

Uniqueness of Decompositions of Skorohod-Semimartingales

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Abstract

In this paper we introduce Skorohod-semimartingales as an expanded concept of classical semimartingales in the setting of Lévy processes. We show under mild conditions that Skorohod-semimartingales similarly to semimartingales admit a unique decomposition.

Key words: Skorohod-semimartingale, white noise, Malliavin calculus.

1 Introduction

Let $X_t = X_t(\omega)$; $t \in [0, T]$, $\omega \in \Omega$ be a stochastic process of the form

$$X_t = \zeta + \int_0^t \alpha(s) ds + \int_0^t \beta(s) \delta B_s + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z) \tilde{N}(dz, \delta s), \quad (1.1)$$

where ζ is a random variable, α is an integrable measurable process, $\beta(s)$ and $\gamma(s, z)$ are measurable processes such that $\beta\chi_{[0,t]}(\cdot)$ and $\gamma\chi_{[0,t]}(\cdot)$ are Skorohod integrable with respect to B_s and $\tilde{N}(dz, ds)$ respectively, and the stochastic integrals are interpreted as Skorohod integrals. Here $B_s = B_s(\omega)$ and $\tilde{N}(dz, ds) = \tilde{N}(dz, ds, \omega)$ is a Brownian motion and an independent Poisson random measure, respectively. Such processes are called *Skorohod-semimartingales*. The purpose of this paper is to prove that the decomposition (1.1) is *unique*, in the sense that if $X_t = 0$ for all $t \in [0, T]$ then

$$\zeta = \alpha(\cdot) = \beta(\cdot) = \gamma(\cdot, \cdot) = 0$$

(see Theorem 3.5).

This is an extension of a result by Nualart and Pardoux [NP], who proved the uniqueness of such a decomposition in the Brownian case (i.e., $\tilde{N} = 0$) and with additional assumption on β .

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We obtain Theorem 3.5 as a special case of a more general decomposition uniqueness theorem for an extended class of Skorohod integral processes with values in the space of generalized random variables \mathcal{G}^* . See Theorem 3.3. Our proof uses white noise theory of Lévy processes. In Section 2 we give a brief review of this theory and in Section 3 we prove our main theorem.

Our decomposition uniqueness is motivated by applications in anticipative stochastic control theory, including insider trading in finance. See [DØPP]

2 A Concise Review of Malliavin Calculus and White Noise Analysis

This Section provides the mathematical framework of our paper which will be used in Section 3. Here we want to briefly recall some basic facts from both Malliavin calculus and white noise theory. See [N], [M] and [DØP] for more information on Malliavin calculus. As for white noise theory we refer the reader to [DØP1], [HKPS], [HØUZ], [K], [LP], [O] and [ØP].

In the sequel denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space on \mathbb{R} and by $\mathcal{S}'(\mathbb{R})$ its topological dual. Then in virtue of the celebrated Bochner-Minlos theorem there exists a unique probability measure μ on the Borel sets of the conuclear space $\mathcal{S}'(\mathbb{R})$ (i.e. $\mathcal{B}(\mathcal{S}'(\mathbb{R}))$) such that

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle \omega, \phi \rangle} \mu(d\omega) = e^{-\frac{1}{2}\|\phi\|_{L^2(\mathbb{R})}^2} \quad (2.1)$$

holds for all $\phi \in \mathcal{S}(\mathbb{R})$, where $\langle \omega, \phi \rangle$ is the action of $\omega \in \mathcal{S}'(\mathbb{R})$ on $\phi \in \mathcal{S}(\mathbb{R})$. The measure μ is called the *Gaussian white noise measure* and the triple

$$(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mu) \quad (2.2)$$

is referred to as (Gaussian) *white noise probability space*.

