Model Risk in Portfolio Management

by

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THESIS

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Contents

Chapter 1

Introduction

Administrators of pension funds manage financial portfolios of considerable amounts. To control the risk involved and have grounds on which to find the best management strategy, it is of interest to have a realistic model of the asset's behavior. Throughout the last decades several attempts have been made of this from a mathematical point of view. Here included is the much used *geometric Brownian motion* (GBM) of Samuelson (1965). A potential problem with the GBM is the normality assumption on the stock's log-returns. Empirical facts tell us that this does not hold on short time spans like a day or a week. The real distribution of log-returns is not symmetric and has heavier tails than the normal distribution.

In this thesis I will study alternatives to the GBM where the log-return follows a *Lévy process*. These processes allow for sudden jumps in the stock price to happen and has been proven to fit the empirical data well. I especially look at two types of Lévy processes; the normal inverse Gaussian (NIG) and the CGMY.

My objective will be on *model risk*. Can we expect significant differences in risk by using such models instead of the GBM? With focus on *Merton's portfolio management problem* [1] of deriving the maximum expected utility of consumption over an infinite time horizon and where the market consists of one stock and one risk-less asset, I will compare the derivation of the optimal portfolio, as well as its behavior, with the GBM and exponential Lévy processes as alternative stock price models. I am particularly interested in the risk involved when believing in the wrong model. As measures of risk I use *Value-at-Risk* (VaR) and *conditional Value-at-Risk* (cVaR). To have a fair comparison of the performance of the optimal portfolios I calibrate the parameters of the models such that the log-returns have equal expectations

[1] First published in 1969 by Robert C. Merton(1944-), an american economist.

and variances.

My results indicate that for a given strategy, the differences in VaR and cVaR of the portfolios with GBM and NIG/CGMY as alternative stock price models will be quite small as the time horizon increases. The difference in risk between the models is mostly due to the derivation of the optimal strategy which in turn depends on parameter estimates. This may well lead to significant differences in investment strategies and thus risk profiles, at least when the interest rate of the risk-less asset is close to the mean rate of return on the stock. However, the parameter estimates are often uncertain and the differences can go either way. Hence the utility of using the more complex Lévy processes instead of the ordinary GBM may not be that big.

Here is an outline of the thesis: In chapter 2 I introduce some basic definitions and properties of Lévy processes. Chapter 3 gives a short presentation of the three stock price models that I apply. In chapter 4 Merton's portfolio management problem and its solution in both the GBM case and the exponential Lévy case is presented. I also present an approximation to the optimal portfolio in the latter case which has a more intuitive form. Chapter 5 provides comparisons of the risk/return profile of the optimal portfolios of the Merton problem in the GBM and exponential NIG cases. Chapter 6 concerns the same topic, but with the CGMY instead of the NIG as driver of the stock price. Finally, in chapter 7, I make some concluding remarks.

Chapter 2

Lévy Processes

In this thesis I have used two types of Lévy processes as the driver of the stock prices; the normal inverse Gaussian and the CGMY. Lévy processes is a class of stochastic processes which includes jump processes. The following definition of a Lévy process is found in Sato [13].

Definition 2.1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space. An adapted stochastic process $\{L_t\}_{t\geq 0}$ taking values in $\mathbb{R}$ is a Lévy process if it satisfies the following properties:*

- *$\{L_t\}$ has independent increments: For any $n \geq 1$ and $0 \leq t_0 \leq \cdots \leq t_n$ we have that $L_{t_0}, L_{t_1} - L_{t_0}, \ldots, L_{t_n} - L_{t_{n-1}}$ are independent*
- *$\{L_t\}$ has stationary increments, that is, the distribution of $L_{s+t} - L_s$ where $s, t \geq 0$, does not depend on s.*
- *$\{L_t\}$ is stochastically continuous, that is, for any $\epsilon > 0$, $P(|L_{s+t} - L_s| > \epsilon) \to 0$ as t tends to s.*
- *$L_0 = 0$ almost surely.*
- *$\{L_t\}$ has a modification with right continuous, left limited paths almost surely,(càdlàg version).*

A random variable X has an infinitely divisible probability distribution if for each $n \geq 1$ there exists n i.i.d. variables $X_1, \ldots, X_n$ such that

$$X \stackrel{d}{=} X_1 + \cdots + X_n$$

or equivalently that

$$\phi_X(u) = (\phi_{X_1}(u))^n.$$

A Lévy process $\{L_t\}$ has the property that for each t, L_t has an infinitely divisible probability distribution. This is a consequence of it having stationary and independent increments. It turns out that any random variable with an infinitely divisible law defines a unique Lévy process.

Theorem 2.1 (Lévy-Khintchine representation)**.** *If a random variable X has an infinitely divisible probability distribution μ, then its characteristic function has the form*

$$\phi_X(u) = \mathbb{E}[\exp(iuX)] = \exp\left(ibu - \frac{1}{2}cu^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{|x|<1}(x)\right)\nu(dx)\right) \tag{2.1}$$

where $i = \sqrt{-1}$, $b \in \mathbb{R}$, $c \geq 0$ and ν is a measure on $\mathbb{R}$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \max\{1, x^2\}\nu(dx) < \infty$. The triplet (b, c, ν) is unique. Conversely, for any triplet satisfying the conditions above, there exists a random variable with infinitely divisible probability distribution having (2.1) as its characteristic function.

A Lévy process L_t is infinitely divisible and thus has a Lévy-Khintchine representation. The triplet (b, c, ν) is called the generating triplet or the Lévy triplet. A unique Lévy process can be defined from it by setting $\phi_{L_1}(u)$ equal to the right hand side of (2.1). Then

$$\phi_{L_t}(u) = (\phi_{L_1}(u))^t.$$

ν is called the Lévy measure. It is a measure on the Borel sets of $\mathbb{R}$ not intersecting 0, $\mathbf{B_0}$. For each $A \in \mathbf{B_0}$, $\nu(A)$ measures the expected number of jumps of size $\triangle L \in A$ occurring in a unit time interval.

From the Lévy-Khintchine representation one can show that L_t equals the sum of four Lévy processes. This is made concrete in the next theorem:

Theorem 2.2 (Lévy-Itô decomposition)**.** *A Lévy process L_t has the decomposition*

$$L_t = bt + \sqrt{c}W_t + \int_0^t \int_{|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN(ds, dz) \tag{2.2}$$

where W_t is a standard Brownian motion, $N(ds, dz)$ is a Poisson random measure with intensity $ds \times \nu(dz)$ and $\tilde{N}(ds, dz)$ is the compensated Poisson random measure $N(ds, dz) - ds \times \nu(dz)$. W_t and $N(t, \cdot)$ are independent.

The Poisson random measure N is defined as

$$N(t, A) = \sum_{0<s\leq t} \mathbf{1}_A(\triangle L_s)$$

where $A \in \mathbf{B_0}$ and $\triangle L_s = L_s - L_{s-}$ is the jump at time s. For a fixed $A \in \mathbf{B_0}$, $N(\cdot, A)$ is a Poisson process of intensity $\nu(A)$. See Øksendal and Sulem [11], thm. 1.5.

The Lévy-Itô decomposition shows that a Lévy process is a Brownian motion with drift plus two jump terms. The càdlàg property secures that the number of jumps of magnitude bigger than one is finite on any finite interval (see Eberlein [9]). However, the Lévy process may have infinitely many jumps of magnitude less than one. That is the reason for using a compensated measure in the third term. Subtracting the average sum of jumps secures convergence of term three. It also makes it a martingale (see Øksendal and Sulem [11]). The Brownian term is the continuous martingale part of L_t. If $c = 0$, L_t is a purely discontinuous Lévy process (Eberlein [9]). It is this together with the frequency of small jumps that decides whether or not the paths have finite variation.

Proposition 2.1. *Let L_t be a Lévy process with triplet (b, c, ν). Then*

(i) Almost all paths have finite variation if $c = 0$ and $\int_{|z|<1} |z|\nu(dz) < \infty$.

(ii) Almost all paths have infinite variation if $c \neq 0$ or $\int_{|z|<1} |z|\nu(dz) = \infty$.

With this in mind we see that a Lévy process is a semimartingale, that is, a sum of a local martingale, a finite variation process and an $\mathcal{F}_0$ measurable random variable. The martingale part consists of the Brownian motion and the compensated integral in (2.2). Since the uncompensated jump term in (2.2) has a finite number of jumps, it has finite variation, as of course is the case for the drift part as well. With this property a Lévy process can be used as an integrator in stochastic integrals.

Remark 2.1. *With $\mathbb{E}[|L_t|] < \infty$, $\int_{|z|\geq 1} |z|\nu(dz) < \infty$ and L_t can be decomposed as*

$$L_t = \mathbb{E}[L_1]t + \sqrt{c}W_t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\tilde{N}(ds, dz)$$

This follows from the fact that both W_t and the compensated integral term are martingales with expectation 0. In this case we see that L_t is a martingale if and only if $\mathbb{E}[L_1] = 0$.

Chapter 3

Stock Price Models

3.1 The Geometric Brownian Motion

A common model in financial mathematics for modelling the price of a stock at time t is the geometric Brownian motion (GBM). It is the solution of the stochastic differential equation

$$dS_t = \alpha S_t dt + \sigma S_t dW_t$$

where α is the drift coefficient and σ is the volatility coefficient of the stock. From Itô's stochastic calculus using Itô's formula the solution can be shown to be

$$S_t = S_0 \exp(\mu_{GBM} t + \sigma W_t)$$

where I have used the notation $\mu_{GBM} := \alpha - \frac{1}{2}\sigma$. We see that the exponent is a drifted Brownian motion with drift μ_{GBM} and volatility σ. Consequently the log-return from a time t_0 to time t_1,

$$\ln\left(\frac{S_{t_1}}{S_{t_0}}\right) = \mu_{GBM}(t_1 - t_0) + \sigma(W_{t_1} - W_{t_0}),$$

is normally distributed with mean $\mu_{GBM}(t_1 - t_0)$ and variance $\sigma^2(t_1 - t_0)\big)$.

In this thesis I shall consider alternative stock price models where the log-returns are distributed by other Lévy processes that contain jump parts, like the normal inverse Gaussian and the CGMY. These models allow for both skewness and heavier tails than the normal distribution.

3.2 The Normal Inverse Gaussian Distribution

The normal inverse Gaussian (NIG) distribution has four parameters; α, β, μ and δ. α is a steepness parameter measuring heaviness of the tails, β is an asymmetry parameter, μ is a location parameter and δ is a scale parameter. The density function of a NIG distribution is:

$$nig(x;\alpha,\beta,\mu,\delta) = \frac{\delta\alpha}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\right)\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}$$

where K_1 is the modified Bessel function of the second kind of index 1 (see Abramowitz/Stegun [1]). The parameters α, β, and δ satisfy the inequalities $\delta > 0$ and $0 \leq |\beta| \leq \alpha$.

Figure 3.1 shows how the density function varies with different parameters. In the first plot we see how the density function becomes steeper with increasing α's. In the second plot I vary the β parameter. With a negative β the distribution is skewed to the left and vice versa if β is negative. In the two plots $\delta = 2$. A higher δ would make the curves flatter as the variance increases.

The NIG distribution has finite moments of all order. The expectation and variance of a NIG random variable L is

$$\mathbb{E}[L] = \mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}, \qquad \mathrm{Var}[L] = \frac{\delta\alpha^2}{(\alpha^2-\beta^2)^{3/2}}.$$

We can model a stock price S_t as an exponential NIG process by writing

$$S_t = S_0\exp(\mu_{NIG}t + L_t)$$

where L_t is a Lévy process with unit increments having the distribution $NIG(\alpha,\beta,0,\delta)$. NIG is closed under convolution of two variables X and Y having the same scale and asymmetry parameters α and β (see Rydberg [12]). This implies that L_t is distributed $NIG(\alpha,\beta,0,\delta t)$.

By Remark 2.1 L_t can be decomposed as

$$L_t = \gamma_c t + \int_0^t\int_{\mathbb{R}\setminus\{0\}} z\tilde{N}(ds,dz)$$

where $\gamma_c = \mathbb{E}[L_1]$ and $\tilde{N}(ds,dz)$ is the compensated Poisson random measure associated to ν_{NIG}. The Lévy measure of the NIG process is

$$\nu_{NIG}(dz) = \frac{\delta\alpha}{\pi|z|}\exp(\beta z)K_1(\alpha|z|)dz$$

For any set of parameters, the NIG model has infinite variation and infinite activity. See e.g. Cont and Tankov [8] for more details.

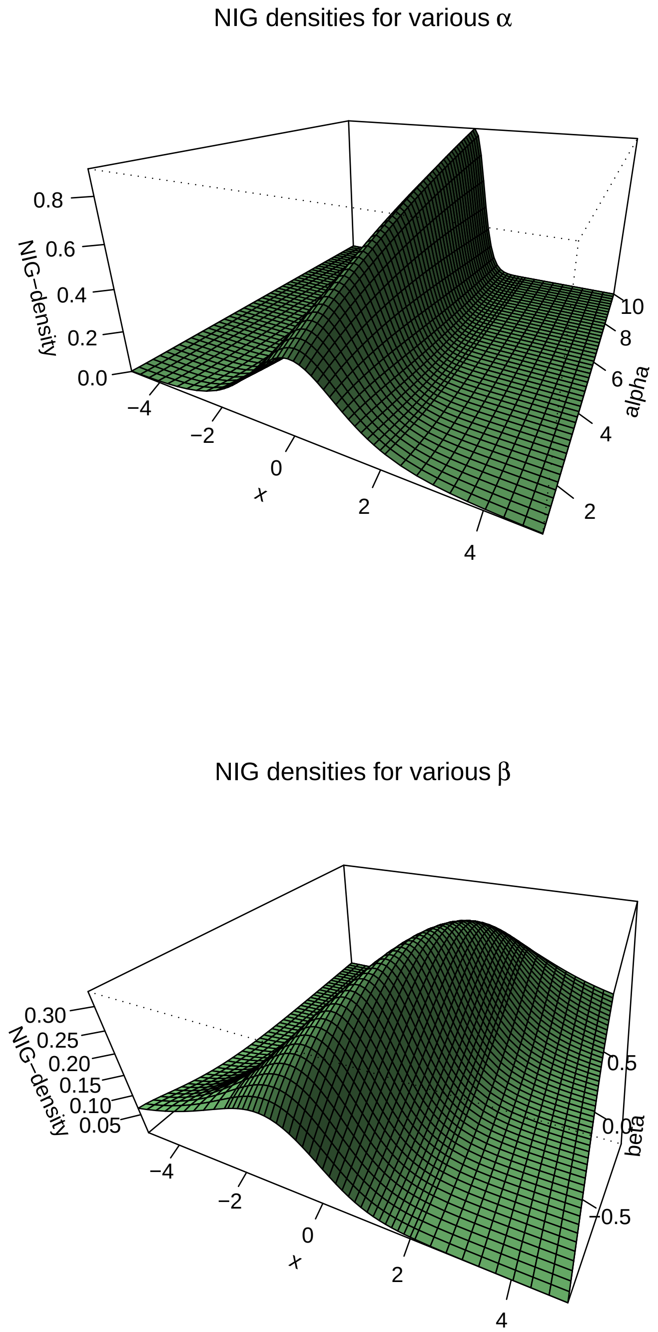


Figure 3.1: *At the top:* $nig(x; \alpha, 0, 0, 2)$ *for* $\alpha = 1, \ldots, 10$*. At the bottom :* $nig(x; 1, \beta, 0, 2)$ *for* $\beta = -0.9, \ldots, 0.9$*.*

3.3 The CGMY Lévy Process

The CGMY Lévy Process is another model used for modelling log-returns of stocks. It is a special form of tempered stable processes presented by Carr, Geman, Madan and Yor in [7]. CGMY like the NIG has no diffusion part, that is $c = 0$ in its Lévy triplet. It does not have a closed form expression of its density function, but is defined via its Lévy density ν_{CGMY}:

$$\nu_{CGMY}(z) = \begin{cases} C\frac{\exp(-G|z|)}{|z|^{1+Y}} & \text{if } z < 0 \\ C\frac{\exp(-Mz)}{z^{1+Y}} & \text{if } z > 0 \end{cases}$$

The parameters must satisfy $C > 0$, $G \geq 0$, $M \geq 0$ and $Y < 2$. C is a parameter that determines the general level of activity. A higher C leads to more jumps. Y is a parameter that defines the structure of the paths. With $Y < 1$, the paths have finite variation. With $Y \geq 1$, they have infinite variation. 0 marks the limit for finite activity. With $Y < 0$, CGMY has finite activity and infinite activity with $Y \geq 0$. G and M are parameters that determine the skewness and kurtosis of the process. $G < M$ means the left tail is heavier and vice versa if $G > M$. With small G and M there is a high probability of large jumps, so the kurtosis decrease with G and M. A CGMY process has finite moments of all orders and can be expressed as

$$L_t = \gamma_c t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\tilde{N}(ds, dz) \tag{3.1}$$

where $\gamma_c = \mathbb{E}[L_1]$ and $\tilde{N}(ds, dz)$ is the compensated Poisson random measure associated to ν_{CGMY}.

With $\nu_{CGMY}(z)$ defined, L_t's characteristic function can be expressed in terms of the Gamma function:

$$\begin{aligned}\phi_{CGMY}(u,t) = \exp\Big(t\big(\gamma_c + C\Gamma(-Y)[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y \\ + iuYM^{Y-1} - iuYG^{Y-1}]\big)\Big).\end{aligned}$$

By differentiating ϕ_{CGMY} we get the variance, skewness and kurtosis of L_t:

$$\mathrm{Var}[L_t] = tC\Gamma(2-Y)\Big[\frac{1}{M^{2-Y}} - \frac{1}{G^{2-Y}}\Big]$$

$$\mathrm{Skew}[L_t] = \frac{tC\Gamma(3-Y)\Big[\frac{1}{M^{3-Y}} - \frac{1}{G^{3-Y}}\Big]}{(\mathrm{Var}[L_t])^{3/2}}$$

$$\mathrm{Kurt}[L_t] = \frac{tC\Gamma(4-Y)\Big[\frac{1}{M^{4-Y}} - \frac{1}{G^{4-Y}}\Big]}{(\mathrm{Var}[L_t])^{2}}$$

If $Y < 1$, the jumps of the process, L_t^j, is a compound Poisson process,

$$L_t^j = \int_0^t \int_{\mathbb{R}\backslash\{0\}} zN(ds, dz), \tag{3.2}$$

with characteristic function

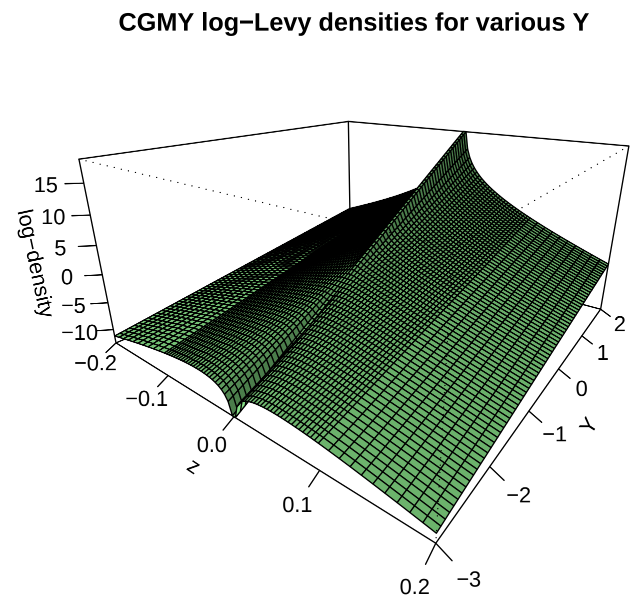
$$\phi_{CGMY,j}(u,t) = \exp\Big(t\big(C\Gamma(-Y)[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y]\big)\Big),$$

implying that $\mathbb{E}[L_t^j] = tCY[G^{Y-1} - M^{Y-1}]$.

We can model a stock price S_t as an exponential CGMY process by writing

$$S_t = S_0 \exp(L_t) \tag{3.3}$$

where L_t is a Lévy process in the form of (3.1) above. Five parameters are needed; (C, G, M, Y) describing the jumps along with the expectation γ_c describing the drift.



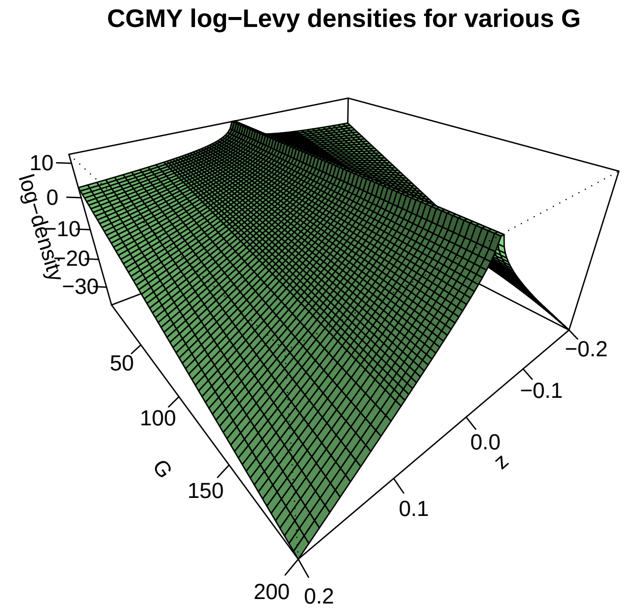


Figure 3.2: *$\nu_{CGMY}(z)$ on logarithmic scale. I let $C = 1$ and $G = M$. Top figure: $G, M = 40$ and Y varies from -3 to 2. Bottom figure: $Y = 1$ and G varies from 1 to 200.*

Chapter 4

Merton's Portfolio Management Problem

4.1 Introduction

I wish to study the portfolio problem, stated by Merton, 1969, in [10], of deriving the optimal investment and consumption strategy in a market with one risk free asset, governed by a fixed interest rate intensity r, and one risky asset following either a geometric Brownian motion or alternatively a more general exponential Lévy process like the two presented in the previous chapter. The investor is assumed to have a HARA (Hyperbolic Absolute Risk Aversion) utility function. I also assume there are no transaction costs in the market.

The solutions to Merton's problem with geometric Brownian motion and exponential Lévy processes with jumps will be compared and an approximation to the latter presented.

