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TOPOLOGICAL ANDRÉ-QUILLEN HOMOLOGY FOR CELLULAR COMMUTATIVE S-ALGEBRAS

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Abstract. Topological André-Quillen homology for commutative S-algebras was introduced by Basterra following work of Kriz, and has been intensively studied by several authors. In this paper we discuss it as a homology theory on CW S-algebras and apply it to obtain results on minimal atomic p-local S-algebras which generalise those of Baker and May for p-local spectra and simply connected spaces. We exhibit some new examples of minimal atomic S-algebras.

Introduction

In this paper we give an account of some results on *topological André-Quillen homology* and *cohomology* for CW commutative A-algebras, where A is a commutative S-algebra. The main goal is to develop arguments based on skeletal filtrations with a view to continuing the work begun in [11] by extending results of [2] to the case of CW commutative S-algebras. Some of this work originally appeared in the second author's PhD thesis [9], but we go further and use it to investigate some examples.

Our main sources on topological André-Quillen (co)homology include [3–6, 15].

For definiteness, we work in the category of commutative S-algebras described in [7]. For a commutative S-algebra A, $\mathscr{M}_A$ denotes the category of A-modules and $\overline{h}\mathscr{M}_A = \mathscr{D}_A$ the derived homotopy category of A-modules. Where necessary we assume that q-cofibrant replacements are made and in this context we use the terminology of [7], where details on suitable model structures for categories of S-algebras can be found. For general notions of model categories, see [10].

We will often consider a map of commutative S-algebras $A \longrightarrow B$ and will use the notation B/A to indicate B viewed as an A-algebra and refer to it as a pair of S-algebras. We remark that this notation is not ideal given the appearance of algebras over and under a given one, but we follow the use made of it by Basterra [3], which in turn follows Kriz in his influential preprint [13] and Quillen [16].

1. Recollections on Topological André-Quillen (co)homology

We recall from [3] that for a pair of commutative S-algebras B/A there is a B-module $\Omega_{B/A}$ for which

$$\overline{h}\mathscr{C}_A/B(B, B \vee M) \cong \overline{h}\mathscr{M}_B(\Omega_{B/A}, M). \tag{1.1}$$

Here $\overline{h}\mathscr{C}_A/B$ denotes the derived category of commutative A-algebras over B. Notice that the identity map on $\Omega_{B/A}$ corresponds to a morphism $B \longrightarrow B \vee \Omega_{B/A}$ which projects onto the *universal derivation*

$$\delta_{B/A} \in \overline{h}\mathscr{M}_A(B, \Omega_{B/A}). \tag{1.2}$$

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The *topological André-Quillen homology* and *cohomology* of B/A with coefficients in a B-module M are defined by

$$\mathrm{TAQ}_*(B/A;M) = \pi_*\Omega_{B/A} \wedge_B M, \tag{1.3a}$$

$$\mathrm{TAQ}^*(B/A;M) = \pi_{-*}F_B(\Omega_{B/A}, M), \tag{1.3b}$$

where F_B denotes the internal function object in $\mathscr{M}_B$.

Associated to an A-algebra map $B \longrightarrow C$, there are natural long exact sequences

$$\begin{aligned}\cdots \longrightarrow \mathrm{TAQ}_k(B/A;M) \longrightarrow \mathrm{TAQ}_k(C/A;M) \longrightarrow \mathrm{TAQ}_k(C/B;M)& \\ \longrightarrow \mathrm{TAQ}_{k-1}(B/A;M) \longrightarrow \cdots&\end{aligned} \tag{1.4a}$$

$$\begin{aligned}\cdots \longrightarrow \mathrm{TAQ}^k(C/B;M) \longrightarrow \mathrm{TAQ}^k(C/A;M) \longrightarrow \mathrm{TAQ}^k(B/A;M)& \\ \longrightarrow \mathrm{TAQ}^{k+1}(C/B;M) \longrightarrow \cdots&\end{aligned} \tag{1.4b}$$

We will be especially interested in the situation where A and B are connective and the map $\varphi\colon A \longrightarrow B$ induces an isomorphism $\pi_0 A \xrightarrow{\cong} \pi_0 B$; we will write $\Bbbk = \pi_0 A = \pi_0 B$. Then there is an Eilenberg-Mac Lane object $H\Bbbk$, which can be taken to be a CW commutative A-algebra or B-algebra, which allows us to define the *ordinary topological André-Quillen homology* and *cohomology* of B/A:

$$\mathrm{HAQ}_*(B/A) = \mathrm{TAQ}_*(B/A;H\Bbbk) = \pi_*\Omega_{B/A} \wedge_B H\Bbbk, \tag{1.5a}$$

$$\mathrm{HAQ}^*(B/A) = \mathrm{TAQ}^*(B/A;H\Bbbk) = \pi_{-*}F_B(\Omega_{B/A}, H\Bbbk). \tag{1.5b}$$

When $\pi_0 C = \Bbbk$, the long exact sequences of (1.4) become long exact sequences as follows.

$$\cdots \longrightarrow \mathrm{HAQ}_k(B/A) \longrightarrow \mathrm{HAQ}_k(C/A) \longrightarrow \mathrm{HAQ}_k(C/B) \longrightarrow \mathrm{HAQ}_{k-1}(B/A) \longrightarrow \cdots \tag{1.6a}$$

$$\cdots \longrightarrow \mathrm{HAQ}^k(C/B) \longrightarrow \mathrm{HAQ}^k(C/A) \longrightarrow \mathrm{HAQ}^k(B/A) \longrightarrow \mathrm{HAQ}^{k+1}(B/A) \longrightarrow \cdots \tag{1.6b}$$

We can also introduce coefficients in a $\Bbbk$-module M by setting

$$\mathrm{HAQ}_*(B/A;M) = \mathrm{TAQ}_*(B/A;HM) = \pi_*\Omega_{B/A} \wedge_B HM, \tag{1.7a}$$

$$\mathrm{HAQ}^*(B/A;M) = \mathrm{TAQ}^*(B/A;HM) = \pi_{-*}F_B(\Omega_{B/A}, HM). \tag{1.7b}$$

An important result on HAQ_* is provided by [3, lemma 8.2]. However it appears that this result is incorrectly stated (although the proof seems to be correct) and should read as follows. For a map of A-modules θ, we denote by C_θ the mapping cone of θ in $\mathscr{M}_A$.

Lemma 1.1 (Basterra [3, lemma 8.2])**.** *Let $\varphi\colon A \longrightarrow B$ be an n-equivalence, where A and B are connective and $n \geqslant 1$. Then $\Omega_{B/A}$ is n-connected and there is a map of A-modules $\tau\colon \mathrm{C}_\varphi \longrightarrow \Omega_{B/A}$ for which*

$$\tau_*\colon \pi_{n+1}\mathrm{C}_\varphi \xrightarrow{\cong} \pi_{n+1}\Omega_{B/A}.$$

An immediate consequence is an analogue of the classical Hurewicz theorem.

Corollary 1.2. *The map τ induces isomorphisms*

$$\tau_*\colon \pi_k\mathrm{C}_\varphi \xrightarrow{\cong} \mathrm{HAQ}_k(B/A) \quad (k \leqslant n+1).$$

Proof. From [7] there is a Künneth spectral sequence for which

$$\mathrm{E}^2_{p,q} = \mathrm{Tor}^{B_*}_{p,q}(\pi_*(\Omega_{B/A}), \Bbbk) \Longrightarrow \pi_{p+q}\Omega_{B/A} \wedge_B H\Bbbk = \mathrm{HAQ}_{p+q}(B/A).$$

For dimensional reasons we have $\mathrm{E}^\infty_{0,n+1} = \mathrm{E}^2_{0,n+1}$ and so, on recalling that $\Bbbk = \pi_0 A = \pi_0 B$,

$$\mathrm{HAQ}_{n+1}(B/A) = [\pi_*(\Omega_{B/A}) \otimes_{B_*} \Bbbk]_{n+1} = \pi_{n+1}\Omega_{B/A} \otimes_{B_0} \Bbbk = \pi_{n+1}\Omega_{B/A}.$$ □

We will need to know the value of HAQ_* on sphere objects. Recall that for any A-module X, there is a free commutative A-algebra on X,

$$\mathbb{P}_A X = \bigvee_{i\geqslant 0} X^{(i)}/\Sigma_i,$$

where $X^{(i)}$ is the i-th smash power over A. We remark that if $A \longrightarrow A'$ is a morphism of commutative S-algebras, then from [7] we have

$$\mathbb{P}_{A'}(A' \wedge_A X) \cong A' \wedge_A \mathbb{P}_A X. \tag{1.8}$$

The A-algebra map $\mathbb{P}_A X \longrightarrow \mathbb{P}_A * = A$ induced by collapsing X to a point makes A into an $\mathbb{P}_A X$-algebra and there is a cofibration sequence of $\mathbb{P}_A X$-modules

$$\mathbb{P}^+_A X \longrightarrow \mathbb{P}_A X \xrightarrow{\varepsilon} \mathbb{P}_A * = A, \tag{1.9}$$

where

$$\mathbb{P}^+_A X = \bigvee_{i\geqslant 1} X^{(i)}/\Sigma_i$$

and ε is the augmentation. For the A-sphere $S^n = S^n_A$ with $n > 0$, we obtain the commutative A-algebra $\mathbb{P}_A S^n$ and augmentation $\mathbb{P}_A S^n \longrightarrow A$; this allows us to view an A-module or algebra as a $\mathbb{P}_A S^n$-module or algebra.

