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Arbitrage and asymmetric information

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Abstract

In this paper we use some ideas of Cornet and de Boisdeffre to study the concept of arbitrage under asymmetric information. The mathematical framework is a separable probability space where the agents' information are represented by σ-algebras. In this setting we formulate some versions of the fundamental theorem of asset pricing (aka the Dalang-Morton-Willinger theorem) for the case of asymmetric information. We also study the revealing properties of no-arbitrage prices and prove that the results of Cornet and de Boisdeffre hold in a more general setting.

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MSC2000: 91B24, 60H30

1 Introduction

In markets subject to asymmetric information, agents will use asset prices to extract information. This is the basis of rational expectations models where the agents use their knowledge of the other agents' characteristics and the observed market prices to update their own probability assessments (cf. e.g. [9]). An alternative approach, suggested in [3] is to assume that agents extract information from asset prices only by analysing *arbitrage opportunities.* Contrary to rational expectations models the agents need no a priori knowledge of the other agents' preferences or behaviour.

The scope of [3] is firstly to extend the concept of arbitrage to the case of asymmetric information and secondly to study how no-arbitrage prices reveal information. In the follow-up paper [4], the authors study how agents can extract information by successively ruling out "arbitrage states" i.e. states in which an arbitrage opportunity would give a strictly positive payoff. The existence of a no-arbitrage equilibrium in a market with asymmetric information is dealt with in [6]. Extensions to the multiperiod case are studied in [1]. All these papers are limited to a finite dimensional state space, and the agents' information are represented by subsets of the state space (sub-trees in the multiperiod case).

The aim of this paper is to use the ideas from [3] in a more mathematically profound analysis of a financial market. Our framework will be a separable probability space and the agents' information will be represented by σ-algebras (filtrations in the multiperiod case). In this framework we can formulate some "asymmetric information versions" of the fundamental theorem of asset pricing (aka the Dalang-Morton-Willinger theorem). As we will see, the results of [3] and [4] hold true in a more general setting. Hopefully, the added mathematical sophistication will enable us to use the powerful tools of mathematical finance for further analysis of the multiperiod (briefly discussed in this paper) and continuous time markets. The issue of existence of a no-arbitrage equilibrium is not dealt with in this paper.

The paper is organised as follows: In Section 2 we introduce information structures and arbitrages in an asymmetric information setting and state some versions of the fundamental theorem of asset pricing. We will see that some information structures do not allow no-arbitrage prices. In Section 3 we introduce the concept of arbitrage-free refinements of information structures. We discuss whether informational asymmetries can prevail in an arbitrage-free market and find that this issue is linked to market completeness. Section 4 deals with information revealed by prices. In Section 5 we study *how* agents can extract information from prices by analysing arbitrage opportunities. In Section 6 we briefly discuss the extension to the multiperiod market.

2 Information and arbitrage

2.1 Framework and notation

Consider the complete probability space $(\Omega, \mathcal{F}, P)$. We assume that $\mathcal{F}$ is separable. There are J assets in the economy, traded at time 0, with the $\mathcal{F}$-measurable $\mathbb{R}^J$-valued time T payoff V. We assume that $\mathcal{F}$ is generated by V. A *portfolio* is a (possibly random) J-dimensional vector whose components denote the holdings of the assets, the payoff of the portfolio z is the random variable $V^\top z$. A *price function* is an $\mathcal{F}$-measurable $\mathbb{R}^J$-valued random variable.

Assumption 2.1. V is bounded and there exist some portfolio $z^* \in \mathbb{R}^J$ such that $V^\top z^* > 0$ a.s. and $\frac{V}{V^\top z^*}$ is integrable.

An agent's information will be represented by a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$. We assume all σ-algebras to be completed. For any σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ we let $P(\cdot|\mathcal{G})$ denote the regular version of the *conditional probability* (cf. [2, Section 33]). The separability of $\mathcal{G}$ ensures that for any $\omega \in \Omega$ and $F \in \mathcal{F}$ the smallest elements in $\mathcal{G}$ that contains ω or F are well-defined as

$$G(\omega,\mathcal{G}) := \bigcap_{\substack{G\in\mathcal{G},\\ \omega\in G}} G, \qquad G(F,\mathcal{G}) := \bigcap_{\substack{G\in\mathcal{G},\\ F\subseteq G}} G.$$

An *information structure* is a collection $(\mathcal{H}_i) := (\mathcal{H}_1,\ldots,\mathcal{H}_I)$ of σ-algebras representing the agents' information. The agents' *pooled* information is given by the *join*

$$\underline{\mathcal{H}} : = \bigvee \mathcal{H}_i,$$

while their *common information* is given by the *meet*

$$\overline{\mathcal{H}} : = \bigwedge \mathcal{H}_i.$$

We can also define the *meet* of a sequence $(\mathcal{G}_i^{(1)}), (\mathcal{G}_i^{(2)}), \ldots$ of information structures as

$$(\textstyle\bigwedge_{k=1,2\ldots} \mathcal{G}_i^{(k)}).$$

The information structure $(\mathcal{H}_i)$ is *symmetric* if all the $\mathcal{H}_i$'s coincide. The information structure $(\mathcal{G}_i)$ is a *refinement* of $(\mathcal{H}_i)$ if $\mathcal{H}_i \subseteq \mathcal{G}_i$ for all i, we also say that $(\mathcal{H}_i)$ is *coarser* than $(\mathcal{G}_i)$. Clearly for any refinement $(\mathcal{G}_i)$, $\underline{\mathcal{H}} \subseteq \underline{\mathcal{G}}$. The refinement is *self-attainable* if $\underline{\mathcal{H}} = \underline{\mathcal{G}}$.

Remark 2.1. In our notation, all the analysis in [3] take place on some fixed $\omega \in \Omega$ and agent i's information is represented by $G(\omega, \mathcal{H}_i)$.

2.2 Arbitrage-free information structures

Definition 2.1 (arbitrage, arbitrage-free σ-algebras and no-arbitrage price functions)**.** Given the price function ϕ, a vector $z \in \mathbb{R}^J$ is a *ϕ-arbitrage* for the σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ at ω if

$$\phi(\omega)^\top z \le 0, \qquad V^\top z \ge 0\, P(\cdot|\mathcal{G})(\omega)\text{-a.s. and} \quad P(V^\top z > 0|\mathcal{G})(\omega) > 0. \tag{2.1}$$

The price function ϕ is a *no-arbitrage price function* for $\mathcal{G}$ and $\mathcal{G}$ is *ϕ-arbitrage-free* if $\mathcal{G}$ is ϕ-arbitrage-free at ω for any ω outside a set of measure 0. The set of no-arbitrage price functions for $\mathcal{G}$ is denoted $\Phi(\mathcal{G})$.

Remark 2.2. Strictly speaking we should also refer to V in the definition and say that $\mathcal{G}$ is ϕ-arbitrage-free *for V* and that ϕ is a no-arbitrage price *for V* and $\mathcal{G}$ etc. But in the one-period market, V is fixed and we need not refer to it untill we treat the multiperiod market.

Remark 2.3. Note that we do *not* assume that the price function is $\mathcal{G}$-measurable. This may seem odd when thinking of $\mathcal{G}$ as the agent's information: clearly the agent will observe the asset prices. But as the asset prices can depend on information that is not available to the agent we cannot assume that the asset price as a *mapping* $\phi : \Omega \to \mathbb{R}^J$ is $\mathcal{G}$-measurable.

Remark 2.4. For the case of a $\mathcal{G}$-measurable price function the property of no-arbitrage in Definition 2.1 coincides with the standard definition, namely that there is no $\mathcal{G}$-measurable $\mathbb{R}^J$-valued random variable ξ such that

$$\phi^\top \xi \leq 0, \qquad V^\top \xi \geq 0 \, P\text{-a.s. and} \quad P(V^\top z > 0) > 0.$$

For a proof, see [5, Lemma 2.3].

