A CONSTRUCTIVE INDUCTIVE PROOF
OF THE HIRSCH CONJECTURE

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ABSTRACT. Warren M. Hirsch posed the conjecture which bears his name in a letter of 1957 to George B. Dantzig. Simply stated in geometric terms, Hirsch proposed that a polytope in dimension \( d \) with \( n \) facets admits a path of at most \((n - d)\) edges connecting any two vertices. Hirsch posed his conjecture in the context of a linear program in \( d \) variables and \( n \) constraints as requiring a maximum of \((n - d)\) pivots — steps of the simplex algorithm — on the shortest path, to achieve an optimum. Over the years the Hirsch conjecture has attracted much research attention, and has been proved in a number of special cases. This article contributes a proof in the general bounded case.

1. INTRODUCTION

Warren M. Hirsch posed the conjecture which bears his name in a letter of 1957 to George B. Dantzig. Simply stated in geometric terms, Hirsch proposed that a polytope in dimension \( d \) with \( n \) facets admits a path of at most \((n - d)\) edges connecting any two vertices. Hirsch posed his conjecture in the context of a linear program in \( d \) variables and \( n \) constraints as requiring a maximum of \((n - d)\) pivots — steps of the simplex algorithm — on the shortest path, to achieve an optimum.

Over the years the Hirsch conjecture has attracted much research attention, and has been proved in a number of special cases. It is known, for example, to be true for polytopes of 2d facets, wherein it is known as the ‘d-step conjecture,’ however to be false for general unbounded polytopes (Klee and Walkup 1967, p. 69, second paragraph). As well it is true for \((0,1)\)-polytopes (Naddef 1989). Hitherto the truth of the conjecture has been unknown for general bounded polytopes. In the literature this concept of the maximum length of a minimal path over all pairs of vertices is frequently named the diameter of a polytope, symbolized \( \Delta_b(n) \), with the roman subscript implying the bounded version.

This article contributes a proof in the general bounded case, and adapts the name Hirsch conjecture (“HC”) to this case, as is now customary. Boundedness is explicitly assumed throughout in the form that any edge of a polytope have two end points. By an argument of (Klee and Walkup 1967, Lemma 2.7, p. 60) it is sufficient to submit a proof for non-singular polytopes — those limited to \((d + 1)\) hyperplanes of dimension \( d \) intersecting at each vertex, the assumption within.

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2. REVIEW OF THE LITERATURE

The literature on the Hirsch conjecture and related matters is extensive. This tract relies not on specific methods of earlier works, except casually in basic geometric concepts, developing instead its methodology ab initio. The history, however, is relevant in context. The corpus of these works cleaves naturally into several categories, though there is some overlap. Here is a list en masse, proceeding chronologically within each category.

- **Hirsch conjecture, specifically:** (Saigal 1969; Lawrence, Jr. 1972; Lawrence, Jr. 1978; Walkup 1978; Naddef 1989; Holt and Klee 1998; Fritzscbe and Holt 1999; Ordóñez 2003; Vershynin 2006; Odaka 2008)
- **Path length on polytopes:** (Klee 1965a; Klee and Walkup 1967; Adler 1974; Adler and Dantzig 1974; Von Hohenbalken 1978; Lindberg and Ölafsson 1984; Kravtsov 1985; Lagarias and Prabhu 1998; Brightwell, Van Den Heuvel, and Stougie 2006)
- **Paths on polytopes in general:** (Gale 1956; Klee 1965b; Klee 1966b; Adler, Dantzig, and Murty 1974)
- **Vertices on polytopes:** (Balinski 1961; Manas and Nedoma 1968; Dyer and Proll 1977)
- **Polytopes in general:** (Grünbaum 1970; Tutte 1971; Philip 1972; Mattheiss 1973; Silverman 1973a; Silverman 1973b; Burdet 1974; Greenberg 1975; Zaslavsky 1975; Klee 1978; Epelman 1979; Grünbaum 2003)
- **Specific problems:** (Klee and Witzgall 1968; Balinski and Russakoff 1972; Bolker 1972; Balinski 1974; Padberg and Rao 1974; Bolker 1976)
- **Linear Programming:** (Klee 1964; Klee 1966a; Klee 1967; Klee and Minty 1972; Klee 1974; Niedringhaus and Steiglitz 1978; Telgen 1978; Kalai 1997; Todd 2002)
- **Probabilistic methods:** (Orden 1980; Kelly and Tolle 1981)