4.2 Assumptions and Problems

Let the wealth process at time t for an investor with consumption rate $c = (c_t)$, fraction of wealth in the stock $\pi = (\pi_t)$ and initial wealth $x \geq 0$ be denoted by $X_t^{c,\pi,x}$. If the stock follows a geometric Brownian motion the wealth process must follow the dynamics

$$dX_t^{c,\pi,x} = [r + (\mu_{GBM} + \frac{1}{2}\sigma^2 - r)\pi_t)]X_t^{c,\pi,x}dt - c_t dt + \sigma\pi_t X_t^{c,\pi,x}dW_t \quad (4.1)$$

For stock models where the log-return follows a Lévy process with jumps, the dynamics of wealth become more complicated. Benth, Karlsen and Reikvam show in [6] that the wealth dynamics for a general exponential Lévy process with Lévy triplet (μ, σ^2, ν) according to (2.1) is:

$$dX_t^{c,\pi,x} = [r + (\hat{\mu} - r)\pi_t]X_t^{c,\pi,x}dt - c_t dt + \pi_{t-}X_{t-}^{c,\pi,x}\int_{\mathbb{R}\setminus\{0\}}(e^z - 1)\tilde{N}(dt, dz)$$

where

$$\hat{\mu} = \mu + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}\setminus\{0\}}(e^z - 1 - z\mathbf{1}_{|z|<1})\nu(dz). \tag{4.2}$$

$\hat{\mu}$ is the mean rate of return of the stock and is assumed to be higher than r. The dynamics also rest on the assumption that

$$\int_{|z|\geq 1}|e^z - 1|\nu(dz) < \infty \tag{4.3}$$

(see Benth et al. [6], p. 450).

The stochastic control problem of the investor is to find an admissible pair of a consumption rate c_t and a fraction of wealth π_t that maximizes the expected utility of consumption over an infinite time horizon. One assumes a discount rate η and a risk aversion coefficient $1 - \gamma$.

The class of admissible controls (c, π) when initial wealth is x, $\mathcal{A}_x$, can be characterized as follows: $(c, \pi) \in \mathcal{A}_x$ if

- c_t is nonnegative, adapted and satisfies $\int_0^t \mathbb{E}[c_s]ds < \infty \quad \forall t \geq 0$.
- $\pi_t \in [0, 1]$, is adapted and has càdlàg paths
- c_t is such that $X_t^{c,\pi,x} \geq 0$ almost surely $\qquad \forall t \geq 0$

The problem is for an initial wealth $x \geq 0$ to find admissible c^* and π^* such that

$$V(x) = \sup_{(c,\pi)\in\mathcal{A}_x} \mathbb{E}\left[\int_0^\infty e^{-\eta t}\frac{c_t^\gamma}{\gamma}\right] = \mathbb{E}\left[\int_0^\infty e^{-\eta t}\frac{(c_t^*)^\gamma}{\gamma}\right]$$

$\gamma \in (0, 1)$ measures the investor's tolerance towards risk. A low γ will give a higher utility of consumption in the early stages of the time period in which wealth is well known, but if the wealth, and thereby implicitly the consumption, should rise to a high level, low γ's will not give as high a utility as big ones. Thus a "risk lover" will have a high γ.

4.3 Solution

To solve this problem we apply the dynamic programming principle leading to the Hamilton-Jacobi-Bellmann (HJB) equation. Given that the wealth at present is x, the equation for the optimal consumption and investment is:

$$\max_{c\geq 0,\pi\in[0,1]} \Big[(r+(\hat{\mu}-r)\pi)xv'(x) - cv'(x) - \eta v(x) + \frac{c^\gamma}{\gamma} + \frac{1}{2}\sigma^2\pi^2x^2v''(x) + \int_{\mathbb{R}\setminus\{0\}} \big(v(x+\pi x(e^z-1)) - v(x) - \pi x v'(x)(e^z-1)\big)\nu(dz)\Big] = 0 \tag{4.4}$$

A good guess for the solution is $V(x) = Kx^\gamma$. Benth et al. show in [6] that this leads to an integral equation for π, which is independent of time:

$$(\hat{\mu}-r) - (1-\gamma)\sigma^2\pi + \int_{\mathbb{R}\setminus\{0\}} \big((1+\pi(e^z-1))^{\gamma-1} - 1\big)(e^z-1)\nu(dz) = 0 \tag{4.5}$$

The optimal consumption has a simpler expression:

$$c = (K\gamma)^{\frac{1}{\gamma-1}}x \tag{4.6}$$

We see that it is optimal at all times to allocate a constant fraction of wealth in the stock and to consume another constant fraction. To find the constant K we substitute (4.6) into (4.4). This yields:

$$K = \frac{1}{\gamma}\Big[\frac{1-\gamma}{\eta-k(\gamma)}\Big]^{1-\gamma}$$

where

$$k(\gamma) = \max_{\pi\in[0,1]} \Big[\gamma\big(r+(\hat{\mu}-r)\pi\big) - \frac{1}{2}\sigma^2\pi^2\gamma(1-\gamma) + \int_{\mathbb{R}\setminus\{0\}} \big((1+\pi(e^z-1))^\gamma - 1 - \gamma\pi(e^z-1)\big)\nu(dz)\Big]. \tag{4.7}$$

To secure that the value function is positive and finite the condition

$$\eta > k(\gamma)$$

will suffice. With this the optimal consumption strategy can be written more compactly as

$$c^*(x) = \frac{\eta-k(\gamma)}{1-\gamma}x.$$

By differentiating the expression which is to be maximized in (4.7) with respect to π, we obtain an expression proportional to the left side of (4.5). It thus follows that the maximizing π^* in (4.7) is nothing but the solution π^* of (4.5).

In the GBM the Lévy measure $\nu_{GBM} \equiv 0$. Thus the optimal allocation in the stock according to (4.5) is easily seen to be

$$\pi^*_{GBM} = \frac{\mu_{GBM} + \frac{1}{2}\sigma^2 - r}{\sigma^2(1-\gamma)} \tag{4.8}$$

and the consumption rate is explicitly given since

$$k_{GBM}(\gamma) = \gamma \left[r + \frac{(\mu + \frac{1}{2}\sigma^2 - r)^2}{2\sigma^2(1-\gamma)} \right]. \tag{4.9}$$

For exponential Lévy models with jump like NIG and CGMY, the solutions are not as straightforward.

4.4 Analysis of the Optimal Stock Allocation π^*

Let us analyze equation (4.5) a bit to see how the solution differs in the geometric Brownian motion case compared to other exponential Lévy models that satisfy the scheme of the Merton problem. I assume the alternative Lévy process L_t driving the stock has no Gaussian part, but a nonzero jump part, i.e. $c = 0$ and $\nu \neq 0$ in its Lévy triplet. Assumption (4.3) implies that L_t can be written as a compensated compound Poisson process,

$$L_t = \gamma_c t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\tilde{N}(ds, dz)$$

as in remark 2.1. Included in this class of processes are the NIG and the CGMY.

For such Lévy processes the variance is (see appendix B):

$$\mathrm{Var}[L_t] = t \int_{\mathbb{R}\setminus\{0\}} z^2 \nu(dz). \tag{4.10}$$

$\hat{\mu}$ in the form of (4.2) is expressed through the standard truncation function $\mathbf{1}_{|z|<1}$ of the Lévy-Khintchine representation. With NIG and CGMY we may instead use the truncation function $\mathbf{1}_{|z|<\infty}$ leading to

$$\hat{\mu} = \gamma_c + \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\nu(dz).$$

This expression is better understood if we substitute the integrand with its Taylor series expansion at $z = 0$. We then get

$$\begin{aligned}\hat{\mu} &= \gamma_c + \int_{\mathbb{R}\setminus\{0\}} \Big(\frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \dots\Big)\nu(dz) \\ &= \gamma_c + \frac{1}{2}\mathrm{Var}[L_1] + \int_{\mathbb{R}\setminus\{0\}} \Big(\frac{z^3}{6} + \frac{z^4}{24} + \dots\Big)\nu(dz)\end{aligned}$$

where I have applied (4.10). Recognizing that $\hat{\mu} = \mu_{GBM} + \frac{1}{2}\sigma^2$ in the GBM we see the resemblance of these.

Regarding the integral part of (4.5) we calculate the first terms in the Taylor series of $\big((1+\pi(e^z-1))^{\gamma-1} - 1\big)(e^z - 1)$ at $z = 0$ to be

$$\pi(\gamma-1)z^2 + \frac{1}{2}(\gamma-1)\pi(2+\pi(\gamma-2))z^3.$$

By (4.10) it is thus possible to extract the term $\pi(\gamma-1)\mathrm{Var}[L_1]$ from the integral, again similar to the term $\pi(\gamma-1)\sigma^2$ in the GBM case. If the log increments of the GBM and the alternative exponential Lévy model have equal mean and variance, the only part of (4.5) that differ is a $\nu(dz)$-integral of a linear combination of terms z^n of order $n \geq 3$. These terms are usually small compared to the other terms of the equation. More on this will follow in the next section.

4.5 Optimal Portfolios under Different Drivers

4.5.1 Geometric Brownian Motion

It is not hard to see that the process of the optimal portfolio under the geometric Brownian motion is another geometric Brownian motion. Substitute the expressions for optimal consumption and investment into (4.1) and we get a model of the form:

$$dX_t^{c^*,\pi^*,x} = a_{GBM} X_t^{c^*,\pi^*,x} dt + \sigma\pi^*_{GBM} X_t^{c^*,\pi^*,x} dW_t \tag{4.11}$$

where

$$a_{GBM} = r + (\mu_{GBM} + \frac{1}{2}\sigma^2 - r)\pi^*_{GBM} - \frac{\eta - k_{GBM}(\gamma)}{1-\gamma}.$$

The optimal portfolio thus follows the geometric Brownian motion

$$X_t^{c^*,\pi^*,x} = x\exp\left((a_{GBM} - \frac{1}{2}\sigma^2{\pi^*_{GBM}}^2)t + \sigma\pi^*_{GBM}W_t\right). \tag{4.12}$$

4.5.2 NIG and CGMY

This section, like section 4.4, will treat the cases where the driver of the stock price is a Lévy process L_t with no Gaussian part, but with a nonzero jump part, that is

$$L_t = \gamma_c t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\tilde{N}(ds, dz)$$

The dynamics of the optimal portfolio is

$$dX_t^{c^*,\pi^*,x} = aX_t dt + \pi^* X_{t-} \int_{\mathbb{R}\setminus\{0\}} (e^z - 1)\tilde{N}(dt, dz) \tag{4.13}$$

where

$$a = r + (\hat{\mu} - r)\pi^* - \frac{\eta - k(\gamma)}{1 - \gamma}.$$

It is possible to obtain a closed solution to this stochastic differential equation by applying Itô's formula for Lévy processes (see Øksendal/Sulem [11], p. 7). From this formula we get that the optimal portfolio at time t can be expressed as:

$$\begin{aligned} X_t^{c^*,\pi^*,x} =& x \exp\Big(a \cdot t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \Big(\ln\big(1 + \pi^*(e^z - 1)\big) - \pi^*(e^z - 1)\Big)\nu(dz)ds \\ &+ \int_0^t \int_{\mathbb{R}\setminus\{0\}} \ln\big(1 + \pi^*(e^z - 1)\big)\tilde{N}(ds, dz)\Big) \end{aligned} \tag{4.14}$$

Let us denote the exponent in this expression $Y_t = Y_t^{c^*,\pi^*,1} = \ln X_t^{c^*,\pi^*,1}$. If we let $\pi^* = 1$, Y_t should be of the same type as L_t and we see that this indeed is true because the integral involving the compensated Poisson random measure $\tilde{N}(ds, dz)$ then becomes $\int_0^t \int_{\mathbb{R}\setminus\{0\}} z\tilde{N}(ds, dz)$, the jump part of L_t.

Another question then arise. Is Y_t close to being a process of the L_t type even when $\pi^* < 1$? To answer this we may look at the Taylor series expansion of the integrand

$$f_\pi(z) := \ln\big(1 + \pi(e^z - 1)\big)$$

at $z = 0$. f_π is infinitely differentiable on $\mathbb{R}$ for any $\pi \in [0, 1]$. Taylors theorem thus states that for any $n \geq 0$:

$$f_\pi(z) = f_\pi(0) + f'_\pi(0)z + \frac{1}{2!}f''_\pi(0)z^2 + \cdots + \frac{1}{n!}f_\pi^{(n)}(0)z^n + R(n+1) \tag{4.15}$$

where the remainder term $R(n+1)$ satisfies

$$R(n+1) = \frac{f_\pi^{(n+1)}(\xi)}{(n+1)!} z^{n+1}$$

for some ξ between 0 and z.

By straightforward differentiation we get

$$f_\pi(z) = \ln\big(1 + \pi(e^z - 1)\big) = \pi z + \frac{1}{2}\pi(1-\pi)z^2 + \frac{1}{6}(2\pi^3 - 3\pi^2 + \pi)z^3 + R(4) \tag{4.16}$$

In figure 4.1 I have plotted $f_\pi(z)$ along with the first term in its Taylor series, πz, and the standard integrand in a Lévy process, z. We see that $|f_\pi(z)| \leq |z| \quad \forall z$ so that Y_t has jumps of smaller magnitude than L_t. $f'_\pi(z) = \frac{\pi e^z}{(1-\pi)+\pi e^z}$ tends to 1 when $z \to +\infty$ and to 0 when $z \to -\infty$. There is a good fit around 0 between $f_\pi(z)$ and πz, and since ν_{NIG} and ν_{CGMY} has most of its mass centered in this area it seems natural to use the approximation πz for $f_\pi(z)$ in the jump part of Y_t. πz gives smaller positive jumps than $f_\pi(z)$, but also negative jumps of greater magnitude. With πz instead of $f_\pi(z)$ the process X_t can be approximated as

$$X_t^{c^*,\pi^*,x} \approx \hat{X}_t^{c^*,\pi^*,x} := x \exp\left((a+i)t + \pi^* \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\tilde{N}(ds, dz)\right) \tag{4.17}$$

where $i = \int_{\mathbb{R}\setminus\{0\}} \left(\ln\left(1 + \pi^*(e^z - 1)\right) - \pi^*(e^z - 1)\right)\nu(dz)$. The exponent has a drift term and the same jump term as L_t, only scaled by the factor π^*.

I will now explain the similarities and differences of X_t in the GBM and the NIG/CGMY cases. Like in section 4.4 we can write the mean rate of return as

$$\hat{\mu} = \gamma_c + \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\nu(dz).$$

The deterministic part (and expected value) of Y_1 is then

$$\begin{aligned}
a + i = \quad & r + (\hat{\mu} - r)\pi^* - c + \int_{\mathbb{R}\setminus\{0\}} \Big(\ln\big(1 + \pi^*(e^z - 1)\big) - \pi^*(e^z - 1)\Big)\nu(dz) \\
= \quad & r(1 - \pi^*) - c + \pi^*\Big(\gamma_c + \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\nu(dz)\Big) \\
& + \int_{\mathbb{R}\setminus\{0\}} \Big(\ln\big(1 + \pi^*(e^z - 1)\big) - \pi^*(e^z - 1)\Big)\nu(dz) \\
= \quad & r(1 - \pi^*) - c + \pi^*\gamma_c + \int_{\mathbb{R}\setminus\{0\}} \Big(\ln\big(1 + \pi^*(e^z - 1)\big) - \pi^* z\Big)\nu(dz) \\
= \quad & r(1 - \pi^*) - c + \pi^*\gamma_c + \int_{\mathbb{R}\setminus\{0\}} \Big(\frac{1}{2}\pi^*(1 - \pi^*)z^2 + R(3)\Big)\nu(dz)
\end{aligned}$$

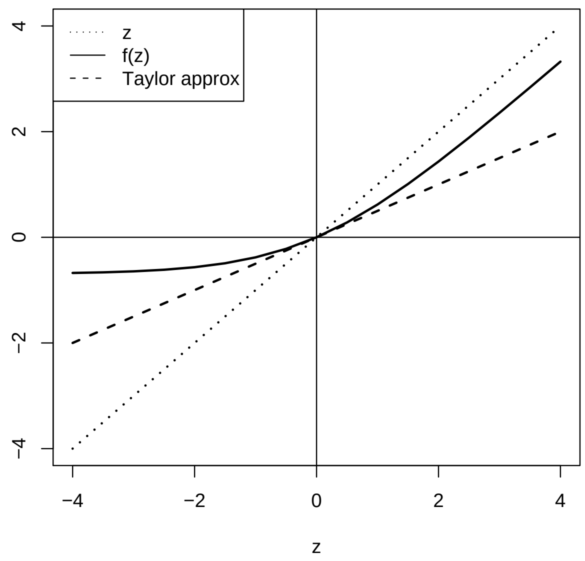


Figure 4.1: *The integrand $f_\pi(z)$ in the jump part of Y_t and its approximation πz when $\pi = 0.5$.*

where c denotes the consumption rate $\frac{\eta - k(\gamma)}{1-\gamma}$. In the final step I have substituted $f_{\pi^*}(z)$ by its Taylor expansion.

Let us compare the expected value of Y_t in the GBM case (i), and with L_t (NIG/CGMY) as the driver (ii):

(i) $\Big[r(1-\pi^*) - c + \pi^* \mu_{GBM} + \frac{1}{2}\sigma^2 \pi^*(1-\pi^*)\Big]t$

(ii) $\Big[(r(1-\pi^*) - c + \pi^*\gamma_c + \int_{\mathbb{R}\setminus\{0\}} \big(\frac{1}{2}\pi^*(1-\pi^*)z^2 + R(3)\big)\nu(dz)\Big]t$

Apart from the last terms, (i) and (ii) have the same form. In the next two chapters I shall compare the optimal portfolio with exponential NIG and CGMY respectively as alternatives to GBM. I then calibrate the parameters such that the mean and variance of $\ln S_t$ in both models coincide, i.e. $\mu_{GBM} = \gamma_c$ and $\sigma^2 = \mathrm{Var}[L_1]$. If we apply the same optimal (c^*, π^*) as well then the only difference between (i) and (ii) is the last term.

For any Lévy process we have

$$\mathrm{Var}[L_t] = t \int_{\mathbb{R}\setminus\{0\}} z^2 \nu(dz).$$

Thus if $\sigma^2 = \text{Var}[L_1]$, the only difference between (i) and (ii) is the term $t\int_{\mathbb{R}\setminus\{0\}} R(3)\nu(dz)$ in (ii). As we will see in the next section, integrals of this sort are often small compared to the other terms. $\nu_{NIG}(z)$ grows proportional to $|z|^{-2}$ and $\nu_{CGMY}(z)$ grows proportional to $|z|^{-(1+Y)}$ as $|z| \to 0$ (see appendix A). Thus in both NIG and CGMY $z^3\nu(z) \to 0$ as $z \to 0$.

We know that the stochastic terms of Y_t in cases (i) and (ii) are

(i) $\sigma\pi^* W_t$ and

(ii) $\int_0^t \int_{\mathbb{R}\setminus\{0\}} f(z)\tilde{N}(ds, dz) \approx \pi^* \int_0^t \int_{\mathbb{R}\setminus\{0\}} z\tilde{N}(ds, dz)$.

The variance in case (i) is $\pi^{*2}\sigma^2 t$ since $\text{Var}[W_t] = t$. In case (ii) we apply theorem B of appendix B. Let us put $\gamma(z) = f_\pi(z)$. Then since f is deterministic and $|f_\pi(z)| \le |z| \quad \forall z$, the conditions in theorem B are satisfied. Thus:

$$\begin{aligned}\text{Var}[Y_t] &= t\int_{\mathbb{R}\setminus\{0\}} f_{\pi^*}(z)^2\nu(dz) = t\int_{\mathbb{R}\setminus\{0\}} \Big(\ln\big(1+\pi^*(e^z-1)\big)\Big)^2\nu(dz) \\ &= t\int_{\mathbb{R}\setminus\{0\}} \Big(\pi^{*2}z^2 + \pi^{*2}(1-\pi^*)z^3 + R(4)\Big)\nu(dz) \\ &= \pi^{*2}\text{Var}[L_1]t + t\int_{\mathbb{R}\setminus\{0\}} \Big(\pi^{*2}(1-\pi^*)z^3 + R(4)\Big)\nu(dz)\end{aligned} \tag{4.18}$$

where I have substituted $\big(\ln\big(1+\pi^*(e^z-1)\big)\big)^2$ with its Taylor expansion. The variances in (i) and (ii) have similar forms except for the integral in (4.18). This term is not present in the approximation case. Thus if $\sigma^2 = \text{Var}[L_1]$, the log-returns have equal variance.

4.5.3 Numerical Examples

To verify the results of the previous section, and see how well approximation (4.17) works, we look at some examples where $\sigma^2 = \text{Var}[L_1]$ and the controls (c, π) are the same in both models.

Example A: Let $L_t \sim \text{NIG}(56.16,\ 2.641,\ -0.0006t,\ 0.015t)$. This process will be the main example in the next chapter and has a realistic set of parameters for stock log-return. For simplicity let us put $t = 1$. By the formula for variance of a NIG distributed variable in chapter 3.2, $\text{Var}[L_1] = 2.680 \cdot 10^{-4}$. Thus the variance of the approximation $\hat{Y}_1$ with $\pi = 0.5$ is $0.5^2 \cdot 2.680 \cdot 10^{-4} =$

$\pi = 0.5$	Real/Appr.	Var (+)	Var (-)	Var tot
Example A	Real	$3.59 \cdot 10^{-5}$	$3.12 \cdot 10^{-5}$	$6.708 \cdot 10^{-5}$
	Approx.	$3.55 \cdot 10^{-5}$	$3.15 \cdot 10^{-5}$	$6.700 \cdot 10^{-5}$
Example B	Real	$5.70 \cdot 10^{-5}$	$4.14 \cdot 10^{-5}$	$9.843 \cdot 10^{-5}$
	Approx.	$5.58 \cdot 10^{-5}$	$4.21 \cdot 10^{-5}$	$9.791 \cdot 10^{-5}$
Example C	Real	$3.75 \cdot 10^{-5}$	$6.69 \cdot 10^{-5}$	$1.044 \cdot 10^{-4}$
	Approx.	$3.68 \cdot 10^{-5}$	$6.91 \cdot 10^{-5}$	$1.059 \cdot 10^{-4}$

Table 4.1: *Decomposition of variance of one-day portfolio log-returns in the real process, X_t and in the approximated one, $\hat{X}_t$ due to positive (+) and negative (-) jumps.*

$6.700 \cdot 10^{-5}$, which is the variance in the GBM case as well. The variance of the real process Y_1 in the NIG case is calculated from (4.18) to be $6.708 \cdot 10^{-5}$, just slightly higher. Looking at variance coming from positive and negative jumps separately, the variance of positive jumps is lower in the approximation than in the real model (see table 4.1). With negative jumps it is the opposite. This can be explained by looking at figure 4.1 where we see that the approximation has smaller positive jumps, but also negative jumps of greater magnitude. The difference of the expectations of Y_1 in the GBM and exponential NIG cases is of order 10^{-9}, and indicates that the integral of the remainder term R(3) with respect to ν_{NIG} in Y_1 is very small.

