TIGHT PACKING A HYPERSPHERE WITH OTHERS OF LIKE KIND

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ABSTRACT. All are familiar with the exercise of surrounding a circle with six circles of the same size. A similar exercise — surrounding a hypersphere with hyperspheres of the same size tightly packed — is not possible in any higher dimension. This note demonstrates the equivalence of this exercise to that of constructing specialized simplicial complexes, and then provides a necessary condition for surrounding a hypersphere, proving the condition is violated for hyperspheres of dimension greater than 1. An implication for the general packing of hyperspheres follows.

1. INTRODUCTION

Modern theory of packing spheres began with the two papers (Levenshtein 1979) and (Odlyzko and Sloane 1979). More recent developments are chronicled in (Conway and Sloane 1999), (Cohn and Elkies 2003), and (Cohn 2002). The last two are a sequence, which, like Beethoven’s first two piano concerti, became published in reverse order. These are good foundational studies for the present œuvre, and are recommended with references within.

We begin with some rudimentary definitions. The $n$-sphere is the sphere of dimension $n$ considered embedded in $\mathbb{R}^{n+1}$. The 1-sphere is the circle; the 2-sphere is the common sphere; all others carry the dimension as part of the name. Unless otherwise qualified, all spheres have radius one.

Think of the circle $S_1$ with six circles around it. One could form six equilateral triangles — 2-simplexes — joining the center of $S_1$ with the centers of the surrounding circles, joining those centers to form a hexagon. The insight looking forward is that the interior angles of these triangles — $\pi/3$ each — exactly complete a circle. In this case note that $n=1$, and that $\text{arcsec}(n+1)$ divides $2\pi$ six times. The concept of tightness appears, meaning that any circle touching $S_1$, say $T_1$, also touches two others.

Think next of the 2-sphere $S_2$, trying first to surround another 2-sphere $T_2$ touching $S_2$, with others of like kind also touching $S_2$. One finds by simple examination that five spheres will fit, but that a sixth will not. The curvature of $S_2$ now plays a role. The concept of tightness here is that any sphere touching $T_2$ and $S_2$ also touches two others.

As in the first example, construct now a series of 3-simplexes, or tetrahedra, joining the center of $S_2$ with the center of $T_2$ and with the centers of any two adjacent spheres to it. In analogy to the fitting of spheres, five of these 3-simplexes will fit without consuming a complete circle. Here the angle of relevance is the one joining the center of the common edge of the 3-simplexes to the outer vertices of any one of the simplexes. Such vertices lie in a plane, for all are equidistant from the two common vertices of the tetrahedra, and thus lie in

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the intersection of two 2-spheres, which is a circle. This angle, easily determined from the
basic geometry of simplexes and an appeal to the law of cosines, is \( \text{arcsec}(3) = \text{arcsec}(n+1) \),
which does not divide \( 2\pi \), so \( T_2 \) cannot be surrounded exactly by spheres of like kind.

2. THE GENERAL CASE

Now the plan for generalizing becomes apparent, but first here is a brief digression on
the geometry of simplexes. An inductive argument reveals that the height of a standard
n-simplex, the convex hull of \((n+1)\) points joined by edges of unit length, is \( \sqrt{(n+1)/(2n)} \).
Here the height is measured as the distance from any vertex to the barycenter of the \((n-1)\)
simplex opposite. Also, the angle between any two vertices to the barycenter of the \((n-2)\)
simplex opposite is \( \text{arcsec}(n) \). In the examples above the simplexes have edges of length
two, but our primary interest is the angle inscribed, which is invariant to scaling.

Following is the principal result, stated formally.

**Theorem 2.1.** Tight packing a hypersphere with others of like kind is not possible in dimensions
greater than 1.

**Proof.** It shall be demonstrated that a single sphere attached to another of like kind cannot
be surrounded by an integer number of spheres of like kind in dimensions greater than 1,
implying the result.

In the general case, one attempts to surround an \( n \)-sphere \( T_n \) attached to another, \( S_n \), by
starting with \( n \) other \( n \)-spheres, \([U^1_n, \ldots, U^n_n]\), all touching. Together their centers describe an
\((n+1)\)-simplex \( \Delta^1_{n+1} \). At each iteration one attaches another \((n+1)\)-simplex to the previous
one on a common \( n \)-simplex boundary by adding a new vertex. Let these simplexes be
\([\Delta^2_{n+1}, \Delta^3_{n+1}, \ldots] \).

Vertices added this way are co-planar, meaning in the same 2-plane, because they are
equidistant from a set of \( n \) distinct vertices, to wit, the centers of \([S_n, T_n, U^1_n, \ldots, U^{n-2}_n] \) (reducing to \([S_1, T_2, T_2, \ldots] \) for \( n = 1 \) and \( n = 2 \), respectively) and thus are in the intersection
of \( n - n \)-spheres, which is a circle.

The angle of relevance within this circle is the one joining two vertices of a simplex by the
barycenter of the common \((n-1)\)-simplex opposite. This angle is \( \text{arcsec}(n+1) \). If and only
if this angle divides \( 2\pi \) can one surround \( T_n \) by spheres of like kind.

The issue then devolves upon finding a solution in natural numbers \( n \) and \( k \) such that
\[
    k \cdot \text{arcsec}(n+1) = 2\pi
\]
The only solution to this equation is the familiar \( n = 1 \) and \( k = 6 \), the circle with six circles
around. \( \square \)

**Remark.** An interesting conclusion involves finding the [irrational] dimension for which \( k = 5 \). We know the solution for \( k = 6 \), and \( k = 4 \) cannot occur because
\[
    \lim_{n \to \infty} \text{arcsec}(n+1) = \frac{\pi}{2},
\]
ever attained. An easy analysis reveals that \( k = 5 \) at \( n = \sqrt{5} \). Implications for comparisons
with the golden ratio and the Fibonacci numbers tempt, but will be set aside.

Finally, one sees that a tight packing of \( n \)-spheres into any \((n+1)\)-dimensional manifold
is not possible if the manifold admits the inclusion of an \( n \)-sphere of radius 4. The demonstration
is left to the reader.
References


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