Dept. of Math./CMA University of Oslo
Pure Mathematics No 26
ISSN 0806–2439 December 2008

A Malliavin calculus approach to general stochastic differential games with partial information

Yeliz Yolcu Okur[1)] Bernt Øksendal[1),2)] Ta Thi Kieu An[1)]

19 December 2008

Abstract

In this paper we consider a general partial information stochastic differential game where the state process is a controlled Itô-Lévy process. We use Malliavin calculus to derive a maximum principle for general stochastic differential games. The results are applied to solve a worst case scenario portfolio problem in finance.

Mathematics Subject Classification 2000: 49N70, 60G51, 60G57, 60HXX, 93E20, 91A15, 90C47, 91B28.

Key words: Malliavin calculus, stochastic differential game, stochastic control, jump diffusion, partial information, optimal worst case scenario portfolio.

[1)] Center of Mathematics for Applications (CMA), University of Oslo, Box 1053 Blindern, N-0316 Oslo, Norway. Email: <atkieu@math.uio.no>, <yelizy@math.uio.no>, <oksendal@math.uio.no>.

[2)] Norwegian School of Economics and Business Administration (NHH), Helleveien 30, N-5045 Bergen, Norway.

1 Introduction

Suppose the dynamics of a state process $X(t) = X^{(u_0,u_1)}(t,\omega)$; $t \geq 0$, $\omega \in \Omega$, is a controlled Itô-Lévy process in $\mathbb{R}$ of the form

$$
(1.1)\qquad \begin{cases} dX(t) = b(t, X(t), u_0(t), \omega)dt + \sigma(t, X(t), u_0(t), \omega)dB(t) \\ \qquad\qquad + \int_{\mathbb{R}_0} \gamma(t, X(t^-), u_0(t^-), u_1(t^-, z), z, \omega)\tilde{N}(dt, dz); \\ X(0) = x \in \mathbb{R}. \end{cases}
$$

where the coefficients $b : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$ and $\gamma : [0,T] \times \mathbb{R} \times U \times K \times \mathbb{R}_0 \times \Omega$ are given $\mathcal{F}_t$-predictable processes and U, K are given open convex subsets of $\mathbb{R}^2$ and $\mathbb{R} \times \mathbb{R}_0$ respectively. Here $\mathbb{R}_0 = \mathbb{R} - \{0\}$, $B(t) = B(t,\omega)$, and $\eta(t) = \eta(t,\omega)$, given by

$$
(1.2)\qquad \eta(t) = \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz);\ t \geq 0,\ \omega \in \Omega,
$$

are a 1-dimensional Brownian motion and an independent pure jump Lévy martingale, respectively, on a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Thus

$$
(1.3)\qquad \tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt
$$

is the *compensated Poisson jump measure* of $\eta(\cdot)$, where $N(dt, dz)$ is the *Poisson jump measure* and $\nu(dz)$ is the *Lévy measure* of the pure jump Lévy process $\eta(\cdot)$. For simplicity, we assume that

$$
(1.4)\qquad \int_{\mathbb{R}_0} z^2 \nu(dz) < \infty.
$$

The processes $u_0(t)$ and $u_1(t,z)$ are the control processes and have values in a given open convex set U and K respectively for a.a. $t \in [0,T]$, $z \in \mathbb{R}_0$ for a given fixed $T > 0$. Also, $u_0(\cdot)$ and $u_1(\cdot)$ are càdlàg and adapted to a given filtration $\{\mathcal{E}_t\}_{t\geq 0}$, where

$$
\mathcal{E}_t \subseteq \mathcal{F}_t, \qquad t \in [0,T].
$$

$\{\mathcal{E}_t\}_{t\geq 0}$ represents the information available to the controller at time t. For example, we could have

$$
\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+}; \quad t \in [0,T],\ \delta > 0 \text{ is a constant},
$$

meaning that the controller gets a delayed information compared to $\mathcal{F}_t$. We refer to [YZ] and [ØS] for more information about stochastic control of Itô diffusions and jump diffusions, respectively.
Let $f : [0,T] \times \mathbb{R} \times U \times K \times \Omega \to \mathbb{R}$ and $g : \mathbb{R} \times \Omega \to \mathbb{R}$ are given $\mathbb{F}$-adapted stochastic processes. Suppose there are two players in the stochastic differential game and the given performance functionals for players are as follows:

$$J_i(u_0, u_1) = \mathbb{E}^x\Big[\int_0^T \int_{\mathbb{R}_0} f_i(t, X(t), u_0(t), u_1(t,z), z, \omega)\mu(dz)dt + g_i(X(T), \omega)\Big], \quad i = 1, 2,$$

where μ is a measure on the given measurable space $(\Omega, \mathcal{F})$ and $\mathbb{E}^x = \mathbb{E}^x_P$ denotes the expectation with respect to P given that $X(0) = x$. Suppose that the controls $u_0(t)$ and $u_1(t,z)$ have the form

$$u_0(t) = (\pi_0(t), \theta_0(t)); \quad t \in [0,T] \tag{1.5}$$

$$u_1(t,z) = (\pi_1(t,z), \theta_1(t,z)); \quad (t,z) \in [0,T] \times \mathbb{R}_0. \tag{1.6}$$

Let $\mathscr{A}_\Pi$ and $\mathscr{A}_\Theta$ denote the given family of controls $\pi = (\pi_0, \pi_1)$ and $\theta = (\theta_0, \theta_1)$ such that they are contained in the set of $\mathscr{E}_t$-adapted controls, (1.1) has a unique strong solution up to time T and

$$\mathbb{E}^x\Big[\int_0^T \int_{\mathbb{R}_0} |f_i(t, X(t), \pi_0(t), \pi_1(t,z), \theta_0(t), \theta_1(t,z), z, \omega)|\mu(dz)dt + |g_i(X(T), \omega)|\Big] < \infty, \quad i = 1, 2.$$

The partial information non-zero-sum stochastic differential game problem we consider is the following:

Problem 1.1 *Find* $(\pi^*, \theta^*) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ *(if it exists) such that*

(i) $J_1(\pi, \theta^*) \le J_1(\pi^*, \theta^*)$ *for all* $\pi \in \mathscr{A}_\Pi$

(ii) $J_2(\pi^*, \theta) \le J_2(\pi^*, \theta^*)$ *for all* $\theta \in \mathscr{A}_\Theta$.

Such a control (π^*, θ^*) is called a *Nash Equilibrium* (if it exists). The intuitive idea is that there are two players, Player I and Player II. While Player I controls π, Player II controls θ. Each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only his or her own strategy (i.e., by changing unilaterally). Player I and Player II are in *Nash Equilibrium* if each player is making the best decision she can, taking into the account of the other player's decision. Note that since we allow

b, σ, γ, f and g to be stochastic processes and also because our controls are $\mathscr{E}_t-$adapted, this problem is not of Markovian type and hence cannot be solved by dynamic programming. Our paper is related to the recent paper [AØ] and [MØZ], where a maximum principle for stochastic differential games with partial information and maximum principle are deal with respectively. However, in [AØ] the existence of a solution of the adjoint equations is assumed. This is an assumption which often fails in the partial information case. We handle this problem by using Malliavin calculus techniques. We use Malliavin calculus to obtain a maximum principle for this general non-Markovian stochastic differential game with partial information.

2 A brief review of Malliavin calculus for Lévy processes

In this section, first we recall the basic definition and properties of Malliavin calculus for the space of functionals of Gaussian space. Second, we recall some fundamental properties and definitions of Malliavin calculus for Lévy processes related to this paper, for reader's convenience. For further information about Malliavin calculus, see [N], [BDLØP] and [DØP].

According to the Lévy–Itô decomposition theorem, any Lévy process $Y(t)$ with

$$\mathbb{E}[Y^2(t)] < \infty \quad \text{for all} \quad t$$

can be written

$$Y(t) = at + bB(t) + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz)$$

with constants a and b, so it suffices to deal with Malliavin calculus for $B(\cdot)$ and for

$$\eta(\cdot) := \int_0 \int_{\mathbb{R}_0} z\tilde{N}(ds, dz)$$

separately.

2.1 Malliavin calculus for $B(\cdot)$

A natural starting point is the Wiener-Itô chaos expansion theorem, which states that any $F \in L^2(\mathcal{F}_T, P)$ can be written

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{2.1}$$

for a unique sequence of symmetric deterministic functions $f_n \in L^2(\lambda^n)$, where λ is Lebesgue measure on $[0,T]$ and

$$I_n(f_n) = n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \cdots, t_n) dB(t_1) dB(t_2) \cdots dB(t_n) \tag{2.2}$$

(the n-times iterated integral of f_n with respect to $B(\cdot)$) for $n = 1, 2, \ldots$ and $I_0(f_0) = f_0$ when f_0 is a constant.

Moreover, we have the isometry

$$\mathbb{E}[F^2] = ||F||^2_{L^2(p)} = \sum_{n=0}^{\infty} n! ||f_n||^2_{L^2(\lambda^n)}. \tag{2.3}$$

Definition 2.1
Let $\mathscr{D}^{(B)}_{1,2}$ be the space of all $F \in L^2(\mathcal{F}_T, P)$ such that its chaos expansion (2.1) satisfies

$$||F||^2_{\mathscr{D}^{(B)}_{1,2}} := \sum_{n=1}^{\infty} n n! ||f_n||^2_{L^2(\lambda^n)} < \infty. \tag{2.4}$$

For $F \in \mathscr{D}^{(B)}_{1,2}$ and $t \in [0,T]$, we define the *Malliavin derivative* of F at t with respect to $B(\cdot)$, $D_t F$, by

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \tag{2.5}$$

where $I_{n-1}(f_n(\cdot,t))$ is the $(n-1)$-times iterated Itô integral to the first $n-1$ variables of $f_n(t_1, t_2, \cdots, t_n)$ and keep the last variable $t_n = t$ as a parameter.

One can easily check that

$$\mathbb{E}\Big[\int_0^T (D_t F)^2 dt\Big] = \sum_{n=1}^{\infty} n n! ||f_n||^2_{L^2(\lambda^n)} = ||F||^2_{\mathscr{D}^{(B)}_{1,2}}, \tag{2.6}$$

so if $F \in \mathscr{D}^{(B)}_{1,2}$, then $(t, \omega) \to D_t F(\omega)$ belongs to $L^2(\lambda \times P)$.