Consider the Doleans-Dade exponential

$$\tilde{e}(\phi, \omega) = e^{\langle \omega, \phi \rangle - \frac{1}{2}\|\phi\|_{L^2(\mathbb{R})}^2}, \quad (2.3)$$

which is holomorphic in ϕ around zero. Hence there exist *generalized Hermite polynomials* $H_n(\omega) \in ((\mathcal{S}(\mathbb{R}))^{\hat{\otimes} n})'$ (i.e. dual of n -th completed symmetric tensor product of $\mathcal{S}(\mathbb{R})$) such that

$$\tilde{e}(\phi, \omega) = \sum_{n \geq 0} \frac{1}{n!} \langle H_n(\omega), \phi^{\otimes n} \rangle \quad (2.4)$$

for all ϕ in a neighborhood of zero in $\mathcal{S}(\mathbb{R})$. One verifies that the orthogonality relation

$$\int_{\mathcal{S}'(\mathbb{R})} \langle H_n(\omega), \phi^{(n)} \rangle \langle H_m(\omega), \psi^{(m)} \rangle \mu(d\omega) = \begin{cases} n! (\phi^{(n)}, \psi^{(n)})_{L^2(\mathbb{R}^n)}, & m = n \\ 0 & m \neq n \end{cases} \quad (2.5)$$

is fulfilled for all $\phi^{(n)} \in (\mathcal{S}(\mathbb{R}))^{\widehat{\otimes} n}$, $\psi^{(m)} \in (\mathcal{S}(\mathbb{R}))^{\widehat{\otimes} m}$. From this relation we obtain that the mappings $(\phi^{(n)} \mapsto \langle H_n(\omega), \phi^{(n)} \rangle)$ from $(\mathcal{S}(\mathbb{R}))^{\widehat{\otimes} n}$ to $L^2(\mu)$ have unique continuous extensions

$$I_n : \widehat{L}^2(\mathbb{R}^n) \longrightarrow L^2(\mu),$$

where $\widehat{L}^2(\mathbb{R}^n)$ is the space of square integrable symmetric functions. It turns out that $L^2(\mu)$ admits the orthogonal decomposition

$$L^2(\mu) = \sum_{n \geq 0} \oplus I_n(\widehat{L}^2(\mathbb{R}^n)). \quad (2.6)$$

Note that that $I_n(\phi^{(n)})$ can be considered an n -fold iterated Itô integral $\phi^{(n)} \in \widehat{L}^2(\mathbb{R}^n)$ with respect to a Brownian motion B_t on our white noise probability space. In particular

$$I_1(\varphi\chi_{[0,T]}) = \langle H_1(\omega), \varphi\chi_{[0,T]} \rangle = \int_0^T \varphi(t) dB_t, \varphi \in L^2(\mathbb{R}). \quad (2.7)$$

Let $F \in L^2(\mu)$. It follows from (2.6) that

$$F = \sum_{n \geq 0} \langle H_n(\cdot), \phi^{(n)} \rangle \quad (2.8)$$

for unique $\phi^{(n)} \in \widehat{L}^2(\mathbb{R}^n)$. Further require that

$$\sum_{n \geq 1} nn! \left\| \phi^{(n)} \right\|_{\widehat{L}^2(\mathbb{R}^n)}^2 < \infty. \quad (2.9)$$

Then the *Malliavin derivative* D_t of F in the direction B_t is defined by

$$D_t F = \sum_{n \geq 1} n \langle H_{n-1}(\cdot), \phi^{(n)}(\cdot, t) \rangle.$$

Denote by $\mathbb{D}_{1,2}$ the stochastic Sobolev space which consists of all $F \in L^2(\mu)$ such that (2.9) is satisfied. The Malliavin derivative D is a linear operator from $\mathbb{D}_{1,2}$ to $L^2(\lambda \times \mu)$ (λ Lebesgue measure). The adjoint operator δ of D as a mapping from $Dom(\delta) \subset L^2(\lambda \times \mu)$ to $L^2(\mu)$ is called *Skorohod integral*. The Skorohod integral can be regarded as a generalization of the Itô integral and one also uses the notation

$$\delta(u\chi_{[0,T]}) = \int_0^T u(t) \delta B_t \quad (2.10)$$

for Skorohod integrable (not necessarily adapted) processes $u \in L^2(\lambda \times \mu)$ (i.e. $u \in Dom(\delta)$).

