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SDEs driven by VMLV processes and ambit fields

A study of existence and uniqueness

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The front page depicts a section of the root system of the exceptional Lie group E_8, projected into the plane. Lie groups were invented by the Norwegian mathematician Sophus Lie (1842–1899) to express symmetries in differential equations and today they play a central role in various parts of mathematics.

Abstract

The main topic of this thesis is SDEs driven by VMLV and VMBV processes. Further study is done on their respective subclasses of processes, abbreviated as LSS and BSS processes. Secondarily, SPDEs driven by ambit fields in an infinite-dimensional Hilbert space are studied. In both cases, the aim is to find conditions ensuring the existence and uniqueness of solutions. Moreover, processes abbreviated as fBSS processes are analyzed. This is (to my knowledge) a new process in the sense that it is defined in this thesis. We also look at SDEs driven by such fBSS processes and attempt to define integrals where the integrator is a fBSS process. Finally, some properties of integrals against VMLV and VMBV processes and ambit fields are obtained.

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Contents

CHAPTER 1

Introduction

The subject of this thesis is stochastic differential equations (SDEs), that is, differential equations with at least one stochastic term. The study of SDEs makes up a large and varied field whose applications are many and wide-ranging, like the evolution of interest rates in finance or physical phenomena like temperature. The subject was birthed by Kiyosi Itô in his 1946 paper, [Itô46], where he studied SDEs on the form

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s, \tag{1.1}$$

where B is a standard Brownian motion. Terminologically, 1.1 is said to be a Brownian motion driven SDE. The first integral term is called the drift and the second integral term is called the noise, therefore, it is also common to say that equation 1.1 is a SDE with Brownian noise. Equations with this type of noise are by far the most studied. Interest rate models with Brownian noise are, for instance, studied in [CT06]. Other types of noise are possible, such as processes with jumps. Prominent examples of this are Lévy processes, and, in general, semimartingales with jumps.

This thesis will focus on a particular type of SDE where the noise is not necessarily a semimartingale, specifically, the focus will be on equations driven by volatility modulated Lévy-driven Volterra (VMLV) processes, which, if without drift, take on the following form

$$\int_0^t G(t, s)\sigma(s)dL(s), \tag{1.2}$$

where L is a Lévy process, G is a deterministic function called the kernel function, and σ is a stochastic process called the volatility. In case L is a Brownian motion, then the process defined by (1.2) is called a volatility modulated Brownian-driven Volterra (VMBV) process. These two classes of processes are rather large, and particular attention is devoted to the subclass of VMLV processes called Lévy semistationary (LSS) processes and the subclass of VMBV processes called Brownian semistationary (BSS) processes. The core of this thesis is dedicated to the study of the existence and uniqueness of solutions to VMLV and VMBV driven equations.

The above discussion refers to real-valued SDEs, but another interesting subject is stochastic partial differential equations (SPDEs) set on an infinite-dimensional Hilbert space, that is, equations of the form

$$X_t = X_0 + \int_0^t AX_s ds + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB(s),$$

where A is an operator generating a C_0-semigroup and, being a little informal, B is a Brownian motion on the Hilbert space. The terminology from the real-valued case is otherwise preserved. The theory on SPDEs is newer, but by no means unstudied, and those interested in interest rate models are again referred to [CT06], as they also treat infinite-dimensional SPDEs modeling interest rates.

As for the real-valued case, our study of SPDEs is focused on equations with a particular type of noise, known as ambit fields. VMBV processes are real-valued, but, keeping the informal tone, one could think of Hilbert-valued processes defined by

$$\int_0^t G(t, s)\sigma(s)dB(s),$$

as a Hilbert-valued VMBV process. An examination of these equations is also included, and the focal point is again the existence and uniqueness of solutions.

Differential equations, both stochastic and non-stochastic, are often written in differential form, however, this hides an important part of the study of S(P)DEs, which is ensuring that the noise term is even defined. Hence, integral form has been chosen above to highlight that stochastic integration theory is also an important part of the analysis of S(P)DEs. VMLV, or ambit field-driven equations is no exception, and one must therefore know the meaning of expressions of the sort:

$$\int_0^t X(s)dY(s), \tag{1.3}$$

where Y is a VMLV process or an ambit field, depending on the context. As defining such expressions is non-trivial, the thesis is also comprised of a healthy portion of relevant background theory. Most importantly, an introduction to Malliavin calculus is given.

The final chapter is devoted to the study of fractional Brownian semistationary (fBSS) processes, which are processes defined by

$$\int_0^t g(t - s)\sigma(s)dB^H(s),$$

where B^H is a fractional Brownian motion. Defining this integral requires special attention, and a big theme in this chapter is how this should be done. The chapter is finished with an attempt at defining integrals of the form 1.3 when Y is an fBSS process.

1.1 Structure of the thesis

The thesis has a total of eight chapters, all covering different theory. Chapters 2-5 are pretty much all background material borrowed from other sources and clearly referenced. Chapters 6 and 8 consists partly of background theory and partly of my own results. Finally, Chapter 7 is almost exclusively my own work. However, the proof of some of the results draws upon certain results from chapter 5, the extent of this will be made clear when we get there. In more detail, the contents of each chapter are the following:

- **Chapter 2:** A short introduction to the basic concepts of stochastic analysis needed in this thesis is given. Section 2.1 defines the Itô integral and states the important Itô isometry. Section 2.2 gives a similar introduction for stochastic integrals with respect to compensated Poisson random measures, as this is a more technical concept, the introduction will be correspondingly longer. Finally, section 2.3 is a collection of definitions and results that, in my opinion, does not fit naturally into any other section of this thesis.
- **Chapter 3:** The subject of this chapter is somewhat more exotic than the theory presented in chapter 2, and therefore a rather comprehensive introduction to the important concepts of real-valued Malliavin calculus is given. Each section in this chapter considers a version of Malliavin calculus. Section 3.1 considers the continuous case, specifically the case for Brownian motions. And section 3.2 considers the discontinuous case of a pure jump Lévy process. In both cases, the most important parts of the sections are the definitions of the Malliavin derivative and the definition of the Skorohod integral, but some important properties of these definitions and their interplay are also treated.
- **Chapter 4:** Many of the same concepts as in chapter 2 and 3 are here introduced, but this time for Hilbert spaces. Section 4.1 starts off with the basic operator theoretical concepts of trace and Hilbert-Schmidt norm. The following section, section 4.2, is a also a "deterministic" section and introduces the Bochner integral and Fréchet derivative. The purpose of these sections is also to declare notation for the following sections. Sections 4.2 and 4.3 corresponds to chapter 2 in the real-valued case and introduces Hilbert-valued stochastic processes and stochastic integration respectively. Finally, section 4.5 institutes Hilbert-valued Malliavin calculus. Of all the sections on background theory, section 4.3, 4.4 and 4.5 have, in increasing order of difficulty, been the hardest sections to write about. In part because the mathematical level of these sections is above what I have learned as part of the courses I have taken, and in part because finding references that treat these topics rigorously have been difficult to obtain.
- **Chapter 5:** This chapter is meant as an appetizer on S(P)DEs and as preparation for what is to come in chapter 7. The collection of results presented in this chapter has been chosen with the aim of introducing various solution techniques, solution concepts, and types of uniqueness of solutions that we will encounter in chapter 7. We state some of the most well-known theorems on S(P)DEs, but also some lesser-known results that we will profit from later on. Mainly, these results are concentrated on the existence and uniqueness of various S(P)DEs, but a result on the Malliavin differentiability of a solution is also stated. The structure of this

chapter is simple: section 5.1 deals with real-valued SDEs, and section 5.2 deals with Hilbert-valued SPDEs. Unlike the other preparatory chapters, this one has a few proofs. In part, these proofs are given with the same rationale as above. That is, they are given as an introduction to various concepts encountered later on. But they are also given for the readers' convenience, as certain results in chapter 7 apply these proofs.

- **Chapter 6:** After a fairly thorough inquiry into relevant but more general background theory, this chapter is more specific and utilizes the concepts of chapters 2, 3 and 4. The starting point in section 6.1 is a formal introduction to volatility modulated Lévy/Brownian-driven Volterra processes. Thereafter, section 6.2 defines integration with respect to VMLV/VMBV processes and looks at some properties of these integrals. And, as in chapter 5, the Hilbert-valued case is split into its own singular section. Hence, section 6.3 defines ambit fields, integration with respect to ambit fields, and looks at some properties of this integral. The chapter is rounded off by section 6.4, where further properties of integrals with respect to VMLV/VMBV processes and ambit fields are studied.

- **Chapter 7:** This is the main destination of this thesis, and all the preceding chapters play their part in ensuring that our study in this chapter is well grounded. The subject is VMLV/VMBV/ambit field-driven S(P)DEs. This is by far the longest chapter of this thesis, and structurally, it follows the same recipe as chapter 5 and 6. That is, it begins with the study of real-valued SDEs driven by VMLV/VMBV processes and ends with the study of Hilbert-valued SPDEs driven by ambit fields. More concretely, section 7.1 studies equations with linear drift, section 7.2 studies equations with nonlinear drift, and section 7.3 studies nonlinear equations, all in the real-valued context. Finally, section 7.4 studies equations set on a Hilbert space.

- **Chapter 8:** The final chapter of this thesis is devoted to the study of what I have termed a fractional Brownian semistationary process. The name is chosen because the integrand is the same as for a BSS process, but the integrator is a fractional Brownian motion (fBM) instead of a standard Brownian motion. Since integration with respect to a fBM is not as easily defined as integration with respect to a standard Brownian motion, the structure of this chapter is determined by the three approaches chosen for defining such an integral. Section 8.1 considers the easiest definition, where the integral is defined as a Wiener integral, section 8.2 considers the more complicated route of a pathwise integral, and lastly, we consider the route via Skorohod integration, which is a generalization of the Wiener integral approach of section 8.1.

1.2 Contributions

The main contribution of this thesis is chapter 7. Every statement and every proof in this section is done by myself. However, certain proofs follow almost immediately by proofs found in chapter 5, when this is the case, it is clearly referenced. There are contributions in other chapters as well, we therefore list them explicitly. By contribution, I mean any theorem, proposition, lemma, corollary, definition or remark that I have not been able to find in the literature.

- **Chapter 3:**
  - Remark 3.1.13, on moving deterministic functions in and out of the Malliavin derivative in the Brownian motion case.
  - Remark 3.1.15, on how a chain rule implies a more general product rule. This is mentioned in [DØP09], but they have not written it out.
  - Proposition 3.1.22, on the commutation of the Malliavin derivative and an integral with respect to a finite signed measure in the Brownian motion case.
  - Remark 3.2.11, on moving deterministic functions in and out of the Malliavin derivative in the pure jump Lévy case.
  - Proposition 3.2.17, on the commutation of the Malliavin derivative and an integral with respect to a finite signed measure in the pure jump Lévy case.
- **Chapter 6:** The subsection of section 6.1 on continuous modifications.
  - Theorem 6.1.9, on the continuous modification of a BSS process.
  - Theorem 6.1.10, on the continuous modification of a LSS process.
  - Example 6.1.11, on examples of functions satisfying the conditions of theorem 6.1.10.
  - Proposition 6.2.7, on the integration by parts formula for VMLV integrals. This is a result stated in [BBV18], but their formula was wrong and this result corrects that mistake.
  - Section 6.4, all results in this section are my own. This is also why I have put these results in its own section.
- **Chapter 7:**
  - Section 7.1, all results are my own, but Theorem 7.1.1 is inspired by Proposition 25 in [BBV18].
  - Section 7.2, all results are my own, but the proof of certain results rely on proofs given in chapter 5.
  - Section 7.3, all results are my own.
  - Section 7.4, all results are my own, but Theorem 7.4.1 and Theorem 7.4.2 are in part proved by applying proofs given in chapter 5.
- **Chapter 8:**
  - Definition 8.1.2, on the definition of a fBSS process through Wiener integration.
  - Proposition 8.1.3, on the autocovariance of a fBSS process.
  - Theorem 8.1.5, on continuous modifications of fBSS processes.

- Theorem 8.1.6, on existence and uniqueness of a fBSS driven SDE, but the proof relies on the result by [Eva14] given in chapter 5.
- Definition 8.2.5, on the definition of a fBSS process through pathwise integration.
- Theorem 8.2.6, on the existence and uniqueness of a fBSS driven SDE, but this follows almost immediately by a proof in [NR02]
- Definition 8.3.2, on the definition of a fBSS process through Skorohod integration.
- Definition 8.3.3, on the derivation of an integral w.r.t. a fBSS process, but the derivation of this definition is inspired by the derivation of Definition 17 on page 120 in [BBV18]
- Proposition 8.3.4-8.3.6, on properties of the integral defined in 8.3.3, but these are inspired by Lemma 12, Proposition 21 and Proposition 23, respectively, found in section 4.3 in [BBV18]
- Proposition 8.3.7-8.3.8, on further properties of the integral defined in 8.3.3.

1.3 Comments on terminology and notation

Most of the notation used in each chapter is clarified at the beginning of it. I have tried my best to remain as consistent as possible, but some things might cause confusion. Firstly, I have attempted to use Brownian motion in the real-valued setting and Wiener process in the Hilbert-valued setting. There are several different types of these processes, and they are therefore listed below:

- B denotes a real-valued Brownian motion.
- $\bar{B}$ denotes a two-sided real-valued Brownian motion.
- B^H denotes a fractional Brownian motion
- W denotes a Q-Wiener process
- $\tilde{W}$ denotes a cylindrical Brownian motion.

Secondly, a source of confusion might be the notation used for the Skorohod integral in the pure jump Lévy case. Here, I have failed to remain consistent. The three following notations are all found in this thesis, but they all mean the same thing:

$$\tilde{N}(\delta t, dz) = \tilde{N}(\delta z, \delta t) = \tilde{N}(\delta z, dt).$$

In general, if there is a "δ" inside the parenthesis of $\tilde{N}(\cdot,\cdot)$, then it is meant to denote the Skorohod integral. If there is not, then it denotes the compensated Poisson random measure, that is, $\tilde{N}(dt, dz) = \tilde{N}(dz, dt)$.

Lastly, the term "integration by parts" is used in different ways. For this reason, I have tried to consistently reference the specific formula in question, although I do believe that this should be clear from the context.

CHAPTER 2

Stochastic calculus

This chapter will introduce the most basic concepts relevant for this thesis. We will also declare the notation used for the real-valued part of this thesis, the Hilbert-valued part will be introduced in chapter 4. We assume that the reader is familiar with the most basic concepts of probability theory, measure theory and real analysis.

The framework is a complete filtered probability space, that is $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where Ω is a sample space, $\mathcal{F}$ is a complete σ-algebra on Ω, $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$ is a filtration on the the measurable space $(\Omega, \mathcal{F})$ and P is a probability measure. $\mathbb{F}$ is assumed to be complete and right-continuous throughout this thesis. Respectively, this means that, for any t, $\mathcal{F}_t$ is augmented for all null sets (w.r.t. P), and

$$\mathcal{F}_t = \sigma\Big\{\bigcup_{s<t} \mathcal{F}_s\Big\}.$$

On a a bounded interval $[0, T]$, the filtration is simply $\mathbb{F} = \{F_t\}_{t\in[0,T]}$.

A Stochastic process is a parameterized collection of random variables $\{X_t\}_{t\geq 0}$ or $\{X(t)\}_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, P)$. If the stochastic process is defined on a bounded interval $[0, T]$ as follows $X : [0, T] \times \Omega \to \mathbb{R}$, $(t, \omega) \mapsto X(t, \omega)$, we write $\{X_t\}_{t\in[0,T]}$ or $\{X_t\}_{0\leq t\leq T}$. $\{X(t)\}_{t\geq 0}$ will be assumed to be $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^+ := [0, \infty)$, and we often set $X := \{X_t\}_{t\in[0,T]}$. Furthermore if the dependence on ω is not needed we usually write X_t or $X(t)$ for $X(t, \omega)$. We call a stochastic process $\mathbb{F}$-adapted if for all $t \geq 0$, the random variable $X(t) = X(t, \omega)$, $\omega \in \Omega$, is $\mathcal{F}_t$-measurable.

We refer to [Øks13] for details on the above terminology and definitions. The rest of this chapter relies on [Øks03], [DØP09] and [App09].

2.1 The Itô integral for Brownian motions

The natural starting point for the preliminary theory is to define Itô integrals and look at some basic properties. First we define Brownian motions, sometimes also referred to as Wiener processes. Without making any guarantees, we have attempted to consistently use the terminology Brownian motion in the real-valued context and Wiener process in the Hilbert-valued context.

Definition 2.1.1 (Brownian motion, [App09])**.** A stochastic process $\{B(t)\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$ is a Brownian motion if

i) $B(0) = 0$ P-a.s.,

ii) $B(t+s) - B(t)$ is independent of $B(u)$ for $u \leq s$,

iii) B has centered Gaussian increments, i.e. $B(t) - B(s) \sim \mathcal{N}(0, t-s)$,

iv) B has continuous paths.

With the above definition at hand we can define integrals with Brownian motions as integrator. The version we present is called the Itô integral, and for the Itô integral of a function to be defined it must satisfy the following criteria.

Definition 2.1.2 ([Øks03])**.** Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions $f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ such that

i) $(t,\omega) \mapsto f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $\mathbb{R}^+$.

ii) $\omega \mapsto f(t,\omega)$ is $\mathcal{F}_t$-measurable for all $0 \leq t \leq T$,

iii) $E\Big[\int_S^T f(t,\omega)^2 dt\Big] < \infty$.

Where $\mathcal{F}_t$ is the σ-algebra generated by the random variables $\{B(s)\}_{0 \leq s \leq t}$.

In the following and in the rest of this thesis we let χ denote the indicator (characteristic) function, meaning that for some set X and some subset $A \subset X$ we define

$$\chi_A(x) := \begin{cases} 1, & \text{for } x \in A \\ 0, & \text{for } x \notin A \end{cases}$$

The construction of the Itô integral is done via the limit of simple (elementary) functions, in principle similar to the construction of the Lebesgue integral.

Definition 2.1.3 (Simple functions, [Øks03])**.** A function $\phi \in \mathcal{V}$ is called simple if it is on the following form

$$\phi(t,\omega) = \sum_{j=1}^{\infty} e_j(\omega) \cdot \chi_{[t_j, t_{j+1})}(t),$$

where the random variables e_j must be $\mathcal{F}_{t_j}$-measurable for all $j \in \mathbb{N} \cup \{0\}$. For simple functions the Itô integral is defined by

$$\int_S^T \phi(t,\omega) dB_t(\omega) := \sum_{j=1}^{\infty} e_j(\omega)[B_{t_{j+1}} - B_{t_j}](\omega).$$

We now give the general definition of the Itô integral, and some nice properties.

Definition 2.1.4 (The Itô integral, [Øks03])**.** Let $f \in \mathcal{V}(S,T)$. Then the Itô integral of f (from S to T) is defined by

$$\int_S^T f(t) dB(t) = L^2(P) - \lim_{n\to\infty} \int_S^T \phi_n(t) dB(t)$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E\bigg[\int_S^T (f(t) - \varphi_n(t))^2 dt\bigg] \to 0 \quad \text{as}\, n \to \infty.$$

The Itô integral can be extended to a larger class of functions, we settle with a small remark and refer to [KS98] for details.

Remark 2.1.5 ([Øks03], [KS98]). The condition iii) in definition 2.1.2 can be relaxed to

$$P\Big(\int_0^T f(s,\omega)^2 dt < \infty\Big) = 1.$$

Let L^2_{prob} denote the set of measurable, adapted processes satisfying the above integrability condition. For $\{X\}_{t\in[0,T]} \in L^2_{prob}$ one can define the Itô integral through the use of localization. This extends the Itô integral of 2.1.4 in the sense that $\mathcal{V}$ is a proper subset of L^2_{prob}, and in the sense that for any $Y \in \mathcal{V}$ the two integrals coincide.

The following properties hold for the Itô integral defined in 2.1.4, but not necessarily for the extension of the Itô integral defined in the above remark.

Theorem 2.1.6 (Properties of the Itô integral, [Øks03]). *Let $f, g \in \mathcal{V}(0,T)$ and let $0 \leq S < U < T$. Then*

(i) $\int_S^T f(t)dB_t = \int_S^U f(t)dBt + \int_U^T f(t)dB_t$ *P-almost surely (P-a.s.),*

(ii) $\int_S^T (cf(t) + g(t))dB_t = c\int_S^T f(t)dB_t + \int_S^T g(t)dB_t$ *P-a.s., where c is a constant in $\mathbb{R}$,*

(iii) $E[\int_S^T f(t)dB_t] = 0,$

(iv) $\int_S^T f(t)dB(t)$ *is $\mathcal{F}_T$-measurable.*

Finally, we state the Itô isometry, which is crucial not only to the construction of the Itô integral via simple functions, but also in applications of the integral as it connects the stochastic Itô integral with a deterministic integral in $L^2(P)$ sense. This, and other versions of the Itô isometry for different stochastic integrals, will be of great importance to us later on.

Theorem 2.1.7 (The Itô isometry, [Øks03]).

$$E\bigg[\Big(\int_S^T f(t)dB(t)\Big)^2\bigg] = E\bigg[\int_S^T f(t)^2 dt\bigg] \quad \textit{for all } f \in \mathcal{V}(S,T)$$

2.2 The stochastic integral for compensated Poisson measures

This section will give the definition of the stochastic integral with respect to the compensated Poisson measure, for that we need a few definitions. We begin with some notational preliminaries.

We define

- $f(x-) := \lim_{y\uparrow x} f(y)$ for $y < x$ to be the left limit of a function
- and $f(x+) := \lim_{y\downarrow x} f(y)$ for $y > x$ to be the right limit of a function.

We will occasionally use the terms càdlàg and càglàd, since these terms will be used already in the definition of Lévy processes we define them here.

Definition 2.2.1 (Càdlàg and càglàd, [App09])**.** Let $I = [a, b]$ be an interval in $\mathbb{R}^+$. A mapping $f : I \to \mathbb{R}$ is said to be càdlàg if, for all $t \in [a, b]$, f has a left limit at t and f is right-continuous at t, i.e.

- for all sequences $\{t_n\}_{n\in\mathbb{N}}$ in (a, b) with each $t_n < t$ and $\lim_{n\to\infty} t_n = t$ we have that $\lim_{n\to\infty} f(t_n)$ exists;
- for all sequences $\{t_n\}_{n\in\mathbb{N}}$ in (a, b) with each $t_n \geq t$ and $\lim_{n\to\infty} t_n = t$ we have that $\lim_{n\to\infty} f(t_n) = f(t)$;
- for the end-points we stipulate that f is right continuous at a and has a left limit at b.

A càglàd function (i.e. one that is left-continuous with right limits) is defined similarly.

With these definitions at hand we can begin the main part of this section, and defining Lévy processes is a natural starting point.

Definition 2.2.2 (Lévy process, [DØP09])**.** A one-dimensional Lévy process is a stochastic process $L = \{L(t)\}_{t\geq 0}$:

$$L(t) = L(t, \omega), \quad \omega \in \Omega,$$

with the following properties:

i) $L(0) = 0$ P-a.s.,

ii) L has independent increments, that is, for all $t > 0$ and $h > 0$, the increment $L(t + h) - L(t)$ is independent of $L(s)$ for all $s \leq$t,

iii) L has stationary increments, that is, for all $h > 0$ the increment $L(t + h) - L(t)$ has the same probability law as $L(h)$,

iv) It is stochastically continuous, that is, for every $t \geq 0$ and $\epsilon > 0$ then $\lim_{s\to t} P\{|L(t) - L(s)| > \epsilon\} = 0$,

v) L has càdlàg paths, that is, the trajectories are right-continuous with left limits.

A process L satisfying (i)-(iv) is a Lévy process in law, and it can be shown that such processes L always has a version with càdlàg paths.

The jump of L at time t is defined by

$$\Delta L(t) := L(t) - \lim_{s\uparrow t} L(t)$$

Set $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and let $\mathcal{B}(\mathbb{R}_0)$ denote the σ-algebra generated by the family of all Borel subsets $U \subset \mathbb{R}$, such that the closure $\bar{U} \subset \mathbb{R}_0$.

Defining the Poisson random measure is now possible. It measures the amount of jumps of a certain size in a specific time interval for the Lévy process in question.

Definition 2.2.3 (Poisson random measure, [DØP09])**.** If $U \in \mathcal{B}(\mathbb{R}_0)$ with $\bar{U} \subset \mathbb{R}_0$ and $t > 0$, we define

$$N(t,U) := \sum_{0 \le s \le t} \chi_U(\Delta L(s)),$$

the Poisson random measure on $B(0,\infty) \times \mathcal{B}(\mathbb{R}_0)$ is given by

$$N((a,b] \times U) = N(b,U) - N(a,U), \quad 0 < a \le b,\, U \in \mathcal{B}(\mathbb{R}_0).$$

For our purposes the Poisson random measure is mainly important as a step in defining the next two measures. We begin with the Lévy measure, which measures the expected amount of jumps of a certain size in a time interval of length 1.

Definition 2.2.4 (Lévy measure, [DØP09])**.** If $U \in \mathcal{B}(\mathbb{R}_0)$ with $\bar{U} \subset \mathbb{R}_0$ and $t > 0$, we define the Lévy measure ν of L to be

$$\nu(U) := E[N(1,U)], \quad U \in \mathcal{B}(\mathbb{R}_0).$$

Remark 2.2.5. [DØP09] The Lévy measure always satisfies the following integrability condition

$$\int_{\mathbb{R}_0} \min(1, z^2)\nu(dz) < \infty.$$

This is a consequence of the Lévy-Khintchine formula which we will state later.

Finally, we can define the compensated Poisson random measure (cPrm), which is what will be integrated against in the forthcoming stochastic integral. It is defined as the difference between the Poisson random measure and the Lévy measure.

Definition 2.2.6 (Compensated Poisson random measure, [DØP09])**.** We define the compensated Poisson measure $\tilde{N}$, by

$$\tilde{N}(dt,dz) := N(dt,dz) - \nu(dz)dt.$$

The following formula, called the Lévy-Khintchine formula, allows one to compute the characteristic function of any infinitely divisible process (see page 5 in [BBV18]) in a simple way. We only state the one dimensional version for Lévy processes.

Theorem 2.2.7 (Lévy-Khintchine Formula for Lévy Processes, [DØP09])**.** *Let L be a Lévy process in law. Then*

$$E[e^{iuL(t)}] = e^{i\psi(u)}, \quad u \in \mathbb{R} \quad (i = \sqrt{-1}), \tag{2.1}$$

with the characteristic exponent

$$\psi(u) := i\alpha u - \frac{1}{2}\beta^2 u^2 + \int_{|z|<1} (e^{iuz} - 1 - iuz)\nu(dz) + \int_{|z|\ge 1} (e^{iuz} - 1)\nu(dz) \tag{2.2}$$

where the parameters $\alpha \in \mathbb{R}$ *and* $\beta^2 \geq 0$ *are constants and* $\nu = \nu(dz)$, $z \in \mathbb{R}_0$, *is a* σ*-finite measure on* $\mathcal{B}(\mathbb{R}_0)$ *satisfying*

$$\int_{\mathbb{R}_0} \min(1, z^2)\nu(dz) < \infty. \tag{2.3}$$

It follows that ν *is the Lévy measure of* L. *Conversely, given the constants* $\alpha \in \mathbb{R}$ *and* β^2 *and the* σ*-finite measure* ν *on* $\mathcal{B}(\mathbb{R}_0)$ *such that* (2.3) *holds, then there exists a process* L *(unique in law) such that* (2.1) *and* (2.2) *hold. The process* L *is a Lévy process in law.*

The second stochastic integral relevant for this thesis is now introduced. The integrator is a compensated Poisson random measure instead of a Brownian motion, and for this to work we need a different kind of probability space than in section 2.1. So, for any t, let $\mathcal{F}_t$ be the complete σ-algebra generated by the random variables $B(s)$ and $\tilde{N}(ds, dz)$, $z \in \mathbb{R}_0$, $s \leq t$. Further, equip the probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$.

Definition 2.2.8 (cPrm Itô integral, [DØP09])**.** For any $\mathbb{F}$-adapted process θ such that

$$E\left[\int_0^T \int_{\mathbb{R}_0} \theta^2(t, z)\nu(dz)dt\right] < \infty \quad \text{for some } T > 0,$$

we define the stochastic integral w.r.t. the compensated Poisson random measure by

$$\int_0^T \int_{\mathbb{R}_0} \theta(s, z)\tilde{N}(ds, dz) = \lim_{n \to \infty} \int_0^T \int_{|z| \geq 1/n} \theta(s, z)\tilde{N}(ds, dz), \quad 0 \leq t \leq T,$$

where the limit is to be taken in $L^2(P)$. This integral is also linear, centered and $\mathcal{F}_T$-measurable, see Theorem 4.2.3 in [App09].

In the case where $\theta(s, z) = \theta(s)z$, which is the case when working with VMLV processes, we also write

$$\int_0^T \theta(s)dL(s) := \int_0^T \int_{\mathbb{R}_0} z\theta(s)\tilde{N}(ds, dz).$$

Remark 2.2.9. As in the Brownian motion case we can extend the cPrm Itô integral to all processes $\theta(t, z), z \in \mathbb{R}_0, t \in [0, T]$ such that

$$\int_0^T \int_{\mathbb{R}_0} \theta(s, z)\nu(dz)ds < \infty, \quad P - a.s.$$

We refer to [Pro10] for details.

There exist an isometry for these integrals which is analogous to the isometry for Itô integrals in the Brownian motion case.

Theorem 2.2.10 (cPrm Itô isometry, [DØP09])**.**

$$E\left[\left(\int_0^T \int_{\mathbb{R}_0} \theta(t, z)\tilde{N}(dt, dz)\right)^2\right] = E\left[\int_0^T \int_{\mathbb{R}_0} \theta(t, z)^2\nu(dz)dt\right].$$

By comparing the definitions of a Brownian motion and a Lévy process one can see that a Brownian motion is a special case of a Lévy process. Furthermore, all Lévy processes can be represented in the following way.

Theorem 2.2.11 (Lévy-Itô decomposition, [DØP09]). *Let $L = \{L(t)\}_{t\geq 0}$ be a Lévy process. Then L, admits the following integral representation*

$$L(t) = at + \sigma B(t) + \int_0^t \int_{|z|<1} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} zN(ds, dz).$$

for some constants $a, \sigma \in \mathbb{R}$.

This section is concluded with the Itô formula for Itô-Lévy processes, the definition of these processes are motivated by the Lévy-Itô decomposition above.

Definition 2.2.12 (Lévy-Itô process, [DØP09]). Lévy-Itô processes are defined as the process $X = \{X_t\}_{t\geq 0}$ given by

$$X(t) = x + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z)\tilde{N}(ds, dz), \qquad (2.4)$$

where $\{\alpha_t\}_{t\geq 0}$, $\{\beta_t\}_{t\geq 0}$, $\{\gamma_t\}_{t\geq 0}$ are predictable processes satisfying the following integrability condition

$$\int_0^t \Big[|\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}_0} \gamma^2(s, z)\nu(dz)\Big]ds < \infty, \quad P - a.s.,$$

for all $t > 0$, $z \in \mathbb{R}_0$. Assuming this integrability condition the integrals are well-defined and the Lévy-Itô process X is a local martingale. If we strengthen the integrability condition to

$$E\bigg[\int_0^t \Big[|\alpha(s)| + \beta^2(s) + \int_{\mathbb{R}_0} \gamma^2(s, z)\nu(dz)\Big]ds < \infty\bigg],$$

for all $t > 0$, $z \in \mathbb{R}_0$, then X is a martingale. In differential form, (2.4) looks like

$$dX(t) = \alpha(t)dt + \beta(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t, z)\tilde{N}(dt, dz), \quad X_0 = x.$$

Now the Itô formula follows, which we will need in section 7.1 where we express the solution of an SDE with linear coefficients via the Itô formula. Denote by $C^{1,2}(A \times B)$ the space of functions on $A \times B$ differentiable at least once in the first variable and at least twice in the second variable.

Theorem 2.2.13 (Itô formula, [DØP09]). *Let $X = \{X(t)\}_{t\geq 0}$, be the Lévy-Itô process given by*

$$X(t) = x + \int_0^t \alpha(s)ds + \int_0^t \beta(s)dW(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s, z)\tilde{N}(ds, dz)$$

and let $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a function in $C^{1,2}((0, \infty) \times \mathbb{R})$ and define

$$Y(t) := f(t, X(t)), \quad t \geq 0.$$

Then the process $Y = \{Y(t)\}_{t\geq 0}$, is also a Lévy-Itô process and its differential form is given by

$$dY(t) = \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))\alpha(t)dt \tag{2.5}$$

$$+ \frac{\partial f}{\partial x}(t, X(t))\beta(t)dW(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X(t))\beta^2(t)dt \tag{2.6}$$

$$+ \int_{\mathbb{R}_0} \Big[f(t, X(t) + \gamma(t,z)) - f(t, X(t)) - \frac{\partial f}{\partial x}(t, X(t))\gamma(t,z)\Big]\nu(dz)dt \tag{2.7}$$

$$+ \int_{\mathbb{R}_0} \Big[f(t, X(t-) + \gamma(t,z)) - f(t, X(t-))\Big]\tilde{N}(dt, dz). \tag{2.8}$$

2.3 Useful results and definitions

This section is a collection of theorems and definitions which will be needed, but that does not fit naturally in any other section. We start with an inequality that will be used many times.

Lemma 2.3.1 (Elementary inequality, [Mit64])**.** *For $1 \leq i \leq n \in \mathbb{N}$, let $a_i \in \mathbb{R}$, then*

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 \leq n \sum_{i=1}^{n} a_i^2.$$

In chapter 5 and 7 we will come across two different ways in which two distinct stochastic processes can be the same.

Definition 2.3.2 (Modification, Indistinguishable, [Øks03])**.** Suppose that $X = \{X(t)\}_{t\geq 0}$ and $Y = \{Y(t)\}_{t\geq 0}$ are stochastic processes on $(\Omega, \mathcal{F}, P)$. Then we say that X is a modification of (or a version of) Y if

$$P(\{X(t,\omega) = Y(t,\omega)\}) = 1 \quad \text{for all } t.$$

A stronger sense of "sameness" is indistinguishability, it should be clear from the following definition that two indistinguishable processes are also modifications of each other.

The two processes X and Y are said to be indistinguishable (or pathwise equal) if

$$P(\{X(t,\omega) = Y(t,\omega) \text{ for all } t \geq 0\}) = 1.$$

We end this section with some well known results. Firstly, Kolmogorov's continuity theorem, which are used in sections 6.1 and 8.1 where we show that BSS, LSS and fBSS processes admits continuous modifications under certain regularity conditions on g and σ. Secondly, the classical Fubini theorem as well as a stochastic version of it are presented, these will be used in chapter 3 and chapter 7.

Theorem 2.3.3 (Kolmogorov's continuity theorem, [Øks03])**.** *Suppose that the process $X = \{X_t\}_{t\geq 0}$ satisfies the following condition: For all $T > 0$ there exist positive constants α, β, D such that*

$$E[|X_t - X_s|^{\alpha}] \leq D \cdot |t-s|^{1+\beta}; \quad 0 \leq s, t \leq T.$$

Then there exists a continuous version of X.

The famous classical Fubini theorem is now stated.

Theorem 2.3.4 (Fubini's Theorem, [MW12]). *Suppose that* $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ *bare* σ*-finite measure spaces. Let* f *be a complex-valued* $\mathcal{A} \times \mathcal{B}$ *measurable function on* $X \times Y$ *such that at least one of the integrals*

- $\int_{X\times Y} |f(x,y)| d(\mu \times \nu)(x,y)$
- $\int_X \int_Y |f(x,y)| d\mu(x) d\nu(y)$
- $\int_Y \int_X |f(x,y)| d\nu(x) d\mu(y)$

are finite, then the following equalities hold.

$$\begin{aligned}\int_{X\times Y} f(x,y) d(\mu \times \nu)(x,y) &= \int_X \int_Y f(x,y) d\mu(x) d\nu(y) \\ &= \int_Y \int_X f(x,y) d\nu(x) d\mu(y)\end{aligned}$$

Lastly, in this chapter, we give the stochastic Fubini theorem of [Pro10]. There are plenty of other versions, but this is sufficient for our purposes.

Theorem 2.3.5 (Stochastic Fubini Theorem, [Pro10]). *Let* X *be a semimartingale, let* $H(a,t,\omega)$ *be* $\mathcal{A} \otimes \mathcal{P}$ *measurable, let* μ *be a finite positive measure on* $\mathcal{A}$*, and assume*

$$\Big(\int_A H^2(a,t) d\mu(a) \Big)^{1/2}$$

is X *integrable. Letting* $Z_t^a = \int_0^t H(a,s) dX_s$ *be* $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ *measurable and* Z^a *càdlàg for each* a*, then* $Y_t = \int_A Z_t^a d\mu(a)$ *exists and is a càdlàg version of* $H \cdot X$*, where* $H_t = \int_A H(a,t) d\mu(a)$*.*

See [Pro10] for details on what is meant by X integrable, as it is a general definition and a somewhat lengthy construction we will not go into that. But in connection with Remark 2.1.5 we point out that in the specific instance where the semimartingale X is a Lévy process (see page 55 in [Pro10]), then a sufficient condition for an adapted process $\{u_s\}_{s\in[0,T]}$ being X integrable is the following

$$\int_0^T u^2(s) ds < \infty \quad a.s.$$

CHAPTER 3

Malliavin calculus

Building upon the concepts of stochastic calculus introduced in the last section, we can explore the subject of Malliavin calculus. This topic is crucial for this thesis, as the definition of integrals against VMBV/VMLV processes involves both the Skorohod integral and the Malliavin derivative. Malliavin calculus is a more exotic subject than stochastic calculus, and the presentation is therefore more detailed. In particular, the construction of the concepts of Skorohod integral and Malliavin derivative is included.

There are two ways of defining the Malliavin derivative and the Skorohod integral in the real-valued case. Since we will consider integration against both VMBV and VMLV processes and therefore need Malliavin calculus in both the Brownian motion case and the pure jump Lévy case, we have chosen the chaos expansion approach. As we shall see, these two cases are very similar in their constructions. Alternatively, one could define the Malliavin derivative as a stochastic gradient via directional derivatives. See [DØP09] for a more detailed discussion on the pros and cons of these two approaches.

The definitions and results in this section are mainly gathered from chapters 1-3 and 10-12 in [DØP09], but [Nua06] has also been consulted for a different point of view.

As mentioned, there are two slightly different cases and they will be considered separately. We present the Brownian motion case first and then the pure jump Lévy case.

3.1 Brownian motion case

Let $B = \{B_t\}_{t\in[0,T]}$ be a one-dimensional Brownian motion on the complete probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_t$ be the complete σ-algebra generated by $B(s)$, $0 \leq s \leq t$, for all $t \geq 0$. The corresponding filtration is denoted by $\mathbb{F} = \{\mathcal{F}\}_{t\in[0,T]}$.

Skorohod integrals and Malliavin derivatives are defined using the Wiener-Itô chaos expansion, but to get there we must first define iterated Itô integrals and symmetric functions.

Definition 3.1.1 (Symmetric function, [DØP09])**.** A real function $g : [0,T]^n \to \mathbb{R}$ is called symmetric if

$$g(t_{\sigma_1}, ..., t_{\sigma_n}) = g(t_1, ..., t_n)$$

for all permutations $\sigma = (\sigma_1, ..., \sigma_n)$ of $(1, 2, ..., n)$. Let $L^2([0,T]^n)$ be the standard space of square integrable Borel real functions on $[0,T]^n$ such that

$$\|g\|^2_{L^2([0,T]^n)} := \int_{[0,T]^n} g^2(t_1, ..., t_n)dt_1 \cdots dt_n < \infty.$$

Not all functions are symmetric, but all functions can be symmetrized, and for the definition of the Skorohod integral such a symmetrization will be necessary.

Let $\tilde{L}^2([0,T]^n) \subset L^2([0,T]^n)$ be the space of real symmetric square integrable Borel functions on $[0,T]^n$.

Definition 3.1.2 (Symmetrization of a function, [DØP09])**.** If f is a real function on $[0,T]^n$, then its symmetrization $\tilde{f}$ is defined by

$$\tilde{f}(t_1, ..., t_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, ..., t_{\sigma_n}),$$

where the sum is taken over all permutations σ of $(1, ..., n)$.

The definition of iterated Itô integrals is now viable and as will be immediately clear, they are defined in a natural way considering the name.

Definition 3.1.3 (Iterated Itô integral, [DØP09])**.** Let f be a deterministic function defined on

$$S_n = \{(t_1, ..., t_n) \in [0,T]^n : 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T\}$$

such that

$$\|f\|^2_{L^2(S_n)} := \int_{S_n} f^2(t_1, ..., t_n)dt_1 \cdots dt_n < \infty.$$

Then we can define the n-fold iterated Itô integral as

$$J_n(f) := \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1, ..., t_n)dB(t_1)dB(t_2) \cdots dB(t_{n-1})dB(t_n).$$

If $g \in \tilde{L}^2([0,T]^n)$ we define

$$I_n(g) := \int_{[0,T]^n} g(t_1, ..., t_n)dB(t_1) \cdots dB(t_n) := n!J_n(g). \tag{3.1}$$

We also call $I_n(g)$ the n-fold iterated Itô integral of g.

By the Itô isometry the following property holds.

Proposition 3.1.4 ([DØP09])**.** *If $g \in \tilde{L}^2([0,T]^m)$ and $h \in \tilde{L}^2([0,T]^n)$, we have*

$$E[J_m(g)J_n] = (g, h)_{L^2(S_n)} \begin{cases} 0, & m \neq n \\ (g, h)_{L^2(S_n)}, & m = n \end{cases} \tag{3.2}$$

where

$$(g, h)_{L^2(S_n)} := \int_{S_n} g(t_1, ..., t_n)h(t_1, ..., t_n)dt_1 \cdots dt_n.$$

Upcoming is the definition of the Wiener-Itô chaos expansion of a random variable, this is a sum of iterated Itô integrals. The definitions of the Skorohod integral and the Malliavin derivative relies fully on this construction.

Theorem 3.1.5 (The Wiener-Itô chaos expansion, [DØP09])**.** *Let F be an $\mathcal{F}_T$-measurable random variable in $L^2(P)$. Then there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$ of functions $f_n \in \tilde{L}^2([0,T]^n)$ such that*

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where the convergence is in $L^2(P)$. Moreover, we have the isometry

$$\|F\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n!\|f_n\|^2_{L^2([0,T]^n)}.$$

As we define the Skorohod integral, the necessity of presenting the previous definitions of this section becomes clear. This integral is fundamental for this thesis as many of the SDEs we consider contains at least one term which involves the Skorohod integral.

Definition 3.1.6 (The Skorohod integral, [DØP09])**.** Let $\{u(t)\}_{t\in[0,T]}$, be a measurable stochastic process such that for all $t \in [0,T]$ the random variable $u(t)$ is $\mathcal{F}_T$-measurable and $E[\int_0^T u^2(t)dt] < \infty$. Let its Wiener–Itô chaos expansion be

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t)).$$

Then we define the Skorohod integral of u by

$$\delta(u) := \int_0^T u(t)\delta B(t) := \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n)$$

when convergent in $L^2(P)$. Here $\tilde{f}_n$, $n = 1, 2, ...$ are the symmetric functions derived from $f_n(\cdot,t)$, $n = 1, 2, ...$ We say that u is Skorohod integrable, and we write $u \in Dom(\delta)$ if the series converges in $L^2(P)$.

Note that a stochastic process u is in $Dom(\delta)$ if and only if

$$E[\delta(u)^2] = \sum_{n=1}^{\infty} (n+1)!\|\tilde{f}\|^2_{L^2[0,T]^{n+1}} < \infty. \tag{3.3}$$

The Skorohod integral satisfies several properties which are usually considered desirable for integrals, however, there is one exception which will be pointed out.

Proposition 3.1.7 (Properties of the Skorohod integral, [DØP09])**.** *Let $u, v \in Dom(\delta)$, $\alpha, \beta \in \mathbb{R}$ and $t \in [0,T]$, then*

(i) $\chi_{(0,t]}u \in Dom(\delta)$ *and* $\chi_{(t,T]}u \in Dom(\delta)$,

(ii) $\int_0^t u(s)\delta B(s) = \int_0^T \chi_{(0,t]}(s)u(s)\delta B(s)$ *and*
$\int_t^T u(s)\delta B(s) = \int_0^T \chi_{(t,T]}(s)u(s)\delta B(s)$,

(iii) $\int_0^T u(s)\delta B(s) = \int_0^t u(s)\delta B(s) + \int_t^T u(s)\delta B(s)$

(iv) $\delta(\alpha u + \beta v) = \alpha\delta(u) + \beta\delta(v)$,

(v) $E[\delta(u)] = 0$.

Remark 3.1.8 ([DØP09]). Let $u \in Dom(\delta)$ and G be a $\mathcal{F}_T$-measurable random variable with $Gu \in Dom(\delta)$. For Skorohod integrals we have in general

$$G\int_0^T u(t)\delta B(t) \neq \int_0^T Gu(t)\delta B(t),$$

whereas if u was Itô integrable we would be able to pull G in and out of the integral, that is

$$G\int_0^T u(t)dB(t) = \int_0^T Gu(t)dB(t).$$

See example 2.4 in [DØP09] for an example of this occurrence.

Since the Skorohod integral is defined as a sum of iterated Itô integrals it makes sense for there to be some kind of correspondence between the two, in fact, we even have equality for all adapted integrands.

Theorem 3.1.9 (The Skorohod integral extends the Itô integral, [DØP09]). *Let $u = \{u(t)\}_{t\in[0,T]}$, be a measurable $\mathcal{F}$-adapted stochastic process such that*

$$E\left[\int_0^T u^2(t)dt\right] < \infty.$$

Then $u \in Dom(\delta)$ and its Skorohod integral coincides with the Itô integral

$$\int_0^T u(t)\delta B(t) = \int_0^T u(t)dB(t).$$

Many of the SDEs under consideration in chapter 7 also contain at least one term involving a Malliavin derivative, this definition too, relies fully on the Wiener-Itô chaos expansion.

Definition 3.1.10 (Malliavin derivative, [DØP09]). Let $F \in L^2(P)$ be $\mathcal{F}_T$-measurable with chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_n \in \tilde{L}^2([0,T]^n)$, $n = 1, 2, ...$

i) We say that $F \in \mathbb{D}_{1,2}$ if

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{n=1}^{\infty} nn!\|f_n\|^2_{L^2([0,T]^n)} < \infty.$$

ii) If $F \in \mathbb{D}_{1,2}$ we define the Malliavin derivative D_tF of F at time t as the expansion

$$D_tF = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t)), \quad t \in [0,T],$$

where $I_{n-1}(f_n(\cdot,t))$ is the $(n-1)$-fold iterated integral of $f_n(t_1,...,t_{n-1},t)$ with respect to the first $n-1$ variables $t_1,...,t_{n-1}$ and $t_n = t$ left as a parameter.

The next property of the Malliavin derivative is crucial for the proof of several of the results listed below.

Theorem 3.1.11 (Closability of the Malliavin derivative, [DØP09])**.** *Suppose $F \in L^2(P)$ and $F_k \in \mathbb{D}_{1,2}$, $k = 1, 2, ...$, such that*

(i) *$F_k \to F$, $k \to \infty$, in $L^2(P)$*

(ii) *$\{D_t F_k\}_{k=1}^{\infty}$ converges in $L^2(P \times \lambda)$.*

Then $F \in \mathbb{D}_{1,2}$ and $D_t F_k \to D_t F$, $k \to \infty$, in $L^2(P \times \lambda)$.

The Malliavin derivative, like the Skorohod integral, inherits linearity from the Wiener-Itô chaos expansion, this is used frequently throughout this thesis.

Furthermore, the Malliavin derivative satisfies a product rule and a chain rule similar to the classical derivative. We denote by $\mathbb{D}^0_{1,2}$ the set of all $F \in L^2(P)$ whose chaos expansion has finitely many terms.

Theorem 3.1.12 (Product rule, [DØP09])**.** *Suppose $F_1, F_2 \in \mathbb{D}^0_{1,2}$. Then $F_1, F_2 \in \mathbb{D}_{1,2}$ and also $F_1F_2 \in \mathbb{D}_{1,2}$ with*

$$D_t(F_1F_2) = F_1 D_t F_2 + F_2 D_t F_1.$$

The above product rule only holds for random variables with finite chaos expansions, therefore, its direct use is limited. However, the set of all $G \in D^0_{1,2}$ is dense in $L^2(P)$. Hence, it is often possible to use the product rule in combination with a density argument to prove general results that hold for all $G \in \mathbb{D}_{1,2}$. For many of our applications we escape such a density argument and the contents of the following remark is often sufficient.

Remark 3.1.13. We will on several occasions want to move something deterministic outside of the Malliavin derivative, this is therefore proved.

Let $\{X_t\}_{t\in[0,T]}$ be a stochastic process with chaos expansion $X(s) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,s))$, and let g be a deterministic function, then

$$\begin{aligned}
D_t(g(s)X(s)) &= D_t\Big(g(s)\sum_{n=0}^{\infty} I_n(f_n(\cdot,s))\Big) = D_t\Big(\sum_{n=0}^{\infty} g(s)I_n(f_n(\cdot,s))\Big) \\
&= D_t\Big(\sum_{n=0}^{\infty} I_n(g(s)f_n(\cdot,s))\Big) = \sum_{n=1}^{\infty} nI_{n-1}(g(s)f_n(\cdot,t,s)) \\
&= g(s)\sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t,s)) = g(s)D_t\Big(\sum_{n=0}^{\infty} I_n(f_n(\cdot,s))\Big) = g(s)D_t(X(s))
\end{aligned}$$

for all $s, t \in [0,T]$. For simplicity, we refer to the product rule instead of this remark, since a product rule allowing for infinite chaos expansions would have implied the above result.

