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On the div-curl lemma in a Galerkin setting

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Abstract

Given a sequence of Galerkin spaces X_h of curl conforming vector fields, we state necessary and sufficient conditions under which it is true that the scalar product $u_h \cdot u'_h$ of two sequences of vector fields $u_h, u'_h \in X_h$ converging weakly in L^2, converges in the sense of distributions to the right limit, whenever u_h is discrete divergence free and $\operatorname{curl} u_h$ is precompact in H^{-1}. The conditions on X_h are related to super-approximation and discrete compactness results for mixed finite elements, and are satisfied for Nédélec's edge elements. We also provide examples of sequences of discrete divergence free edge element vector fields converging weakly to 0 in L^2 but whose divergence is not precompact in H^{-1}_{loc}.

1 Introduction

The div-curl lemma of Murat [4] and Tartar [6] comes in many variants. For instance it can be formulated for scalar products of differential forms and, more generally still, for a quadratic form applied to vector-valued functions. We shall be content with the following version, which captures much of its essence.

We say that a set U is precompact in a topological space X, and write $U \Subset X$, if it is included in X and its closure in X is compact. We say that a sequence is precompact in a topological space X if its set of elements is precompact in X. For an open subset S of Euclidean space we denote by $\mathcal{D}(S)$ the set of smooth functions whose support is precompact in S.

In the following, let S be the interior of a bounded convex polyhedron in the Euclidean space $\mathbb{R}^3$. All integrals will be on S equipped with Lebesgue measure. Sequences will be indexed by a countable set of positive reals accumulating only at 0 and the index variable denoted h.

Lemma 1.1 (div-curl). *Suppose (u_h) and (u'_h) are sequences of $\mathrm{L}^2(S)$ vector fields such that:*

- *(u_h) converges weakly in $\mathrm{L}^2(S)$ to some $u \in \mathrm{L}^2(S)$ and $(\operatorname{div} u_h)$ is precompact in $\mathrm{H}^{-1}(S)$,*
- *(u'_h) converges weakly in $\mathrm{L}^2(S)$ to some $u' \in \mathrm{L}^2(S)$ and $(\operatorname{curl} u_h)$ is precompact in $\mathrm{H}^{-1}(S)$.*

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Then for any $\phi \in \mathcal{D}(S)$ we have:

$$\lim_{h\to 0} \int (u_h \cdot u'_h)\phi = \int (u \cdot u')\phi. \tag{1}$$

When (1) holds for all $\phi \in \mathcal{D}(S)$ we shall say simply that $(u_h \cdot u'_h)$ converges vaguely to $u \cdot u'$. This is also often referred to as (weak-star) convergence in the sense of distributions. In this lemma, precompactness (resp. weak convergence) in a Sobolev space X can be replaced by precompactness (resp. weak convergence) in X_{loc}.

This result is optimal in many senses. For instance we have the following reciprocal which sheds light on the results we shall prove. Suppose (u_h) converges weakly to u in $\mathrm{L}^2(S)$. Suppose furthermore that for any (u'_h) which converges weakly in $\mathrm{L}^2(S)$ to say u' and has the property that $(\mathrm{curl}\, u'_h)$ is precompact in $\mathrm{H}^{-1}(S)$, we have that $(u_h \cdot u'_h)$ converges vaguely to $u \cdot u'$. Then $(\mathrm{div}\, u_h)$ is precompact in $\mathrm{H}^{-1}_{loc}(S)$ (use $u'_h = \mathrm{grad}\, p_h$ for (p_h) weakly converging in $\mathrm{H}^1(S)$ to obtain this).

We are interested in extending the above results to a Galerkin setting. Suppose we have a family (X_h) of closed subspace of the space of vector fields:

$$\mathrm{H}_{\mathrm{curl}}(S) = \{u \in \mathrm{L}^2(S) \ : \ \mathrm{curl}\, u \in \mathrm{L}^2(S)\}, \tag{2}$$

equipped with the graph norm. Define closed subspaces Y_h and $\tilde{Y}_h$ of $\mathrm{H}^1(S)$ by:

$$Y_h = \{p \in \mathrm{H}^1(S) \ : \ \mathrm{grad}\, p \in X_h\}, \tag{3}$$

$$\tilde{Y}_h = Y_h \cap \mathrm{H}^1_0(S). \tag{4}$$

In all of the following we will assume that:

$$\forall u \in \mathrm{L}^2(S) \ \lim_{h\to 0} \inf_{u_h \in X_h} \|u - u_h\|_{\mathrm{L}^2} = 0, \tag{5}$$

$$\forall p \in \mathrm{H}^1(S) \ \lim_{h\to 0} \inf_{p_h \in Y_h} \|p - p_h\|_{\mathrm{H}^1} = 0, \tag{6}$$

$$\forall p \in \mathrm{H}^1_0(S) \ \lim_{h\to 0} \inf_{p_h \in \tilde{Y}_h} \|p - p_h\|_{\mathrm{H}^1} = 0. \tag{7}$$

