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Financial Markets with Delay

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Abstract

This thesis focuses on the study of financial markets with delay and in a broader sense, the study of stochastic differential equations with delay. We describe a continuous model, argue the existence, uniqueness and positivity of a solution to the stochastic differential equation chosen. We then prove non-arbitrage property as well as the completeness of the market. Numerical approaches are developed: the classic Euler-Maruyama scheme for delay equations and a so-called logarithmic Euler-Maruyama scheme.

A pure jump model is then considered. Existence, uniqueness and positivity of the solution to the stochastic differential equation describing the stock price dynamics are proven. We prove the non-arbitrage property and the incompleteness of the market. An extension of the classic Euler-Maruyama scheme with jumps is developed and an approach to hedging in such cases is then discussed.

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Contents

Introduction

The classical Black-Scholes model (1973) came as a revolution with its elegant option pricing formula. It is, however, not without faults. The assumption of constant volatility has been questioned and turns out to be quite far from reality. Stochastic volatility models such as the Heston or GARCH model tried to correct the model to fit the "skew" visible in the data. Another critique of the model is that there is no account of the past in the model. It is assumed that the current price of a stock gives us all the information that we could ever want. This critique (and hence all subsequent improvements on this front) is the main drive behind this (draft) thesis. It is natural to want to add such dependence into the model, say to represent the decisions of buyers and sellers basing their decision on previous prices or other phenomenon.

We will very briefly give a bit of background on the original model discovered by Black, Scholes and Merton in 1973 and then move onto models with memory in the dynamics of the price of the risky asset.
This will first be done in the continuous case. We will need to go back to mathematical fundamentals such as the existence of a solution to the functional differential equation modeling the dynamics of the risky asset as well as the conditions for its positivity. This will be followed by a study of the financial market where the risky asset follows these dynamics, such as the lack of arbitrage opportunities and the completeness of the market as a whole.

A numerical approach will be developed, a classic Euler-Maruyama type method as well as a different logarithmic Euler-Maruyama method. The latter guarantees positivity of each of the approximations, which is certainly a desirable trait. However, we have found that this kind of approximation is too restrictive. The growth rate of the coefficients of the functional differential equation need to be of the order $\log \log x$, among other things. This is by nature of the approximation and sadly we have not found a way to lower this requirement any further. Rate and order of convergence are also discussed and compared.

Jumps will then be added to the model. More specifically, we will then focus on a pure jump model. We will see that under this setting, the market loses its completeness property. This follows from the general theory and is not unique to markets where delay is introduced in the dynamics of the risky asset. We will once again first study the existence of a solution, and the conditions for its positivity. This will naturally lead into a market analysis where we will see exactly why the completeness is lost. Some claims are no longer replicable.

A numerical approach will also be developed for this case, but the logarithmic model was not adaptable to this setting. The hypotheses on the coefficients are too strong to get anything out of it in the jump case. A regular Euler-Maruyama method is preferred and developed. Rate and order of convergence are also discussed and compared.

Our last chapter will focus on the natural continuation of the market analysis. While harder, there are alternatives to the pricing and hedging issues encountered. We discuss the basics of pricing in a jump driven market and then discuss minimal variance hedging. If a perfect hedge cannot be constructed, we may try to get as close as possible (in L^2). The practical meaning of this is to focus on minimalizing the risk of the hedge. We will find this possible and discuss the shape of said hedge portfolio.

Notations

• $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space. These will be assumed to be complete, $\mathbb{P}$-augmented. Moreover, they will usually be equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$.

• B denotes a Brownian motion.

• a and b both denote time delays. R denotes the maximum delay, i.e $\max(a, b)$.

• $\mathbb{1}$ denotes the indicator function.

• ξ denotes the price of the riskless asset.

• S denotes the price of the risky asset.

• T denotes the time horizon.

• π often denotes a portfolio.

• $\mathbb{Q}$ often denotes an equivalent measure.

• $V(t)$ denotes the wealth process.

• Φ denotes the distribution of a $\mathcal{N}(0, 1)$ random variable.

• Z denotes a Lévy process.

• $\mathbb{D}_{1,2}$ denotes the subspace of L^2 of $\mathcal{F}_T$-measurable random variable possessing a chaos expansion (of finite norm).

• $\mathcal{D}F$ denotes the non-anticipative derivative of F.

• $D_{t,z}F$ denotes the Malliavin derivative of F.

Some Preliminaries and Definitions

In the more classical case, we consider a market made up of a risk-free asset of price $\xi(t)$ at time t and a risky asset (or assets) of price $S(t)$ at time t. We demand that the market is fair in the sense that we cannot see into the future, i.e that (ξ, S) is adapted with regards to a filtration $\mathbb{F}$, representing the information available to us. We consider derivatives based on those assets, namely call and put options, and are interested in a pricing formula for those. The problem, however, comes in the inherent risky nature of such derivatives and how the potential payoff is based on information we do not have: the price of the assets at the time horizon T.

This is where the Black-Scholes formula comes into play, it gives us a pricing formula at time $t = 0$ of both calls and puts. Let $S = (S(T) - K)^+$ (resp. $(K - S(T))^+$) denote the payoff of the call option of exercise price K (resp. put). Suppose that both ξ and S are regulated by the SDEs

$$\begin{cases} d\xi(t) &= r\xi(t)dt, \qquad t \in [0, T] \\ \xi(0) &= 1 \end{cases}$$

$$\begin{cases} dS(t) &= \mu dt + \sigma dB(t), \qquad t \in [0, T] \\ S(0) &= x, \ x > 0 \end{cases}$$

where $r, \mu, \sigma \neq 0$ are constants. Then the price of a call (resp. put) at time t is given by

$$V(t) = S(t)\Phi(\alpha_2) - Ke^{-r(T-t)}\Phi(\alpha_1)$$

(resp.)

$$V(t) = Ke^{-r(T-t)}\Phi(-\alpha_1) - S(t)\Phi(-\alpha_2)$$

where Φ is the cumulative distribution function of a centered normal random variable of variance 1,

$$\alpha_1 = \frac{1}{\sigma\sqrt{(T-t)}}\Big(\log\frac{x}{K} + \big(r + \frac{\sigma^2}{2}\big)(T-t)\Big)$$

and

$$\alpha_2 = \alpha_1 - \sigma\sqrt{(T-t)}$$

Note that a more general SDE can be considered for the dynamics of the risky asset. One can also go a step further and consider the hedging of these claims (lower bounded $\mathcal{F}_T$-measurable random variables), that is finding the portfolio replicating the claim. As we will see, in complete markets such as this one, the existence of such a portfolio is guaranteed, however its shape isn't necessarily easy to find. Before we go on, let us give definitions for the terms we have just used. We first need to define a self-financing and an admissible portfolio

• We say that a portfolio $\pi = (\pi_\xi, \pi_S)$ is self-financing if $\int_0^t \pi(u)\, dS(u)$ exists for all $t \in [0, T]$ and if

$$V_\pi(t) := (\pi_\xi, \pi_S(t)) \cdot (\xi(t), S(t))^T = V_0 + \int_0^t \pi_\xi(u)\, d\xi(u) \int_0^t \pi_S(u)\, dS(u)$$

i.e that the changes in the portfolio depend only on the changes in the asset prices themselves and not on external factors. This means that once the initial wealth of the portfolio V_0 is chosen, we are not able to withdraw or put in more money in the wealth process. It is however, possible, given the amount π_S invested in the risky asset, possible to choose V_0 and π_ξ such that π is self-financing. We can now define an admissible portfolio.

• A portfolio is called admissible if it is self-financing and if there exists a constant C_π such that

$$V_\pi(t) > -C_\pi$$

for all $t \in [0, T]$, i.e there is a threshold of debt that the one can take.

• Finally, we say that a claim F is replicable if there exists an admissible portfolio π and an initial value V_0 such that

$$F = V_\pi(T) = V_0 + \int_0^t \pi_\xi(u)\, d\xi(u) \int_0^t \pi_S(u)\, dS(u)$$

The replicability and hedging of a claim are important and at the heart of many studies in stochastic calculus.
An arbitrage opportunity is the existence of a portfolio π such that $V_0 = 0$, $V_\pi(T) \geq 0$ and $\mathbb{P}(V_\pi(T) > 0) > 0$. If there are no such π, then we say that the market does not allow for arbitrage. Moreover, if every claim is replicable, the market is called complete.

This paper has for goal to study these concepts in different financial markets with delay, i.e in various delay settings. We will for the most part not re-explain these concepts as they remain the same and we will assume that the reader is fairly familiar with them.

Chapter I

A Delayed Black-Scholes Model

I.1. Overview of the model

Throughout this chapter, we present models first developed by Arriojas et al. (see [AHMP07] and [AHMP06]).

1.1. General Framework

Consider the probability space equipped with a filtration $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F}$ is the natural filtration of a one-dimensional Brownian motion B. The integration with respect to B will be done in the classical Ito sense. Let T, R, b be positive constants with the only condition that $R \geq b$ and equip $\mathcal{C} = C([-R, 0], \mathbb{R})$ with the supremum norm. Let $\eta \in \mathcal{C}$ be an $\mathcal{F}_0$-measurable process and define the stochastic memory S_t by $S_t(s) := S(t+s)$, $s \in [-R, 0]$. We consider an SFDE of the form below

$$\begin{cases} dS(t) & = f(t, S_t)S(t)dt + g(S(t-b))S(t)dB(t), \qquad t \in [0, T] \\ S(t) & = \eta(t), \qquad t \in [-R, 0] \end{cases} \tag{1.1}$$

Remark. When writing $f(t, S_t)$, since $S(t) = S_t(0)$, this does not exclude a drift depending on $S(t)$. Technically, if we defined the stochastic memory for $s \in [-R, 0)$, then the drift would be written as $f(t, S(t), S_t)$. For readability's sake, we choose not to here.

Remark. Notice that the drift coefficient is taken to be quite general. One can imagine that potentially multiple delays of the form $S(t - a_1) \dots S(t - a_n)$ can be considered and even delays in the form of integrals of S_t. For this chapter however, we assume a general drift $f(t, S_t)$ as the following results are not exclusive to particular types of drift delays and the drift does not in fact play a big role in the main results of this chapter. In particular cases where a specific drift is chosen we will consider delays of the form $f(S(t-a))$, a drift with a single delay $a > 0$.

Remark. Also note that while our volatility coefficient may also be generalized somewhat, since it plays a much bigger role in the computations, we have chosen for readability's sake a drift with a single discrete delay. The results we present however are also valid for delays of the form $S(t-b_1)\dots S(t-b_n)$ or extendable for different forms of delays.

To ensure that our model for the stock price exists, it is necessary to determine under what hypotheses (if any) a solution to (1.1) exists. Moreover, for it to have any meaning, it is also necessary that said solution is not only unique but positive at all times as stock prices cannot be negative. Let us first list a familiar set of assumptions under which we will see that (1.1) is well-posed.

Hypotheses (H)

(i) (Global Linear Growth in the drift coefficient) $\exists L > 0$ s.th $\forall (t,\varphi) \in [0,T] \times \mathcal{C}$,

$$||f(t,\varphi)|| \leq L(1 + ||\varphi||)$$

(ii) (Local Lipschitzianity of the drift coefficient) $\forall n \geq 1$, $\exists L_n > 0$ s.th $\forall (t,\varphi^i) \in [0,T] \times \mathcal{C},\ i = 1,2$,

$$||f(t,\varphi^1) - f(t,\varphi^2)|| \leq L_n ||\varphi^1 - \varphi^2||$$

(iii) (Continuity and Boundedness of the volatility coefficient) The function $g : \mathbb{R} \to \mathbb{R}$ is continuous, bounded and $g \neq 0$.

Theorem 1.1.1. *(Existence and uniqueness theorem) Under the hypotheses (H), the SFDE (1.1) has a pathwise unique solution.*

Proof. Begin by limiting the range of t to $[0, \min(a,b)]$ where $a > 0$ is the delay in the stochastic coefficient. The reason for this is to replace the stochastic, troublesome $f(t,S_t)$ and $g(S(t-b))$ by the much easier to work with $f(t,\eta(t))$ and $g(\eta(t-b))$ since they coincide on the interval $[0,\min(a,b)]$. Define the martingale

$$M(t) := \int_0^t f(t,\eta(t))\,dt + \int_0^t g(\eta(t-b))\,dB(t)$$

then

$$dS(t) = S(t)dM(t) \tag{1.2}$$

and we recognize a well known stochastic exponential, of which a unique solution exists. Thus, we know the shape of the solution, that is

$$S(t) = \eta(0)exp\Big(\int_0^t f(t,\eta(t)) - \frac{1}{2}g^2(\eta(t-b))\,dt + \int_0^t g(\eta(t-b))\,dB(t)\Big) \tag{1.3}$$

We repeat this process for time steps of length $\min(a,b)$ and the proof is complete.

□

Remark. To emphasize the dependence on the initial condition η, we sometimes use an anticipated super index ${}^{\eta}S(t)$.

Remark. As is reviewed in Appendix A, one might apply theorem I.1 of [Moh98] to (1.1). However, obtaining the uniqueness via this theorem does not offer any insight on what sufficient conditions to add to ensure positivity of the solution. The shape of the solution is key and necessary. As we will see in the next result, this proof allows us to glean a sufficient positivity condition.

Corollary 1.1.1. *(A Sufficient Condition for Positivity) Consider the SFDE (1.1). If $\eta(0) > 0$, then the unique solution $S(t)$ is strictly positive for all $t \geq 0$.*

Proof. We use the form of the solution we showed in (1.3). Clearly, if $\eta(0) > 0$, then $S(t) > 0$. □

Remark. We are treating a very special case of drift in this paper, mainly that of $f(t, S_t)S(t)$. While unnecessary for the scope of this paper, one might imagine that the solution is not of exponential shape and that the drift is simply $f(t, S_t)$. In this case, how then do we proceed? Once again, it is necessary to construct the solution in proving its existence and uniqueness, so as to prove a sufficient condition for positivity. The way to construct the solution is a bit trickier. In this case, one would consider the martingale

$$\begin{cases} d\psi(t) &= \psi(t)dM(t), \qquad t \in [0, T] \\ \psi(t) &= 1, \qquad t \in [-R, 0] \end{cases}$$

and then the SFDE

$$\begin{cases} dy(t) &= \psi(t)^{-1} f(t, \psi_t y_t)\, dt \\ y(t) &= \eta, \qquad t \in [-R, 0] \end{cases}$$

We need not know what the shape of y is, only that it exists. Hence, we use theorem I.1 of [Moh98] to prove existence and uniqueness of y. Then, we define $S(t) := \psi(t)y(t)$ and in fact, by construction, we have that $S(t)$ is indeed a solution of our original SFDE. Moreover, by uniqueness of ψ and y, then $S(t)$ is also unique. Finally, we see that if $f > 0$, then y is increasing. Because $\psi(t) > 0$ for all $t \geq 0$, it is easy to see that then $\eta(0) > 0$ implies $S(t) > 0$. Hence, we keep positivity of the solution but under an additional sufficient condition, mainly that f is strictly positive.

1.2. A complete Market

A natural question following our inquiry about whether or not (I.1) is a well-posed model for the stock price is to verify whether or not arbitrage is allowed in our modeled market and whether or not that market is complete. In this section, we aim to those questions as well in a somewhat classical way, that is by applying Girsanov's theorem.
As mentioned before, we will here restrict ourselves to one case of drift, namely $f(t, S_t)S(t)$. Hence, we suppose that the market is regulated by the SFDE (1.1), i.e

$$\begin{cases} dS(t) &= f(t, S_t)S(t)dt + g(S(t-b))S(t)dB(t), \qquad t \in [0, T] \\ S(t) &= \eta, \qquad t \in [-R, 0] \end{cases}$$

where $S(t)$ represents the price of the risky assets and ξ represents the price of the riskless asset, of constant interest rate $r > 0$.

Theorem 1.1.2. *The model does not allow for arbitrage. Moreover, the market is complete.*

Proof. • Let

$$\widetilde{S}(t) := e^{-rt}\, S(t)$$

denote the discounted stock price. Applying Ito's formula to the function $g(t, x) = e^{-rt}x$, we find that

$$\begin{aligned} d\widetilde{S}(t) &= e^{-rt}\, dS(t) + S(t)\, d(e^{-rt}) = e^{-rt}\big(f(t, S_t)S(t)dt + g(S(t-b))S(t)dB(t)\big) - re^{-rt}\, S(t)\, dt \\ &= \underbrace{e^{-rt}\, S(t)}_{=\widetilde{S}(t)}\big((f(t, S_t) - r)\, dt + g(S(t-b))\, dB(t)\big) \\ &= \widetilde{S}(t)\big((f(t, S_t) - r)\, dt + g(S(t-b))\, dB(t)\big) \end{aligned} \tag{1.6}$$

This beckons us to define the process

$$\widehat{S}(t) = \int_0^t f(t, S_u) - r\, dt + \int_0^t g(S(u-b))\, dB(u) \tag{1.6}$$

Then

$$d\widetilde{S}(t) = \widetilde{S}(t)\, d\widehat{S}(t) \tag{1.7}$$

i.e

$$\widetilde{S}(t) = \phi(0) + \int_0^t \widetilde{S}(t)\, d\widehat{S}(t) \tag{1.8}$$

Now define

$$\phi(t) = -\frac{f(t, S_t) - r}{g(S(t-b))}$$

and

$$Z(t) = \int_0^t \phi(u)\, dB(u) - \frac{1}{2}\int_0^t \phi^2(u)\, du$$

Note that to ensure that those two are well defined, we must impose conditions so that the denominator of $\phi(t)$ does not vanish. Since $S(t) > 0$ by theorem 1.1, it suffices to impose the condition that $g(x) \neq 0$ if $x > 0$. The aim is to apply Girsanov's theorem to $d\mathbb{Q} := Z(T)\, dP$. It is however necessary to first verify that the conditions for the theorem are met, i.e that $\phi \in M^2_{\text{loc}}([0, T])$ and that Z is a $\mathbb{P}$-martingale.

The process ϕ is progressively measurable, as $\phi(t)$ is $\mathcal{F}$-adapted (even predictable if the drift coefficient doesn't include the present price) and right-continuous. The square integrability of ϕ comes from pathwise continuity, which implies that it is almost surely

bounded on the finite interval $[0, T]$. Moreover, the fact that g is non-zero on the given interval ensures that $\frac{1}{g}$ also stays bounded on finite intervals, hence ϕ belongs to $L^2([0, T])$, as $[0, T]$ is obviously finite.
It remains to show that Z is a $\mathbb{P}$-martingale, which requires a bit more work. If once again the delay in the drift coefficient is $a > 0$, one can use the predictability of the process with respect to the filtration $\mathcal{F}_{t-l}$ where $l = \min(a, b)$. We condition on the past with time steps l, using that $\int_{T-l}^{T} \phi(u)\, dB(u)$ is centered and normally distributed of variance $\int_{T-l}^{T} \phi^2(u)\, du$. Then, it is possible to prove that $\mathbb{E}[Z(t)] = 1$ i.e the Novikov condition is satisfied and Z is indeed a $\mathbb{P}$-martingale. We refer to [AHMP07].

With the conditions for Girsanov's theorem being verified, it follows that the process

$$\widetilde{B}(t) = B(t) + \int_0^t \phi(u)\, du$$

is a standard Brownian motion w.r.t $\mathbb{Q}$. Finally, thanks to the way we have defined ϕ, we have

$$dB(t) = d\widetilde{B}(t) - \frac{f(t, S_t) - r}{g(S(t-b))} dt$$

and hence, (1.7) can be rewritten as

$$\widehat{S}(t) = \int_0^t f(u, S_u) - r\, du - \int_0^t \cancel{g(S(u-b))} \frac{f(t, S_u) - r}{\cancel{g(S(u-b))}}\, dt + \int_0^t g(S(u-b))\, d\widetilde{B}(u) = \int_0^t g(S(u-b))\, d\widetilde{B}(u)$$

i.e $\widehat{S}(t)$ can be expressed as an Ito integral w.r.t a standard Brownian motion under the measure $\mathbb{Q}$. It is thus a $\mathbb{Q}$-local martingale (and even a $\mathbb{Q}$-martingale). By (1.7), it follows that $\widetilde{S}$ is also a $\mathbb{Q}$-local martingale. By the fundamental theorem of asset pricing, we can thus deduce that the market does not allow for arbitrage.

• It remains to show that the market is complete under the measure $\mathbb{Q}$. Take a contingent claim (i.e an option) of payoff function X, that is a $\mathcal{F}_t$-measurable random variable, and define

$$U(t) := \mathbb{E}_{\mathbb{Q}}[e^{-rt} X | \mathcal{F}_t]$$

Thus, U is a $\mathbb{Q}$-martingale and hence we can apply the martingale representation theorem. There exists a ($\mathcal{F}_t$-)predictable, L^2 process H such that

$$U(t) = \mathbb{E}_{\mathbb{Q}}[e^{-rt} X] + \int_0^t H(u)\, d\widetilde{B}(u) \tag{1.9}$$

Next, define the strategy for the risky assets like so:

$$\pi_S(t) := \frac{H(t)}{\widetilde{S}(t) g(S(t-b))} \tag{1.10}$$

and compensate the portfolio with the riskless asset as usual:

$$\pi_\xi(t) := U(t) - \pi_S(t)\widetilde{S}(t) \tag{1.11}$$

If $V(t)$ now denotes the value of the portfolio at time t, then

$$V(t) = \pi_\xi(t)e^{rt} + \pi_S(t)S(t) = e^{rt}U(t) - \pi_S \underbrace{e^{rt}\,\widetilde{S}(t)}_{=S(t)} + \pi_S S(t) = e^{rt}U(t) \tag{1.12}$$

Taking $t = T$, it is clear that

$$V(T) = e^{rt}\mathbb{E}_{\mathbb{Q}}[e^{-rt}X|\mathcal{F}_T] = \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_T] = X\mathbb{E}_{\mathbb{Q}}[1|\mathcal{F}_T] = X$$

where we used that X is $\mathcal{F}_T$-measurable. Moreover, by the Ito product rule applied to (1.12),

$$dV(t) = U(t)de^{rt} + e^{rt}dU(t)$$

Taking it one term at a time, we have by definition

$$U(t)de^{rt} = \pi_\xi(t)de^{rt} + \pi_S(t)\tilde{S}(t)de^{rt} = \pi_\xi(t)de^{rt} + r\pi_S(t)S(t)dt$$

As for the second term, things get slightly more complicated. By (1.9), we have

$$e^{rt}dU(t) = e^{rt}H(t)d\widetilde{B}(t) = e^{rt}H(t)dB(t) + e^{rt}H(t)\frac{f(t, S_t) - r}{g(S(t-b))}dt$$

By (1.11), we have $H(t) = \pi_S(t)\tilde{S}(t)g(S(t-b))$ and hence

$$e^{rt}dU(t) = e^{rt}\pi_S(t)\tilde{S}(t)g(S(t-b))dB(t) + e^{rt}\pi_S(t)\tilde{S}(t)f(t, S_t)dt - re^{rt}\pi_S(t)\tilde{S}(t)dt$$

Grouping up the terms, using that $S(t) = e^{rt}\tilde{S}(t)$, we find that

$$e^{rt}dU(t) = \pi_S(t)\underbrace{\big(f(t, S_t)S(t)dt + g(S(t-b))S(t)dB(t)\big)}_{=dS(t)} - r\pi_S(t)S(t)dt$$

Hence, both leftover terms from A and B cancel out and we get

$$dV(t) = \pi_\xi(t)\underbrace{de^{rt}}_{=d\xi(t)} + \pi_S(t)\,dS(t)$$

i.e, the change in the value of the portfolio only depends on the changes in price of the risky and riskless assets: the portfolio is self-financing. Hence the market is complete.