The following theorem, which corresponds to the equivalence statement in [3, Definition 2.2], is a version of the *fundamental theorem of asset pricing* (cf. e.g. [8, Theorem 1.6]):

Theorem 2.1. *The price function ϕ is a no-arbitrage price function for $\mathcal{G}$ if and only if for almost all ω there exist some probability measure $P^{(0)} \sim P$ on $(\Omega, \mathcal{F})$ such that*

$$\frac{\phi(\omega)}{\phi(\omega)^\top z^*} = E^{(0)}\Big[\frac{V}{V^\top z^*}\big|\mathcal{G}\Big](\omega). \tag{2.2}$$

Proof. Fix some ω and put $\phi(\omega) = q$. The absence of arbitrage implies that $q^\top z^* > 0$ so that the representation (2.2) is well-defined. It is easy to see that this representation implies absence of arbitrage. For the converse, consider

$$\mathcal{C} := \Big\{ E^{(0)}\Big[\frac{V}{V^\top z^*}\big|\mathcal{G}\Big](\omega) - \frac{q}{q^\top z^*};\ P^{(0)} \sim P \Big\} \subseteq \mathbb{R}^J.$$

Clearly, (2.2) holds if and only if $0 \in \mathcal{C}$. If $0 \notin \mathcal{C}$, then by the separating hyperplane theorem (e.g. [8, Proposition A.1]) there exist some $\zeta \in \mathbb{R}^J$ such that

$$x^\top \zeta \geq 0, \qquad \text{for all } x \in \mathcal{C} \tag{2.3a}$$

and

$$x_0^\top \zeta > 0, \qquad \text{for some } x_0 \in \mathcal{C}. \tag{2.3b}$$

Consequently

$$\frac{V^\top \zeta}{V^\top z^*} \geq \frac{q^\top \zeta}{q^\top z^*} \qquad P(\cdot|\mathcal{G})(\omega)\text{-a.s.}$$

and

$$P\Big(\frac{V^\top \zeta}{V^\top z^*} > \frac{q^\top \zeta}{q^\top z^*}\big|\mathcal{G}\Big)(\omega) > 0.$$

Hence,

$$z := \begin{cases} \frac{q^\top \zeta}{q^\top \zeta - q^\top z^*} z^* - \frac{q^\top z^*}{q^\top \zeta - q^\top z^*}\zeta, & q^\top \zeta < q^\top z^*, \\ \zeta - z^*, & q^\top \zeta = q^\top z^*, \\ -\frac{q^\top \zeta}{q^\top \zeta - q^\top z^*} z^* + \frac{q^\top z^*}{q^\top \zeta - q^\top z^*}\zeta, & q^\top \zeta > q^\top z^* \end{cases}$$

represents a ϕ-arbitrage at ω. □

As a direct consequence we have that:

Corollary 2.1.1. *For every σ-algebra $\mathcal{G}$, there exist some $\mathcal{G}$-measurable no-arbitrage price function.*

Later on we will need the following even stronger result:

Lemma 2.1. *If $\phi \in \Phi(\mathcal{G})$ then for any $\mathcal{F}' \supseteq \mathcal{G}$ we have that $E[\phi|\mathcal{F}'] \in \Phi(\mathcal{G})$.*

Proof. Suppose not and let z be an arbitrage for $E[\phi|\mathcal{F}']$ at ω, i.e.

$$E[\phi|\mathcal{F}'](\omega)^\top z \leq 0, \tag{2.4}$$

$$V^\top z \geq 0\, P(\cdot|\mathcal{G})(\omega)\text{-a.s. and} \quad P(V^\top z > 0|\mathcal{G})(\omega) > 0. \tag{2.5}$$

As ϕ is a no-arbitrage price function, (2.5) implies that $\phi(\omega)^\top z > 0$. For (2.4) to hold we must have that for any $F \in \mathcal{F}'$ containing ω there must be some $F' \subset F$ such that $P(F') > 0$ and $\phi^\top z < 0$ on F'. By (2.5) there must be some $G \in \mathcal{G}$ containing ω such that $V^\top z \geq 0$ on G. As $\mathcal{G} \subseteq \mathcal{F}'$, $G \in \mathcal{F}'$ and there must be some $G' \subset G$ such that $P(G') > 0$ and $\phi^\top z < 0$ on G'. Hence ϕ cannot be a no-arbitrage price function for $\mathcal{G}$ at ω. ☐

Absence of arbitrage can also be defined for information structures:

Definition 2.2 (common no-arbitrage price function, arbitrage-free information structure)**.** The price function ϕ is a *common no-arbitrage price function* for $(\mathcal{H}_i)$ and $(\mathcal{H}_i)$ is ϕ-arbitrage-free if all the $\mathcal{H}_i$'s are ϕ-arbitrage-free. The set of common no-arbitrage price functions for $(\mathcal{H}_i)$ is denoted

$$\Phi_c((\mathcal{H}_i)) := \bigcap \Phi(\mathcal{H}_i).$$

$(\mathcal{H}_i)$ is *arbitrage-free* if there exist some common no-arbitrage price function, i.e. $\Phi_c((\mathcal{H}_i)) \neq \emptyset$.

Note that the concept of an arbitrage-free information structure has no counterpart in Definition 2.1, because one can always find some no-arbitrage prices for any σ-algebra (cf. Corollary 2.1.1). These may not coincide for different σ-algebras, but clearly as in [3, Proposition 2.1]:

Corollary 2.1.2. *Any symmetric information structure is arbitrage-free.*

As a direct consequence of Theorem 2.1 we have an "asymmetric information version" of the fundamental theorem of asset pricing:

Corollary 2.1.3. *The price function ϕ is a common no-arbitrage price function for $(\mathcal{H}_i)$ if and only if for almost all ω, there exist some collection of measures $P^{(1)}, \ldots, P^{(I)} \sim P$ such that*

$$\frac{\phi(\omega)}{\phi(\omega)^\top z^*} = E^{(i)}\Big[\frac{V}{V^\top z^*}\Big|\mathcal{H}_i\Big](\omega), \qquad i = 1, \ldots, I.$$

It would be natural to assume that asset prices are $\underline{\mathcal{H}}$-measurable. As the following result shows, this does not affect the existence of common no-arbitrage price functions.

Proposition 2.1. *If there exists some common no-arbitrage price function for $(\mathcal{H}_i)$, there exists some $\underline{\mathcal{H}}$-measurable common no-arbitrage price function for $(\mathcal{H}_i)$.*

Proof. By Lemma 2.1, if $\phi \in \Phi_c((\mathcal{H}_i))$, then $E[\phi|\underline{\mathcal{H}}] \in \Phi_c((\mathcal{H}_i))$. □

Example 2.1 (cf. Section 2.3 [3]). Suppose $I = 2,\ J = 3,\ \mathcal{F} := \sigma\{F_1, \ldots, F_4\}$,

$$V(\omega) := \begin{cases} \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^\top, & \omega \in F_1, \\ \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top, & \omega \in F_2, \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top, & \omega \in F_3, \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^\top, & \omega \in F_4, \end{cases}$$

and

$$\mathcal{H}_1 := \sigma\{F_4\} \text{ and } \mathcal{H}_2 := \sigma\{F_3\}.$$

Then

$$\Phi(\mathcal{H}_1) = \Big\{ \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}^\top \chi_{F_4^C} + \begin{bmatrix} 0 & q_2 & 0 \end{bmatrix}^\top \chi_{F_4};$$
$$p_1 < p_2,\ p_2 > 0,\ p_3 > 0,\ q_2 > 0 \Big\},$$
$$\Phi(\mathcal{H}_2) = \Big\{ \begin{bmatrix} p_1 & p_2 & 0 \end{bmatrix}^\top \chi_{F_3^C} + \begin{bmatrix} 0 & 0 & q_3 \end{bmatrix}^\top \chi_{F_3};\ p_1 < p_2,\ p_2 > 0,\ q_3 > 0 \Big\}.$$