We shall say that a sequence of elements $u_h \in X_h$ is discrete divergence free when for all h:

$$\forall p_h \in \tilde{Y}_h \ : \ \int u_h \cdot \mathrm{grad}\, p_h = 0. \tag{8}$$

Discrete divergence free vector fields need not be truly divergence free, but any weak $\mathrm{L}^2(S)$ limit of such vector fields must be, due to (7).

We would like to obtain necessary and sufficient conditions on (X_h) for the following to hold:

–*Galerkin div-curl lemma* $(\star)$: For any discrete divergence free sequence of elements $u_h \in X_h$ converging weakly in $\mathrm{L}^2(S)$ to u and any sequence of elements $u'_h \in X_h$ converging weakly in $\mathrm{L}^2(S)$ to u' and having precompact curl in $\mathrm{H}^{-1}(S)$, $(u_h \cdot u'_h)$ converges vaguely to $u \cdot u'$.

In a previous paper [1] we obtained sufficient conditions expressed as follows.

Define W_h and V_h by:

$$W_h = \{u_h \in X_h \ : \ \operatorname{curl} u_h = 0\} = \{\operatorname{grad} p_h \ : \ p_h \in Y_h\}, \tag{9}$$

$$V_h = \{u_h \in X_h \ : \ \forall w_h \in W_h \ \int u_h \cdot w_h = 0\}. \tag{10}$$

–*Super-approximation* (SA) is the following property. For all $\phi \in \mathcal{D}(S)$ we have:

$$\lim_{h \to 0} \sup_{p_h \in Y_h} \inf_{\tilde{p}_h \in \tilde{Y}_h} \|p_h - \phi \tilde{p}_h\|_{\mathrm{H}^1(S)} / \|p_h\|_{\mathrm{H}^1(S)} = 0. \tag{11}$$

–*Uniform norm equivalence* (UNE) is the following property. There is $C > 0$ such that:

$$\forall h \ \forall v_h \in V_h \ \|v_h\|_{\mathrm{L}^2(S)} \leq C \| \operatorname{curl} v_h\|_{\mathrm{H}^{-1}(S)}. \tag{12}$$

In [1] we proved that these two conditions are sufficient and that they are satisfied when X_h are Nédélec's edge element spaces [5] of given polynomial degree, attached to quasi-uniform meshes $\mathcal{T}_h$ of mesh-width h. (SA) is appears in super convergence results [7]. We also related (UNE) to discrete compactness in the sense of Kikuchi [3]. Moreover we asked the question, if there are discrete divergence free sequences $u_h \in X_h$ converging weakly in $\mathrm{L}^2(S)$, whose divergence is not precompact in $\mathrm{H}^{-1}(S)$, but were not able to answer it.

In this paper we introduce weaker versions of (SA) and (UNE):

–*Weak super-approximation* (WSA) is the following property. For all $\phi \in \mathcal{D}(S)$ we have:

$$\lim_{h \to 0} \sup_{u_h \in X_h} \sup_{p_h \in Y_h} \inf_{\tilde{p}_h \in \tilde{Y}_h} | \int u_h \cdot \operatorname{grad}(\phi p_h - \tilde{p}_h) \, | / (\|p_h\|_{\mathrm{H}^1(S)} \|u_h\|_{\mathrm{L}^2(S)}) = 0. \tag{13}$$

–*Local uniform norm estimate* (LUNE) is the following property. For any nonempty open subset S' of S which is precompact in S, there is $C > 0$ such that:

$$\forall h \ \forall v_h \in V_h \ \|u_h\|_{\mathrm{L}^2(S')} \leq C \| \operatorname{curl} u_h\|_{\mathrm{H}^{-1}(S)}. \tag{14}$$

We shall prove that (WSA) and (LUNE) together are necessary and sufficient for ($\star$) to hold. Moreover for the case when (X_h) are Nédélec's edge elements, we shall provide a big supply of sequences of vector fields $u_h \in X_h$ which are discrete divergence free and converge weakly to 0 in $\mathrm{L}^2(S)$ but nevertheless have the property that for some precompact subsets S' of S we have:

$$\liminf_{h \to 0} \| \operatorname{div} u_h\|_{\mathrm{H}^{-1}(S')} > 0, \tag{15}$$

which essentially says that $(\operatorname{div} u_h)$ is *not* precompact in $\mathrm{H}^{-1}(S')$.