Example B: Let $L_t \sim \text{NIG}(32.50, 3.560, -0.0015t, 0.0125t)$. This is a positively skewed distribution with high kurtosis (see section 5.7). The difference of the expectations of Y_1 in the two models is of order less than 10^{-8}. Other results are similar to example A and displayed below.

Example C: Let $L_t \sim \text{NIG}(25.85, -6.262, 0.003t, 0.01t)$. This is a very negatively skewed distribution with high kurtosis. See section 5.7 for more details. $\text{Var}[L_1] = 4.236 \cdot 10^{-4}$ and thus with, $\pi = 0.5$, the variance of Y_1 in the GBM case (and the approximation case) is $0.5^2 \cdot 4.236 \cdot 10^{-4} = 1.059 \cdot 10^{-4}$. $\text{Var}[Y_1]$ in the NIG case is calculated from (4.18) to be $1.044 \cdot 10^{-4}$. The difference of the expectations of Y_1 in the two models is of order 10^{-7}.

The tables 4.2-4.4 show variance, skewness and kurtosis of the logarithm of X_1 and $\hat{X}_1$ for different stock allocations π. The variance in example A and B is higher in the real process than in its approximation. That is because the Lévy process has positive skewness pushing more weight to the positive side where the real jumps are higher than the approximated ones. In example C,

Stock	Real/Approx.	$\pi = 0.1$	$\pi = 0.5$	$\pi = 0.9$
Example A	Real	$2.687 \cdot 10^{-6}$	$6.708 \cdot 10^{-5}$	$2.171 \cdot 10^{-4}$
	Approx.	$2.680 \cdot 10^{-6}$	$6.700 \cdot 10^{-5}$	$2.171 \cdot 10^{-4}$
Example B	Real	$3.958 \cdot 10^{-6}$	$9.843 \cdot 10^{-5}$	$3.175 \cdot 10^{-4}$
	Approx.	$3.916 \cdot 10^{-6}$	$9.791 \cdot 10^{-5}$	$3.172 \cdot 10^{-4}$
Example C	Real	$4.133 \cdot 10^{-6}$	$1.046 \cdot 10^{-4}$	$3.420 \cdot 10^{-4}$
	Approx.	$4.236 \cdot 10^{-6}$	$1.059 \cdot 10^{-4}$	$3.431 \cdot 10^{-4}$

Table 4.2: *Variance of portfolio log-returns in the real model and in the approximated one.*

Stock	Real/Approx.	$\pi = 0.1$	$\pi = 0.5$	$\pi = 0.9$
Example A	Real	0.233	0.198	0.163
	Approx.	0.154	0.154	0.154
Example B	Real	0.720	0.629	0.539
	Approx.	0.517	0.517	0.517
Example C	Real	-1.11	-1.26	-1.41
	Approx.	-1.45	-1.45	-1.45

Table 4.3: *Skewness of portfolio log-returns in the real model and in the approximated one.*

the process has negative skewness, it is the other way. The variances do not deviate much and converge as π increases.

Skewness and kurtosis in the approximation is invariant of π. It is not so in the real process. As we saw in figure 4.1 jumps are higher in the real process than in the approximation. Thus the real skewness is also higher, but it steadily decreases and converges to the approximation's when π increases. Kurtosis is similar to the variance in the sense that positive skewness makes the kurtosis of the real process higher and the opposite if skewness is negative.

The values in tables 4.2-4.4 tells us that the approximation (4.17) is good, at least when π is close to 1. Some further evidence of this is the QQ-plots in figure 4.2. I have simulated 200 thousand scenarios of 5 days portfolio log-return with both the real process X_5 and the approximation $\hat{X}_5$. The stock allocation is $\pi = 0.5$. (See the next chapter for simulation details.) Real log-returns appear to have slightly higher quantiles in both ends, especially in the skewness direction. This is probably due to it having higher jumps than the approximation. Still the conclusion must be that the approximation is good. For comparison I have also made QQ-plots of $\ln X_5$ in the NIG and GBM cases where the tail quantiles differ more.

Stock	Real/Approx.	$\pi = 0.1$	$\pi = 0.5$	$\pi = 0.9$
Example A	Real	3.69	3.64	3.60
	Approx.	3.60	3.60	3.60
Example B	Real	8.53	8.13	7.84
	Approx.	7.78	7.78	7.78
Example C	Real	12.68	13.42	14.45
	Approx.	14.77	14.77	14.77

Table 4.4: *Kurtosis of portfolio log-returns in the real model and in the approximated one.*

To sum up: The small differences in both the expectations and variances of Y_t indicates that the Lévy integral terms z^n of order $n \geq 3$ is rather insignificant. I have presented an approximation to the optimal portfolio X_t in the Lévy case, expressed quite similarly to the GBM case, and perhaps more easily interpreted. This approximation was shown to be quite good for different kind of stocks.

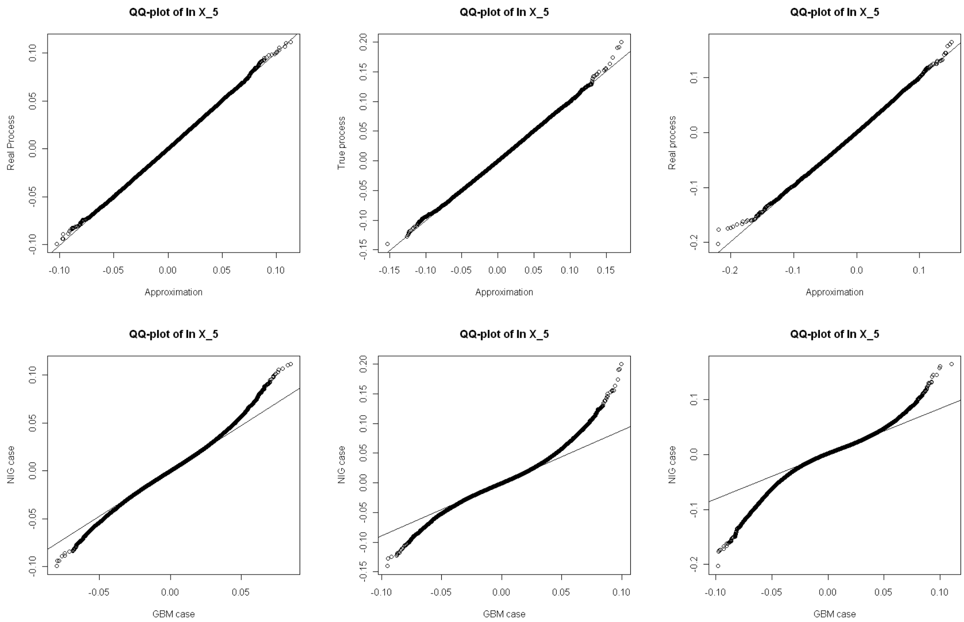


Figure 4.2: *QQ plot of portfolio log-returns over 5 days in examples A (left), B (center) and C (right). Stock allocation* $\pi = 0.5$. *Top: The real* $\ln X_5$ *in NIG case vs its approximation* $\ln \hat{X}_5$. *Bottom:* $\ln X_5$ *in the NIG vs GBM.*

Chapter 5

GBM vs. Exponential NIG

5.1 Introduction

In this chapter I will analyze how the choice of stock price model affects the risk/return profile of the optimal portfolio of the Merton problem. Comparisons will be made between the geometric Brownian motion and the exponential NIG model. I will employ the risk measures Value-at-Risk and conditional Value-at-Risk (expected shortfall).

5.2 Risk Measures

There are several definitions of a risk measure. Acerbi and Tasche defines in [2] a risk measure to be a mapping $\rho : V \mapsto \mathbb{R}$, where V is a given set of random variables, such that ρ is

(i) (monotonous) $X \in V, X \geq 0 \quad \Rightarrow \quad \rho(X) \leq 0$

(ii) (sub-additive) $X, Y, X + Y \in V \quad \Rightarrow \quad \rho(X + Y) \leq \rho(X) + \rho(Y)$

(iii) (positively homogeneous) $X \in V, h > 0, hX \in V \quad \Rightarrow \rho(hX) = h\rho(X)$

(iv) (translation invariant) $X \in V, a \in \mathbb{R} \quad \Rightarrow \quad \rho(X + a) = \rho(X) - a$

This is really a definition of a family of risk measures called *coherent risk measures*. Not all risk measures are coherent. Value-at-Risk does not satisfy the second property of sub-additivity. I will still use VaR as an example of a risk measure since it satisfies the the other three conditions and is a market norm for measuring risk.

5.2.1 Value-at-Risk (VaR)

Value-at-risk is a commonly used measure of financial risk. In this thesis I will use the following definition of VaR:

Definition 5.1. *Let X_t be a stochastic variable representing the value of a portfolio at time t. Then we say that the Value-at-Risk of the portfolio at risk level q is:*

$$VaR_q(X_t) = \inf\{x \in \mathbb{R} : P(X_0 - X_t > x) \leq q\} \tag{5.1}$$

where X_0 is the initial value of the portfolio.

Hence the probability of a greater loss than VaR_q is exactly q. Value-at-Risk is not a coherent risk measure, since it does not satisfy the condition of subadditivity.

5.2.2 Conditional Value-at-Risk (cVaR)

Value-at-Risk has been criticized of not providing info of how bad things really *can* go when they do. A way to measure this is the conditional Value-at-Risk (cVaR), also called expected shortfall or expected tail loss. I will use the definition given in Acerbi/Tasche [2].

Definition 5.2. *Let X_t be a stochastic variable representing the value of a portfolio at time t. Then we say that the conditional Value-at-Risk of the portfolio at risk level q is:*

$$cVaR_q(X_t) = \mathbb{E}[X_0 - X_t | X_t \leq x_q] \tag{5.2}$$

where x_q is the q-quantile in the distribution of X_t.

The cVaR is the mean of the losses that are worse that the VaR. It it therefore more sensitive to the extreme values in the tail of the loss distribution.

5.3 Problems

I assume there exists a set of log-return data to which both a normal distribution, used in the geometric Brownian motion, and a NIG distribution used in the exponential NIG model are fitted. Given an interest rate intensity r and a discount rate η the fitted models leads to sets of optimal controls (c^*_{GBM}, π^*_{GBM}) and (c^*_{NIG}, π^*_{NIG}) for every risk aversion coefficient $1 - \gamma$. I want to answer two main questions concerning risk/return:

1. What is the difference in VaR and cVaR when we compare portfolios with the same pair (c^*, π^*), but with different underlying models for stock return?

2. What is the difference in VaR and cVaR when we compare portfolios with different optimal controls (c^*_{GBM}, π^*_{GBM}) and (c^*_{NIG}, π^*_{NIG}), but with a fixed underlying model?

5.4 Methods

I have implemented results from the article [5] by Benth et al. where they compare solutions to the Merton problem under the GBM and the exponential NIG. To obtain a set of realistic parameters, they use maximum likelihood methods to fit both a NIG model and a normal one to daily log-return data of Norsk Hydro on the New York Stock Exchange (NYSE) ranging from 1990 through 1998. I will use this example throughout the entire chapter. The estimated NIG-parameters were: $\alpha = 56.16, \beta = 2.641, \mu_{NIG} = -0.0006$ and $\delta = 0.015$. This giving an expected daily rate of return of $\hat{\mu} = 2.403 \cdot 10^{-4}$ (6.01% yearly assuming 250 trading days). The distribution has a skewness of 0.15 and a kurtosis of 3.60 (see appendix B for the definition of these).

The MLE for the normal distribution were $\mu_{GBM} = 0.000101$ and $\sigma = 0.0166$ leading to an expected rate of return of $2.388 \cdot 10^{-4}$ (5.97% yearly).

The interest rate intensity was taken to be $r = 2 \cdot 10^{-4}$ and $\eta = 0.06$ though not necessarily true for the given period. This easily leads to π^*_{GBM} and c^*_{GBM} through the formulas (4.8) and (4.9). It it is harder though to calculate π^*_{NIG} and c^*_{NIG} considering (4.5) and (4.7).

5.4.1 Deriving the Optimal NIG Portfolio

For deriving the optimal NIG portfolio there are integrals involving the Lévy measure $\nu_{NIG}(dz)$ over $\mathbb{R} \setminus \{0\}$ that needs to be evaluated. The measure has most of its mass centered around zero as is seen in figure 5.1. I used the R function "integrate" with a cutoff at ± 2 and the build in Bessel function. The cutoff should provide a good degree of precision, see the appendix for a detailed discussion of this. To solve (4.5), which here takes the form

$$f(\pi) = (\hat{\mu} - r) + \int_{\mathbb{R}\setminus\{0\}} \big((1 + \pi(e^z - 1))^{\gamma-1} - 1\big)(e^z - 1)\nu_{NIG}(dz) = 0 \tag{5.3}$$

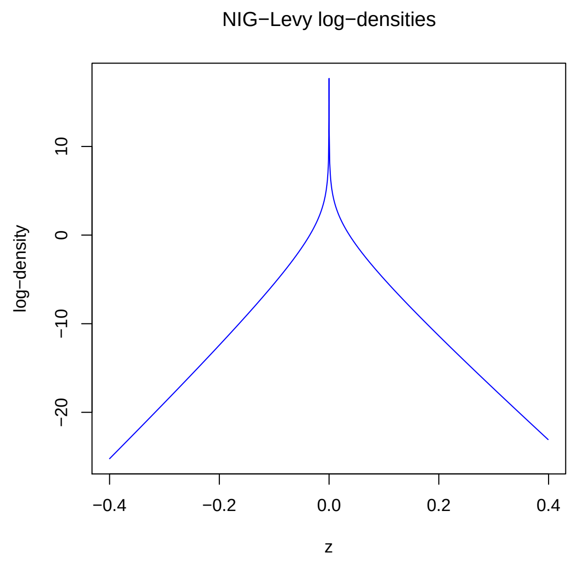


Figure 5.1: *Density of the Lévy measure $\nu(z)$ on log-scale for the parameters of Norsk Hydro.*

where

$$\hat{\mu} = \mu_{NIG} + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}} + \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z)\nu_{NIG}(dz) \tag{5.4}$$

I use the method of bisection. The interval $[0, 1]$ was bisected 14 times to reach a precision of $2^{-14} = 6.1 \cdot 10^{-5}$ of π^*_{NIG}.

5.5 Simulation

Throughout this chapter I let the initial wealth X_0 of the investor be 1 so that VaR and cVaR are the fraction of wealth that is lost. To evaluate the VaR and cVaR of the NIG portfolio, I employ Monte Carlo simulation of the stock returns. Simulation of NIG variables has been well developed, e.g. by Rydberg [12] , and will not be a topic in this thesis. I use the R package called "fBasics" for generating NIG variables. Figure 5.2 indicates that the MC simulated NIG variables follow the wanted NIG distribution well. To have a reasonable approximation to the continuous re-allocation of the optimal portfolio I choose to draw log-returns and re-allocate with one-day intervals. This was found to be of sufficient precision.

To obtain a fair comparison of the portfolio performance under the GMB and the exponential NIG model in question 1 I calibrate the models such

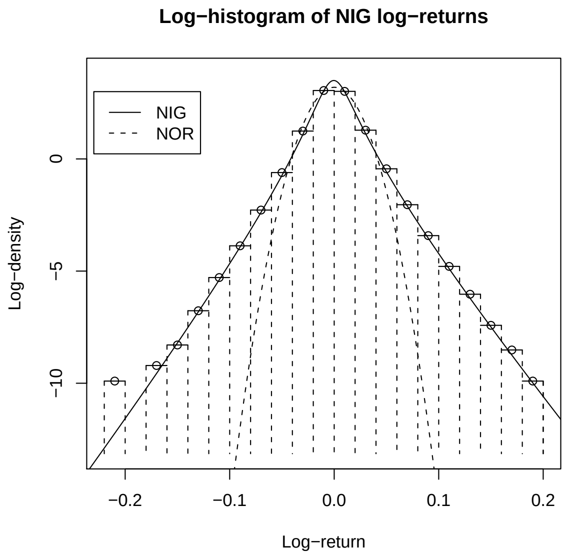


Figure 5.2: *Log-histogram of simulated daily log-returns for Norsk Hydro under the NIG model. The full line shows the true log-density. The dashed line is the corresponding normal log-density curve.*

that the mean and variance of log-returns coincide. I simply exchange the ML estimates μ_{GBM} and σ with the mean and standard deviation of the estimated NIG distribution. With these new parameters in the GBM I simulate the development of a portfolio under both models. Below is the results when the optimal investment proportion was π^*_{GBM} as an investor believing in the GBM would apply. Since the time horizon is infinite consumption rates are small, ranging from 0.024% for $\gamma = 0.10$ to 0.036% for $\gamma = 0.95$. The investor believing in the GBM had a slightly higher c^*, most noticeable for γ's close to one. The differences however were small, and I have therefore chosen not to include consumption in the results. The tables below contain VaR and cVaR at levels 1% and 5 %. I have also included results on the right tail behavior of the return distribution. P_q denotes the upper q quantile and TP_q ("Tail Profit") denotes the mean of the outcomes better than P_q.

For the 5 days horizon we see that there are distinct differences in both $\mathrm{VaR}_{0.01}$ and $\mathrm{cVaR}_{0.01}$ when looking at question 1. With all wealth invested in the stock $\mathrm{VaR}_{0.01}$ is 8.45% with NIG and 8.14% with GBM, a relative difference of 4%. The cVaR at this level is 10.00% and 9.30% respectively, a relative difference of 7.5%. We see from figure 5.3 that this difference distributes itself smoothly over all the γ's, which is to be expected from the results of section 4.5.2. At level 5% we notice that the VaR is actually slightly higher for the

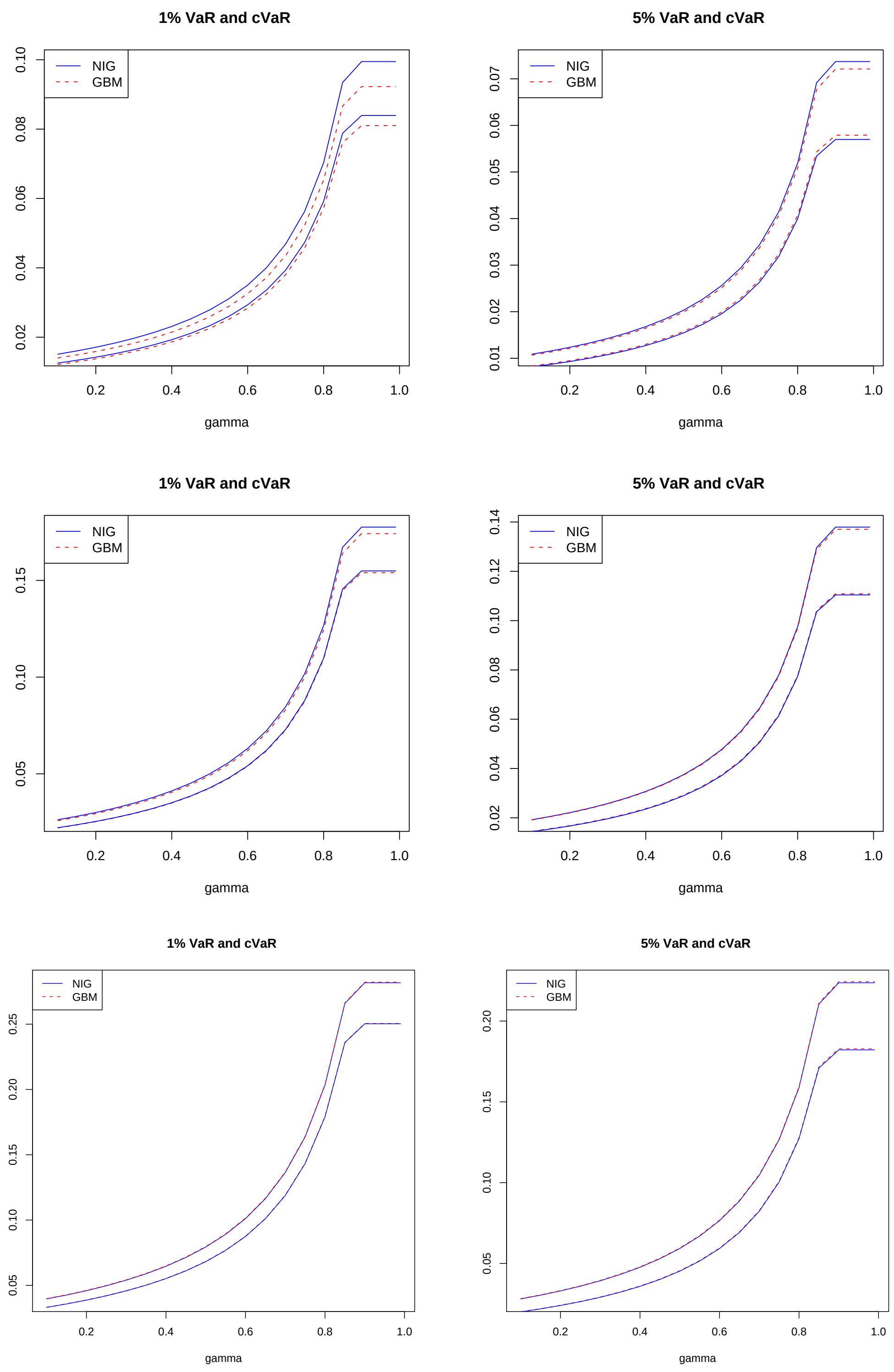


Figure 5.3: *The same weights, but different underlying models, (question 1). Left side:* $VaR_{0.01}$ *and* $cVaR_{0.01}$*. Right side:* $VaR_{0.05}$ *and* $cVaR_{0.05}$*. Top: 5 days horizon. Center: 20 days. Bottom: 60 days. Exponential NIG is the full line and GBM is the dashed line.*