□

Remark. The measure $\mathbb{Q} = {}^{\eta}\mathbb{Q}$ is, by construction, dependent on the initial condition. A legitimate question to have would be to study exactly how strong the dependence truly is. For this study of the sensitivity of the initial condition, we will refer to [BDNHP16].

Remark. The choice of interest rate for the riskfree asset can be extended to a deterministic $r(t)$. With a stochastic delay, one loses the Markov property however. This is discussed in [Moh84].

Remark. We remind the reader that to satisfy the no-arbitrage property, contingent claims of payoff X must necessarily have a unique price at time t given by

$$X_0 = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}X|\mathcal{F}_t]$$

One way to verify this is to run the numbers and observe that in the case that the price is higher than that, a broker has an arbitrage opportunity on the riskfree asset. A similar computation can be run for an investor if the price is lower than the above one. This not only ties in with the proof of the previous theorem giving us a hedging strategy for the contingent claim, but it will also be very important for our next theorem as this is the starting point from which we derive a Black-Scholes formula for European options in our model.

1.3. A Pricing Formula for the European Call and Put options, Hedging

Theorem 1.1.3. *(A Pricing Formula for the European Call Option) Assume that* $f = f_1$*. Let* X *represent the payoff of a European call option of exercise price* K *with maturity* T*, that is* $X = (S(T) - K)^+$*. Then the value (fair price) of that option at time* t *is given by*

$$V(t) = \begin{cases} e^{rt}\mathbb{E}_{\mathbb{Q}}\big[\Sigma\big(\widetilde{S}(T-b), -\frac{1}{2}\int_{T-b}^{T} g^2(S(u-b))\,du, \int_{t-b}^{T} g^2(S(u-b))\,du\big)\big|\mathcal{F}_t\big] & , t \in [0, T-b) \\ S(t)\Phi(\beta_+(t)) - Ke^{-r(T-t)}\Phi(\beta_-(t)) & , t \in [T-b, T] \end{cases}$$

where Φ *is the distribution of a* $\mathcal{N}(0,1)$ *random variable, where*

$$\beta_\pm(t) := \frac{\frac{\log S(t)}{K} + r(T-t) \pm \frac{1}{2}\int_t^T g^2(S(u-b))\,du}{\big(\int_t^T g^2(S(u-b)))\,du\big)^{\frac{1}{2}}}$$

and where

$$\Sigma(x, m, \sigma^2) := \mathbb{E}_{\mathbb{Q}}\big[(xe^{m+\sigma N} - Ke^{-rT})^+\big],\ \sigma, x > 0,\ m \in \mathbb{R}$$

Note that this pricing formula is only really explicit for the last delay segment $[T-b, T]$ (note as well that this is not dependent on the drift delay!) which is certainly useful, but not quite perfect. Indeed, before this period, it is very hard to compute analytically and it is usually approximated using Monte Carlo methods for example. For examples of numerical approximations, we refer to the next section.
Before we prove the theorem, let us get rid of the most tedious computational details first in the form of the following lemma.

Lemma 1.1.1. *Let* ξ $\mathcal{N}(0,1)$ *and define the map* Σ *by*

$$\Sigma(x, m, \sigma^2) := \mathbb{E}_{\mathbb{Q}}\big[(xe^{m+\sigma\xi} - Ke^{-rT})^+\big]$$

then

$$\Sigma(x, m, \sigma^2) = x^{m+\frac{\sigma^2}{2}\Phi(\alpha_2)} - xKe^{-rT}\Phi(\alpha_1)$$

where

$$\alpha_1 = \alpha_1(x, m, \sigma) := \frac{1}{\sigma}\Big(\log\frac{x}{K} + m + rT\Big)$$

and

$$\alpha_2 = \alpha_2(x, m, \sigma) = \alpha_1 + \sigma$$

Proof. The positivity condition in the expression of Σ is equivalent to

$$\log\frac{x}{K} + m + \sigma\xi + rT \geq 0$$

which can be written as

$$\alpha_1 + \xi \geq 0$$

where

$$\alpha_1 = \alpha_1(x, m, \sigma) := \frac{1}{\sigma}\Big(\log\frac{x}{K} + m + rT\Big)$$

Thus, we have

$$\Sigma(x, m, \sigma^2) = \mathbb{E}_{\mathbb{Q}}[(xe^{m+\sigma\xi} - Ke^{-rT})\mathbb{1}_{\{\alpha_1+\xi\geq 0\}}] = \int_{-\alpha_1}^{\infty}\big(xe^{m+\sigma y} - Ke^{-rT}\big)\frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}\,dy$$
$$= \int_{-\infty}^{\alpha_1}\big(xe^{m-\sigma y} - Ke^{-rT}\big)\frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}\,dy \tag{1.14}$$

where we have used that $\xi \sim \mathcal{N}(0, 1)$. Now, separating (1.14) integral as the difference of two integrals, we have

$$\Sigma(x, m, \sigma^2) = x\int_{-\infty}^{\alpha_1} e^{m+\sigma y}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy - xKe^{-rT}\int_{-\infty}^{\alpha_1}\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy = \mathcal{I}_1 + \mathcal{I}_2$$

From a first glance, we see that

$$\mathcal{I}_2 = xKe^{-rT}\Phi(\alpha_1)$$

where Φ is the cumulative distribution function of a centered normal variable with variance 1. As for $\mathcal{I}_1$, executing the change of variable $z = y + \sigma$, we find

$$\mathcal{I}_1 = x\int_{-\infty}^{\alpha_1} e^{m+\frac{\sigma^2}{2}}\frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\,dz = xe^{m+\frac{\sigma^2}{2}}\Phi(\alpha_2)$$

where $\alpha_2 = \alpha_2(x, m, \sigma) := \alpha_1 + \sigma$. Thus,

$$\Sigma(x, m, \sigma^2) = xe^{m+\frac{\sigma^2}{2}}\Phi(\alpha_2) - xKe^{-rT}\Phi(\alpha_1)$$

□

With this out of the way, we now proceed to prove the pricing formula for European options.

Proof of theorem 1.1.4. We will only detail the case $X = (S(T) - K)^+$ as the other case is analogous. We start this proof from the previous remark, i.e that

$$V(t) = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}_t] = e^{rt}\mathbb{E}_{\mathbb{Q}}[(\widetilde{S}(T) - e^{-rT}K)^+|\mathcal{F}_t]$$

Because of the stochastic exponential shape of $\tilde{S}(t)$ in (1.6), we have

$$\widetilde{S}(t) = \phi(0)exp\Big(\int_t^T g(S(u-b))\,d\widetilde{B}(u) - \frac{1}{2}\int_t^T g(S(u-b))^2\,du\Big)$$

First of all, $S(t)$ is $\mathcal{F}_t$-measurable by definition and so is $\widetilde{S}(t)$. Suppose now that $t \in [T-b, T]$. The same then follows for the delayed $S(t-b)$ and thus $\int_t^T g(S(u-b))^2\,du$. Let us now study the remaining term. Given the information at time t, i.e when conditioning w.r.t $\mathcal{F}_t$, for $t \in [T-b, T]$, the function $g(S(t-b))$ becomes "deterministic" and hence $\int_t^T g(S(u-b))\,d\widetilde{B}(u)$ follows a centered normal distribution with $\sigma^2 = \int_t^T g(S(u-b))^2\,du$. This allows the computation of the conditional expectation (which can be expressed analytically!). Thus, we conclude that

$$V(t) = e^{rt}\Sigma\Big(\widetilde{S}(t), -\frac{1}{2}\int_t^T g(S(u-b))^2\,du, \int_t^T g(S(u-b))^2\,du\Big)$$

where

$$\Sigma(x, m, \sigma^2) := \mathbb{E}_{\mathbb{Q}}[(xe^{m+\sigma\xi} - Ke^{-rT})^+]$$

and where $\xi \sim \mathcal{N}(0, 1)$. A few computational tricks similar to the original Black-Scholes proof are required to transform this expression into what we originally promised. With our original values for m and σ, we have $m + \frac{\sigma^2}{2} = 0$ and so, plugging back in the original values for each of these in the expression for $V(t)$ as well as using the above lemma, we have

$$\begin{aligned}\alpha_1\Big(\widetilde{S}(t), -\frac{1}{2}\int_t^T g(S(u-b))^2\,du, \int_t^T g(S(u-b))^2\,du\Big) &= \frac{\log\frac{\widetilde{S}(t)}{K} + rT - \frac{1}{2}\int_t^T g(S(u-b))^2\,du}{\sqrt{\int_t^T g(S(u-b))^2\,du}} \\ &= \frac{\log\frac{S(t)}{K} + r(T-t) - \frac{1}{2}\int_t^T g(S(u-b))^2\,du}{\sqrt{\int_t^T g(S(u-b))^2\,du}}\end{aligned}$$

Similarly,

$$\alpha_2\Big(\widetilde{S}(t), -\frac{1}{2}\int_t^T g(S(u-b))^2\,du, \int_t^T g(S(u-b))^2\,du\Big) = \frac{\log\frac{S(t)}{K} + r(T-t) + \frac{1}{2}\int_t^T g(S(u-b))^2\,du}{\sqrt{\int_t^T g(S(u-b))^2\,du}}$$

Hence

$$V(t) = \underbrace{e^{rt}\widetilde{S}(t)}_{=S(t)}\Phi(\beta_+) - Ke^{-r(T-t)}\Phi(\beta_-) = S(t)\Phi(\beta_+) - Ke^{-r(T-t)}\Phi(\beta_-) \tag{1.14}$$

where

$$\beta_\pm = \frac{\log\frac{S(t)}{K} + r(T-t) \pm \frac{1}{2}\int_t^T g(S(u-b))^2\,du}{\sqrt{\int_t^T g(S(u-b))^2\,du}}$$

which finishes the first part of our proof. It now remains to consider what happens if $t < T-b$. In which case, one can similarly deduce that

$$V(t) = e^{rt}\mathbb{E}_{\mathbb{Q}}\Big[\Sigma\Big(\widetilde{S}(T-b), -\frac{1}{2}\int_{T-b}^T g^2(S(u-b))\,du, \int_{t-b}^T g^2(S(u-b))\,du\Big)\Big|\mathcal{F}_t\Big]$$

using the representation of $\widetilde{S}(T)$ as below

$$\widetilde{S}(T) = \widetilde{S}(T-b)exp\Big(\int_{T-b}^T g(S(u-b))\,d\widetilde{B}(u) - \frac{1}{2}\int_{T-b}^T g(S(u-b))^2\,du\Big)$$

which concludes the proof for the call option. A similar proof can be constructed for a put option.

□

Remark. We quickly make use of the put-call parity to alternatively determine the price of the put option for $t \in [T-b, T]$ without having to go through the above proof all over again. We reminder the reader that if $V_C(t)$ and $V_P(t)$ are respectively the prices of the call and put option, then

$$V_P(t) = V_C(t) + Ke^{-r(T-t)} - S(t)$$

Using the above theorem for $t \in [T-b, T]$. We see that

$$V_P(t) = S(t)\Big(\Phi(\beta_+(t)) - 1\Big) + Ke^{-r(T-t)}\Big(1 - \Phi(\beta_-(t))\Big)$$

Using the symmetry of the Φ function we have

$$\Phi(\beta_+(t)) - 1 = -\frac{1}{\sqrt{2\pi}}\int_{\beta_+(t)}^\infty = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\beta_+(t)} = -\Phi(-\beta_+(t))$$

Likewise, $1 - \Phi(\beta_-(t)) = \Phi(-\beta_-(t))$. Hence,

$$V_P(t) = Ke^{r(T-t)}\Phi(-\beta_-(t)) - S(t)\Phi(-\beta_+(t)) \tag{1.15}$$

Theorem 1.1.4. *The hedging strategy for a European call (resp. put) option, for* $t \in [T - b, T]$*, is given by*

$$\pi_S(t) = \Phi(\beta_+(t)), \;\; \pi_\xi(t) = -Ke^{-rT}\Phi(\beta_-(t))$$

(resp.)

$$\pi_S(t) = \Phi(-\beta_+(t)), \;\; \pi_\xi(t) = -Ke^{-rT}\Phi(-\beta_-(t))$$

Proof. The proof follows from (1.14) and (1.15). Indeed, we can rewrite (1.14) into

$$V(t) = S(t)\Phi(\beta_+) - Ke^{-rT}\Phi(\beta_-)\underbrace{e^{rt}}_{=\xi}$$

and the same holds in an analog manner for (1.15). We then repeat an exact analog of the latter part of the proof of theorem 1.1.2 using the Ito product rule exactly as in theorem 1.1.2. We then see that

$$dV(t) = \Phi(\beta_+)dS(t) - Ke^{-rT}\Phi(\beta_-)de^{rt}$$

and the proof is complete. The same can be done for the case of the put option. □

I.2. Numerical Approximations

As we have seen in the previous section, when proving the existence and uniqueness of our model, constructing a solution of a given SFDE is quite tricky. One proceeds by steps of length equal to the delay to make use of the initial condition. Moreover, the pricing formula we obtained after much hard work is not perfect as the price before the time $T - b$, where T is the time horizon and b the volatility delay, is in the form of a conditional expectation. This motivates us to develop a numerical scheme for SFDEs which one can approximate the true solution.

Because of the positivity condition in financial markets, we propose to approximate the stock price via a logarithmic Euler-Maruyama scheme tailored to our model. The logarithmic Euler-Maruyama scheme, as the name suggests, is an adaptation of the Euler-Maruyama method for SFDEs. Essentially, we use the Ito formula to write the process as an exponential, and we approximate the process by approximating the exponential argument.
This raises questions of L^p integrability (to ensure the solution and its approximation do not blow up even when taken as exponential arguments!). This is of course assuming that the classic Euler-Maruyama approximations do indeed converge even for SFDEs, which is not a given and we will need to be proven.

As this paper is not focused on numerical results, we will only briefly give the main results and more often than not sketches of proofs. Some computational details may be skipped and in those cases, we will refer to the original works.
We will see that we are able to ensure exponential integrability if we add a new assumption on the coefficients of our SFDE and how, under this new assumption, the convergence of the logarithmic scheme is tied to the convergence of the classical Euler-Maruyama scheme for SFDEs. Said convergence will be proven under global and local Lipschitz hypotheses both for a general class of SFDEs. We will then apply this convergence result and discuss the order of convergence.

2.1. Convergence of the Euler-Maruyama method for SFDEs

Let us begin by defining the classic Euler-Maruyama approximation scheme and its convergence. For generality's sake, consider a more general form of SFDE. We remind the reader that as we have said before, the plan is not to apply this scheme directly to our model, but rather to its logarithm.

$$\begin{cases} dS(t) & = F(t, S_t)dt + G(S_t)dB(t), \qquad t \in [0, T] \\ S(t) & = \eta(t), \qquad t \in [-R, 0] \end{cases} \tag{2.1}$$

Assume the following hypotheses

Hypotheses (H)

(i) (Global Lipschitzianity of the drift and volatility coefficients) The functions F and G are jointly continuous and uniformly (i.e globally) Lipschitz in the space (second) coordinate, in the sense that there exists a deterministic, left-continuous and non-decreasing function $\mu : [-R, 0] \to \mathbb{R}_+$ such that for all $t \in [0, T]$ and $\varphi_1, \varphi_2 \in L^p(\Omega, \mathcal{C})$,

$$||F(t, \varphi_1) - F(t, \varphi_2)||_{L^p(\Omega, \mathbb{R}^d)} + ||G(\varphi_1) - G(\varphi_2)||_{L^p(\Omega, \mathbb{R}^{d \times m})} \leq \int_{-R}^{0} |\varphi_1(u) - \varphi_2(u)|^p \, d\mu(u)$$

(ii) ($\mathbb{F}$-Adaptedness of the coefficients) If φ is adapted, so are $F(t, \varphi(t))$ and $G(\varphi(t))$.

(iii) (Continuity requirement on the initial condition) We have $\eta \in \mathcal{C}$ and $\mathcal{F}_0$-measurable. Moreover, $\eta \in L^p([-R, 0]; \mathbb{R}^n)$ for $p > 2$ and there exists a constant $\alpha > 0$ such that

$$\mathbb{E}\big[|\eta(t) - \eta(s)|^p\big] \leq \alpha |t - s|$$

for all $t, s \in [-R, 0]$.

Remark. Let $L := \mu(0) - \mu(-R)$. Then (H)(i) turns into the "usual" Lipschitz condition. That is, note that this is a slightly stricter condition than the usual Lipschitz condition.

Define the (discrete) Euler-Maruyama approximation of (2.1) to be

$$\begin{cases} \overline{S}(t_{k+1}) & = \overline{S}(t_k) + F(t_k, \overline{S}_{t_k})\Delta t + G(\overline{S}_{t_k})\Delta B_t, \qquad t_k \geq 0 \\ \overline{S}(t_k) & = \eta(t_k), \qquad t_k \in [-R, 0] \end{cases} \tag{2.2}$$

for a given partition π of the time axis $[-R, T]$, of step $\Delta t = \frac{T}{N}$, N being the number of steps and $t_k := k\Delta t$. Meanwhile, ΔB_t represents the increments the increments of the Brownian motion and

$$\overline{S}_{t_k}(\theta) := \frac{t_{i+1} - \theta}{\Delta t}\overline{S}(t_{k+i}) - \frac{t_i - \theta}{\Delta t}\overline{S}(t_{k+i+1}) \tag{2.3}$$

for $t_i \leq \theta \leq t_{i+1}$. This last equality is simply a linear interpolation. If the delay does not correspond to a time step, i.e $t_j - \theta$ does not correspond to some other t_i, $i < j$, then a linear approximation of the stochastic memory on the two closest gridpoints is taken.

Remark. We have only defined the process and its stochastic memory on the gridpoints but we can easily extend the processes to all $[0, T]$. That is, for any $t \in [0, T]$, we define

$$\overline{S}(t) := \sum_{j=0}^{N} \overline{S}(t_j)\mathbb{1}_{t_j, t_{j+1}}(t)$$

and

$$\overline{S}_t(\theta) := \sum_{j=0}^{N} \overline{S}_{t_j}(\theta)\mathbb{1}_{t_j, t_{j+1}}(t)$$

Finally, define

$$\begin{cases} d\zeta(t) & = F(t, \overline{S}_t)dt + G(\overline{S}_t)dB(t), \qquad t \in [0, T] \\ \zeta(t) & = \eta(t), \qquad t \in [-R, 0] \end{cases} \tag{2.4}$$

The latter process will be referred to as the continuous Euler-Maruyama approximation of S. Notice that the continuous approximation we have defined coincides with the discrete approximation on all of the t_k. Before we prove the convergence of the scheme, let us prove a few lemmas from [Mao03].

Lemma 1.2.2. *We have*

$$\mathbb{E}\Big[\sup_{t\in[-R,T]} |\zeta(t)|^p\Big] \leq C$$

where $C = C_{\eta,p,T}$ is independent of Δt.

Proof. See [Mao03], lemma 3.2, for a detailed proof. From the definition of $\zeta(t)$, one has that

$$|\zeta(t)|^p \leq 3^{p-1}\left(|\eta(0)|^p + \Big|\int_0^t F(u, \overline{S}_u)\, du\Big|^p + \Big|\int_0^t G(\overline{S}_u)\, dB(u)\Big|^p\right)$$

Now, by the Hölder inequality, we have

$$\Big|\int_0^t F(u, \overline{S}_u)\, du\Big|^p \leq t^{p-1}\int_0^t |F(u, \overline{S}_u)|^p\, du$$

which, using global linear growth of f, one can bound by

$$\Big|\int_0^t F(u, \overline{S}_u)\, du\Big|^p \leq C_1 \int_0^t \mathbb{E}\Big[\sup_{s\in[-R,u]} |\zeta(t)|^p\Big]\, du$$

Likewise, using the Burkholder-Davis-Gundy inequality, one can obtain

$$\Big|\int_0^t G(\overline{S}_u)\, dB(u)\Big|^p \leq \mathbb{E}\Big[\sup_{s\in[0,t]}\Big|\int_0^s G(\overline{S}_u)\, dB(u)\Big|^p\Big] \leq C_2\int_0^t \mathbb{E}\Big[\sup_{s\in[-R,u]} |\zeta(t)|^p\Big]\, du$$

Hence,

$$\mathbb{E}\Big[\sup_{u\in[0,t]} |\zeta(t)|^p\Big] \leq C_3 + (C_1 + C_2)\int_0^t \mathbb{E}\Big[\sup_{s\in[-R,u]} |\zeta(t)|^p\Big]\, du$$

and an application of the Grönwall inequality gives the desired result. □

We require one more lemma before we go on to prove the convergence of the method. This lemma is unique to SFDEs, as we are estimating the L^p norm difference of the memory of the continuous approximation and of the discrete approximation.

Lemma 1.2.3. *There is $\beta > 0$ independent of Δt such that*

$$\mathbb{E}[|\zeta_t(\theta) - \overline{S}_t(\theta)|^p\Big] \leq \beta\Delta t^{\frac{p}{2}}$$

for $\theta \in [-R, 0]$ and where $\zeta_t(s) = \zeta(t+s)$ for $s \in [-R, 0]$.

Proof. See [Mao03], lemma 3.3, for a detailed proof of the case $p = 2$. The result is technical and long, and hence we will only cover it heuristically. Although the proofs do not differ much and jumps are included, one might also want to check the proof of lemma 3.2 in [BBMY11] where the case of a general $p > 2$ is treated. Let $t \in [0, T]$ and let $\theta \in [-R, 0]$.

Let k_t and k_θ be integers such that $t \in [t_{k_t}, t_{k_t+1}]$ and $\theta \in [t_{k_\theta}, t_{k_\theta+1}]$. Using (2.3), one can rewrite $\overline{S}_t$ like such

$$\overline{S}_t(\theta) := \overline{S}_{t_k}(\theta) = \overline{S}(t_{k_t+k_\theta}) + \frac{\theta - t_{k_\theta}}{\Delta t}\left(\overline{S}(t_{k_t+k_\theta+1}) - \overline{S}(t_{k_t+k_\theta})\right)$$

We can then bound our desired norm difference by

$$\mathbb{E}[|\zeta_t(\theta) - \overline{S}_t(\theta)|^p] \leq C_1\mathbb{E}\left[|\zeta_t(\theta) - \overline{S}(t_{k_t+k_\theta})|^p\right] + C_2\mathbb{E}\left[|\overline{S}(t_{k_t+k_\theta+1}) - \overline{S}(t_{k_t+k_\theta})|^p\right] = E_1 + E_2$$

• Let us focus on the second term first, that is E_2. Then, one must consider two cases. the case where $k_s + k_\theta \geq 0$ and the the case where $k_s + k_\theta < 0$.