For a pair B/A, the B-module $\Omega_{B/A}$ is defined in the homotopy category $\mathscr{D}_B$ by

$$\Omega_{B/A} = \mathrm{L}Q_B \mathrm{R}I_B(B^c \wedge_A B). \tag{1.10}$$

Here we are using the following notation in keeping with [3]. $(-)^c$ is a cofibrant replacement functor. $\mathrm{R}I_B$ is the right derived functor of the augmentation ideal functor on the category of B-algebras augmented over B; the targets of I_B and $\mathrm{R}I_B$ are the category of B-nucas (non-unital B-algebras) and its homotopy category. $\mathrm{L}Q_B$ is the left derived functor of Q_B which is defined by the following strict pushout diagram in the category of B-modules.

$$\begin{array}{ccc} N \wedge_B N & \longrightarrow & * \\ \downarrow & & \downarrow \\ N & \longrightarrow & Q_B(N) \end{array} \tag{1.11}$$

Proposition 1.3. *Let X be a cell A-module, so that $\mathbb{P}_A X$ is q-cofibrant as an A-algebra. Then we have*

$$\Omega_{\mathbb{P}_A X/A} = \mathbb{P}_A X \wedge_A X.$$

The same result was obtained in the case of $A = S$ in lemma 3.6 and example 3.8 of Kuhn [14] using stabilisation. Our proof uses the following Lemma.

Lemma 1.4. *Let A be a commutative S-algebra, let N be an A-nuca and let B be an A-algebra. Then*

$$Q_B(B \wedge_A N) = B \wedge_A Q_A(N).$$

Proof. This follows from the fact that the functor $B \wedge_A (-)$ is a left adjoint in the category of modules, hence it respects colimits, together with the following identities:

$$B \wedge_A * = *,$$
$$B \wedge_A N \wedge_B B \wedge_A N \cong B \wedge_A N \wedge_A N. \qquad \square$$

For an A-module M and A-nuca N, the A-nuca $\mathbb{A}X = \mathbb{P}^+_A X$ satisfies the adjunction isomorphism

$$\mathscr{M}_A(X, N) \cong \mathscr{N}_A(\mathbb{A}X, N),$$

where $\mathscr{N}_A$ denotes the category of A-nucas. The idea for the proof of our next result was due to Maria Basterra.

Lemma 1.5. *For each A-module M, there is a natural isomorphism*

$$\mathscr{M}_A(Q_A(\mathbb{A}X), M) \cong \mathscr{M}_A(X, M),$$

hence $Q_A(\mathbb{A}X) \cong X$.

Proof. Given the natural isomorphism, Yoneda's lemma implies that $Q_A(\mathbb{A}X) \cong X$.

Recall Basterra's functor $Z\colon \mathscr{M}_A \longrightarrow \mathscr{N}_A$ which assigns to each A-module M the same module $Z(M)$ with the trivial product $Z(M) \wedge_A Z(M) \longrightarrow Z(M)$. Then for any A-nuca N we have

$$\mathscr{M}_A(Q_A(N), M) \cong \mathscr{N}_A(N, Z(M)),$$

since an A-module map $Q_A(N) \longrightarrow M$ fits into a commutative diagram

$$\begin{array}{ccccc} N \wedge_A N & \longrightarrow & * & & \\ \downarrow & & \downarrow & \searrow & \\ N & \longrightarrow & Q_A(N) & & \\ & \searrow & & \searrow & \\ & & & & M \end{array}$$

in which the rectangle is a pushout diagram. Hence there is a factorisation

$$\begin{array}{ccc} N \wedge_A N & \longrightarrow & N \\ \downarrow & & \downarrow \\ M \wedge_A M & \xrightarrow{0} & M \end{array}$$

of the products, while for an A-nuca map $N \longrightarrow Z(M)$ there is a factorisation

$$\begin{array}{ccccc} N \wedge_A N & & \longrightarrow & & N \\ \downarrow & & & & \downarrow \\ Z(M) \wedge_A Z(M) & \longrightarrow & * & \longrightarrow & Z(M) \end{array}$$

showing that the map factors through a module map $Q_A(N) \longrightarrow M$.

Now for any A-module X, we have

$$\mathscr{M}_A(Q_A(\mathbb{A}X)), M) \cong \mathscr{N}_A(\mathbb{A}X, Z(M)) \cong \mathscr{M}_A(X, M),$$

where the second isomorphism is a consequence of the universal property of $\mathbb{A}X$. □

Proof of Proposition 1.3. We start by describing an explicit q-cofibrant replacement for $\mathbb{P}_A X$ in $\mathscr{C}_A/\mathbb{P}_A X$. Recall that $\mathbb{P}_A$ preserves pushouts, so for any two A-modules X, Y, there is an isomorphism of A-algebras

$$\mathbb{P}_A(X \vee Y) \cong \mathbb{P}_A X \wedge_A \mathbb{P}_A Y.$$

Let X' denote a copy of X and let $f\colon X' \longrightarrow X$ be the composition

$$f\colon X' \xrightarrow{\text{pinch}} X \vee X \xrightarrow{\text{id} \vee *} X \vee *.$$

Notice that f is homotopic to the identity map and so it is a homotopy equivalence and the induced map of A-algebras $\mathbb{P}f\colon \mathbb{P}_A X' \longrightarrow \mathbb{P}_A X$ is a homotopy equivalence. If X is a q-cofibrant A-module then $\mathbb{P}_A X$ and $\mathbb{P}_A X'$ are q-cofibrant A-algebras.

Now consider the 'multiplication' map

$$\mathbb{P}_A X \wedge_A \mathbb{P}_A X' \xrightarrow{\text{id} \wedge \mathbb{P}f} \mathbb{P}_A X \wedge_A \mathbb{P}_A X \xrightarrow{\text{mult}} \mathbb{P}_A X,$$

which is a map of left $\mathbb{P}_A X$-algebras.

We can also consider another copy of X which we will label X''. We can map this to X by the trivial map and so obtain a map of left $\mathbb{P}_A X$-algebras

$$\mathbb{P}_A X \wedge_A \mathbb{P}_A X'' \xrightarrow{\mathrm{id}\wedge *} \mathbb{P}_A X \wedge_A \mathbb{P}_A * = \mathbb{P}_A X.$$

The point is that there are isomorphisms

$$\mathbb{P}_A X \wedge_A \mathbb{P}_A X' \underset{\mathbb{P}\beta}{\overset{\mathbb{P}\alpha}{\rightleftarrows}} \mathbb{P}_A X \wedge_A \mathbb{P}_A X''$$

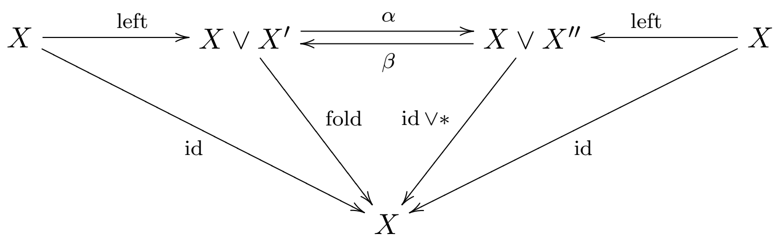
in $\mathscr{C}_A/\mathbb{P}_A X$ which are obtained by applying the functor $\mathbb{P}_A$ to the following compositions of q-cofibrant A-modules and using the fact that $\mathbb{P}_A$ preserves pushouts:

$$\alpha\colon X \vee X' \xrightarrow{\mathrm{id}\vee \mathrm{pinch}} X \vee X \vee X \xrightarrow{\mathrm{fold}\vee \mathrm{id}} X \vee X'',$$

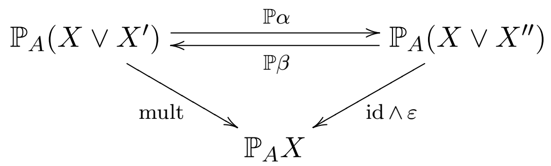
$$\beta\colon X \vee X'' \xrightarrow{\mathrm{id}\vee \mathrm{pinch}} X \vee X \vee X \xrightarrow{\mathrm{id}\vee(-\mathrm{id})\vee \mathrm{id}} X \vee X \vee X \xrightarrow{\mathrm{fold}\vee \mathrm{id}} X \vee X'.$$

It is straightfoward to verify that these are inverse homotopy equivalences and in the following diagram, the outermost triangles commutes and the central one is homotopy commutative.

(1.12)



Now the maps α and β induce A-algebra self maps of $\mathbb{P}_A(X \vee X)$, viewed in turn as $\mathbb{P}_A(X \vee X')$ $\mathbb{P}_A(X \vee X'')$. Using (1.12), we see that these are $\mathbb{P}_A X$-algebra maps, where we use the inclusion of the left copy of X to induce the algebra structure, and furthermore they are a pair of inverse homotopy equivalences. There is a strictly commuting diagram



in which the left hand arrow is the multiplication map and the right hand one uses the augmentation on the right hand factor.