Hence

$$\Phi_c((\mathcal{H}_1, \mathcal{H}_2)) = \emptyset.$$

2.3 Future arbitrages

An alternative approach to arbitrage, not dealing explicitly with asset prices, is the following:

Definition 2.3 (future arbitrage opportunity)**.** An allocation $(z^{(i)}) \in (\mathbb{R}^J)^I$ is a *future arbitrage opportunity* for $(\mathcal{G}_i)$ at ω if

$$\sum z^{(i)} = 0, \tag{2.6a}$$

$$V^\top z^{(i)} \geq 0 \quad P(\cdot|\mathcal{H}_i)(\omega)\text{-a.s.} \qquad \text{for all } i = 1, \ldots, I \tag{2.6b}$$

and

$$P(V^\top z^{(j)} > 0|\mathcal{H}_j)(\omega) > 0 \qquad \text{for some } j \in \{1, \ldots, I\}. \tag{2.6c}$$

The two arbitrage concepts are related:

Proposition 2.2. *The information structure $(\mathcal{H}_i)$ is arbitrage-free if and only if there are no future arbitrage opportunities for $(\mathcal{H}_i)$ at any ω outside a set of measure 0.*

Proof. Using Theorem 2.1, it is easily seen that absence of arbitrage ensures that there are no future arbitrage opportunities. Conversely, fix ω and consider

$$\mathcal{C} := \Bigg\{ \Big\{ E^{(i)}\Big[\frac{V}{V^\top z^*}\big|\mathcal{H}_i\Big](\omega) - E^{(I)}\Big[\frac{V}{V^\top z^*}\big|\mathcal{H}_I\Big](\omega);\ i = 1.\dots, I-1 \Big\}; \\ P^{(j)} \sim P,\ j = 1,\dots, I \Bigg\} \subseteq \mathbb{R}^{J(I-1)}.$$

Clearly $(\mathcal{H}_i)$ is arbitrage-free at ω if and only if $0 \in \mathcal{C}$. By the separating hyperplane theorem, if $0 \notin \mathcal{C}$, there exists some $\zeta \in \mathbb{R}^{J(I-1)}$ such that (2.3) hold. Hence there exist some collection $\zeta^{(1)}, \dots, \zeta^{(I-1)} \in \mathbb{R}^J$ and some $x \in \mathbb{R}^{I-1}$ such that

$$\frac{V^\top \zeta^{(i)}}{V^\top z^*} \geq x_i\ P(\cdot|\mathcal{H}_i)(\omega)\text{-a.s. and } \frac{V^\top \zeta^{(i)}}{V^\top z^*} \leq x_i\ P(\cdot|\mathcal{H}_I)(\omega)\text{-a.s.}$$

for all $i = 1, \dots, I-1$ and for some $j \in 1, \dots I-1$

$$P\Big(\frac{V^\top \zeta^{(j)}}{V^\top z^*} > x_j\big|\mathcal{H}_j\Big)(\omega) > 0 \text{ or } P\Big(\frac{V^\top \zeta^{(j)}}{V^\top z^*} < x_i\big|\mathcal{H}_I\Big)(\omega) > 0.$$

In any case, taking

$$z^{(j)} := \begin{cases} \frac{1}{1-x_j}\zeta^{(j)} - \frac{x_j}{1-x_j}z^*, & x_j < 1, \\ \zeta^{(j)} - z^*, & x_j = 1, \\ -\frac{1}{1-x_j}\zeta^{(j)} + \frac{x_j}{1-x_j}z^*, & x_j > 1, \end{cases}$$

$z^{(I)} = -z^{(j)}$ and all other portfolios 0 results in a future arbitrage at ω. □

Example 2.1 (continued) The allocation

$$z^{(1)} := \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} =: -z^{(2)}$$

is a future arbitrage opportunity on $F_1 \cup F_2$. While

$$z^{(1)} := \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} =: -z^{(2)}$$

is a future arbitrage opportunity on F_3.

3 Arbitrage-free refinements

If the refinement $(\mathcal{G}_i)$ of the information structure $(\mathcal{H}_i)$ is arbitrage free, we refer to $(\mathcal{G}_i)$ as an *arbitrage-free refinement* of $(\mathcal{H}_i)$. Clearly, by Corollary 2.1.2 the *pooled refinement,* where $\mathcal{G}_i = \underline{\mathcal{H}}$ for all i is arbitrage-free and self-attainable. But as the following example shows, the agents do not necessarily have to share *all* their information to find a no-arbitrage price.

Example 2.1 (continued) The information structure

$$\mathcal{G}_1 := \mathcal{H}_1 \bigvee \mathcal{H}_2 = \sigma\{F_3, F_4\},$$
$$\mathcal{G}_2 := \mathcal{H}_2 = \sigma\{F_3\},$$

is clearly a self-attainable refinement of $(\mathcal{H}_1, \mathcal{H}_2)$. We have

$$\Phi(\mathcal{G}_1) = \Big\{ \begin{bmatrix} p_1 & p_2 & 0 \end{bmatrix}^\top \chi_{F_1 \cup F_2} + \begin{bmatrix} 0 & 0 & q_3 \end{bmatrix}^\top \chi_{F_3} + \begin{bmatrix} 0 & r_2 & 0 \end{bmatrix}^\top \chi_{F_4}; \quad p_1 < p_2,\, p_2 > 0,\, q_3 > 0,\, r_2 > 0 \Big\}$$

and

$$\Phi(\mathcal{G}_2) \equiv \Phi(\mathcal{H}_2) \supset \Phi(\mathcal{G}_1).$$

Hence,

$$\Phi_c((\mathcal{G}_1, \mathcal{G}_2)) = \Phi(\mathcal{G}_1).$$

Proposition 3.1. *For any information structure there exists a unique coarsest refinement that is arbitrage-free. Moreover, this refinement is self-attainable.*

To prove this we proceed as in the proof of [3, Proposition 3.2].

Lemma 3.1. *The meet of a countable sequence of self-attainable arbitrage-free refinements of an information structure is also a self-attainable arbitrage-free refinement.*

Proof. The meet is clearly a self-attainable refinement. The meet of a countable sequence of σ-algebras or information strucures can be formed by successively taking the meet of two σ-algebras or information structures. Hence it is sufficient to prove that if $(\mathcal{G}_i^{(1)})$ and $(\mathcal{G}_i^{(2)})$ are arbitrage-free, then so is $(\mathcal{G}_i)$ defined by

$$\mathcal{G}_i := \mathcal{G}_i^{(1)} \bigwedge \mathcal{G}_i^{(2)}, \qquad i = 1, \ldots, I.$$