Suppose the first case for now. The term E_2 can be handled by using the linear growth condition and a previous lemma to bound the second moment of our approximation. Indeed, by the triangular inequality and lemma 1.2.2

$$E_2 \leq C_2\left(\mathbb{E}\left[|F(t_{k_t+k_\theta} - \overline{S}_{k_t+k_\theta})\Delta t|^p\right] + \mathbb{E}\left[|G(\overline{S}(t_{k_t+k_\theta}))\Delta B_t|^p\right]\right)$$

where we have used that $\mathbb{E}[\Delta t \Delta B_t] = 0$ to separate the terms into their own square. Then, using the quadratic variation of the Brownian motion, i.e that $\mathbb{E}\left[(\Delta B_t)^2\right] = \Delta t$ and taking out the Δ terms, we get that

$$E_2 \leq C_2 \Delta t^{\frac{p}{2}}\left(\Delta t^{\frac{p}{2}}\mathbb{E}\left[|F(t_{k_t+k_\theta} - \overline{S}_{k_t+k_\theta})|^p\right] + \mathbb{E}\left[|G(\overline{S}(t_{k_t+k_\theta}))|^p\right]\right)$$

and

$$E_2 \leq C_2 K \Delta t^{\frac{p}{2}}(1 + \mathbb{E}[||\overline{S}_{k_t+k_\theta}||^2]) \leq C_2 K \Delta t^{\frac{p}{2}}$$

by global linear growth and lemma 1.2.2.

Suppose now that $k_s + k_\theta < 0$. Then we conclude directly with an application of lemma 1.2.3.

•. The term E_1 is handled with more care: we need to even more carefully consider the cases $k_s + k_\theta \geq 0$, $k_s + k_\theta \leq 2$ and $k_s + k_\theta = 1$. The latter case is trickiest. We refer to lemma 3.3 in [Mao03] for the computational details. The idea of the proof is to use that in that range, one identifies the memory to the initial condition, which enables us to use (H)(iii) and the same techniques as in our analysis of E_2 which in turn gives us the estimate we need.

□

Remark. Notice that the convergence rate depends on the Brownian motion and the volatility part. We have used here the quadratic variation of the Brownian motion to show that instead of Δt^p, we end up with $\Delta t^{\frac{p}{2}}$. However, in the jump case, we will see that we need to consider yet another variation, the jump variation. As we will see, this lowers the order of convergence by nature of the increments of the Poisson process, which is bound by Δt.

We now state the convergence result.

Theorem 1.2.5 (Convergence of the Euler-Maruyama method under Global Lipschitz hypotheses). *Under the earlier hypotheses (H), we have*

$$\lim_{\Delta t\to 0} \mathbb{E}\Big[\sup_{t\in[0,T]} |S(t)-\zeta(t)|^p\Big] = 0$$

Moreover, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |S(t)-\zeta(t)|^p\Big] = O(\Delta t^{\frac{p}{2}})$$

i.e, the order of convergence of the method is exactly $\frac{1}{2}$*.*

Proof. Once again, since this is not the main topic of the thesis, we will not be giving all the details of the proof. The standard proof we are following can be found in [BBMY11], theorem 3.1. The idea of the proof is to once again write

$$\begin{aligned}\mathbb{E}\Big[\sup_{u\in[0,t]} |S(u)-\zeta(u)|^p\Big] \leq & 3^{p-1}\Big(\mathbb{E}\Big[\sup_{u\in[0,t]}\Big|\int_0^u F(s,S_s)-F(s,\overline{S}_s)\,ds\Big|^p\Big] \\ & + \mathbb{E}\Big[\sup_{u\in[0,t]}\Big|\int_0^u G(S_s)-G(\overline{S}_s)\,dB(s)\Big|^p\Big]\Big) \end{aligned} \tag{2.5}$$

Now, let us treat each integral term separately. On one hand,

$$\Big(\mathbb{E}\Big[\sup_{u\in[0,t]}\Big|\int_0^u F(s,S_s)-F(s,\overline{S}_s)\,ds\Big|^p\Big] \leq C_1^{p,T}\int_0^t \mathbb{E}[|F(s,S_s)-F(s,\overline{S}_s)|^p]\,ds$$

We add and substract ζ in the integrand to split it like so

$$\begin{aligned} C^1_{p,T}\int_0^t \mathbb{E}[|F(s,S_s)-F(s,\overline{S}_s)|^p]\,ds &\leq C^1_{p,T}\Big(\int_0^t \mathbb{E}[|F(s,S_s)-F(s,\zeta_s)|^p]\,ds + \int_0^t \mathbb{E}[|F(s,\zeta_s)-F(s,\overline{S}_s)|^p]\,ds\Big) \\ &= I_1 + I_2 \end{aligned}$$

The goal here is to exploit lemma 1.2.3 on the second integral I_2 of the right-hand side using the global Lipschitz condition. As for the first integral I_1, one might apply Hölder's inequality to make the following bound appear

$$\int_0^t \mathbb{E}[|F(s,S_s)-F(s,\zeta_s)|^p]\,ds \leq C^2_{p,T,L}\int_0^t \mathbb{E}\Big[\sup_{u\in[0,s]}|S(u)-\zeta(u)|^p\Big]\,ds$$

To summarize,

$$\Big(\mathbb{E}\Big[\sup_{u\in[0,t]}\Big|\int_0^u F(s,S_s)-F(s,\overline{S}_s)\,ds\Big|^p\Big] \leq C^3_{p,T}\Delta t^{\frac{p}{2}} + C^2_{p,T}\int_0^t \mathbb{E}\Big[\sup_{u\in[0,s]}|S(u)-\zeta(u)|^p\Big]\,ds$$

We now turn our attention to the second integral in (2.5). Using the same idea and process, although we supplement the use of Hölder's inequality with the Burkholder-Davis-Gundy inequality, we have likewise

$$\mathbb{E}\Big[\sup_{u\in[0,t]}\Big|\int_0^u G(S_s)-G(\overline{S}_s)\,dB(s)\Big|^p\Big] \leq C^5_{p,T}\Delta t^{\frac{p}{2}} + C^4_{p,T}\int_0^t \mathbb{E}\Big[\sup_{u\in[0,s]}|S(u)-\zeta(u)|^p\Big]\,ds$$

Hence, put together, one has

$$\mathbb{E}\Big[\sup_{u\in[0,t]}|S(u)-\zeta(u)|^p\Big] \leq C^7_{p,T}\Delta t^{\frac{p}{2}} + C^6_{p,T}\int_0^t \mathbb{E}\Big[\sup_{u\in[0,s]}|S(u)-\zeta(u)|^p\Big]\,ds$$

Finally, we once again conclude with Grönwall's inequality, which gives us the desired convergence result and order of convergence both.

□

As promised, with the result proven for global Lipschitz hypotheses, we will now weaken it to local hypotheses supplemented by linear growth conditions. Replace (H)(i) by the following hypotheses

Hypotheses (H')

(i) (Local Lipschitzianity of the drift and volatility coefficients) The coefficients F and G are locally Lipschitz in the second variable, i.e for all $n \geq 1$ there exists a determinstic, left-continuous and non-decreasing function $\mu_n : [-R,0] \to \mathbb{R}_+$ such that for all $\varphi_1, \varphi_2 \in \mathcal{C}_n := \{\varphi \in \mathcal{C},\ ||\varphi||_{\mathcal{C}} < n\}$, we have

$$||F(t,\varphi_1)-F(t,\varphi_2)||_{L^p(\Omega,\mathbb{R}^d)} + ||G(\varphi_1)-G(\varphi_2)||_{L^p(\Omega,\mathbb{R}^{d\times m})} \leq \int_{-R}^0 |\varphi_1(u)-\varphi_2(u)|^p\,d\mu_n(u)$$

(ii) (Global Linear Growth of the drift and volatility coefficients) There exists a constant $K > 0$ such that for all $\varphi \in \mathcal{C}$, we have

$$||F(t,\varphi)||_{L^p(\Omega,\mathbb{R}^d)} + ||G(\varphi)||_{L^p(\Omega,\mathbb{R}^{d\times m})} \leq K(1+||\varphi||_{\mathcal{C}})$$

(iii) (Bound on the Local Lipschitz coefficients) Let $L_n := \mu_n(0) - \mu_n(-R)$ where μ_n refers to the function in (H')(i). Then there exists a constant $\alpha > 0$ and $\varepsilon > 0$ such that $L_n^{1+\varepsilon} \leq \alpha \log(n)$.

Remark. Note that (H')(i), just like (H)(i) is stricter than the "usual" Lipschitz condition. Indeed, if we pick $L_n := \mu_n(0) - \mu_n(-R)$ then we readily see that L_n are local Lipschitz constants for the coefficients.

Theorem 1.2.6 (Convergence of the Euler-Maruyama method under Local Lipschitz Hypotheses). *Under the hypotheses (H'), (H')(ii), (H')(iii) and (H)(iii), we have*

$$\lim_{\Delta t\to 0}\mathbb{E}\Big[\sup_{t\in[0,T]}|S(t)-\zeta(t)|^p\Big] = 0$$

Moreover, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|S(t)-\zeta(t)|^p\Big] = O(\Delta t^{\frac{p}{2}-\varepsilon})$$

where $\varepsilon > 0$ is an arbitrary small number, i.e the order of convergence of the method is $\frac{1}{2} - \varepsilon$.

Proof. We will proceed similarly to the extension of the general existence and uniqueness theorem found in Appendix A, i.e using a truncating procedure. Consider the truncated coefficients outside some open ball $\mathcal{C}_n$, i.e define

$$F_n(t,x) = \begin{cases} F(t,x) & \text{if } ||x|| < n \\ F(t, \frac{x}{||x||}n) & \text{if } ||x|| \geq n \end{cases}$$

and

$$G_n(x) = \begin{cases} F(x) & \text{if } ||x|| < n \\ G(\frac{x}{||x||}n) & \text{if } ||x|| \geq n \end{cases}$$

In some open ball $\mathcal{C}_n$, by the local Lipschitz hypothesis, these coefficients verify a global Lipschitz hypothesis. Hence, in $\mathcal{C}_n$, the SFDE with coefficients f_n and g_n has a unique solution S^n approximated via $S^{n,k}$. Moreover, by the definition of the truncation, for two open balls $\mathcal{C}_i$ and $\mathcal{C}_j$, the coefficients F_i, F_j and G_i, G_j agree on their intersection $\mathcal{C}_i \cap \mathcal{C}_j$, the smallest of the two balls. Hence, the same goes for the processes S^i and S^j.
Likewise, since the coefficients in $\mathcal{C}_n$ verify a global Lipschitz hypothesis, there exists a Euler-Maruyama approximation ζ^n of S^n, which exists and converges in $\mathcal{C}_n$ with order of convergence $\frac{1}{2}$.
Consider the stopping times $\tau_n = T \wedge \inf\{t \in [0,T], |S^n(t)| \vee |\zeta^n(t)| \ > n\}$. By the previous remark, we easily see that $\tau_n \to T$ when $n \to \infty$. Moreover, we have

$$|S(t) - \zeta(t)|^p = \sum_{j=0}^{\infty} |S^j(t) - \zeta^j(t)|^p \mathbb{1}_{[\tau_{j-1},\tau_j]}(t) \tag{2.6}$$

and thus,

$$\mathbb{E}\big[\sup_{t\in[0,T]} |S(t) - \zeta(t)|^p\big] \leq \sum_{j=0}^{\infty} \mathbb{E}\big[\sup_{t\in[0,T]} |S^j(t) - \zeta^j(t)|^p \mathbb{1}_{[\tau_{j-1},\tau_j]}(t)\big] \tag{2.7}$$

We now wish to isolate the norm difference of S^j and ζ^j. Let $\varepsilon > 0$. Then applying Hölder's inequality with $p' = \frac{p+\varepsilon}{p}$ and $q' = \frac{p+\varepsilon}{\varepsilon}$, we get

$$\mathbb{E}\big[\sup_{t\in[0,T]} |S^j(t) - \zeta^j(t)|^p \mathbb{1}_{[\tau_{j-1},\tau_j]}(t)\big] \leq \mathbb{E}\big[\sup_{t\in[0,T]} |S^j(t) - \zeta^j(t)|^{p+\varepsilon}\big]^{\frac{p}{p+\varepsilon}} \mathbb{E}\big[\mathbb{1}_{[\tau_{j-1},\tau_j]}(t)\big]^{\frac{\varepsilon}{p+\varepsilon}} \tag{2.8}$$

On one side, we can estimate

$$\mathbb{E}\big[\mathbb{1}_{[\tau_{j-1},\tau_j]}(t)\big]^{\frac{\varepsilon}{p+\varepsilon}} = \mathbb{P}(\sup_{[0,T]} |S^j(t)| \vee \sup_{[0,T]} |\zeta^j(t)| \geq j-1)^{\frac{\varepsilon}{p+\varepsilon}} \leq C_q\Big(\frac{j}{p}\Big)^{\frac{-q\varepsilon}{p+\varepsilon}}$$

for some $q \geq 2$ using lemma 1.2.2 and lemma 1.2.3 (see also [BBMY11], theorem 4.1). Now, on the side, we know from before theorem 1.2.5 that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |S^j(t) - \zeta^j(t)|^{p+\varepsilon}\Big]^{\frac{p}{p+\varepsilon}} = C_j \Delta t^{\frac{p}{2}-\varepsilon}$$

Moreover, as is detailed in [BBMY11], the shape of C_j is in fact exponential because of Grönwall's inequality. Indeed, if we consider an ε, δ approach to the limit, for some sufficiently small ε there exists δ_1, δ_2 such that we have

$$C_j = e^{(\delta_1+\delta_2)L_j^{1+\frac{\varepsilon}{2}}}$$

By (H')(iii) there exists α such that $L_j^{1+\varepsilon} \leq \alpha \log j$ for all j, we can bound the above C_j by $j^{\alpha(\delta_1+\delta_2)}$. Hence, we indeed have that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |S^j(t) - \zeta^j(t)|^{p+\varepsilon}\Big]^{\frac{p}{p+\varepsilon}} = O(\Delta t^{\frac{p}{2}-\varepsilon})$$

even with a local lipschitz condition and not a global one.
Combining both of these estimates, we deduce that the order of convergence is the same even under a local Lipschitz hypothesis. That is, the approximations converge and with order $\frac{1}{2} - \varepsilon$ and the proof is complete.

□

Remark. We will see that the condition $L_n^{1+\varepsilon} \leq \alpha \log(n)$ is indeed a strong condition. In the next subsection, we will try to apply this local method to prove convergence of the logarithmic method, which involves using the convergence under local Lipschitz hypotheses for certain exponentially shaped coefficients which do not, by nature, verify the condition. Hence, only a specific kind of these coefficients will be admissible which is rather restrictive.

2.2. A Logarithmic Euler-Maruyama method

Now that the convergence for classical scheme is proven, we are now ready to detail and prove the logarithmic Euler-Maruyama method. For a detailed look at general logarithmic numerical approximations of this type, we refer to [YHZ21].

We model the stock price by the SFDE per the model presented in the previous section. That is,

$$\begin{cases} dS(t) &= f(t, S_t)S(t)dt + g(S(t-b))S(t)dB(t), \qquad t \in [0,T] \\ S(t) &= \eta(t), \qquad t \in [-R, 0] \end{cases} \tag{2.9}$$

The hypotheses on the coefficients are the same as previously, i.e global linear growth and local Lipschitzianity of the drift coefficient f as well as continuity and boundedness of g. For a given partition of the time axis π of $[-R, 0]$, of step $\Delta t = \frac{T}{N}$, N being the number of steps, let $t_k := k\Delta t$ (here, k can take negative values!). Then we can approximate the solution to this SFDE by a discrete scheme of the sort

$$\begin{cases} \delta(t_{k+1}) &= \delta(t_k)exp\Big(\big(f(t_k,\delta_{t_k}) - \frac{1}{2}g^2(\delta_{t_k}(b))\big)\Delta t + g(\delta(t_k - b))\Delta B_k\Big), \qquad t_k \geq 0 \\ \delta(t_k) &= \eta(t_k), \qquad t_k \in [-R, 0] \end{cases} \tag{2.10}$$

where $\Delta B_k = B(t_{k+1}) - B(t_k)$ and where

$$\delta_{t_k}(\theta) := \frac{t_{i+1} - \theta}{\Delta t}\delta(t_{k+i}) - \frac{t_i - \theta}{\Delta t}\overline{S}(t_{k+i+1})$$

for $t_i \leq \theta \leq t_{i+1}$ represents the memory part. All this last random variable ensures is that if the delay chosen does not have to be a multiple of the chosen time step. For example, if the memory in the SFDE requires the value of δ in between t_i and t_j, which is only calculated at t_i and t_j, then we will take the linear approximation between the (consecutive) times t_i and t_j to be that value we need.

Remark. Just like before, note that we have so far only been approximating at the grid points, i.e at discrete points. This is easily extendable though. Indeed, define

$$\delta(t) := \sum_{j=0}^{\infty} \delta(t_j)\mathbb{1}_{t_j,t_{j+1}}(t)$$

and

$$\delta_t(\theta) := \sum_{j=0}^{\infty} \delta_{t_j}(\theta)\mathbb{1}_{t_j,t_{j+1}}(t)$$

Remark. Note that to ensure $\mathbb{F}$-adaptedness of the approximation, while we take the liberty to implement linear approximations of the memory of the process, we cannot directly use linear approximations of $\delta(t_k)$ and $\delta(t_{k+1})$ and instead must give to the interval $[t_k, t_{k+1}]$ the left-hand most value $S(t_k)$.

While the convergence of the Euler-Maruyama method is well known even for SFDEs, and we will prove its convergence and order of convergence later, assume convergence of said method for now. As we are essentially approximating the logarithm of the solution, it is not given that this approximation holds through the passage to the exponential function. For this, it is enough to invoke the mean-value theorem. Indeed, we use the fact that

$$|e^x - e^y| \leq (e^x + e^y)|x - y|$$

Applying Cauchy-Schwarz's inequality, we then have, for any $t > 0$,

$$\mathbb{E}[|S(t) - \delta(t)|^p] = \mathbb{E}[|e^{\log S(t)} - e^{\log \delta(t)}|^p] \leq C_p \mathbb{E}[|\log S(t) - \log \delta(t)|^{2p}]^{\frac{1}{2}} \underset{k\to\infty}{\to} 0$$

where

$$C_p = \mathbb{E}\big[e^{p|\log S(t)|}\big] + \mathbb{E}\big[e^{p|\log \delta(t)|}\big]$$

Given an exponential integrability condition, i.e if both $\mathbb{E}\big[e^{p|\log S(t)|}\big] < \infty$ and $\mathbb{E}\big[e^{p|\log \delta(t)|}\big] < \infty$, then C_p is clearly a finite constant depending only on p. This proves convergence in a

regular L^p space (of course, given that the solution and approximation both belong to that space). We refer to theorem 4.3 of [YHZ21] which we refer to for a detailed proof. We will however state the result and quickly give the general idea of the proof shortly after having defined some new notations.
If we write $Y(t) = \log S(t)$, then the Ito formula applied to the logarithm tells us that

$$dY(t) = \Big(f\big(t, e^{Y_t}\big) - \frac{1}{2}g^2\big(e^{Y(t-b)}\big)\Big)dt + g\big(e^{Y(t-b)}\big)dB(t)$$

To make things slightly easier for the reader, we shall henceforth denote the drift and volatility coefficients of the above by $f_{\log}(t, Y_t)$ and $g_{\log}(Y_t)$ respectively, i.e

$$f_{\log}(t, Y_t) := f\big(t, e^{Y_t}\big) - \frac{1}{2}g^2\big(e^{Y(t-b)}\big) = F(t, Y_t)$$

and

$$g_{\log}(Y_t) := g(e^{Y(t-b)}) = G(Y_t)$$

We have indicated F and G so as to be clear what we shall apply the results of the previous section to. Here, these are the coefficients we will be working with. We now proceed to give integrability results justifying the earlier claim that C_p is in fact a finite constant. Let us first show that $Y(t) = \log S(t)$ has an exponential p-th moment.

Lemma 1.2.4. *Suppose that, for all $t \in [0, T]$ and for any $m \in [1, 2)$,*

$$xf_{\log}(t, x) + \frac{p-1}{2}g^2_{\log}(x) \leq Kp^m \tag{2.11}$$

then the random variable $\log S(t)$ has an exponential p-th moment, i.e if $\mathbb{E}\big[e^{p|\log S(t)|}\big] < \infty$, for all $p > 1$.

Proof. We will only give a general idea of the proof here. Applying the Ito formula to the function $x \mapsto x^q$, $q > 1$, we have

$$dY^q(t) = qY^{q-2}(t)\big(Y(t)f_{\log}(t, Y_t) + \frac{q-1}{2}g^2_{\log}(Y_t)\big)dt + qY^{q-1}(t)g_{\log}(Y_t)dB(t)$$

Define the stopping time $\tau := \inf\{t \in [0, T] : |Y(t)| \geq L\}$ for some $L > 0$. Moreover, from (2.11), it is possible to prove that

$$\begin{aligned}
\mathbb{E}\big[Y^{2q}(t\wedge\tau)\big] &= \log\eta^{2q}(0) + \mathbb{E}\Big[\int_0^{t\wedge\tau} 2qY^{2q-2}(u)\big(Y(u)f_{\log}(u, Y_u) + \frac{2q-1}{2}g^2_{\log}(Y_u)\big)du\Big] \\
&\leq \log\eta^{2q}(0) + 2Kq\mathbb{E}\Big[\int_0^{t\wedge\tau}(2q)^m Y^{2q-2}(u)du\Big] \\
&\leq \log\eta^{2q}(0) + 2t(2q)^{mp} + Cq\int_0^{t\wedge\tau}\mathbb{E}\big[Y(u\wedge\tau)^{2q}\big]du
\end{aligned}$$

for a constant C. Applying the Grönwall inequality with $\alpha(t) = \log\eta^{2q}(0) + 2t(2q)^{mp}$ and $\beta(t) = Cq\mathbb{E}\big[Y(u\wedge\tau)^{2q}\big]$, one can deduce that

$$\mathbb{E}\big[Y^{2q}(t\wedge\tau)\big] \leq C_{q,t}(2q)^{mp}$$

and using the definition it is also possible to prove that

$$\mathbb{P}(\tau \leq t) \leq \frac{C_{q,t}(2p)^{mq}}{L^{2q}}$$

where $C_{q,t}$ are the same constants depending on q and t in the last two equations. Hence, as $L \to \infty$, we see that $\mathbb{P}(\tau > t) = 1$. Thus, using Fatou's lemma, it is possible to extend this bound of $\mathbb{E}\big[Y(t \wedge \tau)^{2q}\big]$ to $\mathbb{E}\big[Y^{2q}(t)\big]$ with a constant $C_{q,T}$. Then, Jensen's inequality says that

$$\mathbb{E}[|Y(t)|^q] \leq \mathbb{E}[Y(t)^{2q}]^{\frac{1}{2}} \leq C_{q,T}\, q^{\frac{mp}{2}}$$

Now, writing the exponential as a series, we know that for any $p > 0$

$$\mathbb{E}\big[e^{p|Y(t)|}\big] = \sum_{j=0}^{\infty} \frac{p^j |Y(t)|^j}{j!} = \sum_{j=0}^{\infty} \frac{p^j \mathbb{E}[|Y(t)|^j]}{j!}$$

On one hand, we know that the numerator grows roughly at the speed of $j^{\frac{mj}{2}}$ while, by Stirling's approximation formula, $j!$ acts as $j^{j+\frac{1}{2}}$. Since $m < 2$, the general terms grows at most at the speed of $\frac{1}{j^2}$ and hence the series converges, i.e $\mathbb{E}\big[e^{p|Y(t)|}\big] < \infty$ and $\log S(t)$ has a p-th moment which completes the proof.