By this non-compactness result, the Galerkin div-curl lemma cannot be immediately deduced from the continuous one. In view of the above mentioned reciprocal to the continuous div-curl lemma it is therefore crucial that we have the condition $u'_h \in X_h$ in the Galerkin version. On the other hand the continuous div-curl Lemma 1.1 is a special case of the Galerkin one, corresponding simply to the choice $X_h = \mathrm{H}_{\mathrm{curl}}(S)$ for all h. For this choice, (SA) is trivial and (UNE) is quite standard (see Proposition 2.1 in [1] and the appended remark).

2 Necessary and sufficient conditions

We introduce some notations. Define spaces V and W of vector fields by:

$$W = \{u \in \mathrm{L}^2(S) \ : \ \operatorname{curl} u = 0\} = \{\operatorname{grad} p \ : \ p \in \mathrm{H}^1(S)\}, \tag{16}$$

$$V = \{u \in \mathrm{L}^2(S) \ : \ \forall w \in W \ \int u \cdot w = 0\}. \tag{17}$$

Recall that there is $C > 0$ such that:

$$\forall u \in V \ \|u\|_{\mathrm{L}^2(S)} \leq C \|\operatorname{curl} u\|_{\mathrm{H}^{-1}(S)}. \tag{18}$$

This is a reformulation of the fact that $\operatorname{curl} : \mathrm{L}^2(S) \to \mathrm{H}^{-1}(S)$ has closed range. Let P_V be the L^2 orthogonal projection onto V. Its kernel is W, so it preserves the curl.

We shall first prove that (WSA) and (LUNE) are sufficient for $(\star)$ to hold.

Proposition 2.1. *Suppose (WSA) holds. Suppose $u_h \in X_h$ converges weakly in $\mathrm{L}^2(S)$ to u and is discrete divergence free. Suppose $p_h \in Y_h$ converges weakly to p in $\mathrm{H}^1(S)$. Then $(u_h \cdot \operatorname{grad} p_h)$ converges vaguely to $u \cdot \operatorname{grad} p$.*

Proof. Pick $\phi \in \mathcal{D}(S)$ and write:

$$\int (u_h \cdot \operatorname{grad} p_h)\phi = \int u_h \cdot \operatorname{grad}(\phi p_h) - \int (u_h \cdot \operatorname{grad} \phi) p_h, \tag{19}$$

$$= \int u_h \cdot \operatorname{grad}(\phi p_h - \tilde{p}_h) - \int (u_h \cdot \operatorname{grad} \phi) p_h. \tag{20}$$

The first term can be made to tend to 0. By Rellich compactness $\mathrm{H}^1(S) \to \mathrm{L}^2(S)$ and the divergence freeness of u, the second converges to:

$$- \int (u \cdot \operatorname{grad} \phi) p = \int (u \cdot \operatorname{grad} p) \phi. \tag{21}$$

This completes the proof. □

Proposition 2.2. *Suppose (LUNE) holds. Suppose $u_h \in X_h$ converges weakly in $\mathrm{L}^2(S)$ to u, that:*

$$\forall h \ \forall p_h \in Y_h \int u_h \cdot \operatorname{grad} p_h = 0, \tag{22}$$

and that $(\operatorname{curl} u_h)$ is precompact in $\mathrm{H}^{-1}(S)$. Then (u_h) converges to u in $\mathrm{L}^2_{loc}(S)$.

Proof. Remark that u must be an element of V by (6). Moreover since $(\operatorname{curl} u_h)$ is precompact in $\mathrm{H}^{-1}(S)$ and converges weakly in this space, the convergence is actually strong. Therefore $(P_V u_h)$ converges to u in $\mathrm{L}^2(S)$.