Since (1.9) is a homotopy fibre sequence, so is

$$\mathbb{P}_A X \wedge_A \mathbb{P}_A^+ X \longrightarrow \mathbb{P}_A X \wedge_A \mathbb{P}_A X \xrightarrow{\mathrm{id}\wedge\varepsilon} \mathbb{P}_A X,$$

and in the homotopy category $\overline{h}\mathscr{M}_{\mathbb{P}_A X}$ this gives

$$\mathrm{R}I_{\mathbb{P}_A X}(\mathbb{P}_A X \wedge_A \mathbb{P}_A X) \cong \mathbb{P}_A X \wedge_A \mathbb{P}_A^+ X.$$

In an analogous fashion to the outcome in algebra, Lemma 1.5 implies that $Q_A(\mathbb{A}X) = X$. Also, by Lemma 1.4,

$$Q_{\mathbb{P}_A X}(\mathbb{P}_A X \wedge_A \mathbb{P}_A^+ X) = \mathbb{P}_A X \wedge_A X.$$

Now we must identify $\mathrm{L}Q_{\mathbb{P}_A X}(\mathbb{P}_A X \wedge_A \mathbb{P}_A^+ X)$. Since X is a cell A-module, $\mathbb{P}_A X \wedge_A \mathbb{P}_A^+ X$ is a cofibrant $\mathbb{P}_A X$-nuca. Furthermore we have $\mathrm{L}Q(-) \cong Q((-)^c)$ (*i.e.*, cofibrant replacement composed with Q) and hence

$$\mathrm{L}Q_{\mathbb{P}_A X}(\mathbb{P}_A X \wedge_A \mathbb{P}_A^+ X) = Q_{\mathbb{P}_A X}(\mathbb{P}_A X \wedge_A \mathbb{P}_A^+ X) = \mathbb{P}_A X \wedge_A X. \qquad \square$$

On taking $B = \mathbb{P}_A X$ and $C = A$, the cofibration sequence of [3, proposition 4.2] yields the cofibration sequence of A-modules

$$\Omega_{\mathbb{P}_A X/A} \wedge_{\mathbb{P}_A X} A \longrightarrow \Omega_{A/A} \longrightarrow \Omega_{A/\mathbb{P}_A X}$$

in which $\Omega_{A/A} \simeq *$. Hence as A-modules,

$$\Omega_{A/\mathbb{P}_A X} \simeq \Sigma\Omega_{\mathbb{P}_A X/A} \wedge_{\mathbb{P}_A X} A \simeq \Sigma X, \tag{1.13}$$

where the second $\simeq$ comes from Proposition 1.3.

Proposition 1.6. *For any $\mathbb{P}_A S^n$-module M we have*

$$\mathrm{TAQ}_*(\mathbb{P}_A S^n/A; M) \cong M_{*-n}, \quad \mathrm{TAQ}^*(\mathbb{P}_A S^n/A; M) \cong M^{*-n}.$$

In particular,

$$\mathrm{HAQ}_k(\mathbb{P}_A S^n/A) = \mathrm{HAQ}^k(\mathbb{P}_A S^n/A) = \begin{cases} \Bbbk & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Taking $X = S^n$ in Proposition 1.3 we obtain

$$\mathrm{TAQ}_*(\mathbb{P}_A S^n/A; M) = \pi_*\Omega_{\mathbb{P}_A S^n/A} \wedge_{\mathbb{P}_A S^n} M = \pi_* S^n \wedge M \cong M_{*-n},$$
$$\mathrm{TAQ}^*(\mathbb{P}_A S^n/A; M) = \pi_{-*}F_{\mathbb{P}_A S^n}(\Omega_{\mathbb{P}_A S^n/A}, M) = \pi_{-*}F(S^n, M) \cong M^{*-n}.$$

When $M = H\Bbbk$ with $\Bbbk = \pi_0 A$ this gives

$$\mathrm{HAQ}_k(\mathbb{P}_A S^n/A) = \mathrm{HAQ}^k(\mathbb{P}_A S^n/A) = \begin{cases} \Bbbk & \text{if } k = n, \\ 0 & \text{otherwise,} \end{cases}$$

as claimed. □

Proposition 1.7. *We have*

$$\mathrm{HAQ}_k(A/\mathbb{P}_A S^n) = \mathrm{HAQ}^k(A/\mathbb{P}_A S^n) = \begin{cases} \Bbbk & \text{if } k = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Setting $X = S^n$ in (1.13) we have

$$\Omega_{A/\mathbb{P}_A S^n} \wedge_A H\Bbbk \simeq \Sigma\Omega_{\mathbb{P}_A S^n/A} \wedge_{\mathbb{P}_A S^n} A \wedge_A H\Bbbk \simeq \Sigma S^n \wedge_A H\Bbbk,$$
$$F_A(\Omega_{A/\mathbb{P}_A S^n}, H\Bbbk) \simeq F_A(\Sigma\Omega_{\mathbb{P}_A S^n/A} \wedge_{\mathbb{P}_A S^n} A, H\Bbbk) \simeq F_A(\Sigma S^n, H\Bbbk).$$

Now using Proposition 1.6, the result is immediate. □

For a connective commutative S-algebra A and $n > 0$, there is a Quillen adjunction

$$\mathscr{M}_S(S^n, A) \rightleftarrows \mathscr{C}_S(\mathbb{P}_S S^n, A)$$

which gives rise to an isomorphism

$$\pi_n A = \overline{h}\mathscr{M}_S(S^n, A) \cong \overline{h}\mathscr{C}_S(\mathbb{P}_S S^n, A).$$

So to each homotopy class $[f] \in \pi_n A$ we may assign the homotopy class of $f' \in \mathscr{C}_S(\mathbb{P}_S S^n, A)$. Then the induced homomorphism

$$\mathbb{Z} \xrightarrow{\cong} \mathrm{HAQ}_n(\mathbb{P}_S S^n/S) \xrightarrow{f'_*} \mathrm{HAQ}_n(A/S)$$

gives rise to a homomorphism

$$\theta\colon \pi_n A \longrightarrow \mathrm{HAQ}_n(A/S), \quad \theta([f]) = f'_*(1). \tag{1.14}$$

This is an analogue of the Hurewicz homomorphism for ordinary TAQ homology. An alternative (and perhaps more direct) description of this uses the universal derivation of (1.2) to define for any A-algebra B, a morphism of A-modules

$$B \xrightarrow{\delta_{B/A}} \Omega_{B/A} = A \wedge_A \Omega_{B/A} \longrightarrow H\Bbbk \wedge_A \Omega_{B/A} \longrightarrow H\Bbbk \wedge_B \Omega_{B/A}, \tag{1.15}$$

and then take homotopy. It then follows that the following diagram commutes for any morphism of commutative A-algebras $\varphi\colon B \longrightarrow C$, where the morphisms are the evident ones.

$$\begin{array}{ccccc} B & \xrightarrow{\delta_{B/A}} & \Omega_{B/A} & \longrightarrow & H\Bbbk \wedge_B \Omega_{B/A} \\ \downarrow & & \downarrow & & \downarrow \\ C & \xrightarrow{\delta_{C/A}} & \Omega_{C/A} & \longrightarrow & H\Bbbk \wedge_C \Omega_{C/A} \\ & \searrow^{\delta_{C/B}} & \downarrow & & \downarrow \\ & & \Omega_{C/B} & \longrightarrow & H\Bbbk \wedge_C \Omega_{C/B} \end{array} \tag{1.16}$$

We will use this diagram later with $A = S$.

We say that the S-algebra A is *simply connected* if it is connective and $\pi_0 A = \mathbb{Z}$. Our next result is an analogue of a standard result on connective spectra. Here we consider $\mathrm{HAQ}_*(A/S) = \mathrm{HAQ}_*(A/S; H\mathbb{Z})$.

Proposition 1.8. *Let $\varphi\colon A \longrightarrow B$ be a map of simply connected commutative S-algebras. Then φ is an equivalence if and only if $\varphi_*\colon \mathrm{HAQ}_*(A/S) \longrightarrow \mathrm{HAQ}_*(B/S)$ is an isomorphism. In particular, the unit $S \longrightarrow B$ of such an S-algebra is a weak equivalence if and only if $\mathrm{HAQ}_*(B/S) = 0$.*

Proof. Let $n \geqslant 0$. Then φ is an n-equivalence if and only if the mapping cone C_φ is n-connected. But on combining Corollary 1.2 with the long exact sequence of (1.5a), we see that C_φ is n-connected if and only if $\varphi_*\colon \mathrm{HAQ}_k(A/S) \longrightarrow \mathrm{HAQ}_k(B/S)$ is an isomorphism for all $k \leqslant n$. Since this holds for all n, the result follows. □

As a corollary we have an analogue of the Hurewicz isomorphism theorem.

Corollary 1.9. *Let A be a commutative S-algebra whose unit $\eta\colon S \longrightarrow A$ is an n-equivalence. Then the Hurewicz homomorphism $\theta\colon \pi_{n+1} A \longrightarrow \mathrm{HAQ}_{n+1}(A/S)$ induces a monomorphism*

$$\theta'\colon \pi_{n+1} A/\eta_*\pi_{n+1} S \longrightarrow \mathrm{HAQ}_{n+1}(A/S)$$

which is an isomorphism if $\eta_*\colon \pi_n S \longrightarrow \pi_n A$ is a monomorphism.