The chain rule holds more generally and, as we shall see, it actually implies a more general product rule as well.

Theorem 3.1.14 (Chain rule, [DØP09]). *Assume $F_1, ..., F_m \in L^2(P)$ is Hida–Malliavin differentiable in $L^2(P)$. Suppose that $\varphi \in C^1(\mathbb{R}^m)$, $D_t F_i \in L^2(P)$, for all $t \in \mathbb{R}$, and $\frac{\partial \varphi}{x_i}(F) D.F_i \in L^2(P \times dt)$ for $i = 1, ..., m$, where $F = (F_1, ..., F_m)$. Then $\varphi(F)$ is Hida–Malliavin differentiable and*

$$D_t \varphi(F) = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i}(F) D_t F_i.$$

The Hida-Malliavin derivative is defined for a larger class of random variables and coincides with the Malliavin derivative we have defined earlier when they are both defined, since we never use the Hida-Malliavin derivative we refer to [DØP09] for details.

Remark 3.1.15. Note that 3.1.14 extends the product rule. Choose $F_1, F_2 \in L^2(P)$ with $D_t F_i \in L^2(P)$, for $i = 1, 2$, and define $\varphi \in C^1(\mathbb{R}^2)$ by $\varphi(F_1, F_2) = F_1 F_2$. If we also assume

$$\frac{\partial \varphi}{\partial x_i}(F) D.F_i = F_j D.F_i \in L^2(P \times dt),$$

for $i, j = 1, 2$ with $i \neq j$, we get

$$D_t \varphi(F_1, F_2) = \frac{\partial \varphi(F_1, F_2)}{\partial x_1} D_t F_1 + \frac{\partial \varphi(F_1, F_2)}{\partial x_2} D_t F_2 = F_2 D_t F_1 + F_1 D_t F_2.$$

In particular, the above chain rule implies the following more simple chain rule

Theorem 3.1.16 (Chain rule, [DØP09]). *Let $G \in \mathbb{D}_{1,2}$ and $g \in C^1(\mathbb{R})$ with bounded derivative. Then $g(G) \in \mathbb{D}_{1,2}$ and*

$$D_t g(G) = g'(G) D_t G.$$

Here $g'(x) = \frac{d}{dx} g(x)$.

Another interesting property is that the Malliavin derivative preserves adaptedness.

Theorem 3.1.17 (Malliavin derivative and adaptedness, [DØP09]). *Let $u = \{u(s)\}_{s \in [0,T]}$, be an $\mathbb{F}$-adapted stochastic process and assume that $u(s) \in \mathbb{D}_{1,2}$ for all s. Then*

(i) *$D_t u(s)$, $s \in [0, T]$, is $\mathbb{F}$-adapted for all t;*

(ii) *$D_t u(s) = 0$, for $t > s$.*

We end this subsection with some important formulas for the relation between the Skorohod integral and the Malliavin derivative.

Theorem 3.1.18 (Duality formula, [DØP09]). *Let $F \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$-measurable and let u be a Skorohod integrable stochastic process. Then*

$$E\left[F \int_0^T u(t) \delta B(t)\right] = E\left[\int_0^T u(t) D_t F dt\right].$$

Note that this duality, together with the closability of the Malliavin derivative, implies that the Skorohod integral is closable in the sense that if

$$\int_0^T u_n(t)\delta B(t), \quad n \in \mathbb{N}$$

converges in $L^2(P)$ and

$$\lim_{n\to\infty} u_n = 0, \quad \text{in } L^2(P \times dt),$$

where dt denotes the Lebesgue measure on $[0,T]$, then

$$\lim_{n\to\infty} \int_0^T u_n(t)\delta B(t) = 0 \quad \text{in } L^2(P).$$

Another correspondence between the Malliavin derivative and the Skorohod integral is the integration by parts formula. Recalling Remark 3.1.8, this formula seems reasonable.

Theorem 3.1.19 (Integration by parts, [DØP09]). *Let $\{u(t)\}_{t\in[0,T]}$, be a Skorohod integrable stochastic process and let $F \in \mathbb{D}_{1,2}$ such that the product $Fu(t)$, $t \in [0,T]$, is Skorohod integrable. Then*

$$F\int_0^T u(t)\delta B(t) = \int_0^T Fu(t)\delta B(t) + \int_0^T u(t)D_t F dt.$$

Furthermore, one might wonder what happens if we take the Malliavin derivative of the Skorohod integral. The next theorem will quench this curiosity.

Theorem 3.1.20 (The fundamental theorem of calculus, [DØP09]). *Let $u = \{u(s)\}_{s\in[0,T]}$, be a stochastic process such that*

$$E\left[\int_0^T u^2(s)ds\right] < \infty$$

and assume that, for all $s \in [0,T]$, $u(s) \in \mathbb{D}_{1,2}$ and that, for all $t \in [0,T]$, $D_t u \in Dom(\delta)$. Assume also that

$$E\left[\int_0^T \big(\delta(D_t u)\big)^2 dt\right] < \infty$$

Then $\int_0^T u(s)\delta B(s)$ is well-defined and belongs to $\mathbb{D}_{1,2}$ and

$$D_t\Big(\int_0^T u(s)\delta B(s)\Big) = \int_0^T D_t u(s)\delta B(s) + u(t)$$

The final correspondence result of this section is often more useful than the $L^2(P)$ equality (3.3), and will be used several times in chapter 7.

Theorem 3.1.21 (Isometry for Skorohod intgral, [DØP09]). *Let u be a measurable process such that $u(s) \in \mathbb{D}_{1,2}$ for a.a. s and*

$$E\left[\int_0^T u^2(t)dt + \int_0^T\int_0^T |D_t u(s) D_s u(t)| ds dt\right] < \infty.$$

Then u is Skorohod integrable and

$$E\left[\left(\int_0^T u(s)\delta B(s)\right)^2\right] = E\left[\int_0^T u^2(t)dt + \int_0^T \int_0^T D_t u(s) D_s u(t) ds dt\right]. \tag{3.4}$$

Notice that if u is also assumed to be $\mathbb{F}$-adapted, then by Theorem 3.1.17, the Skorohod isometry (3.4) reduces to the Itô isometry 2.1.7.

In chapter 7 we will on many occasions want to commute the Malliavin derivative and an integral which integrates against a Lebesgue-Stieltjes measure. I have been unsuccessful in my attempts at finding a result on this in the literature, so the following result is proved by me.

Proposition 3.1.22 (Commutation of Malliavin derivative and integral)**.** *Let μ be a finite signed measure. Let $X = \{X_t\}_{t\in[0,T]}$ be a stochastic process in $L^2(P)$ such that X_s is $\mathcal{F}_T$-measurable for all $s \in [0,T]$. Further we assume that X is Malliavin differentiable and the integrability conditions*

$$\int_0^u E\big[(X_s)^2\big] d\mu(s) < \infty \tag{3.5}$$

and

$$\int_0^u \int_0^T E\big[(D_t X_s)^2\big] dt d\mu(s) < \infty \tag{3.6}$$

for all $u \in [0,T]$. Then, $\int_0^u X_s d\mu(s) \in \mathbb{D}_{1,2}$ and

$$D_r \int_0^u X_s d\mu(s) = \int_0^u D_r X_s d\mu(s).$$

Proof. For starters, note that by the linearity of the Malliavin derivative we have

$$\begin{aligned} D_r\Big(\int_0^u X_s d\mu(s)\Big) &= D_r\Big(\int_0^u X_s d\mu^+(s) - \int_0^u X_s d\mu^-(s)\Big) \\ &= D_r \int_0^u X_s d\mu^+(s) - D_r \int_0^u X_s d\mu^-(s), \end{aligned}$$

where μ^+ and μ^- are positive measures stemming from the Jordan decomposition of μ, see [MW12] for details on the Jordan decomposition theorem. We can therefore, without loss of generalization, assume μ is a positive measure.

The rest of this proof has two parts, we first show that

$$D_r \int_0^u X_s^m d\mu(s) = \int_0^u D_r X_s^m d\mu(s), \tag{3.7}$$

where $X_s^m = \sum_{n=0}^m I_n(f_n(\cdot,s))$ converges to $X = \sum_{n=0}^\infty I_n(f_n(\cdot,s))$ in $L^2(P)$. Such a chaos expansion of X is guaranteed to exist by Theorem 3.1.5. To obtain (3.7) we need to swap the μ integral with the iterated Itô integrals, consequently we apply the stochastic Fubini theorem 2.3.5. This theorem requires the integrand to be predictable and for a integrability condition to be satisfied. Even though we do not assume that X itself is predictable, this is not a problem since the Itô integrals in the chaos expansions are predictable.

Furthermore, by (3.2) and Theorem 2.3.4, we have

$$
\begin{aligned}
&\int_0^u E\Big[\Big(\sum_{n=0}^{m} n! J_n(f_n(\cdot,s))\Big)^2\Big] d\mu(s) \\
&= \int_0^u \sum_{n=0}^{m} n! \int_{S_n} (f_n(t_1,...,t_n,s))^2 dt_1 \cdots dt_n d\mu(s) \\
&= \sum_{n=0}^{m} n! \int_0^u \int_0^T \cdots \int_0^{t_2} f_n(t_1,...,t_n,s)^2 dt_1 \cdots dt_n d\mu(s) \\
&= \cdots = \sum_{n=0}^{m} n! \int_0^T \cdots \int_0^{t_2} \int_0^u f_n(t_1,...,t_n,s)^2 d\mu(s) dt_1 \cdots dt_n < \infty,
\end{aligned}
$$

which is finite by assumption 3.5. Therefore, applying the stochastic Fubini theorem 2.3.5 is justified, and we get

$$
\begin{aligned}
&\int_0^u D_r X_s^m d\mu(s) = \int_0^u D_r \sum_{n=0}^{m} I_n(f_n(\cdot,s)) d\mu(s) \\
&= \int_0^u \sum_{n=1}^{m} n I_{n-1}(f_n(\cdot,r,s)) d\mu(s) = \sum_{n=1}^{m} n \int_0^u I_{n-1}(f_n(\cdot,r,s)) d\mu(s) \\
&= \sum_{n=1}^{m} n \int_0^u \int_0^T \cdots \int_0^T f_n(t_1,\cdots,t_{n-1},r,s) dB_{t_1} \cdots dB_{t_{n-1}} d\mu(s) \\
&= \cdots = \sum_{n=1}^{m} n \int_0^T \cdots \int_0^T \int_0^u f_n(t_1,\cdots,t_{n-1},r,s) d\mu(s) dB_{t_1} \cdots dB_{t_{n-1}} \\
&= \sum_{n=1}^{m} n I_{n-1}\Big(\int_0^u f_n(\cdot,r,s) d\mu(s)\Big).
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
&D_r \int_0^u X_s^m d\mu(s) = D_r \int_0^u \sum_{n=0}^{m} I_n(f_n(\cdot,s)) d\mu(s) \\
&= \cdots = D_r \sum_{n=0}^{m} I_n\Big(\int_0^u f_n(\cdot,s) d\mu(s)\Big) = \sum_{n=1}^{m} n I_{n-1}\Big(\int_0^u f_n(\cdot,r,s) d\mu(s)\Big).
\end{aligned}
$$

Hence, equality (3.7) holds. For infinite chaos expansion we can not automatically swap the sum and the integrand, we will instead take the limit in equality (3.7). We have

$$
\begin{aligned}
&E\Big[\int_0^T \Big(\int_0^u D_r X_s d\mu(s) - D_r \int_0^u X_s^m d\mu(s)\Big)^2 dr\Big] \\
&E\Big[\int_0^T \Big(\int_0^u D_r X_s - D_r X_s^m d\mu(s)\Big)^2 dr\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mu([0,u]) \int_0^u E\Big[\int_0^T \big(D_r X_s - D_r X_s^m\big)^2 dr\Big] d\mu(s) \\
&= \mu([0,u]) \int_0^u E\Big[\int_0^T \Big(\sum_{n=m+1}^{\infty} I_{n-1}(f(\cdot,s))\Big)^2 dr\Big] d\mu(s), \qquad (3.8)
\end{aligned}
$$

which goes to 0 as $m \to \infty$ by assumption (3.6). Lastly, we have

$$
E\Big[\Big(\int_0^u X_s - X_s^m d\mu(s)\Big)^2\Big] \leq \mu([0,u]) \int_0^u E\big[(X_s - X_s^m)^2\big] d\mu(s) \qquad (3.9)
$$

which goes to 0 by (3.5). Summing up, we know that $\int_0^u X_s^m d\mu(s)$ converges to $\int_0^u X_s d\mu(s)$ in $L^2(P)$ by (3.9), and that $D_r \int_0^u X_s^m$ converges in $L^2(P \times dt)$ by (3.8), these are the conditions of Theorem 3.1.11. Therefore, by the closability of the Malliavin derivative, we have that $\int_0^u X_s d\mu(s) \in \mathbb{D}_{1,2}$ and that

$$
\begin{aligned}
&\int_0^u D_r X_s d\mu(s) = \lim_{m\to\infty} \int_0^u D_r X_s^m d\mu(s) \\
&= \lim_{m\to\infty} D_r \int_0^u X_s^m d\mu(s) = D_r \int_0^u X_s d\mu(s),
\end{aligned}
$$

where the limit is taken in $L^2(P \times dt)$.

■

3.2 Pure jump Lévy case

This section will give a similar summary for Malliavin calculus as the last section, but this time with respect to the pure jump Lévy case, that is, a process $L = \{L(t)\}_{t\geq 0}$ of the form

$$
L(t) = \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(dz, ds).
$$

As this case is very similar to the Brownian motion case we keep it shorter, but some differences will be pointed out.

Let $L^2((\lambda \times \nu)^n) = L^2(([0,T] \times \mathbb{R}_0)^n)$ be the space of deterministic real functions f such that

$$
\|f\|_{L^2((\lambda\times\nu)^n)} = \Big(\int_{([0,T]\times\mathbb{R}_0)^n} f^2(t_1, z_1, ..., t_n, z_n) dt_1 \nu(dz_1) \cdots dt_n \nu(dz_n)\Big)^{1/2} < \infty,
$$

where λ denotes the Lebesgue measure on $[0, T]$.

Definition 3.2.1 (Symmetrization of a function, [DØP09]). The symmetrization $\tilde{f}$ of f is defined by

$$
\tilde{f}(t_1, z_1, ..., t_n, z_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, z_{\sigma_1}, ..., t_{\sigma_n}, z_{\sigma_n}),
$$

the sum being taken over all permutations $\sigma = (\sigma_1, ..., \sigma_n)$ of $\{1, ..., n\}$. Note that the symmetrization is over the n pairs $(t_1, z_1), ..., (t_n, z_n)$ and not over the $2n$ variables $t_1, z_1, ..., t_n, z_n$.

Definition 3.2.2 (Symmetric function, [DØP09]). A function $f \in L^2((\lambda \times \nu)^n)$ is called symmetric if $f = \tilde{f}$. We denote the space of all symmetric functions in $L^2((\lambda \times \nu)^n)$ by $\tilde{L}^2((\lambda \times \nu)^n)$.

Definition 3.2.3 (Iterated integral, [DØP09]). Let f be a deterministic function defined on

$$G_n = \{(t_1, z_1, ..., t_n, z_n) : 0 \le t_1 \le ... \le t_n \le T,\ z_i \in \mathbb{R}_0),\ i = 1, ..., n\}$$

such that

$$\|f\|^2_{L^2(G_n)} := \int_{G_n} f^2(t_1, z_1, ..., t_n, z_n) dt_1 \nu(dz_1) \cdots dt_n \nu(dz_n) < \infty.$$

Then we can define the n-fold iterated integral as

$$J_n(f) := \int_0^T \int_{\mathbb{R}_0} \cdots \int_0^{t_2^-} \int_{\mathbb{R}_0} f(t_1 z_1, ..., t_n, z_n) \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n).$$

If $g \in \tilde{L}^2((\lambda \times \nu)^n)$ we define

$$I_n(g) := \int_{([0,T] \times \mathbb{R}_0)^n} g(t_1 z_1, ..., t_n, z_n) \tilde{N}^{\otimes n}(dt, dz) := n! J_n(g).$$

We also call $I_n(g)$ the n-fold iterated integral of g.

Observe that the iterated integrals are taken with respect to the compensated Poisson random measure associated to the Lévy process L and not with respect to the process itself.

Theorem 3.2.4 (The Wiener-Itô chaos expansion, [DØP09]). *Let F be an $\mathcal{F}_T$-measurable random variable in $L^2(P)$. Then there exists a unique sequence $\{f_n\}_{n=0}^\infty$ of functions $f_n \in \tilde{L}^2((\lambda \times \nu)^n)$ such that*

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where the convergence is in $L^2(P)$. Moreover, we have the isometry

$$\|F\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2((\lambda \times \nu)^n)}.$$

Definition 3.2.5 (The Skorohod integral, [DØP09]). Let $X = X(t, z)$, $0 \le t \le T$, $z \in \mathbb{R}_0$, be a stochastic process (more precisely a random field) such that $X(t, z)$ is an $\mathcal{F}_T$-measurable random variable for all $(t, z) \in [0, T] \times \mathbb{R}_0$ and

$$E\left[\int_0^T \int_{\mathbb{R}_0} X^2(t, z) \nu(dz) dt \right] < \infty.$$

Then for each (t, z), the random variable $X(t, z)$ has an expansion of the form

$$X(t, z) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t, z)), \quad \text{where } f_n(\cdot, t, z) \in \tilde{L}^2((\lambda \times \nu)^n).$$

Let $\tilde{f}_n(t_1, z_1, ..., t_n, z_n, t_{n+1}, z_{n+1})$ be the symmetrization of $f_n(t_1, z_1, ..., t_n, z_n, t, z)$ as a function of the $n+1$ pairs $(t_1, z_1), ..., (t_n, z_n), (t, z) = (t_{n+1}, z_{n+1})$. Suppose that

$$\sum_{n=0}^{\infty}(n+1)!\|\tilde{f}_n\|^2_{L^2((\lambda\times\nu)^{n+1})} < \infty.$$

Then we say that X is Skorohod integrable and we write $X \in Dom(\delta)$. We define the Skorohod integral $\delta(X)$ of X with respect to $\tilde{N}$ by

$$\delta(X) = \int_0^T \int_{\mathbb{R}_0} X(t,z)\tilde{N}(\delta t, dz) := \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n).$$

Furthermore, the following $L^2(P)$ equality holds

$$E[\delta(X)^2] = \sum_{n=1}^{\infty}(n+1)!\|\tilde{f}\|^2_{L^2((\lambda\times\nu)^{n+1})} < \infty.$$

Proposition 3.2.6 (Properties of the Skorohod integral, [DØP09]). *Let $u, v \in Dom(\delta)$ and $\alpha, \beta \in \mathbb{R}$, then*

i) $\delta(\alpha u + \beta v) = \alpha\delta(u) + \beta\delta(v)$,

ii) $E[\delta(u)] = 0$.

Theorem 3.2.7 (The Skorohod integral extends the Itô integral, [DØP09]). *Let $X = X(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}_0$, be a predictable process such that*

$$E\left[\int_0^T \int_{\mathbb{R}_0} X^2(t,z)\nu(dz)dt\right] < \infty.$$

Then X is both Itô and Skorohod integrable with respect to $\tilde{N}$ and

$$\int_0^T \int_{\mathbb{R}_0} X(t,z)\tilde{N}(\delta t, dz) = \int_0^T \int_{\mathbb{R}_0} X(t,z)\tilde{N}(dt, dz).$$

Note that in the above theorem the requirement is that the integrand is predictable, which is stronger than the assumption in the Brownian case where adaptedness of the integrand was enough.

Definition 3.2.8 (The Malliavin derivative, [DØP09]). Let $F \in L^2(P)$ be $\mathcal{F}_T$-measurable with chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_n \in \tilde{L}^2((\lambda\times\nu)^n)$, $n = 1, 2, ...$

i) We say that $F \in \mathbb{D}_{1,2}$ if

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{n=1}^{\infty} n n!\|f_n\|^2_{L^2((\lambda\times\nu)^n)} < \infty.$$

ii) If $F \in \mathbb{D}_{1,2}$ we define the Malliavin derivative $D_{t,z}F$ of F at $(t,z) \in [0,T] \times \mathbb{R}_0$ as the expansion

$$D_{t,z}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t, z)),$$

where $I_{n-1}(f_n(\cdot, t, z))$ is the $n-1$-fold iterated integral of $f_n(t_1, z_1..., t_{n-1}, z_{n-1}, t, z)$ with respect to the first $n-1$ pairs of variables $(t_1, z_1), ..., (t_{n-1}, z_{n-1}$ and $(t_n, z_n) = (t, z)$ left as a parameter.

Theorem 3.2.9 (Closability of the Malliavin derivative, [DØP09]). *Suppose $F \in L^2(P)$ and F_k, $k = 1, 2, ...,$ are in $\mathbb{D}_{1,2}$ and that*

(i) $F_k \to F$, $k \to \infty$ *in* $L^2(P)$.

(ii) $D_{t,z}F_k$, $k = 1, 2, ...,$ *converges in* $L^2(P \times \lambda \times \nu)$.

Then $F \in \mathbb{D}_{1,2}$ and $D_{t,z}F_k \to D_{t,z}F$, $k \to \infty$, in $L^2(P \times \lambda \times \nu)$.

In the Brownian case we saw that the Malliavin derivative satisfies a product rule and a chain rule similar to that of the classical derivative. In the pure jump Lévy case however, both the product rule and the chain rule takes on a different form. We denote by $\mathbb{D}_{1,2}^{\exp}$ the set of all linear combinations of exponentials of the form

$$\exp\left\{ \int_0^T \int_{\mathbb{R}_0} h(s) z \tilde{N}(ds, dz) \right\},$$

where $h \in L^2[0,T]$.

Theorem 3.2.10 (Product rule, [DØP09]). *Suppose $F_1, F_2 \in \mathbb{D}_{1,2}^{\exp}$. Then $F_1F_2 \in \mathbb{D}_{1,2}^{\exp}$ and*

$$D_{t,z}(F_1F_2) = F_1 D_{t,z}F_2 + F_2 D_{t,z}F_1 + D_{t,z}F D_{t,z}G.$$

As in the Brownian motion case this product rule is of limited direct use and we remark again that we can move deterministic functions in and out of the Malliavin derivative.

Remark 3.2.11. Let $X(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}_0$ be a stochastic process with chaos expansion $X(s,z) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, s, z))$, and let g be a deterministic function, then

$$\begin{aligned} D_{t,y}(g(s,z)X(s,z)) &= \sum_{n=1}^{\infty} nI_{n-1}(g(s,z)f_n(\cdot, t, y, s, z)) \\ &= g(s,z)D_t\Big(\sum_{n=0}^{\infty} I_n(f_n(\cdot, s, z))\Big) = g(s,z)D_{t,y}(X(s,z)) \end{aligned}$$

for all $t \in [0,T]$, $y \in \mathbb{R}_0$. Unlike in the Brownian case we will refer to this remark when it is used, since this product rule does not share its form with the product rule for a traditional derivative.

There is another alternative product rule..

Theorem 3.2.12 (Alternative product rule, [Di +05]). *Let $F, G \in \mathbb{D}_{1,2}$ with G bounded. Then $FG \in \mathbb{D}_{1,2}$ and we have*

$$D_{t,z}(FG) = FD_{t,z}G + GD_{t,z}F + D_{t,z}FD_{t,z}G, \quad \lambda \times \nu - a.e.$$

The chain rule holds more generally.

Theorem 3.2.13 (Chain rule, [DØP09]). *Let $F \in \mathbb{D}_{1,2}$ and let g be a real continuous function on $\mathbb{R}$. Suppose $g(F) \in L^2(P)$ and $g(F + D_{t,z}F) \in L^2(P \times \lambda \times \nu)$. Then $g(F) \in \mathbb{D}_{1,2}$ and*

$$D_{t,z}g(F) = g(F + D_{t,z}F) - g(F).$$

In opposition to the product rule and the chain rule, the duality formula is analogous to the Brownian motion case.

Theorem 3.2.14 (Duality formula, [DØP09]). *Let $X(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}$, be Skorohod integrable and $F \in \mathbb{D}_{1,2}$. Then*

$$E\left[F \int_0^T \int_{\mathbb{R}_0} X(t,z)\tilde{N}(\delta t, dz)\right] = E\left[\int_0^T \int_{\mathbb{R}_0} X(t,z)D_{t,z}F\nu(dz)dt\right]$$

The integration by parts formula is different from the Brownian case. This is a direct consequence of the product rule, see [DØP09] for proof.

Theorem 3.2.15 (Integration by parts, [DØP09]). *Let $X(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}$, be a Skorohod integrable stochastic process and $F \in \mathbb{D}_{1,2}$ such that the product $X(t,z) \cdot (F + D_{t,z}F)$, $t \in [0,T]$, $z \in \mathbb{R}$, is Skorohod integrable. Then*

$$\begin{aligned} &F \int_0^T \int_{\mathbb{R}_0} X(t,z)\tilde{N}(\delta t, dz) \\ &= \int_0^T \int_{\mathbb{R}_0} X(t,z)\big(F + D_{t,z}F\big)\tilde{N}(\delta t, dz) + \int_0^T \int_{\mathbb{R}_0} X(t,z)D_{t,z}F\nu(dz)dt. \end{aligned}$$

The fundamental theorem of calculus takes on the same form as seen before.

Theorem 3.2.16 (The fundamental theorem of calculus, [DØP09]). *Let $X = X(s,y), (s,y) \in [0,T] \times \mathbb{R}_0$, be a stochastic process such that*

$$E\left[\int_0^T \int_{\mathbb{R}_0} X^2(s,y)\nu(dy)ds\right] < \infty.$$

Assume that $X(s,y) \in \mathbb{D}_{1,2}$ for all $(s,y) \in [0,T] \times \mathbb{R}_0$, and that $D_{t,z}X(\cdot,\cdot)$ is Skorohod integrable with

$$E\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_0^T \int_{\mathbb{R}_0} D_{t,z}X(s,y)\tilde{N}(\delta s, dy)\right)^2 \nu(dz)dt\right] < \infty.$$

Then

$$\int_0^T \int_{\mathbb{R}_0} X(s,y)\tilde{N}(\delta s, dy) \in \mathbb{D}_{1,2}$$

and

$$D_{t,z}\int_0^T\int_{\mathbb{R}_0} X(s,y)\tilde{N}(\delta s,dy)=\int_0^T\int_{\mathbb{R}_0} D_{t,z}X(s,y)\tilde{N}(\delta s,dy)+X(t,z).$$

In particular, if $X(s,\ y) = Y(s)y$, *then*

$$D_{t,z}\int_0^T\int_{\mathbb{R}_0} Y(s)\delta L(s)=\int_0^T\int_{\mathbb{R}_0} D_{t,z}Y(s)\delta L(s)+zY(t).$$

Theorem 3.2.17 (The Lévy-Skorohod isometry, [DØP09])**.** *Let* $X\in L^2(P\times\lambda\times\nu)$ *and* $DX\in L^2(P\times(\lambda\times\nu)^2)$. *Then the following isometry holds*

$$E\Big[\Big(\int_0^\infty\int_{\mathbb{R}_0} X(t,z)\tilde{N}(\delta t,dz)\Big)^2\Big]=E\Big[\int_0^\infty\int_{\mathbb{R}_0} X^2(t,z)\nu(dz)dt$$
$$+\int_0^\infty\int_{\mathbb{R}_0}\int_0^\infty\int_{\mathbb{R}_0} D_{t,z}X(s,y)D_{s,y}X(t,z)\nu(dy)ds\nu(dz)dt\Big]$$

We end this chapter with the same type of commutation result as in the last section.

Proposition 3.2.18 (Commutation of Malliavin derivative and integral)**.** *Let* μ *be a finite signed measure. Let* $X = X(s,z)$, $t\in[0,T]$, $z\in\mathbb{R}_0$ *be a stochastic process in* $L^2(P)$ *such that* $X(s,z)$ *is* $\mathcal{F}_T$*-measurable for all* $(s,z)\in[0,T]\times\mathbb{R}_0$. *Further we assume that* X *is Malliavin differentiable and the integrability conditions*

$$\int_0^u E\big[(X(s,z))^2\big]d\mu(s)<\infty$$

and

$$\int_0^u\int_0^T\int_{\mathbb{R}_0} E\big[(D_{t,y}X(s,z)^2\big]\nu(dy)dtd\mu(s)<\infty$$

for all $u\in[0,T]$. *Then we have* $\int_0^u X(s,z)d\mu(s)\in\mathbb{D}_{1,2}$ *and*

$$D_{r,y}\int_0^u X(s,z)d\mu(s)=\int_0^u D_{r,y}X(s,z)d\mu(s).$$

Proof. The result follows by the same procedure as for Proposition 3.1.22. ■

CHAPTER 4

Hilbert space preliminaries

Whereas the last two chapters have focused on the real-valued case of stochastic analysis and Malliavin calculus, this chapter will introduce Malliavin calculus and other necessary preliminaries in the Hilbert-valued case. Basics in functional analysis will, for the most part, be assumed known, but relevant terminology will be declared. This chapter is used in both chapter 5, 6 and 7, where we will be looking at SDEs in a Hilbert space and integrals with ambit fields as integrators.

Throughout this chapter we let $\mathcal{H}, \mathcal{G}$ be real separable Hilbert spaces with inner product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ respectively. The norm of an element $h \in \mathcal{H}$ will be denoted by $\|h\|_{\mathcal{H}}$.

We begin by looking at two important classes of operators.

4.1 Hilbert-Schmidt and trace class operators

This section is based on appendices in [PZ07] and [Fol16]. Let $L(\mathcal{H}, \mathcal{G})$ be the space of bounded linear operators from $\mathcal{H} \to \mathcal{G}$ equipped with the operator norm

$$\|T\|_{L(\mathcal{H},\mathcal{G})} = \sup_{x \in \mathcal{H}, x \neq 0} \frac{\|Tx\|_{\mathcal{G}}}{\|x\|_{\mathcal{H}}}.$$

If $\mathcal{H} = \mathcal{G}$ we write $L(\mathcal{H}) := L(\mathcal{H}, \mathcal{H})$, and we define the adjoint of $T \in L(\mathcal{H}, \mathcal{G})$ to be the unique operator $T^* \in L(\mathcal{G}, \mathcal{H})$ satisfying

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \quad \text{for all } x \in \mathcal{H}, y \in \mathcal{G}.$$

$T \in L(\mathcal{H}, \mathcal{G})$ is called symmetric or self-adjoint if

$$\langle Tx, y \rangle_{\mathcal{G}} = \langle x, Ty \rangle_{\mathcal{H}}, \quad \text{for all } x, y \in \mathcal{H}$$

that is $T = T^*$. Furthermore, we say that $T \in L(\mathcal{H})$ is nonnegative if

$$\langle Tx, x \rangle_{\mathcal{H}} \geq 0, \quad \text{for all } x \in \mathcal{H}.$$

Note that a symmetric operator and a self-adjoint operator is generally not the same and that in functional analysis the above definition is usually the definition of a self-adjoint operator. However, when the domain of T is H, as is the case above, then they are equivalent definitions. Since the term symmetric seems to be the most commonly used in books concerned with stochastic analysis we will stick to this terminology.

In chapter 6 we will encounter Hilbert-Schmidt operators, we therefore define them here. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$.

Definition 4.1.1 (Trace class operator, [Fol16]). An operator $T \in L(\mathcal{H})$ is called trace-class if the nonnegative operator $|T| := (T^*T)^{1/2}$ has finite trace, that is

$$Tr(|T|) = \sum_{i=1}^{\infty} \langle |T|e_i, e_i\rangle_{\mathcal{H}} < \infty.$$

Definition 4.1.2 (Hilbert-Schmidt operator, [PZ07], [Fol16]). An operator $T \in L(\mathcal{H}, \mathcal{G})$ is called Hilbert-Schmidt if T^*T is trace class, i.e.

$$Tr(T^*T) = \sum_{i=1}^{\infty} \|Te_i\|_{\mathcal{G}}^2 < \infty.$$

The space of all Hilbert–Schmidt operators $L_2(\mathcal{H}, \mathcal{G})$ acting from $\mathcal{H}$ into $\mathcal{G}$ is itself a separable Hilbert space with inner product

$$\langle T, S\rangle_2 = \sum_{i=1}^{\infty} \langle Te_i, Se_i\rangle_{\mathcal{G}}$$

The corresponding Hilbert–Schmidt norm is denoted by $\|\cdot\|_2$.

Moreover, the following useful estimates holds for the composition of operators in the Hilbert-Schmidt norm. These will be useful in section 7.4.

Proposition 4.1.3 (Hilbert-Schmidt norm estimates, [LR15]). *Let $\mathcal{U}$ be another Hilbert space and assume $S_1 \in L(\mathcal{H}, \mathcal{G})$, $S_2 \in L(\mathcal{G}, \mathcal{U})$, $T \in L_2(\mathcal{U}, \mathcal{H})$. Then $S_1T \in L_2(\mathcal{U}, \mathcal{G})$ and $TS_2 \in L_2(\mathcal{G}, \mathcal{H})$ and*

$$\|S_1T\|_{L_2(\mathcal{U},\mathcal{G})} \leq \|S_1\|_{L(\mathcal{H},\mathcal{G})}\|T\|_{L_2(\mathcal{U},\mathcal{H})}$$
$$\|TS_2\|_{L_2(\mathcal{G},\mathcal{H})} \leq \|T\|_{L_2(\mathcal{U},\mathcal{H})}\|S_2\|_{L(\mathcal{G},\mathcal{U})}$$

4.2 The Bochner integral and the Fréchet derivative

This section will construct the Bochner integral based on the material found in section 2.6 in [HE15]. The framework is the measure space $(X, \mathcal{A}, \mu)$ and a Banach space $\mathcal{B}$ with norm $\|\cdot\|$, omitting the subscript $\mathcal{B}$ for simplicity. For the function $f : X \to \mathcal{B}$, the Bochner integral is defined as $\int_X f d\mu$. We formalize the definition in the following. The definition of the Fréchet derivative is also included.

The construction of the Bochner integral closely resembles the construction of the Lebesgue integral. We begin with the definition of simple Banach valued functions and the definition of Bochner integrals for simple integrands.

Definition 4.2.1 (Simple function, [HE15]). A function $f : E \to \mathcal{B}$ is called simple if it can be represented as

$$f(x) = \sum_{i=1}^{n} \chi_{A_i}(x)h_i$$

for some $n \in \mathbb{N}$ and for $A_i \in \mathcal{A}$ and $h_i \in \mathcal{B}$ for $1 \leq i \leq n$.

The definition of the Bochner integral for simple functions now follow, and it takes on the same form as the Lebesgue integral with simple integrands.

Definition 4.2.2 (Bochner integral for simple function, [HE15])**.** A function $f : E \to \mathcal{B}$ is called simple if it can be represented as

$$f(x) = \sum_{i=1}^{n} \chi_{A_i}(x) h_i$$

for some $n \in \mathbb{N}$ and for $A_i \in \mathcal{A}$ and $h_i \in \mathcal{B}$ for $1 \leq i \leq n$.

The general Bochner integral is now viable, we will need it in the definition of integrals against ambit fields in section 6.3 and in section 7.4 where we consider Hilbert valued SPDEs.

Definition 4.2.3 (Bochner integral, [HE15])**.** A measurable function f is said to be Bochner integrable if there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple and Bochner integrable functions such that

$$\lim_{n\to\infty} \int_X \|f_n - f\| d\mu = 0,$$

where the integral on the left hand side of the equality is an ordinary Lebesgue integral. In this case, the Bochner integral of f is defined as

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

This limit exist, as by the triangle inequality applied twice, with $\{f_n\}_{n\in\mathbb{N}}$ a sequence of simple functions, we get

$$\begin{aligned} \left\| \int_X f_n d\mu - \int_X f_m d\mu \right\| &\leq \int_X \|f_n - f_m\| d\mu \\ &\leq \int_X \|f_n - f\| d\mu + \int_X \|f - f_m\| d\mu \end{aligned}$$

where both terms converge to 0 as $n, m \to \infty$ by assumption. Hence, $\{\int_X f_n d\mu\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and its limit must exist by the completeness of $\mathcal{B}$. Lastly, the limit is independent of the approximating sequence and therefore the integral $\int_X f d\mu$ is well-defined.

Such an approximating sequence is not guaranteed to exist in general, but the following condition ensures it does.

Theorem 4.2.4 ([HE15])**.** *Let f be a measurable function from X to $\mathcal{B}$ with*

$$\int_X \|f\| d\mu < \infty.$$

Suppose that for each n there exists a finite-dimensional subspace $\mathcal{B}_n$ of $\mathcal{B}$ such that

$$\lim_{n\to\infty} \int_X \|f - g_n\| d\mu = 0$$

for some measurable sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ taking values in $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$. Then, there exist a sequence of simple and Bochner integrable functions $\{f_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \int_X \|f_n - f\| d\mu = 0.$$

Our work is done in separable Hilbert spaces, and fortunately, in this case the integrability condition $\int_X \|f\| d\mu < \infty$ ensures the existence of such a sequence $\{g_n\}_{n\in\mathbb{N}}$.

Theorem 4.2.5 (Bochner Integral in separable Hilbert space, [HE15])**.** *Suppose $\mathcal{B}$ is a separable Hilbert space and f is a measurable function from X to $\mathcal{B}$ with $\int_X \|f\| d\mu < \infty$. Then, f is Bochner integrable.*

This section is concluded with the definition of the Fréchet derivative, which will be needed when we study the integrals of section 6.3. This definition is borrowed from Definition 6.1.1 and Definition 6.1.3 in [Lin17].

Theorem 4.2.6 (Fréchet derivative, [Lin17])**.** *Let X, Y be normed spaces and U an open set of X. A function $f : U \to Y$ is Fréchet differentiable at $x \in U$ if there exists a bounded linear operator $A : X \to Y$ such that*

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Ah\|_Y}{\|h\|_X} = 0.$$

We call $f'(x) := A$ the Fréchet derivative of f and say that f is Fréchet differentiable on U if f is Fréchet differentiable at each $x \in U$.

4.3 Stochastic processes

This section is mainly a collection of definitions of Hilbert-valued stochastic processes which will be needed in the forthcoming section. Throughout this section $(U, \langle \cdot, \cdot \rangle_U)$ is a Hilbert space and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a filtered probability space satisfying the usual conditions.

We start with the definition of a Hilbert-valued martingale, and, as we shall immediately see, its definition is analogous to the definition of real-valued martingales.

Definition 4.3.1 (Martingale, [DZ14])**.** Let B be a separable Banach space with norm $\|\cdot\|_B$ and I an interval of $\mathbb{R}$. An integrable and adapted E-valued process $X = \{X(t)\}_{t\in I}$, is said to be a martingale if

i) X is integrable, that is $E\|X(t)\|_B < +\infty$ for all $t \in I$.

ii) $E(X(t) \mid \mathcal{F}_s) = X(s), \quad P-$a.s. for arbitrary $t, s \in I$, $t \geq s$.

There are several different ways of defining Wiener processes and Lévy processes on a Hilbert space, so we begin with the one that most closely resembles the real-valued case.

Definition 4.3.2 (Lévy process, [PZ07])**.** Let B be a Banach space. A stochastic process $L = \{L(t)\}_{t\geq 0}$ taking values in B and that satisfies the following conditions

(i) $L(0) = 0$,

(ii) L has independent and stationary increments,

(iii) L is stochastically continuous,

is called a Lévy process.

In the following, let Q be a trace class, symmetric and nonnegative operator on the Hilbert space U. Then, the Wiener processes on a Hilbert space called Q-Wiener processes can be defined as follows.

Definition 4.3.3 (Q-Wiener process, [DZ14])**.** A U-valued stochastic process $\{W(t)\}_{t\geq 0}$ is called a Q-Wiener process if

(i) $W(0) = 0$,

(ii) W has continuous trajectories,

(iii) W has independent increments,

(iv) $(W(t) - W(s)) \sim N(0, (t-s)Q)$, $t \leq s \leq 0$.

Let now $\{f_j\}_{j=1}^{\infty}$ be an ONB in U diagonalizing Q, and let the corresponding eigenvalues be $\{\lambda_j\}_{j=1}^{\infty}$. Let $\{w_j(t)\}_{t\geq 0}$, $j = 1, 2, ...$, be a sequence of independent (real-valued) Brownian motions defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, P)$. A Q-Wiener $\{W\}_{t\geq 0}$ process admits the decomposition

$$W_t = \sum_{n=1}^{\infty} \lambda_j^{1/2} w_j(t) e_j.$$

Finally, we define the generalized Wiener process and the special case of a cylindrical Wiener process which is what we will need in this thesis.

Definition 4.3.4 (Generalized Wiener process, [GM11], [DZ14])**.** We call a family $\{\tilde{W}(t, \cdot)\}_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ a generalized Wiener process in a Hilbert space U if:

(1) For an arbitrary $t \geq 0$, the mapping $\tilde{W}_t : U \to L^2(\Omega, \mathcal{F}, P)$ is linear;

(2) For an arbitrary $u \in U$, $\tilde{W}(t, u)$ is an $\mathcal{F}_t$-Brownian motion;

(3) For arbitrary $u, u' \in U$ and $t \geq 0$,

$$E(\tilde{W}(t, u)\tilde{W}(t, u')) = t\langle Qu, u'\rangle_U. \tag{4.2}$$

Where the operator Q is symmetric and nonnegative.

If Q equals the identity operator the generalized Wiener process is called a cylindrical Wiener process.

For an arbitrary self-adjoint and nonnegative definite operator Q, a generalized Wiener process $a \to \tilde{W}(a, \cdot)$ satisfying (4.2) can be constructed.

Let $\{e_j\}_{j=1}^{\infty}$ be a complete and orthonormal basis in U and $\{B_j\}_{j=1}^{\infty}$ a sequence of independent real valued Brownian motions. Then, the following defines the required generalized Wiener process

$$\tilde{W}(t, a) = \sum_{j=1}^{\infty} \langle Q^{1/2} e_j, a\rangle B_j(t), \quad t \geq 0,\ a \in U.$$

Furthermore, we can use the same orthonormal basis and the same sequence of Wiener processes to define a Q-Wiener process.

Proposition 4.3.5 ([DZ14]). *Let U_1 be a Hilbert space such that $U_0 = Q^{1/2}(U)$ is embedded into U_1 with a Hilbert–Schmidt embedding J. Then the formula*

$$W(t) = \sum_{j=1}^{\infty} Q^{1/2} e_j B_j(t), \quad t \geq 0,$$

defines a U_1-valued Q-Wiener process. Moreover, if Q_1 is the covariance of W then the spaces $Q_1^{1/2}(U_1)$ and $Q^{1/2}(U)$ are identical.

4.4 Stochastic integration

A natural continuation of the last section is to define stochastic integration on Hilbert spaces. The theory is collected from chapter 8 in [PZ07] and chapter 4 in [DZ14]. There are two separate cases that will be considered, one is integrals with respect to generalized Wiener processes and the other is integrals with respect to càdlàg square integrable martingales, this includes the special cases of Q-Wiener processes and Lévy processes as defined in the last section.

Let $\mathcal{M}^2(U)$ denote the space of all càdlàg square integrable martingales in U with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. The space of trace class (or nuclear) operators on U will be denoted by $L_1(U)$ and the space of symmetric nonnegative trace class operators on U will be denoted by $L_1^+(U)$.

Assume $M, N \in \mathcal{M}^2(U)$, then $\langle M|N\rangle$ denotes the unique predictable process, with trajectories having locally bounded variation, such that

$$\langle M(t), N(t)\rangle_U - \langle M|N\rangle_t, \quad t \geq 0,$$

is a martingale. $\langle M|N\rangle$ is called the angle bracket. We will also introduce the operator-valued angle bracket $\langle\langle M|N\rangle\rangle$. To avoid any confusion we underscore that $\langle \cdot \mid \cdot\rangle$ is not the same thing as the inner product $\langle \cdot, \cdot\rangle$.

For $x, y, z \in U$, we define the linear operator $x \otimes y$ by $(x \otimes y)(z) = \langle y, z\rangle_U x$.

Definition 4.4.1 (Martingale covariance, [PZ07]). Let $M \in \mathcal{M}^2(U)$. Then there is a unique right-continuous $L_1^+(U)$-valued increasing predictable process $\langle\langle M|M\rangle\rangle_t$, $t \geq 0$ such that $\langle\langle M|M\rangle\rangle_0 = 0$ and the process $(M(t) \otimes M(t) - \langle\langle M|M\rangle\rangle_t, t \geq 0)$ is an $L_1(U)$-valued martingale. Moreover, there exists a predictable $L_1^+(U)$-valued process $(Q_t, t \geq 0)$ such that

$$\langle\langle M|M\rangle\rangle_t = \int_0^t Q_s d\langle M|M\rangle_s, \quad \forall t \geq 0 \tag{4.1}$$

The $L_1^+(U)$-valued process Q satisfying (4.1) is called the martingale covariance of M.

As usual, the stochastic integral is constructed through the aid of simple processes.

Definition 4.4.2 (Simple process, [PZ07]). An $L(U, H)$-valued stochastic process ψ is said to be simple if there exist a sequence of non-negative numbers $t_0 = 0 < t_1 < \cdots < t_m$, a sequence of operators $\psi_j \in L(U, H)$, $j = 1, ..., m$, and a sequence of events $A_j \in \mathcal{F}_{t_j}$, $j = 0, ..., m-1$, such that

$$\psi(s) \sum_{j=0}^{m-1} \chi_{A_j} \chi_{(t_j, t_{j+1}]}(s) \psi_j, \quad s \geq 0.$$

The class of all simple processes with values in $L(U,H)$ is denoted by $\mathcal{S} := \mathcal{S}(U,H)$. For a simple process ψ, we set

$$I_t^M(\psi) = \sum_{j=0}^{m-1} \chi_{A_j}\psi(M(t_{j+1}\wedge t) - M(t_j \wedge t), \quad t \geq 0.$$

Let $T < \infty$ and equip $\mathcal{S} = \mathcal{S}(U,H)$ with the seminorm

$$\|\psi\|_{M,T}^2 := E\int_0^T \|\psi(s)Q_s^{1/2}\|_{L_2(U,H)}^2 d\langle M|M\rangle_s.$$

If some $\phi \in \mathcal{S}$ satisfies $\|\psi - \phi\|_{M,T} = 0$, then ϕ and ψ are identified with each other. Let $\mathcal{L}_{M,T}^2(H)$ be the completion of $(\mathcal{S}, \|\cdot\|_{M,T})$ with norm denoted by $\|\cdot\|_{M,T}$. We also introduce the space $\mathcal{L}_{M,T,U}^2(H)$ to be the class of all $L(U,H)$-valued processes belonging to $L_{M,T}^2(H)$.

Theorem 4.4.3 (Martingale stochastic integral, [PZ07])**.**

(i) *For any $t \in [0,T]$, there is a unique extension of I_t^M to a continuous linear operator, denoted also by I_t^M, from $(\mathcal{L}_{M,T}^2(H), \|\cdot\|_{M,T})$ into $L^2(\Omega, \mathcal{F}, P; H)$, so for $\psi \in \mathcal{L}_{M,T}^2(H)$ we have*

$$I_t^M(\psi) := \int_0^t \psi(s)dM(s).$$

Furthermore for any $0 \leq r \leq t \leq T$ we have the isometry

$$E\left[\left\|\int_r^t \psi(s)dM(s)\right\|_H^2\right] = E\int_r^t \|\psi(s)Q_s^{1/2}\|_{L_2(U,H)}^2 d\langle M|M\rangle_s$$

and the inequality

$$\|\chi_{(r,t]}\psi\|_{M,T}^2 \leq \|\psi\|_{M,T}^2$$

(ii) *For any $\psi \in \mathcal{L}_{M,T}^2(H)$, $(I_t^M(\psi), t \in [0,T])$ is an H-valued martingale. It is square integrable and mean-square continuous, meaning*

$$\lim_{t\to t_0} E\left[\|I_t^M(\psi) - I_{t_0}^M(\psi)\|_H^2\right] = 0.$$

Also $I_0^M(\psi) = 0$.

(iii) *Let A be a bounded linear operator from H into a Hilbert space V. Then, for every $\psi \in \mathcal{L}_{M,T}^2(H)$, $A\psi \in \mathcal{L}_{M,T}^2(V)$ and $AI^M(\psi) = I^M(A\psi)$.*

The following lemma facilitates the characterization of the space $\mathcal{L}_{M,T}^2(H)$ by granting a decomposition of the covariance operator Q_t. Let $\mathcal{P}_{[0,T]}$ denote the σ-algebra of predictable sets in $\Omega \times [0,T]$.

Lemma 4.4.4 ([PZ07])**.** *There are predictable real-valued processes $\gamma_n = \gamma_n(t,\omega)$ and predictable U-valued processes $g_n = g_n(t,\omega)$, $n \in \mathbb{N}$, such that*

$$Q_t(\omega) = \sum_{n=1}^{\infty} \gamma_n(t,\omega) g_n(t,\omega) \otimes g_n(t,\omega), \quad t \geq 0, \omega \in \Omega.$$

Define the Hilbert space $\mathcal{H}$, with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$, let $\tilde{T}(\omega, t)$ be the unique continuous linear operator from $Q^{1/2}(\omega)U$ into $\mathcal{H}$ satisfying

$$\tilde{T}(\omega, t)\sqrt{\gamma_n(\omega, t)}g_n(\omega, t) = e_n, \quad n \in \mathbb{N}.$$

On the space $([0, T] \times \Omega, \times \mathcal{P}_{[0,T]})$, introduce the σ-finite measure given by

$$\mu_M(d\omega, dt) = d\langle M|M\rangle_t(\omega)P(d\omega).$$

We can now characterize the space $\mathcal{L}^2_{M,T}(H)$.