Example 2.2 If $F = \int_0^T f(t) dB(t)$ with $f \in L^2(\lambda)$ deterministic, then

$$D_t F = f(t) \text{ for } a.a.\, t \in [0,T].$$

More generally, if $u(s)$ is Skorohod integrable, $u(s) \in \mathscr{D}_{1,2}$ for $a.a.$ s and $D_t u(s)$ is Skorohod integrable for $a.a.$ t, then

$$D_t\Big(\int_0^T u(s)\delta B(s)\Big) = \int_0^T D_t u(s) \delta B(s) + u(t) \text{ for a.a. } (t,\omega), \tag{2.7}$$

where $\int_0^T \psi(s)\delta B(s)$ denotes the Skorohod integral of ψ with respect to $B(\cdot)$. (See [N], page 35–38 for a definition of Skorohod integrals and for more details.)

Some other basic properties of the Malliavin derivative D_t are the following:

(i) **Chain rule** ([N], page 29)
Suppose $F_1, \dots, F_m \in \mathscr{D}_{1,2}^{(B)}$ and that $\psi : \mathbb{R}^m \to \mathbb{R}$ is C^1 with bounded partial derivatives. Then $\psi(F_1, \cdots, F_m) \in \mathscr{D}_{1,2}$ and

$$D_t\psi(F_1, \cdots, F_m) = \sum_{i=1}^{m} \frac{\partial \psi}{\partial x_i}(F_1, \cdots, F_m) D_t F_i. \tag{2.8}$$

(ii) **Integration by parts/duality formula** ([N], page 35)
Suppose $u(t)$ is $\mathcal{F}_t-$adapted with $E[\int_0^T u^2(t)dt] < \infty$ and let $F \in \mathscr{D}_{1,2}^{(B)}$. Then

$$\mathbb{E}[F \int_0^T u(t)dB(t)] = \mathbb{E}[\int_0^T u(t) D_t F dt]. \tag{2.9}$$

2.2 Malliavin calculus for $\tilde{N}(\cdot)$

The construction of a stochastic derivative/Malliavin derivative in the pure jump martingale case is similar to the Brownian motion case. In this case the corresponding Wiener-Itô chaos expansion theorem states that any $F \in L^2(\mathcal{F}_T, P)$ (where in this case $\mathcal{F}_t = \mathcal{F}_t^{(\tilde{N})}$ is the $\sigma-$algebra generated by $\eta(s) := \int_0^s \int_{\mathbb{R}_0} z\tilde{N}(dr, dz);\ 0 \le s \le t$) can be written as

$$F = \sum_{n=0}^{\infty} I_n(f_n);\ f_n \in \hat{L}^2((\lambda \times \nu)^n), \tag{2.10}$$

where $\hat{L}^2((\lambda \times \nu)^n)$ is the space of functions $f_n(t_1, z_1, \dots, t_n, z_n);\ t_i \in [0, T],\ z_i \in \mathbb{R}_0$ such that $f_n \in L^2((\lambda \times \nu)^n)$ and f_n is symmetric with respect to the pairs of variables $(t_1, z_1), \dots, (t_n, z_n)$. It is important to note that in this case the $n-$times iterated integral $I_n(f_n)$ is taken with respect to $\tilde{N}(dt, dz)$ and not with respect to $d\eta(t)$. Thus, we define

$$I_n(f_n) = n! \int_0^T \int_{\mathbb{R}_0} \int_0^{t_n} \int_{\mathbb{R}_0} \cdots \int_0^{t_2} \int_{\mathbb{R}_0} f_n(t_1, z_1, \cdots, t_n, z_n) \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n) \tag{2.11}$$

for $f_n \in \hat{L}^2((\lambda \times \nu)^n)$.

The Itô isometry for stochastic integrals with respect to $\tilde{N}(dt, dz)$ then gives the following isometry for the chaos expansion:

$$||F||_{L^2(P)}^2 = \sum_{n=0}^{\infty} n! ||f_n||_{L^2((\lambda \times \nu)^n)}^2. \tag{2.12}$$

As in the Brownian motion case we use the chaos expansion to define the Malliavin derivative. Note that in this case there are two parameters t, z, where t represents time and $z \neq 0$ represents a generic jump size.

Definition 2.3 **(Malliavin derivative $D_{t,z}$)** [BDLØP], [DMØP], [DØP] Let $\mathscr{D}_{1,2}^{(\tilde{N})}$ be the space of all $F \in L^2(\mathcal{F}_T, P)$ such that its chaos expansion (2.10) satisfies

$$||F||^2_{\mathscr{D}_{1,2}^{(\tilde{N})}} := \sum_{n=1}^{\infty} nn!||f_n||^2_{L^2((\lambda\times\nu)^2)} < \infty. \tag{2.13}$$

For $F \in \mathscr{D}_{1,2}^{(\tilde{N})}$, we define the Malliavin derivative of F at (t,z) (with respect to $\tilde{N(\cdot)}$), $D_{t,z}F$, by

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t,z)), \tag{2.14}$$

where $I_{n-1}(f_n(\cdot,t,z))$ means that we perform the $(n-1)-$times iterated integral with respect to $\tilde{N}$ to the first $n-1$ variable pairs $(t_1,z_1),\cdots,(t_n,z_n)$, keeping $(t_n,z_n)=(t,z)$ as a parameter.

In this case we get the isometry.

$$\mathbb{E}[\int_0^T \int_{\mathbb{R}_0} (D_{t,z}F)^2\nu(dz)dt] = \sum_{n=0}^{\infty} nn!||f_n||^2_{L^2((\lambda\times\nu)^n)} = ||F||^2_{\mathscr{D}_{1,2}^{(\tilde{N})}}. \tag{2.15}$$

(Compare with (2.6).)

Example 2.4 If $F = \int_0^T \int_{\mathbb{R}_0} f(t,z)\tilde{N}(dt,dz)$ for some deterministic $f(t,z) \in L^2(\lambda\times\nu)$, then

$$D_{t,z}F = f(t,z) \text{ for } a.a.\ (t,z).$$

More generally, if $\psi(s,\zeta)$ is Skorohod integrable with respect to $\tilde{N}(\delta s, d\zeta)$, $\psi(s,\zeta) \in \mathscr{D}_{1,2}^{(\tilde{N})}$ for $a.a.\ s,\zeta$ and $D_{t,z}\psi(s,\xi)$ is Skorohod integrable for $a.a.\ (t,z)$, then

$$D_{t,z}(\int_0^T \int_{\mathbb{R}} \psi(s,\zeta)\tilde{N}(\delta s,d\zeta)) = \int_0^T \int_{\mathbb{R}} D_{t,z}\psi(s,\zeta)\tilde{N}(\delta s,d\zeta) + u(t,z) \text{ for } a.a.\ t,z, \tag{2.16}$$

where $\int_0^T \int_{\mathbb{R}} \psi(s,\zeta)\tilde{N}(\delta s,d\zeta)$ denotes the *Kabanov-Skorohod integral* of ψ with respect to $\tilde{N}(\cdot,\cdot)$. (See [DØP] for a definition of such Skorohod integrals and for more details.)

The properties of $D_{t,z}$ corresponding to the properties (2.8) and (2.9) of D_t are the following:

(i) **Chain rule ([DØP])** Suppose $F_1,\cdots,F_m \in \mathscr{D}_{1,2}^{(\tilde{N})}$ and that $\phi : \mathbb{R}^m \to \mathbb{R}$ is continuous and bounded. Then $\phi(F_1,\cdots,F_m) \in \mathscr{D}_{1,2}^{(\tilde{N})}$ and

$$D_{t,z}\phi(F_1,\cdots,F_m) = \phi(F_1 + D_{t,z}F_1,\ldots,F_m + D_{t,z}F_m) - \phi(F_1,\ldots,F_m). \tag{2.17}$$

(ii) **Integration by parts/duality formula [DØP]** Suppose $\Psi(t,z)$ is $\mathcal{F}_t$-adapted and $\mathbb{E}[\int_0^T \int_{\mathbb{R}_0} \psi^2(t,z)\nu(dz)dt] < \infty$ and let $F \in \mathscr{D}_{1,2}^{(\tilde{N})}$. Then

$$\mathbb{E}\Big[F \int_0^T \int_{\mathbb{R}_0} \Psi(t,z)\tilde{N}(dt,dz)\Big] = \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} \Psi(t,z) D_{t,z} F \nu(dz)dt\Big]. \tag{2.18}$$

3 The general maximum principle for the stochastic differential games

We now return to Problem 1.1 given in the introduction. We make the following assumptions:

Assumption 3.1

(3.1) The functions $b : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$, $\gamma : [0,T] \times \mathbb{R} \times U \times K \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$, $f_i : [0,T] \times \mathbb{R} \times U \times K \times \Omega \to \mathbb{R}$ and $g_i : \mathbb{R} \times \Omega \to \mathbb{R}$ for $i = 1,2$ are all continuously differentiable (C^1) with respect to $x \in \mathbb{R}$ and $u_0 \in U$, $u_1 \in K$ for each $t \in [0,T]$ and a.a. $\omega \in \Omega$.

(3.2) For all $s, r, t \in (0,T)$, $t \leq r$, and all bounded $\mathscr{E}_t-$measurable random variables $\alpha = \alpha(\omega)$, $\xi = \xi(\omega)$ the controls $\beta_\alpha(s) := (0, \beta^i_\alpha(s))$ and $\eta_\xi(s) := (0, \eta^i_\xi)$ for $i = 1,2$ with

$$\beta^i_\alpha(s) = \alpha^i(\omega)\chi_{[t,r]}(s); \quad s \in [0,T]$$

and

$$\eta^i_\xi(s) = \xi^i(\omega)\chi_{[t,r]}(s); \quad s \in [0,T]$$

belong to $\mathscr{A}_\Pi$ and $\mathscr{A}_\Theta$ respectively. Also, we will denote the transposed of the vectors β and η by β^*, η^* respectively.

