\mathfrak{A}_{t^-}\big) = \lim_{t^-\to t} E\big(\mathfrak{D}\xi(\theta,t)|\mathfrak{A}_{t^-}\big) = E\big(\mathfrak{D}\xi(\theta,t)|\mathfrak{A}_t\big),$$

since $D\xi(\theta,t) \in L_2(\Omega)$ is an $\mathfrak{A}_t$-measurable random variable and $\lim_{t^-\to t}\mathfrak{A}_{t^-} = \mathfrak{A}_t$, see (2.4). Thus the formula (4.5) for all the random variables $\xi = F$ with the functions (4.6) of the values $\xi_j = \mu(\Lambda_j)$, $j = 1,...,m$, on disjoint simple sets.

Actually the formula holds for Λ_j, $j = 1,...,m$, as general *disjoint* sets in $\Theta \times \mathbb{T}$. Indeed let us take the approximations (2.9) into account: $\Lambda_j = \lim_{n\to\infty}\Lambda_j^{(n)}$, $j = 1,...,m$, by disjoint simple sets $\Lambda_j^{(n)}$ which can be chosen in such a way that the limits

$$\mu(\Lambda_j) = \lim_{n\to\infty}\mu(\Lambda_j) \quad (j = 1,...,m)$$

in $L_2(\Omega)$ also hold for almost all $\omega \in \Omega$. Let ξ and $\xi^{(n)}$, $n = 1,2,...$, be the random variables corresponing to the funtion (4.6) of the values $\xi_j = \mu(\Delta_j)$ and $\xi_j^{(n)} = \mu(\Delta_j^{(n)})$, $n = 1,2,...$, $j = 1,...,m$. Then $\xi = \lim_{n\to\infty}\xi^{(n)}$ in $L_2(\Omega)$ and also

$$D\xi = \lim_{n\to\infty} D\xi^(n) \quad \text{and} \quad \mathfrak{D}\xi = \lim_{n\to\infty}\mathfrak{D}\xi^(n)$$

in $L_2(\Theta\times\mathbb{T}\times\Omega)$. Being $D\xi(\theta,t) = E\big(\mathfrak{D}\xi(\theta,t)|\mathfrak{A}_t\big)$ proved, then the limits above imply

$$D\xi(\theta,t) = E\big(\mathfrak{D}\xi(\theta,t)|\mathfrak{A}_t\big),$$

for almost all $(\theta,t) \in \Theta\times\mathbb{T}$, namely (4.5) holds for $\mathfrak{D}\xi$ as an element of $L_2^I(\Theta\times\mathbb{T}\times\Omega) \subseteq L_2(\Theta\times\mathbb{T}\times\Omega)$.

Clearly, for all random variables $\xi = F$ of type (4.1) where F is a linear combination of these functions (4.6) above, the corresponing limit (4.7) holds as well, resulting formula (4.5).

Then the formula (4.5) can be extended on all the random variables $\xi = F$ with functions F characterized in the scheme (4.1)-(4.4). This can be done by standard approximation arguments applied to the scalar functions

$$\partial\xi := \sum_{j=1}^{m}\partial_j^x F\, 1_{\Lambda_j} \tag{4.8}$$

on the product space $\mathbb{R}\times\Lambda\times\Omega$ where $\Lambda := \bigsqcup_{j=1}^m \Lambda_j$, with the *finite* product-type measure

$$L_0(dx,\theta,t)\times M(d\theta dt)\times P(d\omega), \quad (x,\theta,t,\omega) \in \mathbb{R}\times\Lambda\times\Omega.$$

The component L_0 involved is the *finite* measure on $\mathbb{R}$ equal to $\sigma^2(\theta,t)$ at the atom $x = 0$ and $x^2\, L(dx,\theta,t)$ on $\mathbb{R}_0$. The functions

$$\partial\xi = \partial\xi(x,\theta,t,\omega), \qquad (x,\theta,t,\omega) \in \mathbb{R}\times\Lambda\times\Omega,$$

are elements of the standard space $L_2(\mathbb{R}\times\Lambda\times\Omega)$ with the norm

$$\|\partial\xi\|_{L_2} = \Big(\iiint_{\mathbb{R}\times\Lambda\times\Omega}|\partial\xi|^2 L_0(dx,\theta,t)M(d\theta dt)P(d\omega)\Big)^{1/2},$$

and we have $\|\partial\xi\|_{L_2} = |||\mathfrak{D}\xi|||_{L_2}$ for

$$\mathfrak{D}\xi(\theta, t, \omega) = \int_{\mathbb{R}} \partial\xi(x, \theta, t, \omega) L_0(dx, \theta, t), \quad (\theta, t, \omega) \in \Lambda \times \Omega.$$

See (4.2)-(4.4).