Suppose that $(\mathcal{G}_i)$ is not arbitrage-free on ω and let $(z^{(i)})$ be an allocation such that (2.6) hold. Defining

$$F := \big\{\omega' \in G(\omega, \mathcal{G}_j);\ V(\omega')^\top z^{(j)} > 0\big\},$$

we have that $P(F) > 0$. Clearly

$$V^\top z^{(i)} \equiv 0 \quad P(\cdot|\mathcal{G}_i^{(k)})(\omega_0)\text{-a.s.} \qquad \text{for all } i, k = 1, 2,$$

as $(\mathcal{G}_i^{(1)})$ and $(\mathcal{G}_i^{(2)})$ are arbitrage-free. Hence

$$\omega \notin G(F, \mathcal{G}_j^{(k)}), \qquad k = 1, 2.$$

If $G(F, \mathcal{G}_j^{(1)})$ and $G(F, \mathcal{G}_j^{(2)})$ coincide they would be $\mathcal{G}_j$-measurable and would not be contained in $G(\omega, \mathcal{G}_j)$. Hence the sets cannot coincide, and by the same argument

$$G(F, \mathcal{G}_j^{(1)}) \notin \mathcal{G}_j^{(2)} \text{ and } G(F, \mathcal{G}_j^{(2)}) \notin \mathcal{G}_j^{(1)}.$$

Hence

$$G(\omega, \mathcal{G}_j) \bigcap G(F, \mathcal{G}_j^{(1)})^C \notin \mathcal{G}_j^{(2)} \tag{3.1a}$$

and

$$G(\omega, \mathcal{G}_j) \bigcap G(F, \mathcal{G}_j^{(2)})^C \notin \mathcal{G}_j^{(1)}. \tag{3.1b}$$

Recall that $(\mathcal{G}_i^{(1)})$ and $(\mathcal{G}_i^{(2)})$ have the same pooled information, i.e. $\underline{\mathcal{G}}^{(1)} = \underline{\mathcal{G}}^{(2)}$. Hence by (3.1)

$$G(\omega, \mathcal{G}_j) \bigcap G(F, \mathcal{G}_j^{(1)})^C \in \mathcal{G}_i^{(2)}, \qquad \text{for some } i \neq j \tag{3.2a}$$

and

$$G(\omega, \mathcal{G}_j) \bigcap G(F, \mathcal{G}_j^{(2)})^C \in \mathcal{G}_i^{(1)}, \qquad \text{for some } i \neq j. \tag{3.2b}$$

Without loss of generality we assume that

$$F' := G(F, \mathcal{G}_j^{(1)}) \bigcap G(F, \mathcal{G}_j^{(2)})^C$$

(shaded area in Figure 1) is non-empty, and as $F' \notin \mathcal{G}_j^{(1)}$ we must have that $P(F') > 0$, by completeness. Clearly $V^\top z^{(j)} \equiv 0$ on F', but we also have that

$$P(V^\top z^{(j)} > 0 | \mathcal{G}_j^{(1)})(\omega') > 0, \qquad \omega' \in F'.$$

However, by (3.2b) we have

$$V^\top z^{(j)} \equiv 0,\ P(\cdot | \mathcal{G}_i^{(1)})(\omega')\text{-a.s.} \qquad \text{for some } i \neq j.$$

Hence the allocation $\hat{z}$ with $\hat{z}^{(j)} = z^{(j)} = -\hat{z}^{(i)}$ and all other elements 0 is a future arbitrage opportunity for $(\mathcal{G}_i^{(1)})$ on F', which is a contradiction. □

Proof of Proposition 3.1. Since the pooled refinement is self-attainable and arbitrage-free, the set of self-attainable arbitrage-free refinements is non-empty. By Lemma 3.1 the clearly unique meet of all the arbitrage-free refinements is itself a self-attainable and arbitrage-free refinement. □

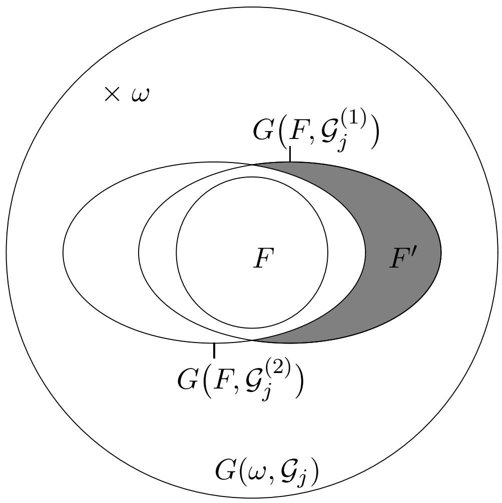

Figure 1

If the agents have to share all their information to reach an arbitrage-free information structure, we say that the information structure is *revealing* (cf. [3, Definition 3.1]):

Definition 3.1 (full revelation)**.** An information structure is *fully revealing* if every self-attainable arbitrage-free refinement is symmetric.

Remark 3.1. The assertion in Definition 3.1 is equivalent to

- the coarsest arbitrage-free refinement is symmetric
- the pooled refinement and the coarsest arbitrage-free refinement coincide

As we shall see, the revealing properties of an information structure is linked to *market completeness.*

Definition 3.2 (contingent claims, attainable claims, completeness)**.** A *contingent $\mathcal{F}$-claim* is a nonnegative and finite-valued random variable X on $(\Omega, \mathcal{F}, P)$. Such a claim is *attainable for the σ-algebra $\mathcal{G} \subseteq \mathcal{F}$* if there exists some $\mathcal{G}$-measurable portfolio z such that

$$V^\top z = X \qquad \text{a.s.} \tag{3.3}$$

We say that the market $\{(\Omega, \mathcal{F}, P),\, V\}$ is *complete* for $\mathcal{G}$ if every contingent $\mathcal{F}$-claim is attainable for $\mathcal{G}$.

As $\mathcal{F}$ is separable it suffices that for every $F \in \mathcal{F}$ there exists some ($\mathcal{G}$-measurable) z such that

$$V^\top z = \chi_F \qquad \text{a.s.} \tag{3.4}$$

for the market to be complete.

Lemma 3.2. *If for any $F \in \mathcal{F}$ there exists some $\mathcal{G}$-measurable portfolio z such that $V^\top z \equiv 0$ on F^C and $V^\top z > 0$ on F, the market $\{(\Omega, \mathcal{F}, P),\, V\}$ is complete for $\mathcal{G}$.*

Proof. Suppose that for any $n; F_{n1}, \ldots, F_{nK_n}$ is a *partition* of $F \in \mathcal{F}$, i.e.

$$F \bigsqcup_{k=1}^{K_n} F_{nk},$$

that any F_{nk} is the (disjoint) union of sets of the form $F_{(n+1)\cdot}$, that $F = F_{11}$ and that

$$\{F_{nk};\ k = 1, \ldots, K_n,\ n = 1, 2 \ldots\}$$

generates the restriction of $\mathcal{F}$ to F. Suppose moreover that $V^\top z_{nk} \equiv 0$ on F_{nk}^C and $V^\top z_{nk} > 0$ on F_{nk} and that

$$\sup_{F_{nk}} V^\top z_{nk} = 1.$$

Define z as the limit of the recursive scheme

$$\begin{aligned} z_1 &:= z_{11}, \\ z_n &:= z_{n-1} + \sum_{k=1}^{K_n} x_{nk} z_{nk}, \qquad n > 1 \end{aligned}$$

where

$$x_{nk} := 1 - \sup_{F_{nk}} V^\top z_{n-1}.$$

Clearly $V \equiv 0$ on F^C and $V^\top z \leq 1$ on F. Moreover, if $V^\top z(\omega) < 1$ for some $\omega \in F$, we must have that for *all* sets in $F' \in \mathcal{F}$ containing ω there must be some $\omega' \in F'$ such that $V^\top z(\omega') = 1$, which is clearly a contradiction. Hence (3.4) holds. ☐

Example 3.1 Suppose $I = 2,\ J = 2,\ \mathcal{F} := \sigma\{F_1, F_2, F_3\}$ with

$$V(\omega) := \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix}^\top, & \omega \in F_1, \\ \begin{bmatrix} 0 & 1 \end{bmatrix}^\top, & \omega \in F_2, \\ \begin{bmatrix} 1 & 1 \end{bmatrix}^\top, & \omega \in F_3. \end{cases} \tag{3.5}$$

This market is complete for any σ-algebra generated by any one of the sets.