Finally, we will also use the following observation which is a straightforward consequence of the definition, also see the discussion around diagram (4.3).

Proposition 1.10. *The* HAQ *Hurewicz homomorphism factors as*

$$\theta\colon \pi_n A \xrightarrow{\text{Hurewicz}} H_n(A) \longrightarrow \mathrm{HAQ}_n(A/S).$$

2. Topological André-Quillen homology for cell S-algebras

We will apply the results of Section 1 to the case of a CW commutative S-algebra R which is the colimit of a sequence of q-cofibrations of q-cofibrant S-algebras:

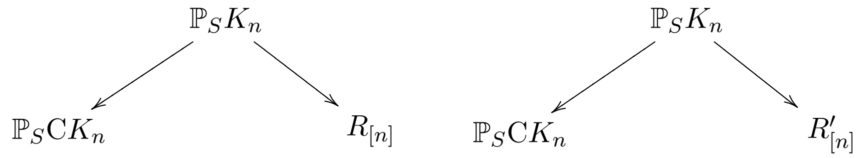
$$S = R_{[0]} \xrightarrow{i_0} \cdots \xrightarrow{i_{n-1}} R_{[n]} \xrightarrow{i_n} R_{[n+1]} \xrightarrow{i_{n+1}} \cdots .$$

Notice that each map $i_n\colon R_{[n]} \longrightarrow R_{[n+1]}$ is also a q-cofibration of S-modules. We will also assume that only cells of degree greater than 1 are attached, thus $R_{[1]} = R_{[0]} = S$ and $\pi_0 R = \pi_0 S$, so these S-algebras are simply connected. The $(n+1)$-skeleton $R_{[n+1]}$ is obtained by attaching a wedge of $(n+1)$-cells to $R_{[n]}$ using a map $k_n\colon K_n \longrightarrow R_{[n]}$ from a wedge of n-spheres K_n and its extension to a map of S-algebras $\mathbb{P}_S K_n \longrightarrow R_{[n]}$. To make this work properly, we need to use cofibrant models at appropriate places.

So assume as an induction hypothesis inductively that $R_{[n]}$ is defined and is a q-cofibrant S-algebra. Let $R'_{[n]}$ be a q-cofibrant replacement for $R_{[n]}$ as an $\mathbb{P}_S K_n$-algebra, thus there is an acyclic fibration $R'_{[n]} \longrightarrow R_{[n]}$.

Now define $R_{[n+1]}$ and $R''_{[n+1]}$ to be the pushouts of the diagrams

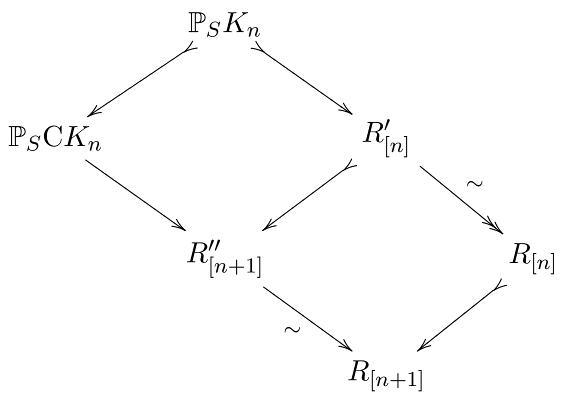
(2.1)



which are given by

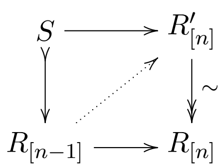
(2.2) $$R_{[n+1]} = \mathbb{P}_S \mathrm{C}K_n \wedge_{\mathbb{P}_S K_n} R_{[n]}, \quad R''_{[n+1]} = \mathbb{P}_S \mathrm{C}K_n \wedge_{\mathbb{P}_S K_n} R'_{[n]}.$$

Now consider the diagram of commutative $\mathbb{P}_S K_n$-algebras (and hence of S-algebras)



in which the upper and composite parallelograms are pushout diagrams. Here we use the standard notions $\twoheadrightarrow$, $\rightarrowtail$ and $\sim$ to denote fibrations, cofibrations and weak equivalences, respectively. By the Cofibration Hypothesis of [7], the left hand q-cofibration forces the parallel ones to be q-cofibrations. Also, by [7, proposition VII.7.4], the functor $\mathbb{P}_S \mathrm{C}K_n \wedge_{\mathbb{P}_S K_n} (-)$ preserves weak equivalences between q-cofibrant S-algebras, so the lower left arrow is also a weak equivalence.

Notice that for each n, there is a lifting diagram of the form



so it does no harm to assume that at each stage we have replaced $R_{[n]}$ by $R'_{[n]}$ in what follows.

Now by [3, proposition 4.6], for a q-cofibrant S-algebra A and q-cofibrant A-algebras $A \longrightarrow B$ and $A \longrightarrow C$, we have

(2.3) $$\Omega_{B\wedge_A C/C} \simeq \Omega_{B/A} \wedge_A C.$$

For $n \geqslant 1$ this gives

$$\Omega_{\mathbb{P}_S \mathrm{C}K_n \wedge_{\mathbb{P}_S K_n} R_{[n]}/R_{[n]}} \simeq \Omega_{\mathbb{P}_S \mathrm{C}K_n/\mathbb{P}_S K_n} \wedge_{\mathbb{P}_S K_n} R_{[n]}$$

and hence there is a long exact sequence derived from (1.6a),

$$\cdots \longrightarrow \mathrm{HAQ}_k(R_{[n]}/S) \longrightarrow \mathrm{HAQ}_k(R_{[n+1]}/S) \longrightarrow \mathrm{HAQ}_k(R_{[n+1]}/R_{[n]}) \longrightarrow \mathrm{HAQ}_{k-1}(R_{[n]}/S) \longrightarrow \cdots$$

which by (2.3) becomes

$$\cdots \longrightarrow \mathrm{HAQ}_k(R_{[n]}/S) \longrightarrow \mathrm{HAQ}_k(R_{[n+1]}/S) \longrightarrow \mathrm{HAQ}_k(\mathbb{P}_S\mathrm{C}K_n/\mathbb{P}_S K_n) \longrightarrow \mathrm{HAQ}_{k-1}(R_{[n]}/S) \longrightarrow \cdots$$

in which there is an equivalence of $\mathbb{P}_S K_n$-algebras

$$\mathbb{P}_S\mathrm{C}K_n \simeq \mathbb{P}_S * = S.$$

Hence we obtain the following long exact sequence

$$\cdots \longrightarrow \mathrm{HAQ}_k(R_{[n]}/S) \longrightarrow \mathrm{HAQ}_k(R_{[n+1]}/S) \longrightarrow \mathrm{HAQ}_k(S/\mathbb{P}_S K_n) \longrightarrow \mathrm{HAQ}_{k-1}(R_{[n]}/S) \longrightarrow \cdots \tag{2.4}$$

Using Proposition 1.7, we can now give an estimate for the size of $\mathrm{HAQ}_*(R/S)$ when R is a finite dimensional CW commutative S-algebra.

Proposition 2.1. *Let R be a CW commutative S-algebra with cells only in degrees at most n. Then $\mathrm{HAQ}_k(R/S) = 0$ when $k > n$.*

Corollary 2.2. *If R has only finitely many cells, then*

$$\sum_{k=0}^{n} \operatorname{rank} \mathrm{HAQ}_k(R/S) \leqslant \text{number of cells.}$$

In the category of S-modules there are three cofibration sequences that will concern us. We have the two cofibration sequences

$$K_n \xrightarrow{k_n} R_{[n]} \longrightarrow \mathrm{C}_{k_n}, \quad R_{[n]} \xrightarrow{i_n} R_{[n+1]} \longrightarrow \mathrm{C}_{i_n}.$$

From the proof of [3, lemma 8.2], there is a homotopy commutative diagram

$$\begin{array}{ccccc}
R_{[n]} & \xrightarrow{i_n} & R_{[n+1]} & \longrightarrow & \mathrm{C}_{i_n} \\
\downarrow & & \downarrow & & \downarrow \tau_n \\
R_{[n]} & \xrightarrow{i_n} & R_{[n+1]} & \xrightarrow{u_n} & \Omega_{R_{[n+1]}/R_{[n]}}
\end{array}$$

which we claim extends to a homotopy commutative diagram of the following form.

$$\begin{array}{ccccc}
R_{[n]} & \longrightarrow & \mathrm{C}_{k_n} & \longrightarrow & \Sigma K_n \\
\downarrow = & & \downarrow & & \downarrow h_n \\
R_{[n]} & \xrightarrow{i_n} & R_{[n+1]} & \longrightarrow & \mathrm{C}_{i_n} \\
\downarrow = & & \downarrow = & & \downarrow \tau_n \\
R_{[n]} & \xrightarrow{i_n} & R_{[n+1]} & \xrightarrow{u_n} & \Omega_{R_{[n+1]}/R_{[n]}}
\end{array} \tag{2.5}$$