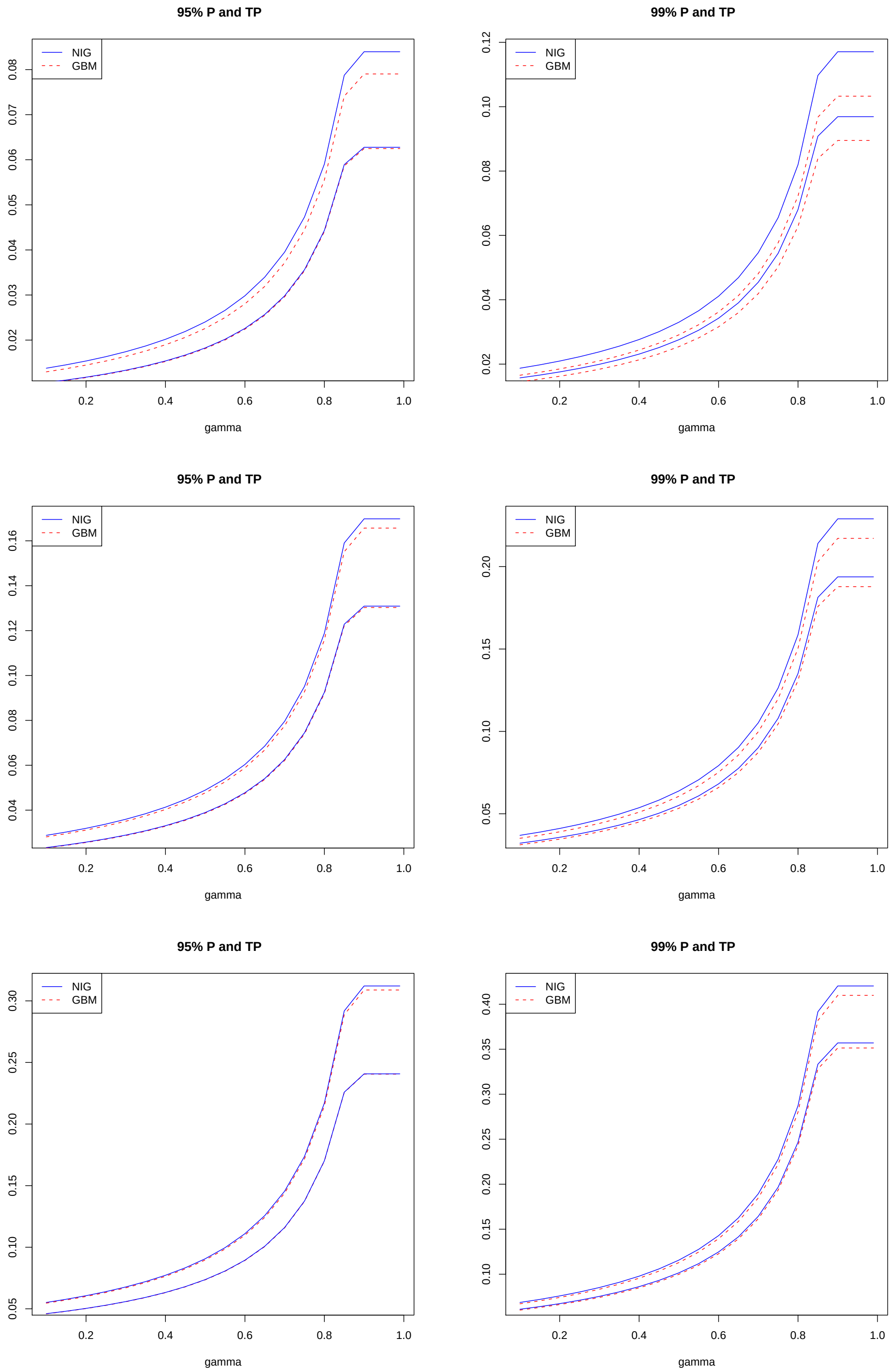


Figure 5.4: *Left side: $P_{0.95}$ and $TP_{0.95}$. Right side: $P_{0.99}$ and $TP_{0.99}$. Top: 5 days horizon. Center: 20 days. Bottom: 60 days. Exponential NIG is the full line and GBM is the dashed line.*

	$\pi = 15.6\%$		$\pi = 56.3\%$		$\pi = 100\%$	
$t = 5$	VaR_q	cVaR_q	VaR_q	cVaR_q	VaR_q	cVaR_q
GBM, q=0.01	1.22	1.41	4.59	5.27	8.14	9.30
NIG, q=0.01	1.27	1.52	4.77	5.66	8.45	10.00
GBM, q=0.05	0.83	1.07	3.23	4.07	5.78	7.22
NIG, q=0.05	0.81	1.09	3.17	4.16	5.69	7.39
	P_q	TP_q	P_q	TP_q	P_q	TP_q
GBM, q=0.95	1.05	1.30	3.55	4.46	6.27	7.93
NIG, q=0.95	1.06	1.37	3.56	4.72	6.27	8.38
GBM, q=0.99	1.46	1.66	5.03	5.78	8.98	10.35
NIG, q=0.99	1.57	1.87	5.45	6.56	9.70	11.73

Table 5.1: *Simulation results answering question 1 (where the underlying models are different, but we apply the same weights). Time horizon is 5 days. The three weights correspond to* $\gamma = 0.10, 0.75$ *and* $\gamma > 0.90$ *respectively. All results are in %.*

	$\pi = 15.6\%$		$\pi = 56.3\%$		$\pi = 100\%$	
$t = 20$	VaR_q	cVaR_q	VaR_q	cVaR_q	VaR_q	cVaR_q
GBM, q=0.01	2.22	2.60	8.77	10.04	15.44	17.51
NIG, q=0.01	2.23	2.64	8.84	10.20	15.58	17.81
GBM, q=0.05	1.46	1.92	6.20	7.78	11.13	13.78
NIG, q=0.05	1.45	1.93	6.16	7.81	11.07	13.84
	P_q	TP_q	P_q	TP_q	P_q	TP_q
GBM, q=0.95	2.32	2.81	7.41	9.30	13.04	16.59
NIG, q=0.95	2.33	2.87	7.43	9.49	13.06	16.92
GBM, q=0.99	3.12	3.51	10.46	12.00	18.79	21.75
NIG, q=0.99	3.21	3.68	10.76	12.57	19.31	22.76

Table 5.2: *Simulation results answering question 1 (where the underlying models are different, but we apply the same weights). Time horizon is 20 days. The three weights correspond to* $\gamma = 0.10, 0.75$ *and* $\gamma > 0.90$ *respectively. All results are in %.*

	$\pi = 15.6\%$		$\pi = 56.3\%$		$\pi = 100\%$	
$t = 60$	VaR_q	cVaR_q	VaR_q	cVaR_q	VaR_q	cVaR_q
GBM, q=0.01	3.33	3.98	14.34	16.42	25.05	28.19
NIG, q=0.01	3.32	3.97	14.33	16.40	25.03	28.21
GBM, q=0.05	2.02	2.82	10.09	12.71	18.29	22.43
NIG, q=0.05	2.00	2.81	10.04	12.66	18.22	22.37
	P_q	TP_q	P_q	TP_q	P_q	TP_q
GBM, q=0.95	4.59	5.46	13.72	17.18	24.05	30.88
NIG, q=0.95	4.61	5.52	13.75	17.38	24.08	31.21
GBM, q=0.99	6.01	6.71	19.35	22.22	35.14	40.99
NIG, q=0.99	6.10	6.86	19.67	22.78	35.70	42.03

Table 5.3: *Simulation results answering question 1 (where the underlying models are different, but we apply the same weights). Time horizon is 60 days. The three weights correspond to* $\gamma = 0.10, 0.75$ *and* $\gamma > 0.90$ *respectively. All results are in %.*

GBM than for the NIG, 5.78% vs 5.69% at $\pi = 1$. However, when looking at the cVaR, the heavier tail of the NIG push this difference the opposite way. From figure 5.3 we see that the VaR and cVaR quickly converge when we increase the time horizon. At the bottom, where the time horizon is 60 days, or a quarter of a year, the differences between the curves are barely noticeable. This can also be seen in table 5.3.

Figure 5.4 shows the potential for high return under the two models. I have plotted the upper 95% and 99% quantiles of the return distribution along with the associated tail profits as defined above. At the 95% level, the two models are very similar, but NIG has higher tail profit. The difference is still small, but more distinct at level 99%. This is also shown in tables 5.1-5.3 where we see for example that $\text{P}_{0.99}$ is 35.70% for NIG and 35.14% for GBM at 60 days and $\pi = 1$. Regarding question 1 we may conclude that for the Norsk Hydro stock we don't have to look at long time horizons before the differences in risk/return gets very small indeed.

What about question 2? In figures 5.5 and 5.6 I have plotted VaR, cVaR, P and TP when different optimal weights is applied to the same underlying model (NIG). This leads to a horizontal shift between the curves of the GBM and the exponential NIG. Table 5.4 compares the results when $\gamma = 0.75$. Here $\pi^*_{NIG} = 60.0\%$ and $\pi^*_{GBM} = 56.3\%$. The relative differences stays almost constant with t as opposed to the situation in question 1. For $t = 5$ $\text{VaR}_{0.01}$ is 5.06% with NIG and 4.74% with GBM, a 7% relative difference. For $t = 60$ the values are 15.33% and 14.33%, also a 7% relative difference.

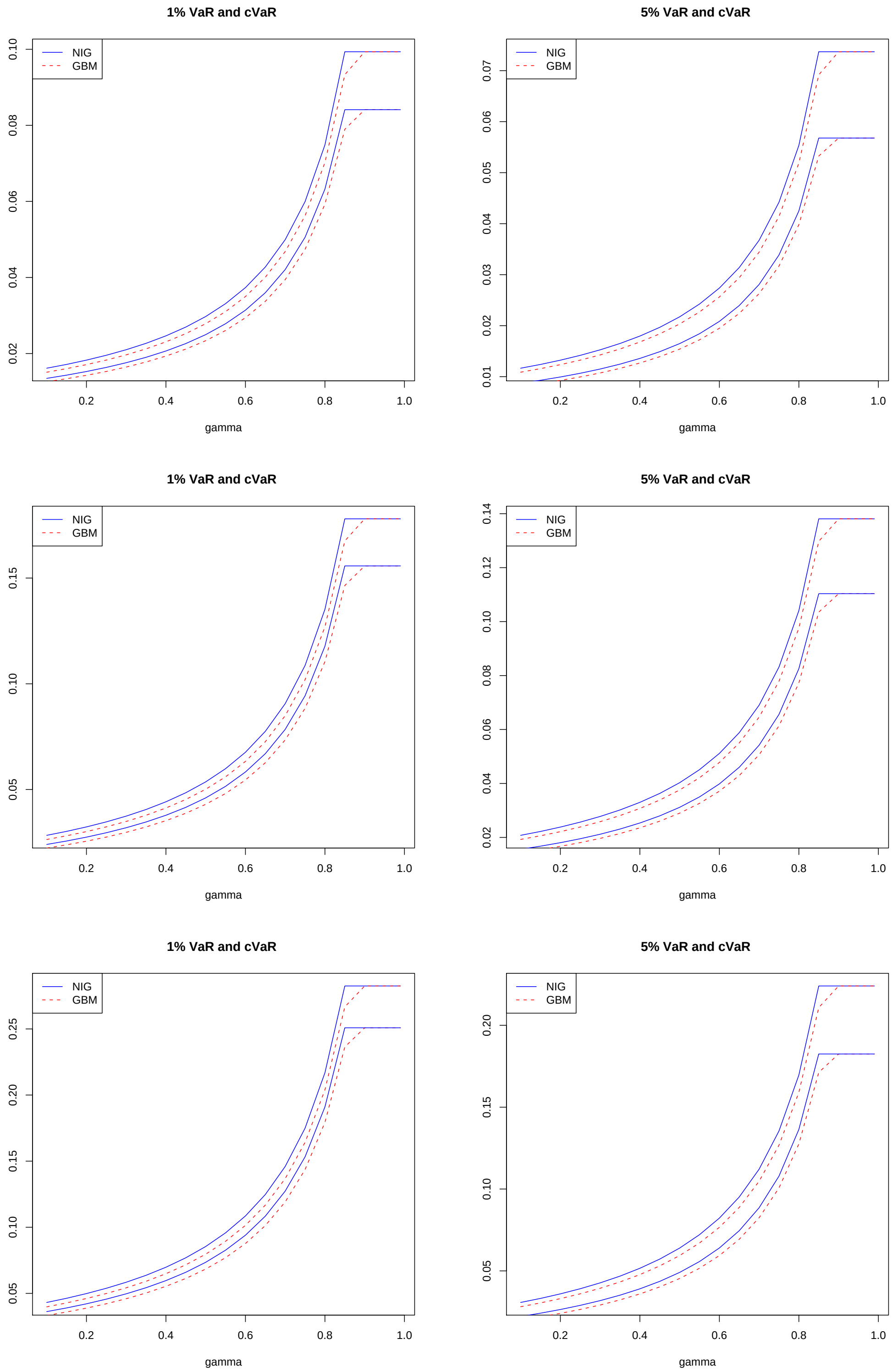


Figure 5.5: *Results with different weights, but the same underlying model. Left side:* $VaR_{0.01}$ *and* $cVaR_{0.01}$*. Right side:* $VaR_{0.05}$ *and* $cVaR_{0.05}$*. Top: 5 days horizon. Center: 20 days. Bottom: 60 days. Exponential NIG is the full line and GBM is the dashed line.*

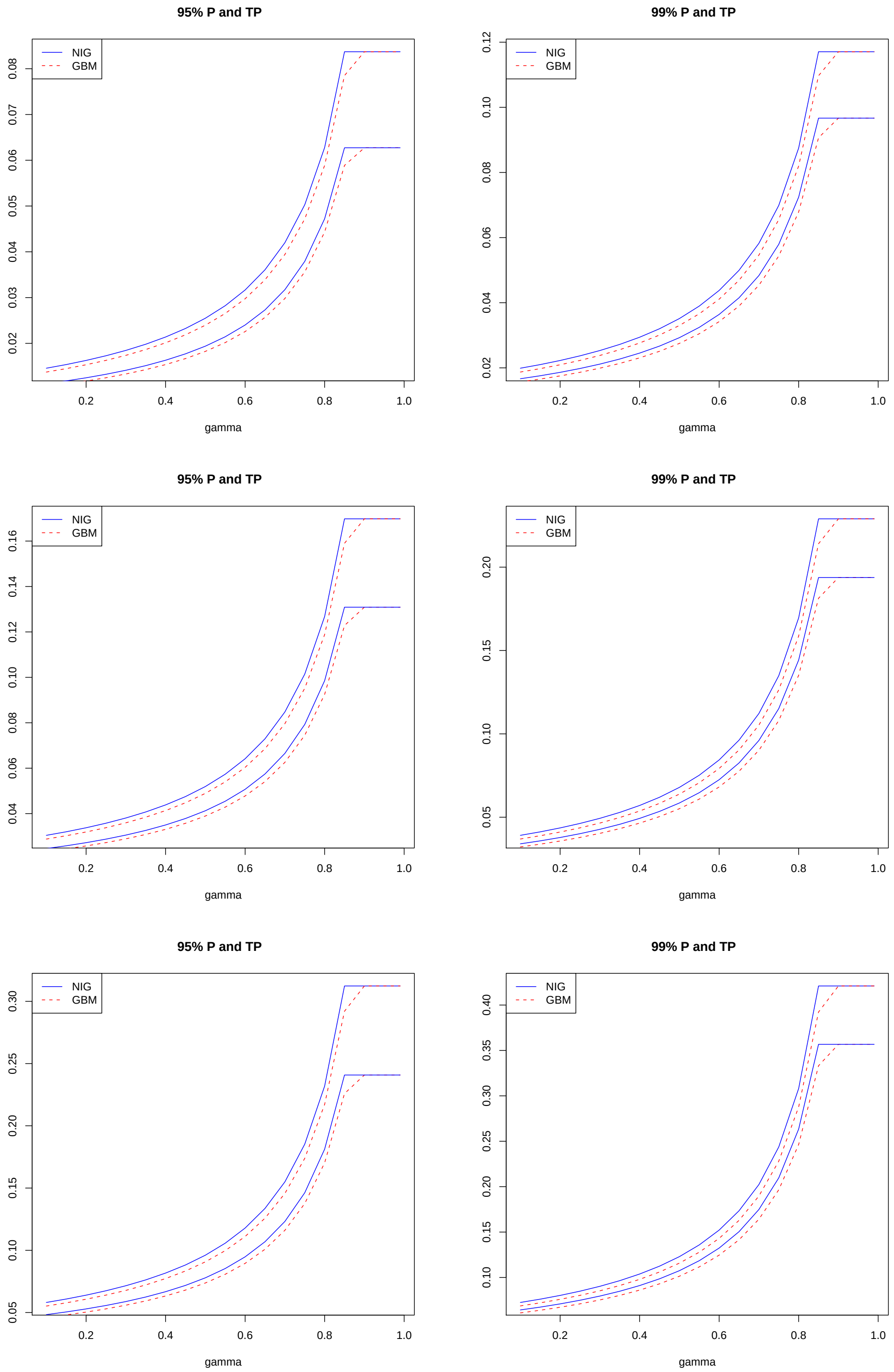


Figure 5.6: *Left side: $P_{0.95}$ and $TP_{0.95}$. Right side: $P_{0.99}$ $TP_{0.99}$. Top: 5 days horizon. Center: 20 days. Bottom: 60 days. Exponential NIG is the full line and GBM is the dashed line.*

$\pi_{GBM} = 56.3\%$	$t = 5$		$t = 20$		$t = 60$	
$\pi_{NIG} = 60.0\%$	VaR_q	cVaR_q	VaR_q	cVaR_q	VaR_q	cVaR_q
GBM, q=0.01	4.74	5.62	8.84	10.19	14.33	16.37
NIG, q=0.01	5.06	6.00	9.43	10.87	15.33	17.50
GBM, q=0.05	3.17	4.14	6.14	7.79	10.07	12.68
NIG, q=0.05	3.39	4.42	7.45	8.32	10.79	13.55
	P_q	TP_q	P_q	TP_q	P_q	TP_q
GBM, q=0.95	3.56	4.72	7.45	9.52	13.76	17.39
NIG, q=0.95	3.79	5.03	7.93	10.14	14.62	18.53
GBM, q=0.99	5.44	6.55	10.80	12.64	19.66	22.81
NIG, q=0.99	5.80	6.99	11.52	13.49	20.97	24.33

Table 5.4: *Simulation results answering question 2 (with the same underlying model, but different weights). Time horizon is 5, 20 and 60 days. The two weights correspond to* $\gamma = 0.75$*. All results are in %.*

This fact is also evident for cVaR, P, and TP at all levels. For all of these categories the values with π^*_{NIG} is about 6.5-7% higher relative to the values with π^*_{GBM}. This applies to all time horizons all well. The reason for this difference is obviously that $\pi^*_{NIG} > \pi^*_{GBM}$, making the portfolio more risky. π^*_{NIG} is actually about 6.6% higher than π^*_{GBM}. Based on the answer to question 1 we can say that for time periods longer than a few days, the effect of weights being different far outweighs the effect of the underlying models being different. We shall later see that with an interest rate closer to the mean rate of return, weights would differ more they do in this example.

5.6 Convergence to the Normal Distribution

The NIG distribution has the property of being closed under convolution. If $L_1 \sim NIG(\alpha, \beta, \mu, \delta)$ then $L_t = NIG(\alpha, \beta, \mu t, \delta t)$. As a consequence of the central limit theorem, the distribution of L_t tends to a normal one with the same mean and variance when t increases. For large t we have that

$$L_t \sim N\left((\mu + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}}) \cdot t, \frac{\delta\alpha^2}{(\alpha^2 - \beta^2)^{3/2}} \cdot t \right) \tag{5.5}$$

Figure 5.7 shows the convergence of the distribution NIG(56.16, 2.641, $-0.0006t$, $0.015t$) to the normal distribution with the same mean and variance. From the plots of one day log-returns ($t = 1$) we see that the NIG distribution clearly has heavier tails and more mass near the mean than the

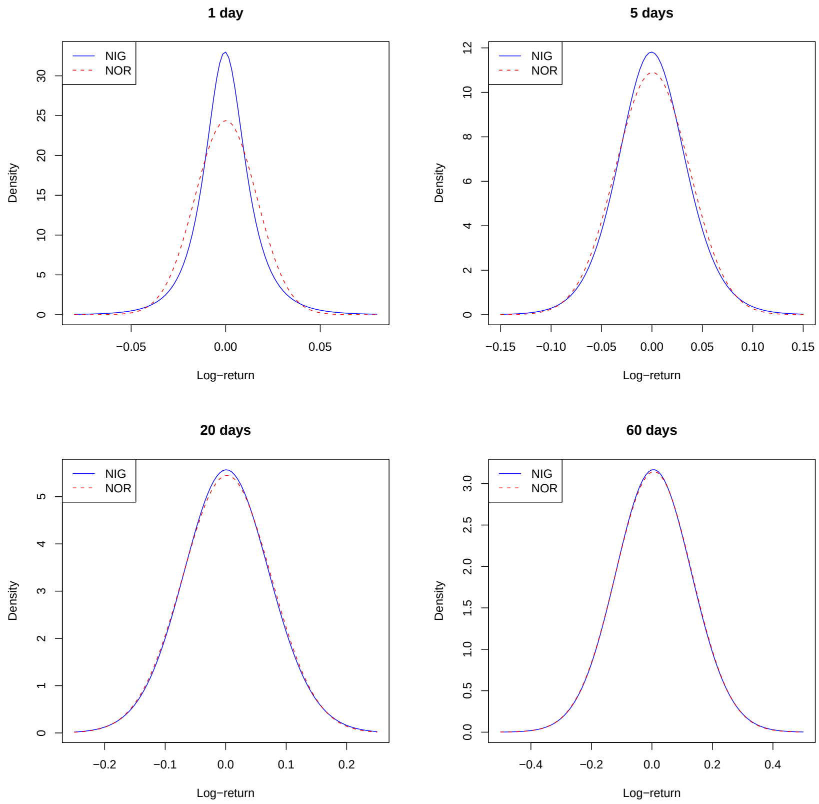


Figure 5.7: *Log-return distribution for Norsk Hydro under the exponential NIG model and the GBM.*

normal distribution. But the distribution converges quite quickly. For $t = 20$ the difference is seemingly very low, and even more so for $t = 60$. Figure 5.8, where log-densities are plotted, gives a better impression of the tail behavior of the two distributions.

Another view on convergence speed is illustrated in figures 5.9 and 5.10. Figure 5.9 shows the difference between the quantiles of the NIG and the corresponding normal distribution for $t = 1, 5, 20$ and 125. This difference is in absolute terms, and we see that the scales on the y-axes do not change much. It seems like increasing t provides a shift of same magnitude for both distributions. Notice how the range of quantiles where the NIG is the larger (where the graph is above the horizontal line) shift downwards. This is due to the positive skewness of this NIG distribution, implying that the NIG median is below the normal one.