3. PLAN OF THE PAPER

The paper presents its argument for an arbitrary $d$ as an induction on $n$. For each pair $(d, n)$ one assumes a bounded polytope of dimension $d$ having $(n+1)$ facets, then withdraws a selected facet, leaving a residual bounded polytope of dimension $d$ having only $n$ facets, for which the HC is assumed to hold by the inductive hypothesis.

**Remark.** To ensure that the residual polytope be bounded it may be necessary first to perturb some of the facets of the chosen $(d, n+1)$ polytope without altering its topology (meaning to retain a homeomorphic polytopic image) so that some subset of $(d+1)$ facets form a simplex. Then any remaining facet not in the simplex could be removed, retaining boundedness, because all such facets serve only to restrict further the already bounded simplex. Such perturbation, e.g., would be necessary for a hypercube. Throughout it is assumed that a chosen polytope represents its class of topologically equivalent examples, and further, that one can apply perturbations in other contexts as necessary in the circumstances.

A generic construction ensues for the purpose of reestablishing the facet withdrawn, first allowing it to delete (render infeasible) a single vertex of the residual polytope and thereafter to delete *seriatim* additional vertices from the sequence of reconstructed polytopes until the once withdrawn facet returns to its original position. At the initial step of this construction
the proof demonstrates that a path of the residual polytope could have its length increased by one edge. At each subsequent step, however, path length cannot increase, and in fact may decrease by one edge. The HC, therefore, continues to hold for the pair \((d, n + 1)\).

The insight of the construction is that it allows paths to break, and provides a way to repair (reconnect) them without increase in length for all steps following the first. The way is clear, therefore, to position the returning facet anywhere in the residual polytope while remaining faithful to the HC. In fact, almost paths local to the construction break in this manner as \(d\) increases, if one assumes a counting measure. This is the statement of Corollary 4.4.

One can view this progressive construction as a finite-step induction, for the results of each step depend on the previous step. This induction differs not at all in quality from any standard induction proof, but in quantity it does. A necessity is a construction sufficiently generic at each of its steps so that it has universal applicability. The alternative is impossible, to require a specific construction at each step for an arbitrarily large number of steps. The special circumstances of the construction herein allow such universality, and thus enable the proof. The author calls this type of proof a constructive-inductive or ‘CI’ proof.

Two lemmas establish the preliminary results, an ‘initial-step’ lemma and a ‘subsequent-step’ lemma, as the processes suggest. Following is the statement of the Hirsch conjecture with its proof, which simply invokes the lemmas, and tidies a few ends.

## 4. The Lemmas

In the context of this discussion a polytope \(P\) is a bounded convex polyhedron of dimension \(d\) having \(n\) facets of dimension \((d - 1)\). All dimensional components of a polytope are finite in number. Conventionally, \((d - 2)\)-dimensional components are ridges and \((d - 3)\)-dimensional components are peaks. If in the lower dimensions facets, ridges, or peaks coincide with \([3\text{-dimensional}]\) cells, \([2\text{-dimensional}]\) faces, \([1\text{-dimensional}]\) edges, or \([0\text{-dimensional}]\) vertices, then they, respectively, shall be cells, faces, edges, or vertices. A path between two designated vertices is a sequence of connected edges. The number of such edges is the path length.