Recalling that $R_{[n+1]}$ is a pushout for the diagram of commutative S-algebras (2.1), the map $\mathrm{C}_{k_n} \longrightarrow R_{[n+1]}$ exists since C_{k_n} is defined as a pushout for the diagram of S-modules

$$\begin{array}{ccc}
& K_n & \\
{\scriptstyle k_n}\swarrow & & \searrow \\
R_{[n]} & & \mathrm{C}K_n
\end{array}$$

and the evident composition

$$\mathrm{C}K_n \longrightarrow \mathbb{P}_S\mathrm{C}K_n \longrightarrow R_{[n+1]}$$

gives rise to a commutative diagram of S-modules of the following form.

$$\begin{array}{ccccc} & & K_n & & \\ & \overset{k_n}{\swarrow} & & \searrow & \\ R_{[n]} & & & & \mathrm{C}K_n \\ & \underset{i_n}{\searrow} & & \swarrow & \\ & & R_{[n+1]} & & \end{array}$$

Using (2.3), we see that there are equivalences of modules over $R_{[n+1]} = R_{[n]} \wedge_{\mathbb{P}_S K_n} \mathbb{P}_S CK_n$:

$$\Omega_{R_{[n+1]}/R_{[n]}} \simeq R_{[n]} \wedge_{\mathbb{P}_S K_n} \Omega_{S/\mathbb{P}_S K_n} \simeq R_{[n]} \wedge_{\mathbb{P}_S K_n} S \wedge \Sigma K_n. \tag{2.6}$$

Now smashing over $R_{[n+1]}$ with $H\mathbb{Z}$ we obtain

$$H\mathbb{Z} \wedge_{R_{[n+1]}} \Omega_{R_{[n+1]}/R_{[n]}} \simeq H\mathbb{Z} \wedge \Sigma K_n, \tag{2.7}$$

On smashing $\tau_n \circ h_n$ with $H\mathbb{Z}$ we obtain a map

$$H\mathbb{Z} \wedge \Sigma K_n \xrightarrow{\mathrm{id} \wedge \tau_n \circ h_n} H\mathbb{Z} \wedge \Omega_{R_{[n+1]}/R_{[n]}}$$

and following this with the natural map

$$H\mathbb{Z} \wedge \Omega_{R_{[n+1]}/R_{[n]}} \longrightarrow H\mathbb{Z} \wedge_{R_{[n+1]}} \Omega_{R_{[n+1]}/R_{[n]}} \simeq H\mathbb{Z} \wedge \Sigma K_n \tag{2.8}$$

yields a self map $f_n\colon H\mathbb{Z} \wedge \Sigma K_n \longrightarrow H\mathbb{Z} \wedge \Sigma K_n$. Since K_n is a wedge of n-spheres, $H\mathbb{Z} \wedge K_n$ is a wedge of copies of $H\mathbb{Z}$. In fact, the map of (2.8) induces an isomorphism on $\pi_{n+1}(\quad)$.

Lemma 2.3. *The map $f_n\colon H\mathbb{Z} \wedge \Sigma K_n \longrightarrow H\mathbb{Z} \wedge \Sigma K_n$ is a weak equivalence. Equivalently, the following maps are isomorphisms:*

$$\pi_{n+1}\Sigma K_n \xrightarrow{(h_n)_*} \pi_{n+1}\,\mathrm{C}_{i_n}, \quad \pi_{n+1}\Sigma K_n \xrightarrow{(h_n \circ \tau_n)_*} \pi_{n+1}\Omega_{R_{[n+1]}/R_{[n]}}.$$

Proof. The pairs $(\mathrm{C}_{k_n}, R_{[n]})$ and $(R_{[n+1]}, R_{[n]})$ occurring in (2.5) are relative cell complexes which have the same cells in degrees up to $2n+1$. The cells in degree $n+1$ correspond to those on ΣK_n and therefore $(h_n)_*\colon \pi_{n+1}\Sigma K_n \longrightarrow \pi_{n+1}\,\mathrm{C}_{i_n}$ is an isomorphism. For a discussion of cellular structures in this context, see [7, VII 3, X 2].

It now follows from the Hurewicz isomorphism theorem that f_n induces an isomorphism on $\pi_{n+1}(H\mathbb{Z} \wedge \Sigma K_n)$ which agrees with $H_{n+1}(\Sigma K_n)$. □

Applying homotopy to the diagram of (2.5), we obtain a diagram of groups, a part of which is

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \pi_{n+1}R_{[n]} & \longrightarrow & \pi_{n+1}\,\mathrm{C}_{k_n} & \longrightarrow & \pi_{n+1}\Sigma K_n & \longrightarrow & \pi_n R_{[n]} & \longrightarrow & \cdots \\ & & \downarrow{=} & & \downarrow & & \cong\downarrow(h_n)_* & & \downarrow{=} & & \\ \cdots & \longrightarrow & \pi_{n+1}R_{[n]} & \xrightarrow{(i_n)_*} & \pi_{n+1}R_{[n+1]} & \longrightarrow & \pi_{n+1}\,\mathrm{C}_{i_n} & \longrightarrow & \pi_n R_{[n]} & \longrightarrow & \cdots \\ & & \downarrow{=} & & \downarrow{=} & & \cong\downarrow(\tau_n)_* & & \downarrow{=} & & \\ \cdots & \longrightarrow & \pi_{n+1}R_{[n]} & \xrightarrow{(i_n)_*} & \pi_{n+1}R_{[n+1]} & \xrightarrow{(u_n)_*} & \pi_{n+1}\Omega_{R_{[n+1]}/R_{[n]}} & \longrightarrow & \pi_n R_{[n]} & \longrightarrow & \cdots \end{array} \tag{2.9}$$

and in which the top two rows are exact. In the portion shown, the bottom row is also exact because $(\tau_n)_*$ is an isomorphism on $\pi_{n+1}(\quad)$.

Using the definition in terms of (1.15), the Hurewicz homomorphism

$$\theta_{n+1}\colon \pi_{n+1}R_{[n+1]} \longrightarrow \mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]})$$

is induced from the composition

$$R_{[n+1]} \xrightarrow{\delta_{R_{[n+1]}/R_{[n]}}} \Omega_{R_{[n+1]}/R_{[n]}} \longrightarrow H\mathbb{Z} \wedge \Omega_{R_{[n+1]}/R_{[n]}} \longrightarrow H\mathbb{Z} \wedge_{R_{[n+1]}} \Omega_{R_{[n+1]}/R_{[n]}},$$

and using (1.16) it extends to a diagram

$$\begin{array}{ccccc} \pi_{n+1}R_{[n+1]} & \longrightarrow & \pi_{n+1}\Omega_{R_{[n+1]}/R_{[n]}} & \longrightarrow & \pi_n R_{[n]} \\ \downarrow{\scriptstyle \theta_{n+1}} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \theta_n} \\ \mathrm{HAQ}_{n+1}(R_{[n+1]}/S) & \longrightarrow & \mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]}) & \longrightarrow & \mathrm{HAQ}_n(R_{[n]}/S) \end{array} \tag{2.10}$$

in which the bottom row is a portion of the usual long exact sequence (1.6) for $A = S$, $B = R_{[n]}$ and $C = R_{[n+1]}$. Furthermore, these diagrams are compatible for varying n.

Using the evident natural transformation $\mathrm{HAQ}_n(\ \ ) \longrightarrow \mathrm{HAQ}_n(\ \ ;\mathbb{F}_p)$, we can map the bottom row of (2.10) into the exact sequence

$$0 \to \mathrm{HAQ}_{n+1}(R_{[n+1]}/S;\mathbb{F}_p) \longrightarrow \mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]};\mathbb{F}_p) \longrightarrow \mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p)$$

to obtain the commutative diagram

$$\begin{array}{cccccc} & \pi_{n+1}R_{[n+1]} & \longrightarrow & \pi_{n+1}\Sigma K_n & \longrightarrow & \pi_n R_{[n]} \\ & \downarrow{\scriptstyle \bar\theta_{n+1}} & & \downarrow{\scriptstyle \mathrm{epi}} & & \downarrow{\scriptstyle \bar\theta_n} \\ 0 \to & \mathrm{HAQ}_{n+1}(R_{[n+1]}/S;\mathbb{F}_p) & \longrightarrow & \mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]};\mathbb{F}_p) & \longrightarrow & \mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p) \end{array} \tag{2.11}$$

in which the rows are exact and the middle vertical arrow is an epimorphism.

Remark 2.4. All of the above works as well if we replace S by the p-local sphere for some prime p, or more generally by a connective commutative S-algebra A with $\pi_0 A$ a localisation of $\pi_0 S$.

3. Minimal atomic p-local commutative S-algebras

From now on we fix a prime p. We work with p-local spectra, suppressing indications of localization from the notation when convenient. Thus S stands for $S_{(p)}$ and so on. When discussing cell structures we always mean these to be taken p-locally, for example in the category of S-modules or S-algebras.