Figure 5.10 provides more details on the tail behavior. In the upper part I have plotted the difference of the lowest ten per cent quantiles. With $t = 1$

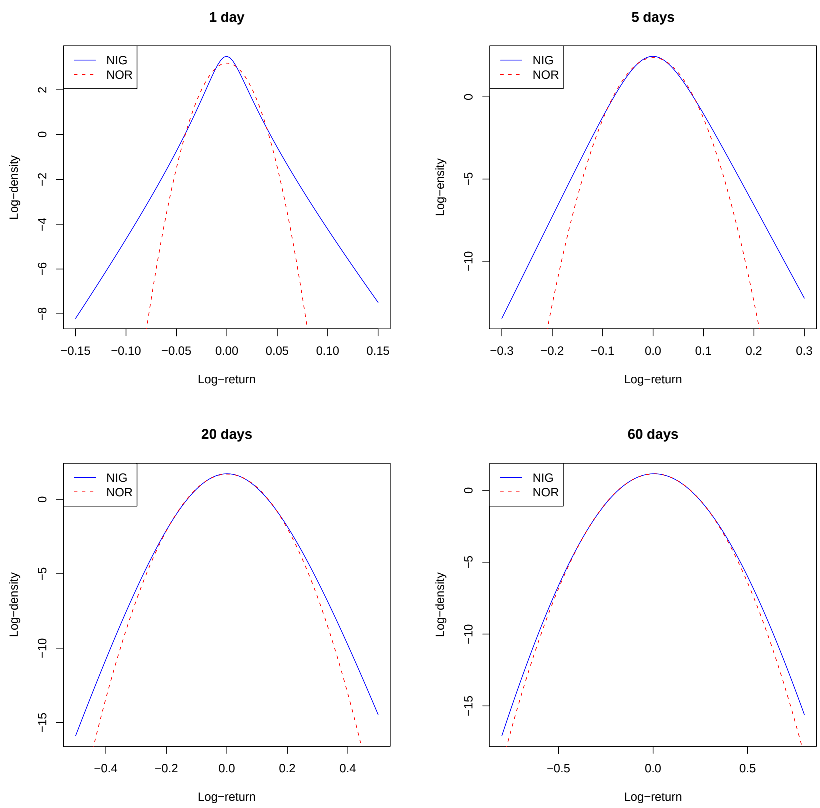


Figure 5.8: *Log-return log-density for Norsk Hydro under the exponential NIG model and the GBM.*

the NIG distribution has the most extreme quantile for levels 3.2% and lower. With $t = 125$ the same limit is 0.3%. This is interesting as it determines which of the two models has the highest Value-at-Risk. For long time horizons and with the parameters as in the Norsk Hydro example, you must go to extremely low levels q of VaR_q to find the portfolio following the NIG model to be the most risky.

I have also plotted the difference of the highest 15 per cent quantiles. Since the NIG in this example has positive skewness, you don't have to go as far out to see the NIG quantile being the most extreme. If the NIG model had had negative skewness instead, circumstances would have been the opposite. More of the lower NIG quantiles would have been smaller than the normal ones. We shall see examples of this in the next section. In "normal times" of the financial world most stocks probably has positive skewness, see Benth [4].

Figure 5.10 shows the difference in quantiles relative to the magnitude

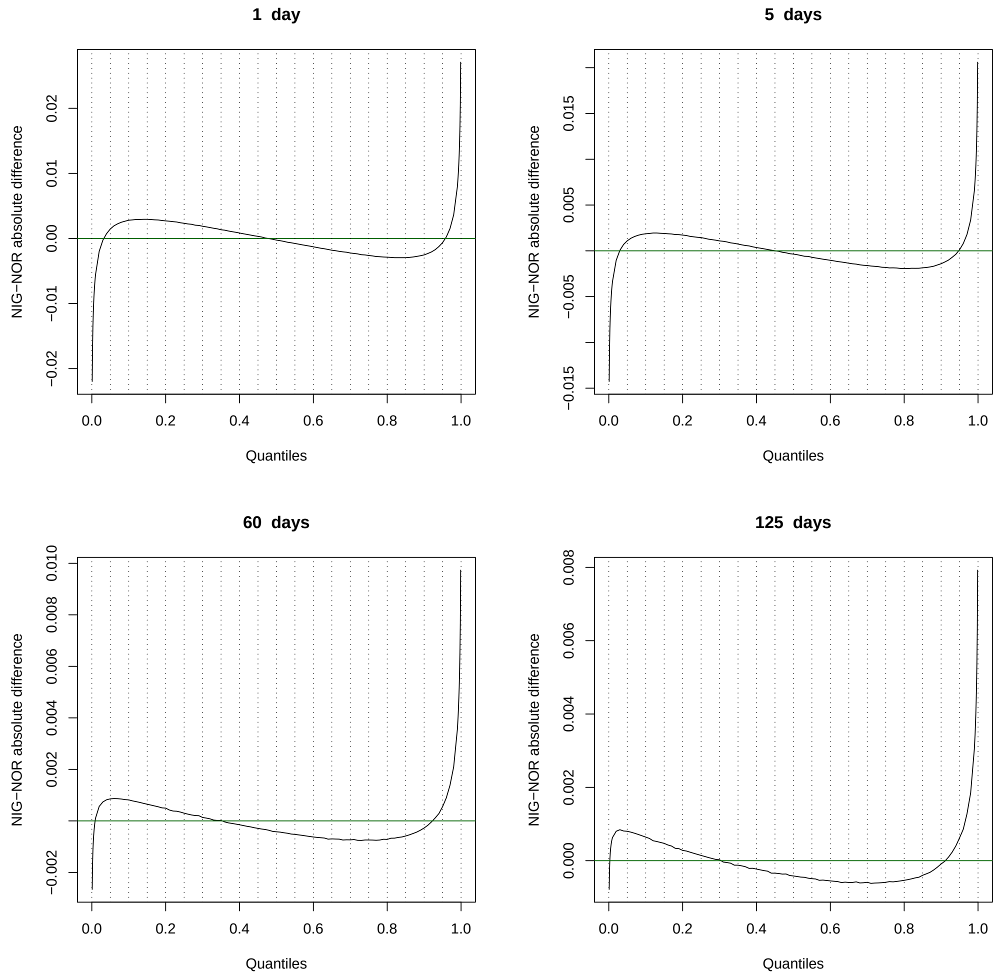


Figure 5.9: *Absolute difference in quantiles of log-returns under the exponential NIG model and the GBM. Mind the scales on the y-axes! Quantiles included: 0.1%-99.9%.*

(absolute value) of the normal quantile. Notice on the scales on the Y-axes how fast the quantiles converge. If we consider the 99% quantiles the NIG is 20.8% higher for $t = 1$, 7.6% higher for $t = 5$, 2.7% higher for $t = 20$ and 0.5% higher for $t = 200$. The convergence is even faster in the other end. With the 0.01% quantiles the NIG is 70.7% lower for $t = 1$, 21.3% lower for $t = 5$, 5.6% lower for $t = 20$ and 0.5% lower for $t = 125$. This indicates once again that for longer time horizons, it does not matter much weather it's the GBM or the NIG-model that the stock follows. It all comes down to the strategy we put in and with that the calculation of optimal weights π^*.

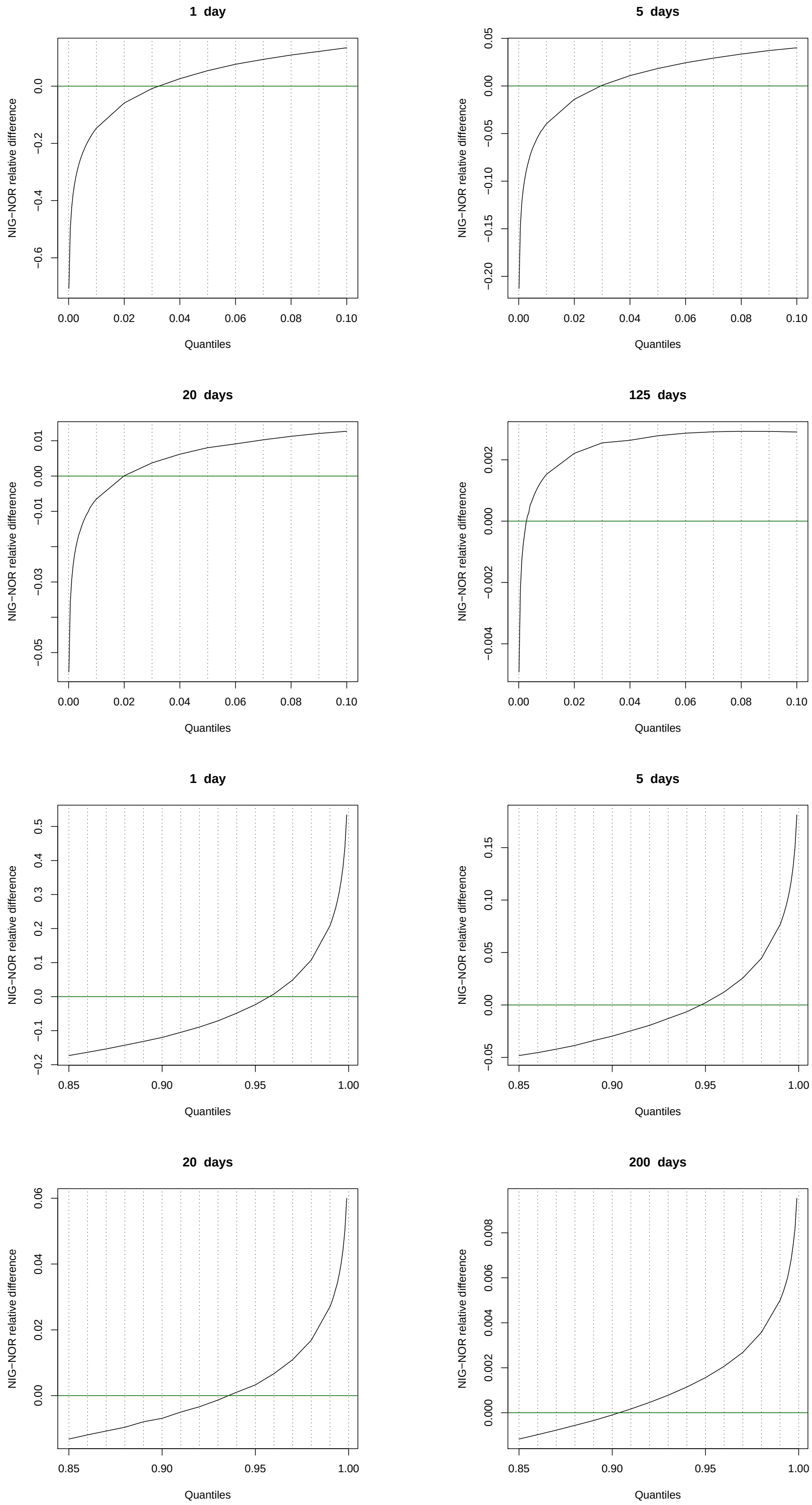


Figure 5.10: *Relative difference in tail quantiles of log-returns under the exponential NIG model and the GBM. Mind the scales on the y-axes!*

	α	β	μ	δ
Stock 1	56.16	2.641	-0.0006	0.0150
Stock 2	32.50	3.560	-0.0015	0.0125
Stock 3	49.07	-10.10	0.0060	0.0250
Stock 4	25.85	-6.262	0.0030	0.0100

Table 5.5: *Parameters of the NIG Lévy processes driving the stocks in section 5.7.*

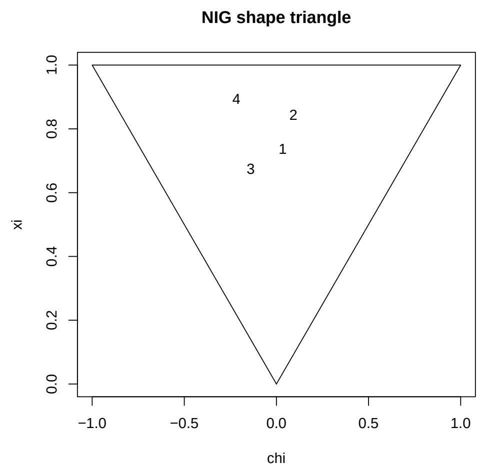


Figure 5.11: *NIG shape triangle with the coordinates given by stocks 1-4.*

5.7 Results for Other Stocks

Let us see if the results obtained in the previous sections are confirmed when we look at stocks of various levels of skewness and kurtosis. We will focus on four stocks with parameters given in table 5.5. Stock 1 is Norsk Hydro introduced earlier. Stock 2 has a fairly high positive skewness and kurtosis. Stock 3 has a fairly high negative skewness, but low kurtosis while Stock 4 has a very high negative skewness and kurtosis. The "NIG shape triangle" (see e.g. Rydberg [12]) provides a nice way to illustrate the shape of a NIG distribution. Stocks 1-4 are shown in the "NIG shape triangle" in figure 5.11 and should cover most shapes that are common for stocks.

Since both the normal distribution and the NIG distribution has explicitly given distribution functions it is possible to obtain the exact Value-at-Risk and cVaR for a portfolio with all wealth invested in the stock. To compare the difference in the exponential NIG and GBM I again calibrate sets of normal parameters such that the mean and variance of log-returns of the stocks

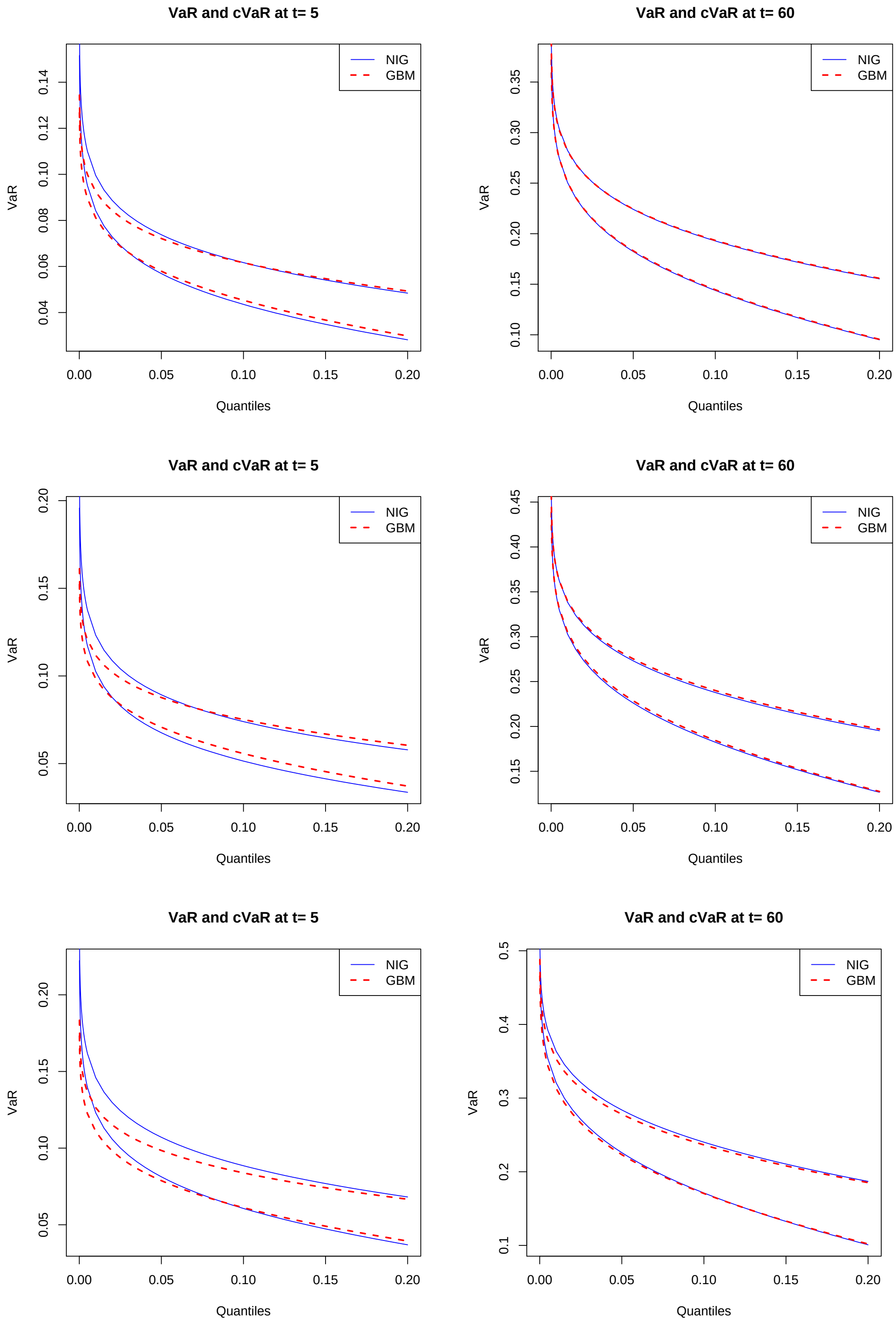


Figure 5.12: *VaR_q and $cVaR_q$ for different levels of q with the exponential NIG model (full lines) and the corresponding GBM (dashed lines). Top: Stock 1, Center: Stock 2, Bottom: Stock 3.*

	Mean	Std dev (yearly)	Skewness	Kurtosis
Stock 1	0.00011	0.0164 (25.9%)	0.15	3.60
Stock 2	-0.00012	0.0198 (31.3%)	0.52	7.79
Stock 3	0.00074	0.0233 (36.9%)	-0.56	2.92
Stock 4	0.00050	0.0206 (32.5%)	-1.45	14.77

Table 5.6: *Mean, standard deviation, skewness and kurtosis of unit time increments of the NIG Lévy processes driving the stocks in section 5.7.*

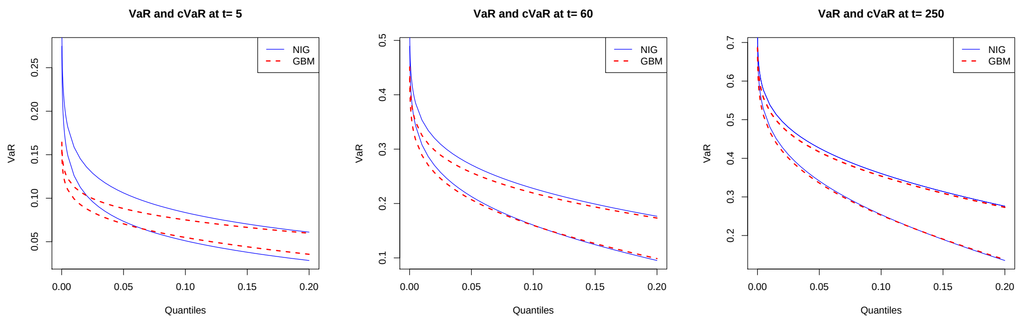


Figure 5.13: *Stock 4: VaR_q and $cVaR_q$ for different levels of q.*

under the two models coincide. The plots on the left hand side in figure 5.12 show VaR_q and cVaR_q of stocks 1-3 on a 5 days horizon for different levels q. Stock 3 has a negative skewness which make the VaR and cVaR in the NIG case higher for low q's. In the plots on the right hand side the time horizon is 60 days. The differences here are clearly smaller, especially for the stocks with positive skewness (see the table below where the values of VaR, cVaR in the GBM are given in fraction of the NIG values). Stock 4 has somewhat more extreme levels of skewness and kurtosis. The evolvement of VaR and cVaR with time is shown in figure 5.13. At $t = 5$ the $\text{VaR}_{0.01}$ and $\text{cVaR}_{0.01}$ of the corresponding GBM model are just 78.8% respectively 71.2% of those of the NIG model. However the convergence of the two models are quite fast. At $t = 60$ the $\text{VaR}_{0.01}$ in the GBM case is 93.7% of that in the NIG and at $t = 250$ (1 year) the $\text{VaR}_{0.01}$ in GBM is 97.3% of $\text{VaR}_{0.01}$ in the NIG. We see that even with NIG driven stocks that have NIG parameters far from normality, with high skewness and/or kurtosis, the levels of VaR and cVaR still approach the VaR and cVaR of the corresponding GBM at a fast pace.

t	5		60		250	
NIG	$\mathrm{VaR}_{0.01}$	$\mathrm{cVaR}_{0.01}$	$\mathrm{VaR}_{0.01}$	$\mathrm{cVaR}_{0.01}$	$\mathrm{VaR}_{0.01}$	$\mathrm{cVaR}_{0.01}$
Stock 1	8.42	9.95	25.06	28.24	43.71	48.28
GBM in %	96.3	92.8	100.0	99.8	100.1	100.1
Stock 2	10.25	12.32	30.29	33.85	52.92	57.43
GBM in %	95.9	90.6	100.7	100.3	100.5	100.4
Stock 3	12.31	14.60	32.08	36.36	49.47	55.32
GBM in %	90.1	86.6	97.6	97.1	98.9	98.8
Stock 4	12.60	15.88	30.81	35.35	48.13	53.84
GBM in %	78.8	71.2	93.7	92.0	97.3	96.8

Table 5.7: *$VaR_{0.01}$ and $cVaR_{0.01}$ in per cent at time horizons t=5, 60 and 250 days. GBM values displayed as % of NIG values.*

5.8 Parameter Dependence

5.8.1 Explaining the Different Weights

With the NIG and normal parameters in the Norsk Hydro case and with daily interest rate $r = 2 \cdot 10^{-4}$, I obtained optimal weights π^*_{GBM} and π^*_{NIG} shown in table 5.8. π^*_{NIG} is here around 6.6% higher than π^*_{GBM}. Benth et al. show in [5] that the relative difference in weights increases as r tends to the mean rate of return of the stocks, $\hat{\mu}$.

It might be tricky to get hold of this difference by comparing $f(\pi) = 0$ in the NIG and GBM cases, the latter with solution

$$\pi^*_{GBM} = \frac{\mu_{GBM} + \frac{1}{2}\sigma^2 - r}{\sigma^2(1-\gamma)}. \tag{5.6}$$

In section 4.4 I pointed out the similarities of these two equations. Results indicated that putting μ and σ, corresponding to the mean and standard deviation of the NIG distribution, into (5.6) would lead to well approximated NIG-weights, at least if skewness and kurtosis are small. This is the case with Norsk Hydro (see column three of table 5.8). In the GBM $\hat{\mu}_{GBM} = 2.3878 \cdot 10^{-4}$. With NIG, $\bar{\mu}$:=mean+ $\frac{1}{2}$ sd^2 $= 2.4017 \cdot 10^{-4}$. In addition the NIG distribution has positive skewness giving an extra contribution to the mean rate of return $\hat{\mu}$. We have $\hat{\mu} = 2.4029 \cdot 10^{-4} > 2.4017 \cdot 10^{-4} = \bar{\mu}$. This difference is low compared do the difference $\hat{\mu} - \hat{\mu}_{GBM}$, but might cause the mentioned approximation of $f(\pi) = 0$ to be less precise for high r. With $r = 2 \cdot 10^{-4}$ this is not a problem.