(3.3) For all $\pi, \beta \in \mathscr{A}_\Pi$ with bounded β , there exists $\delta > 0$ such that

$$\pi + y\beta \in \mathscr{A}_\Pi \quad \text{for all } y \in (-\delta, \delta)$$

and such that the family

$$\Big\{ \frac{\partial f_1}{\partial x}(t, X^{(\pi+y\beta,\theta)}(t), \pi + y\beta, \theta, z) \frac{d}{dy} X^{(\pi+y\beta,\theta)}(t) + \nabla_\pi f_1(t, X^{(\pi+y\beta,\theta)}(t), \pi + y\beta, \theta, z)\beta^*(t) \Big\}_{y\in(-\delta,\delta)}$$

is $\lambda \times \nu \times P$-uniformly integrable and the family

$$\Big\{ g_1'(X^{\pi+y\beta}(T)) \frac{d}{dy} X^{(\pi+y\beta,\theta)}(T) \Big\}_{y\in(-\delta,\delta)}$$

is P-uniformly integrable. Similarly, for all $\theta, \eta \in \mathscr{A}_\Theta$ with η bounded, there exists $\delta > 0$ such that

$$\theta + \upsilon\eta \in \mathscr{A}_\Theta \quad \text{for all } \upsilon \in (-\delta, \delta)$$

and such that the family

$$\Big\{ \frac{\partial f_2}{\partial x}(t, X^{(\pi,\theta+\upsilon\eta)}(t), \pi, \theta + \upsilon\eta, z) \frac{d}{dy} X^{(\pi,\theta+\upsilon\eta)}(t) + \nabla_\theta f_2(t, X^{(\pi,\theta+\upsilon\eta)}(t), \pi(t), \theta + \upsilon\eta, z)\eta^*(t) \Big\}_{\upsilon\in(-\delta,\delta)}$$

is $\lambda \times \nu \times P$-uniformly integrable and the family

$$\Big\{ g_2'(X^{\pi,\theta+\upsilon\eta}(T)) \frac{d}{dy} X^{(\pi,\theta+\upsilon\eta)}(T) \Big\}_{\upsilon\in(-\delta,\delta)}$$

is P-uniformly integrable.

(3.4) For all $\pi, \beta \in \mathscr{A}_\Pi$ and $\theta, \eta \in \mathscr{A}_\Theta$ with β, η bounded the processes $Y(t) = Y^{(\beta)}(t) = \frac{d}{dy} X^{(\pi + y\beta, \theta)}(t)|_{y=0}$, $V(t) = V^{(\eta)}(t) = \frac{d}{dv} X^{(\pi, \theta + v\eta)}(t)|_{v=0}$ exist and satisfy the following equations, respectively:

$$\begin{aligned} dY(t) &= Y(t^-)\Big[\frac{\partial b}{\partial x}(t, X(t), \pi_0(t), \theta_0(t))dt + \frac{\partial \sigma}{\partial x}(t, X(t), \pi_0(t), \theta_0(t))dB(t) \\ &\quad + \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(t, X(t^-), \pi_0(t^-), \pi_1(t^-, z), \theta_0(t^-), \theta_1(t^-, z), z)\tilde{N}(dt, dz)\Big] \\ &\quad + \beta^*(t^-)\Big[\nabla_\pi b(t, X(t), \pi_0(t), \theta_0(t))dt + \nabla_\pi \sigma(t, X(t), \pi_0(t), \theta_0(t))dB(t) \\ &\quad + \int_{\mathbb{R}_0} \nabla_\pi \gamma(t, X(t^-), \pi_0(t^-), \pi_1(t^-, z), \theta_0(t^-), \theta_1(t^-), z)\tilde{N}(dt, dz)\Big]; \\ Y(0) &= 0, \end{aligned}$$

and

$$\begin{aligned} dV(t) &= V(t^-)\Big[\frac{\partial b}{\partial x}(t, X(t), \pi_0(t), \theta_0(t))dt + \frac{\partial \sigma}{\partial x}(t, X(t), \pi_0(t), \theta_0(t))dB(t) \\ &\quad + \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(t, X(t^-), \pi_0(t^-), \pi_1(t^-, z), \theta_0(t^-), \theta_1(t^-, z), z)\tilde{N}(dt, dz)\Big] \\ &\quad + \eta^*(t^-)\Big[\nabla_\theta b(t, X(t), \pi_0(t), \theta_0(t))dt + \nabla_\theta \sigma(t, X(t), \pi_0(t), \theta_0(t))dB(t) \\ &\quad + \int_{\mathbb{R}_0} \nabla_\theta \gamma(t, X(t^-), \pi_0(t^-), \pi_1(t^-, z), \theta_0(t^-), \theta_1(t^-, z), z)\tilde{N}(dt, dz)\Big]; \\ V(0) &= 0. \end{aligned}$$

(3.5) For all $\pi \in \mathscr{A}_\Pi$ and $\theta \in \mathscr{A}_\Theta$ the following processes

$$\begin{aligned} K_i(t) &:= g_i'(X(T)) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial f_i}{\partial x}(s, X(s), \pi, \theta, z_1)\mu(dz_1)ds, \\ D_t K_i(t) &:= D_t g_i'(X(T)) + \int_t^T \int_{\mathbb{R}_0} D_t \frac{\partial f_i}{\partial x}(s, X(s), \pi, \theta, z_1)\mu(dz_1)ds, \\ D_{t,z} K_i(t) &:= D_{t,z} g_i'(X(T)) + \int_t^T \int_{\mathbb{R}_0} D_{t,z} \frac{\partial f_i}{\partial x}(s, X(s), \pi, \theta, z_1)\mu(dz_1)ds, \\ H_i^0(s, x, \pi, \theta) &:= K_i(s) b(s, x, \pi_0, \theta_0) + D_s K_i(s)\sigma(s, x, \pi_0, \theta_0) \\ &\quad + \int_{\mathbb{R}_0} D_{s,z} K_i(s)\gamma(s, x, \pi, \theta, z)\nu(dz), \\ G(t, s) &:= \exp\Big(\int_t^s \Big\{\frac{\partial b}{\partial x}(r, X(r), \pi_0(r), \theta_0(r)) - \frac{1}{2}\Big(\frac{\partial \sigma}{\partial x}\Big)^2(r, X(r), \pi_0(r), \theta_0(r))\Big\}dr \\ &\quad + \int_t^s \frac{\partial \sigma}{\partial x}(r, X(r), \pi_0(r), \theta_0(r))dB(r) \\ &\quad + \int_t^s \int_{\mathbb{R}_0} \ln\Big(1 + \frac{\partial \gamma}{\partial x}(r, X(r^-), \pi(r^-, z), \theta(r^-, z), z)\Big)\tilde{N}(dr, dz) \end{aligned}$$

$$+\int_t^s \int_{\mathbb{R}_0} \Big\{ \ln\Big(1+\frac{\partial\gamma}{\partial x}(r,X(r),\pi,\theta,z)\Big) - \frac{\partial\gamma}{\partial x}(r,X(r),\pi,\theta,z)\Big\}\nu(dz)dr\Big);$$

(3.6)

$$p_i(t) := K_i(t) + \int_t^T \frac{\partial H_i^0}{\partial x}(s,X(s),\pi_0(s),\pi_1(s,z),\theta_0(s),\theta_1(s,z))G(t,s)ds,$$

(3.7)

$$q_i(t) := D_t p_i(t)\,, \qquad \text{and}$$

(3.8)

$$r_i(t,z) := D_{t,z} p_i(t)$$

all exist for $i=1,2$, $0 \le t \le s \le T,\ \ z_1, z \in \mathbb{R}_0$.

We now define the *Hamiltonian* for this general problem:

Definition 3.2 (The general stochastic Hamiltonian) *The general stochastic Hamiltonians for the stochastic differential game in Problem 1.1 are the functions*

$$H_i(t,x,\pi,\theta,\omega) : [0,T]\times\mathbb{R}\times U\times K\times\Omega \to \mathbb{R}, \quad i=1,2$$

defined by

$$\begin{aligned} H_i(t,x,\pi,\theta,\omega) = &\int_{\mathbb{R}_0} f_i(t,x,\pi,\theta,z,\omega)\mu(dz) + p_i(t)b(t,x,\pi_0,\theta_0,\omega) + q_i(t)\sigma(t,x,\pi_0,\theta_0,\omega) \\ &+ \int_{\mathbb{R}_0} r_i(t,z)\gamma(t,x,\pi,\theta,z,\omega)\nu(dz), \quad i=1,2 \end{aligned} \tag{3.9}$$

where $\pi=(\pi_0,\pi_1)$ *and* $\theta=(\theta_0,\theta_1)$.

Remark 3.3 In the classical Markovian case, the Hamiltonian $H_i^* : [0,T]\times\mathbb{R}\times U\times K\times\mathbb{R}\times\mathbb{R}\times\mathcal{R}\to\mathbb{R}$ is defined by

$$\begin{aligned} H_i^*(t,x,\pi,\theta,p,q,r) = &\int_{\mathbb{R}_0} f_i(t,x,\pi,\theta)\mu(dz) + p_i\, b(t,x,\pi_0,\theta_0) + q_i\,\sigma(t,x,\pi_0,\theta_0) \\ &+ \int_{\mathbb{R}_0} r_i(t,z)\gamma(t,x,\pi,\theta,z)\nu(dz), \end{aligned} \tag{3.10}$$

where $\mathcal{R}$ is the set of functions $r_i : \mathbb{R}\times\mathbb{R}_0\to\mathbb{R}$; $i=1,2$ see [FØS]. Thus the relation between H_i^* and H_i is that:

$$H_i(t,x,\pi,\theta,\omega) = H_i^*(t,x,\pi,\theta,p(t),q(t),r(t,\cdot)), \quad i=1,2 \tag{3.11}$$

where $p(\cdot), q(\cdot)$ and $r(\cdot,\cdot)$ are given by (3.6)–(3.8).

We can now formulate our general stochastic maximum principle for zero-sum games:

Theorem 3.4 (Maximum principle for non-zero-sum games)

(i) Suppose $(\hat{\pi}, \hat{\theta}) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ *is a Nash equilibrium, i.e.*

(i) $J_1(\pi, \hat{\theta}) \leq J_1(\hat{\pi}, \hat{\theta})$ *for all* $\pi \in \mathscr{A}_\Pi$
and

(ii) $J_2(\hat{\pi}, \theta) \leq J_2(\hat{\pi}, \hat{\theta})$ *for all* $\theta \in \mathscr{A}_\Theta$.