Not surprisingly, in complete markets information structures are revealing (cf. [3, Proposition 3.3]):

Proposition 3.2. *The following are equivalent*

A. *The market $\{(\Omega, \mathcal{F}, P), V\}$ is complete for $\mathcal{G}$.*

B. *Any arbitrage-free information structure $(\mathcal{G}_i)$ with $\mathcal{G}_i \supseteq \mathcal{G}$ for all i is symmetric.*

Proof.

- **A $\implies$ B**
  Let $(\mathcal{G}_i)$ be an asymmetric information structure. Then there must exist some $F \in \mathcal{F}$, i, j such that $F \in \mathcal{G}_i$ but $F \notin \mathcal{G}_j$. By completeness, there exist some $\mathcal{G}$-measurable z such that $V^\top z = \chi_F$. But then the allocation $z^{(j)} = z,\ z^{(i)} = -z$ and the other $z^{(\cdot)}$'s zero constitutes a future arbitrage on $G(F, \mathcal{G}_j)\backslash F$. Hence $(\mathcal{G}_i)$ cannot be arbitrage-free.

- **B $\implies$ A**
  If **B** holds then for any $F \in \mathcal{F}$, $F \notin \mathcal{G}$ there is some $z \in \mathbb{R}^J$ such that $V^\top z \equiv 0$ on $G(F, \mathcal{G})\backslash F$ and $V^\top z > 0$ on F, but then, by Lemma 3.2, the market must be complete.

□

Corollary 3.2.1. *An information structure $(\mathcal{H}_i)$ is fully revealing if the market is complete for the agents' common information.*

As Example 3.2 shows, an information structure need not be fully revealing even if the market is complete for every agent.

Example 3.2 Suppose $I = 2,\ J = 2,\ \mathcal{F} := \sigma\{F_1, \dots, F_4\}$ with

$$V(\omega) := \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix}^\top, & \omega \in F_1, \\ \begin{bmatrix} 2 & 0 \end{bmatrix}^\top, & \omega \in F_2, \\ \begin{bmatrix} 0 & 1 \end{bmatrix}^\top, & \omega \in F_3, \\ \begin{bmatrix} 1 & 1 \end{bmatrix}^\top, & \omega \in F_4. \end{cases}$$

For the information structure

$$\mathcal{H}_1 := \sigma\{F_1 \cup F_3\}, \qquad \mathcal{H}_2 := \sigma\{F_2 \cup F_3\},$$

the common information is the trivial σ-algebra for which the market is not complete. Any price vector ϕ with $\phi_1 > \phi_2 > 0$ a.s. belongs to $\Phi_c(\mathcal{H}_1, \mathcal{H}_2)$. Hence the information structure is *not* fully revealing.

3.1 Informed vs uninformed agents

Suppose that there are two types of agents in the market: informed agents whose information is represented by the σ-algebra $\mathcal{G}^{(i)}$ and uninformed agents whose information is represented by $\mathcal{G}^{(u)} \subsetneq \mathcal{G}^{(i)}$. It is then natural to ask whether such asymmetries can prevail in a no-arbitrage setting. Corollary 3.2.1 gives a necessary but not sufficient condition for the possibility of having non-revealing arbitrage-free information structures. The following proposition establish a necessary and sufficient condition the possibility of having an arbitrage-free information structure with informed and uninformed agents.

Proposition 3.3. *The information structure* $(\mathcal{G}^{(i)}, \mathcal{G}^{(u)})$ *is arbitrage-free if and only if for every* $H \in \mathcal{G}^{(i)}$ *there exists some probability measure* $\mu \sim P$ *on* $(\Omega, \mathcal{F})$ *such that*

$$\int_H V \, d\mu = \int_{G(H, \mathcal{G}^{(u)}) \setminus H} V \, d\mu. \tag{3.6}$$

Example 3.3 Suppose that $\mathcal{F}$ and V is as in Example 3.1 and that the uninformed agents have access to the trivial σ-algebra only. Hence the market is not complete for the common information. If $\mathcal{G}^{(i)}$ is either $\sigma\{F_1\}$ or $\sigma\{F_2\}$ the information structure is revealing, because there is no probability measure $\mu \sim P$ such that (3.6) holds if $H = F_1$ or $H = F_2$. If $\mu(F_1) = \mu(F_2) = \mu(F_3) = \frac{1}{3}$, then (3.6) holds for $H = F_3$. Hence if $\mathcal{G}^{(i)} = \sigma\{F_3\}$, the information structure is arbitrage-free. In this case any price function of the form

$$\phi := q_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \chi_{F_3} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \chi_{F_1 \cup F_2},$$

with all the q's positive is a common no-arbitrage price.

For the proof of Proposition 3.3 the following lemmas will be useful.

Lemma 3.3. *Suppose that* $(\mathcal{G}, \mathcal{G}_0)$ *with* $\mathcal{G}_0 \subseteq \mathcal{G} \subseteq \mathcal{F}$ *is arbitrage-free and that* $\mathcal{G}_0 \subseteq \mathcal{G}' \subseteq \mathcal{G}$ *then* $(\mathcal{G}', \mathcal{G}_0)$ *is arbitrage-free.*

Proof. Suppose that $(\mathcal{G}', \mathcal{G}_0)$ is not arbitrage-free at ω_0, then by Proposition 2.2 there is a future arbitrage opportunity, i.e. some $z \in \mathbb{R}^J$ such that

$$V^\top z \leq 0 \qquad P(\cdot|\mathcal{G}')(\omega_0)\text{-a.s,} \tag{3.7a}$$

$$V^\top z \geq 0 \qquad P(\cdot|\mathcal{G}_0)(\omega_0)\text{-a.s} \tag{3.7b}$$

and

$$P(V^\top z < 0|\mathcal{G}')(\omega_0) > 0 \tag{3.8a}$$

or

$$P(V^\top z > 0|\mathcal{G}_0)(\omega_0) > 0. \tag{3.8b}$$

As (3.7b) rules out (3.8a), (3.8b) must hold. Further, (3.7a) implies

$$V^\top z \leq 0 \quad P(\cdot|\mathcal{G})(\omega_0)\text{-a.s}$$

i.e. $(-z, z)$ is a future arbitrage opportunity for $(\mathcal{G}, \mathcal{G}_0)$ at ω_0. □

Lemma 3.4. *The information structure $(\mathcal{G}^{(i)}, \mathcal{G}^{(u)})$ is arbitrage-free if and only if $(\mathcal{G}^{(u)} \bigvee \sigma\{H\}, \mathcal{G}^{(u)})$ is arbitrage-free for any $H \in \mathcal{G}^{(i)}$.*

Proof. The "only if" part is a trivial consequence of Lemma 3.3. Suppose that $(\mathcal{G}^{(i)}, \mathcal{G}^{(u)})$ is not arbitrage-free at ω_0. Then, by the same arguments as in the proof of Lemma 3.3 there exists some $z \in \mathbb{R}^J$ such that

$$\begin{aligned} V^\top z &\leq 0 \qquad && P(\cdot|\mathcal{G}^{(i)})(\omega_0)\text{-a.s}, \\ V^\top z &\geq 0 && P(\cdot|\mathcal{G}^{(u)})(\omega_0)\text{-a.s} \end{aligned}$$

and

$$P(V^\top z > 0|\mathcal{G}^{(u)})(\omega_0) > 0.$$

But then $(-z, z)$ is a future arbitrage opportunity for $(\mathcal{G}^{(u)} \bigvee \sigma\{G(\omega_0, \mathcal{G}^{(i)})\}, \mathcal{G}^{(u)})$ at ω_0. □