It is now easy to see that the difference in weights is due to the higher mean rate of return in the NIG model combined with the NIG model having lower variance, $2.68 \cdot 10^{-4}$ vs $2.76 \cdot 10^{-4}$ in the Gaussian one. With $r = 2 \cdot 10^{-4}$ $\frac{\bar{\mu}-r}{\hat{\mu}_{GBM}-r} = \frac{4.017}{3.878} = 1.036$ and $(\frac{\sigma}{\text{sd}[NIG]})^2 = 1.028$. Multiplying these gives 1.065 which is close to the real ratio between π^*_{NIG} and π^*_{GBM}. With r tending to $\hat{\mu}_{GBM}$ the first ratio grows and scales up the differences. With lower values of r, the relation between $\bar{\mu} - r$ and $\hat{\mu}_{GBM} - r$ is more balanced and the difference in variance becomes more significant. If for example the estimated GBM has a larger rate of return, but also larger volatility, then $\pi^*_{GBM} > \pi^*_{NIG}$ for large r, but possibly $\pi^*_{GBM} < \pi^*_{NIG}$ for small r. Such a case was found:

Based on daily closing prices of General Motors (GM) on the New York Stock Exchange from January 2, 1962 to December 31, 2006, giving a total of 11,327 daily log-returns[1], the methods "nigFit" and "nFit" from R's package "fBasics" provided NIG and normal ML-estimates. $\hat{\mu}$ of NIG was $3.79 \cdot 10^{-4}$ (9.47% yearly) and $\hat{\mu}_{GBM}$ was $3.82 \cdot 10^{-4}$ (9.55% yearly). Standard deviation of the estimated NIG was 0.0169 (26.8% yearly) and σ was 0.0171 (27.0% yearly). Skewness and kurtosis was 0.24 and 3.50, quite similar to Norsk Hydro. With $r = 0.0002$ (5 % yearly), π^*_{NIG} is about 0.3% (relative) higher than π^*_{GBM}. With $r = 0.00035$ (8.75% yearly), π^*_{GBM} is about 6.7% higher.

γ	π_{GBM}(%)	π_{NIG} (%)	π_{NIG} (%) approx
0.10	15.64	16.67	16.65
0.20	17.59	18.75	18.73
0.30	20.10	21.43	21.41
0.40	23.46	25.00	24.98
0.50	28.15	30.00	29.98
0.60	35.18	37.51	37.47
0.65	40.21	42.87	42.83
0.70	46.91	50.02	49.96
0.75	56.29	60.04	59.96
0.80	70.37	75.06	74.94
0.85	93.82	>100	99.92
0.90	>100	>100	>100

Table 5.8: *Optimal weights in the risky asset of Norsk Hydro. The third column is the weights given by the approximation method.*

[1] Obtained from Yahoo Finance [14]

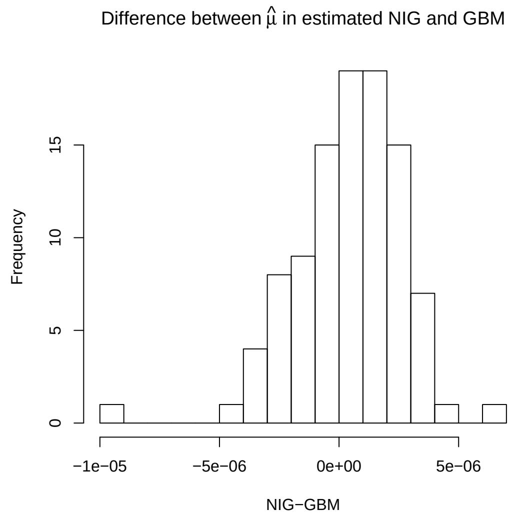


Figure 5.14: *Histogram of* $\hat{\mu} - \hat{\mu}_{GBM}$.

5.8.2 Parameter Uncertainty

How does parameter uncertainty influence π^*_{GBM} and π^*_{NIG}? We know empirical log-returns are fitted well by a NIG distribution and data sets of size 1000 might be common for parameter estimation. I therefore employed the estimated NIG parameters of General Motors to make 100 data sets, each containing 1000 simulated NIG log-returns. For each set I fitted both a NIG and a normal distribution. The 100 estimated $\hat{\mu}_{GBM}$ varied from -0.001 to 0.0015 (-27% to 38% yearly), a huge spread indicating the uncertainty in parameter estimates (especially due to the mean). My focus though is on model deviations. The difference between $\hat{\mu}$ and $\hat{\mu}_{GBM}$ in the 100 estimated exponential NIG and GBM's varied from about $-5 \cdot 10^{-6}$ to $5 \cdot 10^{-6}$ (see histogram). Hence, given that true log-returns really followed the NIG from GM data, both the exponential NIG and the GBM could have been assumed to have the greatest mean rate of return. If the data set utilized for parameter estimation is not yet realized, it thus seems quite random which of the investors, one believing in the GBM, the other the exponential NIG, will have the greatest optimal stock allocation, and thus taking the greatest risk. We also see that (absolute) differences of the mean rates of return in the fitted models are quite small ($\pm$ 0.13% yearly). To have significantly different optimal weights in the Merton problem the interest rate r has to be quite close to the estimated mean rate of return.

Chapter 6

GBM vs. Exponential CGMY

6.1 Introduction

This chapter will treat much of the same problems as the previous chapter, but this time with the CGMY instead of the NIG as driver of the stock price. I will present ways of simulating such processes and discuss in what ways an exponential CGMY model differs from the GBM and also the exponential NIG.

6.2 The Variance Gamma Process

The Variance Gamma (VG) is the member of the family of CGMY processes with $Y = 0$. A VG process L_t (without drift as in (3.2)) can be expressed as the difference of two Gamma stochastic processes or as a time changed Brownian motion with drift where the time change (a nondecreasing Lévy process) follows a Gamma process. I will use the latter form and write

$$L_t = \theta G_t^\nu + \sigma W(G_t^\nu) \tag{6.1}$$

where $\theta \geq 0$, $G_t^\nu \sim$ Gamma(with mean t and variance νt) and W is a Brownian motion. This representation makes it simple to simulate a VG process since Gamma generators are build into most mathematical computer software. A VG process L_t as above has the following expressions for expectation, variance, skewness and kurtosis in terms of θ, ν and σ:

$\mathbb{E}[L_t]$	θt
$\text{Var}[L_t] = \kappa_2$	$(\sigma^2 + \theta^2\nu)t$
$\text{Skewness}[L_t]$	$\kappa_2^{-3/2} \cdot (2\theta^3\nu^2 + 3\sigma^2\theta\nu)t$
$\text{Kurtosis}[L_t]$	$\kappa_2^{-2} \cdot (3\sigma^4\nu + 6\theta^4\nu^3 + 12\sigma^2\theta^2\nu^2)t$

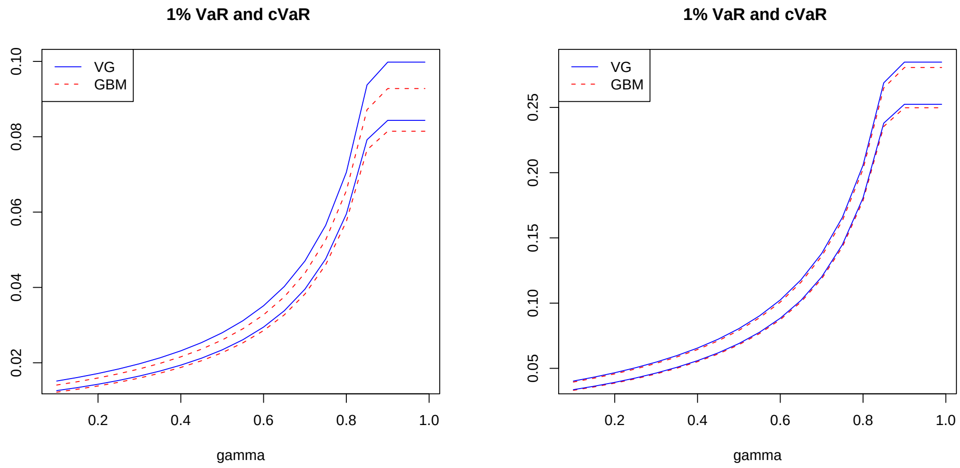


Figure 6.1: *$VaR_{0.01}$ and $cVaR_{0.01}$ at 5 days (left) and 60 days (right). All values are in percent.*

The connection between θ, ν, σ and C, G, M are as follows (see Carr et al. [7]):

$$C = \frac{1}{\nu}, \qquad G = \frac{1}{\eta_n} \qquad \text{and} \qquad M = \frac{1}{\eta_p}$$

where $\eta_n = \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} - \frac{\theta\nu}{2}$ and $\eta_p = \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} + \frac{\theta\nu}{2}$.

By choosing the parameters $\theta = 7.04 \cdot 10^{-4}, \nu = 1.1937, \sigma = 0.01635$ and adding a drift term bt such that $\gamma_c = \theta + b = 1.06 \cdot 10^{-4}$, the resulting VG process has exactly the same expectation, variance, skewness and kurtosis as the NIG process of Norsk Hydro in the previous chapter. Is there then any real difference between the VG in this case and the NIG? To answer this we again turn to the scheme in question 1 of section 5.3 where the same weights π were applied to both the exponential Lévy model and the corresponding GBM. As in the previous chapter I let the daily interest rate r be 0.0002. Figure 6.1 shows the $\text{VaR}_{0.01}$ and $\text{cVaR}_{0.01}$ of the optimal solution to the Merton problem, and table 6.1 shows the values when all wealth is invested in stocks. Results on the upper 99% P and TP are also included. We see that the plots have the same shape as in the NIG case and the values are very similar to the NIG results. The optimal weights π^*_{VG} was found to be almost identical as well.

$t = 5$	$\text{VaR}_{0.01}$	$\text{cVaR}_{0.01}$	$\text{P}_{0.99}$	$\text{TP}_{0.99}$
GBM	8.11	9.24	8.89	10.27
VG	8.46	9.98	9.71	11.72
$t = 60$				
GBM	25.05	28.17	35.13	41.25
VG	25.08	28.21	35.57	42.05

Table 6.1: *$VaR_{0.01}$, $cVaR_{0.01}$ and $P_{0.99}$, $TP_{0.99}$ for portfolios with π=1 in the comparison with VG. Values are in %.*

6.3 CGMY with $Y \neq 0$

6.3.1 Simulation Approach

To simulate a CGMY process L_t with $Y \neq 0$ proved more difficult than the VG case. I have simulated CGMY's with $Y > 0$ and will present the algorithm used. The Lévy-Itô decomposition (2.2) expresses L_t as the sum of a deterministic drift term, a compensated compound Poisson process of small jumps and a compound Poisson process of big jumps (remember the Gaussian part is zero):

$$L_t = bt + \int_0^t \int_{|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN(ds, dz)$$

If $Y < 0$ we remember that L_t has finite activity, i.e. finitely many jumps, which allows us to not compensate for small jumps, but write the jump part of L_t as a pure compound Poisson process. A general algorithm for simulating such processes are as follows (Cont and Tankov [8], p. 174) :

Algorithm 6.1 (Compound Poisson simulation). *If L_t is a pure compound Poisson process one can generate it in this way:*

- *Simulate the number N of jumps in the interval $[0, T]$. N is Poisson distributed with intensity λT, where $\lambda = \nu(\mathbb{R} \setminus \{0\})$.*
- *The N jumps $Y_i, \quad i = 1, \ldots, N$ are i.i.d. with law $\frac{\nu(dz)}{\lambda}$.*
- *For $i = 1, \ldots, N$, let $U_i \sim$ Unif(0, T), where U_i are independent, represent the jump times.*
- *The process at $t \in [0, T]$ is then $L_t = \sum_{i=1}^N \mathbf{1}_{U_i<t} Y_i$.*

For the simulations in this thesis I have chosen to re-allocate with daily intervals. Increments over this time frame are obtained by applying the algorithm with $T = 1$. The actual jump times are insignificant for our purpose so point 3 is omitted. What stands as the difficult part of the algorithm is point 2, i.e. to simulate from the density $\frac{\nu(z)}{\lambda}$. I will not go into this for $Y < 0$, but focus on the cases where $Y > 0$.

For processes with $Y \in (0, 2)$ there are infinitely many small jumps. One then choose a truncation level $\epsilon > 0$ that is close to 0. Jumps lower than ϵ in absolute size is not simulated and the compensated term $\int_0^t \int_{|z|<\epsilon} z\tilde{N}(ds, dz)$ of L_t is thus not included.

It might be smart to split the compound Poisson process into a sum of positive jumps and negative jumps. The expected number of positive jumps greater than ϵ on a unit time interval is

$$\nu_{pos}^{\epsilon} := C \int_{\epsilon}^{\infty} \frac{e^{-Mz}}{z^{Y+1}} dz.$$

To simulate from the distribution with density $p^{\epsilon}(z) := \frac{\nu(z)}{\nu_{pos}^{\epsilon}} \mathbf{1}_{z>\epsilon}$ I use a method described in Cont/Tankov [8], p. 188. Since

$$p^{\epsilon}(z) \leq f^{\epsilon}(z) \frac{Ce^{-M\varepsilon}}{Y\epsilon^{Y}\nu_{pos}^{\epsilon}},$$

where $f^{\epsilon}(z) = \frac{Y\epsilon^{Y}}{z^{Y+1}} \mathbf{1}_{z>\epsilon}$ is a density function, we may apply acceptance-rejection sampling. We simulate X from f^{ϵ} by drawing $U \sim$ Unif(0,1) and letting $X = F^{-1}(1-U) = \epsilon U^{-1/Y}$ where F is f's distribution function. Each X is then accepted with probability $\frac{p^{\epsilon}(X)}{K \cdot f^{\epsilon}(X)}$ where $K = \frac{Ce^{-M\epsilon}}{Y\epsilon^{Y}\nu_{pos}^{\epsilon}}$. An accepted variable X has the distribution p^{ϵ}. Simulating negative jumps is perfectly similar.

To make the approximation better we can use a result from Asmussen and Rosiński [3]. They prove that the error in the approximation of L_t,

$$\begin{aligned} R_t^{\epsilon} &= L_t - \Big[bt + \int_0^t \int_{|z|>\epsilon} zN(ds, dz) - t \int_{\epsilon<|z|<1} z\nu(dz)\Big] \\ &= \int_0^t \int_{|z|<\epsilon} z\tilde{N}(ds, dz), \end{aligned} \tag{6.2}$$

tends to a Brownian motion when the truncation level ϵ tends to 0. We know $\mathbb{E}[R_t^{\epsilon}] = 0$ and from theorem B.1 $\text{Var}[R_t^{\epsilon}] = t \int_{|z|<\epsilon} z^2 \nu(dz)$, but [3] also provides a condition by which the entire distribution is obtained. The precise statement is:

Theorem 6.1. *Let* $\sigma(\epsilon) = \int_{|z|<\epsilon} z^2 \nu(dz)$*. If* $\frac{\sigma(\epsilon)}{\epsilon} \to \infty$ *as* $\epsilon \to 0$ *then*

$$\frac{1}{\sigma(\epsilon)} R_t^\epsilon \to W_t \text{ in distribution as } \epsilon \to 0$$

where W_t *is a Brownian motion.*

With CGMY $\sigma(\epsilon) \sim \epsilon^{1-Y/2}$ near 0, so the condition in the theorem is satisfied with $Y > 0$. The approximation is particularly good when Y is close to 2. In that case the activity just around zero is very high and the variation this bring is well handled by adding the term $\sigma(\epsilon)W_t$.

6.3.2 Some Results

We will look at two examples where I have applied the simulation approach described above.

Example A Let L_t have the parameters $C = 0.11$, $G = 45$, $M = 64$, $Y = 0.5$ and $\gamma_c = 0.00074$. This distribution has about the same mean, standard deviation, skewness and kurtosis as stock 3 of the previous chapter. The precise values are $\text{sd}[L_1] = 0.0227$, $\text{Skew}[L_1] = -0.54$ and $\text{Kurt}[L_1] = 2.93$. With these parameters the daily mean rate of return $\hat{\mu}$ on the stock is $9.957 \cdot 10^{-3}$ (24.89% yearly). To match this high rate of return I choose the daily interest rate $r = 0.0009$ (22.5% yearly). Solving the Merton problem then provides optimal weights π^*_{CGMY} ranging from 20.9% for $\gamma = 0.10$ to 100% for γ's greater than 0.85. The optimal weights in the GBM where log-returns has equal mean and variance is only marginally higher, e.g. 94.2% vs. 93.5% for $\gamma = 0.80$. As before I leave out consumption.

The left side of figure 6.2 shows the resulting $\text{VaR}_{0.01}$ and $\text{cVaR}_{0.01}$ at a 5 days horizon for the different risk levels γ. As was also seen in section 5.7 the negative skewness of the stock provides a greater difference in VaR and cVaR than with positive skewness. At $t = 60$ much of the difference has disappeared as we notice in the plot on the right hand side of figure 6.2. Like the case with VG, the shape is similar to the GBM/NIG plots of the previous chapter. Simulations were done with 500 thousand scenarios and the results resemble those of stock 3 of section 5.7. Table 6.2 displays $\text{VaR}_{0.01}$, $\text{cVaR}_{0.01}$, $\text{P}_{0.99}$ and $\text{TP}_{0.99}$ with a portfolio where all wealth is allocated in the stock. At $t = 5$ $\text{VaR}_{0.01}$ in the CGMY case is 11.94%. The corresponding GBM value is 10.79% which is 90.4% of the CGMY. At $t = 60$ the $\text{VaR}_{0.01}$ in the GBM case is 30.50%, 97.6% of that of the CGMY.

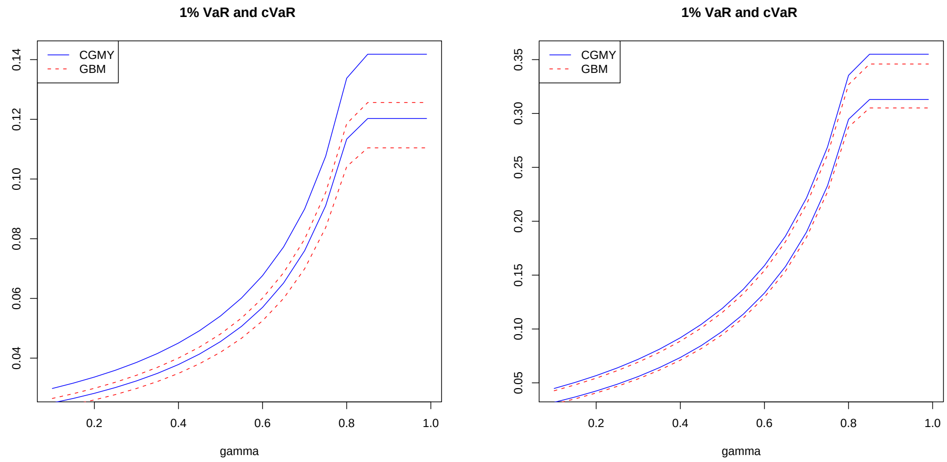


Figure 6.2: *$VaR_{0.01}$ and $cVaR_{0.01}$ at 5 days (left) and 60 days (right) in example A.*

Example B Let L_t have the parameters $C = 0.00044$, $G = 21.6$, $M = 13.7$, $Y = 1.5$ and $\gamma_c = 0$. Standard deviation, skewness and kurtosis of unit time increments $L_{t+1} - L_t \overset{d}{=} L_1$ are then $\text{sd}[L_1] = 0.0195$, $\text{Skew}[L_1] = 0.52$ and $\text{Kurt}[L_1] = 7.60$. This is essentially the values of stock 2 from section 5.7. The mean rate of return on the stock driven by this process is $\hat{\mu} = 1.90 \cdot 10^{-4}$ (4.75% yearly). I choose a daily interest rate intensity $r = 1.5 \cdot 10^{-4}$ (3.75% yearly). Solving the Merton problem then provides optimal weights π^*_{CGMY} ranging from 12.6% for $\gamma = 0.10$ to 100% for γ's greater than 0.90. With the corresponding GBM the optimal stock allocation is a little higher. $\gamma = 0.85$ for example gives $\pi_{CGMY} = 75.6\%$ and $\pi_{GBM} = 69.1\%$. This is due to the positive skewness of the CGMY distribution and the high interest rate compared to $\hat{\mu}$. Going forward as in example A, using equal weights and leaving out consumption, we get the results shown in table 6.3 and figure 6.3. The shape of the curves seems to be consistent with earlier results. With all wealth allocated in stocks the risk measures at level $q = 0.01$ show minor differences. At $t = 5$ VaR and cVaR in the two models differ only slightly. $\text{VaR}_{0.01}$ is 9.62% with GBM and 9.70% with CGMY, a relative difference of just 1%. This is less than with GBM/NIG in section 5.7 where the GBM value was 95.9% of the NIG one. A possible explanation is that with $Y = 1.5$, ν_{CGMY} has more of its mass near zero than ν_{NIG}. This is because $\nu_{CGMY}(z) \sim |z|^{-(1+Y)}$ near zero whereas $\nu_{NIG} \sim |z|^{-2}$ near zero (see appendix A for results on asymptotic behavior of the Bessel function). At $t = 60$ the CGMY portfolios are actually the least risky when looking at $\text{VaR}_{0.01}$ and $\text{cVaR}_{0.01}$. The differences however are small.