□

We now proceed to state a similar estimate for the approximation $\log \delta(t)$, i.e that $\delta(t)$ has an exponential p-th moment.

Lemma 1.2.5. *Suppose once again that, for all $t \in [0, T]$ and for any $m \in [1, 2)$,*

$$x f_{\log}(t, x) + \frac{p-1}{2} g_{\log}^2(x) \leq K p^m \tag{2.12}$$

then the random variable $\log \delta(t)$ has an exponential p-th moment, i.e if $\mathbb{E}\big[e^{p|\log \delta(t)|}\big] < \infty$, for all $p > 1$.

Proof. For a fixed time step δ, using the assumption (2.12), we can bound exponential p-th moment by that of a $\mathcal{N}(0, \sqrt{\delta})$. Then, using lemma 4.6 from [YHZ21], which we refer to for computational details, this estimate is proven to be finite.

□

Theorem 1.2.7 (**Convergence of the Logarithmic Euler-Maruyama method under Local Lipschitz hypotheses**). *Under the hypotheses (H'), (H)(ii) and (H)(iii) for $f_{\log}$ and $g_{\log}$, we have*

$$\lim_{\Delta t \to 0} \mathbb{E}\Big[\sup_{t \in [0,T]} |S(t) - \delta(t)|^p\Big] = 0$$

Moreover, we have

$$\mathbb{E}\Big[\sup_{t \in [0,T]} |S(t) - \delta(t)|^p\Big] = O(\Delta t^{\frac{p}{2} - \varepsilon})$$

i.e, the order of convergence of the method is exactly $\frac{1}{2}\varepsilon$.

Proof. As was explained at the beginning of this subsection, we use the fact that

$$|e^x - e^y| \leq (e^x + e^y)|x - y|$$

and we apply Cauchy-Schwarz's inequality. Then for any $t > 0$,

$$\mathbb{E}[|S(t) - \delta(t)|^p] = \mathbb{E}[|e^{\log S(t)} - e^{\log \delta(t)}|^p] \leq C_p \mathbb{E}[|\log S(t) - \log \delta(t)|^{2p}]^{\frac{1}{2}}$$

where $C_p = \mathbb{E}\big[e^{p|\log S(t)|}\big] + \mathbb{E}\big[e^{p|\log \delta(t)|}\big] < \infty$ by lemma 1.2.4 and lemma 1.2.5. The convergence of the method hence hinges on the convergence of the classic Euler-Maruyama method and the proof is complete.

□

Remark. Note here that it is not given that $f_{\log}$ and $g_{\log}$ verify the given hypotheses!. Both are certainly locally Lipschitz if f and g are, since g is bounded. Hence g^2 is too and locally Lipschitzianity is certainly maintained. The problem lies in the linear growth condition which is not guaranteed because of the exponential term which needs to be balanced, as well as the fact both $g(e^x)$ and $g^2(e^x)$ require to have a linear growth which is a tough ask, even given the assumption in both lemma 1.2.4 and lemma 1.2.5. Moreover, as explained before, because of the exponential term of the argument in $f_{\log}$ and $g_{\log}$, condition (H')(iii) is hard to verify. While possible, and while certainly positivity of all the approximations is desired and is an improvement over the regular method, the drawbacks are too great to use in a general setting. In such cases, our conclusion is that the classical Euler-Maruyama is preferable.

2.3. Simulating Stock Prices using the Logarithmic Euler-Maruyama method

To summarize, we have proven the convergence of the Euler-Maruyama method under global Lipschitz hypotheses as well as under local Lipschitz hypotheses (and linear growth conditions). We have seen that the speed of convergence is in fact the same. We have then developed a logarithmic Euler-Maruyama method based on previous results by Hu, Zao and Yi (see [YHZ21]). This method proves to have flaws in that the choice of coefficients is very restrictive. However, we gain positivity of all the approximation and we do this without altering the order of convergence, which is still $\frac{1}{2} - \varepsilon$.

In this subsection, we will run Monte-Carlo approximations using the logarithmic Euler-Maruyama method and proceed to price options based on these approximations. Note that a regular Euler-Maruyama numerical approximation can be done very similarly and because much work has been done on the latter, we will not proceed to do both.

For test purposes only, we shall imagine a stock with stock price 100 at time $t = 0$. Let $f = 0,\ g(x) = \frac{1}{6} + 0.25 \log\log(x)\cos(\log x)$ and

$$\eta(x) = 99 + \sin\left(\frac{x}{24}\right) + \cos(\frac{x}{42})$$

Note that here that the drift is constant for simplicity's sake. This is not necessary however, and any f can be taken so long as the SFDE and its solution is well defined and so long as the above approximation assumptions hold!

Please also note that these functions are C^1 and hence locally Lipschitz. Moreover, linear growth is trivially verified since we have also asked for growth closer to $\log\log x$ (see the above remark regarding the fact that the logarithmic Euler-Maruyama method requires such slow growth).
We verify the condition regarding the Lipschitz constants (condition (H')(iii)). Let $\varphi_1, \varphi_2 \in C_n := \{\varphi : ||\varphi||_C \leq n\}$. Then

$$\begin{aligned}|g_{\log}(\varphi_1) - g_{\log}(\varphi_2)| &\leq \frac{1}{4}|\log\log e^{\varphi_1(-b)}| \cdot |cos(\log e^{\varphi_1(-b)}) - \log\log e^{\varphi_2(-b)}| \cdot |cos(\log e^{\varphi_2(-b)})| \\ &\leq \frac{1}{4}|\log\varphi_1(-b)\cos(\varphi_1(-b)) + \log\varphi_2(-b)\cos(\varphi_2(-b))| \cdot |\varphi_1(-b) - \varphi_2(-b)| \\ &\leq \frac{1}{2}\log n \cdot |\varphi_1(-b) - \varphi_2(-b)|\end{aligned}$$

i.e $L_n = \frac{1}{2}\log n$ which satisfies (H')(iii).

Finally, we do note a few more things. Our Lipschitz condition required in (H')(i) is slightly stronger than the classical Lipschitz condition. We conjecture that g verifies (H')(i) on the basis that it is C^1. Moreover, while g is not bounded, its growth rate being relative to $\log\log x$, we conjecture that this does not pose an issue. We also mention that we do indeed have $g(x) = 0$ for three values of x being roughly $x = 1.6$, $x = 11.28$ and $x = 68.8$, at least for reasonable values of x, however we also conjecture this not to be an issue in the following computations as these are the only problem points.

We will be doing 1000 approximations of 10 000 steps each, using $b = 0.05$ as delay. Below is the result of those simulations.

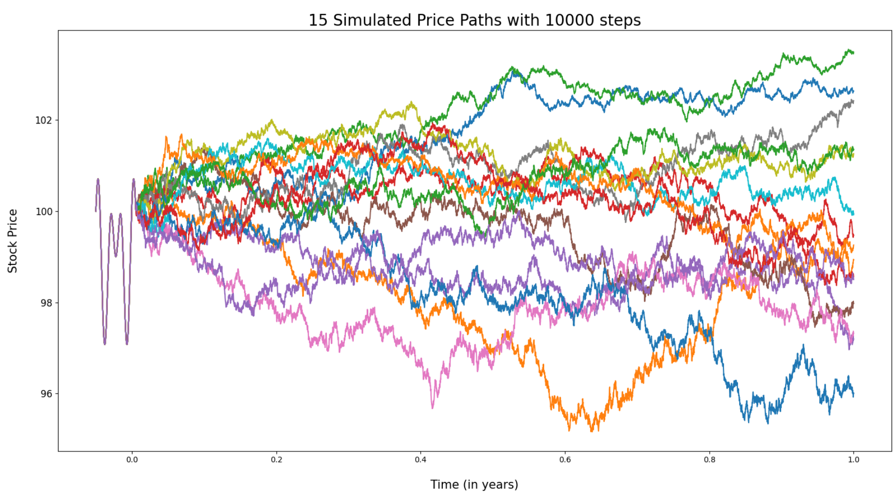

Figure I.1: Stock price simulation using the logarithmic Euler-Maruyama Method over a one year period

We are once again aware of the how restrictive the condition (H')(iii) is. This restrictiveness of our hypotheses is amplified by the fact it needs to be satisfied by $g_{\log}$, the coefficient in the SDE describing the dynamics of $Y(t) = \log S(t)$. While other, more usual coefficients do not satisfy the hypotheses, we conjecture the convergence of the model for $f(x) = 0.1$ and $g(x) = 0.1 + 0.01\cos(x)$. In an effort to showcase the model, and while we have not proven convergence of the model for such coefficients, we have still made some simulations with this different coefficient.

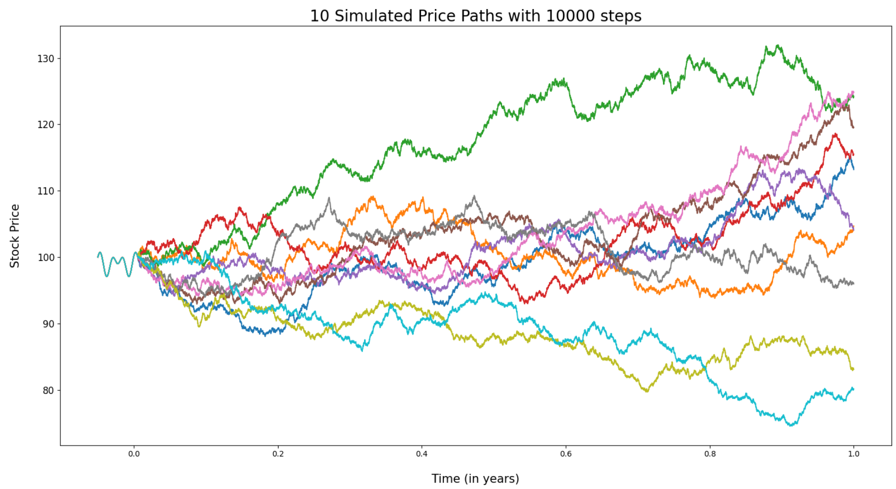

Figure I.2: Stock price simulation using the logarithmic Euler-Maruyama Method over a one year period with a different volatility coefficient

Remark. In an effort to showcase the problem, we mention that using the same method as above, that is, using the mean-value theorem, we find that with this new coeffcient, we have

$$|g_{\log}(\varphi_1) - g_{\log}(\varphi_2)| \leq 0.02e^n |\varphi_1(-b) - \varphi_2(-b)|$$

i.e we have a bound of the order $L_n < e^n$. This is very far from the desired bound, which explains our earlier choice of coefficients.

2.4. Call Option Price comparisons

Using the Monte Carlo approach, we shall now price some options of different strike prices using our delay model approximated with a logarithmic Euler-Maruyama method and the regular Black-Scholes model, using the well-known formula (since the volatility is constant in the classical model and our function $g(x) = 0.1 + 1.5 \log\log(x)\cos(\log x)$ hovers around a mean of roughly 0.1, we take $\sigma = 0.1$ in the case of the classical model). Let $r = 0.05$ and let the numbers of simulated paths be $i = 1000$.
Do however not that to properly price the Europen call option, one needs to first express $dS(t)$ under the measure $\mathbb{Q}$, the risk neutral measure from 2.1.11. Since it was obtained through the Girsanov theorem, the volatility does not change. However, the new drift is

$$f_{\mathbb{Q}} := \cancel{f(t, S_t)} - \frac{g(S(t-b))\cancel{f(t, S_t)}}{\cancel{g(S(t-b)}} + r = r$$

Proceeding with this change, we have

Strike Price of the Call Option	**Price of a European Call Option using the delayed Black-Scholes model**	**Price of a European Call Option using the classic Black-Scholes formula**
\$100	\$8.08	\$6.80
\$105	\$4.20	\$4.04
\$110	\$1.87	\$2.17
\$115	\$1.64	\$1.05
\$120	\$1.29	\$0.46
\$125	\$1.00	\$0.18
\$130	\$0.72	\$0.067
\$135	\$0.46	\$0.022
\$140	\$0.37	\$0.007

We can see that generally speaking, the pricing using the delayed Black-Scholes model is higher to that of the original Black-Scholes price. This may be an effect of the coefficients chosen, however, as they are not carefully chosen. Moreover, we remind the reader that the coefficients chosen technically do not verify all hypotheses for existence, unicity and positivity of the solution as well as the convergence of the logarithmic Euler-Maruyama model. More suitable choices for coefficients are always possible, however the restrictiveness of those conditions makes the model hard to use in practice.

2.5. Code & Algorithms

Below is the code used for the initial stock price simulation and call option comparison using the delayed Black-Scholes model

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

# Setting up the variables
```

```
K = 100  # Strike price
r = 0.05  # Risk-free interest rate
T = 1  # Time in years
N = 10000 #Number of steps
b = 0.05*T  # Delay
deltat = (T-b)/N  # Time step
i = 15  # Number of simulations

S = np.zeros([i, N])  # Stock price
t = np.linspace(-b, T, N)  # Time

# Defining the coefficients and initial data
def f(x):
    return 0

def g(x):
    return 1/6 + 0.25*np.log(np.log(x))*np.cos(np.log(x))

def eta(x):
    return 99 + np.sin(x/24) + np.cos(x/42)

def LinInterS(S,j,k,ib,b):
    return (((ib+1)*deltat - b)/deltat)*S[j, k-ib] - ((ib*deltat - b)/deltat)*
        S[j, k-ib+1]

# Define the call option payoff function
def call_payoff(S, K):
    return np.maximum(S-K, 0)

# Approximation Algorithm
option_prices = []
for j in range(i):
    S[j, 0] = eta(-b)
    for k in range(N - 1):
        if k*deltat > b:
            if b % deltat == 0:
                S[j,k+1] = S[j,k]*np.exp((f(S[j,k-int(b/deltat)])- (0.5*g(S[j,
                    k-int(b/deltat)]))**2)*deltat + g(S[j,k-int(b/deltat)])*np
                    .random.normal(0,np.sqrt(deltat)))
            else:
                ib = int(np.floor(b/deltat))
                S[j,k+1] = S[j,k]*np.exp((f(LinInterS(S,j,k,ib,b))- (0.5*g(
                    LinInterS(S,j,k,ib,b))**2))*deltat + g(LinInterS(S,j,k,ib,
                    b))*np.random.normal(0,np.sqrt(deltat)))
        else:
            S[j, k + 1] = eta(k + 1)

    # Calculate the call option payoff at expiration time T
    call_payoffs = call_payoff(S[j, -1], K)

    # Print the stock price at the maturity date
    print(f"Stock price at maturity for simulation {j+1}: {S[j,-1]}")

    # Discount the call option payoff to the present time
    discounted_payoffs = np.exp(-r * T) * call_payoffs

```

```
    # Calculate the Monte Carlo estimate of the call option price
    option_price = np.mean(discounted_payoffs)
    option_prices.append(option_price)

    # Plot the simulated price path
    plt.plot(t, S[j])

#Plotting all the resulting paths

plt.title('%d Simulated Price Paths with %d steps' % (i,N), fontsize=20)
plt.xlabel('Time (in years)', fontsize=15, labelpad=20)
plt.ylabel('Stock Price', fontsize=15, labelpad=20)
plt.yticks(fontsize=12)
plt.show()