Theorem 4.4.5 (Characterization of admissible integrands, [PZ07]). *For $\psi \in L^2(\Omega \times [0, T], \mathcal{P}_{[0,T]}, \mu_M; L_2(\mathcal{H}, H))$, it follows that*

$$\mathcal{L}^2_{M,T}(H) = \left\{\psi \circ \tilde{T} : \psi \in L^2(\Omega \times [0, T], \mathcal{P}_{[0,T]}, \mu_M; L_2(\mathcal{H}, H))\right\}$$

and

$$\|\psi \circ \tilde{T}\|^2_{M,T} = \int_\Omega \int_0^T \|\psi(\omega, t)\|^2_{L_2(\mathcal{H},H)}\mu_M(d\omega, dt).$$

If we are in the setting where the martingale process M has a constant martingale covariance process Q, then, the space $\mathcal{H}$ equals $Q^{1/2}(U) =: U_0$, which is known as the reproducing kernel space of M. This is the case, for instance, when M is a Lévy process. We have the following corollary to Theorem 4.4.3 and to Theorem 4.4.5.

Corollary 4.4.6 ([PZ07]). *If Q is constant then $\mathcal{L}^2_{M,T}(H) = L^2(\Omega \times [0, T], \mathcal{P}_{[0,T]}, P \times dt; L_2(U_0, H))$. Moreover, for $\psi \in \mathcal{L}^2_{M,T}(H)$, the stochastic integral is a square integrable martingale with*

$$E\left[\left\| \int_0^t \psi(s)dM_s\right\|^2_H\right] = E\left[\int_0^t \left\|\psi(s)\right\|^2_{L_2(U_0,H)}\right]ds.$$

If the Lévy process is a Q-Wiener process, then one can see that the stochastic integral defined above coincides with the definition of [DZ14]. Hence, we employ their theory to define stochastic integrals for generalized Wiener processes.

By the discussion following definition 4.3.4 and proposition 4.3.5, if $\{\tilde{W}(t, \cdot)\}_{t\geq 0}$ is a generalized Wiener process with covariance Q, then there exists a sequence $\{B_j\}_{j=1}^{\infty}$ of independent Brownian motions and an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ in U such that

$$\tilde{W}(a, t) = \sum_{j=1}^{\infty}\langle a, Q^{1/2}e_j\rangle B_j(t), \quad a \in U, t \geq 0.$$

Furthermore, the process defined by

$$W(t) = \sum_{j=1}^{\infty} Q^{1/2}e_jB_j(t), \quad t \geq 0$$

is a Q-Wiener process on any Hilbert space $U_1 \supset U_0$ with Hilbert-Schmidt embedding. We also have the equality

$$\tilde{W}(a, t) = \langle a, W(t)\rangle.$$

The stochastic integral for cylindrical Wiener processes can now be constructed in the following way, recall that $Q = I$ in this case.

Definition 4.4.7 (Cylindrical stochastic integral, [DZ14], [GM11]). Let $\{W_t\}_{t\geq 0}$ be defined as above and let $\psi \in L_2(U_0, H)$, then the stochastic integral with respect to a cylindrical Wiener process is defined as

$$\int_0^t \psi(s)dW(s) := \sum_{j=1}^{\infty} \int_0^t \psi(s)e_j dB_j(s).$$

In addition, we have the same isometry as for the definition of stochastic integrals with respect to Q-Wiener processes

$$E\left[\left\| \int_0^t \psi(s)dW(s)\right\|_H^2\right] = E\left[\int_0^t \|\psi(s)\|^2_{L_2(U,H)}\right].$$

4.5 Malliavin calculus

The last section of this chapter will be devoted to Malliavin calculus on Hilbert spaces, in the real-valued case we used the theory of chaos expansions, but in the Hilbert-valued case this is not an option, and we are forced to take a different route. This is the method mentioned in chapter 2 where the Malliavin derivative is defined, in a sense, like a directional derivative.

First we need the definition of an isonormal Gaussian process.

Definition 4.5.1 (Isonormal Gaussian process, [Nua06]). Let $\mathcal{G}$ be a separable Hilbert space. We say that a stochastic process $W = \{W(h), h \in \mathcal{G}\}$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$ is an isonormal Gaussian process if W is a centered Gaussian family of random variables such that

$$E(W(h)W(g)) = \langle h, g\rangle_{\mathcal{G}}, \quad \text{for all } h, g \in \mathcal{G}.$$

Introduce the following notation: let $\mathcal{G}, \mathcal{G}_1$ be separable Hilbert spaces, let $(\Omega, \mathcal{F}, P)$ be the probability space induced by the isonormal Gaussian process W, and consider the class of Hilbert-valued smooth random elements $F \in L^2(\Omega; \mathcal{G}_1)$ given by $F = f(W(h_1), ..., W(h_n))$ for $h_1, ..., h_n \in \mathcal{G}$, $n \in \mathbb{N}$, and $f : \mathbb{R}_n \to \mathcal{G}_1$ which are infinitely Fréchet differentiable with polynomial bound. These functions f are dense in $L^2(\Omega; \mathcal{G}_1)$. The definition of the Malliavin derivative is now feasible.

Definition 4.5.2 (Malliavin derivative, [BS16][CT06]). The Malliavin derivative of F is defined as

$$DF := \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(h_1), ..., W(h_n)) \otimes h_j.$$

Since $\mathcal{G}_1 \otimes \mathcal{G}$ is isomorphic to $L_2(\mathcal{G}, \mathcal{G}_1)$, the space of Hilbert-Schmidt operators from $\mathcal{G}_1$ to $\mathcal{G}$, we can interpret the Malliavin derivative of smooth F as another random element with values in $L_2(\mathcal{G}, \mathcal{G}_1)$.

We can also extract one-dimensional Malliavin calculus from this definition, simply apply projections onto the coordinates of $\mathcal{G}_1$. Let $l \in \mathcal{G}_1$, we get

$$D^l F := \langle DF, l\rangle_{\mathcal{G}_1} = \sum_{i=1}^{n} \left\langle \frac{\partial f}{\partial x_j}(W(h_1), ..., W(h_n)), l \right\rangle_{\mathcal{G}_1} h_j.$$

The Malliavin derivative D is closable for random smooth elements in $L^2(\Omega;\mathcal{G}_1)$. Hence, D can be extended to the larger domain denoted by $\mathbb{D}^{1,2}(\mathcal{G}_1)$.

Definition 4.5.3 (Extended Malliavin derivative, [BS16], [CT06]). If F is the $L^2(\Omega;\mathcal{G}_1)$ limit of a sequence $\{F_n\}_{n\geq 1}$ of smooth random variables such that $\{DF_n\}_{n\geq 1}$ converges in $L^2(\Omega;L_2(\mathcal{G},\mathcal{G}_1))$, we define DF as

$$DF = \lim_{n\to\infty} DF_n.$$

The subspace $\mathbb{D}^{1,2}(\mathcal{G}_1) \subset L^2(\Omega;\mathcal{G}_1)$ is a separable Hilbert space with the norm

$$\|F\|^2_{\mathbb{D}^{1,2}(\mathcal{G}_1)} = E\big[\|F\|\big]^2_{\mathcal{G}_1} + E\big[\|DF\|\big]^2_{L_2(\mathcal{G},\mathcal{G}_1)}.$$

As the Malliavin derivative is linear and closed we can define an adjoint operator called the divergence operator.

Definition 4.5.4 (Divergence operator, [BS16], [CT06]). We define the divergence operator $\delta_{\mathcal{G}_1} : L^2(\Omega, L_2(\mathcal{G},\mathcal{G}_1)) \to L^2(\Omega,\mathcal{G}_1)$ to be the adjoint of the Malliavin derivative D, that is

$$\big|E\big[\langle DF, G\rangle_{L_2(\mathcal{G},\mathcal{G}_1)}\big]\big| = E\big[\langle F, \delta_{\mathcal{G}_1} G\rangle_{\mathcal{G}_1}\big].$$

For all $F \in \mathbb{D}^{1,2}(\mathcal{G}_1)$ and all $G \in L^2(\Omega, L_2(\mathcal{G},\mathcal{G}_1))$. The divergence $\delta_{\mathcal{G}_1}$ is an unbounded operator and its domain is given by the set of random variables $G \in L^2(\Omega, L_2(\mathcal{G},\mathcal{G}_1))$ s.t.

$$\big|E\big[\langle DF, G\rangle_{L_2(\mathcal{G},\mathcal{G}_1)}\big]\big| \leq C\big(E\|G\|_{\mathcal{G}_1}\big)^{1/2}, \quad F \in \mathbb{D}^{1,2}(\mathcal{G}_1).$$

In the rest of this chapter we uphold the subscript notation, but in future chapters we simply write δ.

The Malliavin derivative on Hilbert spaces defined above satisfies a product rule and a chain rule similar to the real valued case. Let in the following $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ be separable Hilbert spaces.

Theorem 4.5.5 (Product rule, [BS16]). *Let $F \in \mathbb{D}^{1,2}(L_2(\mathcal{G}_1,\mathcal{G}_2))$ and $G \in \mathbb{D}^{1,2}(L_2(\mathcal{G}_2,\mathcal{G}_3))$, then $GF \in \mathbb{D}^{1,2}(L_2(\mathcal{G}_1,\mathcal{G}_3))$ and*

$$D(GF) = (DG)F + GDF,$$

where this equality has to be interpreted as $(D(GF))h = (DG)F(h) + G(DF)(h)$ for all $h \in \mathcal{G}_1$.

Theorem 4.5.6 (Chain rule, [BS16]). *Let $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ be Lipschitz and let $F \in \mathbb{D}^{1,2}(\mathcal{G}_1)$, then $\phi(F) \in \mathbb{D}^{1,2}(\mathcal{G}_2)$ and*

$$D\phi(F) = \bar{\phi}(F)DF,$$

where $\bar{\phi} : \mathcal{G}_1 \to \mathcal{G}_2$ is a random linear operator whose norm is almost surely bounded by the smallest Lipschitz constant of ϕ. In the case where ϕ is Fréchet differentiable we have $\bar{\phi} = \phi'$, where ϕ' is the Fréchet derivative of ϕ.

For the remaining part of this thesis we will focus on a type of isonormal Gaussian process given by cylindrical Wiener integrals. Let $\mathcal{K}$ be yet another separable Hilbert space, then we are interested in processes $\{W_h\}_{h\in\mathcal{G}}$ on the separable Hilbert space $\mathcal{G} := L^2([0,T],\mathcal{K})$ given by

$$W(h) := \int_0^T h(t)d\tilde{W}(t), \quad h \in \mathcal{G},$$

where $d\tilde{W}$ is the representative of the cylindrical Wiener process in $\mathcal{K}$. We can now associate the Malliavin derivative of F with a stochastic process $(D_tF)_{t\in[0,T]}$ with values in $L_2(\mathcal{K},\mathcal{G}_1)$ as follows

$$D_tF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), W(h_2), ..., W(h_n)) \otimes h_i(t).$$

We refer to [BS16] for further details.

The adjoint of the Malliavin derivative is now a special type of the divergence operator called the Skorohod integral.

Definition 4.5.7. For an element $u \in Dom(\delta_{\mathcal{G}_1})$ we write

$$\delta_{\mathcal{G}_1}(u) := \int_0^T u(s)\delta\tilde{W}(s),$$

and call this the Skorohod integral.

The Hilbert-valued Skorohod integral satisfies similar properties as the real-valued Skorohod integral. The first property is the Hilbert space equivalent of the fundamental theorem of calculus. This property is a consequence of the general commutation relationship between the Malliavin derivative and the divergence operator which states that $D\delta(u) = \delta(Du) + u$.

Furthermore, we define the space $\mathbb{L}^{1,2}(\mathcal{G}_1) := L^2([0,T];\mathbb{D}^{1,2}(\mathcal{G}_1))$ with norm given by

$$\|F\|^2_{L^{1,2}(\mathcal{G}_1)} = \int_0^T E\big[\|F_t\|_{\mathcal{G}_1}\big]dt + \int_0^T\int_0^T E\big[\|D_sF_t\|_{L_2(\mathcal{K},\mathcal{G}_1)}\big]dsdt.$$

Theorem 4.5.8 (Fundamental theorem of calculus)**.** *Let u be a stochastic process in $\mathbb{L}^{1,2}(L_2(\mathcal{K},\mathcal{G}_1))$, assume that for all $t \in [0,T]$, the process $\{D_tu(s)\}_{s\in[0,T]}$ is Skorohod integrable and that the process $\{\int_0^T D_tu(s)\delta\tilde{W}(s)\}_{t\in[0,T]}$ has a version which is in $L^2(\Omega \times [0,T]; L^2(\mathcal{K},\mathcal{G}_1))$. Then $\int_0^T u(s)\delta\tilde{W}(s) \in \mathbb{D}^{1,2}(\mathcal{G}_1)$ and for all $t \in [0,T]$*

$$D_t \int_0^T u(s)\delta\tilde{W}(s) = \int_0^T D_tu(s)\delta\tilde{W}(s) + u(t)$$

For the duality formula and the integration by parts formula we introduce the object $Tr_{\mathcal{G}_1}$, defined by

$$Tr_{\mathcal{G}_1}A = \sum_{k=1}^\infty (A(f_k))(f_k), \quad A \in L(\mathcal{G}_1, L(\mathcal{G}_1,\mathcal{G}_3))$$

whenever this sum converges in $\mathcal{G}_3$, and where $\{f_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{G}_1$. Note that this is not the traditional trace defined earlier in this chapter, we use the subscript, in this case $\mathcal{G}_1$, to indicate the difference.

A consequence of the above defined extended trace is that the duality formula is twofolded.

Theorem 4.5.9 (Duality formulas)**.** *From the definition of the Skorohod integral as the adjoint of the Malliavin derivative one has that for all* $u \in L^2(\Omega \times [0,T]; L^2(\mathcal{K}, \mathcal{G}_1))$ *and* $A \in \mathbb{D}^{1,2}(L^2(\mathcal{G}_1, \mathcal{G}_2))$

$$E\bigg[A \int_0^T u(s)\delta\tilde{W}(s)\bigg] = E\bigg[\int_0^T Tr_{\mathcal{G}_1}((D_s A)u(s))ds\bigg],$$

where the integrand is $\mathcal{G}_2$*-valued and the integral is understood as a Bochner integral. Similarly, one could write the trace outside the integral, which would yield an* $L^2(\mathcal{K}, L_2(\mathcal{K}, \mathcal{G}_2))$*-valued integrand and integral, that is*

$$E\bigg[A \int_0^T u(s)\delta\tilde{W}(s)\bigg] = E\bigg[Tr_{\mathcal{K}} \int_0^T (D_s A)u(s)ds\bigg].$$

There is also an integration by parts formula similar to the real-valued case. This is important for the (heuristic) derivation of the definition of the stochastic integral with an ambit field as integrator, see [BBV18].

Theorem 4.5.10 (Integration by parts, [BS16])**.** *Let* $u \in L^2(\Omega \times [0,T]; L_2(\mathcal{K}, \mathcal{G}_1))$ *be in the domain of the Skorohod integral* $\delta_{\mathcal{G}_1}$ *and let* $A \in \mathbb{D}^{1,2}(L_2(\mathcal{G}_1, \mathcal{G}_2))$*. Then* $Au \in Dom(\delta_{\mathcal{G}_2})$ *and*

$$\int_0^t Au(s)\delta\tilde{W}(s) = A\int_0^t u(s)\delta\tilde{W}(s) - Tr_{\mathcal{K}} \int_0^t D_s(A)u(s)ds,$$

for all $t \in [0,T]$*. Note that under the conditions above, the right-hand side of this equality is an element in* $L^2(\Omega; \mathcal{G}_2)$*.*

Finally, in this section, we relate the Skorohod integral to the Itô integral.

Theorem 4.5.11 (The Skorohod integral extends the Itô integral, [CT06])**.** *Let* $u : \Omega \times [0,T] \to L_2(\mathcal{G}_1, \mathcal{G}_2)$ *be a predictable process with*

$$E\bigg[\int_0^T \|u(s)\|^2_{L_2(\mathcal{G}_1,\mathcal{G}_2)} ds\bigg] < \infty$$

then u *is in the domain of the Skorohod integral* $\delta_{\mathcal{G}_2}$ *and*

$$\delta_{\mathcal{G}_2}(u) = \int_0^T u(s)d\tilde{W}(s)$$

where the integral on the right hand side is a cylindrical Wiener integral as defined in section 4.4.

CHAPTER 5

S(P)DE summary

Since the main topic of this thesis is S(P)DEs, it seems natural to summarize some of the existing theory on the subject. We have again two cases, the real-valued case and the Hilbert-valued case. Some of the results presented in this chapter are chosen because they are well known, and some are chosen because they play a crucial role for some of the results we obtain in chapter 7, either in the form of inspiration or where parts of the proof are directly applicable to our situation. There are mainly two strategies for proving the existence of a solution of a S(P)DE: the first is the Picard iteration technique, and the second utilizes Banach's fixed point theorem, the latter also guarantees uniqueness in a certain sense.

5.1 Real valued SDEs

In this section we will look at SDEs with values in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Throughout this chapter we let $\{B_t\}_{t\geq 0}$ denote a multidimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t\}_{t\geq 0}$ an increasing family of σ-algebras such that $\{B_t\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$ Brownian motion. Further specifications regarding dimension and time interval is clarified later on.

There are, as mentioned, two strategies for proving existence of a solution, and contrary to other chapters regarding background material, we also give some proofs.

When applying the Picard iteration technique we are not guaranteed uniqueness, hence, uniqueness must be proved separately. The standard approach for doing so utilizes an inequality called Grönwall's inequality. We therefore start of with this result.

Theorem 5.1.1 (Grönwall's inequality, [App09])**.** *Let $[a, b]$ be a closed interval in $\mathbb{R}$ and $\alpha, \beta : [a, b] \to \mathbb{R}$ be non-negative with α locally bounded and β integrable. If there exists $C \geq 0$ such that, for all $t \in [a, b]$,*

$$\alpha(t) \leq C + \int_a^t \alpha(s)\beta(s)ds$$

then we have

$$\alpha(t) \leq C \exp \int_a^t \beta(s)ds$$

for all $t \in [a, b]$. We will only need the special case when $\beta(s) = 1$ for all $s \in [a, b]$.

We begin our consideration of real-valued SDEs with the following SDE

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s.$$

This equation is well known to have a unique solution by the Picard iteration technique and by Grönwall's inequality. Since we will do proofs in chapter 7 in the same "spirit", the proof of the following theorem is somewhat cut down.

Theorem 5.1.2 (Existence and uniqueness, [Øks03])**.** *Let $T > 0$ and $b(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying*

$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|); \quad x \in \mathbb{R}_n, t \in [0, T]$$

for some constant C, (where $|\sigma|^2 = \sum |\sigma_{ij}|^2$) and such that

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y|; \quad x, y \in \mathbb{R}^n, t \in [0, T]$$

for some constant D. Let Z be a random variable which is independent of the σ-algebra $\mathcal{F}_\infty^{(m)}$ generated by $B_s(\cdot)$, $s \geq 0$ and such that

$$E[|Z|^2] < \infty.$$

Then the stochastic differential equation

$$\begin{aligned} &dXt = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad 0 \leq t \leq T \\ &X_0 = Z \end{aligned}$$

has a unique t-continuous solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and $B_s(\cdot)$; $s \leq t$ and

$$E\left[\int_0^T |X_t|^2 dt\right] < \infty.$$

Proof. **Sketch of the proof of existence and uniqueness**. The existence part of the proof uses the method of Picard iteration. That is, we define $Y_t^0 = X_0$ and

$$Y_t^{k+1} = X_0 + \int_0^t b(s, Y_s^k)ds + \int_0^t \sigma(s, Y_s^k)dB_s.$$

Then by using the assumptions on the coefficients one can show that

$$E[|Y_t^{k+1} - Y_t^k|^2] \leq C_1 \int_0^t E[|Y_s^k - Y_s^{k-1})|^2]ds \tag{5.1}$$

for $k \geq 1$, $t \leq T$ and that,

$$E[|Y_t^1 - Y_t^0|^2] \leq C_2 t$$

where $C_1, C_2 > 0$ are constants. Furthermore, it can be showed that $\{Y_t^k\}_{k=1}^\infty$ is a Cauchy sequence in $L^2(P \times \lambda)$, then by completeness it converges to some X_t. The last step of the existence part of the proof is to show that

$$X_t := L^2(P \times \lambda) - \lim_{k\to\infty} Y_t^k$$

solves the equation, this can be done using the same estimates as in equation (5.1).

For the uniqueness part of the proof we define two solutions $X_t(\omega)$ and $\hat{X}_t(\omega)$ with the same initial condition X_0, then, by again applying the same method as in the derivation of equation (5.1) one obtains

$$E[|X_t - \hat{X}_t|^2] \leq C \int_0^t E[|X_s - \hat{X}_s|^2] ds.$$

The uniqueness now follows by Grönwall's inequality since we get

$$P[|X_t - \hat{X}_t| = 0, \quad \text{for all } t \in \mathbb{Q} \cap [0,T]] = 1,$$

and by continuity of $t \to |X_t - \hat{X}_t|$ it follows that $P[|X_t(\omega) - \hat{X}_t(\omega)| = 0$ for all $t \in [0,T]] = 1$ and we are done. ■

As mentioned, we can also use the Banach fixed point theorem to prove existence and uniqueness of a solution. We state a general version since we also need the Banach fixed point theorem later on, while studying Hilbert-valued SPDEs.

Theorem 5.1.3 (Banach's Fixed Point Theorem, [Eva10])**.** *Let X be Banach space. Assume*

$$A : X \to X$$

is a nonlinear mapping, and suppose that

$$\|Ax - Ay\| \leq \gamma \|x - y\|, \quad x, y \in X$$

for some constant $\gamma < 1$. Then A has a unique fixed point, i.e. there is a unique $x \in X$ with $Ax = x$.

The next result is applied very directly in section 7.2, and a full proof is therefore given for the readers convenience. First we need some notation, the following is taken from [BK81].

Let λ denote the Lebesgue measure and let $\{\mathcal{F}_t\}_{t\geq 0}$ be an increasing family of sigma-algebras such that $\{B_t\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion.

Let $\mathcal{P}$ be the σ-algebra of predictable sets on $\mathbb{R}_+ \times \Omega$. Define the spaces $L^2 = L^2(\mathbb{R}_+ \times \Omega, \mathcal{P}, \lambda \times P)$ and H^2 of predictable processes $X = \{X_t\}_{t\geq 0}$ in $\mathbb{R}^d$ such that

$$\|X\|_2 := \left(E\Big[\int_0^\infty |X_t|^2 dt \Big] \right)^{1/2} < \infty$$

$$\|X\| := \left(\sup_{t\geq 0} E\big[|X_t|^2\big] \right)^{1/2} < \infty$$

respectively.

Let L^2_{loc} (respectively H^2_{loc}) denote the space of processes X such that there is an increasing sequence of $\{\mathcal{F}_t\}_{t\geq 0}$-stopping times $\{T_n\}_{n\in\mathbb{N}} \to \infty$ P-a.s. such that $X\chi_{[0,T_n]}$ belongs to L^2 (respectively the process $X^{T_n^*} := X^{T_n}\chi_{\{T_n \geq 0\}}$ belongs H^2) for all $n \in \mathbb{N}$. The sequence $\{T_n\}_{n\,\in\mathbb{N}}$ is said to be a localizing sequence of X.

Explicitly, this means that L^2_{loc} consists of the following processes

$$L^2_{loc} = \{X \text{ predictable } \mid \int_0^t |X_s|^2 dt < \infty \text{ for all } t \geq 0,\ P - \text{a.s}\}$$

Furthermore, any continuous adapted $\mathbb{R}^d$-valued process C belongs to H^2_{loc}, with localizing sequence given by $T_n := \inf\{t \geq 0 \mid |C_t| \geq n\}$, $n \in \mathbb{N}$. Now a definition necessary for the proof of the theorem.

Proposition 5.1.4 ([BK81])**.** *For any stopping time T, the quotient space H^2_T of H^2 defined by the kernel of the seminorm on H^2*

$$\|X\|_T := \|X^T\|$$

is a Banach space with the norm $\|X\|_T$.

The following is the main result of [BK81].

Theorem 5.1.5 (Existence and uniqueness, [BK81])**.** *Consider on $\mathbb{R}_+ \times \mathbb{R}^d$ the stochastic differential equation*

$$X_t = C_t + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s$$

where the initial process C is continuous adapted with values in $\mathbb{R}^d$, and the coefficients are measurable functions $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $G : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{dm}$ ($d \times m$ - matrix valued) satisfying the following conditions:

1) *Lipschitz condition: There is a function $\ell \in L^2_{loc}(\mathbb{R}^d, \lambda)$ such that*

$$|b(s, x) - b(s, y)| + |\sigma(s, x) - \sigma(s, y)| \leq \ell(t)|x - y|$$

for all $t \in \mathbb{R}_+$, $x, y \in \mathbb{R}^d$.

2) $b(\cdot, x), \sigma(\cdot, x) \in L^2_{loc}(\mathbb{R}^d, \lambda)$ *for some* $x \in \mathbb{R}^d$

Then there exists a global solution $X \in H^2_{loc}$ with continuous paths which is unique up to stochastic equivalence.

Before we write the proof we need a couple lemmas ensuring the integral terms are in H^2_{loc}.

Lemma 5.1.6 ([BK81])**.** *Under the hypotheses of 5.1.5, the process $Y_t := \int_0^t b(s, X_s) ds$, $t \geq 0$, is in H^2_{loc} for every $X \in H^2_{loc}$. For $X \in H^2$, the stopping times $T_n := n$, $n \in \mathbb{N}$, localize Y.*

The stochastic term can also be showed to be in H^2_{loc}, and we refer to [BK81] for proof of both these lemmas.

Lemma 5.1.7 ([BK81])**.** *Under the hypotheses of 5.1.5, the process $Z_t := \int_0^t b(s, X_s) dB_s$, $t \geq 0$, is in H^2_{loc} for every $X \in H^2_{loc}$. For $X \in H^2$, the stopping times $T_: = n$, $n \in \mathbb{N}$, localize Z.*

Note that the above theorem and the following proof uses the terminology of stochastic equivalence, this corresponds to our terminology of modification. That is, two processes are stochastically equivalent if and only if they are modifications of each other.

Proof. Consider the, in general non-linear, operator $S : H^2_{loc} \to H^2_{loc}$ defined by

$$SX_t = C_t + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \quad t \geq 0.$$

The initial process $C \in H^2_{loc}$ is localized by the stopping times $T^n := \inf\{t \geq 0 \mid |C_t| \geq n\} \wedge n$, $n \in \mathbb{N}$, and so is SX for all $X \in H^2$ by Lemma 5.1.6 and 5.1.7. The map $X \to (SX)^{T_n^*}$ therefore defines an operator $S_n : H^2 \to H^2$. For all $X, Y \in H^2$ and $t \geq 0$ we get

$$\begin{aligned}
E\big[|S_n X_t - S_n Y_t|^2\big] &\leq 2E\bigg(T_n \int_0^{t\wedge T_n} |b(s, X_s) - b(s, Y_s)|^2 ds\bigg) \\
&\quad + 2E\bigg(\bigg|\int_0^{t\wedge T_n} b(s, X_s) - b(s, Y_s)dB_s\bigg|^2\bigg) \\
&\leq 2nE\bigg(\int_0^{t\wedge T_n} |b(s, X_s) - b(s, Y_s)|^2 ds\bigg) \\
&\quad + 2E\bigg(\int_0^{t\wedge T_n} |b(s, X_s) - b(s, Y_s)|^2 ds\bigg) \\
&\leq 2(n+1)\int_0^{t\wedge n} E\big[|X_s^{T_n} - Y_s^{T_n}|^2\big]\ell^2(s)ds
\end{aligned}$$

showing in particular that S_n can be considered as an operator on the quotient space $H^2_{T_n}$. Inserting $e^{-\alpha L(s)}e^{\alpha L(s)}$ with $\alpha \geq 0$ and $L(s) := \int_0^s l^2(r)dr$ for $s \geq 0$ yields

$$\begin{aligned}
&E\big[|S_n X_t - S_n Y_t|^2\big] \\
&\leq 2(n+1)\int_0^{t\wedge n} E\big[|X_s^{T_n} - Y_s^{T_n}|^2\big]e^{-\alpha L(s)}e^{\alpha L(s)}\ell^2(s)ds \\
&\leq 2(n+1)\sup_{s\geq 0} E\big[|X_s^{T_n} - Y_s^{T_n}|^2\big]e^{-\alpha L(s\wedge n)}\int_0^{t\wedge n} e^{\alpha L(s)}\ell^2(s)ds \\
&\leq 2(n+1)\|X - Y\|^2_{\sup,T_n}\frac{1}{\alpha}e^{\alpha L(t\wedge n)}
\end{aligned}$$

Where $\|\cdot\|_{\sup,T_n}$ denotes the seminorm on H^2 defined by

$$\|X\|_{\sup,T_n} := \bigg(\sup_{t\geq 0} E\big[|X_t^{T_n}|^2 e^{-\alpha L(t\wedge n)}\big]\bigg)^{1/2}$$

Since $e^{-\alpha L(n)/2}\|X\|_{T_n} \leq \|X\|_{\sup,T_n} \leq \|X\|_{T_n}$, it is equivalent to $\|\cdot\|_{T_n}$ and so it defines an equivalent norm on $H^2_{T_n}$. The above estimate now reads

$$\|S_n X - S_n Y\|^2_{\sup,T_n} \leq \frac{2(n+1)}{\alpha}\|X - Y\|^2_{\sup,T_n}.$$

For a suitable choice of α, e.g. $\alpha = 8(n+1)$, the mapping S_n is a contraction on $H^2_{T_n}$. By the Banach fixed point theorem there is a unique fixed point $X^n \in H^2_{T_n}$. As for $X \in H^2$, $\|X\|_{\sup,T_n} = 0$ is equivalent to $E[|X_t^{T_n}|] = 0$ for all

$t \geq 0$, this fixed point corresponds to a process $X^n \in H^2$ such that for all $t \geq 0$, $(X_t^n)^{T_n}$ is uniquely determined up to stochastic equivalence.

For $m > n$ with the corresponding fixed points $X^m \in H^2_{T_m}$ and $X^n \in H^2_{T_n}$ we have

$$(X^m)^{T_n^*} = (S_m X^m)^{T_n^*} = S_n((X_t^m)^{T_n^*}),$$

in $H^2_{T_n}$, hence by the uniqueness property $(X_t^m)^{T_n^*} = X_t^{n\,T_n}$ P-a.s. for all $t \geq 0$ and any representatives X^m, X^n in H^2, and a global solution $X \in H^2_{loc}$ is determined uniquely up to stochastic equivalence.

By the continuity properties of the Lebesgue integral and of the Itô integral there exist continuous versions of X, and these versions are unique up to stochastic equivalence since it is possible to find a common null set outside of which the trajectories coincide. ∎

Comparing Theorem 5.1.5 and Theorem 5.1.2 we can see that they have a few differences, firstly there is no linear growth assumption on the coefficients in the result of [BK81], there are instead integrability assumptions. Secondly the Lipschitz condition is relaxed to be a predictable, locally square integrable process instead of a constant, this is possible by the use of equivalent norms. Moreover, [BK81] constructs a global solution, that is, a solution not limited to a compact interval $[0, T]$, through the use of localization techniques. Lastly, the uniqueness is not the same, in [Øks03] the solution is unique up to indistinguishability, however, [BK81] only achieves uniqueness up to modification. Hence, we can see that there are pros and cons of both approaches. An extension of Theorem 5.1.2 to a global solution can be found in chapter 5 of [KS98].

We also have similar results for SDEs driven by the compensated Poisson random measure. This result is taken from chapter 6 in [App09], and is a generalization of Theorem 5.1.2 as it includes several noise terms, including a noise term driven by a Brownian motion.

Let $B = \{B(t)\}_{t \geq 0}$ be an r-dimensional standard Brownian motion and N an independent Poisson random measure on $\mathbb{R}_+ \times (\mathbb{R}^d - \{0\})$ with associated compensator $\tilde{N}$ and Lévy measure ν. Assume that B and N are independent of $\mathcal{F}_0$, where $\{\mathcal{F}_t\}_{t \geq 0}$ is the filtration on $(\Omega, \mathcal{F}, P)$. We will look at the following SDE:

$$\begin{aligned} dY(t) &= b(Y(t-))dt + \sigma(Y(t-))dB(t) \\ &\quad + \int_{|z|<c} F(Y(t-), z)\tilde{N}(dt, dz) \\ &\quad + \int_{|z|\geq c} G(Y(t-), z)N(dt, dz), \end{aligned} \tag{5.2}$$

The parameter $c \in [0, \infty]$ specifies what is meant by small and large jumps. Now we state an existence and uniqueness result for an equation involving only small jumps.

Theorem 5.1.8 (Poisson random measure driven SDE with small jumps, [App09])**.** *Consider the SDE*

$$dZ(t) = b(Z(t-))dt + \sigma(Z(t-))dB(t) + \int_{|z|<c} F(Z(t-), z)\tilde{N}(dt, dz), \tag{5.3}$$

with the following assumptions on the coefficients:

(C1) *There exists* $K_1 > 0$ *such that, for all* $y_1, y_2 \in \mathbb{R}^d$,

$$|b(y_1) - b(y_2)|^2 + \|a(y_1, y_1) - 2a(y_1, y_2) + a(y_2, y_2)\| \\ + \int_{|z|<c} |F(y_1, z) - F(y_2, z)|^2 \nu(dz) \leq K_1 |y_1 - y_2|^2$$

(C2) *There exists* $K_2 > 0$ *such that, for all* $y \in \mathbb{R}^d$,

$$|b(y)|^2 + \|a(y, y)\| + \int_{|z|<c} |F(y, z)|^2 \nu(dz) \leq K_2(1 + |y|^2).$$

Where, for each $x, y \in \mathbb{R}^d$ *we introduce the* $d \times d$ *matrix*

$$a(x, y) = \sigma(x)\sigma(y)^T,$$

so that $a^{ik}(x, y) = \sum_{j=1}^{r} \sigma_j^i(x)\sigma_j^k(y)$ *for each* $1 \leq i,\ k \leq d$. *Furthermore,* $\|\cdot\|$ *denotes the matrix seminorm on* $d \times d$ *matrices, given by*

$$\|a\| = \sum_{i=1}^{\infty} |a_i^i|.$$

Then there exists a unique solution $Z = \{(Z(t)\}_{t \geq 0}$, *to the SDE* (5.3) *with the initial condition* $Y(0) = Y_0$ *(a.s.), for which* Y_0 *is* $\mathcal{F}_0$*-measurable. The process* Z *is adapted and càdlàg.*

Proof. The proof uses Picard iteration and is therefore very similar to the proof of Theorem 5.1.2, see [App09]. ■

By uniqueness in the preceding theorem we mean uniqueness up to indistinguishability, which is the same type of uniqueness as in the result by [Øks03]. The solution to the above theorem can be extended to allow for large jumps through the use of what [App09] terms interlacing.

Theorem 5.1.9 (Poisson random measure driven SDE with big jumps, [App09])**.** *Assume that* $c > 0$ *and that the mapping* $y \mapsto G(y, x)$ *is continuous for all* $x \leq c$. *With the conditions of Theorem 5.1.8, there exists a unique càdlàg adapted solution to* (5.2).

Proof. Let $(\tau_n, n \in \mathbb{N})$ be the arrival times for the jumps of the compound Poisson process $(P(t), t \geq 0)$, where each $P(t) = \int_{|z|\geq c} zN(t, dx)$. We then construct a solution to (5.2) as follows:

$$\begin{aligned}
&Y(t) = Z(t) && \text{for } 0 \leq t < \tau_1, \\
&Y(\tau_1) = Z(\tau_1-) + G(Z(\tau_1-), \Delta P(\tau_1)) && \text{for } t = \tau_1, \\
&Y(t) = Y(\tau_1) + Z_1(t) - Z_1(\tau_1) && \text{for } \tau_1 < t < \tau_2, \\
&Y(\tau_2) = Y(\tau_2-) + G(Y(\tau_2-), \Delta P(\tau_2)) && \text{for } t = \tau_2,
\end{aligned}$$

and so on, recursively. This technique is called interlacing. Here Z_1 is the unique solution to (5.3) with initial condition $Z_1(0) = Y(\tau_1)$. Y is clearly adapted, càdlàg and solves (5.3). Uniqueness follows by the uniqueness in Theorem 5.1.8 and the interlacing structure. ■

In chapter 7 we will study SDEs containing Malliavin derivatives, and it therefore seems natural to give a result on the Malliavin differentiability of the solutions found in this section. This theorem is taken from chapter 17 in [DØP09], and for the proof they use the following lemma.

Lemma 5.1.10 ([DØP09]). *Let $\{F_n\}_{n\geq 1}$, be a sequence in $\mathbb{D}_{1,2}$ such that*

$$F_n \to F, \quad n \to \infty,$$

in $L^2(P)$. Further, we require that

$$\sup_{n\geq 1} E\left[\int_0^T \int_{\mathbb{R}_0} |D_{s,z}F_n|^2 \nu(dz)ds\right] < \infty.$$

Then $F \in \mathbb{D}_{1,2}$ and $\{D_{\cdot,\cdot}F_n\}_{n\geq 1}$ converges to $D_{\cdot,\cdot}F$ in the sense of the weak topology of $L^2(P \times \lambda \times \nu)$, where λ denotes the Lebesgue measure and ν denotes the Lévy measure.

The theorem concerning Malliavin differentiability of solutions is now given. Notice that the only added assumption, compared with Theorem 5.1.8, is the integrability assumption on the Lévy measure.

Theorem 5.1.11 (Malliavin differentiability of solutions [DØP09]). *There exists a unique Malliavin differentiable strong solution X to the SDE*

$$X(t) = X_0 + \int_0^T \int_{\mathbb{R}_0} \gamma(s, X(s-), z)\tilde{N}(ds, dz), \quad 0 \leq t \leq T, \tag{5.4}$$

for $X_0 \in \mathbb{R}$, where $\gamma : [0,T] \times \mathbb{R} \times \mathbb{R}_0 \to \mathbb{R}$ satisfies the linear growth condition

$$\int_{\mathbb{R}_0} |\gamma(t,x,z)|^2 \nu(dz) \leq C(1 + |x|^2), \quad 0 \leq t \leq T, x \in \mathbb{R}, \tag{5.5}$$

for a constant $C < \infty$ as well as the Lipschitz condition

$$\int_{\mathbb{R}_0} |\gamma(t,x,z) - \gamma(t,y,z)|^2 \nu(dz) \leq K|x-y|^2, \quad 0 \leq t \leq T, x \in \mathbb{R}. \tag{5.6}$$

On the Lévy measure we impose the following integrability assumption

$$\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty.$$

Proof. The idea of the proof is first to show that the Picard approximations $X_n(t)$, $n \geq 0$, to X given by

$$X_{n+1} = X_0 + \int_0^T \int_{\mathbb{R}_0} \gamma(s, X_n(s-), z)\tilde{N}(ds, dz),$$

are in $\mathbb{D}_{1,2}$ and then to perform the limit $n \to \infty$. Let us first prove by induction on n that

$$X_n(t) \in \mathbb{D}_{1,2} \tag{5.7}$$

and that

$$\phi_{n+1}(t) \leq k_1 + k_2 \int_0^t \phi_n(s)ds \tag{5.8}$$

for all $0 \leq t \leq T$, $n \geq 0$, where k_1, k_2 are constants and

$$\phi_n(t) := \sup_{0 \leq r \leq t} E\Big[\int_{\mathbb{R}_0} \sup_{r \leq s \leq t} (D_{r,\cdot} X_n(s))^2 \nu(dz)\Big] < \infty.$$

One can check that (5.7) and (5.8) are fulfilled for $n = 0$, since

$$D_{t,z} \int_0^T \int_{\mathbb{R}_0} \gamma(s,x,y) \tilde{N}(ds,dy) = \gamma(t,x,z)$$

by the fundamental theorem of calculus 3.2.16. We assume that (5.7) and (5.8) hold for n. Then, the closability of the Malliavin derivative $D_{t,z}$ 3.2.9 and the chain rule 3.2.13 imply that

$$D_{r,z}\gamma(t, X_n(t-), z) = \gamma(t, X_n(t-) + D_{r,z}X_n(t-), z) - \gamma(t, X_n(t-), z)$$

for $r \leq t$ a.e. and ν-a.e. Hence, the fundamental theorem of calculus 3.2.16 gives that $X_{n+1}(t) \in \mathbb{D}_{1,2}$ and

$$\begin{aligned} D_{r,z}X_{n+1}(t) &= \int_0^t \int_{\mathbb{R}_0} D_{r,z}\gamma(s, X_n(s-), y)\tilde{N}(ds,dy) + \gamma(r, X_n(r-), z) \\ &= \int_r^t \int_{\mathbb{R}_0} \Big(\gamma(s, X_n(s-) + D_{r,z}X_n(s-), y) - \gamma(s, X_n(s-), y)\Big)\tilde{N}(ds,dy) \\ &\quad + \gamma(r, X_n(r-), z) \end{aligned}$$

for $r \leq t$ a.e. and ν-a.e. So it follows from (5.5), (5.6), Doob maximal inequality, Fubini theorem, and the Itô isometry that

$$\begin{aligned} & E\Big[\int_{\mathbb{R}_0} \sup_{r \leq s \leq t} (D_{r,\cdot}X_{n+1}(s))^2 \nu(dz)\Big] \\ & \leq 8K \int_r^t E\Big[\int_{\mathbb{R}_0} D_{r,z}X_n(u-)\nu(dz)\Big] du + 2C\big(1 + E[|X_n(r-)|^2]\big) \\ & \leq 8K \int_r^t E\Big[\int_{\mathbb{R}_0} D_{r,z}X_n(u-)\nu(dz)du + 2C(1+\lambda), \end{aligned} \tag{5.9}$$

where

$$\sup_{n \geq 0} E\big[\sup_{0 \leq s \leq T} |X_n(s)|^2\big] < \infty.$$

Note that

$$E\big[\sup_{0 \leq s \leq T} |X_n(s) - X(s)|^2\big] \to 0, \quad n \to \infty,$$

by the Picard iteration scheme. Thus (5.9) shows that (5.7) and (5.8) are valid for $n+1$. Finally, a discrete version of Gronwall inequality applied to (5.11) yields

$$\sup_{n \geq 0} E\Big[\int_0^T \int_{\mathbb{R}_0} |D_{s,z}X_n(t)|^2 \nu(dz)ds\Big] < \infty$$

for all $0 \leq t \leq T$. Then it follows from Lemma 5.1.10 that $X \in \mathbb{D}_{1,2}$. ■

In [DØP09] they claim that it can be proved, with similar arguments, that the solution X to the following the more general equation is also Malliavin differentiable

$$X(t) = X_0 + \int_0^t \alpha(s, X(s))ds + \int_0^t \beta(s, X(s))dW(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s, X(s-), z)\tilde{N}(ds, dz),$$

for $0 \leq t \leq T$, $X_0 \in \mathbb{R}$. Where the coefficients α, β, γ satisfies certain regularity conditions.

We end this section with a slightly different result compared to those considered above. This is a simpler version of the equations we have previously looked at in this chapter and the proof is essentially "non-stochastic" in nature.

Theorem 5.1.12 (Existence and uniqueness of equation with constant noise coefficient [Eva14])**.** *Suppose that* $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ *satisfies*

(i) $|b(s, x)| \leq (1 + |x|)$,

(ii) $|b(s, x) - b(s, y)| \leq L|x - y|$.

Then the SDE

$$X(t) = X_0 + \int_0^t b(s, X(s))ds + W(t), \quad X_0 \in \mathbb{R},$$

has a unique solution up to indistinguishability.

Proof. Define $Z^0(t) = X_0$, for all $t \in [0, T]$ and then

$$Z^{n+1}(t) := X_0 + \int_0^t b(s, Z_n(s))ds + W(t)$$

for $n = 0, 1, ...$ Next write,

$$D_n(t) := \max_{0 \leq s \leq t} |Z^{n+1}(s) - Z^n(s)|$$

and notice that for a given continuous sample path of the Brownian motion, we have

$$D_0(t) = \max_{0 \leq s \leq t} \left| \int_0^s b(r, x)dr + W(s) \right| \leq C$$

for all times $0 \leq t \leq T$, where C depends on ω. We now claim that

$$D^n(t) \leq C \frac{L^n}{n!} t^n$$

for $n = 0, 1, ..., 0 \leq t \leq T$. To see this note that

$$D^n(t) = \max_{0 \leq s \leq t} \left| \int_0^s (b(s, Z^n(s)) - b(s, Z^n(s))dr \right| \leq L \int_0^t D^{n-1}(s)ds$$

$$\leq L \int_0^t C \frac{L^{n-1}}{(n-1)!} t^{n-1} \leq C \frac{L^n}{n!} t^n$$

In view of the claim, for $m \geq n$ we have

$$\max_{0 \leq t \leq T} |Z^m(t) - Z^n(t)| \leq C \sum_{k=n}^{\infty} \frac{L^k T^k}{k!} \to 0, \quad n \to \infty$$

Thus for almost every ω, Z^n converges uniformly for $0 \leq t \leq T$ to a limit process X which, as can be checked, solves the equation. ■

In the above theorem there is no proof of uniqueness, but uniqueness up to indistinguishability follows by Theorem 5.1.2.

5.2 Hilbert valued SPDEs

This section will give a brief summary on some existing results on Hilbert-valued SPDEs. Unlike in the real-valued, where we only considered strong solutions, we will here also look at different solution concepts. We mainly follow chapter 9 in [PZ07], and our introduction of the fairly technical theory on stochastic integrals from the last section will now come in handy.

The definition of what is known as mild solutions depends on the concept of C_0-semigroups, therefore, a natural starting point is to look at its definition and some of its properties.

Definition 5.2.1 (C_0-semigroups, [PZ07])**.** A family $S = (S(t), t \geq 0)$ of bounded linear operators on a Banach space $(B, \|\cdot\|_B)$ is called a C_0-semigroup if

(i) $S(0)$ is the identity operator I,

(ii) $S(t)S(s) = S(t+s)$ for all $t, s \geq 0$,

(iii) $[0, \infty) \ni t \mapsto S(t)z \in B$ is continuous for each $z \in B$.

A C_0-semigroup is generated by a possibly unbounded operator.

Definition 5.2.2 (Generator of C_0-semigroup, [PZ07])**.** Assume that S is a C_0-semigroup on B. We say that an element $z \in B$ is in the domain of the generator of S if

$$\lim_{t \downarrow 0} \frac{(S(t)z - z)}{t} =: Az$$

exists. The set of all such z is denoted by $D(A)$ and Az, $z \in D(A)$, is then a linear operator called the generator of S.

C_0-semigroups has some important properties which come in handy while studying SPDEs.

Theorem 5.2.3 (Properties of C_0-semigroups, [PZ07])**.**

(i) *If S is a C_0-semigroup on B then, for some ω and $M > 0$,*

$$\|S(t)z\|_B \leq e^{\omega t} M \|z\|_B, \quad \forall z \in B, \forall t \geq 0.$$

(ii) *If a densely defined operator A generates a C_0-semigroup S then A is closed and, for any $z \in D(A)$ and $t > 0$,*

$$S(t)z \in D(A) \quad \text{and} \quad \frac{d}{dt}S(t)z = AS(t)z = S(t)Az.$$

The equation under consideration in the first part of this section is the following

$$dX = AXdt + F(X)dt + G(X)dM_t, \quad X(t_0) = X_0 \tag{5.10}$$

where A is the generator of a C_0-semigroup S on a Hilbert space H, and M, defined on another Hilbert space U, is a square integrable martingale. M is assumed to be defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and to satisfy the condition

$$\exists Q \in L_1^+(U) : \forall t \geq s \geq 0, \quad \langle\langle M|M\rangle\rangle_t - \langle\langle M|M\rangle\rangle_s \leq (t-s)Q,$$

where we recall that $L_1^+(U)$ is the space of symmetric nonnegative trace class operators on U. Define $\mathcal{H} := Q^{1/2}(U)$.

We can now look at solution concepts to Hilbert-valued SPDEs of the form (5.10), there are several different types of solutions and we will state two of them, see e.g. chapter 8 in [DZ14] for martingale solutions and appendix G in [LR15] for strong solutions and the relation between strong, mild and weak solutions.

Definition 5.2.4 (Mild solution, [PZ07])**.** Let X_0 be a square integrable $\mathcal{F}_{t_0}$-measurable random variable in H. A predictable process $X : [t_0, \infty) \times \Omega \to H$ is called a mild solution to (5.10) starting at time t_0 from X_0 if

$$\sup_{t\in[t_0,T]} E\|X(t)\|_H^2 < \infty, \quad \forall T \in (t_0, \infty)$$

and

$$X(t) = S(t-t_0)X_0 + \int_{t_0}^t S(t-s)F(X(s))ds$$
$$\int_{t_0}^t S(t-s)G(X(s))dM(s), \quad \forall t \geq t_0.$$

The main theorem of this section now follow, since we give a similar proof in section 7.3, the proof is shortened and we refer to [PZ07] for the full proof.