Proof of Proposition 3.3. The "only if" part stems from the fundamental theorem. For the "if" part it is by Lemma 3.4 sufficient to prove that $(\mathcal{G}^{(u)} \bigvee \sigma\{H\}, \mathcal{G}^{(u)})$ is arbitrage-free for any $H \in \mathcal{G}^{(i)}$. Suppose $\phi \in \Phi(\mathcal{G}^{(u)})$ and define $\hat{\phi} \equiv \phi$ on $G(H, \mathcal{G}^{(u)})^C$ and

$$\hat{\phi}(\omega) = \int_H V d\mu, \quad \omega \in G(H, \mathcal{G}^{(u)})$$

By (3.6), $\hat{\phi} \in \Phi_c(\mathcal{G}^{(u)} \bigvee \sigma\{H\}, \mathcal{G}^{(u)})$ and $(\mathcal{G}^{(u)} \bigvee \sigma\{H\}, \mathcal{G}^{(u)})$ is arbitrage-free. □

3.2 No-arbitrage equilibrium

We now equip every agent i with a strictly increasing utility function $U_i : \mathbb{R} \to \mathbb{R}$ and consider the economy

$$\mathcal{E} := \{V, (\mathcal{H}_i), (U_i)\}.$$

This is a simplification of the economy considered in [3, Section 2.4] which includes consumption goods, spot prices and endowments.

Definition 3.3. A collection $\{(\mathcal{G}_i), (z^{(i)}), \phi\}$, where $\mathcal{G}_i \subseteq \mathcal{F}$ and $z^{(i)} : \Omega \to \mathbb{R}^J$ is $\mathcal{F}$-measurable for all i and ϕ is a price function, constitutes a *no-arbitrage equilibrium* for the economy $\mathcal{E}$ if

- $(\mathcal{G}_i)$ is a self-attainable refinement of $(\mathcal{H}_i)$
- for all i and almost all ω, $z^{(i)}(\omega)$ solves

$$\max_{z \in \mathbb{R}^J} E[U_i(V^\top z)|\mathcal{G}_i](\omega) \text{ subject to } \phi(\omega)^\top z \leq 0 \tag{3.9}$$

- $\sum z^{(i)} = 0$

The following proposition states that an equilibrium price must be a common no-arbitrage price function (cf. [3, Proposition 2.2.]):

Proposition 3.4. *If there is a solution to* (3.9) *for every agent, then* $\phi \in \Phi_c((\mathcal{G}_i))$.

Proof. If $\phi \notin \Phi_c((\mathcal{G}_i))$, then for at least one $F \in \mathcal{F}$ and one agent i, there exists some arbitrage opportunity such that (3.9) has no solution. □

4 Information revealed by prices

We now proceed to study the revealing properties of price functions. The following result is in some sense an analogy of Proposition 3.1 as it deals with the existence of a unique coarsest arbitrage-free σ-algebra for a *given* price function (cf. [3, Lemma 2 and Definition 4.1]).

Proposition 4.1. *Suppose that* $\phi \in \Phi(\mathcal{F}')$ *for some* $\mathcal{F}' \subseteq \mathcal{F}$*, then for any* σ*-algebra* $\mathcal{H} \subseteq \mathcal{F}'$ *there is a unique coarsest* σ*-algebra* $\mathcal{G} \supseteq \mathcal{H}$ *such that* $\phi \in \Phi(\mathcal{G})$. *This* σ*-algebra is referred to as the* σ-algebra revealed by ϕ *and denoted by* $\mathcal{S}(\phi, \mathcal{H})$.

As for Proposition 3.1 the proof is simple once we have established the following:

Lemma 4.1. *Consider a sequence of* σ*-algebras and suppose that* ϕ *is a no-arbitrage price function for each of them. Then* ϕ *is also a no-arbitrage price function for their meet.*

Proof. As in the proof of Lemma 3.1 it suffices to prove that if $\phi \in \Phi(\mathcal{G}^{(1)}) \cap \Phi(\mathcal{G}^{(2)})$ then $\phi \in \Phi(\mathcal{G})$ with $\mathcal{G} := \mathcal{G}^{(1)} \bigwedge \mathcal{G}^{(2)}$. Conversely, suppose that $\mathcal{G}$ is not ϕ-arbitrage-free at ω and $z \in \mathbb{R}^J$ is such that (2.1) holds. Define

$$F := \{\omega' \in G(\omega, \mathcal{G});\ V^\top z > 0\},$$

and note that $P(F) > 0$. The no-arbitrage condition, however, implies that

$$P(F|\mathcal{G}^{(k)})(\omega) = 0, \qquad k = 1, 2.$$

Hence ω does not belong to any of the $G(F, \mathcal{G}^{(k)})$'s. As argued in the proof of Lemma 3.1 these sets cannot coincide and we may without loss of generality assume that

$$F' := G(F, \mathcal{G}^{(1)}) \cap G(F, \mathcal{G}^{(2)})^C$$

is non-empty. Then for any $\omega' \in F'$ we have that

$$V^\top z \geq 0 \; P(\cdot|\mathcal{G}^{(1)})(\omega')\text{-a.s.}, \qquad P(V^\top z > 0|\mathcal{G}^{(1)})(\omega') > 0$$

and

$$V^\top z = 0 \; P(\cdot|\mathcal{G}^{(2)})(\omega')\text{-a.s.}$$

But then $z^{(1)} := z =: -z^{(2)}$ is a future arbitrage for the information structure $(\mathcal{G}^{(1)}, \mathcal{G}^{(2)})$ on F' implying that $\Phi_c((\mathcal{G}^{(1)}, \mathcal{G}^{(2)})) = \emptyset$. □

Proof of Proposition 4.1. By assumption the set of ϕ-arbitrage-free σ-algebras $\mathcal{G} \supseteq \mathcal{H}$ is non-empty. By Lemma 4.1 the clearly unique meet is also ϕ-arbitrage-free. □

Definition 4.1 (no-arbitrage price function)**.** The $\underline{\mathcal{H}}$-measurable function $\phi : \Omega \to \mathbb{R}^J$ is a *no-arbitrage price function* for $(\mathcal{H}_i)$, denoted $\phi \in \Phi_0((\mathcal{H}_i))$ if ϕ is a common no-arbitrage price function for some self-attainable refinement of $(\mathcal{H}_i)$.

Remark 4.1*.* Clearly, by Corollary 2.1.2 $\Phi_0((\mathcal{H}_i)) \neq \emptyset$. We also have that

$$\phi \in \Phi(\underline{\mathcal{H}}) \implies \phi \in \Phi_0((\mathcal{H}_i)).$$

The reverse implication holds if the market is complete for the common information. But it does not hold in general. As pointed out in Example 3.2, any price ϕ with $\phi_1 > \phi_2 > 0$ a.s. is a common no-arbitrage price for $(\mathcal{H}_1, \mathcal{H}_2)$ but is not necessarily a no-arbitrage price for the pooled information $\underline{\mathcal{H}} = \mathcal{F}$.

The following result is the analogue of [3, Proposition 4.2]:

Proposition 4.2. *Given some information structure $(\mathcal{H}_i)$ and price function ϕ the following are equivalent*

- **A***.* $\phi \in \Phi_0((\mathcal{H}_i))$
- **B***.* $\mathcal{S}(\phi, \mathcal{H}_i)$ *exists and* $\mathcal{S}(\phi, \mathcal{H}_i) \subseteq \underline{\mathcal{H}}$ *for all* $i = 1, \ldots, I$.
- **C***.* $(\mathcal{S}(\phi, \mathcal{H}_i))$ *is the coarsest self-attainable refinement of* $(\mathcal{H}_i)$ *that is* ϕ*-arbitrage-free.*

Proof.