$t=5$	$\text{VaR}_{0.01}$	$\text{cVaR}_{0.01}$	$\text{P}_{0.99}$	$\text{TP}_{0.99}$
GBM	10.79	12.30	12.93	14.89
CGMY	11.94	14.13	12.59	14.82
$t=60$				
GBM	30.50	34.42	57.25	66.93
CGMY	31.24	35.45	56.36	65.61

Table 6.2: *$VaR_{0.01}$, $cVaR_{0.01}$ and $P_{0.99}$, $TP_{0.99}$ for portfolios with $\pi=1$ in example A. Values are in %.*

$t=5$	$\text{VaR}_{0.01}$	$\text{cVaR}_{0.01}$	$\text{P}_{0.99}$	$\text{TP}_{0.99}$
GBM	9.62	10.94	10.65	12.30
CGMY	9.70	11.64	11.90	15.47
$t=60$				
GBM	29.57	33.01	41.98	49.60
CGMY	29.22	32.78	43.42	52.47

Table 6.3: *$VaR_{0.01}$, $cVaR_{0.01}$ and $P_{0.99}$, $TP_{0.99}$ for portfolios with $\pi=1$ in example B. Values are in %.*

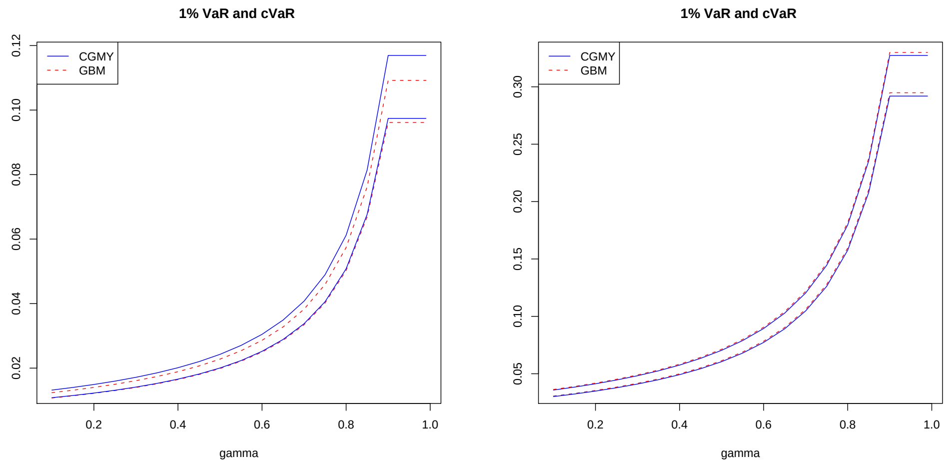


Figure 6.3: *$VaR_{0.01}$ and $cVaR_{0.01}$ at 5 days (left) and 60 days (right) in example B.*

Chapter 7

Conclusion

Alternative stock price models to the GBM where the log-returns follow Lévy processes with jumps have been presented, in particular the NIG and the CGMY. Merton's portfolio management problem, of deriving the maximum expected utility from consumption over an infinite time horizon, has been stated and its solution presented. We have seen that it is optimal to allocate a constant fraction of wealth in the risky asset, a fraction determined by an integral equation involving the Lévy measure ν of the driver.

We have studied some properties of the optimal portfolio and, in particular, stated an approximation to it in terms of the Lévy process driving the stock. Since the Lévy measure of such drivers has most of its mass close to 0, the approximation seemed to work quite well for various stocks.

A great part of the thesis has been devoted to a comparison of the optimal portfolio when the GBM and exponential NIG are the alternative underlying models. If log-returns in both models have the same expectations and variances then results indicated that the difference in risk and return of the optimal portfolio decreases quite rapidly as the time horizon increases. This was noticed with stocks of several shapes and particularly evident in the main example with the Norsk Hydro stock. Essentially the same results was noticed in the case of CGMY log-returns, the distribution of which seemed quite similar to normal inverse Gaussian's. If the stock's log-return had a right-skewed distribution then the lowest levels q of VaR_q tended to be greatest in the case of the geometric Brownian motion and opposite with a left-skewed distribution. But even with very skewed distributions the convergence to normality was rapid.

I found the main source of model risk in the long run to lie in the derivations of optimal strategies. Given an infinite time horizon the consumption rate was low and very similar in the GBM and NIG/CGMY cases, whilst the solution of the mentioned integral equation could differ more. An analysis of the equation told us that interest rates r close to the mean rate of return on the stock, $\hat{\mu}$, could lead to major differences in the stock allocation π^* and thus also the risk and return of the portfolio. An experiment where I fitted simulated log-return data to both a NIG and a normal distribution indicated however that it is quite random which of the GBM or NIG/CGMY models to produce the highest optimal stock allocation given a true underlying model. The experiment also showed that the interest rate r had to be quite close to $\hat{\mu}$ to see any significant differences in weights and thus, risk. Hence the utility of using more complex Lévy processes instead of the ordinary GBM may not be that big in the case of Merton's problem.

It should be noted that the models presented here do not include some important facts of stock behavior. I have not included real phenomena such as positive autocorrelation, transaction costs, stochastic interest rate and volatility. Models including these will probably give higher levels of risk if the same strategy as here is employed. This has to be accounted for when managing risk.

Appendix A

Errors in Estimation of π^*

Equation (4.5) giving the optimal π^* of the Merton problem involves integrals over $\mathbb{R} \setminus \{0\}$ with respect to ν. We remember the form

$$(\hat{\mu} - r) - (1-\gamma)\sigma^2\pi^* + \int_{\mathbb{R}\setminus\{0\}} \big((1+\pi^*(e^z-1))^{\gamma-1} - 1\big)(e^z-1)\nu(dz) = 0 \tag{A.1}$$

where

$$\hat{\mu} = \mu + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} (e^z - 1 - z\mathbf{1}_{|z|<1})\nu(dz).$$

I will explain the error of truncating these integrals at a level $\pm L$.

In the normal inverse Gaussian case the Lévy measure is

$$\nu_{NIG}(dz) = \frac{\delta\alpha}{\pi|z|} e^{\beta z} K_1(\alpha|z|)dz \tag{A.2}$$

The asymptotic behavior of the modified Bessel function of the second kind and index 1, K_1, is as follows (see e.g. Cont and Tankov [8]).

Proposition A.1. *When* $z \to \infty$

$$K_1(z) = e^{-z}\sqrt{\frac{\pi}{2z}}\left[1 + O\left(\frac{1}{z}\right)\right].$$

When $z \to 0$

$$K_1(z) \sim \frac{1}{z}.$$

The integrand in (A.1) is negative for all z when $\gamma \in (0,1)$ and $\pi^* \in [0,1]$. In the NIG case, substituting $K_1(\alpha|z|)$ with $e^{-\alpha|z|}\sqrt{\frac{\pi}{2\alpha|z|}}$ for $|z| > L$ provides an easier form of the integrand. For $z > 0$ the integrand is

$$\begin{aligned}
&\big((1+\pi^*(e^z-1))^{\gamma-1}-1\big)(e^z-1)\frac{\delta\alpha}{\pi z}e^{\beta z}K_1(\alpha z)\\
\geq&\big((1+\pi^*(e^z-1))^{\gamma-1}-1\big)(e^z-1)\frac{\delta\alpha}{\pi z}e^{\beta z}e^{-\alpha z}\sqrt{\frac{\pi}{2\alpha z}}\\
=&\big((1+\pi^*(e^z-1))^{\gamma-1}-1\big)(e^z-1)\delta\sqrt{\frac{\alpha}{2\pi}}z^{-3/2}e^{-(\alpha-\beta)z}\\
\geq&-\delta\sqrt{\frac{\alpha}{2\pi}}z^{-3/2}(e^z-1)e^{-(\alpha-\beta)z}\\
\geq&-\delta\sqrt{\frac{\alpha}{2\pi}}z^{-3/2}e^{-(\alpha-\beta-1)z}
\end{aligned}\tag{A.3}$$

If $\alpha > \beta+1$ the last expression decays faster than exponentially to 0 as z grows. This is true for most stocks (see e.g. Benth [4]). For $z < 0$ we exploit the fact that $(1+\pi^*(e^z-1))^{\gamma-1} \leq e^{z(\gamma-1)}$ to get

$$\begin{aligned}
&\big((1+\pi^*(e^z-1))^{\gamma-1}-1\big)(e^z-1)\frac{\delta\alpha}{\pi|z|}e^{\beta z}K_1(\alpha|z|)\\
\geq&\big((1+\pi^*(e^z-1))^{\gamma-1}-1\big)(e^z-1)\delta\sqrt{\frac{\alpha}{2\pi}}|z|^{-3/2}e^{-(\alpha+\beta)|z|}\\
\geq&\delta\sqrt{\frac{\alpha}{2\pi}}|z|^{-3/2}(e^{z(\gamma-1)}-1)(e^z-1)e^{-(\alpha+\beta)|z|}\\
\geq&-\delta\sqrt{\frac{\alpha}{2\pi}}|z|^{-3/2}e^{-(\alpha+\beta+\gamma-1)|z|}
\end{aligned}\tag{A.4}$$

Again we have an exponentially decreasing integrand. In the NIG model fitted to Norsk Hydro discussed in chapter 5, $\alpha = 56.16$ and $\beta = 2.641$. With this the error of truncating at $\pm L$ is surely less than $\int_{z>L}\delta\sqrt{\frac{\alpha}{2\pi}}z^{-3/2}e^{-(\alpha-\beta-1)z}dz+\int_{z<-L}\delta\sqrt{\frac{\alpha}{2\pi}}|z|^{-3/2}e^{-(\alpha+\beta+\gamma-1)|z|}dz$ which is of order 10^{-16} if $L = 0.5$ and 10^{-26} if $L = 1$. Clearly this is more than sufficient.

In the CGMY case the Lévy measure has the simpler form

$$\nu_{CGMY}(dz) = C\left[\frac{e^{-G|z|}}{|z|^{1+Y}}\mathbf{1}_{z<0}+\frac{e^{-Mz}}{z^{1+Y}}\mathbf{1}_{z>0}\right]dz.$$

Similar calculations as above give

$$\left|\int_{|z|>L} \big((1+\pi^*(e^z-1))^{\gamma-1}-1\big)(e^z-1)\nu_{CGMY}(dz)\right|$$
$$\leq C\left[\int_{z>L} z^{-(1+Y)}e^{-(M-1)z}dz + \int_{z<-L} |z|^{-(1+Y)}e^{-(G+\gamma-1)|z|}dz\right].$$

Typically M and G is greater than 10 which makes the error small even for quite small L. With the parameters of example B of section 6.3 the error is of order 10^{-11} if $L = 1$ and 10^{-17} if $L = 2$.

Appendix B

Cumulants, Skewness and Kurtosis

In this thesis I apply the skewness and kurtosis as measures of asymmetry and heavy tails. These are defined in terms the cumulants, κ_n. The cumulants of a random variable X are derived from its cumulant generating function, i.e. the logarithm of its characteristic function ϕ_X. The cumulants of X, κ_n, are given as

$$\kappa_n(X) = \frac{1}{i^n}\frac{d^n}{du^n}\ln\phi_X(u)\Big|_{u=0}. \tag{B.1}$$

The first cumulant κ_1 is the mean and the second cumulant κ_2 is the variance.

Definition B.1. *If X is a random variable then the skewness of X is defined as*

$$Skew[X] = \frac{\kappa_3(X)}{\kappa_2(X)^{3/2}}.$$

The skewness measures the asymmetry of the distribution of X. Positive skewness means the right of the distribution tail is longer. Outcomes high above the mean is likely.

When I use the term kurtosis in this thesis I always mean the *excess kurtosis*:

Definition B.2. *If X is a random variable then the excess kurtosis of X is defined as*

$$Kurt[X] = \frac{\kappa_4(X)}{\kappa_2(X)^2}.$$

The normal distribution has an excess kurtosis of 0. The kurtosis measures the degree to which the variance is due to infrequent extreme deviations as opposed to frequent small and medium deviations. By that it is a measure of how heavy-tailed the distribution is.

In Lévy process the kurtosis is always positive. Skewness and kurtosis is expressed via its Lévy measures ν. This is a consequence of the following more general result:

Theorem B.1. *If $L_t = \int_0^t \int_{\mathbb{R}\setminus\{0\}} \gamma(z)\tilde{N}(ds, dz)$ where $\gamma : \mathbb{R} \to \mathbb{R}$ is a deterministic function then*

$$\kappa_n(L_t) = t \int_{\mathbb{R}\setminus\{0\}} \gamma(z)^n \nu(dz) \qquad \forall\ n \geq 2 \tag{B.2}$$

provided that the integral is finite.

Proof. From exercise 1.6 of Øksendal/Sulem [11]

$$\phi_{L_t}(u) = \mathbb{E}[\exp(iuL_t)] = \exp\left(\int_0^t \int_{\mathbb{R}\setminus\{0\}} \big(e^{iu\gamma(z)} - 1 - iu\gamma(z)\big)\nu(dz)ds\right).$$

Hence

$$\begin{aligned}\kappa_2(L_t) &= \frac{1}{i^2}\frac{d^2}{du^2}\ln\phi_{L_t}(u)\Big|_{u=0} = \frac{1}{i^2}\frac{d}{du}t\int_{\mathbb{R}\setminus\{0\}} \big(i\gamma(z)e^{iu\gamma(z)} - i\gamma(z)\big)\nu(dz)\Big|_{u=0} \\ &= t\int_{\mathbb{R}\setminus\{0\}} \gamma(z)^2\nu(dz).\end{aligned} \tag{B.3}$$

By induction the result holds for any $n \geq 2$. □

Corollary B.1. *If L_t is a Lévy process with Lévy triplet (b, σ^2, ν) then*

$$Var[L_t] = \kappa_3(L_t) = t\big(\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} z^2\nu(dz)\big).$$

and

$$Skew[L_t] = \frac{\kappa_3(L_t)}{\kappa_2(L_t)^{3/2}}, \qquad Kurt[L_t] = \frac{\kappa_4(L_t)}{\kappa_2(L_t)^2}$$

where

$$\kappa_n(L_t) = t\int_{\mathbb{R}\setminus\{0\}} z^n\nu(dz) \quad \forall\ n \geq 3.$$

Thus the skewness of an increment $L_{t+\triangle} - L_t$ decays proportional to $\triangle^{-1/2}$ while the kurtosis decays as $\triangle^{-2}$.

Appendix C

R Code

C.1 Portfolio Simulation with NIG

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##### Simulation of Portfolio Development with NIG ############################
###############################################################################

## Packages ##
library(fBasics)
library(HyperbolicDist)
require(gplots)

## Parameters from [1], Norsk Hydro ##
muNig=-0.006
alfaNig=56.16
betaNig=2.641
deltaNig=0.015
r=0.0002
muNorMLE=0.000101
sigmaNorMLE=0.0166

## Parameters for normaldistr log-returns ##
muNor=muNig+deltaNig*betaNig/sqrt(alfaNig^2-betaNig^2)
sigmaNor=sqrt(deltaNig*alfaNig^2/(alfaNig^2-betaNig^2)^1.5)
muNor + .5*sigmaNor^2

## Calculation of optimal portfolio weights ##

#Levy density
nu=function(z){
deltaNig*alfaNig/(pi*abs(z))*exp(betaNig*z)*besselK(alfaNig*abs(z),nu=1)
}

ipos=integrate(function(z){(exp(z)-1-z)*nu(z)},0,2,abs.tol=10^(-8))$value
ineg=integrate(function(z){(exp(z)-1-z)*nu(z)},-2,0,abs.tol=10^(-8))$value
ipos+ineg
muhatNig=muNig+deltaNig*betaNig/sqrt(alfaNig^2-betaNig^2)+ipos+ineg
muhatNig

int=function(pigamma){
p=pigamma[1]; gam=pigamma[2];
integrand=function(z){((1+p*(exp(z)-1))^(gam-1)-1)*(exp(z)-1)*nu(z)}
integrate(integrand,-2,0,subdivisions=20000,abs.tol=10^(-8))$value +
```

```
integrate(integrand,0,2,subdivisions=20000,abs.tol=10^(-8))$value
}

int2=function(pigamma){
p=pigamma[1]; gam=pigamma[2];
integrand=function(z){((1+p*(exp(z)-1))^gam-1-gam*p*(exp(z)-1))*nu(z)}
integrate(integrand,-2,0,subdivisions=20000,abs.tol=10^(-8))$value +
integrate(integrand,0,2,subdivisions=20000,abs.tol=10^(-8))$value
}

gamma=c(seq(.10,.95,.05),seq(.96,.99,.01))
npoints=length(gamma)
PiNig=rep(0,npoints)
PiNor=rep(0,npoints)
k_Nig=rep(0,npoints)
k_gbm=rep(0,npoints)
for(i in 1:npoints){
gam=gamma[i]
low=0
up=1
Pistar=(low+up)/2
for(j in 1:14){
left=muhatNig-r+int(c(Pistar,gam))
if(left<0){up=Pistar}else{low=Pistar}
Pistar=(low+up)/2
}
PiNig[i]=Pistar
PiNor[i]=min(1,(muNorMLE+.5*sigmaNorMLE^2-r)/(sigmaNorMLE^2*(1-gam)))
PiNor[i]=max(0,PiNor[i])
k_Nig[i]=gam*(r+(muhatNig-r)*Pistar)+int2(c(Pistar,gam))
k_gbm[i]=gam*(r+(muNorMLE+.5*sigmaNorMLE^2-r)*PiNor[i])-
.5*gam*(1-gam)*sigmaNorMLE^2*PiNor[i]^2
}
eta=0.06/250
c_Nig=(eta-k_Nig)/(1-gamma)
c_Nig_in_percent=100*c_Nig
c_Nor=(eta-k_gbm)/(1-gamma)
c_Nor_in_percent=100*c_Nor
round(cbind(gamma,PiNig,PiNor,c_Nig_in_percent,c_Nor_in_percent),4)

#ConsistencyCheck
eta>k_gbm
eta>k_Nig

## Simulation of log-returns ##
nDays=5
nScenarios=500000
nSim=ndays*nscenarios
lavkNig=matrix(rnig(nSim, mu=muNig, delta=deltaNig,
alpha=alfaNig, beta=betaNig),nDays,nScenarios)
lavkNor=matrix(rnorm(nSim, mean=muNor, sd=sigmaNor), nDays, nScenarios)


par(mfrow=c(2,2))
hist(lavkNig, n=100, probability=TRUE, border="white", col="steelblue",
xlab="log-return",main="Histogram of NIG log-returns")
x = seq(-.3,.3 , 0.0005)
lines(x, dnig(x, alpha=alfaNig, beta=betaNig, delta=deltaNig, mu=muNig))

hist(lavkNor, n=100, probability=TRUE, border="white", col="steelblue")
lines(x, dnorm(x, mean=muNor,sd=sigmaNor))
```

```
logHist(lavkNig)
lines(x, log(dnig(x,alpha=alfaNig,beta=betaNig,delta=deltaNig,mu=muNig)),type="l")
lines(x, log(dnorm(x, mean=muNor,sd=sigmaNor)),type="l")

logHist(lavkNor)
lines(x, log(dnorm(x, mean=muNor,sd=sigmaNor)),type="l")
cbind(c(quantile(lavkNig,0.01),qnig(0.01, alpha=alfaNig, beta=betaNig,
delta=deltaNig, mu=muNig)),c(quantile(lavkNor,0.01),
qnorm(0.01,mean=muNor,sd=sigmaNor)))

theoreticExp=muNig+deltaNig*betaNig/sqrt(alfaNig^2-betaNig^2)
theoreticVar=deltaNig*alfaNig^2/(alfaNig^2-betaNig^2)^1.5
rbind(c(mean(lavkNig),var(lavkNig[1,])),c(theoreticExp,theoreticVar))

## Calculation of portefolio developments ##
P0=1
PNig=matrix(0,nscenarios,npoints)
PNor=matrix(0,nscenarios,npoints)
VaR0.01Nig =0*(1:npoints)
VaR0.05Nig =0*(1:npoints)
VaR0.01Nor =0*(1:npoints)
VaR0.05Nor =0*(1:npoints)
cVaR0.01Nig=0*(1:npoints)
cVaR0.05Nig=0*(1:npoints)
cVaR0.01Nor=0*(1:npoints)
cVaR0.05Nor=0*(1:npoints)
P0.95Nig   =0*(1:npoints)
P0.99Nig   =0*(1:npoints)
P0.95Nor   =0*(1:npoints)
P0.99Nor   =0*(1:npoints)
TP0.95Nig  =0*(1:npoints)
TP0.99Nig  =0*(1:npoints)
TP0.95Nor  =0*(1:npoints)
TP0.99Nor  =0*(1:npoints)

#Daily rebalancing
for(j in 1:nDays){
# weights from normalMLE, development in both models
PNig=PNig%*%diag(1-PiNor)*exp(r)+PNig%*%diag(PiNor)*exp(lavkNig[j,])
PNor=PNor%*%diag(1-PiNor)*exp(r)+PNor%*%diag(PiNor)*exp(lavkNor[j,])
}

for(i in 1:npoints){
Q0.01Nig=quantile(PNig[,i],0.01)
Q0.05Nig=quantile(PNig[,i],0.05)
Q0.95Nig=quantile(PNig[,i],0.95)
Q0.99Nig=quantile(PNig[,i],0.99)
Q0.01Nor=quantile(PNor[,i],0.01)
Q0.05Nor=quantile(PNor[,i],0.05)
Q0.95Nor=quantile(PNor[,i],0.95)
Q0.99Nor=quantile(PNor[,i],0.99)

VaR0.01Nig[i] =1-Q0.01Nig
VaR0.05Nig[i] =1-Q0.05Nig
VaR0.01Nor[i] =1-Q0.01Nor
VaR0.05Nor[i] =1-Q0.05Nor
P0.95Nig[i]   =Q0.95Nig-1
P0.99Nig[i]   =Q0.99Nig-1
P0.95Nor[i]   =Q0.95Nor-1
P0.99Nor[i]   =Q0.99Nor-1
cVaR0.01Nig[i]=1-mean(PNig[PNig[,i]<Q0.01Nig,i])
cVaR0.05Nig[i]=1-mean(PNig[PNig[,i]<Q0.05Nig,i])
```

```
cVaR0.01Nor[i]=1-mean(PNor[PNor[,i]<Q0.01Nor,i])
cVaR0.05Nor[i]=1-mean(PNor[PNor[,i]<Q0.05Nor,i])
TP0.95Nig[i]  =mean(PNig[PNig[,i]>Q0.95Nig,i])-1
TP0.99Nig[i]  =mean(PNig[PNig[,i]>Q0.99Nig,i])-1
TP0.95Nor[i]  =mean(PNor[PNor[,i]>Q0.95Nor,i])-1
TP0.99Nor[i]  =mean(PNor[PNor[,i]>Q0.99Nor,i])-1
}