**Lemma 4.1** (Initial Step). Given a polytope \(P\) of dimension \(d\) having an introduced hyperplane which truncates a single vertex \(v_0\) and no other component in its entirety, any path between two vertices \(q_0\) and \(q_1\) of the residual polytope \(P_0\) having passed through \(v_0\) can be reconnected as a path between those vertices having one additional edge.

**Proof.** (by construction and induction) Assume that \(P\) has \(n\) facets. By the inductive hypothesis \(P\) satisfies the HC. Truncating \(v_0\) removes a simplex \(S_d\) from \(P\), leaving a simplex \(S_{d-1}\) as a new facet of \(P_0\), which now has \((n + 1)\) facets. Any path between \(q_0\) and \(q_1\) having passed through \(v_0\) has its edges formerly incident upon \(v_0\) now terminating in two vertices of \(S_{d-1}\). Inserting the edge connecting those vertices into the path restores it, so augmented. \(\square\)

**Lemma 4.2** (Subsequent Step). Given a polytope \(P\) of dimension \(d\) having an edge \(E\) terminating in vertices \(v_0\) and \(v_1\), let a facet \(f_0\) including \(v_0\) but not \(v_1\) be perturbed so as to truncate \(v_1\), and therefore \(E\) with \(v_0\), but no other edge or vertex of \(P\), to form the new polytope \(P_0\). Then any path between two vertices \(q_0\) and \(q_1\) of \(P\) having had \(E\) as a component either has one lesser edge or can be reconnected so to have the same number of edges in \(P_0\).

**Proof.** (by construction and induction) Assume that \(P\) has \(n\) facets. By the inductive hypothesis \(P\) satisfies the HC. Truncating \(E\) by perturbation of \(f_0\) creates a ridge between the facet \(f_0\) incident to \(v_0\) and the facet \(f_1\).
incident to \(v_1\), each facet not having included \(E\). This ridge is bounded in dimension \(d\) by the \((d-1)\) facets, the \(\{g_i\}\), which defined \(E\), and therefore is a \((d-2)\)-simplex \(S_M\) — the \textit{meta simplex identified with \(E\)}\(^1\). Its facets are simplectic peaks of \(P\). Equivalently, \(S_M\) is bounded in dimension \((d-1)\) by \(\{g_i \cap (f_0 \cap f_1)\}\) \(\equiv \{g_i \cap f_0 \cap f_1\}\).

\textbf{Remark.} See Figures 1, 2, and 3, showing schematic diagrams of the edges before and after truncation of \(E\), in dimensions 3, 4, and 5, respectively. The meta-simplex \(S_M\) is showing in each, and has dimension 1, 2, and 3, respectively. The dimension 3 version, a tetrahedron, is shown in projection.

An alternative description of \(S_M\) devolves from its definition by the \((d-1)\) vertices at pairwise intersections of edges incident to \(v_0\) and \(v_1\), not including \(E\). For the analysis of paths, in particular to determine which paths through \(E\) have their lengths reduced, and which have new edges inserted to replace \(E\), it is desirable to index the cited incident edges in such a way as to identify the pairs.

To this end, number the \(\{g_i\}\) in arbitrary order \(1 \leq i \leq (d-1)\). As \(v_0\) and \(v_1\) are vertices of each \(g_i\), \((d-1)\) edges incident to each of these vertices are contained in \(g_i\), including \(E\). This leaves one edge incident to each of \(v_0\) and \(v_1\) not included in \(g_i\). Label these edges \(e_0^i\) and \(e_1^i\), respectively, and the vertex of \(S_M\) they define, \(w_i\).