Assumption 5.2.5 ([PZ07])**.** We assume that $F : D(F) \to H$ and $G : D(G) \to L(\mathcal{H}, H)$ satisfy Lipschitz-type conditions:

(F) $D(F)$ is dense in H and there is a function $a : (0,\infty) \to (0,\infty)$ satisfying $\int_0^T a(t)dt < \infty$ for all $T < \infty$ such that, for all $t > 0$, and $x, y \in D(F)$

$$\|S(t)F(x)\|_H \leq a(t)(1 + \|x\|_H),$$
$$\|S(t)(F(x) - F(y))\|_H \leq a(t)\|x - y\|_H$$

(G) $D(G)$ is dense in H and there is a function $b : (0, \infty) \to (0, \infty)$ satisfying $\int_0^T b^2(t)dt < \infty$ for all $T < \infty$ such that, for all $t > 0$ and $x, y \in D(G)$,

$$\|S(t)G(x)\|_{L_2(\mathcal{H},H)} \leq b(t)(1 + \|x\|_H),$$
$$\|S(t)(G(x) - G(y))\|_{L_2(\mathcal{H},H)} \leq b(t)\|x - y\|_H.$$

Theorem 5.2.6 (Existence and uniqueness of SPDE driven by square integrable martingale, [PZ07])**.** *Assume that conditions* (*F*) *and* (*G*) *from assumption 5.2.5 are satisfied. Then for all $t_0 \geq 0$ and all $\mathcal{F}_{t_0}$-measurable square integrable random variables X_0 in H there exists a unique (up to modification) solution of* (5.10).

Proof. Given $0 \leq t_0 \leq T < \infty$ we denote by $\mathcal{X}_T$ the space of all predictable processes $Y : [t_0, T] \times \Omega \mapsto H$ such that

$$\|Y\|_T := \Big(\sup_{t \in [t_0,T]} E\big[\|Y(t)\|_H\big]\Big)^{1/2} < \infty.$$

Given $\beta \in \mathbb{R}$ and $Y \in \mathcal{X}_T$, write

$$\|Y\|_{T,\beta} := \Big(\sup_{t \in [t_0,T]} e^{-\beta t} E\big[\|Y(t)\|_H\big]\Big)^{1/2}$$

Clearly $\mathcal{X}_T$ with the norm $\|\cdot\|_T = \|\cdot\|_{T,0}$ is a Banach space. Moreover, the norms $\|\cdot\|_{T,\beta}$, $\beta \in \mathbb{R}$ are equivalent. Note that, from (F) and (G), for all $Y \in \mathcal{X}_T$ and $t \in [t_0, T]$ the integrals

$$I_F(Y)(t) := \int_{t_0}^T S(t-s)F(Y(s))ds$$
$$J_G(Y)(t) := \int_{t_0}^T S(t-s)G(Y(s))dM(s)$$

are well defined. By Proposition 3.21 in [PZ07] they have predictable versions since they are adapted and stochastically continuous. By the Banach fixed-point theorem it suffices to show that, for any $T < \infty$, there are $\beta \in \mathbb{R}$ and a constant $C < 1$ such that

$$\|I_F(Y) + J_G(Y) - I_F(V) - J_G(V)\|_{T,\beta} \leq C\|Y - V\|_{T,\beta}, \quad Y, Z \in \mathcal{X}_T.$$

To this end, we fix $Y, V \in \mathcal{X}_T$. Then

$$\|I_F(Y) + J_G(Y) - I_F(V) - J_G(V)\|^2_{T,\beta}$$
$$\leq 2\|I_F(Y) - I_F(V)\|^2_{T,\beta} + 2\|J_G(Y) - J_G(V)\|^2_{T,\beta}$$

Next, by (F), it can be shown that

$$\|I_F(Y) - I_F(V)\|^2_{T,\beta} \leq C_1\|Y - V\|^2_{T,\beta},$$

and by (G) that

$$\|J_G(Y) - J_G(V)\|^2_{T,\beta} \leq C_2\|Y - V\|^2_{T,\beta},$$

where C_1 and C_2 are positive constants depending on T and β. Then, for sufficiently large β, $C_1 + C_2 < 1$. ■

Interestingly, the linear growth assumptions in 5.2.5 are not actually used for the proof of the theorem, their importance lies in ensuring the integrals in Definition 5.2.4 are well defined. Recall Theorem 5.1.5, which also proved existence by applying the Banach fixed point theorem, but did not assume any linear growth assumption on the coefficients and instead ensured that the integrals were well defined through local integrability conditions on the coefficients.

We also include a couple of explanatory remarks from [PZ07].

Remark 5.2.7 ([PZ07]). Since the domains $D(F)$ and $D(G)$ are dense in H, conditions (F) and (G) imply that, for each $t > 0$, $S(t)F$ and $S(t)G$ have unique extensions to continuous mappings from H to H and from H to $L(\mathcal{H}, H)$, respectively. We also denote these extensions by $S(t)F$ and $S(t)G$. Clearly, for all $t > 0$ and $x, y \in H$,

$$\|S(t)F(x)||_H \leq a(t)(1 + \|x\|_H), \quad \|S(t)(F(x) - F(y))\|_H \leq a(t)\|x - y\|_H$$

and

$$\begin{aligned} &\|S(t)G(x)\|_{L_2(\mathcal{H},H)} \leq b(t)(1 + \|x\|_H), \\ &\|S(t)(G(x) - G(y))\|_{L_2(\mathcal{H},H)} \leq b(t)\|x - y\|_H. \end{aligned}$$

Remark 5.2.8. The function $t \mapsto \|S(t)\|_{L(H,H)}$ is bounded on any finite interval $[0, T]$. Thus, if $F : H \to H$ and $G : H \to L_2(\mathcal{H}, H)$ are Lipschitz continuous, then F and G also satisfies the linear growth assumptions of (F) and (G).

An alternative type of solution of SPDEs is the solution type known as a weak solution.

Definition 5.2.9 (Weak solution, [PZ07]). Assume that (F) and (G) from Assumption 5.2.5 hold. Let $t_0 \geq 0$, and let X_0 be a square integrable $\mathcal{F}_{t_0}$-measurable random variable in H. We say that a predictable H-valued process $\{X(t)\}_{t \geq t_0}$ is a weak solution to (5.4) if

$$\sup_{t \in [t_0, T]} E\|X(t)\|_H^2 < \infty, \quad \forall T \in (t_0, \infty)$$

and, for all $a \in D(A^*)$ and $t \geq t_0$

$$\begin{aligned} \langle a, X(t)\rangle_H = \langle a, X_0\rangle_H + \int_{t_0}^{t} \langle A^* a, X(s)\rangle_H ds \\ + \langle a, F(X(s)\rangle_H ds + \int_{t_0}^{t} \langle G^*(X(s))a, dM(s)\rangle_{\mathcal{H}}. \end{aligned}$$

Under the right assumptions, weak and mild solutions coincide.

Theorem 5.2.10 (Equivalence of weak and mild solutions, [PZ07]). *Assume that (F) and (G) from Assumption 5.2.5 hold. Then X is a mild solution of* (5.10) *if and only if X is a weak solution.*

Cylindrical Wiener case

Above, the noise term was driven by a square integrable martingale, but this is not the only type of noise that can be considered. The equation to be investigated here, is of the same form as (5.10), and with the same assumptions on A, but this time the noise term is driven by a cylindrical Wiener process, that is,

$$dX = AXdt + F(t, X)dt + G(X)d\tilde{W}_t, \quad X(t_0) = X_0. \tag{5.11}$$

We continue to let U, H denote Hilbert spaces and recall that $\mathcal{P}_{[0,T]}$ denotes the predictable σ-algebra on $\Omega_T := [0, T] \times \Omega$. The following is gathered from chapter 7 in [DZ14].

Let $\mathcal{H}_p$, for $p \geq 2$, be the Banach space of all predictable processes Y on H defined on the time interval $[0, T]$ such that

$$\Big(\sup_{t\in[0,T]} E[|Y(t)|^p] \Big)^{1/p} < \infty.$$

The following assumptions on the coefficients of (5.11) are made.

Assumption 5.2.11 ([DZ14]).

(i) The mapping

$$F : [0, T] \times \Omega \times H \to H, \quad (t, \omega, x) \mapsto F(t, \omega, x)$$

is measurable from $(\Omega_T \times H, \mathcal{P}_T \times \mathcal{B}(H))$ into $(H, \mathcal{B}(H))$. Moreover there exists a constant $C > 0$ such that for all $x, y \in H$, $t \in [0, T]$, $\omega \in \Omega$ we have

$$\|F(t, \omega, x) - F(t, \omega, y)\|_H \leq C\|x - y\|_H$$

and

$$\|F(t, \omega, x)\|_H \leq C(1 + \|x\|_H)$$

(ii) G is a strongly continuous mapping from H into $L(U, H)$, that is, for any $u \in U$ the mapping $x \mapsto G(x)u$ from H to H is continuous, such that for any $t > 0$ and $x \in H$, $S(t)G(x)$ belongs to $L_2(U, H)$. We also assume that there exists a mapping $K : [0, +\infty) \to [0, +\infty)$, $t \mapsto K(t)$ satisfying

$$\int_0^T K^2(t)dt < \infty, \quad \forall T < \infty$$

such that

$$\|S(t)G(x)\|_{L_2(U,H)} \leq K(t)(1 + \|x\|_H), \quad t > 0, \ x \in H,$$

and

$$\|S(t)G(x) - S(t)G(y)\|_{L_2(U,H)} \leq K(t)\|x - y\|_H, \quad t > 0, \ x, y \in H.$$

With these assumptions there exist a unique solution to (5.11), formalized in the following theorem. Again, we only provide a short proof and refer to [DZ14] for the full proof.

Theorem 5.2.12 (Existence and uniqueness of SPDE driven by cylindrical Wiener process, [DZ14])**.** *Assume Assumption 5.2.11 and let $p \geq 2$. Then for an arbitrary $\mathcal{F}_0$-measurable initial condition X_0 such that $E[|X_0|^p] < +\infty$ there exists a unique mild solution X of (5.5) in $\mathcal{H}_p$ and there exists a constant C_T, independent of X_0, such that*

$$\sup_{t\in[0,T]} E[|X(t)|^p] \leq C_T(1 + E[|X_0|^p]).$$

Proof. Adopting the notation from the proof of Theorem 5.2.6 and applying assumption (i) and (ii) one can show that, for fixed $U, V \in \mathcal{H}_p$,

$$\begin{aligned}&\sup_{t\in[0,T]} E\big[\|I_F(U)(t) + J_G(U)(t) - I_F(V)(t) - J_G(V)(t)\|_H^p\\&\leq c \sup_{t\in[0,T]} E\big[\|U(t) - V(t)\|_H^p,\end{aligned}$$

where $c > 0$. If T is small enough, then $c < 1$, and consequently, by Banach's fixed point theorem, equation (5.11) has a unique solution in $\mathcal{H}_p$. The case of general $T > 0$ can be treated by considering the equation in intervals $[0, \tilde{T}]$, $[\tilde{T}, 2\tilde{T}]$, ... with $\tilde{T}$ such that $c(\tilde{T}) < 1$. ■

We have slightly simplified the above theorem. They have written the assumptions (i), (ii) for an arbitrary generalized Wiener process, but we have limited ourselves to the case where the covariance operator Q equals the identity operator I on U, as this is the only case we will need.

Note also that the assumptions on the coefficients in 5.2.12 are slightly different from the assumptions on the coefficients in 5.2.6. Hence, it is probably possible to prove these theorems under somewhat different assumptions. Lastly, observe that these two theorems use different techniques for applying Banach's fixed point theorem. Theorem 5.2.6 uses equivalent norms and chooses a sufficiently small β, whereas Theorem 5.2.12 considers the equation on a sufficiently small interval.

CHAPTER 6

VMLV processes and ambit fields

We have now arrived at the core of this thesis. Where the preceding chapters have all been fairly general and focused on giving an overview of relevant theory, this chapter will be more specific and introduces the class of stochastic processes called volatility modulated Lévy-driven Volterra (VMLV) processes and its subclass of volatility modulated Brownian-driven Volterra (VMBV) processes. Special attention will be given to the subclass of VMLV/VMBV processes called Lévy/Brownian semistationary processes.

Integration theory with respect to VMLV processes will also be established. As before, we split this chapter in two, where one part deals with the real-valued case and the other deals with the Hilbert-valued case. In the more general framework of Hilbert spaces, VMLV processes are called ambit fields.

Since the next chapter considers SDEs driven by VMLV processes and SPDEs driven by ambit fields, the definitions in this chapter are crucial. The following is collected from chapters 1, 4, and 7 in [BBV18] and from [BS16].

We begin with the simplest case which is the real-valued one.

6.1 VMBV and VMLV processes

The obvious starting point is to state the definition of VMLV processes and some properties.

Let $(\Omega, \mathcal{F}, P)$ denote a complete probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}}$ satisfying the usual conditions of right-continuity and completeness.

Definition 6.1.1 (Two-sided Lévy process, [BBV18]). Let $L_1 = \{L_1(t)\}_{t\geq 0}$ denote a one-dimensional Lévy process. Further, let L_2 be an independent copy of L_1 having the same characteristic triplet as L_1. The stochastic process $L = \{L(t))\}_{t\in\mathbb{R}}$ defined by

$$L(t) := \begin{cases} L_1(t), & \text{for } t \geq 0, \\ -L_2(-(t-)), & \text{for } t < 0, \end{cases}$$

is called a Lévy process on $\mathbb{R}$.

In the following we consider a Lévy process on $\mathbb{R}$ with respect to $\mathbb{F}$ and with characteristic triplet (ζ, A, ν).

The definitions of VMLV and VMBV processes now follow, later in this chapter, and in chapter 7 we will, in various settings, make several specifications to this rather large class of processes.

Definition 6.1.2 (VMLV/VMBV processes, [BBV18])**.** Let L denote a Lévy process on $\mathbb{R}$ and let $\{\sigma(s-)\}_{s\in\mathbb{R}}$ denote a predictable, càdlàg, nonnegative stochastic process and a a càdlàg stochastic process. Further let $G, Q : \mathbb{R}^2 \to \mathbb{R}$ be measurable deterministic functions with $G(t,s) = Q(t,s) = 0$ for $t < s$, also let $\mu \in \mathbb{R}$. The stochastic process given by

$$X(t) = \mu + \int_{-\infty}^{t} G(t,s)\sigma(s-)dL(s) + \int_{-\infty}^{t} Q(t,s)a(s)ds, \quad t \in \mathbb{R},$$

and satisfying the integrability conditions

$$\int_{-\infty}^{t} AG^2(t,s)\sigma^2(s)ds < \infty, \tag{6.1}$$

$$\int_{-\infty}^{t}\int_{\mathbb{R}_0} \min\Big(1, (G(t,s)\sigma(s)z)^2\Big)\nu(dz)ds < \infty, \tag{6.2}$$

$$\int_{-\infty}^{t} \left| G(t,s)\sigma(s)\zeta + \int_{\mathbb{R}_0} \big[\tau(G(t,s)\sigma(s)z) - G(t,s)\sigma(s)\tau(z)\big]\nu(dz)\right| ds < \infty, \tag{6.3}$$

$$\int_{-\infty}^{t} |Q(t,s)a(s)|ds < \infty, \tag{6.4}$$

is called a volatility modulated Lévy-driven Volterra (VMLV) process.

The functions g, q are called kernel functions and σ is called the volatility/intermittency process. Furthermore, if L is a Brownian motion on $\mathbb{R}$ we call the VMLV process a Brownian-driven Volterra (VMBV) process.

Due to a result of [RR89] the characteristic function of a VMLV process takes on the following form.

Proposition 6.1.3 (Characteristic function of VMLV process, [BBV18])**.** *Suppose that (σ, a) and L are independent and define $\mathcal{F}^\sigma = \{\mathcal{F}^\sigma_t\}_{t\in\mathbb{R}}$ with $\mathcal{F}^\sigma_t = \sigma\{\sigma(s) : s \leq t\}$, and $\mathcal{F}^a = \{\mathcal{F}^a_t\}_{t\in\mathbb{R}}$ where $\mathcal{F}^a_t = \sigma\{a(s) : s \leq t\}$. The conditional characteristic function of $X(t)$ is given by*

$$\begin{aligned} &E[\exp(i\theta X(t)) \mid \mathcal{F}^\sigma \vee \mathcal{F}^a] \\ &\quad = \exp\left(i\theta\mu + i\theta\int_{-\infty}^{t} Q(t,s)a(s)ds + \int_{-\infty}^{t} C(\theta G(t,s)\sigma(s); L(1))ds\right), \end{aligned}$$

where $C(\cdot; L(1))$ denotes the cumulant function of $L(1)$

From the characteristic function one can deduce the second order structure of VMLV processes.

Proposition 6.1.4 (Second order structure of VMLV process, [BBV18])**.** *The conditional second order structure of X is given by*

$$E(X(t) \mid F^\sigma \vee F^a) = \mu + \int_{-\infty}^{t} Q(t,s)a(s)ds + E[L(1)]\int_{-\infty}^{t} G(t,s)\sigma(s)ds,$$

$$Var(X(t) \mid F^\sigma \vee F^a) = Var(L(1))\int_{-\infty}^{t} G^2(t,s)\sigma^2(s)ds,$$

$$Cov((X(t+h), X(t)) \mid F^\sigma \vee F^a) = Var(L(1)) \int_{-\infty}^{t} G(t+h,s)G(t,s)\sigma^2(s)ds,$$

for $t \in \mathbb{R},\ h \geq 0$.

One of the simplifications we will make later on is setting $\sigma = 1$, if we also set $a = 1$ we can see that the unconditional second order structure takes on the same form as above.

This thesis will devote a sizable amount of focus on a subclass of VMLV/VMBV processes called Lévy/Brownian semistationary (LSS/BSS) processes. They are defined by setting $G(t,s) = g(t-s)$ and $Q(t,s) = q(t-s)$.

Definition 6.1.5 (LSS/BSS process, [BBV18])**.** Let L denote a Lévy process on $\mathbb{R}$ and let $\{\sigma(s-)\}_{s\in\mathbb{R}}$ denote a predictable, càdlàg, non-negative stochastic process and a a càdlàg stochastic process. Further let $g, q : (0,\infty) \to \mathbb{R}$ denote deterministic functions and suppose that $\mu \in \mathbb{R}$. The stochastic process given by

$$X(t) = \mu + \int_{-\infty}^{t} g(t-s)\sigma(s-)dL(s) + \int_{-\infty}^{t} q(t-s)a(s)ds, \quad \mu \in \mathbb{R}$$

and satisfying the integrability conditions (6.1), (6.2), (6.3) and (6.4) for $G(t,s) = g(t-s)$ and $Q(t,s) = q(t-s)$, is called a Lévy semistationary (LSS) process. If L is a Brownian motion on $\mathbb{R}$ we call X a Brownian semistationary (BSS) process.

Using the same characteristic function as for VMLV processes one can compute the second order structure of LSS processes, note that by the nature of g and q, a time shift can be performed.

Proposition 6.1.6 (Second order structure of LSS process, [BBV18])**.** *The conditional second order structure of X is given by*

$$E(X(t) \mid F^\sigma \vee F^a) = \mu + \int_0^\infty q(x)a(t-x)dx + E[L(1)] \int_0^\infty g(x)\sigma(t-x)dx,$$

$$Var(X(t) \mid F^\sigma \vee F^a) = Var(L(1)) \int_0^\infty g^2(x)\sigma^2(t-x)dx$$

$$Cov((X(t+h), X(t)) \mid F^\sigma \vee F^a) = Var(L(1)) \int_0^\infty g(x+h)g(x)\sigma^2(t-x)dx,$$

for $t \in \mathbb{R},\ h \geq 0$.

LSS processes are in general not semimartingales, but in the following proposition we can see that under the right conditions they will be.

Proposition 6.1.7 (Semimartingale conditions of LSS process, [BBV18])**.** *Let X be an LSS process and let $\mathbb{F}$ denote a filtration such that L is a semimartingale in that filtration and that both σ and a are adapted to it. Suppose the following conditions hold:*

(i) $E|L(1)| < \infty$.

(ii) *The function values $g(0+)$ and $q(0+)$ exist and are finite.*

(iii) *The kernel function g is absolutely continuous with square integrable derivative g'.*

(iv) *The process $(g'(t-s)\sigma(s-))_{s\in\mathbb{R}}$ is square integrable for each $t \in \mathbb{R}$.*

(v) *The process $(q'(t-s)a(s))_{s\in\mathbb{R}}$ is integrable for each $t \in \mathbb{R}$.*

Then $\{X(t)\}_{t\geq 0}$ is an $\mathbb{F}$-semimartingale with representation

$$X(t) = X(0) + g(0+)\int_0^t \sigma(s-)d\bar{L}(s) + \int_0^t A(s)ds \quad \textit{for } t \geq 0$$

where $\bar{L}(s) = L(s) - E(L(s))$ for $s \in \mathbb{R}$ and

$$\begin{aligned} A(s) &= g(0+)\sigma(s-)E[L(1)] + \int_{-\infty}^{s} g'(s-u)\sigma(u-)dL(u) \\ &+ q(0+)a(s) + \int_{-\infty}^{s} q'(s-u)a(u)du. \end{aligned}$$

A well-known example of an LSS process is the Ornstein-Uhlenbeck (OU) process, this is also an example of a semimartingale under the right condition.

Example 6.1.8 (Ornstein-Uhlenbeck process, [BBV18])**.** Let $g(t-s) = e^{-\lambda(t-s)}$ for $\lambda > 0$ and $s \leq t$, then

$$X(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)}\sigma(s-)dL(s)$$

is a volatility-modulated Lévy driven OU process, which also happens to be a semimartingale if we assume $E[L(1)] < \infty$.

Continuous modifications

We will in this subsection show that BSS processes and LSS processes admit continuous modifications with appropriate assumptions on g and σ. The conditions in the case of an LSS process will be stronger than in the case of a BSS process. This is a natural consequence of the fact that Brownian motions are continuous, whereas Lévy processes are not.

Theorem 6.1.9 (Continuous modification of BSS processes)**.** *Let $Y = \{Y(t)\}_{t\geq 0}$ be a BSS process, that is,*

$$Y(t) = \int_0^t g(t-s)\sigma(s)dB(s).$$

Assume that g is Hölder continuous with exponent $\alpha > 1/4$, i.e.

$$|g(t) - g(s)| \leq |t-s|^{\alpha},$$

also assume the integrability conditions

$$\int_0^T \sigma^2(s)ds < \infty \tag{6.5}$$

$$\int_0^T g(t-s)^{2(2+\epsilon)}\sigma^{2(2+\epsilon)}ds < \infty \tag{6.6}$$

for some small $\epsilon > 0$. Then Y has a continuous modification by Kolmogorov's continuity theorem, see 2.3.3.

Proof. In short, we use 2.3.1 repeatedly to get two separate terms. Then we can then apply the integrability assumption on the first term and the Hölder continuity assumption on the second term. So by applying 2.3.1, and assuming $t \geq u$ we have,

$$|Y(t) - Y(u)|^4 = \Big| \int_0^t g(t-s)\sigma(s)dB(s) - \int_0^u g(u-s)\sigma(s)dB(s) \Big|^4$$
$$\leq \Big(\Big| \int_0^u (g(t-s) - g(u-s))\sigma(s)dB(s) \Big| + \Big| \int_u^t g(t-s)\sigma(s)dB(s) \Big| \Big)^4$$
$$\leq \Big(2 \cdot \Big| \int_0^u (g(t-s) - g(u-s))\sigma(s)dB(s) \Big|^2 + 2 \cdot \Big| \int_u^t g(t-s)\sigma(s)dB(s) \Big|^2 \Big)^2$$
$$\leq 2 \cdot 4 \Big| \int_0^u (g(t-s) - g(u-s))\sigma(s)dB(s) \Big|^4 + 2 \cdot 4 \Big| \int_u^t g(t-s)\sigma(s)dB(s) \Big|^4$$

We can now consider each term separately. Since the integrals are normally distributed we can use the 4th moment of Gaussian variables. By also applying Hölder's inequality with $p = 2 + \epsilon$ and $q = 2 - \delta$ where $1 > \epsilon, \delta > 0$ are such that $1/q + 1/p = 1$, we get

$$E\Big[\Big| \int_u^t g(t-s)\sigma(s)dB(s) \Big|^4 \Big] = 3\Big(\int_u^t g^2(t-s)\sigma^2(s)ds \Big)^2$$
$$\leq 3\Big(\Big(\int_u^t (g^2(t-s)\sigma^2(s))^{2+\epsilon}ds \Big)^{1/(2+\epsilon)} \Big(\int_u^t 1^{2-\delta}ds \Big)^{1/(2-\delta)} \Big)^2$$
$$= (t-u)^{2/(2-\delta)} \cdot 3\Big(\int_0^T g^{2(2+\epsilon)}(t-s)\sigma^{2(2+\epsilon)}(s)ds \Big)^{1/(2+\epsilon)}.$$

For the second term we apply the assumption of Hölder continuity with exponent $\alpha > 1/4$,

$$E\Big[\Big| \int_0^u (g(t-s) - g(u-s))\sigma(s)dB(s) \Big|^4 \Big]$$
$$= 3\Big(\int_0^u (g(t-s) - g(u-s))^2\sigma^2(s)ds \Big)^2$$
$$\leq 3\Big(\int_0^u |(t-s) - (u-s)|^{2\alpha}\sigma^2(s)ds \Big)^2$$
$$= 3(t-u)^{4\alpha}\Big(\int_0^u \sigma^2(s)ds \Big)^2 \leq (t-u)^{4\alpha} \cdot 3\Big(\int_0^T \sigma^2(s)ds \Big)^2$$

where $4\alpha > 1$. Putting everything together gives

$$
\begin{aligned}
&E[|Y(t)-Y(u)|^4] \\
&\le (t-u)^{4\alpha}\Big(8\cdot 3\Big(\int_0^T \sigma^2(s)ds\Big)^2 \\
&\qquad + \frac{(t-u)^{2/(2-\delta)}}{(t-u)^{4\alpha}} 8\cdot 3\Big(\int_0^T g^{2(2+\epsilon)}(t-s)\sigma^{2(2+\epsilon)}(s)ds\Big)^{1/(2+\epsilon)}\Big) \\
&\le (t-u)^{4\alpha}\Big(24\Big(\int_0^T \sigma^2(s)ds\Big)^2 \\
&\qquad + T^{2/(2-\delta)-4\alpha} 24\Big(\int_0^T g^{2(2+\epsilon)}(t-s)\sigma^{2(2+\epsilon)}(s)ds\Big)^{1/(2+\epsilon)}\Big),
\end{aligned}
$$

where we have assumed that $2/(2-\delta) > 4\alpha$, assuming the reverse inequality would work as well.

Hence we can conclude that Y does have a continuous modification with the Hölder continuity assumption and the integrability assumptions (6.5) and (6.6). ■

As already mentioned, the corresponding result for LSS processes requires stronger assumptions, we will therefore give some examples of functions that satisfies these assumptions afterwards.

Theorem 6.1.10 (Continuous modification of LSS processes)**.** *Let $Y = \{Y(t)\}_{t\ge 0}$ be a LSS process, that is*

$$
Y(t) = \int_0^t g(t-s)\sigma(s-)dL(s).
$$

Assume the following

(i) $\int_{\mathbb{R}_0} z^2\nu(dz) \le M < \infty$,

(ii) $g(0) = 0$,

(iii) *g is Hölder continuous with $\alpha > 1/2$, i.e. $|g(s)-g(u)| \le |s-u|^\alpha$,*

(iv) $\int_0^T \sigma^2(s)ds < \infty$.

Then Y has a continuous modification by Kolmogorov's continuity theorem, see 2.3.3.

Proof. The approach follows along the same path as the proof of 6.1.9. We use the elementary inequality 2.3.1 to obtain two terms, and subsequently use the appropriate assumptions to get an expression where we can apply Kolmogorov's continuity theorem. We have,

$$
\begin{aligned}
&|Y(t)-Y(u)|^2 = \Big|\int_0^t g(t-s)\sigma(s-)dL(s) - \int_0^u g(u-s)\sigma(s-)dL(s)\Big|^2 \\
&\le \Big(\Big|\int_0^u (g(t-s)-g(u-s))\sigma(s-)dL_s\Big| + \Big|\int_u^t g(t-s)\sigma(s-)dL(s)\Big|\Big)^2
\end{aligned}
$$

$$\leq 2 \cdot \Big| \int_0^u (g(t-s) - g(u-s))\sigma(s-)dL(s) \Big|^2 + 2 \cdot \Big| \int_u^t g(t-s)\sigma(s-)dL(s) \Big|^2.$$

The first term can be dealt with using the Itô isometry 2.2.10 and assumption (i), (ii) and (iii),

$$\begin{aligned}
&E\Big[\Big| \int_u^t g(t-s)\sigma(s-)dL(s) \Big|^2\Big] = \int_u^t \int_{\mathbb{R}_0} z^2 g^2(t-s)\sigma^2(s)\nu(dz)ds \\
&\leq M \int_u^t (g(t-s) - g(0))^2 \sigma^2(s)ds \leq M \int_u^t (t-s)^{2\alpha}\sigma^2(s)ds \\
&\leq M \int_u^t (t-u)^{2\alpha}\sigma^2(s)ds = M(t-u)^{2\alpha} \int_0^T \sigma^2(s)ds.
\end{aligned}$$

The second term is dealt with using the Itô isometry and assumption (i) and (iii), we get

$$\begin{aligned}
&E\Big[\Big| \int_0^u (g(t-s) - g(u-s))\sigma(s-)dL(s) \Big|^2\Big] \\
&= \int_0^u z^2(g(t-s) - g(u-s))^2\sigma^2(s)\nu(z)ds \\
&\leq M \int_0^u |(t-s) - (u-s)|^{2\alpha}\sigma^2(s)ds \\
&= M \int_0^u |t-u|^{2\alpha}\sigma^2(s)ds \leq (t-u)^{2\alpha} M \int_0^T \sigma^2(s)ds
\end{aligned}$$

Putting everything together gives

$$\begin{aligned}
&E[|Y(t) - Y(u)|^2] \leq (t-u)^{2\alpha}\Big(M \int_0^T \sigma^2(s)ds + M \int_0^T \sigma^2(s)ds\Big) \\
&= (t-u)^{2\alpha} 2M \int_0^T \sigma^2(s)ds
\end{aligned}$$

Hence, by Kolmogorov's continuity theorem, Y does have a continuous modification under the assumptions (i), (ii), (iii), (iv). ■

The assumptions on the kernel function g are rather heavy and the set of kernel functions satisfying (ii) and (iii) is therefore somewhat small, but not empty.

Example 6.1.11 (Examples of functions satisfying (ii) and (iii) from 6.1.10)**.**

(a) g defined by $g(x) = x^\alpha$ satisfies the required assumption for all $\alpha > 1/2$.

(b) If we for simplicity set $\alpha = 1$, we can see that another example of a function satisfying (ii) and (iii) is defined by

$$g(x) = \begin{cases} \ln(x), & \text{if } x \geq 1 \\ 0, & \text{if } x \leq 1 \end{cases}$$

where $\ln(x)$ denotes the natural logarithm of x.

(c) If we again set $\alpha = 1$, then the function defined by $g(x) = e^{-\lambda x} - 1$ or $g(x) = 1 - e^{-\lambda x}$ will also satisfy (ii) and (iii) for $0 < \lambda \leq 1 + c$, where c is some positive constant slightly smaller than 0.4. The process Y will in this case be a "shifted" Ornstein-Uhlenbeck process.

In example (c) it is probably possible to find and explicit relationship between α and λ, we have not looked into that and the constant c was "found" through trial and error.

6.2 Integration with respect to VMBV and VMLV processes

This section will introduce stochastic integration with respect to VMLV and VMBV processes. We consider VMLV/VMBV process without drift, that is, a process X defined by

$$X(t) = \int_0^t G(t,s)\sigma(s-)dL(s).$$

We begin with the case of VMBV processes. Define the operator K_G acting on measurable functions $h : [s,t] \to \mathbb{R}$ for $t \geq s \geq 0$, by

$$K_G(h)(t,s) = h(s)G(t,s) + \int_s^t (h(u) - h(s))G(du,s).$$

Definition 6.2.1 (Integration w.r.t. VMBV process, [BBV18])**.** Suppose that for $s \in \mathbb{R}_+$ the mapping $t \mapsto G(t,s)$ is of bounded variation on $[u,v]$ for all $0 \leq s < u < v < \infty$. Assume that the stochastic process $Y(s)$ on $0 \leq s \leq t$ for fixed $t > 0$ satisfies the following conditions:

(1) For $s \in [0,t]$, the process $u \mapsto (Y(u) - Y(s))$, $s \leq u \leq t$, is integrable with respect to $G(du,s)$ a.s.,

(2) The mapping
$$s \mapsto K_G(Y)(t,s)\sigma(s)\chi_{[0,t]}(s)$$
is Skorohod integrable,

(3) $K_G(Y)(t,s)$ is Malliavin differentiable for $s \in [0,t]$, with
$$s \mapsto D_s\{K_G(Y)(t,s)\}\sigma(s)$$
being Lebesgue integrable on $[0,t]$.

We say that Y is $\mathfrak{L}([0,t])$-integrable, and define

$$\int_0^t Y(s)dX(s) = \int_0^t K_G(Y)(t,s)\sigma(s)\delta B(s) + \int_0^t D_s\{K_G(Y)(t,s)\}\sigma(s)ds.$$

When $G(t,s) = g(t-s)$ the operator K_G can be written in an alternative way, which we will benefit a lot from in chapter 7.

Lemma 6.2.2 ([BBV18]). *Suppose $G(t,s) := g(t-s)$ for $t \geq s$. If $u \mapsto h(u)$ is Lebesgue-Stieltjes integrable on $[s,t]$ with respect to $G(du,s)$, then $u \mapsto h(s+u)$ is Lebesgue-Stieltjes integrable on $[0,t-s]$ with respect to $g(du)$ and*

$$\int_s^t h(u)G(du,s) = \int_0^{t-s} h(u+s)g(du).$$

This allows us to write

$$K_G(h)(t,s) = h(s)g(t-s) + \int_0^{t-s} (h(u+s) - h(s))g(du)$$

for $G(t,s) = g(t-s)$.

In the case of a pure jump Lévy process the stochastic integral looks slightly different, the reason for this is that the integration by parts formula in the Brownian and pure jump Lévy case, as defined in Theorem 3.1.19 and Theorem 3.2.15 respectively, is different (see [BBV18] for details).

Definition 6.2.3 (Integration w.r.t. VMLV process, [BBV18]). Suppose that for $s \in \mathbb{R}_+$ the mapping $t \mapsto G(t,s)$ on $(s,T]$ is of bounded variation. Assume that the stochastic process $Y(s)$ on $0 \leq s \leq t$ for fixed $0 < t \leq T$ satisfies the following conditions:

1. For $s \in [0,t]$, the process $u \mapsto (Y(u) - Y(s))$, $s < u \leq t$, is integrable with respect to $G(du,s)$ a.s.,
2. The mapping
$$(s,z) \mapsto z\big\{K_G(Y)(t,s) + \mathcal{D}_{s,z}(K_G(Y)(t,s))\big\}\sigma(s)\chi_{[0,t]}(s)$$
is Skorohod integrable on $[0,T] \times \mathbb{R}_0$ w.r.t. $\tilde{N}(\delta z, ds)$,
3. $K_G(Y)(t,s)$ is Malliavin differentiable for $(s,z) \in [0,t] \times \mathbb{R}_0$, with
$$(s,z) \mapsto D_{s,z}\{K_G(Y)(t,s)\}\sigma(s)$$
being $\nu(dz)ds$ integrable on $[0,t] \times \mathbb{R}_0$.

We say that Y is $\tilde{\mathfrak{L}}([0,t])$-integrable, and define

$$\begin{aligned}\int_0^t Y(s)dX(s) = &\int_0^t \int_{\mathbb{R}_0} z\big\{K_G(Y)(t,s) + D_{s,z}(K_G(Y)(t,s))\big\}\sigma(s)\tilde{N}(\delta z, ds) \\ &+ \int_0^t \int_{\mathbb{R}_0} D_{s,z}(K_G(Y)(t,s))\sigma(s)\nu(dz)ds.\end{aligned}$$

Note that the above definition does not require $t \mapsto G(t,\cdot)$ to be right continuous. This does however seem to be necessary, and it is definitely necessary for parts of our examination of S(P)DEs in chapter 7. Therefore, we add the assumption that $t \mapsto G(t,\cdot)$ is right continuous and maintain this assumption throughout the rest of this thesis, both in the real-valued case and in the Hilbert-valued case. When $G(t,s) = g(t-s)$, we simply assume that g is right-continuous.

Integrals w.r.t. VMLV/VMBV processes satisfies some of the same desirable properties as classical integrals.

Lemma 6.2.4 ([BBV18]). *Let $0 < t_1 < t_2$ and assume that $s \mapsto Y(s)$ is $\mathfrak{L}([0,t_1])$-integrable (respectively $\tilde{\mathfrak{L}}([0,t_1])$-integrable). Then $s \mapsto Y(s)\chi_{\{s\le t_1\}}(s)$ is($\mathfrak{L}([0,t_2])$-integrable (respectively $\tilde{\mathfrak{L}}([0,t_2])$-integrable) and*

$$\int_0^{t_2} Y(s)\chi_{\{s\le t_1\}}(s)dX(s) = \int_0^{t_1} Y(s)dX(s).$$

The very desirable property of linearity is also satisfied.

Lemma 6.2.5 ([BBV18])). *If Y and Z are two processes which are $\mathfrak{L}([0,t])$-integrable (respectively $\tilde{\mathfrak{L}}([0,t])$-integrable) and $a, b \in \mathbb{R}$ are two constants, then $s \mapsto aY(s) + bZ(s)$ is $\mathfrak{L}([0,t])$-integrable (respectively $\tilde{\mathfrak{L}}([0,t])$-integrable), and*

$$\int_0^t (aY(s) + bZ(s))dX(s) = a\int_0^t Y(s)dX(s) + b\int_0^t Z(s)dX(s).$$

The respective integration by parts formulas of VMLV processes and VMBV processes are also analogous, however there is a mistake in [BBV18] which we point out.

Proposition 6.2.6 (Integration by parts for VMBV integrals, [BBV18]). *Suppose that $s \mapsto Y(s)$ is $\mathfrak{L}([0,t])$-integrable and Z a bounded random variable such that $s \mapsto ZY(s)$ is $\mathfrak{L}([0,t])$-integrable. Then*

$$\int_0^t ZY(s)dX(s) = Z\int_0^t Y(s)dX(s).$$

The corresponding integration by parts formula for VMLV processes given in [BBV18] is wrong, as they forgot a term when applying the product rule 3.2.11.

Proposition 6.2.7 (Integration by parts for VMLV integrals, [BBV18]). *Suppose that $s \mapsto Y(s)$ is $\tilde{\mathfrak{L}}([0,t])$-integrable and Z a bounded random variable such that $s \mapsto ZY(s)$ is $\tilde{\mathfrak{L}}([0,t])$-integrable. Then*

$$\int_0^t ZY(s)dX(s) = Z\int_0^t Y(s)dX(s)$$

For the proof we point out two mistakes.

Proof. Firstly,

$$\begin{aligned} K_G(ZY)(t,s) &= ZY(s)G(t,s) + \int_s^t (ZY(u) + ZY(s))G(du,s) \\ &= Z\Big(Y(s)G(t,s) + \int_s^t (Y(u) + Y(s))G(du,s)\Big) = ZK_G(Y)(t,s). \end{aligned}$$

The first mistake is just a typo, the lower integral bound is supposed to be s, as above, and not 0 as written in [BBV18], see page 137. The second mistake is that there is a missing term in their application of the product rule 3.2.11, see page 138 in [BBV18], the forgotten term is the third term on the right hand side in the following equality:

$$D_{s,z}\{ZK_G(Y)(t,s)\} = D_{s,z}\{Z\}K_G(Y)(t,s) + ZD_{s,z}\{K_G(Y)(t,s)\}$$

$$+ D_{s,z}\{Z\}D_{s,z}\{K_G(Y)(t,s)\}.$$

From here on and out, no mistakes have been found, but the extra term will actually simplify our computations slightly.

Using the above application of the product rule gives

$$\begin{aligned}
&\int_0^t ZY(s)dX(s) \qquad (6.7)\\
&= \int_0^t \int_{\mathbb{R}_0} z\Big(K_G(ZY)(t,s) + D_{s,z}\{K_G(ZY)(t,s)\}\Big)\sigma(s-)\tilde{N}(\delta z, ds)\\
&\quad + \int_0^t \int_{\mathbb{R}_0} zD_{s,z}\{K_G(ZY)(t,s)\}\sigma(s)\nu(dz)ds\\
&= \int_0^t \int_{\mathbb{R}_0} z\Big(Z + D_{s,z}\{Z\}\Big)K_G(Y)(t,s)\sigma(s-)\tilde{N}(\delta z, ds) \qquad (6.8)\\
&\quad + \int_0^t \int_{\mathbb{R}_0} z\Big(Z + D_{s,z}\{Z\}\Big)D_{s,z}\{K_G(Y)(t,s)\}\sigma(s-)\tilde{N}(\delta z, ds) \qquad (6.9)\\
&\quad + \int_0^t \int_{\mathbb{R}_0} zZD_{s,z}\{K_G(Y)(t,s)\}\sigma(s)\nu(dz)ds \qquad (6.10)\\
&\quad + \int_0^t \int_{\mathbb{R}_0} zD_{s,z}\{Z\}K_G(Y)(t,s)\sigma(s)\nu(dz)ds \qquad (6.11)\\
&\quad + \int_0^t \int_{\mathbb{R}_0} zD_{s,z}\{Z\}D_{s,z}\{K_G(Y)(t,s)\}\sigma(s)\nu(dz)ds \qquad (6.12)
\end{aligned}$$

From the integration by parts formula 3.2.15, we have the equality

$$\begin{aligned}
&Z\int_0^t \int_{\mathbb{R}_0} zK_G(Y)(t,s)\sigma(s-)\tilde{N}(\delta z, ds)\\
&= \int_0^t \int_{\mathbb{R}_0} z\Big(Z + D_{s,z}\{Z\}\Big)K_G(Y)(t,s)\sigma(s-)\tilde{N}(\delta z, ds)\\
&\quad + \int_0^t \int_{\mathbb{R}_0} zD_{s,z}\{Z\}K_G(Y)(t,s)\sigma(s)\nu(dz)ds,
\end{aligned}$$

where the right hand side equals the terms (6.8) and (6.11).

Similarly,

$$\begin{aligned}
&Z\int_0^t \int_{\mathbb{R}_0} zD_{s,z}\{K_G(Y)(t,s)\}\sigma(s-)\tilde{N}(\delta z, ds)\\
&= \int_0^t \int_{\mathbb{R}_0} z\Big(Z + D_{s,z}\{Z\}\Big)D_{s,z}\{K_G(Y)(t,s)\}\sigma(s-)\tilde{N}(\delta z, ds)\\
&\quad + \int_0^t \int_{\mathbb{R}_0} zD_{s,z}\{Z\}D_{s,z}\{K_G(Y)(t,s)\}\sigma(s)\nu(dz)ds,
\end{aligned}$$

where the right hand side equals the terms (6.9) and (6.12).

Thus,

$$\int_0^t ZY(s)dX(s) = Z\int_0^t \int_{\mathbb{R}_0} zK_G(Y)(t,s)\sigma(s-)\tilde{N}(\delta z, ds)$$
$$+ Z\int_0^t \int_{\mathbb{R}_0} zD_{s,z}\{K_G(Y)(t,s)\}\sigma(s-)\tilde{N}(\delta z, ds)$$
$$+ Z\int_0^t \int_{\mathbb{R}_0} zD_{s,z}\{K_G(Y)(t,s)\}\sigma(s)\nu(dz)ds$$
$$= Z\int_0^t Y(s)dX(s)$$

■

The integrals with respect to VMLV/VMBV processes also satisfies the property of localization, which basically means that the integral of 0 is 0.

Proposition 6.2.8 (Localization property of VMLV integrals, [BBV18])**.** *Suppose that $s \mapsto Y(s) = 0$ for a.e. $s \leq t$, a.s. Then Y is $\mathfrak{L}([0,t])$-integrable (respectively $\tilde{\mathfrak{L}}([0,t])$-integrable), and*

$$\int_0^t Y(s)dX(s) = 0, \quad a.s.$$

6.3 Ambit fields

The spotlight is now turned to ambit fields, their definition is stated, and integrals against ambit fields are defined. Some properties of this integral will also be presented.

Ambit fields in a Hilbert space is the analogous process to a VMBV process in $\mathbb{R}$. The definition given for ambit fields encompasses less generality than the one given for VMBV processes, we set the lower integrand bound to 0 and exclude the drift term completely. It is however more general in the sense that we do not require predictablity or adaptedness of σ.

Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ denote separable Hilbert spaces.

Definition 6.3.1 (Ambit field, [BBV18])**.** Let $\tilde{W}$ be a cylindrical Wiener process on $\mathcal{H}_1$, $\{\sigma(t)\}_{t\in[0,T]}$ be a stochastic process with values in $L(\mathcal{H}_1, \mathcal{H}_2)$, not necessarily adapted to the Wiener process $\tilde{W}$ and G be a deterministic function depending on two time parameters such that $G(t,s) \in L(\mathcal{H}_2, \mathcal{H}_2)$ for all $0 \leq s \leq t \leq T$. Furthermore, assume that $G(t,\cdot)\sigma(\cdot)$ is Skorohod integrable on $[0,t]$. Then we can define

$$X(t) = \int_0^t G(t,s)\sigma(s)\delta\tilde{W}(s), \tag{6.13}$$

as a random element of $\mathcal{H}_2$.

As for LSS processes there exist conditions ensuring that X, as defined in (6.13), is a semimartingale.

Proposition 6.3.2 (Semimartingale conditions for ambit fields, [BBV18])**.** *For $t > 0$, assume that $G(t,s)$ is well-defined for all $0 \leq s \leq t$. Furthermore, suppose there exists a measurable function $\phi : [0,T] \times [0,T] \to L(\mathcal{H}_2)$ such that*

$$G(t,s) = G(s,s) + \int_s^t \phi(v,s)dv$$

for all $0 \leq s \leq t$, where the integral on the right-hand side is defined in the Bochner sense and

$$\int_0^t \|G(s,s)\|^2_{L(\mathcal{H}_2)}ds, \quad \text{and} \quad \int_0^t \int_0^u \|\phi(v,s)\|^2_{L(\mathcal{H}_2)}dsdv < \infty.$$

We furthermore suppose that σ is adapted to $\tilde{W}$ and pathwise locally bounded almost surely. Then X defined in (6.13) *is a semimartingale with decomposition*

$$X(t) = \int_0^t G(s,s)\sigma(s)d\tilde{W}(s) + \int_0^t \int_0^s \phi(s,u)\sigma(u)d\tilde{W}(u)ds.$$

Integrating against an ambit field takes on the exact same form as in the real-valued case. But first we need the definition of a vector measure, which takes the place of the Lebesgue-Stieltjes measure in the real-valued case.

Definition 6.3.3 (Vector measure, [BBV18])**.** Let $(\Omega, \mathcal{A})$ be a measurable space and B a Banach space. A set function $\mu : \mathcal{A} \to B$ is called a vector measure if

$$\mu(F_1 \cup F_2) = \mu(F_1) + \mu(F_2)$$

for any disjoint $F_1, F_2 \in \mathcal{A}$. If, moreover, for any sequence $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ of pairwise disjoint subsets of Ω , we have $\mu(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mu(F_n)$ (with convergence in the norm topology of B), then the vector measure μ is called countably additive.

The total variation of a vector measure is defined analogously to the total variation of a signed measure.

Definition 6.3.4 (Total variation of vector measures, [BBV18])**.** The total variation $|\mu|$ of a vector measure μ on the measure space $(\Omega, \mathcal{A})$ with values in a Banach space B, is defined as the set function on $(\Omega, \mathcal{A})$ with values in $R_+ \cup \{\infty\}$ by

$$|\mu|(G) = \sup_{\pi} \sum_{A\in\pi} \|\mu(A)\|_B,$$

for $G \in \mathcal{A}$. Here, π is the collection of partitions of G into a finite number of pairwise disjoint sets $A \in \Omega$. If $|\mu(A)| < \infty$, we say that μ is a vector measure of finite variation.

We can now define the integral against an ambit field. The following assumptions are assumed throughout the rest of this thesis.

Assumption 6.3.5 ([BBV18])**.** Suppose that for all $s \in [0,t)$, $t \leq T$, the $L(\mathcal{H}_2)$-valued vector measure $G(du,s)$ is of bounded variation on $[u,v]$ for all $0 \leq s < u < v \leq t$, and that G and σ are such that for all $0 \leq s < t$, $G(t,s)\sigma(s) \in L_2(\mathcal{H}_1, \mathcal{H}_2)$ and $\chi_{[0,t]}(\cdot)G(t,\cdot)\sigma(\cdot)$ is Skorohod integrable for all $t \leq T$.

Recall the extension of the trace $Tr_{\mathcal{H}_1}$ defined by

$$Tr_{\mathcal{H}_1} A = \sum_{k=1}^{\infty} (A(f_k))(f_k), \quad A \in L(\mathcal{H}_1, L(\mathcal{H}_1, \mathcal{H}_3))$$

whenever this sum converges in $\mathcal{H}_3$, and where $\{f_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_1$.