In view of Section 3 we give the construction of the dual pair of spaces $((\mathcal{S}), (\mathcal{S})^*)$, which was first introduced by Hida [H] in white noise analysis: Consider the self-adjoint operator

$$A = 1 + t^2 - \frac{d^2}{dt^2}$$

on $\mathcal{S}(\mathbb{R}) \subset L^2(\mu)$. Then the *Hida test function space* (\mathcal{S}) is the space of all square integrable functionals f with chaos expansion

$$f = \sum_{n \geq 0} \langle H_n(\cdot), \phi^{(n)} \rangle$$

such that

$$\|f\|_{0,p}^2 := \sum_{n \geq 0} n! \left\| (A^{\otimes n})^p \phi^{(n)} \right\|_{L^2(\mathbb{R}^n)}^2 < \infty \quad (2.11)$$

for all $p \geq 0$. We mention that (\mathcal{S}) is a nuclear Fréchet algebra, that is a countably Hilbertian nuclear space w.r.t. the seminorms $\|\cdot\|_{0,p}, p \geq 0$ and an algebra w.r.t. ordinary multiplication of functions. The topological dual (\mathcal{S})^{*} of (\mathcal{S}) is the *Hida distribution space*.

Another useful dual pairing which was studied in [PT] is $(\mathcal{G}, \mathcal{G}^*)$. Denote by N the Ornstein-Uhlenbeck operator (or number operator). The space of smooth random variables \mathcal{G} is the space of all square integrable functionals f such that

$$\|f\|_q^2 := \|e^{qN} f\|_{L^2(\mu)}^2 < \infty \quad (2.12)$$

for all $q \geq 0$. The dual of \mathcal{G} denoted by \mathcal{G}^* is called space of generalized random variables.

We have the following interrelations of the above spaces in the sense of inclusions:

$$(\mathcal{S}) \hookrightarrow \mathcal{G} \hookrightarrow \mathbb{D}_{1,2} \hookrightarrow L^2(\mu) \hookrightarrow \mathcal{G}^* \hookrightarrow (\mathcal{S})^*. \quad (2.13)$$

In what follows we define the *white noise differential operator*

$$\partial_t = D_t|_{(\mathcal{S})} \quad (2.14)$$

as the restriction of the Malliavin derivative to the Hida test function space. It can be shown that ∂_t maps (\mathcal{S}) into itself, continuously. We denote by $\partial_t^* : (\mathcal{S})^* \rightarrow (\mathcal{S})^*$ the adjoint operator of ∂_t . We mention the following crucial link between ∂_t^* and δ :

$$\int_0^T u(t) \delta B_t = \int_0^T \partial_t^* u(t) dt, \quad (2.15)$$

where the integral on the right hand side is defined on (\mathcal{S})^{*} in the sense of Bochner. In fact, the operator ∂_t^* can be represented as Wick multiplication with Brownian white noise $\dot{B}_t = \frac{dB_t}{dt}$, i.e.,

$$\partial_t^* u = u \diamond \dot{B}_t, \quad (2.16)$$

where \diamond represents the Wick or Wick-Grassmann product. See [HØUZ].

We now shortly elaborate a white noise framework for pure jump Lévy processes: Let A be a positive self-adjoint operator on $L^2(X, \pi)$, where $X = \mathbb{R} \times \mathbb{R}_0$ ($\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$) and $\pi = \lambda \times \nu$. Here ν is the Lévy measure of a (square integrable) Lévy process η_t . Assume that A^{-p} is of Hilbert-Schmidt type for some $p > 0$. Then denote by $\mathcal{S}(X)$ the standard countably Hilbert space constructed from A . See e.g. [O] or [HKPS]. Let $\mathcal{S}'(X)$ be the dual of $\mathcal{S}(X)$. In what follows we impose the following conditions on $\mathcal{S}(X)$:

- (i) Each $f \in \mathcal{S}(X)$ has a (π -a.e.) continuous version.
- (ii) The evaluation functional $\delta_t : \mathcal{S}(X) \rightarrow \mathbb{R}; f \mapsto f(t)$ belongs to $\mathcal{S}^1(X)$ for all t .
- (iii) The mapping $(t \mapsto \delta_t)$ from X to $\mathcal{S}^1(X)$ is continuous.

Then just as in the Gaussian case we obtain by the Bochner-Minlos theorem the (*pure jump*) Lévy noise measure τ on $\mathcal{B}(\mathcal{S}^1(X))$ which satisfies

$$\int_{\mathcal{S}^1(X)} e^{i\langle \omega, \phi \rangle} \tau(d\omega) = \exp\left(\int_X (e^{i\phi} - 1)\pi(dx)\right) \quad (2.17)$$

for all $\phi \in \mathcal{S}(X)$.