- **A** $\implies$ **B**
  By **A** there exists some self-attainable arbitrage-free refinement $(\mathcal{G}_i)$, i.e. $\phi \in \Phi_c((\mathcal{G}_i))$. By Proposition 4.1

$$\mathcal{S}(\phi, \mathcal{H}_i) \subseteq \mathcal{G}_i \subseteq \underline{\mathcal{G}} = \underline{\mathcal{H}}, \qquad i = 1, \ldots, I.$$

- **B** $\implies$ **C**
  The refinement $(\mathcal{S}(\phi, \mathcal{H}_i)))$ is by the definition the coarsest ϕ-arbitrage-free refinement and it is self-attainable by **B**.

- **C** $\implies$ **A**
  Obvious

□

These observations motivate the following (cf. [3, Definition 4.3, Proposition 4.4]):

Definition 4.2. The refinement $(\mathcal{S}(\phi, \mathcal{H}_i))$ is referred to as the *refinement revealed by* ϕ. A self-attainable arbitrage-free refinement $(\mathcal{G}_i)$ of $(\mathcal{H}_i)$ is *price-revealable* if there is some price function $\phi \in \Phi_0((\mathcal{H}_i))$ such that for every i, $\mathcal{G}_i = \mathcal{S}(\phi, \mathcal{H}_i)$.

Not all self-attainable arbitrage-free refinements are price-revealable (cf. Example 4.1 below), but the coarsest arbitrage-free refinement is (cf. [3, Proposition 4.3]):

Proposition 4.3. *The coarsest arbitrage-free refinement of an information structure is price-revealable.*

Proof. Let $(\mathcal{G}_i)$ be the coarsest arbitrage-free refinement of the information structure $(\mathcal{H}_i)$ and suppose that $\phi \in \Phi_c((\mathcal{G}_i))$. Clearly $\phi \in \Phi_0((\mathcal{H}_i))$, and by Proposition 4.2

$$\mathcal{S}(\phi, \mathcal{H}_i) \ \subseteq \ \mathcal{G}_i, \qquad i = 1, \ldots, I.$$

But as $(\mathcal{G}_i)$ is the coarsest arbitrage-free refinement of $(\mathcal{H}_i)$, we must also have that

$$\mathcal{S}(\phi, \mathcal{H}_i) \ \supseteq \ \mathcal{G}_i, \qquad i = 1, \ldots, I,$$

i.e.

$$\mathcal{S}(\phi, \mathcal{H}_i) \ = \ \mathcal{G}_i, \qquad i = 1, \ldots, I.$$

□

Example 4.1 Suppose $I = 2,\ J = 2$ and $\mathcal{F}$ and V are as in Example 3.2, and consider the information structure

$$\big(\sigma\{F_1\}, \sigma\{F_1 \cup F_2)\}\big).$$

In this case the coarsest arbitrage free refinement is

$$\big(\sigma\{F_1, F_2\}, \sigma\{F_1 \cup F_2)\}\big).$$

The pooled refinement is not price revealable, because there is no price that will enable agent 2 to distinguish between F_1 and F_2.

5 Reaching an arbitrage free refinement

The follow-up paper [4], analyses how it is possible to reach an arbitrage-free refinement by successively eliminating "arbitrage states". In this framework we define arbitrage sets as sets on which an arbitrage opportunity gives a strictly positive payoff:

Definition 5.1. Given the sets $F \subseteq G \in \mathcal{F}$ and price function ϕ, we say that $F' \in \mathcal{F}$ such that $F' \in G \backslash F$ is a (ϕ, F)-*arbitrage set* in G if there exists some $z \in \mathbb{R}^J$ such that

$$\phi^\top z \leq 0 \text{ on } F,\ V^\top z \geq 0 \text{ on } G, \text{ and } V^\top z > 0 \text{ on } F' \qquad \text{a.s.}$$

The union of all such sets is denoted $A(\phi, F, G)$.

Clearly, there are no (ϕ, F)-arbitrage sets in $G(F, \mathcal{G})$ for any $F \in \mathcal{F}$ if and only if $\mathcal{G}$ is ϕ-arbitrage-free. Noting that $G\backslash A(\phi, F, G)$ could still contain some (ϕ, F)-arbitrage sets, we define recursively

$$\begin{aligned} G^{(0)}(F) &:= G(F, \mathcal{G}), \\ G^{(k)}(F) &:= G^{(k-1)}(F) \backslash A(\phi, F, G^{(k-1)}(F)), \qquad k = 1, 2, \ldots \end{aligned}$$

with the limit

$$G^*(F) := \bigcap_{k=0,1,\ldots} G^{(k)}(F).$$

A similar approach where only some of the arbitrage sets are removed at each step is proved to produce the same result. ([4, Theorem 1]). Though the proof is only stated for the finite case, it also holds in our setting. The following result links the σ-algebra generated by this procedure to the σ-algebra revealed by ϕ (cf. [4, Theorem 2]):

Theorem 5.1. *Suppose $\phi \in \Phi(\mathcal{F}')$ for some $\mathcal{F}' \supseteq \mathcal{G}$. Then*

$$\mathcal{S}^* := \sigma\{G^*(F); F \in \mathcal{F}\}$$

coincides with the σ-algebra revealed by ϕ.

Proof. We first prove that $\mathcal{S}^*$ is ϕ-arbitrage free. Suppose $z \in \mathbb{R}^J$ is an arbitrage portfolio for $\mathcal{S}^*$ at ω. The existence of an arbitrage-free $\mathcal{F}'$ ensures that $A\big(\phi, G(\omega, \mathcal{F}), G(\omega, \mathcal{S}^*)\big)$ is non-empty, which is a contradiction. Hence $\mathcal{S}^*$ is ϕ-arbitrage-free, and accordingly

$$\mathcal{S}(\phi, \mathcal{G}) \subseteq \mathcal{S}^*.$$

Suppose now that there is some $G \in \mathcal{S}^*$ such that $G \notin \mathcal{S}(\phi, \mathcal{G})$. Then there exists some $F' \subseteq G(G, \mathcal{S}(\phi, \mathcal{G}))\backslash G$ that is in $A(\phi, F, G')$ for some $F \in \mathcal{F}$ and some $G' \supseteq G(G, \mathcal{S}(\phi, \mathcal{G}))$. Hence $\mathcal{S}(\phi, \mathcal{G})$ is not arbitrage-free, a contradiction. □

A similar procedure, where ”future arbitrage sets”, defined as sets on which a future arbitrage opportunity yields a strictly positive payoff for the agent, are successively removed, will produce the coarsest arbitrage-free refinement for some information structure (cf. [4, Theorem 3]).

6 The multiperiod case

Now suppose that the asset can be traded at the *trading times* $0, 1, \dots, T$ The information is now modelled as a filtration i.e. an non-decreasing sequence of σ-algebras $\mathbb{F} := \{\mathcal{F}_t;\ t = 0, \dots, T\}$ where $\mathcal{F}_t \subseteq \mathcal{F}$ for all t. As a convention we only consider filtrations with $\mathcal{F}_T = \mathcal{F}$. A *price process* for V is a collection S of $\mathbb{R}^J$-valued $\mathcal{F}$-measurable random variables $S_0, \dots, S_T$ such that $S_T = V$.

We now redefine information structure to mean a set of filtrations $(\mathbb{H}_i)$. We adapt the notions of symmetry, refinement and self-attainability by requiring that these properties hold for $(\mathcal{H}_{i,t})$ for any trading time.