Let P_h be the L^2 orthogonal projection onto X_h and remark that $(P_h u)$ converges to u in $\mathrm{L}^2(S)$ by (5). Moreover $P_h u \in V_h$. Let S' be a non-empty, precompact subset of S. By (LUNE) we have:

$$\|u_h - P_h u\|_{\mathrm{L}^2(S')} \leq C \|\operatorname{curl} u_h - \operatorname{curl} P_h u\|_{\mathrm{H}^{-1}(S)}, \tag{23}$$

$$\leq C \|\operatorname{curl} P_V u_h - \operatorname{curl} P_h u\|_{\mathrm{H}^{-1}(S)}, \tag{24}$$

$$\leq C \|P_V u_h - P_h u\|_{\mathrm{L}^2(S)} \to 0. \tag{25}$$

Hence:

$$\lim_{h\to 0} \|u_h - u\|_{\mathrm{L}^2(S')} = 0. \tag{26}$$

This ends the proof. □

Theorem 2.3. *If (WSA) and (LUNE) hold, then ($\star$) holds.*

Proof. In the proof of Theorem 4.2 in [1], the above Propositions 2.1 and 2.2 are adequate substitutes for Propositions 3.4 and 3.2 of that paper. □

Now we shall prove that (WSA) and (LUNE) are necessary for ($\star$) to hold.

Proposition 2.4. *If ($\star$) holds then (WSA) holds.*

Proof. Let P_h denote the projection in $\mathrm{H}^1(S)$ onto $\tilde{Y}_h$, determined by the scalar product:

$$(p, p') \mapsto \int \operatorname{grad} p \cdot \operatorname{grad} p'. \tag{27}$$

If (WSA) does not hold we have a $\phi \in \mathcal{D}(S)$, an index set G and subsequences $u_h \in X_h$, $p_h \in Y_h$ indexed by G, such that for all h in G:

$$\|u_h\|_{\mathrm{L}^2(S)} \leq 1, \tag{28}$$

$$\|p_h\|_{\mathrm{H}^1(S)} \leq 1, \tag{29}$$

$$\int u_h \cdot \operatorname{grad}(\phi p_h - P_h(\phi p_h)) \geq 1/C, \tag{30}$$

for some $C > 0$. We may suppose in addition that (u_h) is discrete divergence free and, extracting subsequences, that (u_h) converges weakly in $\mathrm{L}^2(S)$ to u, and that (p_h) converges weakly in $\mathrm{H}^1(S)$ to p. For the indices h not in G we let u_h be the $\mathrm{L}^2(S)$ projection on X_h of u (it is discrete divergence free), and p_h be the best $\mathrm{H}^1(S)$ approximation in Y_h of p. We still have weakly convergent sequences. Moreover:

$$\int (u_h \cdot \operatorname{grad} p_h)\phi = \int u_h \cdot \operatorname{grad}(\phi p_h - P_h(\phi p_h)) - \int (u_h \cdot \operatorname{grad} \phi) p_h. \tag{31}$$

As in the proof of Proposition 2.1 the last term converges to:

$$\int (u \cdot \operatorname{grad} p)\phi. \tag{32}$$

Thus by (30) we have a counterexample to ($\star$). □

Proposition 2.5. *If ($\star$) holds then (LUNE) holds.*

Proof. Suppose (LUNE) does not hold. We get a subsequence $(u_h)_{h\in G}$ such that for some (non-negative) $\phi \in \mathcal{D}(S)$ we have:

$$\forall h \ \forall p_h \in Y_h \ \int u_h \cdot \operatorname{grad} p_h = 0, \tag{33}$$

$$\int |u_h|^2 \phi = 1, \tag{34}$$

$$\lim_{h\to 0} \|\operatorname{curl} u_h\|_{\mathrm{H}^{-1}(S)} = 0. \tag{35}$$

We have for any vector field $v \in \mathrm{H}^1_0(S)$ and any scalar field $p \in \mathrm{H}^1(S)$:

$$\int u_h \cdot (\operatorname{curl} v + \operatorname{grad} p) \to 0. \tag{36}$$

From results in [2] it follows that (u_h) converges weakly in $\mathrm{L}^2(S)$ to 0. For indices h not in G define $u_h = 0$. If $(\star)$ were to hold we would have:

$$\lim_{h\to 0} \int |u_h|^2 \phi = 0, \tag{37}$$

which contradicts (34). □

3 A non-compactness result

For the case of Nédélec's edge elements we shall construct sequences $u_h \in X_h$ which are discrete divergence free and converge weakly in $\mathrm{L}^2(S)$ to 0, but whose divergence is not precompact in $\mathrm{H}^{-1}(S')$ for some precompact subsets S' of S. We shall work in dimension 2 rather than 3 since this eases the exposition yet captures the essence of the problem. Once this case is at hand, extension to any higher dimension is easy. We shall use for X_h the lowest order tensor product edge elements on the unit square S, equipped with the uniform Cartesian mesh of width h. Thus our examples hold in the nicest possible setting.