Definition 6.3.6 (Integration w.r.t ambit fields, [BBV18]). Let X be defined as in (6.13) with Assumption 6.3.5 being valid. For fixed $t \geq 0$, the stochastic process $\{Y(s)\}_{s\in[0,t]}$ is integrable with respect to X on $[0,t]$ if

(i) the process $u \mapsto Y(u) - Y(s)$, for $u \in (s,t)$, is integrable with respect to the vector measure $G(du, s)$ almost surely,

(ii) the process $s \mapsto K_G(Y)(t,s)\sigma(s)\chi_{[0,t]}(s)$ is in the domain of the Skorohod integral with respect to $\tilde{W}$, that is, in the domain of the $\mathcal{H}_3$-valued divergence operator $\delta\tilde{W}$, and

(iii) $K_G(Y)(t,s)$ is Malliavin differentiable with respect to D_s, for all $s \in [0,t]$, and the $\mathcal{H}_3$-valued stochastic process $s \mapsto Tr_{\mathcal{H}_1} D_s(K_G(Y)(t,s))\sigma(s)$ is Bochner integrable on $[0,t]$ almost surely.

In this case, we say that $Y \in \mathcal{I}^X(0,t)$ and define the stochastic integral by

$$\int_0^t Y(s)dX(s) = \int_0^t K_G(Y)(t,s)\sigma(s)\delta\tilde{W}(s) + Tr_{\mathcal{H}_1} \int_0^t D_s(K_G(Y)(t,s))\sigma(s)ds.$$

The integral above satisfies many of the same useful properties that integrals against BSS processes satisfy, like linearity and the property of localization, and perhaps more interestingly, a similar integration by parts formula.

Proposition 6.3.7 (Basic calculus rules, [BS16]). *Let $Y, Z \in \mathcal{I}^X(0,t)$ and $a, b \in R$ be two constants, then $aY + bZ \in \mathcal{I}^X(0,t)$ and*

$$\int_0^t (aY(s) + bZ(s))dX(s) = a\int_0^t Y(s)dX(s) + b\int_0^t Z(s)dX(s).$$

If also $0 < t < T$ and $Y \in \mathcal{I}^X(0,t)$ then $Y\chi_{[0,t]} \in \mathcal{I}^X(0,T)$ and

$$\int_0^T \chi_{[0,t]}(s)Y(s)dX(s) = \int_0^t Y(s)dX(s).$$

Now it also follows that for $0 \leq u < v \leq t$ and $Y \in \mathcal{I}^X(0,u) \cap \mathcal{I}^X(0,v)$, that $Y\chi_{[u,v]} \in \mathcal{I}^X(0,t)$ and

$$\int_0^t Y(s)\chi_{[u,v]}(s)dX(s) = \int_0^v Y(s)dX(s) - \int_0^u Y(s)dX(s).$$

The rules in the last proposition can be used to prove the following properties, we refer to [BS16] for proof.

Proposition 6.3.8 (Properties of the ambit field integral, [BS16]). *Let $t > 0$ and assume $Y \in \mathcal{I}^X(0,t)$.*

(i) *(Integration by parts) Let Z be a random linear operator from $\mathcal{H}_3$ to another separable Hilbert space $\mathcal{H}_4$ which is almost surely bounded. Then $ZY \in \mathcal{I}^X(0,t)$ and*

$$Z \int_0^t Y(s)dX(s) = \int_0^t ZY(s)dX(s) \quad \text{almost surely.}$$

(ii) *(Localizedness) The X-integral is local, that is, if $Y = 0$ on a measurable set $A \subset \Omega$, then*

$$\int_0^t Y(s)dX(s) = 0 \quad \text{on } A.$$

(iii) *Let Y be a simple process, that is, $Y = \sum_{j=1}^{n-1} Z_j \chi_{(t_j, t_{j+1}]}$ where Z_j is a random linear operator from $\mathcal{H}_2$ to $\mathcal{H}_3$ which is almost surely bounded for all $j = 1, ..., n-1$ and $0 \leq t_1 < \cdots < t_n \leq t$ is a partition of the interval $[0,t]$. Then $Y \in \mathcal{I}^X(0,t)$ and*

$$\int_0^t Y(s)dX(s) = \sum_{j=1}^{n-1} Z_j(X(t_{j+1}) - X(t_j)).$$

(iv) *Let furthermore σ be Malliavin differentiable. Then the X-integral is a continuous linear operator from $\mathcal{I}^X(0,t)$ to $L^2(\Omega; \mathcal{H}_3)$.*

6.4 Further properties of VMBV/VMLV and ambit field integrals

Since VMBV and VMLV integrals involve both Malliavin derivatives and Skorohod integrals, it would be interesting to see how they behave when imitating some of the properties of the Skorohod integral. We start with imitating the Skorohod isometry 3.1.21 (respectively 3.2.17), then we look at what happens if we imitate the duality formula 3.1.18 (respectively 3.2.14), and finally, the fundamental theorem of calculus 3.1.20 (respectively 3.2.16).

Let Y be a VMBV process, then

$$\begin{aligned}
&E\Big[\Big(\int_0^t u(s)dY(s)\Big)^2\Big] \\
&= E\Big[\Big(\int_0^t K_G(u)(t,s)\sigma(s)\delta B(s)) + \int_0^t D_s\{K_G(u)(t,s)\}\sigma(s)ds\Big)^2\Big] \\
&= E\Big[\Big(\int_0^t K_G(u)(t,s)\sigma(s)\delta B(s))\Big)^2 + \Big(\int_0^t D_s\{K_G(u)(t,s)\}\sigma(s)ds\Big)^2\Big],
\end{aligned}$$

where the last equality follows since $E[\int_0^t K_G(u)(t,s)\sigma(s)\delta B(s)] = 0$.

When Y is a VMLV process we have

$$
\begin{aligned}
&E\Big[\Big(\int_0^t u(s)dY(s)\Big)^2\Big]\\
&= E\Big[\Big(\int_0^t z\Big[K_G(u)(t,s) + D_{s,z}\{K_G(u)(t,s)\}\Big]\sigma(s)\tilde{N}(\delta z, ds)\\
&\quad + \int_0^t D_{s,z}\{K_G(u)(t,s)\}\sigma(s)\nu(dz)ds\Big)^2\Big]\\
&= E\Big[\Big(\int_0^t zK_G(u)(t,s)\sigma(s)\tilde{N}(\delta z, ds)\Big)^2\\
&\quad + \Big(\int_0^t zD_{s,z}\{K_G(u)(t,s)\}\sigma(s)\tilde{N}(\delta z, ds)\Big)^2\\
&\quad + \Big(\int_0^t D_{s,z}\{K_G(u)(t,s)\}\sigma(s)\nu(dz)ds\Big)^2\Big],
\end{aligned}
$$

where the last equality again follows since the Skorohod integral has expectation 0.

Imitating the duality formulas grants nice expressions both in the VMBV case and in the VMLV case.

Proposition 6.4.1. *Let Y be a VMBV process and let u be $\mathfrak{L}([0,t])$-integrable. Assume that the random variable F is in $\mathbb{D}_{1,2}$, and that F and $K_G(u)(t,s)$ satisfies the conditions of 3.1.14, then*

$$E\Big[F\int_0^t u(s)dY(s)\Big] = E\Big[\int_0^t D_s\{K_G(u)(t,s)F\}\sigma(s)ds\Big]$$

Proof.

$$
\begin{aligned}
&E\Big[F\int_0^t u(s)dY(s)\Big]\\
&= E\Big[F\int_0^t K_G(u)(t,s)\sigma(s)\delta B(s) + F\int_0^t D_s\{K_G(u)(t,s)\}\sigma(s)ds\Big]\\
&= E\Big[\int_0^t K_G(u)(t,s)\sigma(s)D_sF ds + \int_0^t D_s\{K_G(u)(t,s)\}F\sigma(s)ds\Big]\\
&= E\Big[\int_0^t D_s\{K_G(u)(t,s)F\}\sigma(s)ds\Big],
\end{aligned}
$$

where the second equality follows by the duality formula 3.1.18 and the third equality follows by 3.1.14. ■

For the VMLV case we get a similar formula.

Proposition 6.4.2. *Let Y be a VMLV process and let $u = u(s,z)$, $s \in [0,t]$, $z \in \mathbb{R}_0$ be $\tilde{\mathfrak{L}}([0,t])$-integrable. Assume that the random variable F is in $\mathbb{D}_{1,2}$, and that F and $K_G(u)(t,s)$ satisfies the conditions of 3.2.10 or of 3.2.12, then*

$$E\Big[F\int_0^t u(s)dY(s)\Big] = E\Big[\int_0^t \int_{\mathbb{R}_0} zD_{s,z}\{K_G(u)(t,s)F\}\sigma(s)\nu(dz)ds\Big]$$

Proof.

$$
\begin{aligned}
&E\Big[F\int_0^t u(s)dY(s)\Big] \\
&= E\Big[F\int_0^t\int_{\mathbb{R}_0} z\Big[K_G(u)(t,s) + D_{s,z}\{K_G(u)(t,s)\}\Big]\sigma(s)\tilde{N}(\delta z, ds) \\
&\quad + F\int_0^t\int_{\mathbb{R}_0} zD_s\{K_G(u)(t,s)\}\sigma(s)ds\Big] \\
&= E\Big[\int_0^t\int_{\mathbb{R}_0} z\Big[K_G(u)(t,s) + D_{s,z}\{K_G(u)(t,s)\}\Big]D_{s,z}\{F\}\sigma(s)\nu(dz)ds \\
&\quad + \int_0^t\int_{\mathbb{R}_0} zD_{s,z}\{K_G(u)(t,s)\}F\sigma(s)\nu(dz)ds\Big] \\
&= E\Big[\int_0^t\int_{\mathbb{R}_0} zD_{s,z}\{K_G(u)(t,s)F\}\sigma(s)\nu(dz)ds\Big],
\end{aligned}
$$

where the second equality follows by the duality formula 3.2.14 and the third equality follows by the product rule 3.2.10 or by 3.2.12. ■

Finally, we imitate the fundamental theorem of calculus. For a particular choice of σ, this is the best imitation, and naming it the fundamental theorem of calculus for VMBV and VMLV integrals seems justified. But first, a couple lemmas.

Lemma 6.4.3. *For a twice Malliavin differentiable random variable F with chaos expansion $\sum_{n=0}^{\infty} I_n(f_n)$ we have*

$$D_rD_s\{F\} = D_sD_r\{F\}$$

and in the jump case

$$D_{r,y}D_{s,z}\{F\} = D_{s,z}D_{r,y}\{F\}$$

Proof. Since the proof follows in the exact same way in both cases, we only proves the jump case.

$$
\begin{aligned}
&D_{r,y}D_{s,z}\{F\} = D_{r,y}\Big(\sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,s,z))\Big) \\
&= \sum_{n=2}^{\infty} n(n-1)I_{n-2}(f_n(\cdot,r,y,s,z)) = \sum_{n=2}^{\infty} n(n-1)I_{n-2}(f_n(\cdot,s,z,r,y)) \\
&= D_{s,z}\Big(\sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,r,y))\Big) = D_{s,z}D_{r,y}\{F\},
\end{aligned}
$$

where the third equality follows since f_n is symmetric. ■

We also separate the manipulation of the Malliavin derivative and the kernel operator K_G into its own lemma.

Lemma 6.4.4. *For the kernel operator*

$$K_G(h)(t,s) = G(t,s)h(s) + \int_s^t (h(u) - h(s))G(du,s)$$

we have

$$\begin{aligned}&D_r\big[D_s\{K_G(h)(t,s)\}\sigma(s)\big]\\&= D_s\{K_G(D_r h)(t,s)\}\sigma(s) + D_s\{K_G(h)(t,s)\}D_r\{\sigma(s)\},\end{aligned}$$

where we assume the necessary conditions on σ and $K_G(h)(t,s)$ such that 3.1.14 is applicable.

Proof. By the product rule 3.1.14 and Lemma 6.4.3 we have

$$\begin{aligned}&D_r\big[D_s\{K_G(h)(t,s)\}\sigma(s)\big]\\&= D_r D_s\{K_G(h)(t,s)\}\sigma(s) + D_s\{K_G(h)(t,s)\}D_r\{\sigma(s)\}\\&= D_s D_r\{K_G(h)(t,s)\}\sigma(s) + D_s\{K_G(h)(t,s)\}D_r\{\sigma(s)\}.\end{aligned}$$

Now, by the commutation of the Malliavin derivative and the Lebesgue-Stieltjes integral we have

$$\begin{aligned}&D_r\{K_G(h)(t,s)\} = D_r\Big\{G(t,s)h(s) + \int_s^t (h(u) - h(s))G(du,s)\Big\}\\&= G(t,s)D_r\{h(s)\} + \int_s^t (D_r\{h(u)\} - D_r\{h(s)\})G(du,s)\\&= K_G(D_r h)(t,s).\end{aligned}$$

Hence,

$$\begin{aligned}&D_s D_r\{K_G(h)(t,s)\}\sigma(s) + D_s K_G(h)(t,s)D_r\{\sigma(s)\}\\&= D_s\{K_G(D_r h)(t,s)\}\sigma(s) + D_s\{K_G(h)(t,s)\}D_r\{\sigma(s)\}\end{aligned}$$

■

With the aid of the two preceding lemmas, the proof of the next theorem is simplified.

Proposition 6.4.5 (The fundamental theorem of calculus for VMBV integrals with constant volatility)**.** *Let Y be a VMBV process with volatility $\sigma = 1$. Assume for all $r \in [0,t]$, that $D_r\{u(\cdot)\}$ is $\mathfrak{L}([0,t])$-integrable, and that the application of Lemma 6.4.4 is justified. Then $\int_0^t u(s)dY(s) \in \mathbb{D}_{1,2}$ and*

$$D_r\Big(\int_0^t u(s)dY(s)\Big) = \int_0^t D_r u(s)dY(s) + K_G(u)(t,r) \tag{6.14}$$

Proof. As it is interesting in its own right, we first write out the case when σ is general and then set $\sigma = 1$ to get (6.14),

$$D_r\Big(\int_0^t u(s)dY(s)\Big)$$

$$
\begin{aligned}
&= D_r\Big(\int_0^t K_G(u)(t,s)\sigma(s)\delta B(s) + \int_0^t D_s\{K_G(u)(t,s)\}\sigma(s)ds\Big)\\
&= \int_0^t D_r\{K_G(u)(t,s)\sigma(s)\}\delta B(s) + K_G(u)(t,r)\sigma(r)\\
&\quad + \int_0^t D_r\big[D_s\{K_G(u)(t,s)\}\sigma(s)\big]ds\\
&= \int_0^t \Big[K_G(D_r u)(t,s)\sigma(s) + K_G(u)(t,s)D_r\{\sigma(s)\}\Big]\delta B(s) + K_G(u)(t,r)\sigma(r)\\
&\quad + \int_0^t \Big[D_s\{K_G(D_r u)(t,s)\}\sigma(s) + D_s\{K_G(u)(t,s)\}D_r\{\sigma(s)\}\Big]ds.
\end{aligned}
$$

Where the second equality use the fundamental theorem of calculus 3.1.20 and the third equality uses Lemma 6.4.4 and 3.1.14.

If now $\sigma = 1$, then $D_r\sigma(s) = 0$ and we have

$$
\begin{aligned}
D_r\Big(\int_0^t u(s)dY(s)\Big) &= \int_0^t \{K_G(D_r u)(t,s)\}\delta B(s) + K_G(u)(t,r)\\
&\quad + \int_0^t D_s\{K_G(D_r u)(t,s)\}ds\Big)\\
&= \int_0^t D_r u(s)dY(s) + K_G(u)(t,r).
\end{aligned}
$$

■

The corresponding result and lemma for VMLV processes take on a more complicated form.

Lemma 6.4.6. *For the kernel operator*

$$K_G(h)(t,s) = G(t,s)h(s) + \int_s^t (h(u) - h(s))G(du,s)$$

we have

$$
\begin{aligned}
&D_{r,y}\big[D_{s,z}\{K_G(h)(t,s)\}\sigma(s)\big] = D_{s,z}\{K_G(D_{r,y}\{h\})(t,s)\}\sigma(s)\\
&\quad + D_{s,z}\{K_G(h)(t,s)\}D_{r,y}\{\sigma(s)\} + D_{s,z}\{K_G(D_{r,y}\{h\})(t,s)\}D_{r,y}\{\sigma(s)\},
\end{aligned}
$$

where we assume the necessary conditions on σ and $K_G(h)(t,s)$ such that either 3.2.10 or 3.2.12 is applicable.

Proof. By the product rule 3.2.10, or by 3.2.12, and Lemma 6.4.3, we have

$$
\begin{aligned}
&D_{r,y}\big[D_{s,z}\{K_G(h)(t,s)\}\sigma(s)\big] = D_{s,z}\{D_{r,y}\{K_G(h)(t,s)\}\}\sigma(s)\\
&\quad + D_{s,z}\{K_G(h)(t,s)\}D_{r,y}\{\sigma(s)\} + D_{s,z}\{D_{r,y}\{K_G(h)(t,s)\}\}D_{r,y}\{\sigma(s)\}.
\end{aligned}
$$

Now, by the commutation of the Malliavin derivative and the Lebesgue-Stieltjes integral (see 3.2.18) we have, similarly to 6.4.4,

$$D_{r,y}\{K_G(h)(t,s)\} = K_G(D_{r,y}h)(t,s).$$

Hence,

$$
\begin{aligned}
&D_{s,z}\{D_{r,y}\{K_G(h)(t,s)\}\}\sigma(s) + D_{s,z}\{K_G(h)(t,s)\}D_{r,y}\{\sigma(s)\} \\
&\quad + D_{s,z}\{D_{r,y}\{K_G(h)(t,s)\}\}D_{r,y}\{\sigma(s)\} \\
&= D_{s,z}\{K_G(D_{r,y}\{h\})(t,s)\}\sigma(s) + D_{s,z}\{K_G(h)(t,s)\}D_{r,y}\{\sigma(s)\} \\
&\quad + D_{s,z}\{K_G(D_{r,y}\{h\})(t,s)\}D_{r,y}\{\sigma(s)\}.
\end{aligned}
$$

■

Even though the fundamental theorem of calculus takes on a more complicated form in the VMLV case, in principle, we get the same formula as for VMBV processes. The difference stems from the difference in their corresponding definitions.

Proposition 6.4.7 (The fundamental theorem of calculus for VMLV integrals with constant volatility)**.** *Let Y be a VMLV process with volatility $\sigma = 1$. Assume for all $r \in [0,t]$, $y \in \mathbb{R}$, that $D_{r,y}\{u(\cdot)\}$ is $\tilde{\mathfrak{L}}([0,t])$-integrable, and that the application of Lemma 6.4.6 is justified. Then $\int_0^t u(s)dY(s) \in \mathbb{D}_{1,2}$ and*

$$
\begin{aligned}
&D_{r,y}\Big(\int_0^t u(s)dY(s)\Big) \\
&= \int_0^t D_{r,y}u(s)dY(s) + y\big(K_G(u)(t,r) + D_{r,y}\{K_G(u)(t,r)\}\big)
\end{aligned}
\tag{6.15}
$$

Proof. Again, we first write out the case when σ is general and then set $\sigma = 1$ to get (6.15),

$$
\begin{aligned}
&D_{r,y}\Big(\int_0^t u(s)dY(s)\Big) \\
&= D_{r,y}\Big(\int_0^t\int_{\mathbb{R}_0} z\Big[K_G(u)(t,s) + D_{s,z}\{K_G(u)(t,s)\}\Big]\sigma(s)\tilde{N}(\delta z, ds) \\
&\qquad + \int_0^t\int_{\mathbb{R}_0} zD_{s,z}\{K_G(u)(t,s)\}\sigma(s)\nu(dz)ds\Big) \\
&= \int_0^t\int_{\mathbb{R}_0} D_{r,y}\Big[z\big(K_G(u)(t,s) + D_{s,z}\{K_G(u)(t,s)\}\big)\sigma(s)\Big]\tilde{N}(\delta z, ds) \\
&\qquad + yK_G(u)(t,r)\sigma(r) + yD_{r,y}\{yK_G(u)(t,r)\}\sigma(r) \\
&\qquad + \int_0^t\int_{\mathbb{R}_0} D_{r,y}\big[zD_{s,z}\{K_G(u)(t,s)\}\sigma(s)\big]\nu(dz)ds \\
&= \int_0^t\int_{\mathbb{R}_0} z\Big(\big[K_G(D_{r,y}u)(t,s) + D_{s,z}\{K_G(D_{r,y}u)(t,s)\}\big]\sigma(s) \\
&\quad + \big[K_G(u)(t,s) + D_{s,z}\{K_G(u)(t,s)\}\big]D_{r,y}\{\sigma(s)\} \\
&\quad + \big[K_G(D_{r,y}u)(t,s) + D_{s,z}\{K_G(D_{r,y}u)(t,s)\}\big]D_{r,y}\{\sigma(s)\}\Big)\tilde{N}(\delta z, ds) \\
&\quad + yK_G(u)(t,r)\sigma(r) + yD_{r,y}\{K_G(u)(t,r)\}\sigma(r) \\
&\quad + \int_0^t\int_{\mathbb{R}_0} z\Big(D_{s,z}\{K_G(D_{r,y}u)(t,s)\}\sigma(s) \\
&\qquad + D_{s,z}\{K_G(u)(t,s)\}D_{r,y}\{\sigma(s)\}
\end{aligned}
$$

$$+ D_{s,z}\{K_G(D_{r,y}u)(t,s)\}D_{r,y}\{\sigma(s)\}\Big)\nu(dz)ds$$

Where the second equality uses the fundamental theorem of calculus 3.2.16 and the third equality uses Lemma 6.4.6 and 3.2.10 or 3.2.12.

As we can see, we get a very ugly formula unless we assume $\sigma = 1$, then $D_{r,y}\sigma(s) = 0$ and we have

$$\begin{aligned}
&D_{r,y}\Big(\int_0^t u(s)dY(s)\Big) \\
&= \int_0^t \int_{\mathbb{R}_0} z\Big[K_G(D_{r,y}u)(t,s) + D_{s,z}\{K_G(D_{r,y}u)(t,s)\}\Big]\tilde{N}(\delta z, ds) \\
&\quad + yK_G(u)(t,r) + yD_{r,y}\{K_G(u)(t,r)\} \\
&\quad + \int_0^t \int_{\mathbb{R}_0} D_{s,z}\{K_G(D_{r,y}u)(t,s)\}\nu(dz)ds\Big) \\
&= \int_0^t D_{r,y}\{u(s)\}dY(s) + y\big(K_G(u)(t,r) + D_{r,y}\{K_G(u)(t,r)\}\big)
\end{aligned}$$

■

Further properties of ambit field integrals

In a similar fashion to the real-valued case, we can calculate the $L^2(P)$ norm of

$$\int_0^t Y(s)dX(s),$$

and imitate the duality formula. Imitating the fundamental theorem of calculus is a little more difficult, we will return to this discussion later. For the $L^2(P)$ norm we get

$$\begin{aligned}
&E\Big[\Big(\int_0^t u(s)dY(s)\Big)^2\Big] \\
&= E\Big[\Big(\int_0^t K_G(u)(t,s)\sigma(s)\delta\tilde{W}(s)) + \int_0^t D_s\{K_G(u)(t,s)\}\sigma(s)ds\Big)^2\Big] \\
&= E\Big[\Big(\int_0^t K_G(u)(t,s)\sigma(s)\delta B(s))\Big)^2 + \Big(\int_0^t D_s\{K_G(u)(t,s)\}\sigma(s)ds\Big)^2\Big],
\end{aligned}$$

where the last equality follows since $E[\int_0^t K_G(u)(t,s)\sigma(s)\delta\tilde{W}(s)] = 0$.

Recall the notation from section 6.3 and let $\mathcal{H}_4$ be another separable Hilbert space.

Lemma 6.4.8. *For $A \in L(\mathcal{H}_1, L(\mathcal{H}_1, \mathcal{H}_3)$ and $F \in L(\mathcal{H}_3, \mathcal{H}_4)$ we have*

$$Tr_{\mathcal{H}_1}(FA) = FTr_{\mathcal{H}_1}(A) \in \mathcal{H}_4$$

Proof. Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}_1$, then

$$((FA)(e_k))(e_k) = (F(A(e_k)))(e_k) = F((A(e_k))(e_k)),$$

since this is the only thing that makes sense. Hence,

$$Tr_{\mathcal{H}_1}(FA) = \sum_{k=1}^{\infty}((FA)(e_k))(e_k) = \sum_{k=1}^{\infty} F((A(e_k))(e_k)) = F\sum_{k=1}^{\infty}(A(e_k))(e_k)$$
$$= FTr_{\mathcal{H}_1}(A)$$

■

The "duality formula" for integrals with respect to ambit fields now follow.

Proposition 6.4.9. *Let u be $\mathcal{I}^X(0,t)$-integrable and let $F \in \mathbb{D}^{1,2}(L_2(\mathcal{H}_3, \mathcal{H}_4))$. Then,*

$$E\left[F\int_0^t u(s)dY(s)\right] = E\left[Tr_{\mathcal{H}_1}\int_0^t D_s\{FK_G(u)(t,s)\}\sigma(s)ds\right]$$

Proof.

$$E\left[F\int_0^t u(s)dY(s)\right]$$
$$= E\left[F\int_0^t K_G(u)(t,s)\sigma(s)\delta\tilde{W}(s) + FTr_{\mathcal{H}_1}\int_0^t D_s\{K_G(u)(t,s)\}\sigma(s)ds\right]$$
$$= E\left[Tr_{\mathcal{H}_1}\int_0^t D_s\{F\}K_G(u)(t,s)\sigma(s)ds + Tr_{\mathcal{H}_1}\int_0^t FD_s\{K_G(u)(t,s)\}\sigma(s)ds\right]$$
$$= E\left[Tr_{\mathcal{H}_1}\int_0^t D_s\{FK_G(u)(t,s)\}\sigma(s)ds\right],$$

where the second equality follows by the duality formula 4.5.9 and the third equality follows by the product rule 4.5.5 and Lemma 6.4.8. ■

The "fundamental theorem of calculus" for ambit field integrals might also be possible to prove, the start off such a proof would be

$$D_r\int_0^t u(s)dY(s) = D_r\Big(\int_0^t u(s)\delta B(s) + Tr_{\mathcal{H}_1}\int_0^t D_s u(s)ds\Big).$$

On the first term, one can apply 4.5.8, but the second term is trickier. To get the same shape as in 6.4.5 one would need to commute the Malliavin derivative and the operator $Tr_{\mathcal{H}_1}$, and secondly D_r and D_s would need to be switched around. Neither of these issues has, to my eyes, obvious solutions. Furthermore, we cannot commute the Malliavin derivative and the Lebesgue-Stieltjes integral in the same way as in the real-valued case, this might be provable as well, but without such a result, one would have to assume that the kernel function G is Fréchet differentiable. The product rule issues we observed in the real-valued case is not an issue here, however, since we have a general product rule.

CHAPTER 7

SDEs and SPDEs driven by VMLV processes and ambit fields

In the last chapter we introduced LSS processes and ambit fields, and integrals with respect to them. This chapter will look at SDEs and SPDEs driven by these processes, that is, in the real-valued case, equations of the form

$$X(t) = X_0 + \int_0^t b(s, X_s)ds + \int_0^t c(s, X_s))dY(s), \tag{7.1}$$

and in the Hilbert-valued case, equations of the form

$$X(t) = X_0 + \int_0^t AX(s)ds + \int_0^t b(s, X_s)ds + \int_0^t c(s, X_s)dY(s).$$

We will concentrate on finding solutions to these equations, and in what sense they are unique, but a couple results on the Malliavin differentiability of the solutions will also be presented. These solutions will be non-adapted whenever the noise coefficient is time dependent. This is caused by the operator K_G since, even if $c : [0, T] \times \Omega \to \mathbb{R}$ is predictable, we have

$$K_G(c)(t, s) = c(s)G(t, s) + \int_s^t (c(u) - c(s))G(du, s)$$

where we can see that the Lebesgue-Stieltjes integral integrates c up to t which is bigger than s, and which implies that the stochastic integral in (7.1) cannot be adapted. Hence, the solution itself can not be adapted either.

In both the real-valued case and the Hilbert-valued case, we will consider linear and non-linear coefficients, and we let the initial condition X_0 be a random variable in $L^2(P)$ with values in either $\mathbb{R}$ or in the appropriate Hilbert space. The framework in the real valued case is the usual complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, P)$, but note that $\mathcal{F}_t$ is either generated by the Brownian motion $\{B(s)\}_{0 \le s \le t}$, or by the Lévy process $\{L(s)\}_{0 \le s \le t}$ depending on whether we are considering the case of a VMBV process or the case of a VMLV process. The framework in the Hilbert-valued case will be specified when we get there.

Before we begin our study, we remark that if Y is an LSS process and satisfies the assumptions of Proposition 6.1.6, so that Y is a semimartingale, then there already exists theory on the existence and uniqueness of solutions to (7.1), see e.g. [Pro10].

7.1 Linear equations

In this section we will study the case when b is linear and σ is constant, then (7.1) takes the form

$$X(t) = X_0 + \int_0^t (aX(s) + c)ds + \int_0^t \beta dY(s). \tag{7.2}$$

We will also express the solution, in differential form, through the use of Itô's formula.

The following theorem is quite similar to Proposition 25 in [BBV18], the difference is that we consider $G(t,s) = g(t-s)$ and that we include an extra drift term. We have also imposed slightly different assumptions.

Theorem 7.1.1 (Linear b, constant σ)**.** *Let $b(x) = ax + c$ and $\sigma(x) = \beta$, where $a, c, \beta \in \mathbb{R}$. Assume*

$$\int_0^t \int_0^s g^2(s-u)\sigma(u)^2 duds < \infty. \tag{7.3}$$

$$\int_0^t \sigma^2(u)du < \infty. \tag{7.4}$$

Also assume that there exist a nonnegative function $\theta \in L^2([0,T])$ such that

$$|v_g|(t) \leq \int_0^t \theta(s)ds, \quad 0 \leq t \leq T \tag{7.5}$$

Then the equation (7.2) *will have an explicit solution of the form:*

$$X(t) = e^{at}\left(X_0 + c\int_0^t e^{-as}ds + \beta \int_0^t e^{-as}dY(s)\right). \tag{7.6}$$

Proof. By inserting the proposed solution above in to the second term of (7.2) we get

$$a\int_0^t X_s ds = a\bigg(X_0 \int_0^t e^{as}ds + c\int_0^t e^{as}\int_0^s e^{-au}duds + \beta\int_0^t e^{as}\int_0^s e^{-au}dY(u)ds\bigg)$$

calculating term wise, we have for the first term

$$aX_0\int_0^t e^{as}ds = X_0(e^{at}-1),$$

the second term

$$\begin{aligned} ac\int_0^t e^{as}\int_0^s e^{-au}du\,ds &= ac\int_0^t e^{as}(-\frac{1}{a}(e^{-as}-1))ds \\ = c\int_0^t (e^{as}-1)ds &= c\bigg(e^{at}\int_0^t e^{-as}ds - \int_0^t ds\bigg), \end{aligned}$$

and the third term

$$
\begin{aligned}
a\beta \int_0^t e^{as} \int_0^s e^{-au} dY(u)ds &= a\beta \int_0^t \int_u^t e^{as} e^{-au} ds\, dY(u) \\
= a\beta \int_0^t \frac{1}{a}[e^{at} - e^{au}]e^{-au} dY(u) &= e^{at}\beta \int_0^t e^{-au} dY(u) - \beta \int_0^t dY(u).
\end{aligned}
$$

where we used the stochastic Fubini theorem 2.3.5 in the second equality. Adding the terms together gives

$$
\begin{aligned}
X(t) &= X_0 + \int_0^t (aX(s) + c)ds + \int_0^t \beta dY(s) \\
&= e^{at}\left(X_0 + c\int_0^t e^{-as} ds + \beta \int_0^t e^{-as} dY(s)\right)
\end{aligned}
$$

■

We now justify the application of the stochastic Fubini theorem.

Lemma 7.1.2. *Let $f(t) := e^{-at}$. Under the conditions of 7.1.1 we have*

$$
\int_0^t \int_0^s e^{2as} K_g^2(f)(s,u)\sigma^2(u) du ds < \infty.
$$

This justifies the use of the stochastic Fubini theorem 2.3.5

Proof. Since $K_g(f)(t,s)$ is deterministic $D_{s,z}\{K_g(f)(t,s)\} = 0$, hence it follows that

$$
\int_0^t e^{as} \int_0^s e^{-au} dY(u) ds = \int_0^t e^{as} \int_0^s K_g(f)(s,u)\sigma(u) dL(u) ds.
$$

Since $K_g(f)(s,u) = f(u)g(s-u) + \int_u^s (f(v) - f(u))g(dv)$, we can check each term separately using the inequality 2.3.1. The first term gives

$$
\int_0^t \int_0^s e^{2as} e^{-2au} g^2(s-u)\sigma^2(u) du ds \leq e^{2at} \int_0^t \int_0^s g^2(s-u)\sigma^2(u) du ds < \infty.
$$

By using Cauchy-Schwarz and (7.5) we get for the second term

$$
\begin{aligned}
&\int_0^t \int_u^t \left(e^{as} \int_u^s [e^{-av} - e^{-au}] g(dv)\right)^2 \sigma^2(u) ds du \\
&\leq \int_0^t \int_u^t \left(e^{as} \int_u^s |e^{-av} - e^{-au}| v_g(dv)\right)^2 \sigma^2(u) ds du \\
&\leq e^{2at} \int_0^t \int_u^t \left(\int_u^s |e^{-av} - e^{-au}| \theta(v) dv\right)^2 ds \sigma^2(u) du \\
&\leq e^{2at} \int_0^t \int_u^t \Big(\int_u^s |e^{-av} - e^{-au}|^2 dv \int_u^s \theta^2(v) dv\Big) ds \sigma^2(u) du \\
&\leq e^{2at} \int_0^t \int_u^t \Big(\int_u^s 1^2 dv \int_0^t \theta^2(v) dv\Big) ds \sigma^2(u) du \\
&\leq e^{2at} t \int_0^t \sigma^2(u) du \int_0^t \theta^2(v) dv
\end{aligned}
$$

which is finite by assumption. ■

Furthermore, we can express this solution using Itô's formula.

Proposition 7.1.3. *Assume that $g(0)$ is defined, that g is differentiable, and that*

$$\int_0^t \int_u^t (g'(s-u))^2 ds du < \infty \tag{7.7}$$

Then we can apply Itô's formula to get the following representation of (7.6)

$$dX(t) = ae^{at} Z(t)dt + e^{at}(ce^{-at} + \beta V(t))dt + \beta \int_{\mathbb{R}} ze^{at} h(t,t) \tilde{N}(dt, dz),$$

where

$$Z(t) = X_0 + c\int_0^t e^{-au} du + \beta \int_0^t e^{-au} dY(u),$$

$V(t) = \int_0^t \frac{\partial}{\partial t} h(t,u) dL(u)$, *and*

$$h(t,u) := K_g(f)(t,u) = \Big(f(u)g(t-u) + \int_u^t [f(v) - f(u)] g(dv) \Big).$$

Proof. Firstly, since g is differentiable we have that $h(t,u)$ is differentiable in the first variable and we can write $h(t,u) = h(u,u) + \int_u^t \frac{\partial}{\partial s} h(s,u) ds$, where

$$\begin{aligned}
&\frac{\partial}{\partial s} h(s,u) = \frac{\partial}{\partial s} K_g(f)(s,u) \\
&= \frac{\partial}{\partial s}\Big(f(u)g(s-u) + \int_u^s (f(v) - f(u)) g(dv) \Big) \\
&= \frac{\partial}{\partial s}\Big(f(u)g(s-u) + \int_u^s (f(v) - f(u)) g'(v) dv \Big) \\
&= f(u)g'(s-u) - (f(s) - f(u)) g'(s).
\end{aligned}$$

Then, since f is bounded by 1 we get directly that

$$\int_0^t \int_u^t \Big(\frac{\partial}{\partial s} h(s,u) \Big)^2 ds du < \infty,$$

hence we can apply Theorem 2.3.5, which gives

$$\begin{aligned}
\int_0^t h(t,u) dL(u) &= \int_0^t h(u,u) dL(u) + \int_0^t \int_u^t \frac{\partial}{\partial s} h(s,u) ds dL(u) \\
&= \int_0^t h(u,u) dL(u) + \int_0^t \int_0^s \frac{\partial}{\partial s} h(s,u) dL(u) ds.
\end{aligned}$$

Now, let $V(s) = \int_0^s \frac{\partial}{\partial s} h(s,u) dL(u)$ and $Z(t) = X_0 + c\int_0^t e^{-au} du + \int_0^t e^{-au} dY(u)$. We then get the expression

$$
\begin{aligned}
\int_0^t e^{-au}dY(u) &= \int_0^t \int_{\mathbb{R}} z\big\{K_g(f)(t,u) + D_{u,z}\{K_g(f)(t,u)\}\big\}\sigma(u)\tilde{N}(\delta z, \delta u) \\
&\quad + \int_0^t \int_{\mathbb{R}} zD_{u,z}\{K_g(f)(t,u)\}\sigma(u)\nu(dz)ds \\
&= \int_0^t \int_{\mathbb{R}} zK_g(f)(t,u)\sigma(u)\tilde{N}(dz,du) \\
&= \int_0^t h(u,u)dL(u) + \int_0^t V(s)ds,
\end{aligned}
$$

where we recall that $K_g(f)(t,u)$ is deterministic, hence predictable and with $D_{u,z}\{K_g(f)(t,u)\} = 0$. With the preceding calculations we can express $Z(t)$ as a Lévy-Itô process

$$
\begin{aligned}
Z(t) &= X_0 + c\int_0^t e^{-au}du + \beta\int_0^t e^{-au}dY(u) \\
&= X_0 + \int_0^t (ce^{-au} + \beta V(u))du + \beta\int_0^t zh(u,u)\tilde{N}(du,dz),
\end{aligned}
$$

where we have just changed the variable of V. Finally we can apply Itô's formula, see 2.2.13. Let $X(t) = F(t, Z(t)) = e^{at}Z(t)$, then

$$
\begin{aligned}
dX(t) &= \frac{\partial F}{\partial t}(t,Z(t))dt + \frac{\partial F}{\partial z}(t,Z(t))\alpha(t)dt + \frac{\partial F}{\partial z}(t,Z(t))\zeta(t)dB(t) \\
&\quad + \frac{1}{2}\frac{\partial^2 F}{\partial z^2}(t,Z(t))\zeta^2(t)dt + \int_{\mathbb{R}} \big[F(t,Z(t)+\gamma(t,z)) - F(t,Z(t)) \\
&\quad - \frac{\partial F}{\partial z}(t,Z(t))\gamma(t,z)\big]\nu(dz)dt \\
&\quad + \int_{\mathbb{R}} \big[F(t,Z(t^-)+\gamma(t,z)) - F(t,Z(t^-))\big]\tilde{N}(dt,dz) \\
&= \frac{\partial F}{\partial t}(t,Z(t))dt + \frac{\partial F}{\partial z}(t,Z(t))\alpha(t)dt \\
&\quad + \int_{\mathbb{R}} \big[F(t,\gamma(t,z)) - \frac{\partial F}{\partial z}(t,Z(t))\gamma(t,z)\big]\nu(dz)dt + \int_{\mathbb{R}} F(t,\gamma(t,z))\tilde{N}(dt,dz) \\
&= ae^{at}Z(t)dt + e^{at}(ce^{-at} + \beta V(t))dt + \beta\int_{\mathbb{R}} ze^{at}h(t,t)\tilde{N}(dt,dz).
\end{aligned}
$$

■

From 7.1.3 we get the following corollary.

Corollary 7.1.4. *Let Y be a BSS process. Assume the assumption of 7.1.3, then* (7.6) *admits the representation*

$$
ae^{at}Z(t)dt + e^{at}(ce^{-at} + \beta V(t))dt + \beta e^{at}h(t,t)dB(t)
$$

Proof. In the proof of 7.1.3 let $L = B$ be a Brownian motion, then we have

$$\begin{aligned} Z(t) &= X_0 + c\int_0^t e^{-as}ds + \beta\int_0^t e^{-as}dY(s) \\ &= X_0 + \int_0^t (ce^{-as} + \beta V(t))dt + \int_0^t h(s,s)dB(s) \end{aligned}$$

and by Itô's formula

$$\begin{aligned} dX(t) = &\frac{\partial F}{\partial t}(t, Z(t))dt + \frac{\partial F}{\partial z}(t, Z(t))\alpha(t)dt + \frac{\partial F}{\partial z}(t, Z(t))\zeta(t)dB(t) \\ &+ \frac{1}{2}\frac{\partial^2 F}{\partial z^2}(t, Z(t))\zeta^2(t)dt \\ = &ae^{at}Z(t)dt + e^{at}(ce^{-at} + \beta V(t))dt + \beta e^{at}h(t,t)dB(t) \end{aligned}$$

■

7.2 Equations with nonlinear drift

This section will deal with equations of the type

$$X(t) = X_0 + \int_0^t b(t, X_s)dt + \int_0^t c(s)dY(s) \tag{7.8}$$

where b satisfies the usual conditions of linear growth and Lipschitzianity, and c is a stochastic process whose properties will be defined later on. Y can be a VMLV or a VMBV process. Showing the existence and uniqueness of solutions to these SDEs can be done in several ways, depending on what we assume on c, and we will draw upon the results in chapter 5.

First we must define what we mean by "solution". Since we will use the same concept of solution in the next section about nonlinear SDEs, we define the solution for the following more general equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t c(s, X_s)dY(s). \tag{7.9}$$

Definition 7.2.1. A process $X = \{X_t\}_{t\in[0,T]}$ is called a solution to 7.9 if

i) $X \in L^2(P)$,

ii) $c(\cdot, X_\cdot)$ is $\mathfrak{L}([0,T])$-integrable,

iii) X satisfies the equation.

The assumptions on b and the initial condition X_0 will be the same throughout this section.

Assumption 7.2.2. Assume that $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the following

(i) $|b(t,x)| \leq C(1 + |x|)$ for all $t \in [0,T]$, $x \in \mathbb{R}$ and some $C > 0$,

(ii) $|b(t,x) - b(t,y)| \leq D|x - y|$ for all $t \in [0,T]$, $x, y \in \mathbb{R}$ and some $D > 0$.

Assumption 7.2.3. Assume that the initial condition X_0 is a $\mathcal{F}_0$-measurable random variable that satisfies

$$E[X_0^2] < \infty.$$

The first case we will look at is the most general. It uses the proof of 5.1.2 and here the assumptions on σ are limited.

Theorem 7.2.4 (Existence and uniqueness via Picard iteration)**.** *Consider the equation defined in* (7.8)*. Let Y be a VMLV process. Let b and X_0 be as in 7.2.2 and 7.2.3 respectively and assume the stochastic process $c = \{c_t\}_{t\in[0,T]}$ satisfies (1)-(3) in 6.2.1 for all $t \in [0,T]$, that is, c is $\tilde{\mathfrak{L}}([0,t])$-integrable for all $t \in [0,T]$. Furthermore, assume that*

$$E\left[\left|\int_0^t c(s)dY(s)\right|^2\right] \leq S < \infty, \quad t \in [0,T]$$

Then there exists a solution, unique up to modification, of (7.8)*.*

Proof. Firstly we have assumed that the integral

$$\int_0^t c(s)dY(s)$$

is defined. Now define $Z_t^0 = X_0$ and Z^k inductively in the Picard iteration way, that is

$$Z^{k+1} = X_0 + \int_0^t b(s, Z_s^k)ds + \int_0^t c(s)dY(s).$$

The goal is to show that $\{Z_t^k\}_{n=0}^\infty$ is a Cauchy sequence in $L^2(P \times dt)$ via induction. For $k = 0$ this means that we have to find an upper bound on $E[|Z_t^1 - Z_t^0|^2]$. We have

$$\begin{aligned} E[|Z_t^1 - Z_t^0|^2] &= E\left[\left|\int_0^t b(s, Z_s^0)ds + \int_0^t c(s)dY(s)\right|^2\right] \\ &\leq 2E\left[\left|\int_0^t b(s, Z_s^0)ds\right|^2\right] + 2E\left[\left|\int_0^t c(s)dY(s)\right|^2\right]. \end{aligned}$$

The first term is standard

$$\begin{aligned} E\left[\left|\int_0^t b(s, Z_s^0)ds\right|^2\right] &\leq tE\left[\int_0^t |b(s, Z_s^0)|^2 ds\right] \\ &\leq TE\left[\int_0^t (1 + |X_0|^2)ds\right] \leq T^2(1 + E[|X_0|^2]) < \infty. \end{aligned}$$

And by assumption we have

$$E\left[\left|\int_0^t c(s)dY(s)\right|^2\right] \leq S < \infty.$$

Set $C := 2(S + T(1 + E[|X_0|^2]))$. Assume now, for $k \geq 1$, that

$$E[|Z_t^k - Z_t^{k-1}|^2] \leq C\frac{T^{k-1}D^{k-1}}{(k-1)!}t^{k-1}.$$

By induction, we have

$$
\begin{aligned}
&E[|Z_t^{k+1} - Z_t^k|^2] = E\left[\Big|\int_0^t b(s, Z_s^k) - b(s, Z_s^{k-1}) ds\Big|^2\right] \\
&\leq tE\left[\int_0^t |b(s, Z_s^k) - b(s, Z_s^{k-1})|^2 ds\right] \leq TD\int_0^t E[|Z_s^k - Z^{k-1}|^2] ds \\
&\leq TD\int_0^t S\frac{T^{k-1}D^{k-1}}{(k-1)!}s^{k-1} ds = C\frac{T^k D^k}{k!}t^k.
\end{aligned}
$$

Furthermore, $\{Z^k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2(P \times dt)$,

$$
\begin{aligned}
&\|Z_t^n - Z_t^m\|_{L^2(P\times dt)} = \Big\|\sum_{k=m}^{n-1} Z_t^{k+1} - Z_t^k\Big\|_{L^2(P\times dt)} \\
&\leq \sum_{k=m}^{n-1} \|Z_t^{k+1} - Z_t^k\|_{L^2(P\times dt)} \leq \sum_{k=m}^{n-1}\left(E\left[\int_0^T |Z_t^{k+1} - Z_t^k|^2 dt\right]\right)^{1/2} \\
&\leq \sum_{k=m}^{n-1}\left(\int_0^T C\frac{T^k D^k}{k!}t^k dt\right)^{1/2} = \sum_{k=m}^{n-1}\left(\frac{T^k D^k}{(k+1)!}T^{k+1}\right)^{1/2} \to 0, \quad n, m \to \infty.
\end{aligned}
$$

Therefore $\{Z_t^k\}_{n=0}^{\infty}$ is a Cauchy sequence in the complete space $L^2(P \times dt)$. We define the limit in $L^2(P \times dt)$ to be $X_t = \lim_{k\to\infty} Z_t^k$. Lastly, X_t satisfies (7.8) since

$$
E\left[\Big(\int_0^t (b(s, X_s) - b(s, Z_s^k)) ds\Big)^2\right] \leq TE\left[\int_0^t (X_s - Z_s^k)^2 ds\right] \to 0, \quad k \to \infty.
$$

For the uniqueness, assume that U and V are two solutions with the same initial condition X_0, then, as above, we get

$$
\alpha(t) := E\big[|U_t - V_t|^2\big] \leq TD\int_0^t E\big[|U_t - V_t|^2\big] dt.
$$

Hence, by Grönwall's inequality 5.1.1, we conclude that $\alpha(t) = 0$ for all $t \geq 0$. This implies that U and V are modifications of each other and therefore that we have uniqueness up to modification.

■

Remark 7.2.5. The last theorem proves that the Picard iteration forms a sequence in $L^2(P \times dt)$, but we actually have an even stronger sense of convergence since by the assumptions of 7.2.4 we get

$$
\begin{aligned}
&E\big[\sup_{0\leq s\leq t} |Z_s^{k+1} - Z_s^k|^2\big] \leq E\left[\sup_{0\leq s\leq t}\Big(s\int_0^s |b(u, Z_u^{k+1}) - b(u, Z_u^k)|^2 du\Big)\right] \\
&\leq tE\left[\int_0^t |b(u, Z_u^{k+1}) - b(u, Z_u^k)|^2 du\Big)\right]
\end{aligned}
$$

which is on the same form as in the proof of 7.2.4. In addition

$$
E\big[\sup_{0\leq s\leq t} |Z_s^1 - Z_s^0|^2\big]
$$

$$\leq 2E\Big[\sup_{0\leq s\leq t}\Big|\int_0^s b(u,X_0)du\Big|^2\Big] + 2E\Big[\sup_{0\leq s\leq t}\Big|\int_0^s c(u)dY(u)\Big|^2\Big]$$
$$\leq 2tE\Big[\int_0^t |b(u,X_0)|^2du\Big] + 2E\Big[\sup_{0\leq s\leq t}\Big|\int_0^s c(u)dY(u)\Big|^2\Big]$$

where the first term is again in the same form as in the proof above and the last term is bounded by assumption. This means that we get

$$E\big[\sup_{0\leq s\leq T}|X_s - Z_s^k|\big] \to 0, \quad n\to\infty.$$

The purpose of this remark is to show that we can employ Theorem 5.1.11 later on to prove that the solution to 7.2.4 is Malliavin differentiable.

Theorem 7.2.4 gives rise to a series of corollaries that follow, more or less, immediately by the above theorem.