In [2], the following notion was introduced. A p-local CW complex Y is *minimal* if its the cellular chain complex with $\mathbb{F}_p$ coefficients has trivial boundaries, so

$$\mathrm{H}_*(Y;\mathbb{F}_p) = \mathrm{C}_*(Y) \otimes \mathbb{F}_p.$$

An alternative formulation in terms of the skeletal inclusion maps $Y_n \longrightarrow Y_{n+1} \longrightarrow Y$ is that for each n the induced epimorphism

$$\mathrm{H}_n(Y_n;\mathbb{F}_p) \longrightarrow \mathrm{H}_n(Y_{n+1};\mathbb{F}_p)$$

is actually an isomorphism and so

$$\mathrm{H}_n(Y_n;\mathbb{F}_p) \xrightarrow{\cong} \mathrm{H}_n(Y_{n+1};\mathbb{F}_p) \xrightarrow{\cong} \mathrm{H}_n(Y;\mathbb{F}_p).$$

In [2, theorem 3.3] it was shown that every p-local CW complex of finite-type is equivalent to a minimal one, so such minimal complexes exist in abundance.

Continuing to take the point of view that HAQ_* is a good substitute for ordinary homology when considering commutative S-algebras, let us consider the analogous notion in this multiplicative situation. We begin with a suitable definition of minimal in this context.

Definition 3.1. Let R be a p-local CW commutative S-algebra with n-skeleton $R_{[n]}$. Then R is *minimal* if for each n the inclusion maps $R_{[n]} \longrightarrow R_{[n+1]} \longrightarrow R$ induce isomorphisms

$$\mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p) \xrightarrow{\cong} \mathrm{HAQ}_n(R_{[n+1]}/S;\mathbb{F}_p)$$

or equivalently

$$\mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p) \xrightarrow{\cong} \mathrm{HAQ}_n(R/S;\mathbb{F}_p).$$

Proposition 2.1 implies that the homomorphisms here are both epimorphisms, as is true for their analogues in ordinary homology.

Theorem 3.2. *Let R be p-local CW commutative S-algebra, with finitely many p-local cells in each degree. Then there is a minimal p-local CW commutative S-algebra R' and an equivalence of commutative S-algebras $R' \longrightarrow R$.*

Proof. The details follow from the proof of [2, theorem 3.3], replacing ordinary homology with $\mathrm{HAQ}_*(\ \ )$, *mutatis mutandis.* □

This allows us to revisit the results of [2] in the context of simply connected p-local CW commutative S-algebras. The notion of a *nuclear algebra* appears in [11, definition 2.7], while that of an *atomic algebra* appears in [11, definition 2.8]. We also use notions from [2, definition 1.1]. Further details and related results appeared in the second author's PhD thesis [9].

First we have an analogue of [2, theorem 3.4]. We remark that it was not pointed out explicitly in [2] that a nuclear complex is always minimal so our next result has a direct analogue in the context of the earlier work even though it did not appear in the published version.

Theorem 3.3. *Let R be a simply connected p-local CW commutative S-algebra. Then R is nuclear if and only if it is minimal and the Hurewicz homomorphism $\theta\colon \pi_n R \longrightarrow \mathrm{HAQ}_n(R/S;\mathbb{F}_p)$ is trivial for all $n > 0$.*

Proof. Suppose that R is nuclear in the discussion of Section 2. Then as there is a factorisation

$$\begin{array}{ccc} \pi_{n+1}R_{[n+1]} & \longrightarrow & \pi_{n+1}\Sigma K_n \\ & \searrow \quad \nearrow & \\ & p\,\pi_{n+1}\Sigma K_n & \end{array}$$

we have $\overline{\theta}_{n+1} = 0$ for $n \geqslant 0$, and also

$$\mathrm{HAQ}_0(R_{[0]}/S;\mathbb{F}_p) = \mathrm{HAQ}_0(S/S;\mathbb{F}_p) = 0.$$

Similarly, the image of the boundary map $\mathrm{HAQ}_{n+1}(R_{[n+1]}/R_{[n]};\mathbb{F}_p) \longrightarrow \mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p)$ is contained in $\operatorname{im}\overline{\theta}_n = 0$, hence it is trivial. This shows that R is minimal in the sense of Definition 3.1, and has no mod p detectable homotopy in the sense that for $n > 0$, the composition

$$\overline{\theta}_n\colon \pi_n R_{[n]} \longrightarrow \mathrm{HAQ}_n(R_{[n]}/S;\mathbb{F}_p) \longrightarrow \mathrm{HAQ}_n(R/S;\mathbb{F}_p)$$

is trivial.

By a similar argument to the proof of [2, theorem 3.4], the converse also holds, *i.e.*, if R is minimal and has no mod p detectable homotopy then it is nuclear. □

We claim that [11, conjecture 2.9] and the analogue of [2, proposition 2.5] are consequences of our next result.

Theorem 3.4. *A nuclear simply connected p-local CW commutative S-algebra is minimal atomic, hence a core of such an algebra is an equivalence.*

Proof. This follows the analogous proof of [2]. □

We also note the following useful result.

Proposition 3.5. *Let R be minimal atomic as an S-module. Then it is minimal atomic as an S-algebra.*

Proof. This follows easily from the fact that for S-modules, being minimal atomic is equivalent to being irreducible.

As an alternative, notice that if R is minimal atomic as an S-module, then the ordinary homology Hurewicz homomorphism $\pi_n R \longrightarrow H_n(R;\mathbb{F}_p)$ is trivial for $n > 0$, so by Proposition 1.10 the mod p HAQ Hurewicz homomorphism $\pi_n R \longrightarrow \mathrm{HAQ}_n(R/S;\mathbb{F}_p)$ is trivial, whence R is a minimal atomic S-algebra. □

4. The TAQ Hurewicz homomorphism for Thom spectra

In order to calculate with Thom spectra arising from infinite loop maps into BF, we need some information on the relevant universal derivations. The next two results are implicit in the proof of [4, theorem 6.1], but unfortunately they are not stated explicitly and we are grateful to Mike Mandell and Maria Basterra for clarifying this material which is due to them.

Let X be an infinite loop space and let $\underline{X}$ be the associated spectrum viewed as an S-module. Following [4, section 6], consider the augmented S-algebra

$$\Sigma^\infty_{S+}X = S \wedge_{\mathcal{L}} \Sigma^\infty X_+.$$

There is a canonical (evaluation) morphism of S-modules $\sigma\colon \Sigma^\infty_{S+}X \longrightarrow \underline{X}$. Now we can consider the bijection of [3, proposition 3.2] with $A = B = S$:

$$\overline{h}\mathscr{C}_S/S(\Sigma^\infty_{S+}X, S \vee M) \cong \overline{h}\mathscr{M}_S(\mathrm{L}Q_S\mathrm{R}I_S(\Sigma^\infty_{S+}X \wedge_S S), M). \tag{4.1}$$

When $M = \mathrm{L}Q_S\mathrm{R}I_S(\Sigma^\infty_{S+}X \wedge_S S)$, the identity morphism on the right hand side corresponds to a morphism $\Sigma^\infty_{S+}X \longrightarrow S \vee \mathrm{L}Q_S\mathrm{R}I_S(\Sigma^\infty_{S+}X \wedge_S S)$ which projects to a universal derivation $\delta_X\colon \Sigma^\infty_{S+}X \longrightarrow \mathrm{L}Q_S\mathrm{R}I_S(\Sigma^\infty_{S+}X \wedge_S S)$.

Proposition 4.1. *In the homotopy category $\overline{h}\mathscr{M}_S$ there is an isomorphism*

$$\theta\colon \underline{X} \xrightarrow{\cong} \mathrm{L}Q_S\mathrm{R}I_S(\Sigma^\infty_{S+}X \wedge_S S)$$

and a commutative diagram

$$\begin{array}{ccc} & \Sigma^\infty_{S+}X & \\ {\scriptstyle\sigma}\swarrow & & \searrow{\scriptstyle\delta_X} \\ \underline{X} & \xrightarrow{\quad\theta\quad} & \mathrm{L}Q_S\mathrm{R}I_S(\Sigma^\infty_{S+}X \wedge_S S) \end{array}$$

and hence σ realises the universal derivation.

Note that the diagonal on X induces a morphism of commutative $\Sigma^\infty_{S+}X$-algebras

$$\Sigma^\infty_{S+}X \wedge_S \Sigma^\infty_{S+}X \xrightarrow{\mathrm{id}\wedge\mathrm{diag}} \Sigma^\infty_{S+}X \wedge_S \Sigma^\infty_{S+}X \wedge_S \Sigma^\infty_{S+}X \xrightarrow{\mathrm{mult}\wedge\mathrm{id}} \Sigma^\infty_{S+}X \wedge_S \Sigma^\infty_{S+}X$$

which is a weak equivalence. This has the effect of interchanging the multiplication map with the augmentation onto the second factor, giving rise to a composition of isomorphisms in $\overline{h}\mathscr{M}_{\Sigma^\infty_{S+}X}$

$$\Omega_{\Sigma^\infty_{S+}X/S} \xrightarrow{\cong} \Sigma^\infty_{S+}X \wedge_S \mathrm{L}Q_S\mathrm{R}I_S(\Sigma^\infty_{S+}X \wedge_S S) \xrightarrow{\cong} \Sigma^\infty_{S+}X \wedge_S \underline{X}. \tag{4.2}$$

For an infinite loop map $f\colon X \longrightarrow BF$, the resulting Thom spectrum Mf (viewed as a commutative S-algebra) has a Thom diagonal $\Delta\colon Mf \longrightarrow Mf \wedge_S \Sigma^\infty_{S+}X$, so that the composition

$$Mf \wedge_S Mf \xrightarrow{\mathrm{id}\wedge\Delta} Mf \wedge_S Mf \wedge_S \Sigma^\infty_{S+}X \xrightarrow{\mathrm{mult}\wedge\mathrm{id}} Mf \wedge_S \Sigma^\infty_{S+}X$$

is a weak equivalence of Mf-algebras. Furthermore, the map σ induces a map of S-algebras

$$\delta_f \colon Mf \xrightarrow{\Delta} Mf \wedge_S \Sigma^\infty_{S+} X \xrightarrow{\mathrm{id}\wedge\sigma} Mf \wedge_S \underline{X}$$

and then we have the following result in which we use the universal derivation $\delta_{Mf/S}$ of (1.2).