# Print the mean call option price
mean_option_price = np.mean(option_prices)
print("Mean call option price: %f" % mean_option_price)
```

Chapter II

Adding jumps into the model

II.1. Overview of the model

We will primarily use [AH20] which has already developed the theory in the case $g = 0$, i.e a pure jump model. They argue that it is sometimes preferred, and a similar model where $g \neq 0$ has already been studied by Zaheer Imdad and Tusheng Zhang.
Thankfully, the theory of SFDEs with jumps has already been researched in [BCDN$^+$19] (see Appendix B) and we see that for suitable linear growth and Lipschitz conditions on the coefficients f and h, one can quite readily guarantee a solution and uniqueness of a solution. Moreover, one could very much develop this model in different ways, adding a suitable Brownian component of coefficient $g \neq 0$ for example, and one would still have existence and uniqueness of a solution as long as suitable conditions are enforced on g. The plan for this chapter is, however, to follow in the footsteps of [AH20]. Before we head off, note that errors were found in [AH20], namely in chapter 3 & 4, and thus while we do endeavor to follow in their footsteps, we do not take the same path.

1.1. A first look at the model: Existence and uniqueness of a solution

Consider two assets as before, a non-risky asset of price $\xi(t)$ at time t of interest rate $r > 0$ and a risky asset of price $S(t)$ at time t. The authors propose the following dynamics for the price of the underlying risky asset

$$\begin{cases} dS(t) = f(S(t-b))S(t)dt + h(S(t-b))S(t_-)dZ(t), & t \in [0, T] \\ (S_0, S(0)) = (\eta, x), & t \in [-R, 0] \end{cases} \tag{3.1}$$

where f and h are two real bounded measurable functions, R denotes the maximal delay as before, $\eta : [-R, 0] \mapsto \mathbb{R}^d$ is a measurable function and Z is a Lévy process (which we will assume bounded from below). We define the image set of $Z(t)$, i.e the possible size of the jumps of Z, to be the interval $\mathbb{J} = [-K, +\infty)$ for some constant $K > 0$. More specifically, we will later choose Z to be the hyper-exponential jump process. We denote the Poisson random measure associated to the process Z by N and the centered or compensated Poisson measure by $\tilde{N}$. Then, by definition of N and $\tilde{N}$, we have that

$$dZ(t) = \int_{\mathbb{J}} z\, \tilde{N}(dt, dz) := \int_{\mathbb{J}} zN(dt, dz) - \int_{\mathbb{J}} z\nu(dz)dt \tag{3.2}$$

Note that the SFDE is specifically chosen to be pure jump as the authors have found that it is sometimes preferable. Also note that we specifically chose $Z(t)$ to be defined as a stochastic integral with respect to the compensated Poisson measure $\tilde{N}$ and not N. This is to ensure that $Z(t)$ is a martingale, which we will use and need in chapter 3, which builds upon the ideas of this chapter. Using the above notation, we can rewrite (3.1) as

$$\begin{cases} dS(t) = f(\eta(t-b))S(t)dt + \int_{\mathbb{J}} zh(\eta(t-b))S(t_-)\tilde{N}(dt,dz), \; t \in [0, \min(a,b)] \\ (S_0, S(0)) = (\eta, x), \qquad t \in [-R, 0] \end{cases} \tag{3.3}$$

As before, the first question that arises when considering this model is the existence and uniqueness of a solution. However, we need to be careful for we also want to guarantee the positivity of the solution. Luckily, the price of the underlying risky asset has quite rigid shape and it is possible to extract the form of the solution as well, which is what we shall explore in our first theorem.

Theorem 2.1.8. *Suppose that $h(x) \leq \frac{1}{K}$ for all $x \in \mathbb{R}$. Then (3.1) has a pathwise unique solution S. Moreover, $S(t) > 0$ a.s for all times $t > 0$ if $\eta(0) > 0$.*

Proof. We only prove the result for $d = 1$, as the proof for a general d is analog. The details can be found in [AH20] p.5 − 6. First consider $t \in [0, b]$. This reduces our problem to the following

$$\begin{cases} dS(t) = f(\eta(t-b))S(t)dt + \int_{\mathbb{J}} zh(\eta(t-b))S(t_-)\tilde{N}(dt,dz), \; t \in [0, b] \\ (S_0, S(0)) = (\eta, x), \qquad t \in [-R, 0] \end{cases} \tag{3.4}$$

which is a classical SDE driven by the Lévy process Z and has no delay. This is an exponential SDE which has a pathwise unique solution. Note that the stochastic exponential form of the coefficients and their non-dependence on S here ensures that both coefficients are Lipschitz in the second variable, hence the existence and uniqueness of the solution is guaranteed (see for example [Pro05] theorem 6 on p.255). Thanks to Ito's formula for pure-jump Lévy processes we moreover have that, for $t \in [0, \min(a,b)]$,

$$\begin{aligned} S(t) = \eta(0)\exp\Big(&\int_0^t f(\eta(u-b))\,du + \int_0^t \int_{\mathbb{R}_0} \log\big(1 + zh(\eta(u-b))\big)\big)\tilde{N}(du,dz) \\ &+ \int_0^t \int_{\mathbb{R}_0} \log\big(1 + zh(\eta(u-b))\big) - zh(\eta(u-b))\Big)\, du\nu(dz) \end{aligned}$$

For this to be well defined, because of the logarithmic terms, it is necessary that $zh(x) > -1$. This is ensured by the condition that $h(x) \leq \frac{1}{K}$.

By the nature of the solution, we see that $\eta(0) > 0$ is a necessary positivity condition of the solution for all times $t > 0$. The condition that Z is bounded from below is also necessary for the positivity condition here. Both together are a sufficient condition.

Repeating this process recursively for all time intervals of size b, one gets the result for all $t \in [0, T]$.

□

1.2. An Incomplete Market

Now that we have established that our model holds at the most basic level, we will now study the possibility of arbitrage as well as the completeness of the market. We will see that while a change of measure is possible, and thus the market does not allow for arbitrage. However, not every claim is replicable. The market is incomplete and hence while we may be able to price simpler claims like call options, the hedging in incomplete markets will prove to be tough.

Consider the market model from before, that is (3.2). While it is possible to consider a general Lévy process, we shall mainly be interested in hyper or double exponential processes (see [CK12] or [LV17] for example). Hence, we will take Z to be a *compensated* compound process defined by

$$Z(t) = \sum_{j=1}^{A(t)} Y_j - \lambda t \int_{\mathbb{J}} f_Y(z) dz$$

where $A(t)$ is a poisson process of intensity $\lambda > 0$ and where the Y_j are i.i.d random variables of density f_Y. For completeness' sake, we briefly mention what we mean by a hyper or double exponential process. We say that the process Z is hyper-exponential if

$$f_Y(x) := \sum_{i=1}^{n} p_i \gamma_i e^{-\gamma_i x} \mathbb{1}_{x \geq 0} + \sum_{j=1}^{m} q_j \beta_j e^{\theta_j x} \mathbb{1}_{x \leq 0}$$

where $\gamma_i, \beta_j > 0$, $p_i, q_j \geq 0$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ and

$$\sum_{i=1}^{n} p_i + \sum_{j=1}^{m} q_j = 1$$

The double exponential process is the case of $n = m = 1$. Moreover, in our case, since we want to ensure that Z is lower bounded, we can for example ensure that $q_j = 0$ for all $j = 1, \ldots, n$ or by truncating each of the exponentials with $q_j = \frac{q_j'}{1 - e^{-\beta_j K_j}} \mathbb{1}_{\{-K_j < x < 0\}}$. In latter case, we have $K = \max_j K_j$. Note that we will in practice consider a simple double exponential.

Remark. In this case, we have

$$\int_{\mathbb{J}} \nu(dz) = \int_{\mathbb{J}} f_Y(z) dz$$

Remark. The reason for the compensation of the compound Poisson process is to make it so that (3.1) is rewritten in the form (3.2), i.e that Z can be represented a stochastic integral with respect to $\tilde{N}$, which implies that Z is a martingale.
Before we present the main results of this section, we will make a few further assumptions. They are as follows:

Hypotheses (H)

(i) (Continuity on the initial condition) η is Hölder-continuous with constants $\gamma \in \left[\frac{1}{2}, 1\right]$ and $\rho > 0$. Moreover, $\eta(0) > 0$ so as to later guarantee positivity.

(ii) (Global Lipschitzianity of the coefficients) f and h are globally Lipschitz (not locally!). Moreover, f is bounded.

(iii)(Conditions for a well defined model and positivity conditions of the model We have $q_j = 0$ for all $j = 1, \ldots, n$ or $q_j = \dfrac{q_j'}{1 - e^{-\beta_j K_j}} \mathbb{1}_{\{-K_j < x < 0\}}$. Moreover, we have $h(x) \leq \frac{1}{K}$.

(iv) (Bound Condition) For all $t \in [0, T]$,

$$\left|\frac{f(t) - r}{h(t)}\right| \leq 1$$

i.e h is bounded from below and $f(t)$ is at most $h(t) + r$.

Remark. These are fairly standard hypotheses, except for $(H)(iv)$, which we are aware is quite restrictive. We have however not been able to lower this hypothesis. This is to ensure the possibility to apply Girsanov's theorem, which is essential to find the risk neutral measure.

For what follows, as we have mentioned we shall utilize Girsanov's theorem for Lévy processes, which for completeness' sake we shall also state. As this result has been studied in previous courses, we will not prove this result. For a detailed study of the result, see [NØP08], p.199, for example or also [ØS05], section 1.4.

Theorem 2.1.9 (**Girsanov's Theorem for Lévy Processes**). *Let $\mathbb{J}$ be the set of possible sizes of jumps as above. Let $\theta(t, z)$ be an $\mathcal{F}$-predictable process such that $\theta(t, z) \leq 1$ for all $t \in [0, T]$ and $z \in \mathbb{R}_0$ and*

$$\int_0^T \int_{\mathbb{J}} |\log(1 + \theta(t, z))| + \theta^2(t, z)\nu(dz)dt < \infty \ \mathbb{P}\text{- } a.e$$

Let $x(t)$ also be a square integrable $\mathcal{F}$-predictable process. Now consider the process

$$X^{\theta}(t) = exp\Big(- \int_0^t x(u)dB(u) - \int_0^t x^2(u)du + \int_0^t \int_{\mathbb{J}} \log(1 - \theta(t, z)) + \theta(t, z)\nu(dz)du + \int_0^t \int_{\mathbb{J}} \log(1 - \theta(u, z))\tilde{N}(du, dz)\Big)$$

and define the measure $\mathbb{Q} := \mathbb{Q}_\theta$ by $d\mathbb{Q} := X^{\theta}(T)\, d\mathbb{P}$. Assume that X^{θ} verifies the Novikov condition, i.e that

$$\mathbb{E}\Big[exp\Big(\frac{1}{2}\int_0^T x(t)dt + \int_0^T \int_{\mathbb{R}_0} (1 - \theta(t, z)) \log(1 - \theta(t, z)) + \theta(t, z)\nu(dz)dt\Big)\Big] < \infty$$

Then $\mathbb{E}[X^{\theta}(T)] = 1$ *and* $\mathbb{Q}$ *is a well-defined probability measure on* $(\Omega, \mathcal{F}_T)$*. Moreover, if we define*

$$\tilde{N}_{\mathbb{Q}}(dt, dz) = \theta(t, z)\nu(dz)dt + \tilde{N}(dt, dz)$$

and

$$dB_{\mathbb{Q}}(t) = x(t)dt + dB(t)$$

then these two are respectively compensated Poissons random measure of N *and Brownian motion under the new measure* $\mathbb{Q}$*.*

Proof. See [NØP08], p.199. □

With this out of the way, we can now state and prove the first major result of this section. The first step is to prove the non-arbitrage property of the market.

Theorem 2.1.10. *The market does not allow for arbitrage.*

Proof. The aim, always, is to find an ELMM, which, as promised, we will accomplish with a well-chosen use of Girsanov's theorem for Lévy-type processes. By definition, we need to find an equivalent probability $\mathbb{Q}$ such $\tilde{S}(t) := e^{-rt}S(t)$ is a martingale under $\mathbb{Q}$. The way we want to define $\mathbb{Q}$ is so that $\widetilde{S}(t)$ can be expressed as a stochastic integral (which under suitable integrability conditions is a martingale). To do so, we want to apply Girsanov's theorem and hence some hypotheses need to be checked.
First, by Ito's formula for Lévy processes, we have

$$d\tilde{S}(t) = \tilde{S}(t_{-})h(S(t-b))\frac{f(S(t-b)) - r}{h(S(t-b))}dt + \tilde{S}(t_{-})h(S(t-b))\int_{\mathbb{J}} z\tilde{N}(dt, dz) \tag{3.5}$$

Now let $\theta(t, z) = \dfrac{f(S(t-b)) - r}{h(S(t-b))\displaystyle\int_{\mathbb{J}} z\theta^*(z)\nu(dz)}\theta^*(z)$ where θ^* is chosen such that

$$\int_{\mathbb{J}} z\theta^*(z)\nu(dz) < \infty$$

and

$$\Big|\frac{\theta^*(z)}{\int_{\mathbb{J}} z\theta^*(z)\nu(dz)}\Big| \leq 1$$

For example, we can take $\theta^*(z) = \mathbb{1}_A$ where $A \subset \mathbb{J}$ is a set such that this integral is finite.
We would like to apply Girsanov's theorem, hence we need to verify that $\theta(t, z) \leq 1$. This is clear by H(v) and by definition of θ^*, as we have

$$|\theta(t, z)| \leq \Big|\frac{f(S(t-b)) - r}{h(S(t-b))}\Big| \cdot \Big|\frac{\theta^*(z)}{\int_{\mathbb{J}} z\theta^*(z)\nu(dz)}\Big| \leq 1$$

Moreover the process $\theta(t, z)$ is also predictable since f and g both are. Indeed, for the same reason, by definition of θ, we have

$$\int_0^T \int_{\mathbb{J}} |\log(1+\theta(u,z))| + \theta^2(u,z)\nu(dz)du < \infty, \ \mathbb{P}\text{-a.e}$$

where we have used the definition of θ and the nature of the measure ν to justify that the quantity is finite (it is in fact even bounded!). Now define

$$S^\theta(t) := exp\Big(\int_0^t \log(1-\theta(u))\tilde{N}(du,dz) + \int_0^t \log(1-\theta(u)) + \theta(u)\nu(dz)du\Big)$$

For us to be able to apply Girsanov's theorem, it only remains to justify that the Novikov condition is verified for S^θ. This is once again a consequence of (H)(v), since for the same reasons,

$$\int_0^T \int_{\mathbb{J}} (1-\theta(u))\log(1-\theta(u)) + \theta(u)\nu(dz)du < \infty$$

where the bound is effectively done on the integral over $[0,T]$ since $\theta(t)$ does not have a space parameter. Hence,

$$\mathbb{E}\Big[exp\Big(\frac{1}{2}\int_0^T \int_{\mathbb{J}} (1-\theta(u))\log(1-\theta(u)) + \theta(u)\nu(dz)du\Big)\Big] < \infty$$

Then, by Girsanov's theorem, $\mathbb{E}[S^\theta(T)] = 1$ and we can define a new measure

$$d\mathbb{Q} := S^\theta(t)d\mathbb{P}$$

where we also define a new (compensated Poisson) random measure $\tilde{N}_{\mathbb{Q}}$ on the new probability space, under $\mathbb{Q}$, defined by

$$\tilde{N}_{\mathbb{Q}}(du,dz) = \theta(t)\nu(dz)du + \tilde{N}(du,dz) \tag{3.6}$$

Hence, going back to (3.5), we end up with

$$d\tilde{S}(t) = \int_{\mathbb{J}} z\, S(t_-)h(S(t-b))\tilde{N}_{\mathbb{Q}}(dt,dz)$$

which is, by definition of the stochastic integral, a martingale under $\mathbb{Q}$ since it is $L^2(\mathbb{Q})$. Since $\tilde{S(t)}$ represents the discounted price at time t, we have that $\mathbb{Q}$ is indeed an ELMM and hence the market has no-arbitrage.

□

Remark. The process θ verifies that the new drift $f_{\mathbb{Q}}$ of $S(t)$ has the form

$$\begin{aligned} f_{\mathbb{Q}}(S_t) =& f(S(t-b)) - \frac{f(S(t-b)) - r}{\cancel{h(S(t-b))\int_{\mathbb{J}} z\theta^*(z)\nu(dz)}} \cancel{h(S(t-b)\int_{\mathbb{J}} z\theta^*(z)\nu(dz)} \\ =& \cancel{f(S(t-b))} - \cancel{f(S(t-b))} + r = r \end{aligned}$$

More generally speaking, any such process θ, no necessarily one of the form we have chosen, must verify this condition for $\mathbb{Q}$ to be an equivalent martingale measure.

Remark. If Z was a general Lévy process, and not assumed to be a compound Poisson process, one could change H(iv) to ensure $\theta(t,z) \leq 1$. The way we have achieved this is by assuming we can separate $\theta(t,z)$ into a product of functions of t and z respectively. Hence, this result, under a different but suitable (although still restrictive, if not more restrictive) assumption H(v), would still hold.

Remark. The measure $\mathbb{Q}$ is in no way unique, that would imply that the market is complete, which as we will see is not the case. There exists an infinity of measures, for example we could consider a θ which cannot be factorized.

The completeness of the market as we have just stated is trickier. The market happens to be incomplete, as often markets driven by Lévy processes with jumps are. The intuition for this, as we will show, is that there must be as many risky assets as there are jumps. Let us proceed to show just that.

Theorem 2.1.11. *The market is incomplete*

Proof. We know from the proof of the non-arbitrage property that

$$d\tilde{S}(t) = \int_{\mathbb{J}} zS(t_-)h(S(t-b))\tilde{N}_{\mathbb{Q}}(dt,dz) = S(t_-)\int_{\mathbb{J}} zh(S(t-b))\tilde{N}_{\mathbb{Q}}(dt,dz) \tag{3.7}$$

Define $Y(t) = \log(\tilde{S}(t))$. Then by Ito's formula for Lévy processes,

$$\begin{aligned} dY(t) &= \int_{\mathbb{J}} \log(\widetilde{S}(t_-)) + zh(S(t-b))\widetilde{S}(t_-) - \log(S(t_-)) - \widetilde{S}^{-1}(t)zh(S(t-b))\widetilde{S}(t_-)\nu_{\mathbb{Q}}(dz)dt \\ &+ \int_{\mathbb{J}} \log(S(t_-)) + zh(S(t-b))S(t_-) - \log(S(t_-))\tilde{N}_{\mathbb{Q}}(dt,dz) \\ &= \int_{\mathbb{J}} \log(1+zh(S(t-b))) - zh(S(t-b))\nu_{\mathbb{Q}}(dz)dt + \int_{\mathbb{J}} \log(1+zh(S(t-b)))\tilde{N}_{\mathbb{Q}}(dt,dz) \end{aligned}$$

i.e

$$Y(t) = \int_0^t \int_{\mathbb{J}} \log(1+zh(S(t-b))) - zh(S(t-b))\nu_{\mathbb{Q}}(dz)dt + \int_0^t \int_{\mathbb{J}} \log(1+zh(S(t-b)))\tilde{N}_{\mathbb{Q}}(dt,dz)$$

and hence,

$$\tilde{S}(t) = \tilde{S}(0)exp\Big(\int_0^t \int_{\mathbb{J}} \log(1+zh(S(t-b))) - zh(S(t-b))\nu_{\mathbb{Q}}(dz)dt + \int_0^t \int_{\mathbb{J}} \log(1+zh(S(t-b))\tilde{N}_{\mathbb{Q}}(dt,dz)\Big)$$

By theorem 2.10 of [ØS05], we see that we must have

$$\log(1+zh(S(t-b))) = e^{-rT}\pi(t)zS(t_-)h(S(t-b))$$

which means that the replicating portfolio has the form

$$\pi(t) = \frac{\log(1 + zh(S(t-b)))}{z\tilde{S}(t_-)h(S(t-b))} \tag{3.8}$$

There is only one way in which this is possible. The clear dependence on z of $\phi(t)$ means that $\nu_{\mathbb{Q}}$ must have for support $\{z_0\}$, $z_0 \in \mathbb{J}$, i.e that there is only one jump size. This is however impossible from our definition of Z. Hence the claim $X = \log(\widetilde{S}(t))$ is not replicable and the market can't be complete.

□

This does not directly imply that none of the claims are replicable however, but it is possible to deduce a guess that the call option is not replicable either in this market. Going back to the expression for $d\tilde{S}(t)$ in (3.7), we can deduce that $\tilde{S}(T)$ is $L^2(\mathbb{Q})$. Indeed, we present part of the argument present in [AH20] and where it fails to give an intuition for this result. Their choice of measure for $\mathbb{Q}$ is $\theta^*(z) = 1$, with a slightly stronger assumption than $(H)(iv)$. Moreover, the compound Poisson process Z is not taken to be compensated. Their argument begins by stating that

$$\tilde{S}^2(T) = exp\Big(\int_0^T\int_{\mathbb{J}} \log(1+zh(S(t-b)))^2 - 2zh(S(t-b))\nu_{\mathbb{Q}}(dz)dt + \int_0^T\int_{\mathbb{J}} \log(1+zh(S(t-b)))^2\tilde{N}_{\mathbb{Q}}(dt,dz)\Big)$$

Let $\tilde{h}(t) = \frac{(1+zh(S(t-b))^4-1}{z}$. We then want to "replace" $h(S(t-b))$ by $\tilde{h}(t)$. This will help us separate $S(T)$ into a product of two terms, one which has mean zero. Since

$$\log(1 + z\tilde{h}(t)) = 2\log(1 + zh(S(t-b)))$$

and by adding and subtracting $\frac{1}{2}z\tilde{h}(t)$, we get

$$\begin{aligned}\tilde{S}^2(T) =& exp\Big(\frac{1}{2}\int_0^T\int_{\mathbb{J}} \log(1 + z\tilde{h}(t) - z\tilde{h}(t))\nu_{\mathbb{Q}}(dz)dt + \frac{1}{2}\int_0^T\int_{\mathbb{J}} \log(1 + z\tilde{h}(t))\tilde{N}_{\mathbb{Q}}(dt,dz)\Big)\\ &\cdot exp\Big(\int_0^T\int_{\mathbb{J}} \frac{z\tilde{h}(t)}{2} - zh(S(t-b))\nu_{\mathbb{Q}}(dz)dt\Big)\end{aligned}$$

We have now separated the part of $S(T)$ with mean zero from the rest. To not only find an upper bound, but to also make use of that fact, we use Hölder's inequality (p = 2). In practice, we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[S^2(T)] \leq& \mathbb{E}_{\mathbb{Q}}\Big[exp\Big(\int_0^T\int_{\mathbb{J}} \log(1 + z\tilde{h}(t) - z\tilde{h}(t))\nu_{\mathbb{Q}}(dz)dt + \int_0^T\int_{\mathbb{J}} \log(1 + z\tilde{h}(t))\tilde{N}_{\mathbb{Q}}(dt,dz)\Big)\Big]^{\frac{1}{2}}\\ &\cdot \mathbb{E}_{\mathbb{Q}}\Big[exp\Big(\int_0^T\int_{\mathbb{J}} z\tilde{h}(t) - 2zh(S(t-b))\nu_{\mathbb{Q}}(dz)dt\Big)\Big]^{\frac{1}{2}}\\ =& \mathbb{E}_{\mathbb{Q}}\Big[exp\Big(\int_0^T\int_{\mathbb{J}} z\tilde{h}(t) - 2zh(S(t-b))\nu_{\mathbb{Q}}(dz)dt\Big)\Big]^{\frac{1}{2}}\end{aligned}$$

Developing the expression of the integrand, we get

$$z\tilde{h}(t) - 2zh(S(t-b)) = z^4h^4(S(t-b)) + 4z^3h^3(S(t-b)) + 6z^2h^2(S(t-b)) + 2zh(S(t-b))$$

With some assumptions, we can go a bit further. Suppose that

$$\int_{\mathbb{J}} z^p \nu_{\mathbb{Q}}(dz) < \infty \tag{3.9}$$

and

$$\int_{\mathbb{J}} h^p(t) \nu_{\mathbb{Q}}(dz) < \infty \tag{3.10}$$

for$p = 1, 2, 3, 4$. Under the assumptions (3.9) and (3.10), the integrand, when integrated over $[0, T]x\mathbb{J}$, is finite. Hence, $\mathbb{E}[S^2(T)] < \infty$. Then, define $U(t) := \mathbb{E}_{\mathbb{Q}}\big[e^{-rT}(S(T) - K)^+ | \mathcal{F}_t\big]$. Since $\tilde{S}(T) \in L^2(\mathbb{Q})$, we have $U(t)$ is square integrable. Therefore, by the martingale representation theorem, there exists a square integrable predictable $\psi(t, z)$ such that

$$U(t) = \mathbb{E}_{\mathbb{Q}}\big[e^{-rT}(S(T) - K)^+\big] + \int_0^t \int_{\mathbb{J}} \psi(u, z)\tilde{N}(du, dz)$$

This is where the original paper by [AH20] fails, as they fail to consider that $\psi(u, z)$ is not necessarily factorized into $\psi_1(u)\psi_2(z)$, which is required in their proposed portfolio for the risky asset, we quote,

$$\pi_S(t) := \frac{\int_{\mathbb{J}} \psi(t, z)\tilde{N}_{\mathbb{Q}}(t, dz)}{\tilde{S}(t)h(S(t - b))}$$

Since this is quite a strong requirement, this is why we can guess as much that the call option is probably not replicable! In which case the pricing and hedging is tougher than it would originally appear. This is the topic of the third chapter, namely a pricing for the call option in an incomplete market (studying the buyer's and seller's prices) as well the minimal variance hedging for the claim.

II.2. Numerical Approximations

2.1. Addition of jumps to the Classical Euler-Maruyama method

Unlike in the first chapter, we will not be discussing a logarithmic Euler-Maruyama method. One was proposed by [AH20], and evoked in [YHZ21], to approximate the stock price. However, this approximation is faulty and the convergence rate is wrong. Hence, we shall only discuss a classical approximation. Luckily [BBMY11] have treated this very topic and hence we shall follow in their footsteps. We refer to their paper for details. Moreover, we have used a similar path to prove the convergence of the continuous case. Hence, only slight modifications to the proofs are needed.

For generality's sake, we will prove the result for a larger class of SFDEs. To ensure the existence and uniqueness of the solution, we shall assume global Lipschitzianity of the coefficients F, H (see appendix B) and ask for adaptedness of the coefficients as well as Hölder-continuity of the initial data. We sum it up by the following hypotheses:

Hypotheses (H)

(i) (Global Lipschitzianity of the drift and volatility coefficients) The functions F and G are jointly continuous and uniformly (i.e globally) Lipschitz in the space (second) coordinate, i.e for all $t \in [0, T]$ and $\varphi_1, \varphi_2 \in L^p(\Omega, \mathcal{C})$, there exists a deterministic, left-continuous and non-decreasing function $\mu : [-R, 0] \to \mathbb{R}_+$ such that

$$||F(\varphi_1) - F(\varphi_2)||_{L^p(\Omega,\mathbb{R}^d)} + ||H(\varphi_1) - H(\varphi_2)||_{L^p(\Omega,\mathbb{R}^{d\times m})} \leq \int_{-R}^{0} |\varphi_1(u) - \varphi_2(u)|^p \, d\mu(u)$$

(ii) ($\mathbb{F}$-Adaptedness of the coefficients) If φ is adapted, so are $F(\varphi(t))$ and $H(\varphi(t))$.

(iii) (Continuity requirement on the initial condition) There exists a constant K and $\gamma \in [\frac{1}{2}, 1]$, such that for all $t, s \in [-b, 0]$ we have

$$|\eta(t) - \eta(s)| \leq K|t - s|^\gamma$$

If we model the stock price by the SFDE per the model presented in the previous section, i.e by approximating its logarithm using a classic Euler-Maruyama model, then for a given partition of the time axis π of step $\Delta t = \frac{T}{N}$ and $t_k = k\Delta t$, we can approximate the solution to this SFDE by a scheme of the sort

$$\begin{cases} \overline{S}(t_{k+1}) & = S(t_k) + F(\overline{S}(t_k)\Delta t + H(\overline{S}(t_k))\Delta Z_k, \qquad t_k \geq 0 \\ \overline{S}(t_k) & = \eta(t_k), \qquad\quad t_k \in [-R, 0] \end{cases} \tag{4.1}$$

where $\Delta Z_k = Z(t_{k+1}) - Z(t_k)$ and where

$$\overline{S}_{t_k}(\theta) := \frac{t_{i+1} - \theta}{\Delta t}\overline{S}(t_{k+i}) - \frac{t_i - \theta}{\Delta t}\overline{S}(t_{k+i+1})$$

for $t_i \leq \theta \leq t_{i+1},\ i = \lfloor -R \rfloor, \dots, -1$, represents the memory part. Just like in chapter 1, section 2, this is to ensure as much freedom on the delay, which does not need to coincide with the gridpoints.

Remark. We have only defined the process and its stochastic memory on the gridpoints but we can easily extend the processes to all $[0, T]$. That is, for any $t \in [0, T]$, we define

$$\overline{S}(t) := \sum_{j=0}^{N} \overline{S}(t_j) \mathbb{1}_{t_j, t_{j+1}}(t)$$

and

$$\overline{S}_t(\theta) := \sum_{j=0}^{N} \overline{S}_{t_j}(\theta) \mathbb{1}_{t_j, t_{j+1}}(t)$$

Finally, define

$$\begin{cases} d\zeta(t) & = F(t, \overline{S}_t)dt + H(\overline{S}_t)dZ(t), \qquad t \in [0, T] \\ \zeta(t) & = \eta(t), \qquad t \in [-R, 0] \end{cases} \tag{4.2}$$

Lemma 2.2.6. *For any $p \geq 1$, there exists K_p depending only on p such that*

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} |S(t)|^p \Big] \vee \mathbb{E}\Big[\sup_{0 \leq t \leq T} |\overline{S}(t)|^p \Big] \leq K_p$$

Proof. We refer to [BBMY11], lemma 3.1, for the proof of this result. The process is almost identical to that of lemma 1.2.2, for both the approximation bound and the solution bound. The difference, there is one more term to estimate. For example, if we want to bound the solution $S(t)$, we need to show (via the use of stopping times!) that

$$\mathbb{E}\Big[\sup_{t \in [0,T]} \Big| \int_0^t H(S(u_-))N(du, dz) \Big|^p \Big] \leq C_1 + C_2 \int_0^t E\Big[\sup_{t \in [0,T]} |S(u_-)|^p \Big] du$$

and we conclude via Grönwall's inequality.

□

We proceed with the proof of the next analogous lemma from chapter 1, i.e the norm of the difference between the memory process, discrete and continuous.

Lemma 2.2.7. *There is $\beta > 0$ independent of Δt such that*

$$\mathbb{E}[|\zeta_t(\theta) - \overline{S}_t(\theta)|^p] \leq \beta \Delta t$$

for $\theta \in [-R, 0]$ and where ζ_t is the stochastic memory of ζ.

Proof. We once again refer to [BBMY11], lemma 3.2, for details. The crux of the change is, again, to estimate an extra term. Here, we use characteristic functions to prove that, for a fixed Δt, then

$$\mathbb{E}\Big[|\Delta Z_k|^p\Big] \leq C\Delta t$$

for some constant C. and the proof is complete.

□

Finally, we are ready to state the convergence result for jumps.

Theorem 2.2.12 (Convergence of the classical Euler-Maruyama method under Global Lipschitz Hypothesis). *Under the earlier hypotheses (H), we have*

$$\lim_{\Delta t \to 0} \mathbb{E}\Big[\sup_{t \in [0,T]} |S(t) - \zeta(t)|^p \Big] = 0$$

Moreover, we have

$$\mathbb{E}\Big[\sup_{t \in [0,T]} |S(t) - \zeta(t)|^p \Big] = O(\Delta t^{\frac{1}{p}})$$

i.e, the order of convergence of the method is exactly $\frac{1}{p}$.

Proof. See [BBMY11] for the computational details. Remember that in the proof of theorem 1.2.6 (see equation (2.5)) we have broken down the proof into two inegral estimates: one for the drift coefficient and one for the volatility coefficient. Here, we have a third integral, which we estimate much like the volatility coefficient from theorem 1.2.6. The Burkholder-Davis-Gundy inequality is used followed by Hölder's estimates. This nets us an estimate of the same shape as the first and second of (2.5), and then we can proceed as before using Grönwall's inequality. The problem, however, is that the estimate is not as sharp, hence why the order of convergence is not $\frac{1}{2}$ but instead $\frac{1}{p}$. This completes the proof.

□

Remark. Note that this result is different from that of [AH20] as they determine the order of convergence to be $\frac{1}{2}$ for all $p \geq 2$, not just $p = 2$. This cannot be, however, as is pointed out by [BBMY11] in remark 3.1. Indeed, if we remind the reader that A is the poisson process in the compound Poisson jump process Z, we have

$$E\Big[|A(k+1) - A(k)|^p\Big] \leq C\Delta t$$

where $C > 0$. This differs from the Brownian motion increments where we would find $\Delta t^{\frac{p}{2}}$ instead. This leads us to conclude that the order of convergence found in [AH20] is wrong.

2.2. Simulating Stock Prices

Now that we have proven the convergence of the classical Euler-Maruyama method for SFDEs, we will run Monte-Carlo approximations using the Euler-Maruyama method and proceed to price options based on these approximations. Imagine a stock with stock price 100 at time $t = 0$ and take $r = 0.05$ with $T = 1$, $h(x) = 0.01 + 0.005\cos(x)$ and $f(x) = r + h(x)$. Moreover, take

$$\eta(x) = 100 + \log(1 + x) + \sin\left(\frac{x}{50}\right)$$

These coefficients certainly verify the hypotheses: they are globally Lipschitz, f is strictly positive and g is certainly bounded. Lastly, we have taken $p = 1$, $q = 0$, $\gamma = 0.5$, $\theta = 0$ and $\lambda = 0.3$. We will be doing 1000 approximations of 1000 steps each, using $b = 0.05$ as delay. Most importantly, as stated before, these coefficients verify hypothesis $(H)(iv)$ by construction. Below is the result of these approximations

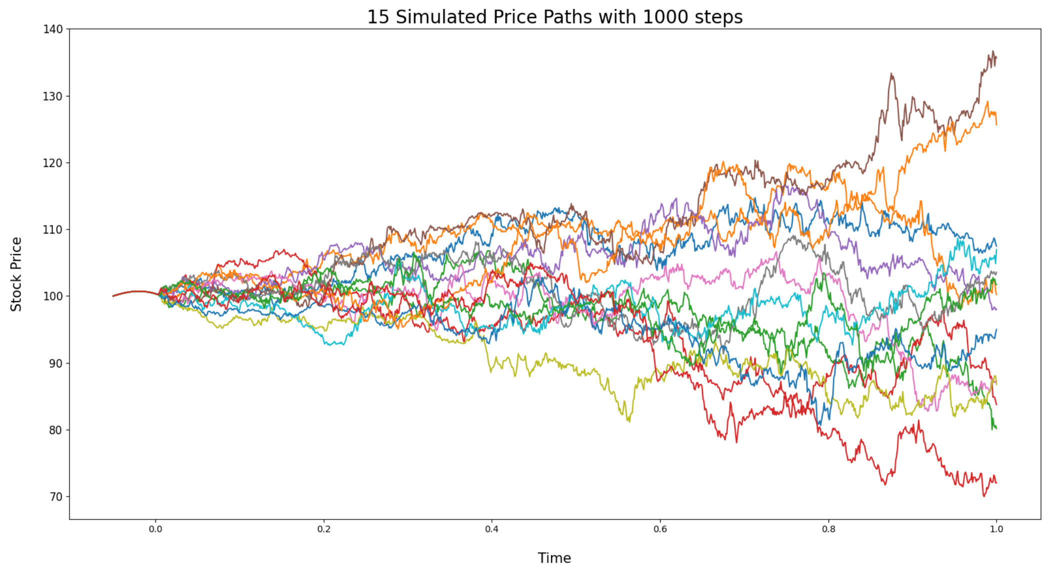

Figure II.1: Stock Price Simulation using the delayed model with jumps over the period of a year

2.3. Call Option Price Comparisons

For test purposes only, we will choose an equivalent martingale measure denoted by $\mathbb{Q}$ defined with

$$\theta(t,z) := \theta(t) = \frac{f(S(t-b)) - r}{h(S(t-b)) \int_{\mathbb{J}} z\nu(dz)}$$

By (3.6), we are able to rewrite the dynamic of S(t) under the measure $\mathbb{Q}$. As always, because the measure was obtained via Girsanov's theorem, the jump dynamics are unchanged. However, very much like in the continuous case, we have a different drift. Indeed, by definition of θ, we have

$$\begin{aligned} f_{\mathbb{Q}}(S_t) =& f(S(t-b)) - \frac{f(S(t-b)) - r}{\cancel{h(S(t-b)) \int_{\mathbb{J}} z\nu(dz)}} \cancel{h(S(t-b))} \cancel{\int_{\mathbb{J}} z\nu(dz)} \\ =& \cancel{f(S(t-b))} - \cancel{f(S(t-b))} + r = r \end{aligned}$$

where r is once again the interest rate. Just like f, by being constant, $f_{\mathbb{Q}}$ verifies all the hypotheses listed previously. Proceeding with this change, i.e replacing f by $f_{\mathbb{Q}}$, we price mock options and compare them to the classical Black-Scholes formula. We have used $\sigma = 0.04$ for the classical model. Below are the results:

Strike Price of the Call Option	Price of a European Call Option using the delayed Black-Scholes model with jumps	Price of a European Call Option using the classic Black-Scholes formula
$100	$7.63	$8.59
$105	$5.46	$6.03
$110	$3.23	$4.07
$115	$2.01	$2.64
$120	$1.26	$9.49
$125	$0.73	$1.00
$130	$0.43	$0.58
$135	$0.11	$0.33
$140	$0.06	$0.18

2.4. Code & Algorithms

Below is the code used for the delayed model with jumps.

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt

#Setting up the variables

K = 100 #strike price
r = 0.05 #risk-free interest rate
T = 1 #time in years
N = 1000
deltat = T/N #time step
b = 0.05*T #delay
i = 1000 #number of simulations
m = 1 #Number of positive exponential paremeter
n = 1 #Number of negative exponential parameter
p = 1 #probability of positive exponential paremeter
q = 0 #probability of negative exponential paremeter
LowerBound = 0 #Lower bound of the Levy process
Gamma = 0.5 #positive exponential coefficient
Beta = 0 #negative exponential coefficient
Lambda = 0.3 #Poisson Coefficient

S = np.zeros([i,N]) #Stock price
t = np.linspace(-b, T, N)  # Time

# Defining the coefficients and initial data
def f(x):
    return r

def h(x):
    return 0.01 + 0.005*np.cos(x)

def eta(x):
    return 99 + np.sin(x/24) + np.cos(x/42)

def LinInterS(S,j,k,ib,b):
```

```
    return (((ib+1)*deltat - b)/deltat)*S[j, k-ib] - ((ib*deltat - b)/deltat)*
        S[j, k-ib+1]

# Define the call option payoff function
def call_payoff(S, K):
    return np.maximum(S-K, 0)

#Inverse transform sampling part

def Yinv(x):
    if x > q:
        return -np.log(1-(x-q)/p)/Gamma
    elif x <= q:
        return 1/Theta*np.log(x/q)

#Definition of the double-exponential Levy process

def Z(u):
    Npoisson = np.random.poisson(Lambda*u)
    if Npoisson != 0:
        Y = np.empty(Npoisson)
        for j in range(Npoisson):
            Y[j] = Yinv(np.random.rand())
        Z = np.sum(Y) - Lambda*u*(p*Gamma + q*Beta*(1-np.exp(-Beta*LowerBound)
            ))  #hyper exponential process
    else:
        Z = 0
    return Z

def deltaZ(u):
    return Z(u+1) - Z(u)

#Approximation Algorithm
option_prices = []
for j in range(0,i-1):
    S[j,0] = eta(0)
    for k in range(0,N-1):
        if k*deltat > b:
            if b % deltat == 0:
                S[j,k+1] = S[j,k] + f(S[j,k-int(b/deltat)])*deltat + h(S[j,k-(
                    b/deltat)])*deltaZ(k)
            else:
                ib = int(np.floor(b/deltat))
                S[j,k+1] = S[j,k] + f(LinInterS(S,j,k,ib,b))*deltat + h(
                    LinInterS(S,j,k,ib,b))*deltaZ(k)
        else:
            S[j,k+1] = eta(k+1)

    # Calculate the call option payoff at expiration time T
    call_payoffs = call_payoff(S[j, -1], K)

    # Print the stock price at the maturity date
    print(f"Stock price at maturity for simulation {j+1}: {S[j,-1]}")

    # Discount the call option payoff to the present time
    discounted_payoffs = np.exp(-r * T) * call_payoffs
```

```
    # Calculate the Monte Carlo estimate of the call option price
    option_price = np.mean(discounted_payoffs)
    option_prices.append(option_price)

    plt.plot(t,S[j])

#Plotting

plt.title('%d Simulated Price Paths with %d steps' % (i,N), fontsize=20)
plt.xlabel('Time', fontsize=15, labelpad=20)
plt.ylabel('Stock Price', fontsize=15, labelpad=20)
plt.yticks(fontsize=12)
plt.show()

# Print the mean call option price
mean_option_price = np.mean(option_prices)
print("Mean call option price: %f" % mean_option_price)
```