The key-point of the approximation argument which will be applied is that for the random variable $\xi = F$ and its approximating sequence $\xi^{(n)} = F^{(n)}$, $n = 1, 2, ...$, the convergences

$$\|\xi - \xi^{(n)}\| \longrightarrow 0, \quad \text{and} \quad \|\partial\xi - \partial\xi^{(n)}\|_{L_2} \longrightarrow 0, \qquad n \to \infty, \tag{4.9}$$

imply

$$\mathfrak{D}\xi = \lim_{n\to\infty} \mathfrak{D}\xi^{(n)}$$

and, being the non-anticipating derivative continuous, $\|\xi - \xi^{(n)}\| \to 0$, $n \to \infty$, implies

$$D\xi = \lim_{n\to\infty} D\xi^{(n)}.$$

Both limits $\mathfrak{D}\xi$ and $D\xi$ are in $L_2(\Theta \times \mathbb{T} \times \Omega)$. Note that we apply *dominated* point-wise convergence with appropriate corresponding majorants to prove the convergences (4.9).

The approximation argument is here organized in three steps in which the corresponding approximating sequences are given. To simplify the notation we present this second part of the proof in the case $m = 1$ ($j = 1$), namely the random variables of type (4.1) involve just the value $\xi_1 = \mu(\Lambda_1)$. Correspondingly, the above function (4.8) is $\partial\xi = \partial\xi(x, \theta, t, \omega) = \partial_1^x F(\xi_1(\omega)) 1_{\Lambda_1}(\theta, t)$, $(x, \theta, t, \omega) \in \mathbb{R} \times \Lambda \times \Omega$.

Set $\xi_1 = \xi(\omega) \in \mathbb{R}$. If $\xi = F$ of type (4.1) is such that $F \in C_0^\infty(\mathbb{R})$, then we can take the partial sums $F^{(n)} = \Phi_n(F)$, $n = 1, 2, ...$, of the Fourier series for F over $|\xi_1| \le h_n$ (for $h_n \to \infty$, $n \to \infty$) as the approximating sequence $\xi^{(n)} = F^{(n)}$, $n = 1, 2, ...$ Then the convergences (4.9) hold. In fact considering the Fourier coefficients, for n big enough ($n \to \infty$), it is

$$\partial_1^x F^{(n)} := \partial_1^x \Phi_n(F) = \Phi_n(\partial_1^x F) \quad (n = 1, 2, ...)$$

whatever $x \in \mathbb{R}$ be.

Next step is to consider $\xi = F$ of type (4.1) with $F \in C_0^1(\mathbb{R})$. Then the approximating sequence $\xi^{(n)} = F^{(n)}$, $n = 1, 2, ...$, is given by the convolutions

$$F^{(n)} := F \star \delta_n = \int_{\mathbb{R}} F(\xi_1 - x_1)\delta_n(x_1) dx_1 \in C_0^\infty(\mathbb{R})$$

with $\delta_n \in C_0^\infty(\mathbb{R})$, $n = 1, 2, ...$, as the standard approximations to the delta-function. Thanks to the fact that

$$\partial_1^x F^{(n)} := \partial_1^x (F \star \delta_n) = (\partial_1^x F) \star \delta_n \quad (n = 1, 2, ...),$$

we deduce that the convergences (4.9) hold.

Finally we can consider the general case $\xi = F$ of type (4.1) with $F \in C^1(\mathbb{R})$. In this case the approximating sequence $\xi^{(n)} = F^{(n)}$, $n = 1, 2, ...$, is given by the truncations $F^{(n)} = F \cdot w_n$, where $w_n \in C_0^1(\mathbb{R})$, $n = 1, 2, ...$, are proper approximations to the unit. Then the convergences (4.9) hold, thanks to

$$\partial_1^x F^{(n)} := \partial_1^x (F w_n) = (\partial_1^x F) w_n + F(\partial_1^x w_n) \quad (n = 1, 2, ...).$$

By this we end the proof. ■

Example. Let us take μ as a *Gaussian-Poisson mixture* on

$$\Theta \times \mathbb{T} = \bigcup_{x \in \mathbb{R}} (\Theta_x \times \mathbb{T})$$

such that on $\Theta_0 \times \mathbb{T}$ the stochastic measure μ is a Gaussian stochastic measure and on $\Theta_x \times \mathbb{T}$, for each $x \neq 0$, it is a centered Poisson stochastic measure multiplied by the scalar factor x. The measures μ on $\Theta_x \times \mathbb{T}$, $x \in \mathbb{R}$, considered are independent. Let $\xi = F$ be a random variable of type (4.1) as a $C^1(\mathbb{R}^m)$ function of the values $\xi_j = \mu(\Lambda_j)$, $j = 1, ..., m$, of the stochastic measure on the *disjoint* sets $\Lambda_j \in \Theta \times \mathbb{T}$, $j = 1, ..., m$. The formula (4.2) can be written as

$$\mathfrak{D}\xi = \sum_{j=1}^{m} \big[\partial_j^0 F \, 1_{\Theta_0 \times \mathbb{T}} + \sum_{x \neq 0} \partial_j^x F \, 1_{\Theta_x \times \mathbb{T}}\big] 1_{\Lambda_j}.$$

Remark. Note that in general formula (4.2) is *not* valid if ξ of form (4.1) is represented as a function $F = F(\xi_1, ..., \xi_m)$ of values $\xi_j = \mu(\Lambda_j)$, $j = 1, ..., m$, on *not disjoint* sets.

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