Proposition 4.2. *In the homotopy category* $\overline{h}\mathscr{M}_{Mf}$ *there is an isomorphism*

$$\Theta \colon Mf \wedge_S \underline{X} \xrightarrow{\cong} \Omega_{Mf/S}$$

and a commutative diagram

$$\begin{array}{ccccc} & & Mf & & \\ & \swarrow^{\delta_f} & & \searrow^{\delta_{Mf/S}} & \\ Mf \wedge_S \underline{X} & & \xrightarrow{\quad\Theta\quad} & & \Omega_{Mf/S} \end{array}$$

and hence δ_f *realises the universal derivation.*

Given this result we can now describe how to calculate the HAQ Hurewicz homomorphism for Mf. Recalling the definition using the map of (1.15), we see that when f factors through BSF, we can take coefficients in any commutative ring R and get homomorphisms

$$\theta \colon \pi_n Mf \longrightarrow \mathrm{HAQ}_n(Mf/S; R) = \mathrm{TAQ}_n(Mf/S; HR),$$

otherwise we may take R to be an $\mathbb{F}_2$-algebra. Then the map of (1.15) factors as in the commutative diagram

(4.3)

$$\begin{array}{ccccccc} Mf & \xrightarrow{\delta_{Mf/S}} & \Omega_{Mf/S} & \longrightarrow & HR \wedge \Omega_{Mf/S} & \longrightarrow & HR \wedge_B \Omega_{Mf/S} \\ \downarrow & & \downarrow & & \downarrow & & \uparrow{\scriptstyle \mathrm{mult}\wedge\mathrm{id}} \\ HR \wedge Mf & \xrightarrow{\delta_{Mf/S}} & HR \wedge \Omega_{Mf/S} & \longrightarrow & HR \wedge HR \wedge_A \Omega_{Mf/S} & \longrightarrow & HR \wedge HR \wedge_{Mf} \Omega_{Mf/S} \end{array}$$

in which the undecorated smash products are taken over S and the downward pointing arrows are obtained by smashing with the unit $S \longrightarrow HR$. Thus θ factors through the usual Hurewicz homomorphism:

$$\theta \colon \pi_n Mf \longrightarrow H_n(Mf; R) \longrightarrow \mathrm{HAQ}_n(Mf/S; R).$$

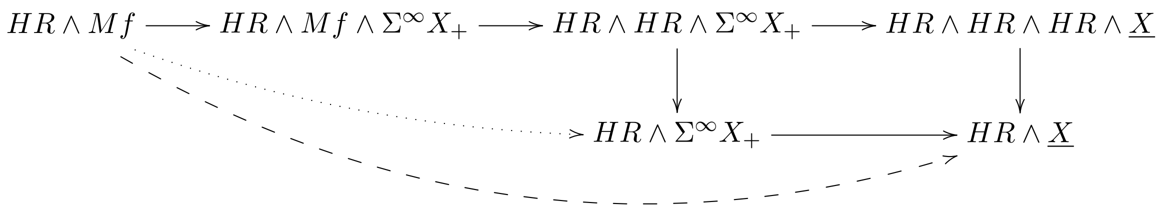
Using Proposition 4.2 we see that θ is equivalent to the homomorphism

$$\theta' \colon \pi_n Mf \longrightarrow H_n(\underline{X}; R) = \pi_n(HR \wedge \underline{X})$$

induced by

$$Mf \xrightarrow{\Delta} Mf \wedge \Sigma^\infty X_+ \xrightarrow{U\wedge\mathrm{id}} HR \wedge \Sigma^\infty X_+ \xrightarrow{\mathrm{id}\wedge\sigma} HR \wedge \underline{X},$$

where $U \in H^0(Mf; R)$ is the orientation class. On smashing with HR, we obtain



where the dashed arrow induces $\theta'' \colon H_n(Mf; R) \longrightarrow \mathrm{HAQ}_n(Mf; R)$ and the dotted arrow induces the Thom isomorphism $H_n(Mf; R) \longrightarrow H_n(X; R)$. Thus θ'' is the Thom isomorphism composed with $\sigma_* \colon H_n(X; R) \longrightarrow H_n(\underline{X}; R)$.

Notice that σ_* factors through the homology suspension

$$H_n(X;R) \xrightarrow{\cong} H_{n+1}(\Sigma X;R) \longrightarrow H_{n+1}(BX;R),$$

where BX is the delooping of X. Hence σ_*, and therefore θ'', annihilates decomposables in the rings on which they are defined.

5. Some examples of Thom spectra

The Thom spectrum MU: By Proposition 4.2, we know that $\Omega_{MU/S} = MU \wedge \Sigma^2 ku$ since BU is the zeroth space in the 1-connected cover of ku which is $\Sigma^2 ku$. Thus we have

$$\mathrm{HAQ}_*(MU/S;R) = H_*(\Sigma^2 ku;R) = H_{*-2}(ku;R).$$

For the prime 2,

$$\mathrm{HAQ}_*(MU/S;\mathbb{F}_2) = H_{*-2}(ku;\mathbb{F}_2),$$

where the right hand side is given by

$$H_*(ku;\mathbb{F}_2) = \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \ldots] \subseteq \mathcal{A}(2)_*$$

and is a sub Hopf algebra of the mod 2 dual Steenrod algebra with the inclusion induced from the natural map $ku \longrightarrow H\mathbb{F}_2$. The generators ζ_r are the conjugates of the generators ξ_r coming from $H_*(B\mathbb{Z}/2;\mathbb{F}_2)$.

For an odd prime p, there is decomposition

$$ku_{(p)} \simeq \bigvee_{1 \leqslant r \leqslant p-1} \Sigma^{2(p-1)r} \ell,$$

where ℓ is the Adams summand for which

$$H_*(\ell;\mathbb{F}_p) = \mathbb{F}_p[\zeta_1, \zeta_2, \zeta_3, \ldots] \otimes \Lambda(\overline{\tau}_2, \overline{\tau}_3, \ldots) \subseteq \mathcal{A}(p)_*,$$

where the embedding into the dual mod p dual Steenrod algebra is induced by a map $\ell \longrightarrow H\mathbb{F}_p$. The generators ζ_r and $\overline{\tau}_r$ are the conjugates of the generators ξ_r and τ_r coming from $H_*(B\mathbb{Z}/p;\mathbb{F}_p)$.

Recall that for a prime p, the image of the mod p Hurewicz homomorphism $h\colon \pi_* MU \longrightarrow H_*(MU;\mathbb{F}_p)$ is

$$h\pi_* MU = \mathbb{F}_p[x_r : r+1 \text{ is not a power of } p] \subseteq H_*(MU;\mathbb{F}_p),$$

where, in the notation of Adams [1],

$$x_r \equiv b_r \quad (\text{mod decomposables}).$$

Of course, x_r is the image of an element of $\pi_{2r} MU_{(p)}$. The Thom isomorphism is

$$\Phi\colon H_*(MU;\mathbb{F}_p) \xrightarrow{\cong} H_*(BU;\mathbb{F}_p); \quad b_n \longleftrightarrow \beta_n,$$

so to determine the Hurewicz homomorphism $\theta\colon \pi_* MU \longrightarrow \mathrm{HAQ}_*(MU/S;\mathbb{F}_p)$ we need to calculate the image of $\sigma_*\colon H_*(BU;\mathbb{F}_p) \longrightarrow H_{*-2}(ku;\mathbb{F}_p)$. Since decomposables are killed by σ_*, it suffices to know its effect on the generators β_n.

When $p = 2$, recall that the map $ku \longrightarrow H\mathbb{F}_2$ induces a monomorphism in $H_*(-;\mathbb{F}_2)$ and the canonical maps give a composition

$$\Sigma^\infty \mathbb{C}\mathrm{P}^\infty \longrightarrow \Sigma^\infty BU \longrightarrow \Sigma^2 ku \longrightarrow \Sigma^2 H\mathbb{F}_2$$

which corresponds to the natural cohomology generator $x \in H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{F}_2)$ and the homology generator β_n dual to x^n (which maps to $\beta_n \in H_{2n}(BU;\mathbb{F}_p)$) maps to the coefficient of t^n in the power series

$$\xi(t) = \sum_{s \geqslant 0} \xi_s^2 t^{2^s}.$$

This shows that

$$\sigma_*(\beta_n)\begin{cases}\neq 0 & \text{if } n \text{ is a power of 2,}\\ = 0 & \text{otherwise.}\end{cases} \tag{5.1}$$

Hence we see that

$$\theta\pi_{2n}MU\begin{cases}\neq 0 & \text{if } n \text{ is a power of 2,}\\ = 0 & \text{otherwise.}\end{cases} \tag{5.2}$$

More precisely, the polynomial generator of π_*MU in degree $2n$ is detected in $\mathrm{HAQ}_{2n}(MU/S;\mathbb{F}_2)$ if and only if n is a power of 2. This shows that $MU_{(2)}$ is not minimal atomic.