Then

$$\mathbb{E}^x[\nabla_\pi \hat{H}_1(t, X^{(\pi,\hat{\theta})}(t), \pi, \hat{\theta}, \omega)|_{\pi=\hat{\pi}} \;|\mathscr{E}_t] = 0, \tag{3.12}$$

and

$$\mathbb{E}^x[\nabla_\theta \hat{H}_2(t, X^{(\hat{\pi},\theta)}(t), \hat{\pi}, \theta, \omega)|_{\theta=\hat{\theta}} \;|\mathscr{E}_t] = 0 \quad \textit{for a.a. } t, \omega, \tag{3.13}$$

where

$$\begin{aligned}
&\hat{X}(t) = X^{(\hat{\pi},\hat{\theta})}(t),\\
&\hat{H}_i(t, \hat{X}(t), \pi, \theta) = \int_{\mathbb{R}_0} f_i(t, \hat{X}(t), \pi, \theta, z)\mu(dz) + \hat{p}_i(t) b(t, \hat{X}(t), \pi_0, \theta_0) + \hat{q}_i(t)\sigma(t, \hat{X}(t), \pi_0, \theta_0)\\
&\qquad + \int_{\mathbb{R}_0} \hat{r}_i(t,z)\gamma(t, \hat{X}(t^-), \pi, \theta, z)\nu(dz),
\end{aligned}$$

with

$$\hat{p}_i(t) = \hat{K}_i(t) + \int_t^T \frac{\partial \hat{H}_i^0}{\partial x}(s, \hat{X}(s), \hat{\pi}(s), \hat{\theta}(s))\, \hat{G}(t,s) ds,$$

$$\hat{K}_i(t) = K_i^{(\hat{\pi},\hat{\theta})}(t) = g_i'(\hat{X}(T)) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial f_i}{\partial x}(s, \hat{X}(s), \hat{\pi}(s,z), \hat{\theta}(s,z), z)\mu(dz) ds,$$

$$\hat{H}_i^0(s, \hat{X}, \hat{\pi}, \hat{\theta}) = \hat{K}_i(s) b(s, \hat{X}, \hat{\pi}_0, \hat{\theta}_0) + D_s \hat{K}_i(s)\sigma(s, \hat{X}, \hat{\pi}_0, \hat{\theta}_0) + \int_{\mathbb{R}_0} D_{s,z}\hat{K}_i(s)\gamma(s, \hat{X}, \hat{\pi}, \hat{\theta}, z)\nu(dz),$$

$$\begin{aligned}
\hat{G}(t,s) := \exp\Big(&\int_t^s \Big\{ \frac{\partial b}{\partial x}(r, \hat{X}(r), \hat{\pi}_0(r), \hat{\theta}_0(r)) - \frac{1}{2}\Big(\frac{\partial \sigma}{\partial x}\Big)^2 (r, \hat{X}(r), \hat{\pi}_0(r), \hat{\theta}_0(r)) \Big\} dr\\
&+ \int_t^s \frac{\partial \sigma}{\partial x}(r, \hat{X}(r), \hat{\pi}_0(r), \hat{\theta}_0(r)) dB(r)\\
&+ \int_t^s \int_{\mathbb{R}_0} \ln\Big(1 + \frac{\partial \gamma}{\partial x}(r, \hat{X}(r^-), \hat{\pi}(r^-, z), \hat{\theta}(r^-, z), z)\Big) \tilde{N}(dr, dz)\\
&+ \int_t^s \int_{\mathbb{R}_0} \Big\{ \ln\Big(1 + \frac{\partial \gamma}{\partial x}(r, \hat{X}(r), \hat{\pi}, \hat{\theta}, z)\Big) - \frac{\partial \gamma}{\partial x}(r, \hat{X}(r), \hat{\pi}, \hat{\theta}, z)\Big\}\nu(dz) dr \Big);
\end{aligned}$$

(3.14)

$$\hat{q}_i(t) := D_t \hat{p}_i(t),$$

and

$$\hat{r}_i(t,z) := D_{t,z} \hat{p}_i(t),$$

for $i = 1, 2$.

(ii) *Conversely, suppose that there exists* $(\hat{\pi}, \hat{\theta}) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ *such that (3.12) and (3.13) hold. Then*

$$\frac{\partial}{\partial y} J_1(\hat{\pi} + y\beta, \hat{\theta}) \mid_{y=0} = 0 \quad \textit{for all } \beta,$$
$$\frac{\partial}{\partial \upsilon} J_2(\hat{\pi}, \hat{\theta} + \upsilon\eta) \mid_{\upsilon=0} = 0 \quad \textit{for all } \eta.$$

In particular, if

$$\pi \longrightarrow J_1(\pi, \hat{\theta})$$

and

$$\theta \longrightarrow J_2(\hat{\pi}, \theta) \quad \textit{are concave,}$$

then $(\hat{\pi}, \hat{\theta})$ *is a Nash equilibrium.*

Proof.

(i) Suppose $(\hat{\pi}, \hat{\theta}) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ is a Nash equilibrium. Since (i) and (ii) hold for all π and θ, $(\hat{\pi}, \hat{\theta})$ is a directional critical point for $J_i(\pi, \theta)$ for $i = 1, 2$ in the sense that for all bounded $\beta \in \mathscr{A}_\Pi$ and $\eta \in \mathscr{A}_\Theta$, there exists $\delta > 0$ such that $\hat{\pi} + y\beta \in \mathscr{A}_\Pi$, $\hat{\theta} + \upsilon\eta \in \mathscr{A}_\Theta$ for all $y, \upsilon \in (-\delta, \delta)$. Then we have

(3.15)

$$\begin{aligned}
0 &= \tfrac{\partial}{\partial y} J_1(\hat{\pi} + y\beta, \hat{\theta})|_{y=0} \\
&= \mathbb{E}^x \big[\textstyle\int_0^T \int_{\mathbb{R}_0} \{ \tfrac{\partial f_1}{\partial x}(t, \hat{X}(t), \hat{\pi}_0(t), \hat{\pi}_1(t,z), \hat{\theta}_0(t), \hat{\theta}_1(t,z), z) \tfrac{d}{dy} X^{(\hat{\pi}+y\beta,\hat{\theta})}(t)|_{y=0} \\
&\quad + \nabla_\pi f_1(t, X^{(\pi,\hat{\theta})}(t), \pi_0(t), \pi_1(t,z), \hat{\theta}_0(t), \hat{\theta}_1(t,z), z)|_{\pi=\hat{\pi}} \beta^*(t) \} \mu(dz) dt \\
&\quad + g_1'(\hat{X}(T)) \tfrac{d}{dy} X^{(\hat{\pi}+y\beta,\hat{\theta})}(T)|_{y=0} \big] \\
&= \mathbb{E}^x \big[\textstyle\int_0^T \int_{\mathbb{R}_0} \{ \tfrac{\partial f_1}{\partial x}(t, \hat{X}(t), \hat{\pi}_0(t), \hat{\pi}_1(t,z), \hat{\theta}_0(t), \hat{\theta}_1(t,z), z) Y(t) \\
&\quad + \nabla_\pi f_1(t, X^{(\pi,\hat{\theta})}(t), \pi_0(t), \pi_1(t,z), \hat{\theta}_0(t), \hat{\theta}_1(t,z), z)|_{\pi=\hat{\pi}} \beta^*(t) \} \mu(dz) dt + g_1'(\hat{X}(T)) Y(T) \big]
\end{aligned}$$

where

(3.16)

$$
\begin{aligned}
Y(t) &= Y^{(\beta)}(t) = \tfrac{d}{dy} X^{(\hat{\pi}+y\beta,\hat{\theta})}(t)|_{y=0} \\
&= \textstyle\int_0^t \{ \frac{\partial b}{\partial x}(s,\hat{X}(s),\hat{\pi}_0(s),\hat{\theta}_0(s))Y(s) + \nabla_\pi b(s, X^{(\pi,\hat{\theta})}(s),\pi_0(s),\hat{\theta}_0(s))|_{\pi=\hat{\pi}}\beta^*(s)\} ds \\
&+ \textstyle\int_0^t \{ \frac{\partial \sigma}{\partial x}(s,\hat{X}(s),\hat{\pi}_0(s),\hat{\theta}_0(s))Y(s) + \nabla_\pi \sigma(s, X^{(\pi,\hat{\theta})}(s),\pi_0(s),\hat{\theta}_0(s))|_{\pi=\hat{\pi}}\beta^*(s)\} dB(s) \\
&+ \textstyle\int_0^t \int_{\mathbb{R}_0} \{ \frac{\partial \gamma}{\partial x}(s,\hat{X}(s^-),\hat{\pi}(s^-),\hat{\theta}(s^-),z)Y(s) \\
&+ \nabla_\pi \gamma(s, X^{(\pi,\hat{\theta})}(s^-),\pi(s^-),\hat{\theta}(s^-),z)|_{\pi=\hat{\pi}}\beta^*(s)\} \tilde{N}(ds,dz).
\end{aligned}
$$

If we use the short hand notation

$$
\frac{\partial f_1}{\partial x}(t,\hat{X}(t),\hat{\pi},\hat{\theta},z) = \frac{\partial f_1}{\partial x}(t,z), \quad \nabla_\pi f_1(t, X^{(\pi,\hat{\theta})}(t),\pi,\hat{\theta},z)|_{\pi=\hat{\pi}} = \nabla_\pi f_1(t,z)
$$

and similarly for $\frac{\partial b}{\partial x}$, $\nabla_\pi b$, $\frac{\partial \sigma}{\partial x}$, $\nabla_\pi \sigma$, $\frac{\partial \gamma}{\partial x}$ and $\nabla_\pi \gamma$, we can write

$$
\begin{aligned}
dY(t) &= \{ \tfrac{\partial b}{\partial x}(t)Y(t) + \nabla_\pi b(t)\beta^*(t)\} dt + \{ \tfrac{\partial \sigma}{\partial x}(t)Y(t) + \nabla_\pi \sigma(t)\beta^*(t)\} dB(t) \\
&+ \textstyle\int_{\mathbb{R}_0} \{ \frac{\partial \gamma}{\partial x}(t)Y(t) + \nabla_\pi \gamma(t,z)\beta^*(t)\} \tilde{N}(dt,dz); \\
Y(0) &= 0.
\end{aligned} \tag{3.17}
$$

By the duality formulas (2.9) and (2.18) and the Fubini theorem, we get

(3.18)

$$
\begin{aligned}
\mathbb{E}^x \Big[\textstyle\int_0^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(t,z)Y(t)\mu(dz)dt \Big] &= \mathbb{E}^x \Big[\textstyle\int_0^T \int_{\mathbb{R}_0} \Big(\int_0^t \Big\{ \frac{\partial f_1}{\partial x}(t,z)[\frac{\partial b}{\partial x}(s)Y(s) + \nabla_\pi b(s)\beta^*(s)] \\
&+ D_s \tfrac{\partial f_1}{\partial x}(t,z)[\tfrac{\partial \sigma}{\partial x}(s)Y(s) + \nabla_\pi \sigma(s)\beta^*(s)] \\
&+ \textstyle\int_{\mathbb{R}_0} D_{s,z_1} \frac{\partial f_1}{\partial x}(t,z)[\frac{\partial \gamma}{\partial x}(s,z_1)Y(s) + \nabla_\pi \gamma(s,z_1)\beta^*(s)] \\
&\nu(dz_1) \Big\} ds \Big) \mu(dz)dt \Big] \\
&= \mathbb{E}^x \Big[\textstyle\int_0^T \Big\{ (\int_s^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(t,z)\mu(dz)dt)[\frac{\partial b}{\partial x}Y(s) + \nabla_\pi b(s)\beta^*(s)] \\
&+ (\textstyle\int_s^T \int_{\mathbb{R}_0} D_s \frac{\partial f_1}{\partial x}(t,z)\mu(dz)dt)[\frac{\partial \sigma}{\partial x}Y(s) + \nabla_\pi \sigma\beta^*(s)] \\
&+ \textstyle\int_{\mathbb{R}_0} (\int_s^T \int_{\mathbb{R}_0} D_{s,z_1} \frac{\partial f_1}{\partial x}(t,z)\mu(dz)dt)[\frac{\partial \gamma}{\partial x}Y(s) + \nabla_\pi \gamma\beta^*(s)] \\
&\nu(dz_1) \Big\} ds \Big]
\end{aligned}
$$