Chapter III

Pricing and Hedging in an Incomplete Delay Market

III.1. Buyer's and Seller's prices: an Estimation for the Pricing of a Call and Put option

We have seen previously in chapter 1 that, in a complete market, a claim has a unique price. However we have also seen in the previous chapter that a market with jumps is only complete if there are as many assets as there are jump sizes and hence the market is incomplete. To be exact, while the non-arbitrage property is guaranteed, the replicability of claims is not. Thus, a question arises: what is a correct price for a claim in an incomplete market? We will answer this question in the case of a European call option, but the same can be done with a European put option. We will follow [ØS05].

Going back to the definition and if need be to simpler discrete models like the Cox-Ross-Rubinstein model, it is easy to see that both parties (the buyer and the seller) each have an idea for a fair price. If F denotes the call option and $\mathcal{A}$ denotes the set of all admissible portfolios, to the buyer, a fair price is

$$p_b(F) = \sup\{x \in \mathbb{R}, \exists \varphi \in \mathcal{A} \text{ s.th } x - \int_0^T \varphi(u)\, dS(u) \leq -F \text{ a.s.}\} \tag{5.1}$$

$$p_s(F) = \inf\{x \in \mathbb{R}, \exists \varphi \in \mathcal{A} \text{ s.th } x + \int_0^T \varphi(u)\, dS(u) \geq F \text{ a.s.}\} \tag{5.2}$$

Theorem 3.1.13 (**Pricing and Hedging in Incomplete Markets**). *We have*

$$p_b(F) \leq \mathbb{E}_{\mathbb{Q}}\left[e^{-rT} F\right] \leq p_s(F) \tag{5.3}$$

Moreover, if F is replicable, then

$$p_b(F) = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT} F\right] = p_s(F) \tag{5.4}$$

where $\mathbb{Q}$ is an equivalent martingale measure as before.

Proof. We follow the proof of [ØS05], theorem 2.14. Let us start by proving the first claim 5.3. Suppose $x \in \mathbb{R}$ and $\varphi \in \mathcal{A}$ s.th

$$x - \int_0^T \varphi(u)\, dS(u) \leq F \tag{5.5}$$

Discounting both sides,

$$x - \int_0^T \varphi(u)\, d\widetilde{S}(u) \leq e^{-rT} F$$

Since φ is admissible, it is lower bounded and hence the left-handside is lower bounded. Moreover, since $\widetilde{S}(t)$ is a martingale under $\mathbb{Q}$, then the stochastic integral w.r.t $\widetilde{S}(u)$ of $\varphi(u)$ is a local martingale under $\mathbb{Q}$. Since a local martingale bounded from below is a supermartingale (see [BB17] p.205 for details on local martingales) then we have that

$$x \leq \mathbb{E}_{\mathbb{Q}}\Big[x - \int_0^T \varphi(u)\, d\widetilde{S}(u)\Big] \leq \mathbb{E}_{\mathbb{Q}}\big[e^{-rT} F\big]$$

Since this is true for all choices of x, φ satisfying (5.4), we conclude that $p_b(F) \leq \mathbb{E}_Q\big[e^{-rT} F\big]$. In an analog fashion, we can do the same for the seller's price to show that

$$p_s(F) \geq \mathbb{E}_{\mathbb{Q}}\big[e^{-rT} F\big]$$

which finishes the first part of the proof and proves (5.3). We now turn our attention to (5.4). Suppose F is replicable. Then there exists $x \in \mathbb{R}$ and $\varphi \in \mathcal{A}$ s.th

$$x - \int_0^T \varphi(u)\, d\widetilde{S}(u) = e^{-rT} F$$

Taking the expectation on both sides and using the fact that $\widetilde{S}(u)$ is a $\mathbb{Q}$-martingale, we get that $x = \mathbb{E}_{\mathbb{Q}}\big[e^{-rT} F\big]$. Thus, $\mathbb{E}_{\mathbb{Q}}\big[e^{-rT} F\big] \leq p_b(F)$ and we (**??**.3) to conclude that

$$p_b(F) = \mathbb{E}_{\mathbb{Q}}\big[e^{-rT} F\big]$$

An exact analog can be done for $p_s(F)$ which proves (5.4).

□

This tells us that although we can indeed not directly give a fair price for the Call (resp. Put) option, as it is not replicable. It is possible, however, to determine what an unfair price would be. Rather, we have a range of prices where there is neither arbitrage for either the seller or the buyer. Rather than having a single, unique price, the matter becomes more complicated when jumps are involved.

III.2. A Risk-Minimizing Approach to Hedging

2.1. A Short Reminder on Malliavin Calculus

We briefly present a few definitions results from [DN01] without proofs, results which will be necessary for the upcoming section. The reason for the lack of proofs is because these results were already studied and hence the author assumes that the reader will be familiar with these concepts.
Start by defining $\tilde{L}^2((\lambda \times \nu)^j)$ to be the space of symmetric functions of $L^2((\lambda \times \nu)^j)$ in the sense of permutation of the arguments. Then we define $\mathbb{D}_{1,2} \subset L^2(\mathbb{P})$ to be the space of all $\mathcal{F}_T$-measurable random variables of $L^2(\mathbb{P})$ such that there exists functions $f_j \in \tilde{L}^2((dt \times \nu)^j)$ such that

$$F = \sum_{j=0}^{\infty} I_j(f_j)$$

where I_j are j iterated stochastic integrals. We call this expansion a chaos expansion. Moreover, for F to be in $\mathbb{D}_{1,2}$, we require that

$$||F||^2_{\mathbb{D}_{1,2}} = \sum_{j=1}^{\infty} jj!||f_j||^2_{L^2((dt\times\nu)^j)} < \infty$$

Then, for $F \in \mathbb{D}_{1,2}$, define the Malliavin derivative of F as

$$D_{t,z}F = \sum_{j=1}^{\infty} jI_{j-1}\big(f_j(t,z)\big)$$

We can finally state the most important result and the one we will rely on for the next section.

Theorem 3.2.14 (Clark-Ocone Formula for pure jump Lévy processes). *Let $F \in \mathbb{D}_{1,2}$. Then*

$$F = \mathbb{E}[F] + \int_0^T \int_{\mathbb{J}} \mathbb{E}\big[D_{t,z}F|\mathcal{F}_t\big]\tilde{N}(dt,dz)$$

where the conditional expectation is chosen to be a predictable version.

Proof. See [NØP08], theorem 12.16.

□

2.2. Minimal Variance Hedging with Delay

We have seen at the end of the last chapter that an application of the Martingale representation theorem would not work and would not yield a portfolio replicating a call option. Moreover, in the previous section, we have seen that this non-replicability yields a range of prices and not a single, unique price for the option. How, then, would one hedge against the call option if it is not replicable?

Just like result (5.3), we will instead choose to see how close we can get instead of trying to get the exact hedging portfolio which does not exist. How "close" we are will be decided by the $L^p(\mathbb{Q})$ norm difference where $\mathbb{Q}$ denotes an equivalent martingale measure (chosen, as there are many). If $p = 2$, the name "minimal variance hedging" takes full meaning. In fact, we will only work in the $p = 2$ case, as we will make use of the Hilbert structure of the $L^2(\mathbb{Q})$ space. We will make full use of Malliavin Calculus although we do not need to extend it to the realm of stochastic functional equations as it will only be used on our claim, per the Clark-Ocone formula (see the previous subsection for a reminder). We will rely on section 12.6 of [NØP08] adapted to our problem as well as [DN01].

Remark. Note that in [DN01], it is specified that the chosen Lévy process must be a martingale under the chosen measure $\mathbb{Q}$, which is the case from the previous chapter since we will be considering $\tilde{S}$. This is the main reason for the change from [AH20] where the Lévy process is not compensated.

Before all else, let us mention that the following result, general though it may be, has the assumption that $f = 0$. While this is important, as we will be considering the Lévy process $\widetilde{S}$ under the measure $\mathbb{Q}$, which, by (3.8) has no drift, this shall not be of concern.

As mentioned previously, we state a result on the geometry of $L^2(\mathbb{Q})$, namely on the orthogonality of a certain family of subsets. Let $H_0 \subset L^2(\mathbb{Q})$ be the space of all F such that $\mathcal{D}F = 0$. Denote by $H_{\widetilde{S}} \subset L^2(\mathbb{Q})$ the subspace of all stochastic integrals with respect to the to the Lévy process $\widetilde{S}$, i.e all processes of the form $\int_0^T \varphi(t)\, d\widetilde{S}(t)$. Finally, denote by $\mathcal{D}$ the non-anticipating derivative. Then we have the following

Lemma 3.2.8. *Let $F \in L^2(\mathbb{Q})$. Then there exists a unique $F_0 \in H_0$ such that*

$$F = F_0 + \int_0^T \mathcal{D}F(t) d\widetilde{S}(t)$$

More generally, we have

$$H = H_0 \oplus H_{\widetilde{S}}$$

Proof. The result and proof can be found in [DN01], p.5. The main idea is to partition H_Z into a direct sum of the spaces $H_{\Delta\widetilde{S}}$, which are defined as the spaces of random variables of the form $F\Delta\widetilde{S}$ where $\Delta\widetilde{S}$ denotes the increments of the Lévy process $\widetilde{S}$ on the intervals $(t, t+h]$ for a time step Δt. If we denote by φ^h the simple functions from the definition of the stochastic integral of a process φ, i.e

$$\int_0^T \varphi(t)\, d\widetilde{S}(t) = \lim_{h\to 0} \int_0^T \varphi^h(t)\, d\widetilde{S}(t)$$

then, for $F = \int_0^T \varphi(t)\, d\widetilde{S}(t) \in H_{\widetilde{S}}$, the projection of F on $H_{\Delta\widetilde{S}}$ is by construction $\varphi^h \Delta\widetilde{S}$. Now, for some fixed $h > 0$, consider

$$H_h := \sum \oplus H_{\Delta\widetilde{S}}$$

Given $F \in L^2(\mathbb{Q})$, then for some $\psi\Delta\widetilde{S}$ we have

$$\mathbb{E}\big[(F - \varphi^h \Delta\widetilde{S})(\psi\Delta\widetilde{S})\big] = 0 \tag{5.1}$$

where the equality is proved by condition with respect to $\mathcal{F}_t$ and using the properties of the Lévy process $\widetilde{S}$. That is, the projection of $F \in L^2(\mathbb{Q})$ onto H_h is

$$\int_0^T \varphi^h(t)\, d\widetilde{S}(t)$$

Letting $h \to 0$, we see that, by construction, we indeed have $H_h \to H_{\widetilde{S}}$. Now once again consider an arbitrary $F \in L^2(\mathbb{Q})$. Take $\varphi = \mathcal{D}F$, we have

$$\varphi^h = \mathbb{E}\big[F \frac{\Delta\widetilde{S}}{\mathbb{E}[|\Delta\widetilde{S}|^2|\mathcal{F}_t]} |\mathcal{F}_t\big]$$

Then

$$F_0 := F - \int_0^T \varphi(t)\, d\widetilde{S}(t)$$

is indeed in H_0 by construction and is orthogonal to any element of $H_{\widetilde{S}}$ by (5.1). Uniqueness is ensured by uniquess of $\mathcal{D}F$, which concludes the proof.

□

Remark. This proof is valid for any martingale Lévy process Z in general. Not that this may not be the Z we have have been working with so far. This is a general result which will be of help for the next result, key of this chapter.