We remark that analogously to the Gaussian case each $F \in L^2(\tau)$ has the unique chaos decomposition

$$F = \sum_{n \geq 0} \langle C_n(\cdot), \phi^{(n)} \rangle \quad (2.18)$$

for $\phi^{(n)} \in \widehat{L}^2(X, \pi)$ (space of square integrable symmetric functions on X). Here $C_n(\omega) \in \left((\mathcal{S}(X))^{\otimes n}\right)'$ are *generalized Charlier polynomials*. Note that $\langle C_n(\cdot), \phi^{(n)} \rangle$ can be viewed the n -fold iterated Itô integral of $\phi^{(n)}$ w.r.t. the compensated Poisson random measure $\tilde{N}(dz, dt) := N(dz, dt) - \nu(dz)dt$ associated with the pure jump Lévy process

$$\eta_t = \langle C_1(\cdot), z\chi_{[0,t]} \rangle = \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(dz, ds). \quad (2.19)$$

Similarly to the Gaussian case we define the (*pure jump*) Lévy-Hida test function space $(\mathcal{S})_\tau$ as the space of all $f = \sum_{n \geq 0} \langle C_n(\cdot), \phi^{(n)} \rangle \in L^2(\tau)$ such that

$$\|f\|_{0,\pi,p}^2 := \sum_{n \geq 0} n! \left\| (A^{\otimes n})^p \phi^{(n)} \right\|_{L^2(X^n, \pi^n)}^2 < \infty \quad (2.20)$$

for $p \geq 0$.

Suppressing the notational dependence on τ we mention that the spaces $(\mathcal{S})^*$, \mathcal{G} , \mathcal{G}^* and the operators $D_{t,z}$, $\partial_{t,z}$, $\partial_{t,z}^*$ can be introduced in the same way as in the Gaussian case. For example (2.15) takes the form

$$\int_0^T \int_{\mathbb{R}_0} u(t, z) \tilde{N}(dz, \delta t) = \int_0^T \int_{\mathbb{R}_0} \partial_{t,z}^* u(t, z) \nu(dz) dt, \quad (2.21)$$

where the left hand side denotes the Skorohod integral of $u(\cdot, \cdot)$ with respect to $\tilde{N}(\cdot, \cdot)$, for Skorohod integrable processes $u \in L^2(\tau \times \pi)$. See e.g. [LP] or [I]. Similar to the Brownian motion case, (see (2.16)), one can prove the representation

$$\partial_{t,z}^* u = u \diamond \check{N}(z, t), \quad (2.22)$$

where $\tilde{N}(z, t) = \frac{\tilde{N}(dz, dt)}{\nu(dz) \times dt}$ is the white noise of \tilde{N} . See [HØUZ] and [ØP].

In the sequel we choose the white noise probability space

$$(\Omega, \mathcal{F}, P) = (\mathcal{S}'(\mathbb{R}) \times \mathcal{S}'(X), \mathcal{B}(\mathcal{S}'(\mathbb{R})) \otimes \mathcal{B}(\mathcal{S}'(X))), \mu \times \tau \quad (2.23)$$

and we suppose that the above concepts are defined with respect to this stochastic basis.

3 Main Results

In this Section we aim at establishing a uniqueness result for decompositions of Skorohod-semimartingales. Let us clarify the latter notion in the following:

Definition 3.1 (Skorohod-semimartingale) *Assume that a process $X_t, 0 \leq t \leq T$ on the probability space (2.23) has the representation*

$$X_t = \zeta + \int_0^t \alpha(s) ds + \int_0^t \beta(s) \delta B_s + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z) \tilde{N}(dz, \delta s) \quad (3.1)$$

for all t . Here we require that $\beta \chi_{[0, t]}(\cdot)$ resp. $\gamma \chi_{[0, t]}(\cdot)$ are Skorohod integrable with respect to B_t resp. $\tilde{N}(dz, dt)$ for all $0 \leq t \leq T$. Further ζ is a random variable and α a process such that

$$\int_0^T |\alpha(s)| ds < \infty \text{ } P\text{-a.e.}$$

Then X_t is called a Skorohod-semimartingale.