Changing notation $s \to t$ and $z_1 \to z$ this becomes

(3.19)

$$\begin{aligned}
\mathbb{E}^x\Big[\int_0^T \int_{\mathbb{R}_0} \tfrac{\partial f_1}{\partial x}(t,z)Y(t)\mu(dz)dt\Big] &= \mathbb{E}^x\Big[\int_0^T \Big\{(\int_t^T \int_{\mathbb{R}_0} \tfrac{\partial f_1}{\partial x}(s,z_1)\mu(dz_1)ds) \\
&[\tfrac{\partial b}{\partial x}(t)Y(t) + \nabla_\pi b(t)\beta^*(t)] \\
&+(\int_t^T \int_{\mathbb{R}_0} D_t \tfrac{\partial f_1}{\partial x}(s,z_1)\mu(dz_1)ds)[\tfrac{\partial \sigma}{\partial x}(t)Y(t) + \nabla_\pi \sigma(t)\beta^*(t)] \\
&+\int_{\mathbb{R}_0}(\int_t^T \int_{\mathbb{R}_0} D_{t,z} \tfrac{\partial f_1}{\partial x}(s,z_1)\mu(dz_1)ds)[\tfrac{\partial \gamma}{\partial x}(t,z)Y(t) \\
&+\nabla_\pi \gamma(t,z)\beta^*(t)]\nu(dz)\Big\}dt\Big].
\end{aligned}$$

On the other hand, by the duality formulas (2.9) and (2.18), we get

$$\begin{aligned}
\mathbb{E}^x\Big[g_1'(\hat{X}(T))Y(T)\Big] &= \mathbb{E}^x\Big[g_1'(\hat{X}(T))\Big(\int_0^T\{\tfrac{\partial b}{\partial x}(t)Y(t) + \nabla_\pi b(t)\beta^*(t)\}dt \\
&+\int_0^T\{\tfrac{\partial \sigma}{\partial x}(t)Y(t) + \nabla_\pi \sigma(t)\beta^*(t)\}dB(t) \\
&+\int_0^T\int_{\mathbb{R}_0}\{\tfrac{\partial \gamma}{\partial x}(t,z)Y(t) + \nabla_\pi \gamma(t,z)\beta(t)\}\tilde{N}(dt,dz)\Big)\Big] \\
&= \mathbb{E}^x\Big[\int_0^T \Big\{g_1'(\hat{X}(T))\tfrac{\partial b}{\partial x}(t)Y(t) + g_1'(\hat{X}(T))\nabla_\pi b(t)\beta^*(t) \\
&+D_t(g_1'(\hat{X}(T)))\tfrac{\partial \sigma}{\partial x}(t)Y(t) + D_t(g_1'(\hat{X}(T)))\nabla_\pi \sigma(t)\beta^*(t) \\
&+\int_{\mathbb{R}_0}[D_{t,z}(g_1'(\hat{X}(T)))\tfrac{\partial \gamma}{\partial x}(t,z)Y(t) \\
&+D_{t,z}(g_1'(\hat{X}(T)))\nabla_\pi \gamma(t,z)\beta^*(t)]\nu(dz)\Big\}dt\Big].
\end{aligned}$$

Recall

$$K_1(t) := g_1'(X(T)) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(s,z_1)\mu(dz_1)\, ds, \tag{3.20}$$

so

$$\hat{K}_1(t) := g_1'(\hat{X}(T)) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial f_1}{\partial x}(s,z_1)\mu(dz_1)\, ds.$$

By combining (3.17)–(3.19), we get

(3.21)

$$\begin{aligned}
&\mathbb{E}^x\big[\int_0^T \big\{\hat{K}_1(t)(\tfrac{\partial b}{\partial x}(t)Y(t) + \nabla_\pi b(t)\beta^*(t)) + D_t\hat{K}_1(t)(\tfrac{\partial \sigma}{\partial x}(t)Y(t) + \nabla_\pi \sigma(t)\beta^*(t)) \\
+&\int_{\mathbb{R}_0} D_{t,z}\hat{K}_1(t)(\tfrac{\partial \gamma}{\partial x}(t,z)Y(t) + \nabla_\pi \gamma(t,z)\beta^*(t))\nu(dz) \\
+&\int_{\mathbb{R}_0} \nabla_\pi f_1(t,z)\beta^*(t)\mu(dz)\big\}dt\big] = 0.
\end{aligned}$$

Now apply this to $\beta = \beta_\alpha \in \mathscr{A}_\Pi$ of the form $\beta_\alpha(s) = \alpha\chi_{[t,t+h]}(s)$, for some $t, h \in (0,T)$, $t+h \leq T$, where $\alpha = \alpha(\omega)$ is bounded and $\mathscr{E}_t$-measurable. Then $Y^{(\beta_\alpha)}(s) = 0$ for $0 \leq s \leq t$ and hence (3.21) becomes

$$A_1 + A_2 = 0, \tag{3.22}$$

where

$$A_1 = \mathbb{E}^x\Big[\int_t^T \{\hat{K}_1(s)\frac{\partial b}{\partial x}(s) + D_s\hat{K}_1(s)\frac{\partial \sigma}{\partial x}(s) + \int_{\mathbb{R}_0} D_{s,z}\hat{K}_1(s)\frac{\partial \gamma}{\partial x}(s)\nu(dz)\}Y^{(\beta_\alpha)}(s)ds\Big],$$

$$\begin{aligned} A_2 &= \mathbb{E}^x\Big[(\textstyle\int_t^{t+h}\big\{\hat{K}_1(s)\nabla_\pi b(s) + D_s\hat{K}_1(s)\nabla_\pi\sigma(s) + \int_{\mathbb{R}_0} D_{s,z}\hat{K}_1(s)\nabla_\pi\gamma(s,z)\nu(dz) \\ &\quad + \textstyle\int_{\mathbb{R}_0}\nabla_\pi f_1(s,z)\mu(dz)\big\}ds)\alpha\Big]. \end{aligned}$$

Note that, by (3.16), with $Y(s) = Y^{(\beta_\alpha)}(s)$ and $s \geq t+h$,

$$dY(s) = Y(s^-)\Big\{\frac{\partial b}{\partial x}(s)ds + \frac{\partial \sigma}{\partial x}(s)dB(s) + \int_{\mathbb{R}_0}\frac{\partial \gamma}{\partial x}(s^-,z)\tilde{N}(ds,dz)\Big\},$$

for $s \geq t+h$. Hence by the Itô formula,

$$Y(s) = Y(t+h)G(t+h,s); \qquad s \geq t+h, \tag{3.23}$$

where, in general, for $s \geq t$,

$$\begin{aligned} G(t,s) = \exp\Big(&\int_t^s \Big\{\frac{\partial b}{\partial x}(r) - \tfrac{1}{2}\Big(\frac{\partial \sigma}{\partial x}\Big)^2(r)\Big\}dr + \int_t^s \frac{\partial \sigma}{\partial x}(r)dB(r) \\ &+ \int_t^s\int_{\mathbb{R}_0}\ln\Big(1+\frac{\partial\gamma}{\partial x}(r^-,z)\Big)\tilde{N}(dr,dz) \\ &+ \int_t^s\int_{\mathbb{R}_0}\Big\{\ln\Big(1+\frac{\partial\gamma}{\partial x}(r,z)\Big) - \frac{\partial\gamma}{\partial x}(r,z)\Big\}\nu(dz)dr\Big). \end{aligned} \tag{3.24}$$

Note that $G(t,s)$ does not depend on h. Put

$$H_1^0(s,x,\pi,\theta) = K_1(s)b(s,x,\pi_0,\theta_0) + D_sK_1(s)\sigma(s,x,\pi_0,\theta_0) + \int_{\mathbb{R}_0} D_{s,z}K_1(s)\gamma(s,x,\pi,\theta,z)\nu(dz), \tag{3.25}$$

and $\hat{H}_1^0(s) = H_1^0(s,\hat{X}(s),\hat{\pi},\hat{\theta})$. Then

$$A_1 = \mathbb{E}^x\Big[\int_t^T \frac{\partial \hat{H}_1^0}{\partial x}(s)Y(s)ds\Big].$$

Differentiating with respect to h at $h=0$ we get

$$\frac{d}{dh}A_1\Big|_{h=0} = \frac{d}{dh}\mathbb{E}^x\Big[\int_t^{t+h}\frac{\partial \hat{H}_1^0}{\partial x}(s)Y(s)ds\Big]_{h=0} + \frac{d}{dh}\mathbb{E}^x\Big[\int_{t+h}^T\frac{\partial \hat{H}_1^0}{\partial x}(s)Y(s)ds\Big]_{h=0}. \tag{3.26}$$

Since $Y(t) = 0$ we see that

$$\frac{d}{dh}\mathbb{E}^x\Big[\int_t^{t+h} \frac{\partial \hat{H}_1^0}{\partial x}(s)Y(s)ds\Big]_{h=0} = 0. \tag{3.27}$$

Therefore, by (3.22),

$$\begin{aligned}\frac{d}{dh}A_1\big|_{h=0} &= \frac{d}{dh}\mathbb{E}^x\Big[\int_{t+h}^T \frac{\partial \hat{H}_1^0}{\partial x}(s)Y(t+h)G(t+h,s)ds\Big]_{h=0} \\ &= \int_t^T \frac{d}{dh}\mathbb{E}^x\Big[\frac{\partial \hat{H}_1^0}{\partial x}(s)Y(t+h)G(t+h,s)\Big]_{h=0} ds \\ &= \int_t^T \frac{d}{dh}\mathbb{E}^x\Big[\frac{\partial \hat{H}_1^0}{\partial x}(s)G(t,s)Y(t+h)\Big]_{h=0} ds. \end{aligned} \tag{3.28}$$