Theorem 3.2.15 (Minimal Variance Hedging of a clain in a pure jump Lévy process driven market). *Let $F \in \mathbb{D}_{1,2} \subset L^2(\mathbb{Q})$ be a claim. Then there exists F^* and $\pi^*(t)$ defined by*

$$F^* := \mathbb{E}[F] + \int_0^T \pi^*(t)\, d\widetilde{S}(t)$$

such that π^* is the minimal variance hedging portfolio, i.e that

$$||F - F^*||_{L^2(\mathbb{Q})} = \min_{\pi} \mathbb{E}\Big[\Big|F - \mathbb{E}[F] - \int_0^T \pi(t)\, d\widetilde{S}(t)\Big|^2\Big]$$

Moreover, we have

$$\pi^*(t) = \frac{\int_{\mathbb{J}} \mathbb{E}\big[D_{t,z}F|\mathcal{F}_t\big] z h(S(t-b))\, \nu(dz)}{\int_{\mathbb{J}} h^2(S(t-b))\, \nu_{\mathbb{Q}}(dz)}$$

Proof. We adapt a proof similar to [NØP08], theorem 12.24. The result can also be found in [BDNL+03], theorem 4.1. Suppose such a π^* and let us define the claim F^* by

$$F^* := \mathbb{E}[F] + \int_0^T \pi^*(t)\, d\widetilde{S}(t) = \mathbb{E}[F] + \int_0^T \int_{\mathbb{J}} \pi^*(t) z h(S(t-b))\, \tilde{N}_{\mathbb{Q}}(dt, dz)$$

Under these notations, we make use of the Hilbert structure of $L^2(\mathbb{Q})$. We know from lemma 3.2.8 that $\mathbb{E}\big[(F - F^*)G\big] = \mathbb{E}\big[F_0 G\big] = 0$ for all $\mathcal{F}_T$-measurable $G \in L^2(\mathbb{Q})$ such that

$$G := \int_0^T \psi(t)\, d\widetilde{S}(t) = \int_0^T \int_{\mathbb{J}} \psi(t) z h(S(t-b))\, \tilde{N}_{\mathbb{Q}}(dt, dz)$$

where ψ denotes an admissible portfolio and $F_0 \in H_0$ is the orthogonal remainder. Using this Hilbert structure, we will be able to confirm the existence of π^* as well as categorize it. Let us first rewrite F using the Clark-Ocone formula for Lévy processes (see theorem 3.2.14).

$$F = \mathbb{E}[F] + \int_0^T \int_{\mathbb{J}} \mathbb{E}\big[D_{t,z}F | \mathcal{F}_t\big]\, \tilde{N}_{\mathbb{Q}}(dt, dz)$$

Thus,

$$\mathbb{E}\big[F_0 G\big] = \mathbb{E}\Big[\Big(\int_0^T \int_{\mathbb{J}} \mathbb{E}\big[D_{t,z}F|\mathcal{F}_t\big] - \pi^*(t) z h(S(t-b))\, \tilde{N}_{\mathbb{Q}}(dt, dz)\Big) \cdot \Big(\int_0^T \int_{\mathbb{J}} \psi(t) h(S(t-b))\, \tilde{N}_{\mathbb{Q}}(dt, dz)\Big)\Big]$$

Developing this expression using the Ito isometry, we get

$$\mathbb{E}\big[F_0 G\big] = \mathbb{E}\Big[\int_0^T \psi(t) \Big(\int_{\mathbb{J}} \mathbb{E}\big[D_{t,z}F|\mathcal{F}_t\big] z h(S(t-b)) - \pi^*(t) z h^2(S(t-b))\Big)\, \nu_{\mathbb{Q}}(dz) dt\Big] = 0$$

Which in turn implies

$$\int_{\mathbb{J}} \mathbb{E}\big[D_{t,z}F|\mathcal{F}_t\big] z h(S(t-b)) - \pi^*(t) z h^2(S(t-b)) \nu_{\mathbb{Q}}(dz) = 0$$

i.e

$$\pi^*(t) = \frac{\displaystyle\int_{\mathbb{J}} \mathbb{E}\big[D_{t,z}F|\mathcal{F}_t\big] z h(S(t-b))\, \nu_{\mathbb{Q}}(dz)}{\displaystyle\int_{\mathbb{J}} h^2(S(t-b))\, \nu_{\mathbb{Q}}(dz)}$$

Hence, such a π^* verifying this sufficient condition does indeed exist and it is indeed the one we sought to find. The proof is complete.

□

Remark. Using lemma 3.2.8, we can go a step further and deduce that $\pi^* = \mathcal{D}F$, the non-anticipative derivative of F.

Conclusion

In this thesis, we have developed two financial market models with delay: a continuous model and a pure jump model. Both have been studied and the results we have proven were somewhat expected. The continuous model is complete while the jump model is incomplete. In the continuous case, we have determined a pricing formula for the call and put options as well as the hedging portfolio. For the study of these markets, the theory of stochastic delay differential equations was studied and developed in the continuous and jump case both. They can be found in the respective appendixes A and B.

In the jump case, while we have determined the lack of arbitrage opportunities, we were not able to price the call and put options. Moreover, this was done with a very special bound condition on the coefficients. In an incomplete market, where the equivalent martingale measure is not unique, the pricing depends on the measure chosen. Given one, we have determine the minimal variance hedging portfolio, that is the closest portfolio (in the $L^2(\mathbb{Q})$ sense) which minimizes the risk. A formula for this portfolio, depending on the measure $\mathbb{Q}$ chosen, was given.

In both cases, we have studied numerical approximations of these models. In the continuous case, two approaches were developed: a classic approach and a logarithmic approach. The latter proved to be extremely restrictive on the coefficients of the stochastic differential equation, although its order of convergence remained unchanged from the classical method. It is important to note that the addition of delays to the latter does not change the order of convergence either (up to a small $\varepsilon > 0$ with local Lipschitz hypothesis). Its implementation requires the classical scheme under locally Lipschitz assumptions on the coefficients, which was therefore developed.

In the jump case, only the classical method was developed under globally Lipschitz assumptions. An extension of the logarithmic scheme was considered and eventually dropped as the hypotheses became so restrictive, more so than in the continuous case, that its usability was in doubt. We have seen that this is a mere extension of the scheme, although the addition of jumps changes the order of convergence from $\frac{1}{2}$ to $\frac{1}{p}$ due to the increments of the Poisson process.

Many papers were researched and used in this thesis. All of them can be found below in the biography. We note that [AH20] was our starting point for the jump case but was found to have many mistakes. A glaring one is the false claim that the jump market is complete.

Moreover, the authors develop a logarithmic numerical approximation which is not clear and of which the convergence rate is also found to be wrong. In particular, the authors claim that the order of convergence is $\frac{1}{2}$ when it can, at most, in fact, only be $\frac{1}{p}$ due to the presence of jumps.

From here, there are many routes one could take to go further in this topic. Plenty of questions remain, such as if it is possible to improve the already presented results or if it is possible to lower the hypotheses we have made. As we have mentioned many times, some hypotheses we have made are quite restrictive. We mention a machine learning approach to portfolio optimization, inspired by [Wan19], which was shortly considered in earlier drafts of the project. Sadly, there was not enough time to study it, develop it and include it in this thesis.

Appendix A

This appendix is dedicated to SFDE's in the Brownian setting and the general existence and uniqueness theorem for such SFDE's. Simply put, an SFDE is a stochastic differential equation where the coefficients can also depend on the memory of the process. Let $\mathcal{C} = C\Big(\big([-R,0],\mathbb{R}^d\big)\Big)$ and let $\eta \in L^2\big(\Omega,\mathcal{C}\big)$. We equip $L^2(\Omega,\mathcal{C})$ with the norm

$$||\eta||_{L^2(\Omega,C)} := \Big(\int_\Omega ||\eta(\omega)||^2_{\mathcal{C}}\, d\mathbb{P}(\omega)\Big)^{\frac{1}{2}}$$

Let $f : [0,T]\times L^2(\Omega,\mathcal{C}) \to L^2(\Omega.\mathbb{R}^d)$ and $g : [0,T]\times L^2(\Omega,\mathcal{C}) \to L^2(\Omega.\mathbb{R}^{d\times n})$ be the drift and volatility coefficients respectively. Then an SFDE is a differential equation of the form:

$$\begin{cases} dS(t) &= f(t,S_t)dt + g(t,S_t)dB(t), \qquad t\in[0,T] \\ S(t) &= \eta(t), \qquad t\in[-R,0] \end{cases} \tag{A.1}$$

where we define the stochastic memory S_t by $S_t(s) := S(t+s)$, $s\in[-R,0]$. It is important to note that while it may not seem apparent at first, the coefficient f and g may also depend on $S(t)$ and not just the memory path. The reason for omitting to specify the dependence on $S(t)$ is, first, to simplify notations and second, that we can write $S(t)$ as $S_t(0)$. Also note here that we are subtly using the fact that there no jumps allowed and that the paths are continuous.

Remark. If f and g are deterministic, which is usually the case, then the space $L^2(\Omega,C)$ is replaced by simply $\mathcal{C}$. For example, this is the case of the earlier chapters of this thesis. However, for generality's sake, we proceed with $L^2(\Omega,C)$.

Let us now suppose the following:

(H)(i) The functions f and g are jointly continuous and uniformly Lipschitz in the space (second) coordinate, i.e for all $t\in[0,T]$ and $\varphi_1,\varphi_2\in L^2(\Omega,\mathcal{C})$, there exists a constant L independent of t such that

$$||f(t,\varphi_1)-f(t,\varphi_2)||_{L^2(\Omega,\mathbb{R}^d)} + ||g(t,\varphi_1)-g(t,\varphi_2)||_{L^2(\Omega,\mathbb{R}^{d\times m})} \le L||\varphi_1-\varphi_2||_{L^2(\Omega,\mathcal{C})}$$

(H)(ii) If $\varphi : [0,T]\to L^2(\Omega,\mathcal{C})$ is adapted, so are $f(t,\varphi(t))$ and $g(t,\varphi(t))$.

Theorem 3.2.16. *(Existence and uniqueness) Suppose* f *and* g *both verify the hypotheses* (H) *above and let* η *be an* $\mathcal{F}_0$*-mesurable element of* $L^2(\Omega, \mathcal{C})$*. Then the SFDE has a unique solution* ${}^{\eta}x : [-r, \infty) \times \Omega \to \mathbb{R}^d$ *with initial condition* η*. Moreover,* ${}^{\eta}x \in L^2\big(\Omega, C\big([-R, T], \mathbb{R}^d\big)\big)$ *for all* $T > 0$.

Proof. We will use a Picard iteration type proof. The goal of this proof is to approximate step by step the solution with a well-chosen sequence of successive approximations. First, we need to establish the space in which those approximations take place. We denote by $L^2_A\big(\Omega, C\big([-R, T], \mathbb{R}^d\big)\big)$ the space of all processes $X \in L^2\big(\Omega, C\big([-R, T], \mathbb{R}^d\big)\big)$ such that $X(t)$ is $\mathcal{F}_0$-measurable for $t \in [-R, 0]$ and $\mathcal{F}_t$-measurable for $t \in [0, T]$. We define suitable Picard iterative processes, check that they are indeed living in this space, and that they form a Cauchy sequence. Specifically, we will want to prove that our approximations X^k will possess the following three properties:

(P)(i) X^k belongs in the approximation space, i.e $X^k \in L^2_A\big(\Omega, C\big([-R, T], \mathbb{R}^d\big)\big)$
(P)(ii) For a fixed time, the stochastic memory is regular enough i.e for each $t \in [0, T]$, $X^k_t \in L^2(\Omega, \mathcal{F}_t; \mathcal{C})$.
(P)(iii) Two successive approximations are "close enough", i.e

$$||X^{k+1} - X^k||_{L^2(\Omega,\mathcal{C})} \leq \left((4L^2)^{(k-1)} \frac{T^{k-1}}{(k-1)!}\right)^{\frac{1}{2}} ||X^2 - X^1||_{L^2(\Omega,\mathcal{C})}$$

and

$$||X^{k+1}_t - X^k_t||_{L^2(\Omega,\mathcal{C})} \leq \left((4L^2)^{(k-1)} \frac{t^{k-1}}{(k-1)!}\right)^{\frac{1}{2}} ||X^2 - X^1||_{L^2(\Omega,\mathcal{C})}$$

where L is the (uniform) Lipschitz constant of g. First define

$$X^1(t, \omega) = \begin{cases} \eta(0, \omega) & \text{if } t \in [0, T] \\ \eta(t, \omega) & \text{otherwise} \end{cases}$$

which corresponds to freezing the initial condition at time $t = 0$. We then define the following approximations for all $k > 1$ by

$$X^{k+1}(t, \omega) = \begin{cases} \eta(0, \omega) + \displaystyle\int_0^t g(u, X^k_u)\, dB(u)(\omega) & \text{if } t \in [0, T] \\ \eta(t, \omega) & \text{otherwise} \end{cases}$$

To prove that our approximation is well chosen, let us proceed by induction. Since the initial condition η is assumed to be an element of $L^2(\Omega, \mathcal{C})$, it follows that ${}^1X \in L^2_A\big(\Omega, C\big([-R, T], \mathbb{R}^d\big)\big)$ by definition of ${}^1X$. Hence, the delayed ${}^1X_t$ also belongs to $L^2\big(\Omega, C\big([-R, T], \mathbb{R}^d\big)\big)$ and it is $\mathcal{F}_t$-mesurable since η is $\mathcal{F}_0$-measurable. Thus, $P(1)(i)$ and $P(1)(ii)$ are both verified. Inputting $k = 1$ in $P(k)(iii)$ trivially yields $P(1)(iii)$.
Let us suppose that the three properties $P(k)$ holds for a certain k. By $P(k)(ii)$, X^k_t is $\mathcal{F}_t$-mesurable and hence, by $H(ii)$, so it $g(t, X^k_t)$. By the adaptability of the Ito integral, this extends to $X^{k+1}(t)$ and its stochastic memory ${}^{(k+1)}X_t$. The L^2 property of the process and

its stochastic memory follow from the definition of g and hence, $P(k+1)(i)$ and $P(k+1)(ii)$ are verified.
Let us now check the last remaining property. By definition of ${}^kX$, we have

$$||X^{k+2}-X^{k+1}||^2_{L^2(\Omega,\mathcal{C})} \leq \mathbb{E}\big[\sup_{t\in[0,T]}\big|\int_0^t g(u,X_u^{k+1})-g(u,X_u^k)\,dB(u)\big|^2\big] < 4\mathbb{E}\big[\big|\int_0^T g(u,X_u^{k+1})-g(u,X_u^k)\,dB(u)\big|^2\big]$$

where we have used Doob's martingale inequality. Using the Ito isometry and Fubini, we obtain

$$\begin{aligned} 4\mathbb{E}\big[\big|\int_0^T g(u,X_u^{k+1})-g(u,X_u^k)\,dB(u)\big|^2\big] &= 4\mathbb{E}\big[\int_0^T \big(g(u,X_u^{k+1})-g(u,X_u^k)\big)^2\,du\big] \\ &= 4\int_0^T \underbrace{\mathbb{E}\big[\big(g(u,X_u^{k+1})-g(u,X_u^k)\big)^2\big]}_{=||g(u,X^{k+1})-g(u,X_u^k)||^2_{L^2(\Omega,\mathbb{R}^{d\times n})}}\,du \\ &= 4\int_0^T ||g(u,X_u^{k+1})-g(u,X_u^k)||^2_{L^2(\Omega,\mathbb{R}^{d\times n})}\,du \end{aligned}$$

By $H(i)$, i.e Lipschitzianity in the second variable, we have that

$$||g(u,X^{k+1})-g(u,X_u^k)||^2_{L^2(\Omega,\mathbb{R}^{d\times n})} < L^2||X^{k+1}-X^k||^2_{L^2(\Omega,\mathcal{C})} < L^2(4L^2)^{k-1}\frac{T^{k-1}}{(k-1)!}||X^2-X^1||^2_{L^2(\Omega,\mathcal{C})}$$

and hence

$$\begin{aligned} 4\int_0^T ||g(u,X^{k+1})-g(u,X_u^k)||^2_{L^2(\Omega,\mathbb{R}^{d\times n})} &\leq 4L^2(4L)^{k-1}\frac{T^{k-1}}{(k-1)!}||X^2-X^1||^2_{L^2(\Omega,\mathcal{C})}\int_0^T u^{k-1}\,du \\ &= (4L^2)^k\frac{T^k}{k!}||X^2-X^1||^2_{L^2(\Omega,\mathcal{C})} \end{aligned}$$

Finally, we note that

$$||X_t^{k+2}-X_t^{k+1}||_{L^2(\Omega,\mathcal{C})} \leq ||X^{k+2}-X^{k+1}||_{L^2(\Omega,\mathcal{C})}$$

which completes our inductory proof of the properties (P) of our approximations. It now remains to check that those approximations indeed converge to a unique solution of our SFDE.
Rewrite X^k as

$$X^k = X^1 + \sum_{j=1}^{k-1}\left({}^jX -{}^{j-1}X\right)$$

Now, if we let $k\to\infty$, the convergence of X^k to some process X is tied to the convergence of the series above, which luckily for us converges. Indeed, using $P(k)(iii)$, for every k,

$$\Big|\Big|\sum_{j=1}^{k-1}\left({}^{j}X-{}^{j-1}X\right)\Big|\Big|_{L^2(\Omega,\mathcal{C})}\leq\sum_{j=1}^{k-1}\big|\big|{}^{j}X-{}^{j-1}X\big|\big|_{L^2(\Omega,\mathcal{C})}$$
$$\leq||X^2-X^1||_{L^2(\Omega,\mathcal{C})}\sum_{j=1}^{k-1}\Big((4L^2)^{j-1}\frac{T^{j-1}}{(j-1)|}\Big)^{\frac{1}{2}}$$

which converges as $k\to\infty$ (by D'Alembert's criterion for example). Moreover, since $\mathcal{F}_t$-measurability is preserved in the limit process, the convergence happens in $L^2_A\big(\Omega, C([-R,T],\mathbb{R}^n)\big)$. We conclude by applying Doob's inequality to $u\mapsto g(u,X^k_u)-g(u,X_u)$:

$$\mathbb{E}\big[\sup_{t\in[0,T]}\big|\int_0^t g(u,X^k_u)-g(u,X_u)\,dB(u)\big|^2\big]\leq 4L^2T||X^k-X||^2_{L^2(\Omega,\mathcal{C})}\to 0$$

Hence, the convergence is preserved by the volatility coefficient, which must mean that our limit process X verifies the original SFDE. Uniqueness follows directly from $H(i)$. □

Remark. It is possible to consider a local Lipschitz hypothesis instead of a global one. The reason for this is that a global Lipschitz hypothesis may often be too restrictive and prevent us from studying more complex phenomenons/models, while a local hypothesis is much less restrictive. To prevent blow ups, we impose an additional linear growth condition, which is not longer guaranteed by the Lipschitz condition since it is now only local. To summarize, it is possible to switch hypothesis (H)(i) for

(H')(i) The coefficients f and g are locally Lipschitz in the second variable, i.e for all $n\geq 1$ there exists $L_n$such that for all $\varphi_1,\varphi_2\in\mathcal{C}_n:=\{\varphi\in\mathcal{C},\ ||\varphi||_{\mathcal{C}}<n\}$, we have

$$||f(t,\varphi_1)-f(t,\varphi_2)||_{L^2(\Omega,\mathbb{R}^d)}+||g(t,\varphi_1)-g(t,\varphi_2)||_{L^2(\Omega,\mathbb{R}^{d\times m})}\leq L_n||\varphi_1-\varphi_2||_{L^2(\Omega,\mathcal{C})}$$

(H')(ii) There exists a constant $K>0$ such that for all $\varphi\in\mathcal{C}$, we have

$$||f(t,\varphi)||_{L^2(\Omega,\mathbb{R}^d)}+||g(t,\varphi)||_{L^2(\Omega,\mathbb{R}^{d\times m})}\leq K(1+||\varphi||_{\mathcal{C}})$$

In essence, given linear growth, it is enough to consider problems on the open balls of $\mathcal{C}$. The proof of the above theorem with this change in hypotheses follows this train of thought. Consider the truncated coefficients outside some open ball $\mathcal{C}_n$, i.e define

$$f_n(t,x)=\begin{cases}f(t,x) & \text{if } ||x||<n\\ f(t,\frac{x}{||x||}n) & \text{if } ||x||\geq n\end{cases}$$

and

$$g_n(t,x)=\begin{cases}g(t,x) & \text{if } ||x||<n\\ g(t,\frac{x}{||x||}n) & \text{if } ||x||\geq n\end{cases}$$

In $\mathcal{C}_n$, by the local Lipschitz hypothesis, these coefficients verify a global Lipschitz hypothesis. Hence, in $\mathcal{C}_n$, the SFDE with coefficients f_n and g_n has a unique solution X^n approximatedvia $X^{n,k}$. Moreover, by the definition of the truncation, for two open balls $\mathcal{C}_i$ and $\mathcal{C}_j$, the coefficients f_i,f_j and g_i,g_j agree on their intersection $\mathcal{C}_i\cap\mathcal{C}_j$, the smallest of the two balls. Hence,

the same goes for the processes S^i and S^j. It then remains to consider the stopping times $\tau_n = T \wedge \inf\{t \in [0,T], |X_n(t)| > n\}$. By the linear growth condition, we have that for some $n*$, we have $\tau_{n*} = T$. Denote the process X^{n*} by X.
Letting $n \to \infty$, we see that $f_n \to f$ and $g_n \to g$. Taking the limit of the process $X(t \wedge \tau_n)$, we end up with a process verifying the SFDE. Moreover, it is unique as it agrees with all X_n and on their respective domains.

Remark. As can be found in [Mao07], one can go a step further. From theorem 4.1 of chapter V of [Mao07] (p.160), thanks to the linear growth condition given to us by the global Lipschitz hypothesis under the assumption of a unique global solution X, one can estimate that, for all $t \geq 0$,

$$\mathbb{E}\big[\sup_{-R\leq u\leq t} |X(u)|^2\big] \leq \frac{3}{2}2(1 + \mathbb{E}\big[||\eta||^2_{\mathcal{C}}\big])e^{Ct}$$

where $C = 4\sqrt{K} + 65K$. In fact, if we now require that η is an element $L^p(\Omega, \mathcal{C})$ for a general $p \geq 2$, then

$$\mathbb{E}\big[\sup_{-R\leq u\leq t} |X(u)|^p\big] \leq \frac{3}{2}2^{\frac{p}{2}}(1 + \mathbb{E}\big[||\eta||^p_{\mathcal{C}}\big])e^{Ct}$$

where $C_p = 2p\sqrt{K} + (33p^2 - p)K$.
This tells us something very strong, that the solution of an SFDE we obtain with the previous theorem has a finite$p$-th moment if the initial condition η is an element of $L^p(\Omega, \mathcal{C})$!