Obviously, the Skorohod-semimartingale is a generalization of semimartingales of the type

$$X_t = \zeta + \int_0^t \alpha(s) ds + \int_0^t \beta(s) dB_s + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z) \tilde{N}(dz, ds),$$

where β, γ are predictable Itô integrable processes w.r.t. to some filtration \mathcal{F}_t and where ζ is \mathcal{F}_0 -measurable. The Skorohod-semimartingale also extends the concepts of the Skorohod integral processes

$$\int_0^t \beta(s) \delta B_s \text{ and } \int_0^t \int_{\mathbb{R}_0} \gamma(s, z) \tilde{N}(dz, \delta s), \quad 0 \leq t \leq T.$$

Further it is worth mentioning that the increments of the Skorohod integral process $Y_t := \int_0^t \beta(s) \delta B_s$ satisfy the following orthogonality relation:

$$E[Y_t - Y_s | \mathcal{F}_{[s, t]^c}] = 0, \quad s < t,$$

where $\mathcal{F}_{[s, t]^c}$ is the σ -algebra generated by the increments of the Brownian motion in the complement of the interval $[s, t]$. See [N] or [PTT]. We point out that Skorohod integral processes may exhibit very rough path properties. For example consider the Skorohod SDE

$$Y_t = \eta + \int_0^t Y_s \delta B_s, \quad \eta = \text{sign}(B_1), \quad 0 \leq t \leq 1.$$

It turns out that the Skorohod integral process $X_t = Y_t - \eta$ possesses discontinuities of the second kind. See [Bu]. Another surprising example is the existence of continuous Skorohod integral processes $\int_0^t \beta(s) \delta B_s$ with a quadratic variation, which is essentially bigger than the expected process $\int_0^t \beta^2(s) ds$. See [BI].

In order to prove the uniqueness of Skorohod-semimartingale decompositions we need the following result which is of independent interest:

Theorem 3.2 *Let ∂_t^* and $\partial_{t,z}^*$ be the white noise operators of Section 2. Then*

(i) ∂_t^* maps $\mathcal{G}^* \setminus \{0\}$ into $(S)^* \setminus \mathcal{G}^*$.

(ii) The operator

$$(u \mapsto \int_{\mathbb{R}_0} \partial_{t,z}^* u(t, z) \nu(dz))$$

maps $\mathcal{G}^* \setminus \{0\}$ into $(S)^* \setminus \mathcal{G}^*$.

(iii)

$$\partial_t^* + \int_{\mathbb{R}_0} \partial_{t,z}^* (\cdot) \nu(dz) : \mathcal{G}^* \setminus \{0\} \times \mathcal{G}^* \setminus \{0\} \longrightarrow (S)^* \setminus \mathcal{G}^*.$$

Proof. Without loss of generality it suffices to show that

$$\partial_t^* \text{ maps } \mathcal{G}^* \setminus \{0\} \text{ into } (S)^* \setminus \mathcal{G}^*.$$

For this purpose consider a $F \in \mathcal{G}^* \setminus \{0\}$ with formal chaos expansion

$$F = \sum_{n \geq 0} \langle H_n(\cdot), \phi^{(n)} \rangle.$$

where $\phi^{(n)} \in \widehat{L}^2(\mathbb{R}^n)$. One checks that $\langle H_n(\cdot), \phi^{(n)} \rangle$ can be written as

$$\langle H_n(\cdot), \phi^{(n)} \rangle = \sum_{|\alpha|=n} c_\alpha \langle H_n(\cdot), \xi^{\widehat{\otimes} \alpha} \rangle$$

where

$$c_\alpha = \left(\phi^{(n)}, \xi^{\widehat{\otimes} \alpha} \right)_{L^2(\mathbb{R}^n)} \quad (3.2)$$

with

$$\xi^{\widehat{\otimes} \alpha} = \xi_1^{\widehat{\otimes} \alpha_1} \widehat{\otimes} \dots \widehat{\otimes} \xi_k^{\widehat{\otimes} \alpha_k}$$

for Hermite functions $\xi_k, k \geq 1$ and multiindices $\alpha = (\alpha_1, \dots, \alpha_k), \alpha_i \in \mathbb{N}_0$. Here $|\alpha| := \sum_{i=1}^k \alpha_i$. By (2.5) we know that

$$\infty > \left\| \langle H_n(\cdot), \phi^{(n)} \rangle \right\|_{L^2(\mu)}^2 = \sum_{|\alpha|=n} \alpha! c_\alpha^2.$$