Corollary 7.2.6. *Let Y in* (7.8) *be an VMBV process and assume that c is $\mathfrak{L}([0,t])$-integrable for all $t\in[0,T]$. On b and X_0 assume 7.2.2 and 7.2.3. Then* (7.8) *has a solution which is unique up to modification.*

Proof. Follows by the same procedure as for the proof of theorem 7.2.4. ∎

Corollary 7.2.7. *Let Y in* (7.8) *be an LSS process or an BSS process and assume Assumption 7.2.2 on b and Assumption 7.2.3 on X_0. Then* (7.8) *has a solution which is unique up to modification.*

Proof. Follows immediately by (7.2.4) in the LSS case, and by (7.2.6) in the BSS case. ∎

If we want adapted and predictable solutions, we will have to restrict ourselves to the case where c is a random variable. Then we get $\int_0^t cdY(s) = cY(t)$, both when Y is an VMLV process and when Y is an VMBV process, see the integration by parts formulas 6.2.7 and 6.2.6 respectively.

Corollary 7.2.8. *Let Y in* (7.8) *be an VMLV process or an VMBV process with predictable or adapted σ respectively, and let c be a random variable. Also assume Assumption 7.2.2 on b and 7.2.3 on X_0. Then* (7.8) *has a solution which is unique up to modification. In this case, the solution will also be adapted.*

Proof. Existence and uniqueness follow immediately by Theorem 7.2.4 and Theorem 7.2.6 respectively. The adaptedness is a result of the adaptedness of each Z^k, which again follows from the fact that $cY(t)$ is adapted by Theorem 63 in [Pro10]. ∎

Obviously, Corollary 7.2.8 holds also in the special case where the VMLV process is a LSS process and the VMBV process is a BSS process.

Corollary 7.2.9. *Let Y in* (7.8) *be an LSS process or an BSS process with predictable or adapted σ respectively, and let c be a random variable. Also assume Assumption 7.2.2 on b and 7.2.3 on X_0. Then* (7.8) *has a solution which is unique up to modification. In this case, the solution will also be adapted.*

Proof. Follows immediately by Corollary 7.2.8. ∎

Banach fixed point

We can also use the Banach fixed point theorem to prove existence and uniqueness for certain choices of c. The Theorem 5.1.5 shows uniqueness in a space of predictable processes. Hence, the process $c = \{c_t\}_{t\in[0,T]}$ cannot be as general as in Theorem 7.2.4 if we want predictable solutions. In this case, we will actually be forced to limit ourselves to the case where c is a random variable and impose certain restrictions on Y depending of whether it is a VMLV or a VMBV process. However, if we consider the spaces in [BK81] and remove the requirement of predictability, we can allow for a more general c.

Recall the notation from chapter 5. Throughout this subsection we assume that $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies

$$|b(t,x) - b(t,y)| \leq \beta(t)|x-y|$$

where $\beta \in L^2_{loc}(\mathbb{R}_+)$, and that

$$\sup_{t\geq 0} E\left[\Big(\int_0^t c(s)dY(s)\Big)^2\right] < \infty. \tag{7.10}$$

Note that, unlike in other parts of this chapter, we do not require that t is in some compact interval $[0,T]$. Furthermore, we define $S : H^2_{loc} \to H^2_{loc}$ by

$$SX(t) = X_0 + \int_0^t b(s, X(s))ds + \int_0^t c(s)dY(s)$$

we then get

$$E[|SX_t - SY_t|^2] = E\left[\left|\int_0^t \big(b(s,X_s) - b(s,Y_s)\big)ds\right|^2\right]. \tag{7.11}$$

This allows us to apply Theorem 5.1.5.

We start off with considering the case where we can directly employ the notation of [BK81], see section 5.1.

Theorem 7.2.10 (Existence and uniqueness via Banach's fixed point theorem)**.** *Let c in* (7.8) *be a random variable and let Y be a LSS process with predictable σ, such that Y has a continuous modification. That is, we assume the conditions of 6.1.10. Then there exists a unique solution of* (7.8)*, up to modification, in the space H^2_{loc}.*

Proof. Since the integrand in $Y(t) = \int_0^t g(t-s)\sigma(s)dL(s)$ is predictable, the integral is adapted by Theorem 63 in [Pro10], and by assumption Y is continuous, hence Y is predictable. The integral is also contained in H^2_{loc} by 7.10. Using 7.11 we can conclude by the proof of 5.1.5. ■

If we assume that Y is a VMBV process we can skip the continuity assumption.

Theorem 7.2.11. *Let c in* (7.8) *be a random variable and let Y be a VMBV process with predictable σ. Then there exist a unique solution, up to modification, in the space H^2_{loc}.*

Proof. Since the integrand in $Y(t) = \int_0^t G(t,s)\sigma(s)dB(s)$ is predictable, the integral is predictable. The integral is also contained in H^2_{loc} by 7.10. Using 7.11 we can conclude by the proof of 5.1.5. ■

Of course, we also get the same corollary as before.

Corollary 7.2.12. *Let c in* (7.8) *be a random variable and let Y be a BSS process with predictable σ. Then there exist a unique solution, up to modification, in the space H^2_{loc}.*

Proof. Follows immediately by Theorem 7.2.11. ■

As mentioned, we can generalize σ and Y if we allow non-predictable solutions. In this case, we must modify the situation of [BS16] in section 5.1 a little bit. We denote by $M^2 = M^2(\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}, dt \times P)$ and G^2 the spaces of processes satisfying

$$\left(E\Big[\int_0^\infty |X_t|^2 dt\Big]\right)^{1/2} < \infty$$

$$\left(\sup_{t\geq 0} E\big[|X_t|^2\big]\right)^{1/2} < \infty$$

respectively. Furthermore, F^2_{loc} and G^2_{loc} denotes the spaces of processes such that there is an increasing sequence of $\{\mathcal{F}_t\}_{t\geq 0}$-stopping times $\{T_n\}_{n\in\mathbb{N}} \to \infty$ P-a.s. such that $X\chi_{[0,T_n]}$ belongs to F^2 and G^2 respectively, for all $n \in \mathbb{N}$. Note that in [BK81] they use $X^{T_n^*} := X^{T_n}\chi_{\{T_n \geq 0\}}$ instead of $X\chi_{[0,T_n]}$ in the definition of H^2_{loc}, this is done to allow for a more general initial condition (see remark 3 in [BK81]), we do not consider such general initial conditions and have therefore left it out.

With the above considerations we can look at a more general solution to (7.8).

Theorem 7.2.13. *Let Y be a VMLV process and let c be a, not necessarily predictable, stochastic process that is $\tilde{\mathfrak{L}}([0,t])$-integrable for all $t \in [0,T]$ such that* (7.10) *holds. Then there exists a unique solution, up to modification, of* (7.8) *in G^2_{loc}.*

Proof. Applying (7.11), the result follows word for word by 5.1.5 if we consider the spaces M^2, M^2_{loc}, G^2, and G^2_{loc} instead of the spaces L^2, L^2_{loc}, H^2, and H^2_{loc} respectively. ■

And the following obvious corollary follows.

Corollary 7.2.14. *Let c be a, not necessarily predictable, stochastic process such that* (7.10) *holds, and let Y be a LSS/VMBV/BSS process. Then there exists a unique solution, up to modification, of* (7.8) *in G^2_{loc}.*

Proof. Follows immediately by 7.2.13. ■

Existence and uniqueness through continuous modifications

The last approach for proving existence and uniqueness only works for LSS processes and BSS processes as it employs their respective continuous modifications found in Chapter 6. The proof relies on Theorem 5.1.12.

Theorem 7.2.15 (Existence and uniqueness through continuous modification)**.** *Assume b satisfies 7.2.2, c is a random variable, and let Y be a BSS process or an LSS processes where the kernel function g and the volatility σ satisfies the assumptions of 6.1.9 and 6.1.10 respectively. Then* (7.8) *has a unique solution up to indistinguishability.*

Proof. This follows directly by 5.1.12, since cY with the assumption in the theorem has continuous sample paths for almost all $\omega \in \Omega$. ∎

There are probably various ways in which we could alter the assumptions of the results in this section, recall some of the discussion in chapter 5.

The solutions found to (7.8) are Malliavin differentiable under suitable assumptions. We only state the result for the most general result, that is, for Theorem 7.2.4.

Malliavin differentiability of solutions

Theorem 7.2.16. *Consider the SDE* (7.8) *and let Y be a VMLV process. Assume the conditions of Assumption 7.2.2 and Assumption 7.2.3 on b and X_0. Also assume that c is $\tilde{\mathfrak{L}}([0,T])$ integrable, and that for all $r \in [0,T]$, $y \in \mathbb{R}$, $D_{r,y}\{c(\cdot)\}$ is $\tilde{\mathfrak{L}}([0,T])$-integrable. Further, we assume that*

$$E\left[|D_{r,y}X_0|^2\right] \le C_1 < \infty$$
$$E\left[\left(D_{r,y}\int_0^s c(u)dY(u)\right)^2\right] \le C_2 < \infty$$

for all $r,s \in [0,T]$, $y \in \mathbb{R}$ and that

$$\frac{\partial b(t,x)}{\partial x} \le M, \quad t \in [0,T], x \in \mathbb{R}.$$

Then the solution X to (7.8) *found in Theorem* (7.2.4) *is Malliavin differentiable. If we also assume that $\sigma = 1$, then $D_{r,y}X_t$ satisfies*

$$\begin{aligned} D_{r,y}X_t = D_{r,y}X_0 &+ \int_0^t \frac{\partial b(s,X_s)}{\partial x} D_{r,y}X_s ds + \int_0^T D_{r,y}c(s)dY(s) \\ &+ y\big(K_G(c)(t,r) + D_{r,y}\{K_G(c)(t,r)\}\big). \end{aligned}$$

Proof. Taking the Malliavin derivative of Z^{k+1} in the Picard iteration of the proof of 7.2.4 we have

$$\begin{aligned} D_{r,y}Z^{k+1}(t) &= D_{r,y}\Big(X_0 + \int_0^t b(t,Z_s^k)dt + \int_0^t c(s)dY(s)\Big) \\ &= D_{r,y}X_0 + \int_0^t D_{r,y}b(s,Z_s^k)ds + D_{r,y}\Big(\int_0^t c(s)dY(s)\Big) \end{aligned}$$

By assumption, we have that X_0 is Malliavin differentiable, and by Theorem 6.4.7 we have that $\int_0^t c(s)dY(s)$ is Malliavin differentiable. Hence we get,

$$
\begin{aligned}
E\big[\sup_{0\le s\le t}(D_{r,y}Z_s^{k+1})^2\big] &\le 3E\big[(D_{r,y}X_0)^2\big] + 3E\left[\sup_{0\le s\le t}\Big(\int_0^s D_{r,y}b(s,Z_u^k)du\Big)^2\right] \\
&\quad + 3E\left[\sup_{0\le s\le t}\Big(D_{r,y}\int_0^s c(u)dY(u)\Big)^2\right] \\
&\le 3E\left[\sup_{0\le s\le t} s\int_0^s \Big(\frac{\partial b(u,Z_u^k)}{\partial z}D_{r,y}Z_u^k du\Big)^2\right] + C_1 + C_2 \\
&\le 3TM^2E\left[\int_0^t \big(D_{r,y}Z_u^k\big)^2 du\right] + C_3,
\end{aligned}
$$

where $C_3 := C_1 + C_2$. This is the form of (5.8) in Theorem 5.1.11, taking remark 7.2.5 into account we can conclude that $X_t \in \mathbb{D}_{1,2}$ for all $t \in [0,T]$ by the proof of 5.1.11. Finally, the equality follows by Proposition 6.4.7

$$
\begin{aligned}
D_{r,y}X_t &= D_{r,y}\Big(X_0 + \int_0^t b(t,X_s)dt + \int_0^t c(s)dY(s)\Big) \\
&= D_{r,y}X_0 + \int_0^t \frac{\partial b(s,X_s)}{\partial x}D_{r,y}X_s ds \\
&+ \int_0^T D_{r,y}c(s)dY(s) + y\big(K_G(c)(t,r) + D_{r,y}\{K_G(c)(t,r)\}\big)
\end{aligned}
$$

■

7.3 Nonlinear equations

This section we will study equations of the form

$$
X(t) = X_0 + \int_0^t b(s,X_s)ds + \int_0^t c(s,X_s)dY(s), \tag{7.12}
$$

and we will mainly focus on the case where Y is a BSS process or a LSS process. We will limit ourselves to the case where the volatility $\sigma = 1$. As one might imagine, this is a significantly more difficult task than the semilinear type of equations considered in the last section. The difficulty stems from the fact that we now have to deal with the Malliavin derivative and Skorohod integral of the function c, which is no longer just time-dependent but also dependent on the solution process X. We will see that heavy assumptions on c are needed for a solution to exist.

The proof of the main theorem of this section is very long, but we have tried to shorten is as much as possible with the aid of a couple lemmas. One of the issues one must deal with is showing that the term

$$
\int_0^t c(s,X_s)dY(s)
$$

is even defined, this is in itself quite a daunting task, and I have been unsuccessful in finding any "clever" solutions to this problem in the literature. The solution we consider is to simply assume that c is such that the term is defined. See, e.g., section 5 in [EPQ97] for the assumption that the noise coefficient is Malliavin differentiable, and sections 3 and 4 in [Buc92] for assumptions on the Skorohod integral of the noise coefficient.

We will use the following lemma numerous times.

Lemma 7.3.1.

$$\int_0^t \int_0^{t-s} f(u+s)g(du)ds = \int_0^t \int_u^t f(s)dsg(du)$$

Proof. By Fubini we have

$$\begin{aligned}
&\int_0^t \int_0^{t-s} f(u+s)g(du)ds = \int_0^t \int_0^t \chi_{[0,t-s]}(u)f(u+s)g(du)ds \\
&= \int_0^t \int_0^t \chi_{[0,t-u]}(s)f(u+s)dsg(du) = \int_0^t \int_0^{t-u} f(u+s)dsg(du) \\
&= \int_0^t \int_u^t f(s)dsg(du)
\end{aligned}$$

■

Furthermore, we will use the Skorohod isometry several times, recall its form

$$E\left[\left(\int_0^T u(s)\delta B(s)\right)^2\right] = E\left[\int_0^T u^2(t)dt + \int_0^T \int_0^T D_t u(s) D_s u(t) ds dt\right].$$

We will need to estimate the second term of this isometry. This estimate is also used several times, so we make it a lemma.

Lemma 7.3.2.

$$E\Big[\int_0^T \int_0^T D_t u(s) D_s u(t) dsdt\Big] \le E\Big[\int_0^T \int_0^T \big(D_t u(s)\big)^2 dsdt\Big]$$

Proof. This follows by repeated use of the Cauchy-Schwarz inequality.

$$\begin{aligned}
&E\Big[\int_0^T \int_0^T D_t u(s) D_s u(t) dsdt\Big] \le \Big|E\Big[\int_0^T \int_0^T D_t u(s) D_s u(t) dsdt\Big]\Big| \\
&\le E\Big[\Big|\int_0^T \int_0^T D_t u(s) D_s u(t) dsdt\Big|\Big] \le E\Big[\int_0^T \Big|\int_0^T D_t u(s) D_s u(t) ds\Big| dt\Big] \\
&\le E\Big[\int_0^T \Big(\int_0^T \big(D_t u(s)\big)^2 ds\Big)^{1/2} \Big(\int_0^T \big(D_s u(t)\big)^2 ds\Big)^{1/2} dt\Big] \\
&\le E\Big[\Big(\int_0^T \int_0^T \big(D_t u(s)\big)^2 dsdt\Big)^{1/2} \Big(\int_0^T \int_0^T \big(D_s u(t)\big)^2 dsdt\Big)^{1/2}\Big] \\
&= E\Big[\int_0^T \int_0^T \big(D_t u(s)\big)^2 dsdt\Big]
\end{aligned}$$

■

Note that the above lemma also holds in the pure jump Lévy setting, from 3.2.17 we can see that we must also integrate with respect to the Lévy measure in this case, but this just means that we would have to apply the Cauchy-Schwarz inequality two times more.

In the remaining part of this section we will need the following inequality for signed measures.

Lemma 7.3.3 ([Jun18])**.** *Let $a \in L^1(|\mu|)$, then*

$$\Big| \int_0^t a(s) d\mu(s) \Big| \leq \int_0^t |a(s)| d|\mu|(s),$$

where μ is a signed measure and $|\mu| := \mu^+ + \mu^-$ is the total variation measure of μ.

Specifically, the above lemma holds when μ is a Lebesgue-Stieltjes measure. We denote the total variation of $g(du)$ by $|v_g|(du)$.

Theorem 7.3.4 (Existence and uniqueness of nonlinear SDE driven by a BSS process)**.** *Assume Y is a BSS process with $\sigma = 1$. Let $T > 0$ and $b : [0,T] \times \mathbb{R} \to \mathbb{R}$, $c : [0,T] \times \mathbb{R} \to \mathbb{R}$ be measurable functions such that, for all $s, t \in [0,T]$, $x, y \in \mathbb{R}$,*

i) $|b(t,x)|^2 + |c(t,x)|^2 \leq C(1 + |x|^2), \quad C > 0.$

ii) $|b(t,x) - b(t,y)| + |c(t,x) - c(t,y)| \leq \Lambda |x - y|, \quad \Lambda > 0.$

iii) $|D_s(c(t,x)) - D_s(c(t,y))| \leq F|x - y|, \quad F > 0.$

iv) $c(t,x) \in \mathbb{D}_{1,2}$.

v) *c is differentiable in its second variable and*

$$\Big| \frac{\partial}{\partial x} c(t,x) \Big| \leq W < \infty.$$

Let the initial condition X_0 satisfy

vi) $E[|X_0|^2] < \infty$

vii) $\int_0^T E[|D_s(X_0)|^2] ds < \infty$

We also assume that the kernel function g and the total variation measure $|v_g|(du)$ of $g(du)$ satisfy

viii) $|g(t)| + |v_g|(T) \leq M$ *for all* $t \in [0,T]$.

Then (7.12) *has a solution, unique up to modification.*

Proof. This is a long proof, but the idea is the same as in Theorem 5.1.2. We use Picard iteration and the assumptions above to show that

$$E[(Z_t^{k+1} - Z_t^k)^2] \leq AE\Big[\int_0^t (Z_s^k - Z_s^{k-1})^2 \Big], \tag{7.13}$$

for some constant $A > 0$, then the existence part of the proof follows by showing that $\{Z_t^k\}_{k=0}^{\infty}$ is a Cauchy sequence in $L^2(P \times dt)$. The uniqueness part uses the same estimate (7.13), and is finished in the same way as in 7.2.4.

Define $Z_t^0 = X_0$ and Z_t^k inductively as follows

$$Z_t^{k+1} = X_0 + \int_0^t b(s, Z_s^k)ds + \int_0^t c(s, Z_s^k)dY_s.$$

Then we get

$$\begin{aligned}
&E[|Z^{k+1}(t) - Z^k(t)|^2] \\
&= E\Bigg[\Big|\int_0^t \big(b(s,Z_s^k) - b(s,Z_s^{k-1})\big)ds + \int_0^t \big(c(s,Z_s^k) - c(s,Z_s^{k-1})\big)dY(s)\Big|^2\Bigg] \\
&= E\Bigg[\Big|\int_0^t \big(b(s,Z_s^k) - b(s,Z_s^{k-1})\big)ds + \int_0^t K_g(c(s,Z_s^k) - c(s,Z_s^{k-1})(t,s)\delta B(s) \\
&\quad + \int_0^t D_s[K_g(c(s,Z_s^k) - c(s,Z_s^{k-1})(t,s)]ds\Big|^2\Bigg] \\
&= E\Bigg[\Big|\int_0^t \big(b(s,Z_s^k) - b(s,Z_s^{k-1})\big)ds + \int_0^t g(t-s)(c(s,Z_s^k) - c(s,Z_s^{k-1}))\delta B_s \\
&\quad + \int_0^t \Big(\int_0^{t-s} \big(c(u+s,Z_{u+s}^k) - c(s,Z_s^k)\big)g(du) \\
&\quad - \int_0^{t-s} \big(c(u+s,Z_{u+s}^{k-1}) - c(s,Z_s^{k-1})\big)g(du)\Big)\delta B(s) \\
&\quad + \int_0^t g(t-s)\big(D_s[c(s,Z_s^k)] - D_s[c(s,Z_s^{k-1})]\big)ds \\
&\quad + \int_0^t \Big(D_s\big[\int_0^{t-s} \big(c(u+s,Z_{u+s}^k) - c(s,Z_s^k)\big)g(du)\big] \\
&\quad - D_s\big[\int_0^{t-s} \big(c(u+s,Z_{u+s}^{k-1}) - c(s,Z_s^{k-1})\big)g(du)\big]\Big)ds\Big|^2\Bigg]
\end{aligned}$$

Now, we rearrange the terms in the last equality and use 3.1.22 to move the Malliavin derivative inside the Lebesgue-Stieltjes integral. By also using the linearity of the Malliavin derivative, the product rule 3.1.12, and the fact that $D_s(g(t-s)) = 0$ we get

$$E\Bigg[\Big|\int_0^t \big(b(s,Z_s^k) - b(s,Z_s^{k-1})\big)ds \tag{7.14}$$

$$+ \int_0^t g(t-s)\big(c(s,Z_s^k) - c(s,Z_s^{k-1})\big)\delta B(s) \tag{7.15}$$

$$+ \int_0^t \Big(\int_0^{t-s} \big(c(u+s,Z_{u+s}^k) - c(u+s,Z_{u+s}^{k-1})\big)g(du)\Big)\delta B(s) \tag{7.16}$$

$$+ \int_0^t \Big(\int_0^{t-s} \big(c(s,Z_s^{k-1}) - c(s,Z_s^k)\big)g(du)\Big)\delta B(s) \tag{7.17}$$

$$+\int_0^t g(t-s)\big(D_s[c(s,Z_s^k)-c(s,Z_s^{k-1})]\big)ds \tag{7.18}$$

$$+\int_0^t \Big(\int_0^{t-s} D_s\big[c(u+s,Z_{u+s}^k)-c(u+s,Z_{u+s}^{k-1})\big]g(du)\Big)ds \tag{7.19}$$

$$+\int_0^t \Big(\int_0^{t-s} D_s\big[c(s,Z_s^{k-1})-c(s,Z_s^k))\big]g(du)\Big)ds\Bigg|^2\Bigg] \tag{7.20}$$

Applying inequality 2.3.1 enables us to consider each term separately. For term (7.14) we use the Lipschitz assumption on b, and the Cauchy-Schwarz inequality,

$$E\left[\left|\int_0^t \big(b(s,Z_s^k)-b(s,Z_s^{k-1})\big)ds\right|^2\right] \le \Lambda^2 TE\left[\int_0^t \big(Z^k(s)-Z^{k-1}(s)\big)^2 ds\right].$$

For term (7.15) we use the Skorohod isometry 3.1.21 coupled with lemma 7.3.2, the boundedness assumption on g and the Lipschitz assumptions on c and its Malliavin derivative,

$$\begin{aligned}
&E\left[\left|\int_0^t g(t-s)\big(c(s,Z_s^k)-c(s,Z_s^{k-1})\big)\delta B(s)\right|^2\right]\\
&\le E\left[\int_0^t g^2(t-s)\big(c(s,Z_s^k)-c(s,Z_s^{k-1})\big)^2 ds\right]\\
&\quad + E\left[\int_0^t\int_0^t \Big(D_r\{g(t-s)\big(c(s,Z_s^k)-c(s,Z_s^{k-1})\big)\}\Big)^2 dsdr\right]\\
&\le M^2\Lambda^2 E\left[\int_0^t \big(Z_s^k-Z_s^{k-1}\big)^2 ds\right]\\
&\quad + M^2F^2TE\left[\int_0^t (Z_s^k-Z_s^{k-1})^2 ds\right]
\end{aligned}$$

For term (7.16) we use the Skorohod isometry together with inequality 7.3.2 and define

$$\phi(t,s) := \int_0^{t-s} \big(c(u+s,Z_{u+s}^k)-c(u+s,Z_{u+s}^{k-1})\big)g(du)$$

to get

$$E\left[\left|\int_0^t \phi(t,s)\delta B(s)\right|^2\right] \le E\left[\int_0^t \phi^2(t,s)ds + \int_0^t\int_0^t \big(D_v\{\phi(t,s)\}\big)^2 dsdv\right].$$

For the first term we appeal to Cauchy-Schwarz, the Lipschitz condition on c, and Lemma 7.3.1 to get

$$
\begin{aligned}
&E\bigg[\int_0^t \Big(\int_0^{t-s} \big(c(u+s, Z^k_{u+s}) - c(u+s, Z^{k-1}_{u+s})\big) g(du)\Big)^2 ds\bigg] \\
&\leq E\bigg[\int_0^t |v_g|((t-s)-0) \int_0^{t-s} \big(c(u+s, Z^k_{u+s}) - c(u+s, Z^{k-1}_{u+s})\big)^2 |v_g|(du) ds\bigg] \\
&\leq M\Lambda^2 E\bigg[\int_0^t \int_0^{t-s} \big(Z^k_{u+s} - Z^{k-1}_{u+s}\big)^2 |v_g|(du) ds\bigg] \\
&= M\Lambda^2 E\bigg[\int_0^t \int_u^t \big(Z^k_s - Z^{k-1}_s(s)\big)^2 ds |v_g|(du)\bigg] \\
&\leq M^2\Lambda^2 E\bigg[\int_0^t \big(Z^k_s - Z^{k-1}_s\big)^2 ds\bigg]
\end{aligned}
$$

Applying 3.1.22, Cauchy-Schwarz, the Lipschitz assumption on the Malliavin derivative of c, and Lemma 7.3.1 to the second term gives

$$
\begin{aligned}
&E\bigg[\int_0^t \int_0^t \bigg(D_v\Big\{\int_0^{t-s} \big(c(u+s, Z^k_{u+s}) - c(u+s, Z^{k-1}_{u+s})\big) g(du)\Big\}\bigg)^2 dsdv\bigg] \\
&\leq ME\bigg[\int_0^t \int_0^t \int_0^{t-s} \big(D_v\{c(u+s, Z^k_{u+s}) - c(u+s, Z^{k-1}_{u+s})\}\big)^2 |v_g|(du)\} dsdv\bigg] \\
&\leq MF^2 E\bigg[\int_0^t \int_0^t \int_0^{t-s} (Z^k_{u+s} - Z^{k-1}_{u+s})^2 |v_g|(du) dsdv\bigg] \\
&\leq MF^2 TE\bigg[\int_0^t \int_u^t (Z^k_s - Z^{k-1}_s)^2 ds |v_g|(du)\bigg] \\
&\leq M^2 F^2 TE\bigg[\int_0^t (Z^k_s - Z^{k-1}_s)^2 ds\bigg].
\end{aligned}
$$

For term (7.17), we once again employ the Skorohod isometry coupled with Lemma 7.3.2,

$$
\begin{aligned}
&E\bigg[\Big|\int_0^t \Big(\int_0^{t-s} \big(c(s, Z^{k-1}_s) - c(s, Z^k_s)\big) g(du)\Big) \delta B(s)\Big|^2\bigg] \\
&= E\bigg[\Big|\int_0^t (g(t-s) - g(0)) \big(c(s, Z^{k-1}_s) - c(s, Z^k_s)\big) \delta B(s)\Big|^2\bigg] \\
&\leq E\bigg[\int_0^t \big(g(t-s) - g(0)\big)^2 \big(c(s, Z^{k-1}_s) - c(s, Z^k_s)\big)^2 ds\bigg] \\
&\quad + E\bigg[\int_0^t \int_0^t \bigg(D_v\big\{\big[g(t-s) - g(0)\big]\big[c(s, Z^{k-1}_s) - c(s, Z^k_s)\big]\big\}\bigg)^2 dsdv\bigg] \\
&\leq 2M^2\Lambda^2 E\bigg[\int_0^t (Z^{k-1}_s - Z^k_s)^2 ds\bigg] + 2M^2 F^2 TE\bigg[\int_0^t (Z^{k-1}_s - Z^k_s)^2 ds\bigg],
\end{aligned}
$$

where the last inequality follows by the boundedness assumption on g, the Lipschitz continuity on c, the product rule 3.1.12 together with the fact that $D_v\{g(t-s)-g(0)\}=0$, and the Lipschitz continuity of the Malliavin derivative of c.

Term (7.18) is dealt with by applying the boundedness of g, the Lipschitz assumption on the Malliavin derivative of c, and the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&E\left[\left|\int_0^t g(t-s)D_s\{c(s,Z_s^k)-c(Z_s^{k-1}\}ds\right|^2\right]\\
&\leq tE\left[\int_0^t g^2(t-s)\big(D_s\{c(s,Z_s^k)-c(Z_s^{k-1}\}\big)^2ds\right]\\
&\leq M^2F^2TE\left[\int_0^t (Z_s^k-Z_s^{k-1})^2ds\right].
\end{aligned}
$$

For term (7.19), we apply the Cauchy-Schwarz inequality twice, the boundedness assumption on g, and Lemma 7.3.1,

$$
\begin{aligned}
&E\left[\left|\int_0^t\Big(\int_0^{t-s}D_s\big\{c(u+s,Z_{u+s}^k)-c(u+s,Z_{u+s}^{k-1})\big\}g(du)\Big)ds\right|^2\right]\\
&\leq MTE\left[\int_0^t\int_0^{t-s}\Big(D_s\big\{c(u+s,Z_{u+s}^k)-c(u+s,Z_{u+s}^{k-1})\big\}\Big)^2|v_g|(du)ds\right]\\
&\leq MF^2TE\left[\int_0^t\int_0^{t-s}(Z_{u+s}^k-Z_{u+s}^{k-1})^2|v_g|(du)ds\right]\\
&= MF^2TE\left[\int_0^t\int_u^t(Z_s^k-Z_s^{k-1})^2ds|v_g|(du)\right]\\
&\leq M^2F^2TE\left[\int_0^t(Z_s^k-Z_s^{k-1})^2ds\right].
\end{aligned}
$$

Finally, term (7.20) is handled with an appeal to Cauchy-Schwarz, the boundedness assumption on g, and the Lipschitz assumption on the Malliavin derivative of c,

$$
\begin{aligned}
&E\left[\left|\int_0^t\Big(\int_0^{t-s}D_s\big\{c(s,Z_s^{k-1})-c(s,Z_s^k))\big\}g(du)\Big)ds\right|^2\right]\\
&\leq TE\left[\int_0^t\big(g(t-s)-g(0)\big)^2\big(D_s\big\{c(s,Z_s^{k-1})-c(s,Z_s^k))\big\}\big)^2ds\right]\\
&\leq 2M^2F^2TE\left[\int_0^t(Z^{k-1}(s)-Z^k(s))^2ds\right].
\end{aligned}
$$

Putting everything together gives

$$
\begin{aligned}
&E[|Z^{k+1}(t) - Z^k(t)|^2] \\
&\leq 7(\Lambda^2 T + (M^2\Lambda^2 + M^2F^2T) + (M^2\Lambda^2 + M^2F^2T) + (2M^2\Lambda^2 \\
&+ 2M^2F^2T) + M^2F^2T + M^2F^2T + 2M^2F^2T) \int_0^t E\big[\big|Z^k(s) - Z^{k-1}(s)\big|^2\big]ds
\end{aligned}
$$

Set $A_1 = 7(\Lambda^2T + 4M^2\Lambda^2 + 8M^2F^2T)$. For $k = 0$, note that

$$
\begin{aligned}
E[|Z^1(t) - Z^0(t)|^2] &= E\Bigg[\Big|\int_0^t b(s, X_0)ds + \int_0^t g(t-s)c(s, X_0)dB(s) \\
&+ \int_0^t \Big(\int_0^{t-s} \big(c(u+s, X_0) - c(s, X_0)\big)g(du)\Big)dB(s) \\
&+ \int_0^t g(t-s)D_s[c(s, X_0)]ds \\
&+ \int_0^t \Big(D_s\big[\int_0^{t-s} \big(c(u+s, X_0) - c(s, X_0)\big)g(du)\big]\Big|^2\Bigg]
\end{aligned}
$$

Applying inequality 2.3.1, so that we can consider each term separately, and then using the Lipschitz assumption on c and its Malliavin derivative gives

$$
\begin{aligned}
&\int_0^{t-s} \big(|c(u+s, X_0) - c(s, X_0)|\big)|v_g|(du) \leq 0 \\
&\int_0^{t-s} \big(|D_s c(u+s, X_0) - D_s c(s, X_0)|\big)|v_g|(du) \leq 0
\end{aligned}
$$

which leaves three terms. These can be dealt with using Cauchy-Schwarz, the linear growth assumptions on b and c, the Itô isometry, the boundedness assumption on g, and the product rule 3.1.12,

$$
\begin{aligned}
&5E\Bigg[\Big|\int_0^t b(s, X_0)ds\Big|^2 + 5E\Bigg[\Big|\int_0^t g(t-s)c(s, X_0)dB(s)\Big|^2\Bigg] \\
&+ 5E\Bigg[\Big|\int_0^t g(t-s)D_s\{c(s, X_0)\}ds\Big|^2\Bigg] \\
&\leq 5CTE\Big[\int_0^t (1 + |X_0|^2)ds\Big] + 5M^2CE\Big[\int_0^t (1 + |X_0|^2)ds\Big] \\
&+ 5M^2TE\Big[\int_0^t \frac{\partial}{\partial x}c^2(s, X_0)\big(D_s\{X_0\}\big)^2 ds\Big] \\
&\leq 5CTt(1 + E[|X_0|^2]) + 5M^2Ct(1 + E[|X_0|^2]) \\
&+ 5M^2W^2Tt\int_0^t E[(D_s\{X_0\})^2]ds \leq A_2 t
\end{aligned}
$$

Where A_2 is a constant depending on C, M, W, T, $E[|X_0|^2]$ and $\int_0^t E\big[\big|D_s\{X_0\}\big|^2\big]ds$. Setting $A_3 = \max\{A_1, A_2\}$ and using induction we get

$$
E[|Z^{k+1}(t) - Z^k(t)|^2] \leq A_3 \int_0^t E[|Z^k(s) - Z^{k-1}(s)|^2]ds
$$

$$\leq A_3 \int_0^t \frac{A_3^k s^k}{k!} ds = \frac{A_3^{k+1} t^{k+1}}{(k+1)!}, \quad k \geq 0,\ t \in [0,T].$$

Furthermore, $\{Z_t^k\}_{n=0}^{\infty}$ is Cauchy in $L^2(P \times dt)$,

$$\|Z_t^n - Z_t^m\|_{L^2(P\times dt)} = \Big\| \sum_{k=m}^{n-1} Z_t^{k+1} - Z_t^k \Big\|_{L^2(P\times dt)}$$

$$\leq \sum_{k=m}^{n-1} \|Z_t^{k+1} - Z_t^k\|_{L^2(P\times dt)} \leq \sum_{k=m}^{n-1} \left(E\left[\int_0^T |Z_t^{k+1} - Z_t^k|^2 dt \right] \right)^{1/2}$$

$$\leq \sum_{k=m}^{n-1} \left(\int_0^T \frac{A_3^{k+1} t^{k+1}}{(k+1)!} dt \right)^{1/2} = \sum_{k=m}^{n-1} \left(\frac{A_3^{k+1} T^{k+2}}{(k+2)!} dt \right)^{1/2} \to 0, \quad n, m \to \infty$$

Define

$$X_t := L^2(P \times dt) - \lim_{n\to\infty} Z_t^n.$$

The last step of the existence part is to check that X_t actually satisfies the SDE. That is, we want

$$E\left[\Big| \int_0^t \big(b(X(s)) - b(Z^n(s)) \big) ds + \int_0^t \big(c(s, X_s) - c(s, Z_s^n) dY(s) \Big|^2 \right] \to 0, \quad n \to \infty \tag{7.21}$$

Splitting this expression up in seven terms as above and using that $Z_t^n \to X_t$ in $L^2(P \times dt)$ shows that (7.21) holds.

The uniqueness part follows in the same way as in 7.2.4 ∎

Remark 7.3.5. Note that unlike in Remark 7.2.5, we cannot achieve the stronger convergence of $E\big[|X_t - Z_t^k|^2\big] \to 0$ as $k \to \infty$ because we would need Doob's L^p inequality (see Theorem 1.7 in chapter 2 of [RY99]). This requires the noise term to be a positive submartingale or a right-continuous martingale, our noise term is neither, hence Doob's L^p inequality is out of bounds.

Under one extra assumption, the same result holds for VMBV process with $\sigma = 1$.

Theorem 7.3.6 (Existence and uniqueness of nonlinear SDE driven by a VMBV process)**.** *Assume Y is a VMBV process with $\sigma = 1$. Let $T > 0$ and $b : [0,T] \times \mathbb{R} \to \mathbb{R}$, $c : [0,T] \times \mathbb{R} \to \mathbb{R}$ be measurable functions satisfying i)-vii) from Theorem 7.3.4. Assume the kernel function G satisfy*

ix) *$t \mapsto G(t,s)$ is absolutely continuous with respect to the Lebesgue measure and the Radon-Nikodym derivative $\partial G(t,s)/\partial t \leq M'$, for all $0 \leq s \leq t \leq T$.*

Then (7.12) has a solution, unique up to modification.

Proof. Note that by assumption (ix) we have

$$K_G(h)(t,s) = h(s)G(t,s) + \int_s^t (h(u) - h(s))\frac{\partial G(u,s)}{\partial u}du.$$

The proof is finished in a very similar manner to the proof of Theorem 7.3.4. ■

The reason for assumption ix) in the above theorem is that we can no longer use Lemma 6.2.2, and therefore, neither Lemma 7.3.1. Hence, assumption ix) enables us to get rid of the Lebesgue-Stieltjes integral, recall that we want to obtain the following equality

$$E\big[|Z_t^{k+1} - Z_t^k|^2\big] \leq C\int_0^t |Z_s^k - Z_s^{k-1}|^2 ds$$

for some constant $C > 0$.

The assumptions on the noise coefficient c are strong but not unparalleled. For example in Proposition 5.3 in [EPQ97] the assumption of a Lipschitz Malliavin derivative was made. Admittedly, they consider a weaker assumption where the Lipschitz "constant" is not a constant but instead a predictable process with an appropriate integrability condition. However, as they also remark, this is a trade-off since they could relax other assumptions if they instead considered a constant as we have done here.

Since the solution of (7.12) found in the theorems above is not itself necessarily Malliavin differentiable, it is interesting to know under what conditions it would be.

Proposition 7.3.7 (Malliavin differentiability of solution)**.** *Under the assumptions of Theorem 7.3.4, plus the following assumption*

i) $|D_t b(s,x)|^2 \leq (1 + |x|^2)$ *for all* $t, s \in [0,T]$, $x \in \mathbb{R}$,

ii) $|D_t c(s,x)|^2 \leq (1 + |x|^2)$ *for all* $t, s \in [0,T]$, $x \in \mathbb{R}$

iii) $|D_r D_t c(s,x)|^2 \leq (1 + |x|^2)$ *for all* $r, t, s \in [0,T]$, $x \in \mathbb{R}$

we have that the solution $X(t)$ *of* (7.12) *is in* $\mathbb{D}_{1,2}$ *for all* $t \in [0,T]$.

Proof. Assume $Z_r^k \in \mathbb{D}_{1,2}$ for all $r \in [0,T]$, we want to show that $Z_r^{k+1} \in \mathbb{D}_{1,2}$ for all $r \in [0,T]$. We get

$$D_t Z_r^{k+1} = D_t X_0 + \int_0^r D_t\{b(s, Z_s^k)\}ds + D_t\int_0^r c(s, Z_s^k)dY(s).$$

Using assumption vii) from Theorem 7.3.4 we have

$$E\bigg[\int_0^T (D_t X_0)^2 dt\bigg] < \infty.$$

The second term gives

$$\int_0^T E\bigg[\bigg(D_t\Big\{\int_0^r b(s, Z_s^k)ds\Big\}\bigg)^2\bigg]dt \leq T\int_0^T E\bigg[\int_0^r \big(D_t\{b(s, Z_s^k)\}\big)^2 ds\bigg]dt$$

$$\leq T\int_0^T E\Big[\int_0^T(1+|Z_s^k|^2)ds\Big]dt \leq T^2(T+\int_0^T E[|Z_s^k|^2]ds) < \infty.$$

Now, by Proposition 6.4.5 we get

$$D_t\int_0^r c(s,Z_s^k)dY(s) = \int_0^r D_t\{c(s,Z_s^k)\}dY(s) + K_g(c(\cdot,Z_\cdot^k))(r,t)$$

$$= \int_0^r g(r-s)D_t\{c(s,Z_s^k)\}\delta B(s) \tag{7.22}$$

$$+ \int_0^r\int_0^{r-s} D_t\{c(u+s,Z_{u+s}^k)\}g(du)\delta B(s) \tag{7.23}$$

$$- \int_0^r\int_0^{r-s} D_t\{c(s,Z_s^k)\}g(du)\delta B(s) \tag{7.24}$$

$$+ \int_0^r g(t-s)D_tD_s\{c(s,Z_s^k)\}ds \tag{7.25}$$

$$+ \int_0^r\int_0^{r-s} D_tD_s\{c(u+s,Z_{u+s}^k)\}g(du)ds \tag{7.26}$$

$$- \int_0^r\int_0^{r-s} D_tD_s\{c(s,Z_s^k)\}g(du)ds \tag{7.27}$$

$$+ K_g(c(\cdot,Z_\cdot^k))(r,t) \tag{7.28}$$

Using the inequality 2.3.1, we can consider each term separately in $L^2(P\times dt)$. Recalling the boundedness assumption on g, and using the Skorohod isometry together with Lemma 7.3.2 and finally the linear growth assumptions on the first and second Malliavin derivatives of c, term (7.22) gives

$$\begin{aligned}
&\int_0^T E\Big[\Big(\int_0^r g(r-s)D_t\{c(s,Z_s^k)\}\delta B(s)\Big)^2\Big]dt\\
&\leq \int_0^T E\Big[\int_0^r\Big(g(r-s)D_t\{c(s,Z_s^k)\}\Big)^2 ds\\
&\qquad + \int_0^r\int_0^r\Big(D_v\big[g(r-s)D_t\{c(s,Z_s^k)\}\big]\Big)^2 dsdv\Big]dt\\
&\leq \int_0^T E\Big[M^2\int_0^T(1+|Z_s^k|^2)ds + M^2\int_0^T\int_0^T(1+|Z_s^k|^2)\big]dsdv\Big]dt\\
&\leq (M^2T+M^2T^2)(T+\int_0^T E[|Z_s^k|^2]ds) < \infty
\end{aligned}$$

For (7.23) we use the same tools as for term (7.22) plus Lemma 7.3.1,

$$\begin{aligned}
&\int_0^T E\Big[\Big(\int_0^r\int_0^{r-s} D_t\{c(u+s,Z_{u+s}^k)\}g(du)\delta B(s)\Big)^2\Big]dt\\
&\leq \int_0^T E\Big[\int_0^r\Big(\int_0^{r-s} D_t\{c(u+s,Z_{u+s}^k)\}g(du)\Big)^2 ds\\
&\qquad + \int_0^r\int_0^r\Big(D_v\Big[\int_0^{r-s} D_t\{c(u+s,Z_{u+s}^k)\}g(du)\Big]\Big)^2 dsdv\Big]dt.
\end{aligned}$$

The first term gives

$$
\begin{aligned}
&\int_0^T E\Big[\int_0^r \Big(\int_0^{r-s} D_t\{c(u+s, Z^k_{u+s})\}g(du)ds\Big\}\Big)^2\Big]dt \\
&\leq M\int_0^T E\Big[\int_0^r \int_0^{r-s} \big(D_t\{c(u+s, Z^k_{u+s})\}\big)^2 |v_g|(du)ds\Big]dt \\
&\leq M\int_0^T E\Big[\int_0^r \int_0^{r-u} (1+|Z^k_s|^2)ds|v_g|(du)\Big]dt \\
&\leq M^2T(T+\int_0^T E[|Z^k_s|]ds) < \infty,
\end{aligned}
$$

and the second term gives

$$
\begin{aligned}
&\int_0^T E\Big[\int_0^r \int_0^r \Big(D_v\Big[\int_0^{r-s} D_t\{c(u+s, Z^k_{u+s})\}g(du)\Big]\Big)^2 dsdv\Big]dt \\
&\leq M\int_0^T E\Big[\int_0^r \int_0^r \Big[\int_0^{r-s} \Big(D_vD_t\{c(u+s, Z^k_{u+s})\}\Big)^2 |v_g|(du)\Big]dsdv\Big]dt \\
&\leq M\int_0^T E\Big[\int_0^r \int_0^r \int_0^{r-u} (1+|Z^k_s|^2)ds|v_g|(du)dv\Big]dt \\
&\leq M^2T^2(T+\int_0^T E[|Z^k_s|]ds) < \infty.
\end{aligned}
$$

Since the remaining terms are all very similar to the above terms, we just list the calculations. Term (7.24):

$$
\begin{aligned}
&\int_0^T E\Big[\Big(\int_0^r \int_0^{r-s} D_t\{c(s, Z^k_s)\}g(du)\delta B(s)\Big)^2\Big]dt \\
&\leq \int_0^T E\Big[\int_0^r \Big(\int_0^{r-s} D_t\{c(s, Z^k_s)\}g(du)\Big)^2 ds \\
&\qquad + \int_0^r \int_0^r \Big(D_v\Big[\int_0^{r-s} D_t\{c(s, Z^k_s)\}g(du)\Big]\Big)^2 dsdv\Big]dt \\
&\leq \int_0^T E\Big[\int_0^r (g(r-u)-g(0))^2[D_t\{c(s, Z^k_s)\}]^2 ds \\
&\qquad + \int_0^r \int_0^r (g(r-u)-g(0))^2[D_vD_t\{c(s, Z^k_s)\}]^2\Big)^2 dsdv\Big]dt \\
&\leq 4M^2T(T+\int_0^T E[|Z^k_s|]ds) + 4M^2T^2(T+\int_0^T E[|Z^k_s|]ds) < \infty.
\end{aligned}
$$

Term (7.25):

$$
\begin{aligned}
&\int_0^T E\Big[\Big(\int_0^r g(t-s)D_tD_s\{c(s, Z^k_s)\}ds\Big)^2\Big]dt \\
&\leq M^2T(T+\int_0^T E[|Z^k_s|^2]) < \infty.
\end{aligned}
$$

Term (7.26):

$$
\int_0^T E\Big[\Big(\int_0^r \int_0^{r-s} D_tD_s\{c(u+s, Z^k_{u+s})\}g(du)ds\Big)^2\Big]dt
$$

$$\leq MT\int_0^T E\Big[\int_0^r\int_0^{r-s}(1+|Z_{u+s}^k)|^2)|v_g|(du)ds\Big]dt$$
$$\leq M^2T^2(T+\int_0^T E[|Z_s^k|^2]) < \infty.$$

Term (7.27):

$$\int_0^T E\Big[\Big(\int_0^r\int_0^{r-s} D_tD_s\{c(s,Z_s^k)\}g(du)ds\Big)^2\Big]dt$$
$$\leq T\int_0^T E\Big[\int_0^r (g(r-s)-g(0))^2[D_tD_s\{c(s,Z_s^k)\}]^2 ds\Big]dt$$
$$\leq 4M^2T^2(T+\int_0^T E[|Z_s^k|^2]) < \infty.$$

Lastly, term (7.28) gives

$$\int_0^T E\Big[\Big(K_g(c(\cdot,Z_\cdot^k))(r,t)\Big)^2\Big]dt$$
$$\leq \int_0^T E\Big[3(g(r-t)c(t,Z_t^k))^2+3\Big(\int_0^{r-t} c(u+t,Z_{u+t}^k)g(du)\Big)^2$$
$$+3\Big(\int_0^{r-t} c(t,Z_t^k)g(du)\Big)^2\Big]dt.$$

Where the first and third term results in

$$\int_0^T E\Big[3(g(r-t)c(t,Z_t^k))^2\Big]dt \leq 3M^2(T+\int_0^T E[|Z_t^k|^2]) < \infty$$

and

$$\int_0^T E\Big[3\Big(\int_0^{r-t} c(t,Z_t^k)g(du)\Big)^2\Big]dt \leq 3\cdot 2M^2(T+\int_0^T E[|Z_t^k|^2]dt) < \infty$$

respectively. Finally, the second term gives

$$\int_0^T E\Big[3\Big(\int_0^{r-t} c(u+t,Z_{u+t}^k)g(du)\Big)^2\Big]dt$$
$$\leq 3M\int_0^T E\Big[\int_0^{T-t}(c(u+t,Z_{u+t}^k))^2|v_g|(du)\Big]dt$$
$$\leq 3ME\Big[\int_0^T\int_u^T(1+|Z_t^k|^2)dtg(du)\Big] \leq 3M^2(T+\int_0^T E[|Z_t^k|^2]dt) < \infty.$$

This shows that $Z_s^k \in \mathbb{D}_{1,2}$, for all $k \geq 0$, since we know that $\int_0^T E[|Z_t^k|^2]dt < \infty$ for all $k \geq 0$. Hence, $\sup_{n\geq 1}\int_0^T E\big[(D_tZ_s^k)\big]dt < \infty$. By the Picard iteration in Theorem 7.3.4, we have

$$\int_0^T E\big[|Z_t - Z_t^k|^2\big] \to 0, \quad k\to\infty,$$

which is stronger than $L^2(P)$ convergence. Hence, we can conclude that the solution X is Malliavin differentiable by Lemma 5.1.10. ■

The last theorem is, in a way, less satisfying than Theorem 5.1.11 since we do not get the same kind of iteration procedure on the Malliavin derivative of Z^k.