Any path between \(q_0\) and \(q_1\) having included \(E\) has also included \(e_0^i\) and \(e_1^j\) for some pair \((i, j)\). If \(i = j\) then the vertex \(w_i\) has replaced \(E\) in the path causing that path length to decrease by 1. If \(i \neq j\), then the path has been severed. Rejoin the path with the edge labeled \(e_{ij}\) of \(S_M\) connecting vertices \(w_i\) and \(w_j\) to leave the path length unchanged. Note that an alternative description of \(e_{ij}\) is the edge determined by the faces including respectively, \(\tilde{e}_0^i\) and \(\tilde{e}_0^j\), and \(\tilde{e}_1^i\) and \(\tilde{e}_1^j\), where the tildes indicate shortened versions of the corresponding edges incident to \(v_0\) and \(v_1\) following the truncation. \(\square\)

The following Corollaries to the Lemma provide specific information about the paths, edges, and vertices.

\textbf{Corollary 4.3.} Following the truncation of \(E\) there are \((d-1)\) distinct direct paths in the sequences \(\{\tilde{e}_0^i, \tilde{e}_1^i\}\), and there are \((d-1)(d-2)\) distinct indirect paths in the sequences \(\{e_0^i, e_{ij}, e_1^j\}\), \(i \neq j\).

\textbf{Corollary 4.4.} Assuming a local counting measure on paths, the fraction of paths broken and repaired without increase in length goes to 1 as \(d\) goes to infinity for given \(n\).

\textbf{Corollary 4.5.} The net increase in the number of edges from truncation is \(d(d-3)/2\), of vertices, \((d-3)\). Their ratio is \(d/2\).  

\footnote{The mnemonic value here is the the meta simplex lies between — and within — each of \(f_0\) and \(f_1\).}
5. THE CONJECTURE

Conjecture 5.1 (Hirsch, 1957). A bounded polytope $P$ in dimension $d$ with $n$ facets admits a path of at most $(n - d)$ edges connecting any two vertices.

Proof. (by construction and induction)

Foundation for the induction on $n$ is that all simplexes $\{S_k\}$ trivially satisfy the Conjecture. Start, therefore, with a bounded polytope $P^*$ of dimension $d$ with $(n+1)$ facets, where $n > d$. Of these facets select $(d+1)$ of them to form a simplex, and withdraw an arbitrary facet $f$ of those which remain, leaving the bounded polytope $P$ with $n$ facets. This polytope satisfies the Conjecture by the inductive hypothesis.

Observe that the vertex set $V$ of $P$ consists of two disjoint sets $\tilde{V}^*$ and $\hat{V}^*$, where $\tilde{V}^* := V^* \setminus V_f$ with the residual $\hat{V}^* := V \setminus \tilde{V}^*$, given $V^*$ as the vertex set of $P^*$ and $V_f$ as the vertex set of $f$.

Position $f$ outside $P$, and move it into $P$, truncating a vertex $v \in \hat{V}^*$. The polytope $P_1$ so formed satisfies the Conjecture by Lemma 4.2.

Next, choose a vertex $v_1 \in \hat{V}^* \setminus v =: \hat{V}^*_1$ connected by an edge to a vertex in $f$, if any. If none exists, then $P_1 = P^*$, which satisfies the Conjecture. Otherwise, move $f$ to truncate $v_1$ and such incident edge to the vertex in $f$, along with that vertex, and no other edge or vertex. This new polytope $P_2$ satisfies the Conjecture by Lemma 4.2.

Continue the induction selecting vertices $\{v_k \mid v_k \in \hat{V}^*_k \setminus v_{k-1} =: \hat{V}^*_k\}$ for truncation, creating a sequence of polytopes $\{P_k\}$, all satisfying the Conjecture, until for some $k$, $P_k = P^*$.

6. CONCLUSION

Long awaited, this proof is ultimately satisfying to the author. Equally so, however, is the implementation of the constructive-inductive proof, which he believes has applicability in other contexts.
Figure 1: Edge truncation, dimension 3
Figure 2: Edge truncation, dimension 4
Figure 3: Edge truncation, dimension 5
(edge labels $e_{14}$, $e_{24}$, $e_{34}$ omitted)
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