Assume that

$$\partial_t^* F \in \mathcal{G}^*. \quad (3.3)$$

Then $\partial_t^* F$ has a formal chaos expansion

$$\partial_t^* F = \sum_{n \geq 0} \langle H_n(\cdot), \psi^{(n)} \rangle.$$

Thus it follows from the definition of ∂_t^* (see Section 2) that

$$\infty > \left\| \langle H_n(\cdot), \psi^{(n)} \rangle \right\|_{L^2(\mu)}^2 = \sum_{|\gamma|=n} \gamma! \left(\sum_{\alpha + \varepsilon^{(m)} = \gamma} c_\alpha \cdot \xi_m(t) \right)^2, \quad (3.4)$$

where the multiindex $\varepsilon^{(m)}$ is defined as

$$\varepsilon^{(m)}(i) = \begin{cases} 1, & i = m \\ 0 & \text{else} \end{cases}.$$

On the other hand we observe that

$$\begin{aligned} & \sum_{|\gamma|=n} \gamma! \left(\sum_{\alpha + \varepsilon^{(m)} = \gamma} c_\alpha \cdot \xi_m(t) \right)^2 \\ &= \sum_{k=1}^n \sum_{\substack{(a_1, \dots, a_k) \in \mathbb{N}^k \\ a_1 + \dots + a_k = n}} a_1! \cdot \dots \cdot a_k! \sum_{i_1 > i_2 > \dots > i_k} \left(\sum_{m \geq 1} c_{a_1 \varepsilon^{(i_1)} + \dots + a_k \varepsilon^{(i_k)} - \varepsilon^{(m)}} \cdot \xi_m(t) \right)^2, \end{aligned}$$

where coefficients are set equal to zero, if not defined. So we get that

$$\begin{aligned} & \left\| \langle H_n(\cdot), \psi^{(n)} \rangle \right\|_{L^2(\mu)}^2 \\ &= \sum_{k=1}^n \sum_{\substack{(a_1, \dots, a_k) \in \mathbb{N}^k \\ a_1 + \dots + a_k = n}} a_1! \cdot \dots \cdot a_k! a_1! \cdot \dots \cdot a_k! \sum_{i_1 > i_2 > \dots > i_k} \left(\sum_{j=1}^k c_{a_1 \varepsilon^{(i_1)} + \dots + a_k \varepsilon^{(i_k)} - \varepsilon^{(i_j)}} \cdot \xi_{i_j}(t) \right)^2. \end{aligned} \quad (3.5)$$

By our assumption there exist $n^* \in \mathbb{N}_0$, $a_2^*, \dots, a_{k_0}^* \in \mathbb{N}$, pairwise unequal $i_2^*, \dots, i_{k_0}^*$, $k_0 \leq n^* - 1$ such that

$$a_2^* + \dots + a_{k_0}^* = n^* - 1$$

and

$$c_{a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)}} \neq 0. \quad (3.6)$$