By (3.16)

$$\begin{aligned} Y(t+h) = \alpha \int_t^{t+h} & \Big\{\nabla_\pi b(r)dr + \nabla_\pi \sigma dB(r) + \int_{\mathbb{R}_0} \nabla_\pi \gamma(r^-,z)\tilde{N}(dr,dz)\Big\} \\ &+ \int_t^{t+h} Y(r^-)\Big\{\frac{\partial b}{\partial x}(r)dr + \frac{\partial \sigma}{\partial x}(r)dB(r) + \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(r^-,z)\tilde{N}(dr,dz)\Big\}. \end{aligned} \tag{3.29}$$

Therefore, by (3.27) and (3.28),

$$\frac{d}{dh}A_1\big|_{h=0} = \Lambda_1 + \Lambda_2, \tag{3.30}$$

where

$$\begin{aligned} \Lambda_1 = \int_t^T \frac{d}{dh}\mathbb{E}^x\Big[\frac{\partial \hat{H}_1^0}{\partial x}(s)G(t,s)\alpha \int_t^{t+h} & \Big\{\nabla_\pi b(r)dr + \nabla_\pi \sigma(r)dB(r) \\ &+ \int_{\mathbb{R}_0} \nabla_\pi \gamma(r^-,z)\tilde{N}(dr,dz)\Big\}\Big]_{h=0} ds \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} \Lambda_2 = \int_t^T \frac{d}{dh}\mathbb{E}^x\Big[\frac{\partial \hat{H}_1^0}{\partial x}(s)G(t,s) \int_t^{t+h} & Y(r^-)\Big\{\frac{\partial b}{\partial x}(r)dr + \frac{\partial \sigma}{\partial x}(r)dB(r) \\ &+ \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(r^-,z)\tilde{N}(dr,dz)\Big\}\Big]_{h=0} ds. \end{aligned} \tag{3.32}$$

By the duality formulae (2.9), (2.18) we have

$$\begin{aligned} \Lambda_1 = \int_t^T \frac{d}{dh}\mathbb{E}^x\Big[\alpha \int_t^{t+h} & \Big\{\nabla_\pi b(r)F_1(t,s) + \nabla_\pi \sigma(r)D_r F_1(t,s) \\ &+ \int_{\mathbb{R}_0} \nabla_\pi \gamma(r,z)D_{r,z}F_1(t,s)\nu(dz)\Big\}dr\Big]_{h=0} ds \end{aligned}$$

(3.33)

$$= \int_t^T \mathbb{E}^x\Big[\alpha\Big\{\nabla_\pi b(t) F_1(t,s) + \nabla_\pi \sigma(t) D_t F_1(t,s) + \int_{\mathbb{R}_0} \nabla_\pi \gamma(t,z) D_{t,z} F_1(t,s)\nu(dz)\Big\}\Big] ds,$$

where we have put

$$F_1(t,s) = \frac{\partial \hat{H}_1^0}{\partial x}(s) G(t,s). \tag{3.34}$$

Since $Y(t) = 0$ we see that

$$\Lambda_2 = 0. \tag{3.35}$$

We conclude that

$$\frac{d}{dh} A_1\big|_{h=0} = \Lambda_1$$

(3.36)

$$= \int_t^T \mathbb{E}^x\Big[\alpha\Big\{F_1(t,s)\nabla_\pi b(t) + D_t F_1(t,s)\nabla_\pi \sigma(t) + \int_{\mathbb{R}_0} D_{t,z} F_1(t,s)\nabla_\pi \gamma(t,z)\nu(dz)\Big\}\Big] ds.$$

Moreover, we see directly that

$$\frac{d}{dh} A_2\big|_{h=0} = \mathbb{E}^x\Big[\alpha\Big\{\hat{K}_1(t)\nabla_\pi b(t) + D_t\hat{K}_1(t)\nabla_\pi\sigma(t) + \int_{\mathbb{R}_0}\{D_{t,z}\hat{K}_1(t)\nabla_\pi\gamma(t,z) + \nabla_\pi f_1(t,z)\}\nu(dz)\Big\}\Big].$$

Therefore, differentiating (3.21) with respect to h at $h = 0$ gives the equation

$$\begin{aligned} &\mathbb{E}^x\Big[\alpha\Big\{\Big(\hat{K}_1(t) + \int_t^T F_1(t,s)ds\Big)\nabla_\pi b(t) + D_t\Big(\hat{K}_1(t) + \int_t^T F_1(t,s)ds\Big)\nabla_\pi\sigma(t) \\ &\quad + \int_{\mathbb{R}_0} D_{t,z}\Big(\hat{K}_1(t) + \int_t^T F_1(t,s)ds\Big)\nabla_\pi\gamma(t,z) + \nabla_\pi f_1(t,z)\nu(dz)\Big\}\Big] = 0. \end{aligned} \tag{3.37}$$

We can reformulate this as follows: If we define, as in (3.6),

$$\hat{p}_1(t) = \hat{K}_1(t) + \int_t^T F_1(t,s)ds = \hat{K}_1(t) + \int_t^T \frac{\partial \hat{H}_1^0}{\partial x}(s) G(t,s)ds, \tag{3.38}$$

then (3.36) can be written

$$\begin{aligned} \mathbb{E}^x\Big[\nabla_\pi\Big\{&\int_{\mathbb{R}_0} f_1(t,\hat{X}(t),\pi,\hat{\theta},z)\mu(dz) + \hat{p}_1(t) b(t,\hat{X}(t),\pi_0,\hat{\theta}_0) + D_t\hat{p}_1(t)\sigma(t,\hat{X}(t),\pi_0,\hat{\theta}_0) \\ &+ \int_{\mathbb{R}_0} D_{t,z}\hat{p}_1(t)\gamma(t,\hat{X}(t),\pi,\hat{\theta},z)\nu(dz)\Big\}_{\pi=(\pi_0(t),\pi_1(t,z))}\alpha\Big] = 0. \end{aligned}$$

Since this holds for all bounded $\mathscr{E}_t$-measurable random variable α, we conclude that

$$\mathbb{E}^x\Big[\nabla_\pi \hat{H}_1(t, X^{(\pi,\hat{\theta})}(t),\pi,\hat{\theta})\ |_{\pi=\hat{\pi}(t)}|\ \mathscr{E}_t\Big] = 0.$$

Similarly, we have

$$
\begin{aligned}
0 &= \tfrac{\partial}{\partial \upsilon} J_2(\hat{\pi}, \hat{\theta}+\upsilon\eta)|_{\upsilon=0} \\
&= \mathbb{E}^x\big[\int_0^T \int_{\mathbb{R}_0}\{\tfrac{\partial f_2}{\partial x}(t, X^{(\hat{\pi},\theta)}(t), \hat{\pi}(t,z), \hat{\theta}(t,z), z)D(t) \\
&+\nabla_\theta f_2(t, X^{(\hat{\pi},\theta)}(t), \hat{\pi}(t,z), \theta(t,z), z)|_{\theta=\hat{\theta}}\eta(t)\}\mu(dz)dt + g_2'(\hat{X}(T))D(T)\big]
\end{aligned} \tag{3.39}
$$

where

(3.40)

$$
\begin{aligned}
D(t) &= D^{(\eta)}(t) = \tfrac{d}{d\upsilon} X^{(\hat{\pi},\hat{\theta}+\upsilon\eta)}(t)|_{\upsilon=0} \\
&= \int_0^t \{\tfrac{\partial b}{\partial x}(s, \hat{X}(s), \hat{\pi}_0(s), \hat{\theta}_0(s))D(s) + \nabla_\theta b(s, X^{(\hat{\pi}_0,\theta_0)}(s), \hat{\pi}_0(s), \theta_0(s))|_{\theta=\hat{\theta}}\eta^*(s)\}ds \\
&+ \int_0^t \{\tfrac{\partial \sigma}{\partial x}(s, \hat{X}(s), \hat{\pi}_0(s), \hat{\theta}_0(s))Y(s) + \nabla_\theta \sigma(s, X^{(\hat{\pi},\theta)}(s), \hat{\pi}_0(s), \theta_0(s))|_{\theta=\hat{\theta}}\eta^*(s)\}dB(s) \\
&+ \int_0^t \int_{\mathbb{R}_0}\{\tfrac{\partial \gamma}{\partial x}(s, \hat{X}(s^-), \hat{\pi}(s^-,z), \hat{\theta}(s^-), z)D(s) \\
&+\nabla_\theta \gamma(s, X^{(\hat{\pi},\theta)}(s^-), \hat{\pi}(s^-,z), \theta(s^-,z), z)|_{\theta=\hat{\theta}}\eta^*(s)\}\tilde{N}(ds,dz).
\end{aligned}
$$

Define

$$D(s) = D(t+h)G(t+h,s); \quad s \geq t+h,$$

where $\hat{G}(t,s)$ is defined as in (3.23). By using similar arguments, we get

$$\mathbb{E}^x\Big[\nabla_\theta \hat{H}_2(t, X^{(\hat{\pi},\theta)}(t), \hat{\pi}, \theta)\ |_{\theta=\hat{\theta}(t)}|\ \mathscr{E}_t\Big] = 0.$$

This completes the proof of (i).