Furthermore, compared to Theorem 5.1.11, we are forced to assume extra conditions on top of an already long list of assumptions, whereas in 5.1.11 they are able to prove Malliavin differentiability of the solution under the typical assumptions of linear growth and Lipschitz continuity. The reason for this is that we have a Malliavin derivative as part of the equation (7.12) we are studying. This is also the cause for assumption iv) in Theorem 7.3.4, as without it one must deal with the problem of showing that the following term from the iteration procedure

$$\int_0^t D_s\{K_g(c(\cdot, Z^k_\cdot)(t,s)\}ds$$

is even defined. The same issue arises for the Skorohod term. Further, note that without assumption iv), using induction to show that Z^k_s is Malliavin differentiable does not work as one then gets

$$D_r Z_t^{k+1} = D_r\Big(X_0 + \int_0^t b(s, Z_s^k)ds + \int_0^t c(s, Z_s^K)dY(s)\Big),$$

where the last term involves a Malliavin derivative, and hence we get the double Malliavin derivative of Z^k_s and then the third Malliavin derivative of Z^k_s and so on in an infinite loop.

Finally, in this section, we look at the case where Y in equation (7.12) is an LSS process. That is, we look at the equation where the noise term is given by

$$\begin{aligned}\int_0^t c(s, X_s)dY(s) = &\int_0^t \int_{\mathbb{R}_0} zK_G(c(\cdot, X.))(t,s)\tilde{N}(\delta z, \delta s)\\ &+ \int_0^t \int_{\mathbb{R}_0} zD_{s,z}\{K_G(c(\cdot, X.))(t,s)\}\tilde{N}(\delta z, \delta s) \qquad (7.29)\\ &+ \int_0^t \int_{\mathbb{R}_0} zD_{s,z}\{K_G(c(\cdot, X.))(t,s)\}\nu(dz)ds.\end{aligned}$$

Theorem 7.3.8 (Existence and uniqueness of SDE driven by LSS process with only small jumps)**.** *Let Y be a LSS process with $\sigma = 1$. We also keep the same assumptions as in Theorem 7.3.4 but adapt them to the jump case. In addition, assume that for all $s, r, t \in [0, T]$, $z, y \in \mathbb{R}_0$,*

(ix) $|D_{s,z}\big[D_{r,y}(c(t, x_1))\big] - D_{s,z}\big[D_{r,y}(c(t, x_2))\big]| \leq K|x_1 - x_2|$, $x_1, x_2 \in \mathbb{R}$, $K > 0$

(x) $E\Big[\int_0^t \int_{\mathbb{R}_0} \big(D_{s,z}[X_0]\big)^4 \nu(dz)ds\Big] < \infty$

(xi) $\int_{\mathbb{R}_0} z^4\nu(dz) \leq E_4 < \infty$.

We also set $\int_{\mathbb{R}_0} \nu(dz) \leq E_0$ and $\int_{\mathbb{R}_0} z^2\nu(dz) \leq E_2$ for some positive constants E_0 and E_2. For simplicity, let K be the Lipschitz constant also for the first Malliavin derivative of c and c itself. Then, (7.12) *has a solution, unique up to modification.*

Proof. We can use the same procedure as in the BSS case. Define $Z_t^0 = X_0$ and Z_t^k inductively as follows:

$$Z_t^{k+1} = X_0 + \int_0^t b(s, Z_t^k)ds + \int_0^t \sigma(s, Z_t^k)dY_s.$$

For the purpose of avoiding unnecessary repetition, we will only look at the extra term (7.29) caused by the difference in the definition of VMLV integrals 6.2.3 versus VMBV integrals 6.2.1. That is not to say that this will be a short proof, however.

Define

$$\varphi(t,s) = K_g(c(\cdot, Z_\cdot^k) - c(\cdot, Z_\cdot^{k-1}))(t,s)$$

We get, by the Skorohod isometry 3.2.17 and 7.3.2

$$\begin{aligned}
&E\left[\left|\int_0^t \int_{\mathbb{R}_0} zD_{s,z}\{\varphi(t,s)\}\tilde{N}(\delta z, \delta s)\right|^2\right] \\
&\le E\left[\int_0^t \int_{\mathbb{R}_0} \big(zD_{s,z}\{\varphi(t,s)\}\big)^2 \nu(dz)ds\right] && (7.30)\\
&\quad + E\left[\int_0^t \int_{\mathbb{R}_0} \int_0^t \int_{\mathbb{R}_0} \Big(D_{r,y}\big[zD_{s,z}\{\varphi(t,s)\}\big]\Big)^2 \nu(dy)dr\nu(dz)ds\right] && (7.31)
\end{aligned}$$

First consider (7.31). By using linearity of the Malliavin derivative and 3.2.18 to move the Malliavin derivative inside the integral, we have

$$\begin{aligned}
&\big|D_{r,y}\big[D_{s,z}\{\varphi(t,s)\}\big]\big| = \big|D_{r,y}\big[D_{s,z}\{K_g(c(\cdot, Z^k(\cdot)) - c(\cdot, Z^{k-1}(\cdot)))(t,s)\}\big]\big| \\
&= \Bigg|D_{r,y}\Bigg(D_{s,z}\Bigg[g(t-s)(c(s, Z^k(s)) - c(s, Z^{k-1}(s))) \\
&\quad + \int_0^{t-s} \big(c(u+s, Z^k(u+s)) - c(u+s, Z^{k-1}(u+s)) \\
&\qquad - (c(s, Z^k(s)) - c(s, Z^{k-1}(s)))\big)g(du)\Bigg]\Bigg)\Bigg| \\
&\le \big|(g(t-s))D_{r,y}\big[D_{s,z}(c(s, Z^k(s)) - c(s, Z^{k-1}(s)))\big]\big| \\
&\quad + \Bigg|\int_0^{t-s} \Bigg(D_{r,y}\big[D_{s,z}c(u+s, Z^k(u+s)) - c(u+s, Z^{k-1}(u+s))\big] \\
&\quad + D_{r,y}\big[D_{s,z}(c(s, Z^{k-1}(s)) - c(s, Z^k(s)))\big]\Bigg)g(du)\Bigg|
\end{aligned}$$

The boundedness condition on g and the Lipschitz assumption on the double Malliavin derivative gives

$$\begin{aligned}
&\big|(g(t-s))D_{r,y}\big[D_{s,z}(c(s, Z^k(s)) - c(s, Z^{k-1}(s)))\big]\big| \\
&\quad + \Bigg|\int_0^{t-s} \Bigg(D_{r,y}\big[D_{s,z}c(u+s, Z^k(u+s)) - c(u+s, Z^{k-1}(u+s))\big] \\
&\quad + D_{r,y}\big[D_{s,z}(c(s, Z^{k-1}(s)) - c(s, Z^k(s)))\big]\Bigg)g(du)\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq M|Z^k(s) - Z^{k-1}(s)| \\
&\quad + \int_0^{t-s} \big|D_{r,y}\big[D_{s,z}c(u+s, Z^k(u+s)) - c(u+s, Z^{k-1}(u+s))\big]\big||v_g|(du) \\
&\quad + |g(t-s) - g(0)|\big|D_{r,y}\big[D_{s,z}(c(s, Z^{k-1}(s)) - c(s, Z^k(s))\big]\big| \\
&\leq MK|Z^k(s) - Z^{k-1}(s)| + \int_0^{t-s} K\big|Z^k(u+s) - Z^{k-1}(u+s)\big||v_g|(du) \\
&\quad + 2MK\big|Z^{k-1}(s) - Z^k(s)\big|
\end{aligned}
$$

Inserting this into (7.31) gives

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}_0} \int_0^t \int_{\mathbb{R}_0} z^2 \Big[3MK|Z^k(s) - Z^{k-1}(s)| \\
&\qquad + K\int_0^{t-s} |Z^k(u+s) - Z^{k-1}(u+s)||v_g|(du)\Big]^2 \nu(dy)dr\nu(dz)ds \\
&\leq 2TE_0E_2 \cdot 3^2M^2K^2 \int_0^t \big[|Z^k(s) - Z^{k-1}(s)|\big]^2 ds \\
&\qquad + 2TE_0E_2K^2 \int_0^t \Big[\int_0^{t-s} |Z^k(u+s) - Z^{k-1}(u+s)||v_g|(du)\Big]^2 ds
\end{aligned}
$$

Applying Cauchy-Schwarz and Lemma 7.3.1 to the second term grants

$$
\begin{aligned}
&\int_0^t \Big[\int_0^{t-s} |Z^k(u+s) - Z^{k-1}(u+s)||v_g|(du)\Big]^2 ds \\
&\leq \int_0^t \int_0^{t-s} |Z^k(u+s) - Z^{k-1}(u+s)|^2|v_g|(du)|v_g|((t-s)-0)ds \\
&= M\int_0^t \int_u^t |Z^k(s) - Z^{k-1}(s)|^2 ds|v_g|(du) \\
&\leq M\int_0^t \int_0^t |Z^k(s) - Z^{k-1}(s)|^2 ds|v_g|(du) \\
&\leq M^2 \int_0^t |Z^k(s) - Z^{k-1}(s)|^2 ds
\end{aligned}
$$

So for the second term, we end up with

$$
\begin{aligned}
&E\Big[\int_0^t \int_{\mathbb{R}_0} \int_0^t \int_{\mathbb{R}_0} \Big(D_{r,y}\big[zD_{s,z}\{\varphi(t,s)\}\big]\Big)^2 \nu(dy)dr\nu(dz)ds\Big] \\
&18TE_0E_2M^2K^2 \int_0^t E\big[|Z^k(s) - Z^{k-1}(s)|\big]^2 ds \\
&\quad + 2TE_0E_2K^2 \int_0^t E\Big[\int_0^{t-s} |Z^k(u+s) - Z^{k-1}(u+s)||v_g|(du)\Big]^2 ds \\
&\leq 20TE_0E_2M^2K^2 \int_0^t E[|Z^k(s) - Z^{k-1}(s)|^2]ds
\end{aligned}
$$

The first term can be estimated in a similar fashion and the details will therefore be skipped.

$$
\begin{aligned}
&E\Big[\int_0^t \int_{\mathbb{R}_0} \big(zD_{s,z}\{\varphi(t,s)\}\big)^2 \nu(dz)ds\Big] \\
&= E\Big[\int_0^t \int_{\mathbb{R}_0} \big(zD_{s,z}\{K_g(c(\cdot,Z^k(\cdot)) - c(\cdot,Z^{k-1}(\cdot)))(t,s)\}\big)^2 \nu(dz)ds\Big] \\
&= E\Big[\int_0^t \int_{\mathbb{R}_0} z^2\Big(D_{s,z}\{g(t-s)(c(s,Z^k(s)) - c(s,Z^{k-1}(s)))\} \\
&\quad + D_{s,z}\Big\{\int_0^{t-s} \big(c(u+s,Z^k(u+s)) - c(u+s,Z^{k-1}(u+s)) \\
&\quad - (c(s,Z^{k-1}(s)) - c(s,Z^k(s)))\big)g(du)\Big\}\Big)^2 \nu(dz)ds\Big] \\
&\le (3E_2K^2M^2 + 3E_2K^2M^2 + 3E_2K^2 \cdot 4M^2)\int_0^t E[|Z^k(s) - Z^{k-1}(s)|^2] \\
&= 18E_2K^2M^2 \int_0^t E[|Z^k(s) - Z^{k-1}(s)|^2]
\end{aligned}
$$

Furthermore, by the Lipschitz condition on c and its Malliavin derivative,

$$
\begin{aligned}
&\int_0^{t-s} |c(u+s,X_0) - c(s,X_0)||v_g|(du) \le 0 \\
&\int_0^{t-s} |D_{s,z}c(u+s,X_0) - D_{s,z}c(s,X_0)||v_g|(du) \le 0,
\end{aligned}
$$

so for $k = 0$ we get

$$
\begin{aligned}
&E[|Z^1(t) - Z^0(t)|^2] \\
&= E\Bigg[\Big|\int_0^t b(s,X_0)ds + \int_0^t \int_{\mathbb{R}_0} zg(t-s)c(s,X_0)\tilde{N}(ds,dz) \\
&\quad + \int_0^t \int_{\mathbb{R}_0} zg(t-s)D_{s,z}[c(s,X_0)]\tilde{N}(ds,dz) \\
&\quad + \int_0^t \int_{\mathbb{R}_0} zg(t-s)D_{s,z}[c(s,X_0)]\nu(dz)ds\Big|^2\Bigg] \\
&\le 4E\Bigg[\Big|\int_0^t b(s,X_0)ds\Big|^2\Bigg] + 4E\Bigg[\Big|\int_0^t \int_{\mathbb{R}_0} zg(t-s)c(s,X_0)\tilde{N}(ds,dz)\Big|^2\Bigg] \\
&\quad + 4E\Bigg[\Big|\int_0^t \int_{\mathbb{R}_0} zg(t-s)D_{s,z}[c(s,X_0)]\tilde{N}(ds,dz)\Big|^2\Bigg] \\
&\quad + 4E\Bigg[\Big|\int_0^t \int_{\mathbb{R}_0} zg(t-s)D_{s,z}[c(s,X_0)]\nu(dz)ds\Big|^2\Bigg]
\end{aligned}
$$

The first term is as before,

$$E\left[\left|\int_0^t b(s,X_0)ds\right|^2\right] \leq C^2T^2(1+E[|X_0|^2])$$

By applying the Itô isometry on the second term, we have

$$\begin{aligned}
&E\bigg[\left|\int_0^t\int_{\mathbb{R}_0} zg(t-s)c(s,X_0)\tilde{N}(ds,dz)\right|^2 \\
&= E\left[\int_0^t\int_{\mathbb{R}_0}\big(zg(t-s)c(s,X_0)\big)^2\nu(dz)ds\right] \\
&\leq M^2E_2E\left[\int_0^t\big(c(s,X_0)\big)^2ds\right] \leq M^2C^2E_2T(1+E[|X_0|)^2].
\end{aligned}$$

For the third term, we use the chain rule 3.2.13, the Itô isometry and Cauchy-Schwarz to get

$$\begin{aligned}
&E\left[\left|\int_0^t\int_{\mathbb{R}_0} zg(t-s)D_{s,z}[c(s,X_0)]\tilde{N}(ds,dz)\right|^2\right] \\
&= E\left[\int_0^t\int_{\mathbb{R}_0}\big(zg(t-s)D_{s,z}[c(s,X_0)]\big)^2\nu(dz)ds\right] \\
&M^2E\left[\int_0^t\Big(\int_{\mathbb{R}_0}z^4\nu(dz)\Big)^{1/2}\Big(\int_{\mathbb{R}_0}\Big(c\big(s,X_0-D_{s,z}[X_0]\big)-c(s,X_0)\Big)^4\nu(dz)\Big)^{1/2}ds\right] \\
&\leq T^{1/2}M^2K^4(E_4)^{1/2}E\left[\int_0^t\int_{\mathbb{R}_0}\big(D_{s,z}[X_0]\big)^4\nu(dz)ds\right].
\end{aligned}$$

For the fourth term, we apply Cauchy-Schwarz twice and the chain rule to get

$$\begin{aligned}
&E\left[\left|\int_0^t\int_{\mathbb{R}_0} zg(t-s)D_{s,z}[c(s,X_0)]\nu(dz)ds\right|^2\right] \\
&\leq tE\left[\int_0^t\left|\int_{\mathbb{R}_0} zg(t-s)D_{s,z}[c(s,X_0)]\nu(dz)\right|^2ds\right] \\
&\leq T\int_{\mathbb{R}_0}z^2\nu(dz)E\left[\int_0^t\int_{\mathbb{R}_0}\big(g(t-s)D_{s,z}[c(s,X_0)]\big)^2\nu(dz)ds\right] \\
&\leq TM^2K^2E_2E\left[\int_0^t\int_{\mathbb{R}_0}\big(D_{s,z}[X_0]\big)^2\nu(dz)ds\right].
\end{aligned}$$

Summing up, we have

$$\begin{aligned}
&E[|Z^1(t)-Z^0(t)|^2] \\
&\leq 4\big(1+E[|X_0|^2)\big(T^2C^2+TM^2C^2E_2\big) \\
&+4E\left[\int_0^t\int_{\mathbb{R}_0}\big(D_{s,z}[X_0]\big)^2\nu(dz)ds\right]TM^2K^2E_2
\end{aligned}$$

$$+ 4E\Big[\int_0^t \int_{\mathbb{R}_0} \big(D_{s,z}[X_0]\big)^4 \nu(dz)ds\Big] T^{1/2} M^2 K^4 (E_4)^{1/2}.$$

We can now finish the proof in the same way as in the BSS case (Theorem 7.3.4). ■

Similarly to the BSS/VMBV case we can generalize Theorem 7.3.8 to VMLV processes.

Corollary 7.3.9. *Let Y be a VMLV process with $\sigma = 1$. Assume the conditions of Theorem 7.3.8 and Theorem 7.3.6. Then* (7.12) *has a solution, unique up to modification.*

Proof. Combining the proof of Theorem 7.3.8 and Theorem 7.3.6 gives the result. ■

Lastly, in this section, we remark that it is possible to prove that the solution of 7.3.8 and the above corollary is Malliavin differentiable under appropriate conditions similar to those of Theorem 7.3.7.

7.4 SPDEs driven by ambit fields

Where the last sections have focused on real-valued SDEs, this section will focus on Hilbert-valued SPDEs. Let H_1, H_2 and H_3 be separable Hilbert spaces.

Let $F : H_2 \to H_2$ and consider the equation

$$dX_t = AX(t)dt + F(X(t))dt + dY(t), \quad X(0) = X_0 \tag{7.32}$$

where A is a possibly unbounded operator on H_2 and Y is an ambit field on H_2, see Definition 6.3.1.

The existence of a solution is proved through the use of Banach's fixed point theorem, and for that we, need an appropriate space for the solution to exist in. In chapter 5, we had predictable solutions, but this is not as easy to achieve for the equations we consider in this chapter, therefore, we modify the situation in chapter 5 a little bit.

Define the space $\mathcal{Y}_T$ of processes $X : [0, T] \times \Omega \to H_3$ such that

$$\|X\|_T := \left(\sup_{t \in [0,T]} E[\|X_t\|^2_{H_3}] \right)^{1/2} < \infty$$

$\mathcal{Y}_T$ with the norm $\|\cdot\|_T$ is a Banach space. For $\gamma \in \mathbb{R}$ and $X \in \mathcal{Y}_T$, we also define the equivalent norms

$$\|X\|_{T,\gamma} := \left(\sup_{t \in [0,T]} e^{-\gamma t} E[\|X(t)\|^2_{H_3}] \right)^{1/2}.$$

Since we will look at non-predictable solutions, in this chapter, we skip the predictability condition in the definition of a mild solution, see 5.2.4, but retain the other conditions. We wish to solve the following integral equation

$$X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)dY(s), \tag{7.33}$$

where S is the C_0-semigroup generated by A. For now, also assume that σ is predictable.

Since S is deterministic we get

$$\int_0^t S(t-s)dY(s) = \int_0^t K_G(S(t-\cdot))(t,s)\sigma(s)\delta\tilde{W}(s) \tag{7.34}$$

$$+ Tr_{H_1} \int_0^t \mathcal{D}_s\{(K_G(S(t-\cdot))(t,s)\}\sigma(s)ds$$

$$= \int_0^t K_G(S(t-\cdot))(t,s)\sigma(s)d\tilde{W}(s) \tag{7.35}$$

Where the last equality follows since $K_g(S)(t,s)$ is deterministic and hence $\mathcal{D}_s\{K_g(S)(t,s)\} = 0$, and the Skorohod integral and the cylindrical Itô integral coincide. Motivated by 7.35 we will consider two cases: one where W is a cylindrical Wiener process as in (7.35) and one where W is a square integrable martingale with trace class covariance operator. To separate these two cases and align ourselves with the notation of chapter 5, we denote, from now on, by M the square integrable martingale and define $\mathcal{H} := Q^{1/2}(H_1)$. First, we consider the martingale case. Let $|v_G|(du,\cdot)$ denote the total variation of $G(du,\cdot)$.

Theorem 7.4.1 (Mild solution of ambit field driven SPDE, martingale case)**.** *Assume on the coefficient F the assumptions of Assumption 5.2.5. Also assume that $G(du,s)$ has bounded total variation on $[u,v]$ for all $0 \le s \le u \le v \le t$, and that*

$$\int_0^t E\big[\|\sigma(s)\|^2_{L_2(\mathcal{H},H_2)}\big]^2 ds < \infty.$$

Then there exists a mild solution of (7.32)*, unique up to modification.*

Proof. Inspecting the proof of Theorem 5.2.6 we can see that the proof holds in this case as well as long as the stochastic integral

$$\int_0^t K_G(S(t-\cdot))(t,s)\sigma(s)dM(s) \tag{7.36}$$

is well defined and predictable.

By Corollary 4.4.6, we can see that (7.39) is well defined if the integrand is predictable and

$$E\left[\int_0^t \|K_G(S(t-\cdot))(t,s)\sigma(s)\|^2_{L_2(\mathcal{H},H_2)}ds\right] < \infty.$$

The predictability of the integrand is clear since $K_G(S)(t,s)$ is deterministic and σ is predictable by assumption.

Now by (i) in Theorem 5.2.3 we have

$$\|S(t)\|_{L(H_2)} = \sup_{\|h_2\|_{H_2}\le 1} \|S(t)h_2\|_{H_2} \le \sup_{\|h_2\|_{H_2}\le 1} e^{\beta t}M\|h_2\|_{H_2} = e^{\beta t}M.$$

And by the elementary inequality 2.3.1 we have

$$
\begin{aligned}
&\|K_G(S(t-\cdot))(t,s)\sigma(s)\|^2_{L_2(\mathcal{H},H_2)} \\
&= \Big\|\Big(S(t-s)G(t,s)+\int_s^t (S(t-u)-S(t-s))G(du,s)\Big)\sigma(s)\Big\|^2_{L_2(\mathcal{H},H_2)} \\
&\leq 3\|S(t-s)G(t,s)\sigma(s)\|^2_{L_2(\mathcal{H},H_2)} + 3\Big\|\int_s^t S(t-u)G(du,s)\sigma(s)\Big\|^2_{L_2(\mathcal{H},H_2)} \\
&\quad + 3\Big\|\int_s^t S(t-s)G(du,s)\sigma(s)\Big\|^2_{L_2(\mathcal{H},H_2)}
\end{aligned}
$$

As usual, we consider these three terms separately, beginning with the first. For all three terms we apply the inequalities of Proposition 4.1.3.

$$
\begin{aligned}
\|S(t-s)G(t,s)\sigma(s)\|^2_{L_2(\mathcal{H},H_2)} &\leq M^2 e^{2\beta(t-s)}\|G(t,s)\|^2_{L(H_2)}\|\sigma(s)\|^2_{L_2(\mathcal{H},H_2)} \\
&\leq M^2 e^{2\beta t}\|G(t,s)\|^2_{L(H_2)}\|\sigma(s)\|^2_{L(\mathcal{H},H_2)}.
\end{aligned}
$$

For the second term we use the inequality on page 209 of [Din00] to get

$$
\begin{aligned}
&\Big\|\int_s^t S(t-u)G(du,s)\sigma(s)\Big\|^2_{L_2(\mathcal{H},H_2)} \\
&\leq \Big\|\int_s^t S(t-u)G(du,s)\Big\|^2_{L(H_2)}\|\sigma(s)\|^2_{L_2(\mathcal{H},H_2)} \\
&\leq \Big(\int_s^t \|S(t-u)\|_{L(H_2)}|v_G|(du,s)\Big)^2\|\sigma(s)\|^2_{L(\mathcal{H},H_2)} \\
&\leq \Big(\int_s^t Me^{\beta t}|v_G|(du,s)\Big)^2\|\sigma(s)\|^2_{L(\mathcal{H},H_2)} \\
&\leq M^2 e^{2\beta t}(|v_G|(t-s,s))^2\|\sigma(s)\|^2_{L(\mathcal{H},H_2)}.
\end{aligned}
$$

The last term follows in a similar way

$$
\begin{aligned}
&\Big\|\int_s^t S(t-s)|v_G|(du,s)\sigma(s)\Big\|^2_{L_2(\mathcal{H},H_2)} \\
&\leq M^2 e^{2\beta t}(|v_G|(t-s,s))^2\|\sigma(s)\|^2_{L(\mathcal{H},H_2)}
\end{aligned}
$$

Then

$$
\begin{aligned}
&E\Big[\int_0^t \|K_G(S(t-\cdot))(t,s)\sigma(s)\|^2_{L_2(\mathcal{H},H_2)}ds\Big] \\
&\leq 3M^2e^{2\beta t}\int_0^t E\big[\|\sigma(s)\|^2_{L(\mathcal{H},H_2)}\big]\Big(\|G(t,s)\|_{L(H_2)} + 2\big[|v_G|(t-s,s)\big]^2\Big)ds \\
&\leq 3M^2e^{2\beta t}\int_0^t E\big[\|\sigma(s)\|^2_{L(\mathcal{H},H_2)}\big]ds\Big(\sup_{0\leq s<t\leq T}\|G(t,s)\|_{L(H_2)} \\
&\quad + 2\big(\sup_{0\leq s\leq t\leq T}|v_G|(t,s)\big)^2\Big) < \infty,
\end{aligned}
$$

where $\sup_{0\leq s\leq t\leq T}\|G(t,s)\|_{L(H_2)}<\infty$ by Definition 6.3.1, and $\sup_{0\leq s\leq t\leq T}|v_G|(t,s)<\infty$ by assumption.

Hence, we conclude that the integral 7.36 is well defined, and by 5.2.6, there exists a mild solution to (7.32), which is unique up to modification. ■

The case where the noise term is driven by a cylindrical Wiener process is very similar, but we can let the drift coefficient be time-dependent, that is, $F:[0,T]\times H_2\to H_2$.

Theorem 7.4.2 (Mild solution of ambit field driven SPDE, cBm case)**.** *Assume on the coefficient F the assumptions of Assumption 5.2.11. Also assume that $G(du,s)$ has bounded variation on $[u,v]$ for all $0\leq s\leq u\leq v\leq t$, and that*

$$\int_0^t E\big[\|\sigma(s)\|^2_{L_2(H_1,H_2)}\big]^2 ds<\infty.$$

Then there exist a mild solution of (7.32), unique up to modification.

Proof. An inspection of the proof of Theorem 5.2.12 reveals that we only need to show that the stochastic integral with respect to a cylindrical Wiener process is well-defined. This follows by the exact same estimation as for Theorem 7.4.1, but this time $Q=I$ and $\mathcal{H}=H_1$. ■

As before, I have, for simplicity, chosen to follow the assumptions of the referenced results, but it is probably possible to work under slightly different assumptions.

Nonlinear noise coefficient

Now we generalize the above results to account for the case of a nonlinear noise coefficient, as in the real-valued case this comes at a cost.

Let H_1,H_2,H_3 denote separable Hilbert spaces, and let $b:H_3\to H_3$ and $c:H_3\to L(H_2,H_3)$. The equation under consideration is the following

$$dX_t=AX(t)dt+b(X(t))dt+c(X(t))dY(t), \tag{7.37}$$

where A is a possibly unbounded operator on H_3, and σ is no longer assumed to be predictable.

We will again look at mild solutions, and in this case that means we must solve the integral equation

$$X(t)=S(t)X_0+\int_0^t S(t-s)b(X(s))+\int_0^t S(t-s)c(X_s)dY(s). \tag{7.38}$$

Note that by Definition 6.3.1, we have

$$\sup_{0\leq s\leq t}\|\sigma(s)\|_{L(H_1,H_2)}\leq M_\sigma<\infty$$
$$\sup_{0\leq s\leq u\leq T}\|G(u,s)\|_{L(H_2)}\leq M_G<\infty,$$

where M_σ and M_G are constants. This will be of use to us in the proof of the next theorem.

Theorem 7.4.3 (Mild solution of nonlinear ambit field driven SPDE)**.** *Consider the equation* (7.37)*. On the coefficients assume the following*

i) *There is a function $\beta : (0,\infty) \to (0,\infty)$ satisfying $\int_0^T \beta(s)ds < \infty$ for all $T < \infty$, such that for all $T \geq t > 0$, $x, y \in H_3$*

$$|S(t)b(x)|_{H_3} \leq \beta(t)(1 + |x|_{H_3}),$$
$$|S(t)(b(x) - b(y))|_{H_3} \leq \beta(t)|x - y|_{H_3}$$

ii) *There is a function $\zeta : (0,\infty) \to (0,\infty)$ satisfying $\int_0^T \zeta^2(s)ds < \infty$ for all $T < \infty$, such that, for all $T \geq t > 0$ and $x, y \in H_3$,*

$$\|S(t)c(x)\|_{L_2(H_2,H_3)} \leq \zeta(t)(1 + |x|_{H_3}),$$
$$\|S(t)(c(x) - c(y))\|_{L_2(H_2,H_3)} \leq \zeta(t)|x - y|_{H_3}$$

iii) *There is a function $\alpha : (0,\infty) \to (0,\infty)$ satisfying $\int_0^T \alpha^2(s)ds < \infty$ for all $T < \infty$, such that, for all $T \geq s, t > 0$ and $x, y \in L_2(H_2, H_3)$,*

$$\|S(t)D_s\{c(x)\}\|_{L_2(H_1,L_2(H_2,H_3))} \leq \alpha(t)(1 + |x|_{H_3}),$$
$$\|S(t)D_s\{c(x) - c(y)\}\|_{L_2(H_1,L_2(H_2,H_3))} \leq \alpha(t)\|x - y\|_{H_3}.$$

Furthermore, we assume that the kernel function G is Fréchet differentiable in the first variable with derivative denoted by $\frac{dG(u,s)}{du}$ for any $0 \leq s < u \leq T$. Lastly, we assume the norm conditions

$$\sup_{0\leq s\leq t} \|D_r\sigma(s)\|_{L_2(H_1,L_2(H_1,H_2))} \leq M'_\sigma < \infty, \quad r \in [0,T]$$
$$\sup_{0\leq s\leq u\leq T} \|\frac{\partial G(u,s)}{\partial u}\|_{L(H_2)} \leq M'_G < \infty,$$

where M'_σ and M'_G are constants. Then there exists a solution in the space $\mathcal{Y}_T$ which is unique up to modification.

Proof. Firstly, by equation (3.7) in [BS16] we get

$$\begin{aligned}
&E\Big[\Big\|\int_0^t S(t-s)c(X(s))dY(s)\Big\|_{H_3}^2\Big] \\
&\leq 2E\Big[\Big\|K_G(S(t-\cdot)c(X(\cdot)))(t,s)\sigma(s)\delta\tilde{W}(s)\Big\|_{H_3}^2\Big] \\
&\quad + 2TE\Big[\int_0^t \|Tr_{H_1}D_s\big\{K_G(S(t-\cdot)c(X(\cdot)))(t,s)\big\}\sigma(s)\|_{H_3}^2 ds\Big] \\
&\leq C_T\big\|K_G(S(t-\cdot)c(X(\cdot)))(t,\cdot)\sigma(\cdot)\big\|_{\mathbb{L}^{1,2}(L_2(H_1,H_3))} \\
&= C_T\int_0^t E\Big[\big\|K_G(S(t-\cdot)c(X(\cdot)))(t,s)\sigma(s)\big\|_{L_2(H_1,H_3)}^2\Big]ds \\
&+ C_T\int_0^t\int_0^t E\Big[\big\|D_r\big\{K_G(S(t-\cdot)c(X(\cdot)))(t,s)\sigma(s)\big\}\big\|_{L_2(H_1,L_2(H_1,H_3))}^2\Big]drds.
\end{aligned}$$

We now consider the second term above. By the product rule 4.5.5 we have

$$\begin{aligned}&D_r\big\{K_G(S(t-\cdot)c(X(\cdot))(t,s)\sigma(s)\big\}\\&=D_r\{K_G(S(t-\cdot)c(X(\cdot)))(t,s)\}\sigma(s)+K_G(S(t-\cdot)c(X(\cdot)))(t,s)D_r\{\sigma(s)\}.\end{aligned}$$

This gives

$$\begin{aligned}&\big\|D_r\big\{K_G(S(t-\cdot)c(X(\cdot)))(t,s)\sigma(s)\big\}\big\|_{L_2(H_1,L_2(H_1,H_3))}\\&\leq\big\|D_r\big\{K_G(S(t-\cdot)c(X(\cdot)))(t,s)\big\}\sigma(s)\big\|_{L_2(H_1,L_2(H_1,H_3))}\\&\quad+\big\|K_G(S(t-\cdot)c(X(\cdot)))(t,s)D_r\{\sigma(s)\}\big\|_{L_2(H_1,L_2(H_1,H_3))}.\end{aligned}$$

And by using the inequalities in 4.1.3 we have

$$\begin{aligned}&\big\|K_G(S(t-\cdot)c(X(\cdot)))(t,s)D_r\{\sigma(s)\}\big\|_{L_2(H_1,L_2(H_1,H_3))}\\&\leq\|K_G(S(t-\cdot)c(X(\cdot)))(t,s)\|_{L(H_2,H_3)}\big\|D_r\sigma(s)\big\|_{L_2(H_1,L_2(H_1,H_2))}\\&\leq\|K_G(S(t-\cdot)c(X(\cdot)))(t,s)\|_{L_2(H_2,H_3)}M'_\sigma.\end{aligned}$$

Similarly,

$$\begin{aligned}&\big\|D_r\{K_G(S(t-\cdot)c(\cdot,X(\cdot)))(t,s)\}\sigma(s)\big\|_{L_2(H_1,L_2(H_1,H_3))}\\&\leq\|D_rK_G(S(t-\cdot)c(\cdot,X(\cdot)))(t,s)\|_{L_2(H_1,L_2(H_2,H_3))}M_\sigma.\end{aligned}$$

This will simplify our estimations, as we shall soon see.

To save space, we occasionally write $L^{2,3}_{HS}:=L_2(H_1,L_2(H_2,H_3))$. Applying the above inequalities results in

$$\begin{aligned}&C_T\int_0^t E\Big[\big\|K_G(S(t-\cdot)c(X(\cdot)))(t,s)\sigma(s)\big\|^2_{L_2(H_1,H_3)}\Big]ds\\&+C_T\int_0^t\int_0^t E\Big[\big\|D_r\big\{K_G(S(t-\cdot)c(X(\cdot)))(t,s)\sigma(s)\big\}\big\|^2_{L_2(H_1,L_2(H_1,H_3))}\Big]drds\\&\leq C_TM_\sigma\int_0^t E\Big[\|K_G(S(t-\cdot)c(X(\cdot)))(t,s)\|^2_{L_2(H_2,H_3)}\Big]ds\\&\quad+2C_TM_\sigma\int_0^t\int_0^t E\Big[\|D_rK_G(S(t-\cdot)c(X(\cdot)))(t,s)\|^2_{L^{2,3}_{HS}}\Big]drds\\&\quad+2C_TtM'_\sigma\int_0^t E\Big[\|K_G(S(t-\cdot)c(X(\cdot)))(t,s)\|^2_{L(H_2,H_3)}\Big]ds.\\&\leq(C_TM\sigma+2C_TTM'_\sigma)\int_0^t E\Big[\|K_G(S(t-\cdot)c(X(\cdot)))(t,s)\|^2_{L_2(H_2,H_3)}\Big]ds\\&\quad+2C_TM_\sigma\int_0^t\int_0^t E\Big[\|D_rK_G(S(t-\cdot)c(X(\cdot)))(t,s)\|^2_{L^{2,3}_{HS}}\Big]drds\end{aligned}$$

We wish to apply the Banach fixed point theorem to find a solution, that is, for some $U,V\in\mathcal{Y}_T$, we want

$$\sup_{t\in[0,T]}e^{-\gamma t}E\Bigg[\Big\|\int_0^t S(t-s)b(U(s))ds+\int_0^t S(t-s)c(U(s))dY(s)ds$$

$$
\begin{aligned}
&- \int_0^t S(t-s)b(V(s)) - \int_0^t S(t-s)c(V(s))dY(s)\Big\|_{H_3}^2\Big] \\
&\leq C \sup_{t\in[0,T]} e^{-\gamma t} E\big[\|U(t)-V(t)\|_{H_3}^2\big]
\end{aligned}
$$

for some $0 < C < 1$.
Expanding the operator K_G coupled with the above estimates gives

$$
\begin{aligned}
&\sup_{t\in[0,T]} e^{-\gamma t} E\Bigg[\Big\|\int_0^t S(t-s)\big(b(U(s)) - b(V(s))\big)ds \\
&\quad + \int_0^t S(t-s)\big(c(U(s)) - c(V(s))\big)dY(s)\Big\|_{H_3}^2\Bigg] \\
&\leq \sup_{t\in[0,T]} e^{-\gamma t}\Bigg(2E\Big[\Big\|\int_0^t S(t-s)\big(b(U(s)) - b(V(s))\big)ds\Big\|_{H_3}^2\Big] && (7.39) \\
&\quad + 2(C_T M\sigma + 2C_T T M'_\sigma)\int_0^t E\Big[\Big\|S(t-s)(c(U(s)) - c(V(s)))G(t,s) && (7.40) \\
&\quad + \int_s^t \Big(S(t-u)\big\{c(U(u)) - c(V(u))\big\} && (7.41) \\
&\quad + S(t-s)\{c(V(s)) - c(U(s))\}\Big)G(du,s)\Big\|_{L_2(H_2,H_3)}^2\Big]ds && (7.42) \\
&\quad + 4C_T M_\sigma \int_0^t\int_0^t E\Big[\Big\|D_r\Big\{S(t-s)\big(c(U(s)) - c(V(s))\big)G(t,s)\Big\}ds && (7.43) \\
&\quad + D_r\Big\{\int_s^t \Big(S(t-u)\big\{c(U(u)) - c(V(u))\big\} && (7.44) \\
&\quad + S(t-s)\{c(V(s)) - c(U(s))\}\Big)G(du,s)\Big\}\Big\|_{L_2(H_1,L_2(H_2,H_3))}^2\Big]drds\Bigg) && (7.45)
\end{aligned}
$$

As before, we will now consider each term separately.

In the following, the inequalities of Proposition 4.1.3 will be used repeatedly. Term (7.39) gives

$$
\begin{aligned}
&\Big\|\int_0^t S(t-s)\big(b(U(s)) - b(V(s))\big)ds\Big\|_{H_3}^2 \\
&\leq \Big(\int_0^t \beta(t-s)\|U(s)) - V(s)\|_{H_3} ds\Big)^2 \\
&\leq \int_0^t \beta(t-s)ds \int_0^t \beta(t-s)\|U(s) - V(s)\|_{H_3}^2 ds
\end{aligned}
$$

By putting $c_1 := \int_0^T \beta(s)ds \geq \int_0^t \beta(t-s)ds$ we get

$$
\begin{aligned}
&\sup_{t\in[0,T]} e^{-\gamma t}E\Big[\Big\|\int_0^t S(t-s)\big(b(U(s))-b(U(s))\big)ds\Big\|^2_{H_3}\Big]\\
&\le c_1 \sup_{t\in[0,T]} e^{-\gamma t}\int_0^t \beta(t-s)E[\|U(s)-V(s)\|^2_{H_3}]ds\\
&\le c_1 \sup_{t\in[0,T]} e^{-\gamma t}\int_0^t \beta(t-s)e^{\gamma s}e^{-\gamma s}E[\|U(s)-V(s)\|^2_{H_3}]ds\\
&\le c_1\|U-V\|^2_{T,\gamma}\sup_{t\in[0,T]}\int_0^t \beta(t-s)e^{-\gamma(t-s)}ds\\
&\le c_1\|U-V\|^2_{T,\gamma}\int_0^T \beta(s)e^{-\gamma s}ds
\end{aligned}
$$

Term (7.40) gives

$$
\begin{aligned}
&\int_0^t \big\|S(t-s)\big(c(U(s))-c(V(s))\big)G(t,s)\big\|^2_{L_2(H_2,H_3)}ds\\
&\le \int_0^t \|G(t,s)\|^2_{L(H_2)}\big\|S(t-s)\big(c(U(s))-c(V(s))\big)\big\|^2_{L_2(H_2,H_3)}ds\\
&\le M_G^2\int_0^t \zeta^2(t-s)\|U(s)-V(s)\|^2_{H_3}ds
\end{aligned}
$$

Applying the same technique as for term (7.39) we end up with

$$
\begin{aligned}
&\sup_{t\in[0,T]} e^{-\gamma t}\int_0^t E\Big[\big\|S(t-s)\big(c(U(s))-c(V(s))G(t,s)\big)\big\|^2_{L_2(H_2,H_3)}\Big]ds\\
&\le \|U-V\|^2_{T,\gamma}M_G^2\int_0^T \zeta^2(s)e^{-\gamma s}ds
\end{aligned}
$$

For the third term (7.41)

$$
\begin{aligned}
&\int_0^t \Big\|\int_s^t S(t-u)\{c(U(u))-c(V(u))\}G(du,s)\Big\|^2_{L_2(H_2,H_3)}ds\\
&=\int_0^t \Big\|\int_s^t S(t-u)\{c(U(u))-c(V(u))\}\frac{dG(u,s)}{du}du\Big\|^2_{L_2(H_2,H_3)}ds\\
&\le \int_0^t \Big(\int_s^t \Big\|S(t-u)\big\{c(U(u))-c(V(u))\big\}\frac{dG(u,s)}{du}\Big\|_{L_2(H_2,H_3)}du\Big)^2ds\\
&\le T\int_0^t\int_s^t \Big\|\frac{dG(u,s)}{du}\Big\|^2_{L(H_2)}\|S(t-u)\{c(U(u))-c(V(u))\}\|^2_{L_2(H_2,H_3)}duds\\
&\le T^2(M_G')^2\int_0^t \zeta^2(t-u)\|U(u)-V(u)\}\|^2_{H_3}du
\end{aligned}
$$

So we have

$$
\sup_{t\in[0,T]} e^{-\gamma t}\int_0^t E\Big[\Big\|\int_s^t S(t-u)\{c(U(u))-c(V(u))\}G(du,s)\Big\|^2_{L_2(H_2,H_3)}\Big]ds
$$

$$\leq \|U-V\|^2_{T,\gamma}T^2(M'_G)^2\int_0^T \zeta^2(s)e^{-\gamma s}ds$$

The fourth term (7.42) is a simpler case of (7.41), we get

$$\begin{aligned}
&\int_0^t \left\| \int_s^t S(t-s)\{c(V(s))-c(U(s))\}G(du,s)\right\|^2_{L_2(H_2,H_3)} ds \\
&\leq T\int_0^t\int_s^t \left\|S(t-s)\big(c(U(s))-c(V(s))\big)\frac{dG(u,s)}{du}\right\|^2_{L_2(H_2,H_3)} duds \\
&\leq T(M'_G)^2\int_0^t \zeta^2(t-s)\|U(s)-V(s)\|^2_{H_3}ds
\end{aligned}$$

And as before this results in

$$\begin{aligned}
&\sup_{t\in[0,T]} e^{-\gamma t}\int_0^t E\left[\left\| \int_0^{t-s} S(t-s)\{c(V(s))-c(U(s))\}G(du,s)\right\|^2_{L_2(H_2,H_3)}\right] ds \\
&\leq \|U-V\|^2_{T,\gamma}(M'_G)^2\int_0^T \zeta^2(s)e^{-\gamma s}ds
\end{aligned}$$

The fifth term (7.43) follows in a similar way after using the product rule 4.5.5 and assumption iv),

$$\begin{aligned}
&\int_0^t\int_0^t \left\|D_r\Big\{S(t-s)\big(c(U(s))-c(V(s))\big)G(t,s)\Big\}\right\|^2_{L^{2,3}_{HS}} dsdr \\
&\leq \int_0^t\int_0^t \|G(t,s)\|^2_{L(H_2)}\left\|D_r\Big\{S(t-s)\big(c(U(s))-c(V(s))\big)\Big\}\right\|^2_{L^{2,3}_{HS}} dsdr \\
&\leq M_G^2\int_0^t\int_0^t \alpha^2(t-s)\big\|U(s)-V(s)\big\|^2_{H_3}dsdr \\
&\leq TM_G^2\int_0^t \alpha^2(t-s)\big\|U(s)-V(s)\big\|^2_{H_3}ds.
\end{aligned}$$

So we get

$$\begin{aligned}
&\sup_{t\in[0,T]} e^{-\gamma t}\int_0^t\int_0^t E\left[\left\|D_r\Big\{S(t-s)\big(c(U(s))-c(V(s))\big)G(t,s)\Big\}\right\|^2_{L^{2,3}_{HS}}\right] dsdr \\
&\leq \|U-V\|^2_{T,\gamma}TM_G^2\int_0^T \alpha^2(s)e^{-\gamma s}ds.
\end{aligned}$$

For the sixth term (7.44) we move the Malliavin derivative inside the integral which gives

$$
\begin{aligned}
&\int_0^t \int_0^t \Big\| D_r \Big[\int_s^t S(t-u)\big\{c(U(u)) - c(V(u))\big\} G(du,s) \Big] \Big\|^2_{L^{2,3}_{HS}} dsdr \\
&= \int_0^t \int_0^t \Big\| \int_s^t D_r \Big(S(t-u)\big\{c(U(u)) - c(V(u))\big\} \Big) \frac{dG(u,s)}{du} du \Big\|^2_{L^{2,3}_{HS}} dsdr \\
&\leq T \int_0^t \int_0^t \int_s^t \Big\| D_r \Big(S(t-u)\big\{c(U(u)) - c(V(u))\big\} \Big) \frac{dG(u,s)}{du} \Big\|^2_{L^{2,3}_{HS}} dudsdr \\
&\leq T \int_0^t \int_0^t \int_s^t \Big\| \frac{dG(u,s)}{du} \Big\|^2_{L(H_2)} \alpha^2(t-u) \big\| U(u) - V(u) \big\|^2_{H_3} dudsdr \\
&\leq T^2 (M'_G)^2 \int_0^t \int_0^t \alpha^2(t-u) \big\| U(u) - V(u) \big\|^2_{H_3} duds \\
&\leq T^3 (M'_G)^2 \int_0^t \alpha^2(t-u) \big\| U(u) - V(u) \big\|^2_{H_3} du.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\sup_{t\in[0,T]} e^{-\gamma t} \int_0^t \int_0^t E\Bigg[\Big\| D_r \Big[\int_s^t S(t-u)\big\{c(U(u)) \\
&\qquad\qquad - c(V(u))\big\} G(du,s) \Big] \Big\|^2_{L^{2,3}_{HS}} \Bigg] dsdr \\
&\leq \|U - V\|^2_{T,\gamma} T^3 (M'_G)^2 \int_0^T \alpha^2(s) e^{-\gamma s} ds.
\end{aligned}
$$

The seventh term (7.45) is a simpler version of (7.44)

$$
\begin{aligned}
&\int_0^t \int_0^t \Big\| D_r \Big[\int_s^t S(t-s)\big\{c(U(s)) - c(V(s))\big\} G(du,s) \Big] \Big\|^2_{L^{2,3}_{HS}} dsdr \\
&\leq T \int_0^t \int_0^t \int_s^t \Big\| D_r \Big[S(t-s)\big\{c(U(s)) - c(V(s))\big\} \Big] \frac{dG(u,s)}{du} \Big\|^2_{L^{2,3}_{HS}} dudsdr \\
&\leq T^2 (M'_G)^2 \int_0^t \int_0^t \alpha^2(t-s) \big\| U(s) - V(s) \big\|^2_{H_3} duds \\
&\leq T^3 (M'_G)^2 \int_0^t \alpha^2(t-s) \big\| U(s) - V(s) \big\|^2_{H_3} ds.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\sup_{t\in[0,T]} e^{-\gamma t} \int_0^t \int_0^t E\Bigg[\Big\| D_r \Big[\int_s^t S(t-s)\big(c(V(s)) \\
&\qquad\qquad - c(U(s))\big) G(du,s) \Big] \Big\|^2_{L^{2,3}_{HS}} \Bigg] dsdr \\
&\leq \|U - V\|^2_{T,\gamma} T^3 (M'_G)^2 \int_0^T \alpha^2(s) e^{-\gamma s} ds.
\end{aligned}
$$

Putting everything together gives

$$
\begin{aligned}
&\sup_{t\in[0,T]} e^{-\gamma t} E\Bigg[\bigg\| \int_0^t S(t-s)\big(b(U(s)) - b(V(s))\big)ds \\
&\quad + \int_0^t S(t-s)\big(c(U(s)) - c(V(s))\big)dY(s)\bigg\|_{H_3}^2\Bigg] \\
&\leq C_\gamma \|U - V\|_{T,\gamma}
\end{aligned}
$$

To conclude the proof, it suffices, by the Banach fixed point theorem, to choose a γ large enough so that the constant C_γ is strictly smaller than 1. Then there exists a solution, unique up to modification.

Lastly, by almost the exact same approach as above, but instead using the linear growth assumption on c and the Malliavin derivative of c, we can show that the stochastic integral in (7.38) is well-defined. ■

Note that in Theorem 5.2.6 they assume that the domain of the coefficients is merely dense in the Hilbert space. I have chosen to skip this generalization for the sake of simplicity and instead assume that the domain equals the Hilbert space H_3.

CHAPTER 8

Fractional BSS processes

Where the previous chapters have focused on Lévy processes, which are semimartingales and satisfy the property of independent increments, this chapter will instead look at a type of processes called fractional Brownian motions (fBm). These processes are not semimartingales, nor do they have independent increments. This leads to complications not seen previously in this thesis. As before, we are concerned with SDEs, but since fractional Brownian motions are not semimartingales, even defining an integral of the form

$$\int_0^t a(s)dB^H(s), \tag{8.1}$$

where B^H denotes a fractional Brownian motion, is not so easy, the very general theory developed in [Pro10] will, for instance, not work. Therefore, there are many ways of defining integrals of the form (8.1). We will limit ourselves to three of these different ways and refer to [Mis08] or [Bia+08] for a treatment of more of these alternatives. As usual, and whenever relevant, we work in a complete probability space $(\Omega, \mathcal{F}, P)$. The fBm is defined as follows.