On the other hand it follows from (3.5) for $n = n^*$ that

$$\begin{aligned}
& \left\| \left\langle H_n(\cdot), \psi^{(n)} \right\rangle \right\|_{L^2(\mu)}^2 \\
& \geq a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \dots, i_{k_0}^*)} \left(\sum_{j=1}^{k_0} c_{\varepsilon^{(i_1^*) + a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_j^*)}} \cdot \xi_{i_j^*}(t) \right)^2 \\
& = a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \dots, i_{k_0}^*)} \sum_{j_1, j_2=1}^{k_0} \left(c_{\varepsilon^{(i_1^*) + a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_1}^*)}} \cdot \xi_{i_{j_1}^*}(t) \right. \\
& \quad \left. \cdot c_{\varepsilon^{(i_1^*) + a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_2}^*)}} \cdot \xi_{i_{j_2}^*}(t) \right) \\
& =: A_1 + A_2 + A_3, \tag{3.7}
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \dots, i_{k_0}^*)} (c_{a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)}})^2 \cdot (\xi_{i_1^*}(t))^2, \\
A_2 &= a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \dots, i_{k_0}^*)} \sum_{j=2}^{k_0} (c_{\varepsilon^{(i_1^*) + a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_j^*)}} \cdot \xi_{i_j^*}(t))^2, \\
A_3 &= a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \dots, i_{k_0}^*)} \sum_{\substack{j_1 \neq j_2 \\ j_1, j_2=1}}^{K_0} \left(c_{\varepsilon^{(i_1^*) + a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_1}^*)}} \cdot \xi_{i_{j_1}^*}(t) \right. \\
& \quad \left. \cdot c_{\varepsilon^{(i_1^*) + a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_2}^*)}} \cdot \xi_{i_{j_2}^*}(t) \right).
\end{aligned}$$

The first term A_1 in (3.7) diverges to ∞ because of (3.6). The second term is positive. The last term A_3 can be written as

$$\begin{aligned}
& A_3 \\
& = a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \dots, i_{k_0}^*)} 2 \sum_{j=2}^{k_0} \left(c_{a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)}} \cdot \xi_{i_1^*}(t) \right. \\
& \quad \left. \cdot c_{\varepsilon^{(i_1^*) + a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_j^*)}} \cdot \xi_{i_j^*}(t) \right) \\
& \quad + a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \dots, i_{k_0}^*)} \sum_{\substack{j_1 \neq j_2 \\ j_1, j_2=1}}^{K_0} \left(c_{\varepsilon^{(i_1^*) + a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_1}^*)}} \cdot \xi_{i_{j_1}^*}(t) \right. \\
& \quad \left. \cdot c_{\varepsilon^{(i_1^*) + a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_2}^*)}} \cdot \xi_{i_{j_2}^*}(t) \right) \\
& =: A_{3,1} + A_{3,2}, \tag{3.8}
\end{aligned}$$

where

$$\begin{aligned}
A_{3,1} &= a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \dots, i_{k_0}^*)} 2 \sum_{j=2}^{k_0} \left(c_{a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)}} \cdot \xi_{i_j^*}(t) \right. \\
&\quad \left. \cdot c_{\varepsilon^{(i_1^*)} + a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_j^*)}} \cdot \xi_{i_1^*}(t) \right), \\
A_{3,2} &= a_2^*! \cdots a_{k_0}^*! \sum_{i_1^* > \max(i_2^*, \dots, i_{k_0}^*)} \sum_{\substack{j_1 \neq j_2 \\ j_1, j_2=1}}^{K_0} \left(c_{\varepsilon^{(i_1^*)} + a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_1}^*)}} \cdot \xi_{i_{j_1}^*}(t) \right. \\
&\quad \left. \cdot c_{\varepsilon^{(i_1^*)} + a_2^* \varepsilon^{(i_2^*)} + \dots + a_{k_0}^* \varepsilon^{(i_{k_0}^*)} - \varepsilon^{(i_{j_2}^*)}} \cdot \xi_{i_{j_2}^*}(t) \right).
\end{aligned}$$

By means of relation (3.2) and the properties of basis elements one can show that the term $A_{3,1}$ in (3.8) converges t -a.e. The other term $A_{3,2}$ with Hermite functions which do not depend on the summation index converges by assumption, too.

We conclude that

$$\left\| \left\langle H_{n^*}(\cdot), \psi^{(n^*)} \right\rangle \right\|_{L^2(\mu)}^2 = \infty,$$

which contradicts (3.4) and it contradicts (3.3), too.