(ii) Conversely, suppose that there exists $(\hat{\pi}, \hat{\theta}) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ such that (3.12) and (3.13) hold. Then by reversing the above arguments, we obtain that (3.22)holds for all $\beta_\alpha(s,\omega) = \alpha(\omega)\chi_{(t,t+h]}(s) \in \mathscr{A}_\Pi$, where

$$A_1 = \mathbb{E}^x\Big[\int_t^T \{\hat{K}_1(s)\frac{\partial b}{\partial x}(s) + D_s\hat{K}_1(s)\frac{\partial \sigma}{\partial x}(s) + \int_{\mathbb{R}_0} D_{s,z}\hat{K}_1(s)\frac{\partial \gamma}{\partial x}(s)\nu(dz)\}Y^{(\beta_\alpha)}(s)ds\Big],$$

$$
\begin{aligned}
A_2 &= \mathbb{E}^x\Big[(\int_t^{t+h}\big\{\hat{K}_1(s)\nabla_\pi b(s) + D_s\hat{K}_1(s)\nabla_\pi \sigma(s) + \int_{\mathbb{R}_0} D_{s,z}\hat{K}_1(s)\nabla_\pi \gamma(s,z)\nu(dz) \\
&+ \int_{\mathbb{R}_0} \nabla_\pi f_1(s,z)\mu(dz)\big\}ds)\alpha\Big],
\end{aligned}
$$

for some $t, h \in [0,T]$ with $t+h \leq T$ and some bounded $\mathcal{E}_t$- measurable α. Similarly,

$$A_3 + A_4 = 0 \tag{3.41}$$

for all $\eta_\xi(s,\omega) = \xi(\omega)\chi_{(t,t+h]}(s) \in \mathscr{A}_\Theta$, where

$$A_3 = \mathbb{E}^x\Big[\int_t^T \{\hat{K}_2(s)\frac{\partial b}{\partial x}(s) + D_s\hat{K}_2(s)\frac{\partial \sigma}{\partial x}(s) + \int_{\mathbb{R}_0} D_{s,z}\hat{K}_2(s)\frac{\partial \gamma}{\partial x}(s)\nu(dz)\}Y^{(\eta_\xi)}(s)ds\Big],$$

$$\begin{aligned} A_4 &= \mathbb{E}^x\Big[(\textstyle\int_t^{t+h}\big\{\hat{K}_2(s)\nabla_\theta b(s) + D_s\hat{K}_2(s)\nabla_\theta\sigma(s) + \int_{\mathbb{R}_0} D_{s,z}\hat{K}_2(s)\nabla_\theta\gamma(s,z)\nu(dz) \\ &\quad + \textstyle\int_{\mathbb{R}_0}\nabla_\theta f_2(s,z)\mu(dz)\big\}ds)\alpha\Big],\end{aligned}$$

for some $t,h \in [0,T]$ with $t+h \le T$ and some bounded $\mathcal{E}_t$- measurable ξ. Hence, these equalities hold for all linear combinations of β_α and η_ξ. Since all bounded $\beta \in \mathscr{A}_\Pi$ and $\eta \in \mathscr{A}_\Theta$ can be approximated pointwise boundedly in (t,ω) by such linear combinations, it follows that (3.22) and (3.41) hold for all bounded $(\beta,\eta) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$. Hence, by reversing the remaining part of the proof above, we conclude that

$$\frac{\partial}{\partial y}J_1(\hat{\pi}+y\beta,\hat{\theta}) \mid_{y=0} = 0,$$

$$\frac{\partial}{\partial v}J_2(\hat{\pi},\hat{\theta}+v\eta) \mid_{v=0} = 0 \quad \text{for all } \beta \text{ and } \eta.$$

□

4 Zero-sum games

Suppose that the given performance functional of Player I is the negative of the Player II, i.e.:

(4.1)

$$J_1(u_0,u_1) = \mathbb{E}^x\Big[\int_0^T\int_{\mathbb{R}_0} f(t,X(t),u_0(t),u_1(t,z),z,\omega)\mu(dz)dt + g(X(T),\omega)\Big] = -J_2(u_0,u_1),$$

where $\mathbb{E}^x = \mathbb{E}_P^x$ denotes the expectation with respect to P given that $X(0)=x$. Suppose that the controls $u_0(t)$ and $u_1(t,z)$ have the form as in the (1.5) and (1.6). Let $\mathscr{A}_\Pi$ and $\mathscr{A}_\Theta$ denote the given family of controls $\pi = (\pi_0,\pi_1)$ and $\theta = (\theta_0,\theta_1)$ such that they are contained in the set of $\mathscr{E}_t$-adapted controls, (1.1) has a unique strong solution up to time T and

$$\text{(4.2)}\quad \mathbb{E}^x\Big[\int_0^T\int_{\mathbb{R}_0} |f(t,X(t),\pi_0(t),\pi_1(t,z),\theta_0(t),\theta_1(t,z),z,\omega)|\mu(dz)dt + |g(X(T),\omega)|\Big] < \infty.$$

Then the partial information zero-sum stochastic differential game problem is the following:

PROBLEM 4.1 *Find $\Phi_{\mathscr{E}} \in \mathbb{R}$, $\pi^* \in \mathscr{A}_\Pi$ and $\theta^* \in \mathscr{A}_\Theta$ (if it exists) such that*

$$\text{(4.3)}\quad \Phi_{\mathscr{E}} = \inf_{\theta\in\mathscr{A}_\Theta}\big(\sup_{\pi\in\mathscr{A}_\Pi} J(\pi,\theta)\big) = J(\pi^*,\theta^*) = \sup_{\pi\in\mathscr{A}_\Pi}\big(\inf_{\theta\in\mathscr{A}_\Theta} J(\pi,\theta)\big).$$

Such a control (π^*, θ^*) is called an *optimal control* (if it exists). The intuitive idea is that while Player I controls π, Player II controls θ. The actions of the players are antagonistic, which means that between Player I and II there is a payoff $J(\pi, \theta)$ which is a reward for Player I and a cost for Player II. Note that since we allow b, σ, γ, f and g to be stochastic processes and also because our controls are $\mathscr{E}_t-$adapted, this problem is not of Markovian type and hence cannot be solved by dynamic programming.

THEOREM 4.2 (MAXIMUM PRINCIPLE FOR ZERO-SUM GAMES)

(i) Suppose $(\hat{\pi}, \hat{\theta}) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ *is a directional critical point for* $J(\pi, \theta)$*, in the sense that for all bounded* $\beta \in \mathscr{A}_\Pi$ *and* $\eta \in \mathscr{A}_\Theta$*, there exists* $\delta > 0$ *such that* $\hat{\pi} + y\beta \in \mathscr{A}_\Pi$*,* $\hat{\theta} + \upsilon\eta \in \mathscr{A}_\Theta$ *for all* $y, \upsilon \in (-\delta, \delta)$ *and*

$$c(y, \upsilon) := J(\hat{\pi} + y\beta, \hat{\theta} + \upsilon\eta), \quad y, \upsilon \in (-\delta, \delta)$$

has a critical point at 0*, i.e.,*

$$\frac{\partial c}{\partial y}(0, 0) = \frac{\partial c}{\partial \upsilon}(0, 0) = 0. \tag{4.4}$$

Then

$$\mathbb{E}^x[\nabla_\pi \hat{H}(t, X^{(\pi,\hat{\theta})}(t), \pi, \hat{\theta}, \omega)|\mathscr{E}_t]_{\pi=\hat{\pi}} = 0, \tag{4.5}$$

$$\mathbb{E}^x[\nabla_\theta \hat{H}(t, X^{(\hat{\pi},\theta)}(t), \hat{\pi}, \theta, \omega)|\mathscr{E}_t]_{\theta=\hat{\theta}} = 0 \quad \text{for a.a. } t, \omega, \tag{4.6}$$

where

$$\hat{X}(t) = X^{(\hat{\pi},\hat{\theta})}(t),$$

$$\begin{aligned} \hat{H}(t, \hat{X}(t), \pi, \theta) \;=\; & \int_{\mathbb{R}_0} f(t, \hat{X}(t), \pi, \theta, z)\mu(dz) + \hat{p}(t)b(t, \hat{X}(t), \pi_0, \theta_0) + \hat{q}(t)\sigma(t, \hat{X}(t), \pi_0, \theta_0) \\ & + \int_{\mathbb{R}_0} \hat{r}(t, z)\gamma(t, \hat{X}(t^-), \pi, \theta, z)\nu(dz), \end{aligned} \tag{4.7}$$

with

$$\hat{p}(t) = \hat{K}(t) + \int_t^T \frac{\partial H^0}{\partial x}(s, \hat{X}(s), \hat{\pi}(s), \hat{\theta}(s))\, \hat{G}(t, s)ds, \tag{4.8}$$

$$\hat{K}(t) = K^{(\hat{\pi},\hat{\theta})}(t) = g'(\hat{X}(T)) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial f}{\partial x}(s, \hat{X}(s), \hat{\pi}(s, z), \hat{\theta}(s, z), z)\mu(dz)ds, \tag{4.9}$$

(4.10)

$$\hat{H}^0(s,\hat{X},\hat{\pi},\hat{\theta}) = \hat{K}(s)b(s,\hat{X},\hat{\pi}_0,\hat{\theta}_0)+D_s\hat{K}(s)\sigma(s,\hat{X},\hat{\pi}_0,\hat{\theta}_0)+\int_{\mathbb{R}_0} D_{s,z}\hat{K}(s)\gamma(s,\hat{X},\hat{\pi},\hat{\theta},z)\nu(dz),$$

$$\begin{aligned}\hat{G}(t,s) := \exp\Big(&\int_t^s \Big\{\frac{\partial b}{\partial x}(r,\hat{X}(r),\hat{\pi}_0(r),\hat{\theta}_0(r)) - \frac{1}{2}\Big(\frac{\partial \sigma}{\partial x}\Big)^2(r,\hat{X}(r),\hat{\pi}_0(r),\hat{\theta}_0(r))\Big\}dr\\ &+\int_t^s \frac{\partial \sigma}{\partial x}(r,\hat{X}(r),\hat{\pi}_0(r),\hat{\theta}_0(r))dB(r)\\ &+\int_t^s\int_{\mathbb{R}_0} \ln\Big(1+\frac{\partial \gamma}{\partial x}(r,\hat{X}(r^-),\hat{\pi}(r^-,z),\hat{\theta}(r^-,z),z)\Big)\tilde{N}(dr,dz)\\ &+\int_t^s\int_{\mathbb{R}_0}\Big\{\ln\Big(1+\frac{\partial \gamma}{\partial x}(r,\hat{X}(r),\hat{\pi},\hat{\theta},z)\Big)-\frac{\partial \gamma}{\partial x}(r,\hat{X}(r),\hat{\pi},\hat{\theta},z)\Big\}\nu(dz)dr\Big);\end{aligned}$$

(4.11)

$$\hat{q}(t) := D_t\hat{p}(t),$$

and

$$\hat{r}(t,z) := D_{t,z}\hat{p}(t).$$

(ii) Conversely, suppose that there exists $(\hat{\pi},\hat{\theta}) \in \mathscr{A}_\Pi \times \mathscr{A}_\Theta$ *such that (4.5) and (4.6) hold. Then* $(\hat{\pi},\hat{\theta})$ *satisfies (4.4).*

5 Application: Worst case scenario optimal portfolio under partial information

We illustrate the results in the previous section by looking at an application to robust portfolio choice in finance:

Consider a financial market with the following two investment possibilities:

(i) A *risk free asset*, where the unit price $S_0(t)$ at time t is

$$dS_0(t) = r(t)S_0(t)dt;\;\; S_0(0)=1;\;\; 0 \le t \le T,$$

where $T > 0$ is a given constant.