First let $N > 1$ be an integer and subdivide the sides of the unit square in N intervals of equal length. The unit square is equipped with the corresponding Cartesian grid. There are $2(N-1)N$ interior edges and $(N+1)^2$ vertices. An edge element function on this grid is truly divergence free iff its degrees of freedom are constant on each horizontal line and each vertical line. Thus the space of truly divergence free edge element vector fields on this grid has dimension $2(N+1)$. It includes the constant vector fields. Denote by Z_N the space of edge element vector fields whose degrees of freedom on the boundary of the square are 0, which are L^2-orthogonal to the gradients of continuous piecewise bilinear functions (possibly non-zero at the boundary vertices), and to the subspace of truly divergence free fields. We have:

$$\dim Z_N \geq 2(N-1)N - (N+1)^2 - 2(N+1) = (N-3)^2 - 12. \tag{38}$$

For $N \geq 7$ the space is non-zero. Pick an element z of Z_7 with L^2 norm 1. We put:

$$\|\operatorname{div} z\|_{\mathrm{H}^{-1}(S)} = \delta > 0. \tag{39}$$

For any domain S' we consider that $\mathrm{H}^1_0(S')$ is equipped with the norm:

$$p \mapsto (\int |\operatorname{grad} p|^2)^{1/2}, \tag{40}$$

and that $\mathrm{H}^{-1}(S')$ is equipped with the dual norm. These norms behave better under scaling than the standard $\mathrm{H}^1(S')$ norm. We shall use that for a vector field u, $\|u\|_{\mathrm{L}^2}$ and $\|\operatorname{div} u\|_{\mathrm{H}^{-1}}$ scale in the same way, under maps $x \mapsto nx$, for $n \neq 0$.

Now let $h = 1/(7n)$ for integer $n > 0$ be the mesh width. Consider the unit square to be filled with n^2 macro squares consisting of 7×7 micro squares of side width h, in the obvious way. Let M_n be the set of macro squares, for a given n. Remark that for any set $\mathcal{S}$ of macro squares in M_n we have an embedding of norm 1:

$$\bigoplus_{s\in\mathcal{S}} \mathrm{H}^1_0(s) \to \mathrm{H}^1_0(\cup\mathcal{S}), \tag{41}$$

which gives an embedding of norm 1:

$$\mathrm{H}^{-1}(\cup\mathcal{S}) \to \bigoplus_{s\in\mathcal{S}} \mathrm{H}^{-1}(s). \tag{42}$$

Here we use the standard Hilbertian direct sum.

For each macro square $s \in M_n$, let z_s be the transported version of z to s which has $\mathrm{L}^2(S)$ norm 1 (the scaled pull-back, by a map of the form $x \mapsto nx+a$). For coefficients $\alpha = (\alpha_s)_{s\in M_n}$ put:

$$u_\alpha = \sum_{s\in M_n} \alpha_s z_s. \tag{43}$$

Then u_α is a discrete divergence free element of X_h. It is L^2 orthogonal to all the vector fields that are constant in each $s \in M_n$ and:

$$\|u_\alpha\|^2_{\mathrm{L}^2(S)} = \sum_{s\in M_n} |\alpha_s|^2. \tag{44}$$

Moreover if S' is an open subset of S:

$$\|\operatorname{div} u_\alpha\|^2_{\mathrm{H}^{-1}(S')} \geq \sum_{s\in M_n\ :\ s\subset S'} |\alpha_s|^2 \|\operatorname{div} z_s\|^2_{\mathrm{H}^{-1}(s)} = \sum_{s\in M_n\ :\ s\subset S'} |\alpha_s|^2\delta^2. \tag{45}$$

Thus any sequence of coefficients $\alpha^n = (\alpha^n_s)_{s\in M_n}$, indexed by $n > 0$, for which:

$$\sum_{s\in M_n} |\alpha^n_s|^2 \quad \text{is bounded}, \tag{46}$$

gives rise to a discrete divergence free sequence of fields $u_h \in X_h$ converging weakly in $\mathrm{L}^2(S)$ to 0, and special choices of α^n will guarantee that $\operatorname{div} u_h|_{S'}$ does not converge to 0 in the $\mathrm{H}^{-1}(S')$ norm. Since $(\operatorname{div} u_h)$ converges weakly to 0 in $\mathrm{H}^{-1}(S)$, this rules out precompactness of $(\operatorname{div} u_h)$ in $\mathrm{H}^{-1}_{loc}(S)$.

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