For an odd prime p, there is splitting of infinite loop spaces

$$BU_{(p)} \simeq W_1 \times \cdots \times W_{p-1},$$

where W_r is the zeroth space of $\Sigma^{2r}\ell$, so that the natural map

$$\Sigma^\infty BU_{(p)} \longrightarrow \Sigma^2 ku_{(p)} \simeq \bigvee_{1\leqslant r\leqslant p-1} \Sigma^{2r}\ell$$

factors through a wedge of maps $\Sigma^\infty W_r \longrightarrow \Sigma^{2r}\ell$. An exercise in using the definition of the Adams splitting together with the fact that lowest degree cohomology class for W_r corresponds to the Newton polynomial in the Chern classes in $H^{2r}(BU;\mathbb{F}_p)$, shows that the induced composition

$$\Sigma^\infty\mathbb{C}\mathrm{P}^\infty \longrightarrow \Sigma^\infty W_r \longrightarrow \Sigma^{2r}H\mathbb{F}_p$$

has the effect on homology of sending β_n to the coefficient of t^n in the series

$$\xi(t)^r = \big(\sum_{s\geqslant 0}\xi_s t^{p^s}\big)^r.$$

In particular, this shows that when $1 \leqslant r \leqslant p-1$, each generator $b_{p^s-1+r} \in H_{2(p^s-1+r)}(MU;\mathbb{F}_p)$ survives to give a non-zero element in $H_{2(p^s-1+r)}(\Sigma^2 ku;\mathbb{F}_p)$, therefore the corresponding generator $x_{p^s-1+r} \in \pi_{2(p^s-1+r)}MU$ also survives to a non-zero element of $H_{2(p^s-1+r)}(\Sigma^2 ku;\mathbb{F}_p)$.

To summarise, the Hurewicz homomorphism $\theta\colon \pi_*MU \longrightarrow \mathrm{HAQ}_*(MU/S;\mathbb{F}_p)$ detects homotopy, so $MU_{(p)}$ is not minimal atomic for any prime p.

A core for $MU_{(2)}$: The problem of identifying a core for $MU_{(p)}$ was first studied in [11] where the following example for the case $p = 2$ appeared. We give a new verification that the domain is indeed minimal atomic. Of course, it would be very interesting to identify cores of MU for the odd primes.

Consider the infinite loop map $BU \longrightarrow BSp$ classifying quaternionification of complex bundles. The fibre is also an infinite loop map $j\colon Sp/U \longrightarrow BU$, where Sp/U is the zeroth space of $\Sigma^2 ko$. Then j induces a Thom spectrum Mj which is a commutative S-algebra and the natural map $Mj \longrightarrow MU$ is a morphism of commutative S-algebras. Furthermore, the associated map in 2-local homology gives an isomorphism onto half of $H_*(MU;\mathbb{Z}_{(2)})$:

$$H_*(Mj;\mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)}[y_{2r-1} : r \geqslant 1] \xrightarrow{\cong} \mathbb{Z}_{(2)}[y'_{2r-1} : r \geqslant 1] \subseteq H_*(MU;\mathbb{Z}_{(2)}),$$

where y_{2r-1}, y'_{2r-1} have degree $2r-2$ and in $H_*(MU;\mathbb{Z}_{(2)})$,

$$y'_{2r-1} \equiv b_{2r-1} \quad (\text{mod decomposables}).$$

There is a similar result for mod 2 homology.

From now on in this example, all spectra are assumed to be localised at 2 and we drop this from the notation.

The argument of [11] shows that Mj is a wedge of suspensions of BP and as the induced map in homology is a monomorphism so is that in homotopy. So to show that $Mj \longrightarrow MU$ is a core we only need to show that Mj is minimal atomic.

This time we need to examine the mod 2 HAQ Hurewicz homomorphism which amounts to a homomorphism

$$\pi_*Mj \longrightarrow H_*(\Sigma^2 ko;\mathbb{F}_2).$$

By work of Stong [18], the map induced by the bottom cohomology class gives a homology monomorphism $H_*(\Sigma^2 ko;\mathbb{F}_2) \longrightarrow A(2)_*$. There is a commutative diagram

$$\begin{array}{ccc} \Sigma^\infty Sp/U & \longrightarrow & \Sigma^\infty BU \\ \downarrow & & \downarrow \\ \Sigma^2 ko & \longrightarrow & \Sigma^2 ku \end{array}$$

and on applying $H_*(-;\mathbb{F}_2)$ and using the calculations described for MU, we see that

$$\sigma_*(y'_{2r-1}) = 0 \quad (r \geqslant 1).$$

Hence the mod 2 HAQ Hurewicz homomorphism is also trivial in positive degrees.

We remark that the morphisms of S-algebras $S \longrightarrow Mj \longrightarrow MU$ give rise to a cofibre sequence of MU-modules

$$MU \wedge \Sigma^2 ko \longrightarrow MU \wedge \Sigma^2 ku \longrightarrow \Omega_{MU/Mj},$$

and this is equivalent to the cofibre sequence

$$MU \wedge \Sigma^2 ko \longrightarrow MU \wedge \Sigma^2 ku \longrightarrow MU \wedge \Sigma^4 ko,$$

where we view ku as $ko \wedge \mathrm{C}_\eta$ for $\eta \in \pi_1 S$ the generator. See [17, proposition 5.3.1] for more on this sequence in a Galois theoretic context. Smashing with $H\mathbb{F}_2$ over MU now gives the usual short exact sequence

$$0 \to H_*(ko;\mathbb{F}_2) \longrightarrow H_*(ku;\mathbb{F}_2) \longrightarrow H_{*-2}(ko;\mathbb{F}_2) \to 0.$$

MSp **2-locally:** Here only the prime $p = 2$ presents an interesting question.

Proposition 5.1. *$MSp_{(2)}$ is minimal atomic as a commutative S-algebra but not as an S-module.*

Proof. We will show that the mod 2 HAQ Hurewicz homomorphism is trivial in positive degrees. It is known that the mod 2 Hurewicz homomorphism $\pi_*MSp \longrightarrow H_*(MSp;\mathbb{F}_2)$ is not trivial in positive degrees, so MSp is not minimal atomic as a 2-local spectrum.

For this example, BSp is the zeroth space of the 3-connected spectrum $\Sigma^4 ko$. From [18], the bottom integral cohomology class induces a monomorphism

$$H_*(ksp;\mathbb{F}_2) \longrightarrow H_*(\Sigma^4 H\mathbb{F}_2;\mathbb{F}_2) = \mathcal{A}(2)_{*-4}.$$

However, the crux of our argument involves a beautiful result of Floyd [8] and we describe this in some detail.

Recall that the natural map $MSp \longrightarrow MO$ induces a mod 2 homology isomorphism onto the 4-th powers:

$$H_*(MSp;\mathbb{F}_2) \xrightarrow{\cong} H_*(MO;\mathbb{F}_2)^{(4)} \subseteq H_*(MO;\mathbb{F}_2). \tag{5.3}$$

In [8, theorems 5.3,5.5], it is shown that π_*MO has a family of polynomial generators $z_r \in \pi_r MO$ ($r+1$ not a power of 2) for which the polynomial subring

$$P_* = \mathbb{F}_2[z_r^{\kappa(r)} : r+1 \text{ not a power of } 2] \subseteq \pi_*MO,$$

satisfies

$$\mathrm{im}(\pi_*MSp \longrightarrow \pi_*MO) \subseteq P_*^{(8)},$$

where

$$\kappa(r) = \begin{cases} 2 & \text{if } r \text{ is odd or a power of 2,} \\ 1 & \text{if } r \text{ is even and not a power of 2.} \end{cases}$$

Kochman [12] shows that these two rings are actually equal, but for our purposes it suffices to have the inclusion.

For our purposes we need only remark that (5.3) together with the fact that the Hurewicz homomorphism $\pi_*MO \longrightarrow H_*(MO; \mathbb{F}_2)$ is monic, combine to show that the Hurewicz homomorphism for MSp has image contained in the squares:

$$\operatorname{im}(\pi_*MSp \longrightarrow H_*(MSp; \mathbb{F}_2)) \subseteq H_*(MSp; \mathbb{F}_2)^{(2)},$$

hence in positive degree this image is contained in the decomposables. Therefore the composition

$$H_*(MSp; \mathbb{F}_2) \xrightarrow{\cong} H_*(BSp; \mathbb{F}_2) \xrightarrow{\sigma_*} H_*(\Sigma^4 ksp; \mathbb{F}_2)$$

is trivial in positive degrees since the second map annihilates decomposables. □

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