Definition 8.0.1 (fractional Brownian motion, [Bia+08])**.** Let H be a constant in $(0,1)$. A fractional Brownian motion $\{B_t^H\}_{t\geq 0}$ of Hurst index H is a continuous and centered Gaussian process with covariance function

$$E[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

For $H = 1/2$, the fBm is then a standard Brownian motion. By the above properties, the following properties can be deduced

1. $B_0^H = 0$ and $E[B_t^H] = 0$ for all $t \geq 0$.
2. B^H has stationary increments, i.e., $B_{t+s}^H - B_s^H$ has the same law as B_t^H for $s, t \geq 0$.
3. B^H is a Gaussian process and $E\big[(B_t^H)^2\big] = t^{2H}$, $t \geq 0$.
4. B^H has continuous trajectories (by Kolmogorov's theorem).

Since Brownian motions are also fractional Brownian motions with $H = 1/2$ it makes sense that fBms with $H \in (0, 1/2)$ and fBms with $H \in (1/2, 1)$ are, in a sense, two different types of processes, that need to be treated separately.

This turns out to be the case rather often, our focus is therefore targeted on fBms with Hurst parameter $H \in (1/2, 1)$, as these are the more well behaved of the two. From a visual point of view, one could say that the trajectories of fBms with $H \in (1/2, 1)$ are less "spiky" than those of fBms with $H \in (0, 1/2)$.

In this chapter we will consider the following SDE

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t g(t-s)\sigma(s)dB^H(s), \tag{8.2}$$

where X_0 is a random variable with values in $\mathbb{R}$. Before we can consider existence and uniqueness of a solution to (8.2) we must define the stochastic integral

$$Y_t := \int_0^t g(t-s)\sigma(s)dB^H(s),$$

where g and σ are defined as in 6.1.5. As we shall see, depending on the definition we must impose further restrictions on g and σ. We will call $\{Y_t\}_t$ a fractional Brownian semistationary (fBSS) process. The ways in which we define this process is, to my knowledge, new, the closest thing I have found is a PhD thesis [Ori15], but their definition is different.

In all three of the following sections we will need what is known as Riemann-Liouville integrals, they are closely connected with fractional Brownian motions as we shall see below. When considering the pathwise case we also need Riemann-Liouville derivatives and we therefore end this "zeroth" section with these definitions.

Definition 8.0.2 (Riemann-Liouville integral, [NR02], [Bia+08], [Mis08])**.** Let f be a deterministic real-valued function that belongs to $L^1(a, b)$, where (a, b) is a finite interval of $\mathbb{R}$. The fractional Riemann Liouville integrals of order $\alpha > 0$ are determined at almost every $x \in (a, b)$ and defined as the

1. Left-sided version:

$$(I^\alpha_{a+}f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)dy.$$

2. Right-sided version:

$$(I^\alpha_{b-}f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)dy.$$

Where $\Gamma(\alpha) = \int_0^\infty s^{\alpha-1}e^{-s}ds$ denotes the Gamma (or Euler) function. The Riemann–Liouville fractional integrals on $\mathbb{R}$ are defined as

$$(I^\alpha_{+}f)(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-y)^{\alpha-1} f(y)dy$$

and

$$(I^\alpha_{-}f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^\infty (y-x)^{\alpha-1} f(y)dy$$

respectively.

The Riemann-Liouville derivative is now defined, and rather naturally considering the name, it is closely linked to the Riemann-Liouville integral.

Definition 8.0.3 (Riemann-Liouville derivative, [NR02], [Bia+08])**.** Consider $\alpha < 1$. We define the fractional Liouville derivatives as

$$D^{\alpha}_{a+} f := \frac{d}{dx} I^{1-\alpha}_{a+} f$$

and

$$D^{\alpha}_{b-} f := \frac{d}{dx} I^{1-\alpha}_{b-} f$$

These definitions imply that for any $f \in L^1(a,b)$

$$D^{\alpha}_{a+} I^{\alpha}_{a+} f = f, \quad D^{\alpha}_{b-} I^{\alpha}_{b-} f = f.$$

The opposite order of operations gives the same result for a slightly smaller class of functions.

Definition 8.0.4 (Weyl representation, [NR02], [Bia+08])**.** We denote by $I^{\alpha}_{a+}(L^p(a,b))$ [respectively, $I^{\alpha}_{b-}(L^p(a,b))$] the family of functions f that can be represented as an I^{α}_{a+}-integral (respectively, I^{α}_{b-}-integral) of some function $\phi \in L^p(a,b)$, $p \geq 1$. Such ϕ is unique (in L^p sense) and coincides with $D^{\alpha}_{a+} f$ (respectively, with $D^{\alpha}_{b-} f$). In particular we denote by I^{α}_{a+} (respectively, I^{α}_{b-}) the map from $L^p(a,b)$ into $I^{\alpha}_{a+}(L^p(a,b))$ [respectively, $I^{\alpha}_{b-}(L^p(a,b))$]. This means that if $f \in I^{\alpha}_{a+}(L^p(a,b))$, we have

$$I^{\alpha}_{a+} D^{\alpha}_{a+} f = f,$$

and if $f \in I^{\alpha}_{b-}(L^p(a,b))$, we have

$$I^{\alpha}_{b-} D^{\alpha}_{b-} f = f.$$

Moreover, given $f \in L^p(a,b)$ the following Weyl representation holds:

$$D^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha-1}} dy \right],$$

and

$$D^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(y-x)^{\alpha-1}} dy \right]$$

for almost every $x \in (a,b)$. The convergence of the integrals at the singularity $y = x$ holds pointwise for almost every $x \in (a,b)$ and moreover in L^p sense if $1 < p < \infty$.

8.1 Wiener integrals

The first definition of (8.1) that will be looked at is the definition through Wiener integrals. This is the simplest definition we consider, but it only allows for deterministic integrands, hence we set $\sigma = 1$.

In [Mis08] they consider a two-sided fBm, that is, a process $\{\bar{B}^H_t\}_{t \in \mathbb{R}}$ with the same properties as in 8.0.1. Furthermore, for $\alpha = H - 1/2$, they define the operator

$$M^H_{\pm} f := \begin{cases} C_H I^{\alpha}_{\pm} f, & H \in (0, 1/2) \cup (1/2, 1), \\ f, & H = 1/2, \end{cases}$$

where,

$$C_H := \Gamma(H+1)\frac{(2H\sin(\pi H)\Gamma(2H))^{1/2}}{\Gamma(H+1/2)}.$$

They also define the space $L^2_H(\mathbb{R}) := \{f : M^H_- f \in L^2(\mathbb{R})\}$ equipped with the norm $\|f\|_{L^2_H(\mathbb{R})} = \|M^H_- f\|_{L^2(\mathbb{R})}$. As mentioned, the fBm is closely linked with Riemann-Louiville integrals and we have the following explicit relationship for any two-sided fBm $\bar{B}^H$ (see [Mis08])

$$\bar{B}^H_t = \int_{\mathbb{R}} (M^H_- \chi^D_{(0,t)})(s)d\bar{B}(s) \tag{8.3}$$

where $\{\bar{B}_s\}_{s\in\mathbb{R}}$ denotes a two-sided standard Brownian motion, and

$$\chi^D_{(a,b)}(t) = \begin{cases} 1, & a \leq t < b \\ -1, & b \leq t < a \\ 0, & \text{otherwise.} \end{cases}$$

The general definition of Wiener integrals w.r.t. fBms now follow.

Definition 8.1.1 (Wiener integral w.r.t. fBm, [Mis08]). Let $f \in L^2_H(\mathbb{R})$. Then the Wiener integral w.r.t. fBm is defined as

$$\int_{\mathbb{R}} f(s)d\bar{B}^H(s) := \int_{\mathbb{R}} (M^H_- f)(s)d\bar{B}(s),$$

where $\bar{B}^H$ and $\bar{B}$ are connected as in (8.3)

This integral is, of course, linear, it is also a centered Gaussian random variable, see Lemma 3.1.3 in [Bia+08].

With the above definition at hand we can define the fractional Brownian "semistationary" process via Wiener integrals. The quotation marks are used to indicate that the term semistationary is not really appropriate when the stochastic volatility σ is deterministic, see Remark 5 on page 23 in [BBV18]. Note also that

$$\begin{aligned}\int_{\mathbb{R}} (M^H_- f\chi_{(0,t)})(s)d\bar{B}(s) &= \int_{\mathbb{R}}\int_s^\infty (r-s)^{\alpha-1} f(r)\chi_{(0,t)}(r)drd\bar{B}(s) \\ &\int_0^t \int_s^t (r-s)^{\alpha-1} f(r)drd\bar{B}(s),\end{aligned}$$

where we have omitted writing the constant C_H from the definition M^H_-, and $1/\Gamma(\alpha)$ from the definition of the Riemann-Liouville integral.

Definition 8.1.2 (fBSS process via Wiener integrals). Let $g \in L^2_H(\mathbb{R})$. Then we define the fractional Brownian "semistationary" process $\{Y_t\}_t$ by

$$Y_t = \int_0^t g(t-s)dB^H(s) = \int_{\mathbb{R}} (M^H_- g(t-\cdot)\chi_{[0,t)})(s)d\bar{B}(s),$$

where we only care about the case where $H \in (1/2, 1)$.

For $H \in (1/2, 1)$, denote by $L^2_{H_2}([0,T])$ the space of functions ϕ satisfying

$$\int_0^T \int_0^T \phi(s)\phi(t)|s-t|^{2H-2} dsdt < \infty.$$

Using the following equality of [MMV01]

$$\begin{aligned} &E\Big[\int_0^T f(s)dB^H(s) \int_0^T g(s)dB^H(s)\Big] \\ &= H(2H-1)\int_0^T \int_0^T f(s)g(t)|s-t|^{2H-2} dsdt, \quad f,g \in L^2_{H_2}([0,T]) \end{aligned} \tag{8.4}$$

we can compute the autocovariance of an fBSS process.

Proposition 8.1.3 (Autocovariance of fBSS processes)**.** *Let $g(t-\cdot) \in L^2_{H_2}([0,T])$, and let X be an fBSS process. Then the autocovariance of X is given by*

$$\begin{aligned} &Cov(X(t+s), X(t)) \\ &= H(2H-1)\int_0^t \int_0^{t+h} g(t+h-s)g(t-u)|h-s+u|^{2H-2} dsdu \end{aligned}$$

Proof. By equality (8.4) we have

$$\begin{aligned} &\mathrm{Cov}(X(t+s), X(t)) = E[X(t+s)X(t)] - E[X(t+s)]E[X(t)] \\ &= E\Big[\int_0^{t+h} g(t+h-s)dB^H(s) \int_0^t g(t-u)dB^H(u)\Big] \\ &= E\Big[\int_0^{t+h} g(t+h-s)dB^H(s) \int_0^{t+h} \chi_{[0,t]}(u)g(t-u)dB^H(u)\Big] \\ &= H(2H-1)\int_0^{t+h}\int_0^{t+h} g(t+h-s)\chi_{[0,t]}(u)g(t-u)|t+h-s-(t-u)|^{2H-2} duds \\ &= H(2H-1)\int_0^{t+h}\int_0^{t} g(t+h-s)g(t-u)|h-s+u)|^{2H-2} duds, \end{aligned}$$

where the second equality follows by the fact that the expectation of fBSS processes are 0. ■

The process defined above admits a continuous modification under suitable conditions on g, but first we need a moment estimate on the Wiener integrals.

Theorem 8.1.4 (Moment estimate for Wiener integrals, [Mis08])**.** *Let $f \in L^{1/H}[a,b]$ and $f = 0$ outside (a,b), for $0 \leq a < b < \infty$. Then we obtain the following estimates: for any $r > 0$, there exists a constant $c(H,r)$, such that for $H \in (1/2, 1)$, it holds that*

$$E\Big[\Big|\int_a^b f(s)dB^H_s\Big|^r\Big] \leq c(H,r)\|f\|^r_{L^{1/H}[a,b]}$$

With the help of this estimate we can prove the existence of a continuous modification of an fBSS process.

Theorem 8.1.5 (Continuous modification of fBSS process)**.** *Let $Y(t) = \int_0^t g(t-s)dB^H(s)$ be defined as above. Assume that*

$$\int_0^t |g(t-s)|^r ds < \infty \tag{8.5}$$

and that g is Hölder continuous with exponent $\alpha = H - 1/r$, where $r > 4$ is even.

Proof. First, we apply inequality 2.3.1 in a similar way as in Theorem 6.1.9 to get

$$\begin{aligned}
|Y(t) - Y(s)|^r] &= \Big| \int_0^t g(t-s)dB_s^H + \int_0^u g(u-s)dB_s^H \Big|^r \\
&\leq C \Big| \int_0^u g(t-s) - g(u-s)dB_s^H \Big|^r + C \Big| \int_u^t g(t-s)dB_s^H \Big|^r .
\end{aligned}$$

Note that the assumption that r is even is necessary for the application of 2.3.1.

Now, (8.5) implies that $g(t-\cdot) \in L^{1/H}[u,t]$ since $r > 1/H$, so we can apply Theorem 8.1.4 and Hölder's inequality on the second term to get

$$\begin{aligned}
&E\bigg[\Big| \int_u^t g(t-s)dB_s^H \Big|^r \bigg] \leq c(H,r) \| g(t-\cdot) \|^r_{L^{1/H}(u,t)} \\
&= c(H,r) \Big(\int_u^t |g(t-s)|^{1/H} ds \Big)^{r \cdot H} \\
&\leq \bigg(\Big(\int_u^t |g(t-s)|^{1/H \cdot rH} ds \Big)^{\frac{1}{rH}} \Big(\int_u^t 1^{\frac{1}{1-\frac{1}{rH}}} ds \Big)^{1-\frac{1}{rH}} \bigg)^{rH} \\
&= c(H,r) \Big(\int_u^t |g(t-s)|^r ds \Big) \Big(\int_u^t ds \Big)^{rH-1} \\
&= (t-u)^{rH-1} c(H,r) \int_u^t |g(t-s)|^r ds
\end{aligned}$$

Estimating the first term with $\alpha = H - 1/r$ gives

$$\begin{aligned}
&E\bigg[\Big| \int_0^u (g(t-s) - g(u-s))dB_s^H \Big|^r \bigg] \leq c(H,r) \| g(t-\cdot) - g(u-\cdot) \|^r_{L^{1/H}(0,u)} \\
&= c(H,r) \Big(\int_0^u |g(t-s) - g(u-s)|^{1/H} ds \Big)^{r \cdot H} \\
&\leq c(H,r) \Big(\int_0^u |(t-s) - (u-s)|^{\alpha/H} ds \Big)^{r \cdot H} \\
&= (t-u)^{r\alpha} E\bigg[\Big| \int_0^u ds \Big|^{rH} \bigg] = (t-u)^{rH-1} \cdot u^{rH} \leq (t-u)^{rH-1} \cdot T^{Hr}
\end{aligned}$$

Where C depends on H and r. Putting everything together gives

$$E[|Y(t)-Y(u)|^r] \leq (t-u)^{rH-1}\Big(CT^{Hr} + c(H,r)\int_u^t |g(t-s)|^r ds\Big).$$

Since $r > 4$, we have that $rH - 1 > 1$, which allows us to apply Kolmogorov's theorem 2.3.3 and conclude that Y does have a continuous modification.

Note also that the above application of Hölder's inequality is legitimate with the assumption that $r > 4$, since that ensures $rH > 1$, which again implies $1/rH < 1$, $1 - 1/rH > 0$ and $1/(1 - 1/rH) > 1$. ■

Applying the above theorem we can proceed in the same fashion as in 7.2.15 to find a solution to (8.2).

Theorem 8.1.6. *Let Y be a fBSS process as defined in 8.1.2 and assume the conditions of Theorem 8.1.5. Then there exists a unique solution, up to indistinguishability, to equation* (8.2) *(with $\sigma = 1$).*

Proof. The existence follows immediately by Theorem 5.1.12 since Y has continuous sample paths under the conditions of Theorem 8.1.5, and the uniqueness follows by Theorem 5.1.2 ■

8.2 Pathwise integrals

In this section we will look at the same SDE, but this time the noise term is defined as a pathwise integral instead of a Wiener integral. The goal is hence to define the integral $\int_0^t g(t-s)\sigma(s)dB_s^H$ pathwise. The construction follows [NR02] and it begins by defining a generalized fractional Lebesgue-Stieltjes integral. But first we need the preliminary definition of what [Bia+08] terms "corrected" functions.

Definition 8.2.1 (Corrected functions, [Bia+08])**.**

$$f_{a+}(x) := \chi_{(a,b)}(f(x) - f(a+)),$$
$$f_{b-}(x) := \chi_{(a,b)}(f(x) - f(b-)),$$

provided that $f(a+) := \lim_{x\to a^+} f(x)$ and $f(b-) := \lim_{x\to b^-} f(x)$ exist.

$$f_{a+}(x) := \chi_{(a,b)}(f(x) - f(a+)),$$
$$f_{b-}(x) := \chi_{(a,b)}(f(x) - f(b-)),$$

Then we can define the generalized Lebesgue-Stieltjes integral.

Definition 8.2.2 (Generalized Lebesgue-Stieltjes integral, [NR02])**.** Suppose that f and g are functions such that $f(a+)$, $g(a+)$ and $g(b-)$ exist, $f_{a+} \in I^a_{\alpha+}(L^p)$ and $g_{b-} \in I^{1-\alpha}_{b-}(L^q)$ for some $p, q \geq 1$, $1/p + 1/q \leq 1$, $0 < \alpha < 1$. Then the generalized Lebesgue-Stieltjes integral of f with respect to g is defined by

$$\int_a^b f dg = \int_a^b D^\alpha_{a+} f_{a+}(x) D^{1-\alpha}_{b-} g_{b-}(x) dx + f(a+)(g(b-) - g(a+)). \quad (8.6)$$

Remark 8.2.3 ([NR02]). If $\alpha p < 1$, under the assumptions of the preceding definition we have $f \in I^{\alpha}_{a+}(L^p)$, and (8.6) can be rewritten as

$$\int_a^b f dg = \int_a^b D^{\alpha}_{a+} f(x) D^{1-\alpha}_{b-} g_{b-}(x) dx$$

which is determined for general functions $f \in I^{\alpha}_{a+}(L^p)$ and $g_{b-} \in I^{1-\alpha}_{b-}(L^q)$.

The following is gathered from appendix D.2 in [Bia+08]. In the following let $1/2 < H < 1$, $1 - H < \alpha < 1/2$. Denote by $W^{\alpha,\infty}_0(0,T)$ the space of measurable functions $f : [0,T] \to \mathbb{R}$ such that

$$\|f\|_{\alpha,\infty} := \sup_{t \in [0,T]} \left(|f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds \right) < \infty.$$

For any $0 < \lambda \leq 1$, denote by $C^{\lambda}(0,T)$ the space of λ-Hölder continuous functions $f : [0,T] \to \mathbb{R}$, equipped with the norm

$$\|f\|_{\lambda} := \|f\|_{\infty} + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{(t-s)^{\lambda}} < \infty$$

where $\|f\|_{\infty} := \sup_{t \in [0,T]} |f(t)|$. We have, for all $0 < \epsilon < \alpha$

$$C^{\alpha+\epsilon}(0,T) \subset W^{\alpha,\infty}_0(0,T) \subset C^{\alpha-\epsilon}(0,T).$$

Moreover, let $W^{1-\alpha,\infty}_T(0,T)$ be the space of measurable functions $g : [0,T] \to \mathbb{R}$ satisfying

$$\|g\|_{1-\alpha,\infty} := \sup_{0 < s < t < T} \left(\frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{(y-s)^{2-\alpha}} dy \right) < \infty.$$

Then, for all $0 < \epsilon < \alpha$,

$$C^{1-\alpha+\epsilon}(0,T) \subset W^{1-\alpha,\infty}_T(0,T) \subset C^{1-\alpha}(0,T).$$

Finally, we define the space $W^{\alpha,1}_0(0,T)$ of measurable functions $h : [0,T] \to \mathbb{R}$ such that

$$\|h\|_{\alpha,1} := \int_0^T \frac{|h(t)|}{t^{\alpha}} dt + \int_0^T \int_0^t \frac{|h(y) - h(t)|}{(t-y)^{\alpha+1}} dy dt < \infty.$$

If $g \in W^{1-\alpha,\infty}_T(0,T)$ and $h \in W^{\alpha,1}_0(0,T)$, then the generalized Stieltjes integral $\int_0^t h dg$ exists for all $t \in [0,T]$. We can now define the pathwise stochastic integral.

Definition 8.2.4 (Pathwise stochastic integral, [NR02])**.** Let $u = \{u_t\}_{t \in [0,T]}$ be a stochastic process whose trajectories belong to the space $W^{\alpha,1}_0(0,T)$, with $1 - H < \alpha < 1/2$, then the pathwise integral

$$\int_0^t u(s) dB^H_s$$

exists in the sense of 8.2.2

In light of the preceding definitions and discussions on fractional Lebesgue-Stieltjes integration we can define the fractional Brownian semistationary process through pathwise integration.

Definition 8.2.5 (Pathwise fractional Brownian semistationary process)**.** Fix $t \in [0,T]$ and assume the trajectories of $s \mapsto g(t-s)\sigma(s)$ is in $W_0^{\alpha,1}(0,T)$, with $1-H<\alpha<1/2$. Then we can define the pathwise fractional Brownian semistationary integral as

$$\int_0^t g(t-s)\sigma(s)dB_s^H,$$

in the sense of Definition 8.2.2

Now that we know that the stochastic integral in (8.2) is well-defined, with typical assumptions on the drift coefficient we can find a unique solution to (8.2).

Theorem 8.2.6 (Existence and uniqueness of pathwise integral driven fBSS process)**.** *Consider the SDE with fBm B_t^H, $H \in (1/2,1)$:*

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t g(t-s)\sigma(s)dB_s^H, \quad t \in [0,T],$$

where the stochastic integral is defined as in Definition 8.2.5. Furthermore, we assume that the coefficient b satisfies the following local Lipschitz property and growth assumption,

i) *for every $N \geq 0$ there exists $L_N > 0$ such that*

$$|b(t,x)-b(t,y)| \leq L_N|x-y|, \quad \forall |x|,|y| \leq N, \forall t \in [0,T],$$

ii) *there exists $b_0 \in L^p(0,T)$, where $p \geq 1/\alpha$ and $\alpha \in (1-H,1/2)$, such that*

$$|b(t,x)| \leq L_0|x| + b_0(t), \quad \forall x \in \mathbb{R}, \forall t \in [0,T].$$

Then there exists a unique solution, up to modification, of (8.2) *belonging to the space $L^0(\Omega,\mathcal{F},P;W^{\alpha,\infty}(0,T))$.*

Proof. Define $A : W_0^{\alpha,\infty}(0,T) \to C^{1-\alpha}(0,T) \subset W_0^{\alpha,\infty}(0,T)$ by

$$AX_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t g(t-s)\sigma(s)dB_s^H.$$

We then get

$$AX_t - AY_t = \int_0^t \big(b(s,X_s)-b(s,Y_s)\big)ds$$

and we can apply Theorem 2.1 in [NR02] to conclude that a solution, unique up to modification, exists. ∎

8.3 Skorohod integrals

Finally, in this chapter we introduce fBSS processes defined through the Skorohod integral, this is an extension of the Wiener integral in section 8.1, as the two integrals will coincide whenever the integrand is deterministic. Using this definition we also attempt to define integrals with respect to fBSS processes. This is meant as a fluid transition into the final chapter where we make suggestions to further work. Before we get to these considerations we need the definition of a Skorohod integral with respect to a fBm.

Recall the notation of section 8.1, in particular, that $\bar{B}^H$ denotes a two-sided fBm.

Definition 8.3.1 (Skorohod integral w.r.t. fBm, [Mis08])**.** Let the stochastic process $X = \{X_t\}_t$ be such that $(M_-^H X)$ exists and belongs to $Dom(\delta)$. Then we define the Skorohod integral with respect to the fBm $\bar{B}^H$ as

$$\int_{\mathbb{R}} X_t \delta \bar{B}^H(t) := \int_{\mathbb{R}} (M_-^H X)(t) \delta \bar{B}(t)$$

for the underlying two-sided Brownian motion $\bar{B}$.

As in section 8.1, we are only interested in the definition on $\mathbb{R}_+$, so from now on we can let B be the one-sided Brownian motion derived from $\bar{B}$.

Definition 8.3.2 (fBSS process via Skorohod integral)**.** Let the the kernel function g and the volatility σ be such that $s \mapsto (M_-^H g(t-\cdot)\sigma\chi_{[0,t]})(s)$ exists and belongs to $Dom(\delta)$, for all $0 \leq s \leq t$. Then we define the fractional semistationary process $X = \{X_t\}_t$ by

$$\begin{aligned} X(t) = \int_0^t g(t-s)\sigma(s)\delta B^H(s) &:= \int_{\mathbb{R}} (M_-^H g(t-\cdot)\sigma\chi_{[0,t)})(s)\delta\bar{B}(s) \\ &= \int_0^t (M_-^H g(t-\cdot)\sigma)(s)\delta B(s). \end{aligned}$$

Considering the above definition and comparing it to Definition 6.1.5, one might wonder what happens if we try to integrate against a fBSS process. In the following, we give one possible answer to this question, it is heavily inspired by the heuristic derivation of the integral with respect to an VMBV process that begins on page 118 in [BBV18], we follow in their path as far as possible. This calculation must also be considered purely heuristic, that is, we assume the computational steps are valid.

Let X be defined as in Definition 8.3.2, and let the process Y have differentiable paths. Then, from page 118-119 in [BBV18] we get,

$$\begin{aligned} \int_0^t Y(s)dX(s) = Y(t)X(t) &- \int_0^t \int_0^s Y'(s)(M_-^H g(s-\cdot)\sigma)(u)\delta B(u)ds \\ &- \int_0^t \int_0^s D_u\{Y'(s)\}(M_-^H g(s-\cdot)\sigma)(u)duds. \end{aligned}$$

From here on, we can no longer follow [BBV18] directly, but after applying both the regular Fubini theorem and the Fubini theorem for Skorohod integrals we end up with something quite similar. For simplicity, we omit writing $1/\Gamma(\alpha)$

from the definition of the Riemann-Liouville integral, and C_H from the definition of M_-^H, this practice is continued throughout the rest of this section whenever we do calculations involving M_-^H. Considering first the second term in the above equality, we have

$$\begin{aligned}
&\int_0^t \int_0^s Y'(s)(M_-^H g(s-\cdot)\sigma)(u)\delta B(u)ds \\
&= \int_0^t \int_u^t Y'(s) \int_u^s g(s-r)\sigma(r)(r-u)^{\alpha-1} dr ds \delta B(u) \\
&= \int_0^t \int_u^t \int_r^t Y'(s)g(s-r)\sigma(r)(r-u)^{\alpha-1} ds dr \delta B(u) \\
&= \int_0^t \int_u^t \int_r^t Y'(s)g(s-r)ds\sigma(r)(r-u)^{\alpha-1} dr \delta B(u)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^t \int_0^s D_u\{Y'(s)\}(M_-^H g(s-\cdot)\sigma)(u)duds \\
&\int_0^t \int_u^t \int_r^t D_u\{Y'(s)\}g(s-r)ds\sigma(r)(r-u)^{\alpha-1} drdu \\
&\int_0^t \int_u^t D_u\Big\{ \int_r^t Y'(s)g(s-r)ds\Big\}\sigma(r)(r-u)^{\alpha-1} drdu
\end{aligned}$$

On the first term we can apply the integration by parts formula 3.1.19, which yields

$$\begin{aligned}
&Y(t)X(t) = Y(t)\int_0^t (M_-^H g(t-\cdot)\sigma)(u)\delta B(u) \\
&= \int_0^t Y(t)(M_-^H g(t-\cdot)\sigma)(u)\delta B(u) + \int_0^t D_u\{Y(t)\}(M_-^H g(t-\cdot)\sigma)(u)du \\
&= \int_0^t \int_u^t Y(t)g(t-r)\sigma(r)(r-u)^{\alpha-1} dr\delta B(u) \\
&\quad + \int_0^t \int_u^t D_u\{Y(t)\}g(t-r)\sigma(r)(r-u)^{\alpha-1} drdu
\end{aligned}$$

Summing up,

$$\begin{aligned}
&\int_0^t Y(s)dX(s) \\
&= \int_0^t \int_u^t \Big[Y(t)g(t-r) - \int_r^t Y'(s)g(s-r)ds\Big]\sigma(r)(r-u)^{\alpha-1} dr\delta B(u) \\
&\quad + \int_0^t \int_u^t D_u\Big\{Y(t)g(t-r) - \int_r^t Y'(s)g(s-r)ds\Big\}\sigma(r)(r-u)^{\alpha-1} drdu.
\end{aligned}$$

From page 119 in [BBV18] we have

$$Y(t)g(t-r) - \int_r^t Y'(s)g(s-r)ds$$

$$= Y(t)g(t-r) - \int_r^t (Y(s) - Y(r)) \frac{\partial g(s-r)}{\partial s} ds.$$

Hence, we actually end up with the same operator K_g as before,

$$K_g(Y)(t,s) = Y(s)g(t-s) + \int_s^t (Y(u) - Y(s)) g(du).$$

Formally, we define integrals with respect to fBSS process as follows.

Definition 8.3.3 (Integral w.r.t. a fBSS process)**.** Suppose that for $s \in \mathbb{R}_+$ the mapping $t \mapsto G(t,s)$ is of bounded variation on $[u,v]$ for all $0 \le s < u < v < \infty$. Assume that the stochastic process $Y(s)$ on $0 \le s \le t$ for fixed $t > 0$ satisfies the following conditions:

1. For $s \in [0,t]$, the process $u \mapsto (Y(u) - Y(s))$, $s \le u \le t$, is integrable with respect to $g(du)$ a.s.

2. The mapping

$$s \mapsto K_g(Y)(t,s)\sigma(s)(s-r)^{\alpha-1}$$

   is Lebesgue integrable on $[0,t]$.

3. The mapping

$$r \mapsto \int_r^t K_g(Y)(t,s)\sigma(s)(s-r)^{\alpha-1} ds$$

   is Skorohod integrable.

4. $K_G(Y)(t,s)$ is Malliavin differentiable for $s \in [0,t]$, with

$$s \mapsto D_r\{K_g(Y)(t,s)\}\sigma(s)(s-r)^{\alpha-1}$$

   being Lebesgue integrable on $[0,t]$.

5. The mapping

$$r \mapsto \int_r^t D_r\{K_g(Y)(t,s)\}\sigma(s)(s-r)^{\alpha-1} ds$$

   is Lebesgue integrable on $[0,t]$.

Then, we define the integral with respect to a fBSS process in the following way

$$\begin{aligned} \int_0^t Y(s) dX(s) = \frac{C_H}{\Gamma(\alpha)} \Big(& \int_0^t \int_r^t K_g(Y)(t,s)\sigma(s)(s-r)^{\alpha-1} ds \delta B(r) \\ & + \int_0^t \int_r^t D_r\{K_g(Y)(t,s)\}\sigma(s)(s-r)^{\alpha-1} ds dr \Big), \end{aligned}$$

and say that Y is $\mathfrak{F}(0,t)$-integrable.

This integral satisfies several of the same properties as the integral in 6.2.1. Firstly, the integral of 1 gives the intuitive result:

$$\int_0^t 1dX(s) = \int_0^t \int_r^t K_g(1)(t,s)\sigma(s)(s-r)^{\alpha-1}ds\delta B(r)$$
$$+ \int_0^t \int_r^t D_r\{K_g(1)(t,s)\}\sigma(s)(s-r)^{\alpha-1}dsdr$$
$$= \int_0^t \int_r^t g(t-s)\sigma(s)(s-r)^{\alpha-1}ds\delta B(r) = \int_0^t (M_-^H g(t-\cdot)\sigma)\delta B(r) = X(t).$$

Furthermore, the integral is linear.

Proposition 8.3.4. *Let $a, b \in \mathbb{R}$ be constants and assume that the two processes Y, Z are $\mathfrak{F}$-integrable, then*

$$\int_0^t (aY(s) + bZ(s)))dX(s) = a\int_0^t Y(s)dX(s) + b\int_0^t Z(s)dX(s)$$

Proof. Follows by the linearity of the operator K_g, the linearity of the Malliavin derivative, the linearity of the Skorohod integral and the linearity of the Lebesgue integral. ∎

We also have the same localization property as the integral in 6.2.1.

Proposition 8.3.5. *Suppose that $s \mapsto Y(s) = 0$ for a.e. $s \leq t$, a.s. Then Y is $\mathfrak{F}(0,t)$-integrable and*

$$\int_0^t Y(s)dX(s) = 0, \quad a.s.$$

Proof. Follows in the exact same way as the proof of Proposition 23 in [BBV18], see page 140. ∎

Moreover, the integration by parts formula is analogous to 6.2.6.

Proposition 8.3.6 (Integration by parts formula for integrals w.r.t. fBSS process)**.** *Assume that $s \mapsto Y(s)$ is $\mathfrak{F}(0,t)$-integrable and let Z a bounded random variable such that $s \mapsto ZY(s)$ is $\mathfrak{F}(0,t)$-integrable. Then*

$$Z\int_0^t Y(s)dX(s) = \int_0^t ZY(s)dX(s).$$

Proof.

$$\int_0^t ZY(s)dX(s) = \int_0^t \int_r^t K_g(ZY)(t,s)\sigma(s)(s-r)^{\alpha-1}ds\delta B(r)$$
$$+ \int_0^t \int_r^t D_r\{K_g(ZY)(t,s)\}\sigma(s)(s-r)^{\alpha-1}dsdr$$

The first term can be written as follows:

$$\int_0^t \int_r^t K_g(ZY)(t,s)\sigma(s)(s-r)^{\alpha-1}ds\delta B(r)$$

$$
\begin{aligned}
&= \int_0^t Z \int_r^t K_g(Y)(t,s)\sigma(s)(s-r)^{\alpha-1} ds\delta B(r) \\
&= Z \int_0^t \int_r^t K_g(Y)(t,s)\sigma(s)(s-r)^{\alpha-1} ds\delta B(r) \\
&\quad - \int_0^t D_r\{Z\} \int_r^t K_g(Y)(t,s)\sigma(s)(s-r)^{\alpha-1} dsdr,
\end{aligned}
$$

where the first equality uses the calculation done in the start of the proof of 6.2.7, and the second equality follows by the integration by parts formula 3.1.19. Using the product rule 3.1.12, the second term can written as

$$
\begin{aligned}
&\int_0^t \int_r^t D_r\{K_g(ZY)(t,s)\}\sigma(s)(s-r)^{\alpha-1} dsdr \\
&= \int_0^t \int_r^t D_r\{Z\}K_g(Y)(t,s)\sigma(s)(s-r)^{\alpha-1} dsdr \\
&\quad + \int_0^t \int_r^t Z D_r\{K_g(Y)(t,s)\}\sigma(s)(s-r)^{\alpha-1} dsdr.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_0^t ZY(s)dX(s) &= Z \int_0^t \int_r^t K_g(Y)(t,s)\sigma(s)(s-r)^{\alpha-1} ds\delta B(r) \\
&+ Z \int_0^t \int_r^t D_r\{K_g(Y)(t,s)\}\sigma(s)(s-r)^{\alpha-1} dsdr = Z \int_0^t Y(s)dX(s)
\end{aligned}
$$

■

The last properties were all heavily inspired by the analogous properties found in section 4.3 in [BBV18]. Next, we prove similar properties to those found in section 6.4.

Proposition 8.3.7. *Let Y be a fBSS process according to definition 8.3.2 and let u be $\mathfrak{F}(0,t)$-integrable. Assume the random variable $F \in \mathbb{D}_{1,2}$, then*

$$
E\left[F \int_0^t u(s)dY(s)\right] = E\left[\int_0^t \int_r^t D_r\{K_G(u)(t,s)F\}\sigma(s)(s-r)^{\alpha-1} dsdr\right]
$$

Proof.

$$
\begin{aligned}
&E\left[F \int_0^t u(s)dY(s)\right] \\
&= E\left[F \int_0^t \int_r^t K_g(u)(t,s)\sigma(s)(s-r)^{\alpha-1} ds\delta B(r)\right. \\
&\quad \left. + F \int_0^t \int_r^t D_r\{K_g(u)(t,s)\}\sigma(s)(s-r)^{\alpha-1} dsdr\right] \\
&= E\left[\int_0^t \int_r^t D_r\{F\}K_g(u)(t,s)\sigma(s)(s-r)^{\alpha-1} dsdr\right. \\
&\quad \left. + \int_0^t \int_r^t F D_r\{K_g(u)(t,s)\}\sigma(s)(s-r)^{\alpha-1} dsdr\right] \\
&= E\left[\int_0^t \int_r^t D_r\{K_G(u)(t,s)F\}\sigma(s)(s-r)^{\alpha-1} dsdr\right],
\end{aligned}
$$

where the second equality follows by the duality formula 3.1.18 and the third equality follows by the product rule 3.1.12. ■

The "fundamental theorem of calculus" also takes on an analogous form to Proposition 6.4.5.

Proposition 8.3.8 (The fundamental theorem of calculus for fBSS integrals with constant volatility)**.** *Let Y be a fBSS process with volatility $\sigma = 1$. Assume that for all $a \in [0,t]$, $D_a\{u(\cdot)\}$ is $\mathfrak{F}(0,t)$-integrable. Then $\int_0^t u(s)dY(s) \in \mathbb{D}_{1,2}$ and*

$$D_a\Big(\int_0^t u(s)dY(s)\Big) = \int_0^t D_a u(s)dY(s) + \int_a^t K_g(u)(t,s)\sigma(s)(s-a)^{\alpha-1}ds \tag{8.7}$$

Proof. We first write out the case when σ is general and then set $\sigma = 1$ to get (8.7),

$$\begin{aligned}
D_a\Big(\int_0^t u(s)dY(s)\Big) &= D_a\Big(\int_0^t\int_r^t K_g(u)(t,s)\sigma(s)(s-r)^{\alpha-1}ds\delta B(r) \\
&+ \int_0^t\int_r^t D_r\{K_g(u)(t,s)\}\sigma(s)(s-r)^{\alpha-1}dsdr\Big) \\
&= \int_0^t\int_r^t D_a\{K_g(u)(t,s)\sigma(s)(s-r)^{\alpha-1}\}ds\delta B(r) \\
&+ \int_a^t K_g(u)(t,s)\sigma(s)(s-a)^{\alpha-1}ds \\
&+ \int_0^t\int_r^t D_a\big[D_r\{K_g(u)(t,s)\}\sigma(s)(s-r)^{\alpha-1}\big]dsdr \\
&= \int_0^t\int_r^t \Big[K_g(D_a u)(t,s)\sigma(s) + K_g(u)(t,s)D_a\{\sigma(s)\}\Big](s-r)^{\alpha-1}ds\delta B(r) \\
&+ \int_a^t K_g(u)(t,s)\sigma(s)(s-a)^{\alpha-1}ds \\
&+ \int_0^t\int_r^t \Big[D_r\{K_g(D_a u)(t,s)\}\sigma(s) \\
&+ D_r\{K_g(u)(t,s)\}D_a\{\sigma(s)\}\Big](s-r)^{\alpha-1}dsdr,
\end{aligned}$$

where the second equality use the fundamental theorem of calculus 3.1.20, and the third equality uses Lemma 6.4.4 and the product rule 3.1.12.

If now $\sigma = 1$, then $D_a\sigma(s) = 0$ and we have

$$\begin{aligned}
D_a\Big(\int_0^t u(s)dY(s)\Big) &= \int_0^t\int_r^t \{K_g(D_a u)(t,s)\}(s-r)^{\alpha-1}ds\delta B(r) \\
&+ \int_a^t K_g(u)(t,s)(s-a)^{\alpha-1}ds + \int_0^t\int_r^t D_r\{K_G(D_a u)(t,s)\}dsdr\Big) \\
&= \int_0^t D_a u(s)dY(s) + \int_a^t K_g(u)(t,s)\sigma(s)(s-a)^{\alpha-1}ds.
\end{aligned}$$

■

By comparing the above properties with the corresponding properties of integrals with respect to VMBV integrals, we can see that all of them takes on the analogous form. The fact that these integrals behave similarly is reasonable given that their respective definitions were derived by almost the exact same procedure.

CHAPTER 9

Summary and suggestions for further work

In chapter 8, we attempted to define fBSS processes and, inspired by the work in [BBV18], we tried to define an integral with respect to fBSS processes. In regards to this chapter there are many possible paths for further study. Most obviously, it seems natural to study fBSS processes in more detail and look at more properties than what is given in chapter 8. One could also define this process through other types of integration theory. Generalizing to a volatility modulated fractional Brownian-driven Volterra process or even to a volatility modulated fractional Lévy-driven Volterra process is also something that could be examined.

Moreover, the integral defined in 8.3.3 could be studied in further detail, and there might be "better" ways of defining such an integral. The study of SDEs driven by fBSS processes is yet another topic that could be explored in more detail. Lastly, it would be interesting to see what applications fBSS processes have, as of now, the research on this seems to be very limited, even on the special case where the kernel function $g(t-s) = e^{-\lambda(t-s)}$, for $\lambda > 0$, which could be termed a fractional Ornstein-Uhlenbeck process. One can consult chapter 7 in [Bia+08] and references therein, for a general discussion on fBms in finance and why its applications are limited.

In chapter 7 we studied S(P)DEs driven by VMLV processes and ambit fields, both in the case of nonlinear noise and the simpler case of purely time-dependent noise, or where the noise coefficient is just a constant. As we saw, the equations driven by purely time-dependent noise were fairly easy to prove existence and uniqueness for, and several proofs followed rather easily from results given in chapter 5. With this in mind, further study on S(P)DEs driven by VMLV processes and ambit fields ought to be done on the case of nonlinear, and not purely time-dependent, noise. Particularly, one might attempt to weaken the very strong conditions imposed on the noise coefficient, as noted, this seems to be very difficult and it might be impossible using Picard iteration, hence, other techniques are likely necessary to achieve this. Reducing the restrictions on the kernel function g is maybe more feasible.

Regarding section 6.4, there are also potential improvements. "The fundamental theorem of calculus" for integrals with respect to ambit fields is perhaps possible to prove, see the discussion at the end of section 6.4. Moreover, the results in the real valued-part of this section, where we assumed that the product rules of section 3.1 and 3.2 were applicable, might be generalized using density arguments.

Bibliography

[App09] Applebaum, D. *Lévy Processes and Stochastic Calculus*. Second. Cambridge University Press, Apr. 2009.

[BBV18] Barndorff-Nielsen, O. E., Benth, F. E. and Veraart, A. E. D. *Ambit Stochastics*. Vol. 88. Probability Theory and Stochastic Modelling. Cham: Springer International Publishing, 2018.

[Bia+08] Biagini, F., Hu, Y., Øksendal, B. and Zhang, T. *Stochastic Calculus for Fractional Brownian Motion and Applications*. Ed. by Gani, J., Heyde, C. C., Jagers, P. and Kurtz, T. G. Probability and Its Applications. London: Springer London, 2008.

[BK81] Barth, T. and Kussmaul, A. U. 'The Banach fixed point method for Ito stochastic differential equations'. In: *Annales scientifiques de l'Université de Clermont. Mathématiques* vol. 69, no. 19 (1981), pp. 1–8.

[BS16] Benth, F. E. and Süss, A. 'Integration Theory for Infinite Dimensional Volatility Modulated Volterra Processes'. In: *Bernoulli* vol. 22, no. 3 (Aug. 2016).

[Buc92] Buckdahn, R. 'Skorohod Stochastic Differential Equations of Diffusion Type'. In: *Probability Theory and Related Fields* vol. 93, no. 3 (Sept. 1992), pp. 297–323.

[CT06] Carmona, R. and Tehranchi, M. *Interest Rate Models: An Infinite Dimensional Stochastic Analysis Perspective*. Springer Finance. Berlin ; New York: Springer, 2006.

[Di +05] Di Nunno, G., Meyer-Brandis, T., Øksendal, B. and Proske, F. 'MALLIAVIN CALCULUS AND ANTICIPATIVE ITÔ FORMULAE FOR LÉVY PROCESSES'. In: *Infinite Dimensional Analysis, Quantum Probability and Related Topics* vol. 08, no. 02 (June 2005), pp. 235–258.

[Din00] Dinculeanu, N. *Vector Integration and Stochastic Integration in Banach Spaces*. Pure and Applied Mathematics. New York: Wiley, 2000.

[DZ14] Da Prato, G. and Zabczyk, J. *Stochastic Equations in Infinite Dimensions*. Second. Cambridge University Press, Apr. 2014.

[DØP09] Di Nunno, G., Øksendal, B. and Proske, F., eds. *Malliavin Calculus for Lévy Processes with Applications to Finance*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009.

[EPQ97] El Karoui, N., Peng, S. and Quenez, M. C. 'Backward Stochastic Differential Equations in Finance'. In: *Mathematical Finance* vol. 7, no. 1 (1997), pp. 1–71.

[Eva10] Evans, L. C. *Partial Differential Equations*. American Mathematical Soc., 2010.

[Eva14] Evans, L. C. *An Introduction to Stochastic Differential Equations*. Providence, Rhode Island: American Mathematical Society, Jan. 2014.

[Fol16] Folland, G. B. *A Course in Abstract Harmonic Analysis*. Zeroth. Chapman and Hall/CRC, Feb. 2016.

[GM11] Gawarecki, L. and Mandrekar, V. *Stochastic Differential Equations in Infinite Dimensions: With Applications to Stochastic Partial Differential Equations*. Probability and Its Applications. Berlin, Heidelberg: Springer Berlin Heidelberg, 2011.

[HE15] Hsing, T. and Eubank, R. L. *Theoretical Foundations of Functional Data Analysis, with an Introduction to Linear Operators*. Wiley Series in Probability and Statistics. Chichester, West Sussex, UK: John Wiley and Sons, Inc, 2015.

[Itô46] Itô, K. 'On a Stochastic Integral Equation'. In: *Proceedings of the Japan Academy, Series A, Mathematical Sciences* vol. 22, no. 2 (Jan. 1946).

[Jun18] Junghenn, H. D. *Principles of Real Analysis: Measure, Integration, Functional Analysis, and Applications*. Chapman and Hall/CRC, Apr. 2018.

[KS98] Karatzas, I. and Shreve, S. E. *Brownian Motion and Stochastic Calculus*. Vol. 113. Graduate Texts in Mathematics. New York, NY: Springer New York, 1998.

[Lin17] Lindstrøm, T. *Spaces: An Introduction to Real Analysis*. Pure and Applied Undergraduate Texts volume 29. Providence, Rhode Island: American Mathematical Society, 2017.

[LR15] Liu, W. and Röckner, M. *Stochastic Partial Differential Equations: An Introduction*. Universitext. Cham: Springer International Publishing, 2015.

[Mis08] Mishura, Y. S. *Stochastic Calculus for Fractional Brownian Motion and Related Processes*. Ed. by Morel, J. .-., Takens, F. and Teissier, B. Vol. 1929. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2008.

[Mit64] Mitrinović, D. S. *Elementary Inequalities,* P. Noordhoff, Jan. 1964.

[MMV01] Mémin, J., Mishura, Y. and Valkeila, E. 'Inequalities for the Moments of Wiener Integrals with Respect to a Fractional Brownian Motion'. In: *Statistics & Probability Letters* vol. 51, no. 2 (Jan. 2001), pp. 197–206.

[MW12] McDonald, J. N. and Weiss, N. A. *A Course in Real Analysis*. 2nd edition. Boston, Mass. ; Amsterdam: Academic Press, Jan. 2012.

[NR02] Nualart, D. and Rascanu, A. 'Differential Equations Driven by Fractional Brownian Motion.' In: *Collectanea Mathematica* vol. 53, no. 1 (2002), pp. 55–81.

[Nua06] Nualart, D. *The Malliavin Calculus and Related Topics*. 2nd ed. Probability and Its Applications. Berlin ; New York: Springer, 2006.

[Ori15] Orimar, S. 'Lévy Semistationary Models with Applications in Energy Markets'. English. PhD thesis. 2015.

[Pro10] Protter, P. E. *Stochastic Integration and Differential Equations: Version 2.1*. 2nd edition, corrected 3rd printing 2005. Stochastic Modelling and Applied Probability 21. Berlin Heidelberg: Springer, 2010.

[PZ07] Peszat, S. and Zabczyk, J. *Stochastic Partial Differential Equations with Levy Noise: An Evolution Equation Approach*. Cambridge: Cambridge University Press, 2007.

[RR89] Rajput, B. S. and Rosinski, J. 'Spectral Representations of Infinitely Divisible Processes'. In: *Probability Theory and Related Fields* vol. 82, no. 3 (Aug. 1989), pp. 451–487.

[RY99] Revuz, D. and Yor, M. *Continuous Martingales and Brownian Motion*. Ed. by Chern, S. S., Eckmann, B., De La Harpe, P., Hironaka, H., Hirzebruch, F., Hitchin, N., Hörmander, L., Knus, M.-A., Kupiainen, A., Lannes, J., Lebeau, G., Ratner, M., Serre, D., Sinai, Y. G., Sloane, N. J. A., Tits, J., Waldschmidt, M., Watanabe, S., Berger, M., Coates, J. and Varadhan, S. R. S. Vol. 293. Grundlehren Der Mathematischen Wissenschaften. Berlin, Heidelberg: Springer Berlin Heidelberg, 1999.

[Øks03] Øksendal, B. *Stochastic Differential Equations*. Universitext. Berlin, Heidelberg: Springer Berlin Heidelberg, 2003.