(ii) A *risky asset*, where the unit price $S_1(t)$ at time t is given by

$$(5.1)\qquad \begin{cases} dS_1(t) = S_1(t^-)[\theta(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0}\gamma_0(t,z)\tilde{N}(dt,dz)],\\ S_1(0) > 0,\end{cases}$$

where $r, \theta, \sigma_0, \gamma_0$ are predictable processes such that

$$\int_0^T \{| \theta(s) | + \sigma_0^2(s) + \int_{\mathbb{R}_0} \gamma_0^2(s,z)\nu(dz)\} ds < \infty \ \ a.s.$$

We assume that θ is adapted to a given subfiltration $\mathscr{E}_t$ and that

$$\gamma_0(t,z,\omega) \geq -1 + \delta \qquad \text{for all } \ t, z, \omega \in [0,T] \times \mathbb{R}_0 \times \Omega$$

for some constant $\delta > 0$.

Let $\pi(t) = \pi(t,\omega)$ be a *portfolio*, representing the amount invested in the risky asset at time t. We require that π be $\mathscr{E}_t$-predictable and self-financing, and hence that the corresponding wealth $X(t) = X^{(\pi,\theta)}(t)$ at time t is given by

$$(5.2) \quad \begin{cases} dX(t) = [X(t) - \pi(t)]r(t)dt + \pi(t)[\theta(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma_0(t,z)\tilde{N}(dt,dz)] \\ X(0) = x > 0. \end{cases}$$

Let us assume that the mean relative growth rate $\theta(t)$ of the risky asset is not known to the trader, but subject to uncertainty. We may regard θ as a *market scenario* or a *stochastic control* of the market, which is playing against the trader. Let $\mathscr{A}_\Pi^\varepsilon$ and $\mathscr{A}_\Theta^\varepsilon$ denote the set of admissible controls π, θ, respectively. *The worst case partial information scenario optimal problem* for the trader is to find $\pi^* \in \mathscr{A}_\Pi^\varepsilon$ and $\theta^* \in \mathscr{A}_\Theta^\varepsilon$ and $\Phi \in \mathbb{R}$ such that

$$\begin{aligned} \Phi &= \inf_{\theta \in \mathscr{A}_\Theta^\varepsilon} (\sup_{\pi \in \mathscr{A}_\Pi^\varepsilon} \mathbb{E}[U(X^{(\pi,\theta)}(T))]) \\ (5.3) \qquad &= \mathbb{E}[U(X^{(\pi^*,\theta^*)(T)})], \end{aligned}$$

where $U : [0,\infty) \to \mathbb{R}$ is a given utility function, assumed to be concave, strictly increasing and $\mathcal{C}^1$ on $(0,\infty)$. We want to study this problem by using Theorem 4.2. In this case we have

$$(5.4) \qquad b(t,x,\pi,\theta) = \pi(\theta - r(t)) + xr(t), \ \ K(t) = U'(X^{(\pi,\theta)}(T))$$

$$\begin{aligned} (5.5)\, H_0(t,x,\pi,\theta) &= U'(X^{(\pi,\theta)}(T))[\pi(\theta - r(t)) + xr(t)] \\ &\quad + D_t(U'(X^{(\pi,\theta)}(T)))\pi\sigma_0(t) + \int_{\mathbb{R}_0} D_{t,z}(U'(X^{(\pi,\theta)}(T)))\pi\gamma(t,z)\nu(dz), \end{aligned}$$

and

$$p(t) = U'(X^{(\pi,\theta)}(T))[1 + \int_t^T r(s)G(t,s)ds],$$

where

$$G(t,s) = \exp(\int_t^s r(v)dv).$$

Hence,

$$\int_t^T r(s)G(t,s)ds = |_t^T \exp(\int_t^s r(v)dv) = \exp(\int_t^T r(v)dv) - 1$$

and

$$p(t) = U'(X^{(\pi,\theta)}(T)) \exp(\int_t^T r(s)ds). \tag{5.6}$$

With this value for $p(t)$ we have

$$\begin{aligned} H(t, X^{(\pi,\theta)}(t), \pi, \theta) &= p(t)[\pi(\theta - r(t)) + r(t)X(t)] \\ &\quad + D_t p(t)\pi\sigma_0(t) + \int_{\mathbb{R}_0} D_{t,z}p(t)\pi\gamma_0(t,z)\nu(dz). \end{aligned} \tag{5.7}$$

Hence equation (4.5) becomes

$$\begin{aligned} &\mathbb{E}[\frac{\partial H}{\partial \pi}(t, \hat{X}(t), \pi, \hat{\theta}) \mid \mathscr{E}_t]_{\pi=\hat{\pi}(t)} \\ &= \mathbb{E}[p(t)(\hat{\theta} - r(t)) + D_t p(t)\sigma_0(t) + \int_{\mathbb{R}_0} D_{t,z}p(t)\gamma_0(t,z)\nu(dz) \mid \mathscr{E}_t] = 0 \end{aligned} \tag{5.8}$$

and equation (4.6) becomes

$$\begin{aligned} &\mathbb{E}[\frac{\partial H}{\partial \theta}(t, \hat{X}(t), \hat{\pi}, \theta) \mid \mathscr{E}_t]_{\theta=\hat{\theta}(t)} \\ &= \mathbb{E}[p(t)\hat{\pi} \mid \mathscr{E}_t] = \mathbb{E}[p(t) \mid \mathscr{E}_t]\hat{\pi}(t) = 0. \end{aligned} \tag{5.9}$$

Since $p(t) > 0$ we conclude that

$$\hat{\pi}(t) = 0. \tag{5.10}$$

This implies that

$$\hat{X}(t) = x \exp(\int_0^t r(s)ds) \tag{5.11}$$

and

$$\hat{p}(t) = U'\left(x \exp(\int_0^T r(s)ds)\right) \exp(\int_t^T r(s)ds); \;\; t \in [0,T]. \tag{5.12}$$

Hence $\hat{p}(t)$ is known, and by (5.8) we get

$$\hat{\theta}(t) = \frac{\mathbb{E}\left[\hat{p}(t)r(t) - D_t\hat{p}(t)\sigma_0(t) - \int_{\mathbb{R}_0} D_{t,z}\hat{p}(t)\gamma_0(t,z)\nu(dz) \mid \mathscr{E}_t\right]}{\mathbb{E}[\hat{p}(t) \mid \mathscr{E}_t]}. \tag{5.13}$$

By Theorem 4.2 we conclude that $(\hat{\pi}, \hat{\theta})$ given by (5.10) and (5.13) is the unique critical point of

$$J(\pi, \theta) = \mathbb{E}[U(X^{(\pi,\theta)}(T))]; \qquad (\pi, \theta) \in (\mathscr{A}_\Pi^\varepsilon, \mathscr{A}_\Theta^\varepsilon).$$

Thus we have proved the following theorem.

THEOREM 5.1 *(Worst case scenario optimal portfolio under partial information)* *Suppose there exists a solution $(\pi^*, \theta^*) \in (\mathscr{A}_\Pi^\varepsilon, \mathscr{A}_\Theta^\varepsilon)$ of the stochastic differential game (5.3). Then*

$$\pi^* = \hat{\pi} = 0 \qquad \text{and} \qquad \theta^* = \hat{\theta} \text{ is given by (5.13).} \tag{5.14}$$

In particular, if $r(s)$ is deterministic, then

$$\pi^* = 0 \qquad \text{and} \qquad \theta^*(t) = r(t). \tag{5.15}$$

REMARK 5.2 *(i) If $r(s)$ is deterministic, then (5.15) states that the worst case scenario is when $\hat{\theta}(t) = r(t)$, for all $t \in [0, T]$, i.e. when the normalized risky asset price*

$$e^{-\int_0^t r(s)ds} S_1(t)$$

is a martingale. In such a situation the trader might as well put all her money in the risk free asset, i.e. choose $\pi(t) = \hat{\pi}(t) = 0$. This trading strategy remain optimal if $r(s)$ is not deterministic, but now the worst case scenario $\hat{\theta}(t)$ is given by the more complicated expression (5.13).

(ii) This is a new approach to, and a partial extension of, Theorem 2.2 in [ØS2] and Theorem 4.1 in the subsequent paper [AØ]. Both of these papers consider the case with deterministic $r(t)$ only. On the other hand, in these papers the scenario is represented by a probability measure and not by the drift.

References

[AØ] T. T. K. AN AND B. ØKSENDAL: *A maximum principle for stochastic differential games with partial information.* J. of Optim. Theory and Appl. 139 (2008), 463–483.

[B] A. BENSOUSSAN: *Stochastic Control of Partially Observable Systems.* Cambridge University Press 1992.

[BDLØP] F. E. Benth, G. Di Nunno, A. Løkka, B. Øksendal, and F. Proske: *Explicit representation of the minimal variance portfolio in markets driven by Lévy processes*. Mathematical Finance 13 (2003), 55–72.

[BØ] F. Baghery and B.Øksendal: *A maximum principle for stochastic control with partial information*. Stoch. Anal. Appl. 25 (2007), 705–717.

[DMØP] G. Di Nunno, T. Meyer-Brandis, B. Øksendal and F. Proske: *Malliavin calculus and anticipative Itô formulae for Lévy processes.* Inf. Dim. Anal. Quant. Probab. 8 (2005), 235-258.

[DØP] G.Di Nunno, B.Øksendal and F. Proske: *Malliavin Calculus for Lévy Processes and Applications to Finance.* Universitext, Springer 2008.

[KO] I. Karatzas and D. Ocone: *A generalized Clark representation formula, with application to optimal portfolios*. Stoch. and Stoch. Reports 34 (1991), 187–220.

[KX] I. Karatzas and X. Xue: *A note on utility maximization under partial observations.* Mathematical Finance 1 (1991), 57–70.

[L] P. Lakner: *Optimal trading strategy for an inverstor: the case of partial information.* Stochastic Processes and Their Applications 76 (1998), 77–97.

[MØZ] T. Meyer-Brandis, B. Øksendal and X.Y. Zhou: *A Malliavin calculus approach to the maximum principle for stochastic control*. Preprint, University of Oslo, 10/2008.

[N] D. Nualart: *Malliavin Calculus and Related Topics.* Springer. Second Edition 2006.

[ØS] B. Øksendal and A. Sulem: *Applied Stochastic Control of Jump Diffusions.* Springer.Second Edition 2007.

[ØS2] B. Øksendal and A. Sulem: *A game theoretic approach to martingale measures in incomplete markets.* Surveys of Applied and Industrial Mathematics 15 (2008), 18–24.

[PQ] H. Pham and M.-C. Quenez: *Optimal portfolio in partially observed stochastic volatility models*. Ann. Appl. Probab. 11 (2001), 210–238.

[YZ] J. Yong and X.Y.Zhou: *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer 1999.