Definition 6.1 (no-arbitrage price process, arbitrage-free filtration)**.** The price process S is a *no-arbitrage price process* for $\mathbb{G}$ and $\mathbb{G}$ is *S-arbitrage-free* if for any $t = 0, \dots, T-1$, S_t is a no-arbitrage price for S_{t+1} and $\mathcal{G}_t$. The set of no-arbitrage price processes for $\mathbb{G}$ is denoted by $\mathbf{S}(\mathbb{G})$.

Remark 6.1. Suppose that the price process S is $\mathbb{G}$-adapted. A *trading strategy* or *portfolio process* for $\mathbb{G}$ is an $\mathbb{R}^J$-valued process $\xi := \{\xi_0, \dots, \xi_T\}$ that is $\mathbb{G}$-predictable, i.e. ξ_0 is constant and ξ_t is $\mathcal{G}_{t-1}$ measurable for $t \geq 1$. A trading strategy is *self-financing* if

$$S_t^\top(\xi_{t+1} - \xi_t) = 0 \text{ a.s.}, \qquad t = 0, \dots, T-1.$$

According to standard theory a self-financing trading strategy is an arbitrage (cf. e.g. [8, Definition 5.10]) if

$$S_0^\top \xi_0 \leq 0, \qquad V^\top \xi_T \geq 0 \text{ a.s., and } \quad P(V^\top \xi_T > 0) > 0.$$

It is further proved ([8, Proposition 5.11]) that there exists an arbitrage opportunity if and only if for some t there exists some $\mathcal{G}_t$-measurable $\mathbb{R}^J$-valued η such that

$$S_t^\top \eta \leq 0, \qquad S_{t+1}^\top \eta \geq 0 \text{ a.s., and } \quad P(S_{t+1}^\top \eta > 0) > 0.$$

Hence, by Remark 2.4, in the case of a $\mathbb{G}$-adapted price process our notion of arbitrage coincides with the standard definition.

Definition 6.2 (common no-arbitrage price process, arbitrage-free information structure)**.** A price process S is a *common no-arbitrage price process* for $(\mathbb{H}_i)$ if it is a no-arbitrage price process for every $\mathbb{H}_i$. The set of common no arbitrage price processes for $(\mathbb{H}_i)$ is denoted by

$$\mathbf{S}_c((\mathbb{H}_i)) := \bigcap \mathbf{S}(\mathbb{H}_i).$$

$(\mathbb{H}_i)$ is an *arbitrage-free information structure* if $\mathbf{S}_c((\mathbb{H}_i)) \neq \emptyset$.

The multiperiod version of the fundamental theorem of asset pricing (Corollary 2.1.3) is:

Theorem 6.1. *The price process S is a common no-arbitrage price process for $(\mathbb{H}_i)$ if and only if at every $\omega \in \Omega$ outside a set of measure 0, there exist some collection of measures $P^{(1)}, \dots, P^{(I)} \sim P$ such that*

$$\frac{S_t}{{S_t}^\top z^*}(\omega) = E^{(i)}\Big[\frac{S_{t+1}}{{S_{t+1}}^\top z^*}\big|\mathcal{H}_{i,t}\Big](\omega), \qquad i = 1, \dots, I, \ t = 0, \dots, T-1.$$

As an immediate extension of Proposition 4.1 we have:

Proposition 6.1. *Suppose that $S \in \mathbf{S}(\mathbb{F})$. Then, for any $\mathbb{G} \subseteq \mathbb{F}$ there exists a unique coarsest filtration $\mathbb{G}' \supseteq \mathbb{G}$ such that $S \in \mathbf{S}(\mathbb{F})$. This filtration is referred to as the* filtration revealed by S.

Definition 6.3. Suppose that the price process S is $\mathbb{F}$-adapted and a no-arbitrage price process for $\mathbb{F}$. We say that the market $\{(\Omega, \mathcal{F}, P),\, S, \mathbb{F}\}$ is *complete* for $\mathbb{G} \subseteq \mathbb{F}$ if at any trading time t and for every $\mathcal{F}_t$-measurable random variable X there exists some $\mathcal{G}_{t-1}$-measurable $z \in \mathbb{R}^J$ such that

$$S_t^\top z = X \text{ a.s.}$$

Remark 6.2. The classical definition of a complete multiperiod market is that for every contingent $\mathcal{F}$-claim X there exists some self-financing trading strategy (cf Remark 6.1) such that

$$V^\top \xi_T = F \qquad \text{a.s.}$$

(cf. e.g. [8, Defintion 5.37]). This is clearly the case if $\{(\Omega, \mathcal{F}, P),\ S, \mathbb{F}\}$ is complete for $\mathbb{F}$. For the converse result, see e.g. [8, Theorem 5.40].

The multiperiod version of Proposition 3.2 is:

Proposition 6.2. *Suppose that S is $\underline{\mathbb{H}}$-adapted. The following are equivalent:*

- **A**. *The market $\{(\Omega, \mathcal{F}, P),\ S, \underline{\mathbb{H}}\}$ is complete for $\mathbb{G}$.*
- **B**. *Any S-arbitrage-free information structure with $\mathbb{G}_i \supseteq \mathbb{G}$ for all i is symmetric.*

Example 6.1 Consider a market with $T = 2$, $I = 2$, $J = 2$ and $\mathcal{F}$ and V as in Example 3.1 and

$$\mathbb{H}_1 := \big\{\{\emptyset, \Omega\},\ \sigma\{F_3\}\big\}, \qquad \mathbb{H}_2 := \big\{\{\emptyset, \Omega\},\ \{\emptyset, \Omega\}\big\}.$$

Then S is a common no-arbitrage price process if and only if it is of the form

$$S_0 := \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad S_1 := \begin{bmatrix} a \\ b \end{bmatrix} \chi_{F_1 \cup F_2} + c \begin{bmatrix} 1 \\ 1 \end{bmatrix} \chi_{F_3},$$

with all constants strictly positive and $a > b \Leftrightarrow \alpha > \beta$ and $a = b \Leftrightarrow \alpha = \beta$. If $a \neq b$ the market $\{(\Omega, \mathcal{F}, P),\ S, \underline{\mathbb{H}}\}$ is complete for $\mathbb{H}_1$, but not for $\mathbb{H}_2$.

The following example illustrates that we cannot establish the revealing properties of a multiperiod information structure without taking prices into account.

Example 6.2 Consider a market with $T = 2$, $I = 2$, $J = 3$, $\mathcal{F}$ and V as in Example 2.1 and

$$\mathbb{H}_1 := \big\{\sigma\{F_4\},\ \sigma\{F_4\}\big\}, \qquad \mathbb{H}_2 := \big\{\{\emptyset, \Omega\},\ \sigma\{F_3\}\big\}.$$

From the single period market we have that for any arbitrage-free refinement we must have $F_3 \in \mathcal{G}_{1,1}$ and S_1 must be of the form

$$S_1 := \begin{bmatrix} q_1 \\ q_2 \\ 0 \end{bmatrix} \chi_{F_1 \cup F_2} + \begin{bmatrix} 0 \\ 0 \\ q_3 \end{bmatrix} \chi_{F_3} + \begin{bmatrix} 0 \\ q_2' \\ 0 \end{bmatrix} \chi_{F_4},$$

whith $q_1 \in \mathbb{R}$, the other constants positive and $q_2 > q_1$. If $q_1 = 0$ the information structure

$$\big\{\mathcal{H}_{1,0},\ \sigma\{F_3, F_4\}\big\}, \qquad \mathbb{H}_2$$

is arbitrage-free. If on the other hand $q_1 \neq 0$ we have to refine $\mathbb{H}_1$ as well and

$$\{\mathcal{H}_{1,0},\ \sigma\{F_3, F_4\}\}, \qquad \{\sigma\{F_4\},\ \mathcal{H}_{2,1}\}$$

is the coarsest arbitrage-free refinement.

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