It follows that

$$\partial_t^* \text{ maps } \mathcal{G}^* \setminus \{0\} \text{ into } (S)^* \setminus \mathcal{G}^*.$$

The proofs of (ii) and (iii) are similar. ■

We are now ready to prove the main result of this paper:

Theorem 3.3 [Decomposition uniqueness for general Skorohod processes]

Consider a stochastic process X_t of the form

$$X_t = \zeta + \int_0^t \alpha(s) ds + \int_0^t \beta(s) \delta B_s + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z) \tilde{N}(dz, \delta s),$$

where $\beta \chi_{[0,t]}$, $\gamma \chi_{[0,t]}$ are Skorohod integrable for all t . Further require that $\alpha(t)$ is element in \mathcal{G}^* a.e. and that α is Bochner-integrable w.r.t. \mathcal{G}^* on the interval $[0, T]$. Suppose that

$$X_t = 0 \text{ for all } 0 \leq t \leq T.$$

Then

$$\zeta = 0, \alpha = 0, \beta = 0, \gamma = 0 \text{ a.e.}$$

Proof. Because of (2.15) and (2.21) it follows that

$$\begin{aligned}
X_t &= \zeta + \int_0^t \alpha(s) ds + \int_0^t \partial_s^* \beta(s) ds + \int_0^t \int_{\mathbb{R}_0} \partial_{s,z}^* \gamma(s, z) \nu(dz) ds \\
&= 0, \quad 0 \leq t \leq T.
\end{aligned}$$

Thus

$$\alpha(t) + \partial_t^* \beta(t) + \int_{\mathbb{R}_0} \partial_{t,z}^* \gamma(t, z) \nu(dz) = 0 \text{ a.e.}$$

Therefore

$$\partial_t^* \beta(t) + \int_{\mathbb{R}_0} \partial_{t,z}^* \gamma(t, z) \nu(dz) \in \mathcal{G}^* \text{ a.e.}$$

Then Theorem 3.2 implies

$$\beta = 0, \gamma = 0 \text{ a.e.}$$

■

Remark 3.4 We mention that Theorem 3.3 is a generalization of a result in [NP] in the Gaussian case, when $\beta \in \mathbb{L}^{1,2}$, that is

$$\|\beta\|_{1,2}^2 := \|\beta\|_{L^2(\lambda \times \mu)}^2 + \|D.\beta\|_{L^2(\lambda \times \lambda \times \mu)}^2 < \infty.$$

As a special case of Theorem 3.3, we get the following:

Theorem 3.5 [Decomposition uniqueness for Skorohod-semimartingales]

Let X_t be a Skorohod-semimartingale of the form

$$X_t = \zeta + \int_0^t \alpha(s) ds + \int_0^t \beta(s) \delta B_s + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z) \tilde{N}(dz, \delta s),$$

where $\alpha(t) \in L^2(P)$ for all t . Then if

$$X_t = 0 \text{ for all } 0 \leq t \leq T.$$

we have

$$\zeta = 0, \alpha = 0, \beta = 0, \gamma = 0 \text{ a.e.}$$

Example 3.6 Assume in Theorem 3.3 that $\gamma \equiv 0$. Further require $\alpha(t) \in L^p(\mu)$ $0 \leq t \leq T$ for some $p > 1$. Since $L^p(\mu) \subset \mathcal{G}^*$ for all $p > 1$ (see [PT]) it follows from Theorem 3.3 that if $X_t = 0$, $0 \leq t \leq T$ then $\zeta = 0$, $\alpha = 0$, $\beta = 0$ a.e.

Example 3.7 Denote by $L_t(x)$ the local time of the Brownian motion. Consider the Donsker delta function $\delta_x(B_t)$ of B_t , which is a mapping from $[0, T]$ into \mathcal{G}^* . The Donsker delta function can be regarded as a time-derivative of the local time $L_t(x)$, that is

$$L_t(x) = \int_0^t \delta_x(B_s) ds$$

for all x a.e. See e.g. [HKPS]. So we see from Theorem 3.3 that the random field

$$X_t = \zeta + L_t(x) + \int_0^t \beta(s) \delta B_s + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z) \tilde{N}(dz, \delta s)$$

has a unique decomposition. We remark that we obtain the same result if we generalize $L_t(x)$ to be a local time of a diffusion process (as constructed in [PSu]) or the local time of a Lévy process (as constructed in [MØP]). Finally, we note that the unique decomposition property carries over to the case when X_t has the form

$$X_t = \zeta + A_t + \int_0^t \beta(s) \delta B_s + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z) \tilde{N}(dz, \delta s),$$

where A_t is a positive continuous additive functional with the representation

$$A_t = \int_{\mathbb{R}} L_t(x) m(dx),$$

where m is a finite measure. See [B] or [F].

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