#1%
plot(gamma,cVaR0.01Nig,type="l",col="blue",main="1% VaR and cVaR ",ylab="")
smartlegend(x="left",y="top",inset=0,legend=c("NIG", "GBM"),
lty=1:2,col=c("blue","red"),bg = "white")
lines(gamma,cVaR0.01Nor,col="red",lty=2)
lines(gamma,VaR0.01Nor,col="red",lty=2)
lines(gamma,VaR0.01Nig,col="blue")
#5%
plot(gamma,cVaR0.05Nig,type="l",col="blue",main="5% VaR and cVaR ",ylab="")
smartlegend(x="left",y="top",inset=0,legend=c("NIG", "GBM"),
lty=1:2,col=c("blue","red"),bg = "white")
lines(gamma,cVaR0.05Nor,col="red",lty=2)
lines(gamma,VaR0.05Nor,col="red",lty=2)
lines(gamma,VaR0.05Nig,col="blue")
#95%
plot(gamma,TP0.95Nig,type="l",col="blue",main="95% P and TP  ",ylab="")
smartlegend(x="left",y="top",inset=0,legend=c("NIG", "GBM"),
lty=1:2,col=c("blue","red"),bg = "white")
lines(gamma,TP0.95Nor,col="red",lty=2)
lines(gamma,P0.95Nor,col="red",lty=2)
lines(gamma,P0.95Nig,col="blue")
#99%
plot(gamma,TP0.99Nig,type="l",col="blue",main="99% P and TP  ",ylab="")
smartlegend(x="left",y="top",inset=0,legend=c("NIG", "GBM"),
lty=1:2,col=c("blue","red"),bg = "white")
lines(gamma,TP0.99Nor,col="red",lty=2)
lines(gamma,P0.99Nor,col="red",lty=2)
lines(gamma,P0.99Nig,col="blue")

print(100*c(VaR0.01Nor[1],VaR0.01Nig[1],cVaR0.01Nor[1],cVaR0.01Nig[1]),4)
print(100*c(VaR0.05Nor[1],VaR0.05Nig[1],cVaR0.05Nor[1],cVaR0.05Nig[1]),4)
print(100*c(VaR0.01Nor[14],VaR0.01Nig[14],cVaR0.01Nor[14],cVaR0.01Nig[14]),4)
print(100*c(VaR0.05Nor[14],VaR0.05Nig[14],cVaR0.05Nor[14],cVaR0.05Nig[14]),4)
print(100*c(VaR0.01Nor[22],VaR0.01Nig[22],cVaR0.01Nor[22],cVaR0.01Nig[22]),4)
print(100*c(VaR0.05Nor[22],VaR0.05Nig[22],cVaR0.05Nor[22],cVaR0.05Nig[22]),4)

print(100*c(P0.95Nor[1],P0.95Nig[1],TP0.95Nor[1],TP0.95Nig[1]),5)
print(100*c(P0.99Nor[1],P0.99Nig[1],TP0.99Nor[1],TP0.99Nig[1]),5)
print(100*c(P0.95Nor[14],P0.95Nig[14],TP0.95Nor[14],TP0.95Nig[14]),5)
print(100*c(P0.99Nor[14],P0.99Nig[14],TP0.99Nor[14],TP0.99Nig[14]),5)
print(100*c(P0.95Nor[22],P0.95Nig[22],TP0.95Nor[22],TP0.95Nig[22]),5)
print(100*c(P0.99Nor[22],P0.99Nig[22],TP0.99Nor[22],TP0.99Nig[22]),5)

## Testing the approximation of the optimal portfolio ##
l1=log(PNig[,14])
a=rnig(nScenarios, mu=nDays*muNig, delta=nDays*deltaNig, alpha=alfaNig,beta=betaNig)
approx1=mean(l1)+PiNor[14]*(a-mean(a))
qqplot(approx1,l1,xlab="Approximation",ylab="True process",main="QQ-plot of ln X_5")
plot.qqline(approx1,l1)
qqplot(log(PNor[,14]),l1, xlab="GBM case", ylab="NIG case",main="QQ-plot of ln X_5")
plot.qqline(log(PNor[,14]),l1)

## Testing Gaussian aggregation of log-returns ##
aggr=matrix(0,2,nScenarios)
```

```
for(t in 1:nScenarios){
aggr[1,t]=sum(lavkNig[,t])
aggr[2,t]=sum(lavkNor[,t])
}

#Comparing mean, st dev and quantiles
c(mean(aggr[1,]),mean(aggr[2,]),mean(aggr[1,])/mean(aggr[2,]))
c(sd(aggr[1,]),sd(aggr[2,]),sd(aggr[1,])/sd(aggr[2,]))
quantile(aggr[1,],0.99)
quantile(aggr[2,],0.99)
plot(density(aggr[1,]),col="blue",xlab="blue-NIG, red-normal")
lines(density(aggr[2,]),col="red")
qqnorm(aggr[1,])
qqline(aggr[1,])

#[1] Benth,Karlsen,Reikvam: "Non Gaussian Portfolio Management"
```

C.2 Portfolio Simulation with CGMY

```
##### Simulation of portfolio development with CGMY #############################
##################################################################################

## Parameters ##
E=0.000074
sigmaNor=0.01637

## Case Y=0: Variance Gamma ##
theta=0.000703585                    # Drift of Brownian motion
ny=1.19365                           # Variance rate of Gamma subordinator
sigma=sqrt(sigmaNor^2-theta^2*ny)    # Volatility of Brownian motion
C=  1/ny
G=  1/(sqrt(theta^2*ny^2/4+sigma^2*ny/2)-theta*ny/2)
M=  1/(sqrt(theta^2*ny^2/4+sigma^2*ny/2)+theta*ny/2)
Y=  0
c(C,G,M,Y)

## CGMY ##
C<-  .00044
G<-  21.6
M<-  13.7
Y<-  1.5

sigmaNor=sqrt(C*gamma(2-Y)*(G^(Y-2)+M^(Y-2)))
ExpJumps=C*gamma(-Y)*Y*(G^(Y-1)-M^(Y-1))
r=0.00015

##### Weights ####################################################################
# CGMY Levy measure
nuCGMY=function(x){ (abs(x)>0)*(C*exp(-M*x*1*(x>0)+G*x*1*(x<0))/abs(x)^(1+Y))}
muNorMLE=E
sigmaNorMLE=sigmaNor

#The rest of the calculations are similar to the NIG case.

####### Functions ################################################################

##   VG simulation   ##
VGunitIncr=function(par){    #Returns simulated unit increment of VG
```

```
n=par[1]; theta=par[2]; sigma=par[3]; ny=par[4];
dGamma=rgamma(n,shape=(1/ny),scale=ny)
sigma*rnorm(n)*sqrt(dGamma)+theta*dGamma
}

## Compound Poisson simulation ##
eps=0.008
divisorPos=integrate(nuCGMY,eps,Inf)$value
divisorNeg=integrate(nuCGMY,-Inf,-eps)$value
tail=integrate(nuCGMY,-Inf,-1)$value +integrate(nuCGMY,1,Inf)$value
KPos=C*eps^(-Y)/(Y*divisorPos)*exp(-M*eps)
KNeg=C*eps^(-Y)/(Y*divisorNeg)*exp(-G*eps)
f=function(x){Y*eps^Y/abs(x)^(Y+1)*(abs(x)>eps)}

CGMYunitIncrComPois=function(n){       #Returns simulated unit increment of CGMY
nJumpsPos=rpois(n,divisorPos)
nJumpsNeg=rpois(n,divisorNeg)

jumpSizesPos=array(0,sum(nJumpsPos))
jumpSizesNeg=array(0,sum(nJumpsNeg))
xtraFactor=max(KPos,KNeg)*1.05
WPos=runif(round(sum(nJumpsPos)*xtraFactor))
WNeg=runif(round(sum(nJumpsNeg)*xtraFactor))
VPos=runif(round(sum(nJumpsPos)*xtraFactor))
VNeg=runif(round(sum(nJumpsNeg)*xtraFactor))
XcanPos=eps*WPos^(-1/Y)
XcanNeg=-eps*WNeg^(-1/Y)

TPos=nuCGMY(XcanPos)/(divisorPos*KPos*f(XcanPos))
TNeg=nuCGMY(XcanNeg)/(divisorNeg*KNeg*f(XcanNeg))
XPos=XcanPos[VPos<TPos]
XNeg=XcanNeg[VNeg<TNeg]
jumpSizesPos=XPos[1:sum(nJumpsPos)];
jumpSizesNeg=XNeg[1:sum(nJumpsNeg)];
acceptanceRatio=length(XPos)/length(XcanPos)

startP=1
startN=1
Out=0*(1:n)
merk=(1:n)*(nJumpsPos>0)
index=merk[merk>0]
merkN=(1:n)*(nJumpsNeg>0)
indexN=merkN[merkN>0]
for(o in 1:length(index)){
Out[index[o]]=sum(jumpSizesPos[startP:(startP+nJumpsPos[index[o]]-1)])
startP=startP+nJumpsPos[index[o]];
}
for(o in 1:length(indexN)){
Out[indexN[o]]=Out[indexN[o]]
+sum(jumpSizesNeg[startN:(startN+nJumpsNeg[indexN[o]]-1)]);
startN=startN+nJumpsNeg[indexN[o]];
}
Out
} #end function returning simulations

######### Simulation of log-returns ##############################################
nDays=5
nScenarios=5*10^5
nSim=nDays*nScenarios
lavkNor=matrix(rnorm(nSim, mean=E, sd=sigmaNor),nDays, nScenarios)

## Case Y=0 ##
```

```
if(Y==0){sample=E-theta+VGunitIncr(c(nSim,theta,sigma,ny))}

## Case 0<Y<1: Finite variation, infinite activity, compound Poisson ##
ExpectedSmallJumps=integrate(function(x){C*(exp(-M*x)-exp(-G*x))/x^Y},0,eps)$value
sigmaepsSquared=integrate(function(x){x^2*nuCGMY(x)},0,eps)$value
+integrate(function(x){x^2*nuCGMY(x)},-eps,0)$value
sigmaeps=sqrt(sigmaepsSquared)
if(Y>0 && Y<1){sample=E-ExpJumps+CGMYunitIncrComPois(nSim)
+ExpectedSmallJumps+sigmaeps*rnorm(nSim)}

## Case 1=<Y<2    Infinite variation, infinite activity ##
sigmaepsSquared=integrate(function(x){x^2*nuCGMY(x)},0,eps)$value
+integrate(function(x){x^2*nuCGMY(x)},-eps,0)$value
sigmaeps=sqrt(sigmaepsSquared)
drift=E-integrate(function(x){x*nuCGMY(x)},eps,Inf)$value
-integrate(function(x){x*nuCGMY(x)},-Inf,-eps)$value
if(Y>1 && Y<2){sample=drift+CGMYunitIncrComPois(nSim)+sigmaeps*rnorm(nSim)}

lavkCGMY=matrix(sample, nDays, nScenarios)

########### Calculation of portfolio developments ###############################
# Similar to the NIG case
```

C.3 Exact VaR and cVaR for GBM and Exponential NIG

```
library(fBasics)
library(gplots)

########### NIG parameters ######################################################
p=matrix(0,4,5)
p[,1]=c(56.16, 2.641 ,0.0150,-0.0006)    # Norsk Hydro
p[,2]=c(32.50, 3.560 ,0.0125,-0.0015)    # High pos skewness and kurtosis
p[,3]=c(49.07,-10.10 ,0.0250,  .0060)    # Negative skewness, moderate kurtosis
p[,4]=c(25.85,-6.262 ,0.0100,  .0030)    # Big negative skewness, high kurtosis

choice=2
a<-p[1,choice]
b<-p[2,choice]
d<-p[3,choice]
m<-p[4,choice]
qrange=c(seq(0.0001,0.005,0.0001),seq(0.01,0.2,0.005))

## Mean,sd,skewness,kurtosis ##
info=c(m+d*b/sqrt(a^2-b^2),sqrt(d*a^2/(a^2-b^2)^1.5),
3*b/(a*sqrt(d*sqrt(a^2-b^2))),3*(1+4*b^2/a^2)/(d*sqrt(a^2-b^2)))
round(info,5)

## Exact VaR and cVaR of stock in the NIG case ##
nig=function(u,t){d*t*a/pi*exp(d*t*sqrt(a^2-b^2)+b*(u-m*t))
*besselK(a*sqrt((d*t)^2+(u-m*t)^2),1)/sqrt((d*t)^2+(u-m*t)^2)}
VaRNIG=function(t){1-exp(qnig(qrange,mu=m*t,alpha=a,beta=b,delta=d*t))}

cVaRNIG=function(t){                     # Returns cVaR of stock at time t.
resvec=array(0,length(qrange))
nig_t=function(x){nig(x,t)}
for(i in 1:length(qrange)){
div=qrange[i]
```

```
integrand=function(x){exp(x)*nig_t(x)/div}
resvec[i]=1-integrate(integrand,-10,qnig(qrange[i],
mu=m*t,alpha=a,beta=b,delta=d*t))$value
}
resvec
}

########### Normal parameters ##################################################
#mu=0.000101
#sigma=0.01637
mu=m+d*b/sqrt(a^2-b^2)
sigma=sqrt(d*a^2/(a^2-b^2)^1.5)

## Exact VaR and cVaR of stock with time horizon t
VaRNor=function(t){1-exp(sigma*sqrt(t)*qnorm(qrange)+t*mu)}

cVaRNor=function(t){
resvec=array(0,length(qrange))
for(i in 1:length(qrange)){
div=qrange[i]
integrandNor=function(x){exp(sigma*sqrt(t)*x)*dnorm(x)/div}
resvec[i]=1-exp(mu*t)*integrate(integrandNor,-Inf,qnorm(qrange[i]))$value
}
resvec
}

###########  VaR plots  ########################################################
days=60
plot(qrange,VaRNIG(days),type="l",col="blue",
main=paste("VaR and cVaR at t=",days),ylab="VaR",xlab="Quantiles")
smartlegend(x="right",y="top",inset=0,legend=c("NIG","GBM"),lwd=1:2,
lty=1:2,col=c("blue","red"),bg = "white")
lines(qrange,VaRNor(days),lty=2,col="red",lwd=2)
abline(v=c(seq(0,0.2,.05)),lty=2)

##  cVaR plot   ##
plot(qrange,cVaRNIG(days),type="l",col="blue",
main=paste("Conditional Value at Risk at t=",days),ylab="cVaR",xlab="Quantiles")
lines(qrange,cVaRNor(days),lty=2,lwd=2,col="red")
abline(v=c(seq(0,0.2,.05)),lty=2)
lines(qrange,cVaRNIG(days),lty=1,col="blue")

######## NIG shape triangle ####################################################
xi=array(0,4)
chi=array(0,4)
for(i in 1:4){
xi[i]=(1+p[3,i]*sqrt(p[1,i]^2-p[2,i]^2))^-.5
chi[i]=xi[i]*p[2,i]/p[1,i]
}
points=c(0,-1,1,0)
end=c(0,1,1,0)
plot(chi,xi,xlim=c(-1,1),ylim=c(0,1),xlab="chi",ylab="xi",pch=c("1","2","3","4"),
main="NIG shape triangle")
lines(points,end)
```

C.4 General Motors Calculations

```
#### Packages ####
library(fBasics)
```

```
library(HyperbolicDist)

#### Read file and find log-returns ####
tabell=read.csv("GM.csv")     #csv file downloaded from YahooFinance
date=tabell[,1]
price=rev(tabell[,7])         #stock prices
Days=length(kurs)
logretGM=log(price[2:Days]/price[1:(Days-1)])
plot(logretGM)
hist(logretGM)

##### Maximum Likelihood #####
## NormalMLE
nFit(logretGM)
#Estimated Parameter(s):
#       mean           sd
#0.0002356632 0.0170982911

## NIGMLE
nf=nigFit(logretGM)
#Parameter Estimates:   55.53495 4.134795 0.01581245 -0.0009449254

## Shape triangle:
nigShapeTriangle(nf)
#$chi 0.05436322       $zeta 0.7301592

##### Simulation of GM log-returns and ML-estimatimation ##############################
#######################################################################################

library(fBasics)
est=matrix(0,100,4)
estnor=matrix(0,100,2)

# Simulate 100 sets of log-returns, 1000 data in each set
# Fits the data to both a NIG and a normal distribution
for(i in 1:100){
data=rnig(1000,alpha=55.53,beta=4.1348,delta=0.015812,mu=-.0009449)
nf=nigFit(data)
norf=nFit(data)
est[i,]=slot(nf,name="fit")$estimate
estnor[i,]=slot(norf,name="fit")$estimate
}

NIG=matrix(0,100,6)
NOR=rep(0,100)
# Calculates the mean rate of return
for(i in 1:100){
NIG[i,1]=est[i,4]+est[i,3]*est[i,2]/sqrt(est[i,1]^2-est[i,2]^2            # Mean
NIG[i,2]=sqrt(est[i,3]*est[i,1]^2/(est[i,1]^2-est[i,2]^2)^1.5)             # St dev
NIG[i,3]=3*est[i,2]/(est[i,1]*sqrt(est[i,3]*sqrt(est[i,1]^2-est[i,2]^2)))  # Skewness
NIG[i,4]=3*(1+4*est[i,2]^2/est[i,1]^2)/                                    # Kurtosis
(est[i,3]*sqrt(est[i,1]^2-est[i,2]^2))
nu=function(z){
est[i,3]*est[i,1]/(pi*abs(z))*exp(est[i,2]*z)*besselK(est[i,1]*abs(z),nu=1)
}
ipos=integrate(function(z){(exp(z)-1-z)*nu(z)},0,2,abs.tol=10^(-8))$value
ineg=integrate(function(z){(exp(z)-1-z)*nu(z)},-2,0,abs.tol=10^(-8))$value
NIG[i,6]=NIG[i,1]+ipos+ineg                               # Mean rate of return NIG
NOR[i]=estnor[i,1]+ .5*estnor[i,2]^2                      # Mean rate of return GBM
}
print(cbind(NIG[,6],NOR))
hist(NIG[,6])
```

```
hist(NIG[,6]-NOR,breaks=12,xlab="NIG-GBM" ,
main=expression(paste("Difference between ",hat(mu)," in estimated NIG and GBM")))
```

C.5 Calculations on the Portfolio Approx. (Mathematica)

```
Taylor series
Series[Log[1+(Exp[z]-1)p],{z,0,5}]
p z+1/2 (p-p^2) z^2+1/6 (p-3 p^2+2 p^3) z^3+1/24 (p-7 p^2+12 p^3-6 p^4) z^4
+ 1/120 (p-15 p^2+50 p^3-60 p^4+24 p^5) z^5+O[z]^6

Comparing the real portfolio process with its approximation:
Norsk Hydro:
{\[Mu]NIG,\[Delta],\[Alpha],\[Beta]}={0.0002,-0.006,0.015,56.16,2.641};
Stock 2:
{\[Mu]NIG,\[Delta],\[Alpha],\[Beta]}={0.0002,0.0015,0.0125,32.5,3.56};
Stock 4:
In[9]:= {\[Mu]NIG,\[Delta],\[Alpha],\[Beta]}={0.003,0.01,25.85,-6.262};
In[11]:=nu[z_]:=(\[Delta]\[Alpha])/(\[Pi]Abs[z]) Exp[\[Beta]z] BesselK[1,\[Alpha] Abs[z]]

p={.1,.5,.9};

Difference in expectation:
In[14]:=
NIntegrate[(Log[1+(Exp[z]-1) p]-p z-1/2 p (1-p) z^2)nu[z],{z,0.0001,5}]+
NIntegrate[(Log[1+(Exp[z]-1) p]-p z-1/2 p (1-p) z^2)nu[z],{z,-5,-0.0001}];

Decomposed variance, skewness and kurtosis for(1): NIG approx and
(2): the real optimal portfolio :
In[28]:= TableForm[{{"Positive","Negative","Sum"},
v1=Append[vec={NIntegrate[(z p)^2 nu[z],{z,0,5}],
NIntegrate[(z p)^2 nu[z],{z,-5,0}]},Total[vec]]}]

Out[28]//TableForm= Positive Negative Sum
1.42011*10^-6   1.25971*10^-6   2.67982*10^-6
0.0000355028    0.0000314928    0.0000669956
0.000115029     0.000102037     0.000217066

skew1=NIntegrate[(z p)^3 nu[z],{z,0,5}]/(NIntegrate[(z p)^2 nu[z],{z,0,5}]
+NIntegrate[(z p)^2 nu[z],{z,-5,0}])^1.5 + NIntegrate[(zp)^3nu[z],{z,-5,0}]/
(NIntegrate[(z p)^2 nu[z],{z,0,5}]+NIntegrate[(z p)^2 nu[z],{z,-5,0}])^1.5

Out[26]= {0.153796,0.153796,0.153796}

In[29]:= kurt1=NIntegrate[(z p)^4 nu[z],{z,0,5}]/(NIntegrate[(z p)^2 nu[z],{z,0,5}]
+NIntegrate[(z p)^2 nu[z],{z,-5,0}])^2 +NIntegrate[(z p)^4 nu[z],{z,-5,0}]/
(NIntegrate[(z p)^2 nu[z],{z,0,5}]+NIntegrate[(z p)^2 nu[z],{z,-5,0}])^2;

TableForm[{{"Positive","Negative","Sum"},
v2=Append[vec2={NIntegrate[Log[1+(Exp[z]-1) p]^2 nu[z],{z,0,5}],
NIntegrate[Log[1+(Exp[z]-1) p]^2 nu[z],{z,-5,0}]},Total[vec2]]}]

Out[30]//TableForm= Positive  Negative  Sum
1.45131*10^-6      1.23573*10^-6      2.68704*10^-6
0.0000359291       0.0000311549       0.000067084
0.000115301        0.000101815        0.000217115
```

```
In[31]:= skew2=NIntegrate[Log[1+(Exp[z]-1) p]^3 nu[z],{z,0,5}]/
(NIntegrate[Log[1+(Exp[z]-1) p]^2 nu[z],{z,0,5}]+
NIntegrate[Log[1+(Exp[z]-1) p]^2 nu[z],{z,-5,0}])^1.5 +
NIntegrate[Log[1+(Exp[z]-1) p]^3 nu[z],{z,-5,0}]/
(NIntegrate[Log[1+(Exp[z]-1) p]^2 nu[z],{z,0,5}] +
NIntegrate[Log[1+(Exp[z]-1) p]^2 nu[z],{z,-5,0}])^1.5

Out[31]= {0.233187,0.197697,0.162548}

In[33]:= kurt2=NIntegrate[Log[1+(Exp[z]-1) p]^4 nu[z],{z,0,5}]/
(NIntegrate[Log[1+(Exp[z]-1) p]^2 nu[z],{z,0,5}] +
NIntegrate[Log[1+(Exp[z]-1) p]^2 nu[z],{z,-5,0}])^2 +
NIntegrate[Log[1+(Exp[z]-1) p]^4 nu[z],{z,-5,0}]/
(NIntegrate[Log[1+(Exp[z]-1) p]^2 nu[z],{z,0,5}] +
NIntegrate[Log[1+(Exp[z]-1) p]^2 nu[z],{z,-5,0}])^2;

In[22]:= NIntegrate[z^2 nu[z],{z,0,5}]+NIntegrate[z^2 nu[z],{z,-5,0}]
Out[22]= 0.000267982

In[23]:= NIntegrate[z^4 nu[z],{z,0,5}]+NIntegrate[z^4 nu[z],{z,-5,0}]
Out[23]= 2.58298*10^-7
```

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