Theorem 3.2.17. *(Markov Property) Suppose f and g are jointly continuous and (uniformly) globally Lipschitz in the second variable with respect to the first. Let ${}^{\eta}X^{t_1}$ denote the solution of the SFDE*

$$\begin{cases} dX^{t_1}(t) = h(t, X^{t_1}_t)\,dt + g(t,, X^{t_1}_t)\,dB(t) & \textit{if } t > t_1 \\ X^{t_1}(t) = \eta(t - t_1) & \textit{if } t_1 - r \leq t \leq t_1 \end{cases}$$

Then

$$\mathbb{P}(X(t_2) \in B|\mathcal{F}_{t_1}) = \mathbb{P}(X(t_2) \in B|X(t_1))$$

Proof. We will show that both sides of the equation equal $\mathbb{P}({}^{\eta}X^{t_1}_{t_2})$. Let us first show that $\mathbb{P}(X(t_2) \in B|\mathcal{F}_{t_1}) = \mathbb{P}({}^{\eta}X^{t_1}(t_2) \in B)$. Define the mappings

$$T^{t_1}_{t_2} : L^2(\Omega, \mathcal{F}_{t_1}; \mathcal{C}) \to L^2(\Omega, \mathcal{F}_{t_2}; \mathcal{C})$$
$$\eta \mapsto T^{t_1}_{t_2}(\eta) := {}^{\eta}X^{t_1}(t_2)$$

Note that by the previous theorem, by uniqueness of solutions with such drift and volatility coefficients, we have $T^{t_1}_{t_2} \circ T^0_{t_1} = T^0_{t_2}$. Then, using the definition of conditional probability, one can rewrite the condition into

$$\int_A \mathbb{1}_B({}^{\eta}X^0_{t_2}(\omega))\,d\mathbb{P}(\omega) = \int_A \int_\Omega \mathbb{1}_B\big(T^{t_1}_{t_2}(T^0_{t_1}(\eta)(\omega'))(\omega)\big)\,d\mathbb{P}(\omega)\,d\mathbb{P}(\omega') \tag{A.2}$$

for all $A \in \mathcal{F}_t$ and Borel sets $B \in \mathcal{B}(\mathcal{C})$. In [Moh84], p.51, it is proven that this relation holds when $\mathbb{1}_B$ is replace by an arbitrary uniformly continuous and bounded function $f : \mathcal{C} \to \mathbb{R}$. The argument mainly involves approximation by simple functions and the continuity of our mappings. We now want to prove the initial relationship for all open sets $B \subset \mathcal{C}$ by approximation $\mathbb{1}_B$ with uniformly continuous, bounded functions. We proceed using the fact that $\mathcal{C}$ admits uniformly continuous partitions of unity. Hence, we can pick a family of sets $\{B_k\}_{k=1}^{\infty}$ and associated uniformly continuous functions $\{f_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} f_k = 1$. Introduce the sequence of functions $\{g_j\}_{j=1}$

$$g_j(\eta) = \begin{cases} \sum_{k=1}^{m} f_k(\eta) & \text{if } \eta \in B \\ 0 & \text{otherwise} \end{cases}$$

The g_j are clearly uniformly continuous as a finite sum of uniformly continuous functions and bounded by definition, hence the previous step applies. Moreover, we have $g_j \to \mathbb{1}_B$ as $j \to \infty$ and thus, using dominated convergence, we have (1) for open sets $B \subset \mathcal{C}$. The remaining argument to generalize this to any Borel set is measure-theoretic in nature.
Set $\nu(B)$ to be the right-hand side of (1). It is in fact a (Borel) measure and in fact finite. Moreover, it is also regular since $\mathcal{C}$ is metric. Hence, by regularity

$$\int_A \mathbb{1}_B({}^{\eta}X^0(t_2)(\omega))\, d\mathbb{P}(\omega) := \nu^*(B) = \nu(B)$$

for any Borel set $B \subset \mathcal{C}$ and this part of the proof is over.

It remains to show that $\mathbb{P}({}^{\eta}X^{t_1}(t_2) \in B) = \mathbb{P}(X(t_2) \in B | X(t_1))$. For this we refer to [Moh84], p.56-58.

□

Appendix B

This appendix focuses on the theory of SFDEs with jumps. For this, we will mainly follow [BCDN$^+$19].

Consider a usual complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let B be an m-dimensional, $\mathcal{F}$-adapted Brownian motion. Let $N = (N^1, \dots, N^n)$ be the Poisson random measures associated with n independent $\mathcal{F}$-adapated Lévy processes and let $\nu = (\nu_1, \dots, \nu_n)$ denote the respective Lévy measures of these processes. Finally, consider the compensated Poisson random measure defined by

$$\tilde{N}(dt, dz) = (N^1(dt, dz) - \nu_1(dz)dt, \dots, N^n(dt, dz) - \nu_n(dz)dt)$$

Unlike in the previous appendix and in chapter 1, it is now important to differentiate the dependance $X(t)$ and X_t as well as represent the initial value by a pair of a continuous function η as before and a point x representing $X(0)$. The reasoning for this is that there may be a jump at 0 (or at a given point t for that matter), which it is now necessary to take into account. Hence, define the new space $M^p := L^p([-r, 0], \mathbb{R}^d) \times \mathbb{R}^d$. Also define $L^p_{\mathcal{F}} := [0, T] x L^p(\Omega, M^p)$ the product space of processes such that $(\psi, v) \in L^p(\Omega, \mathcal{F}_t, M^p)$ is $\mathcal{F}_t$-measurable. Now consider the following differential equation

$$\begin{cases} dX(t) &= f(t, X(t), X_t)dt + g(t, X(t), X_t)dB(t) + \displaystyle\int_{\mathbb{R}_0} h(t, X_t, X(t))(z)\tilde{N}(dt, dz), \qquad t \in [0, T] \\ (X_0, X(0)) &= (\eta, x) \qquad t \in [-R, 0] \end{cases}$$

where R is the maximum delay considered and $(\eta, x) \in L^p(\Omega, \mathcal{F}_0; M^p)$. Moreover, we ask that the coefficients f, g, h are as follows

$$\begin{aligned} f &: L^p_{\mathcal{F}} \to L^p(\Omega, \mathbb{R}^d) \\ g &: L^p_{\mathcal{F}} \to L^p(\Omega, \mathbb{R}^{d \times m}) \\ h &: L^p_{\mathcal{F}} \to L^p(\Omega, L^2(\nu, \mathbb{R}^{d \times n})) \end{aligned}$$

Finally, as we want the respective integrals to be well defined, we will ask that the map $(t, \omega, z) \mapsto h(t, Y_t, Y(t))(w)(z)$ has a predictable version and that the two maps

$$(t,\omega)\to f(t,X(t),X_t)(\omega)$$
$$(t,\omega)\to g(t,X(t),X_t)(\omega)$$

have progressively measurable versions.
To extract existence of a solution and its uniqueness, we will need to impose further conditions on the coefficients of the differential equation, as one usually does. However, before we attack the statement and proof of said theorem, we must first prove intermediary results.

Lemma 3.2.9. *(Ito's formula for semimartingales) Let X be a semimartingale and let $F\in C^2(\mathbb{R}^d)$. Then $F(X)$ is also a semimartingale and*

$$F(X(t))=F(X_0)+\int_{0_+}^t F'(X(u_-))\,dX_u+\frac{1}{2}\int_{0_+}^t F''(X(u_-)d[X,X]_u^c+\sum_{0\le u\le t}\big[F(X(u))-F(X(u_-))-F'(X(u_-))\Delta X(u)\big]$$

where $\Delta X(u)$ denotes the jumps of the process X and $[X,X]_u^c$ refers to the continuous part of $[X,X]_u$.

Proof. We refer to [Pro05], theorem 32, p. 78-81 or alternatively [Kun04], theorem 1.12, p.12 for the proof of this result.

□

Lemma 3.2.10. *(Kunita-Watanabe Formula) Let X be a semimartingale and let $F\in C^2(\mathbb{R}^d)$. Suppose that X can be written component-wise like so*

$$X^i(t):=X(0)+A(t)+M^i(t)+\int_0^t\int_{\mathbb{R}_0}g(u,z)\tilde{N}(du,dz)+\int_0^t\int_{\mathbb{R}_0}h(u,z)N(du,dz)$$

such that $A^i(t)$ is a continuous, adapted process of finite variation, $M(t)$ is a continuous local martingale and $|g|\cdot|h|=0$. Then

$$F(X(t))=F(X(0))+\sum_{j=1}^d\int_0^t\frac{\partial F}{\partial x_j}(X(u_-))dA^j(u)+\sum_{j=1}^d\int_0^t\frac{\partial F}{\partial x_j}(X(u_-))\,dM^j(u)+\frac{1}{2}\sum_{i,j=1}^d\int_0^t\frac{\partial^2 F}{\partial x_i\partial x_j}(X(u))\,d\langle M^j,M^j\rangle_u$$
$$+\int_0^t\int_{\mathbb{R}_0}F(X(u_-)+g(u,z))-F(X(u_-))\tilde{N}(du,dz)+\int_0^t\int_{\mathbb{R}_0}F(X(u)+h(u,z))-F(X(u_-))N(du,dz)$$
$$+\int_0^t\int_{\mathbb{R}_0}F(X(u_-)+g(u,z))-F(X(u_-))-\sum_{j=1}^d g^j(u,z)\frac{\partial F}{\partial x_j}(X(u_-))du\nu(dz)$$

Proof. We refer to [Kun04], theorem 2.5, p.20-22 for the proof of this result.

□

Lemma 3.2.11. *(Kunita's Inequality) Let $p>1$ and suppose that f,g and h are predictable processes taking values in $\mathbb{R}^d$, $\mathbb{R}^{d\times m}$ and $\mathbb{R}^{d\times n}$ respectively. Consider the Lévy-Ito process*

$$X(t)=X_0+\int_0^t f(u)\,du+\int_0^t g(u)\,dB(u)+\int_0^t\int_{\mathbb{R}_0}h(u,z)\tilde{N}(du,dz)$$

Then there exists a constant $C=C(q,d,m,n,T)$ such that for all $t\le T$,

$$\mathbb{E}\big[\sup_{0\le u\le t}|X(t)|^p\big]\le C\Big[||Y_0||^p_{L^p(\Omega,\mathbb{R}^d}+\int_0^t\big(||f(u)||^p_{L^p(\Omega,\mathbb{R}^d)}+||g(u)||^p_{L^p(\Omega,\mathbb{R}^{d\times m})}+||h(u)||^p_{L^p(\Omega,L^p(\nu))}+||h(u)||^p_{L^p(\Omega,L^2)\nu))}\big)\,du\Big]$$

Proof. The detailed proof of the result can be found in [Kun04]. We will, however, give an overview of this proof. Start by proving the result for $n = 1$. Define the process

$$Y^1(t) := \sum_{j=1}^{d} \int_0^t g^j(u)\, dB^j(u)$$

Notice that $Y(t)$ is in fact a continuous martingale by definition of the stochastic integral. Hence, applying Ito's formula for semimartingales to $Y(t)$ and $x \mapsto |x|^p$ for $p > 1$, the final term vanishes and we get

$$|Y^1(t)|^p = p \int_0^t |Y^1(u)|^{p-2} Y^1(u) dY^1(u) + \frac{1}{2}p(p-1) \int_0^t |Y^1(u)|^{p-2} g(u)^2 du$$

Parts of the proof rely on local martingales moving forward, hence we define an increasing sequence of stopping times τ_k such that $\mathbb{P}(\tau_k < T) \to 0$ as $k \to \infty$. Via Doob's inequality, followed by Hölder's inequality and using the expression above, we have

$$\begin{aligned}\mathbb{E}\big[\sup_{0 \le u \le t\wedge\tau_k} |Y^1(u)|^p\big] &\le p^q \mathbb{E}[|Y^1(t \wedge \tau_k)|^p] \\ &\le \frac{1}{2}p^{q+1}(p-1)\mathbb{E}\big[\sup_{0 \le u \le t\wedge\tau_k} |Y^1(u)|^p\big]^{\frac{p-2}{p}} \mathbb{E}\Big[\Big(\int_0^{t\wedge\tau_k} g(u)^2\Big)^{\frac{p}{2}}\Big]^{\frac{2}{p}}\end{aligned}$$

Now define the pure jump process

$$Y^2(t) := \int_0^t \int_{\mathbb{R}_0} h(u,z)\tilde{N}(du,dz)$$

Once again, applying Ito's formula for semimartingales to $Y^2(t)$ and $x \mapsto |x|^p$ for $p > 1$, all the terms but two vanish and we get

$$\begin{aligned}|Y^2(t)|^p = &\int_0^t \int_{\mathbb{R}_0} |Y^2(u_-) + h(u,z)|^p - |Y^2(u_-)|^p|\tilde{N}(du,dz) + \\ &\int_0^t \int_{\mathbb{R}_0} |Y^2(u_-) + h(u,z)|^p - |Y^2(u_-)|^p| - p\, h(u,z)|Y^2(u_-)|^{p-2}Y^2(u_-)du\nu(dz)\end{aligned}$$

where we simplified $p|Y^2(u_-)|^{p-1}\text{sign}(Y^2(u_-)) = p|Y^2(u_-)|^{p-2}Y^2(u_-)$. We now aim to estimate $\mathbb{E}[\sup_{0 \le u \le t} |Y^2(u_-)|^p]$ and hence, one might remember that the stochastic integral with respect to a compensated Poisson measure is a martingale and hence vanishes in our estimate. It then remains to estimate the second term. Applying Taylor-Lagrange's theorem to $f(x) = |x|^p$ (with integral remainder and where we view the absolute value here as a norm to simplify the computations) of order 1, we have

$$|Y^2(u_-)+h(u,z)|^p - |Y^2(u_-)|^p| - p\, h(u,z)|Y^2(u_-)|^{p-2}Y^2(u_-) = \frac{1}{2}p(p-1)|Y^2(u_-)+\theta h(u,z)|^{p-2}h^2(u,z)$$

for some parameter $|\theta| < 1$. Knowing this, we can bound the right-hand side like so

$$\frac{1}{2}p(p-1)|Y^2(u_-)| + \theta|h(u,z)|^{p-2}h^2(u,z) \leq C_1|Y^2(u_-)|^{p-2}h^2(u,z) + C_2|g|^p$$

where the two constants depend on θ. Hence

$$E[|Y^2(t)|^p] \leq C_1\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}_0}|Y^2(u_-)|^{p-2}h^2(u,z)\,du\nu(dz)\Big] + C_2\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}_0}|h(u,z)|^p\,du\nu(dz)\Big]$$

Let us estimate the first and most troublesome term in this expression. By Hölder's inequality,

$$\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}_0}|Y^2(u_-)|^{p-2}h^2(u,z)\,du\nu(dz)\Big] \leq \mathbb{E}\big[\sup_{0\leq u\leq t}|Y^2(u_-)|^p\big]^{1-\frac{2}{p}}\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}_0}h^2(u,z)\,du\nu(dz)\Big]^{\frac{2}{p}}$$

Using Young's inequality with $p' = \frac{p}{2}$, we then find

$$\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}_0}|Y^2(u_-)|^{p-2}h^2(u,z)\,du\nu(dz)\Big] \leq \frac{p}{p-2}\mathbb{E}\big[\sup_{0\leq u\leq t}|Y^2(u_-)|^p\big] + \frac{2}{p}\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}_0}h^2(u,z)\,du\nu(dz)\Big]$$

Taking the supremum in our original expression, and using Doob's inequality, one finds

$$\begin{aligned}\mathbb{E}\big[\sup_{0\leq u\leq t}|Y^2(u)|^p\big] \leq& \frac{q^p p C_1}{p-2}\mathbb{E}\big[\sup_{0\leq u\leq t}|Y^2(u_-)|^p\big] + \frac{q^p 2C_1}{p}\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}_0}h^2(u,z)\,du\nu(dz)\Big]\\ &+ q^p C_2\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}_0}|h(u,z)|^p\,du\nu(dz)\Big]\end{aligned}$$

Choose C_1 such that $\frac{q^p p C_1}{p-2} < 1$ and subtract the first term on both sides. Using the fact that $\sup_{0\leq u\leq t}|Y^2(u)|^p = \sup_{0\leq u\leq t}|Y^2(u_-)|^p$ holds a.s for all $t\in[0,T]$, we then have

$$\mathbb{E}[\sup_{0\leq u\leq t}|Y^2(u)|^p \leq C_3\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}_0}g^2(u,z)\,du\nu(dz)\Big] + C_4\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}_0}|g(u,z)|^p\,du\nu(dz)\Big]$$

where C_3 and C_4 are lengthy constants coming simply from division by the resulting constant from the subtraction on the left side. Combined with our original result for $Y^1(t)$, we have almost proven the original result we were aiming for. One last verification remains, but since

$$\mathbb{E}\Big[\sup_{0\leq u\leq t}\Big|\int_0^t f(u)du\Big|^p\Big] \leq \mathbb{E}\Big[\Big(\int_0^t|f(u)|du\Big)^p\Big]$$

we indeed have the desired result.

□

With the notations clear, we are ready to present the existence and uniqueness result for SFDEs with jumps. First, however, we must establish a few hypotheses on the coefficients.

(H)(i) The functions f, g and h are uniformly Lipschitz in the second and third variable, i.e we require that there exists $L > 0$ such that for all $t \in [0,T]$ and for all $(\eta_1, x_1), (\eta_2, x_2) \in$

$L^p(\Omega, \mathcal{F}_t; M^p)$, then

$$||f(t,\eta_1,x_1) - f(t,\eta_2,x_2)||^p_{L^p(\Omega,\mathbb{R}^d)} + ||g(t,\eta_1,x_1) - g(t,\eta_2,x_2)||^p_{L^p(\Omega;\mathbb{R}^{d\times n})}$$
$$+ ||h(t,\eta_1,x_1) - h(t,\eta_2,x_2)||^p_{L^p(\Omega,L^p(\nu))} + ||h(t,\eta_1,x_1) - h(t,\eta_2,x_2)||^p_{L^p(\Omega,L^2(\nu))}$$
$$\leq L||(\eta_1,x_1) - (\eta_2,x_2)||^p_{L^p(\Omega,M^p)}$$

(H)(ii) The functions f, g and h have at most global linear growth, i.e there exists $K > 0$ such that for all $t \in [0,T]$ and for all $(\eta, x) \in L^p(\Omega, \mathcal{F}_t; M^p)$, then

$$||f(t,\eta,x)||_{L^p(\Omega,\mathbb{R}^d)} + ||g(t,\eta,x)||_{L^p(\Omega;\mathbb{R}^{d\times n})} + ||h(t,\eta,x)||_{L^p(\Omega;L^p(\nu))} + ||h(t,\eta,x)||_{L^p(\Omega;L^2(\nu))}$$
$$\leq K\big(1 + ||(\eta,x)||_{L^p(\Omega;M^p)}\big)$$

Theorem 3.2.18. *(Existence and uniqueness) (i) Let $\eta, x \in L^p(\Omega, \mathcal{F}_0; M^p)$ such that η is càdlàg $\mathbb{P}$-a.s. Let $X \in L^p_A(\Omega, L^p([-r,T], \mathbb{R}^d))$ be a strong solution of the SFDE. Then the process $(t,\omega) \mapsto (X_t(\omega), X(t,\omega))$ has a càdlàg modification.*

(ii) Assuming (H)(i) and (H)(ii), there exists a strong solution X of the SFDE. Moreover, there exists $K > 0$ such that for all $t \in [-r, T]$,

$$||X||_{L^p_A(\Omega,L^p([-r,t]))} \leq e^{Kt}(Kt + ||(\eta,x)||_{L^p(\Omega,M^p)})$$

(iii) If $X, Y \in L^p_A(\Omega, L^p([-r,T], \mathbb{R}^d))$ are two strong solutions of the SFDE, then $X = Y$ $\mathbb{P}$-a.s.

Proof. We use a Picard iteration type proof. The goal of this proof is to approximate step by step the solution with a well-chosen sequence of successive approximations. First, we need to establish the space in which those approximations take place. We denote by $L^p_A(\Omega, L^p([-r,T], \mathbb{R}^d))$ the space of all processes $X \in L^p(\Omega, L^p([-r,T], \mathbb{R}^d))$ such that $X(t)$ is $\mathcal{F}_0$-measurable for $t \in [-R, 0]$ and $\mathcal{F}_t$-measurable for all $t \in [0,T]$. We define suitable Picard iterative processes, check that they are indeed living in this space, and that they form a Cauchy sequence. Specifically, we will want to prove that our approximations X^k will possess the following properties:

(P)(i) X^k belongs in the approximation space, i.e $X^k \in L^p_A(\Omega, L^p([-r,T], \mathbb{R}^d))$
(P)(ii) Two successive approximations are "close enough", i.e

$$||X^{k+1} - X^k||_{L^p_A(\Omega,L^p([-r,T],\mathbb{R}^d))} \leq \frac{(LCt)^{k-1}}{k!}||X^2 - X^1||_{L^p_A(\Omega,L^p([-r,T],\mathbb{R}^d))}$$

where C is a constant and L is the (uniform) Lipschitz constant of in $(H)(i)$. We begin by defining $X^1(t) = \eta(0)$ and $X^1_0 = \eta$. Then, for all $k > 1$, define

$$X^{k+1}(t) = \eta(0) + \int_0^t f(u, X^k(u), X_u^k)\, du + \int_0^t g(u, X^k(u), X_u^k)\, dB(s) + \int_0^t h(u, X^k(u), X_u^k)(z)\tilde{N}(du, dz)$$
$$X_t^{k+1} = \eta$$

This process is well defined by all the assumptions written above. Moreover, the process is integrable for every $k \geq 1$ by (H)(ii), as the integral over $[0, T]$ can for example be bounded by $KT(1+||(X_t^k, X^k(t))||^p_{L^p(\Omega, M^p)})$ which is finite. By a direct application of Kunita's inequality, we have a finite bound and the process belongs to $L^p_A(\Omega, L^p([-r, T], \mathbb{R}^d))$. Hence $P(i)$ is verified. We now proceed to investigate $P(ii)$. First notice that the property trivially holds for $k = 1$ by $(H)(i)$. Let us now suppose that the property holds for an arbitrary $k \geq 1$. Then,

$$\begin{aligned}
||X^{k+2} - X^{k+1}||^p_{L^p_A(\Omega, L^p([-r,T],\mathbb{R}^d))} &\leq \mathbb{E}\big[\sup_{t\in[0,T]} |X^{k+2} - X^{k+1}|^p\big] \\
&= \mathbb{E}\big[\sup_{t\in[0,T]} \big| \int_0^t f(u, X^{k+1}(u), X_u^{k+1}) - f(u, X^k(u), X_u^k)\, du \\
&+ \int_0^t g(u, X^{k+1}(u), X_u^{k+1}) - g(u, X^k(u), X_u^k)\, dB(u) \\
&+ \int_0^t h(u, X^{k+1}(u), X_u^{k+1})(z) - h(u, X^k(u), X_u^k)(z)\, \tilde{N}(du, dz)\big|^p\big]
\end{aligned}$$

The process of which the supremum is taken on the right-hand side being a semi-martingale, by definition of the stochastic integrals, we can use Kunita's inequality which gives us a constant $C > 0$ such that

$$\begin{aligned}
||X^{k+2} - X^{k+1}||^p_{L^p_A(\Omega, L^p([-r,T],\mathbb{R}^d))} \leq& C \int_0^t \Big(||f(u, X^{k+1}(u), X_u^{k+1}) - f(u, X^k(u), X_u^k)||^p_{L^p(\Omega,\mathbb{R}^d)} \\
&+ ||g(u, X^{k+1}(u), X_u^{k+1}) - g(u, X^k(u), X_u^k)||^p_{L^p(\Omega,\mathbb{R}^{d\times n})} \\
&+ ||h(u, X^{k+1}(u), X_u^{k+1}) - h(u, X^k(u), X_u^k)||^p_{L^p(\Omega,L^p(\nu))} \\
&+ ||h(u, X^{k+1}(u), X_u^{k+1}) - h(u, X^k(u), X_u^k)||^p_{L^p(\Omega,L^2(\nu))}\Big)\, du \\
&\leq LC \int_0^t \underbrace{||X_u^{k+1} - X_u^k||^p_{L^p(\Omega;M^p)}}_{\leq ||X^{k+1}-X^k||^p_{L^p_A(\Omega,L^p([-r,T],\mathbb{R}^d))}}\, du \\
&\leq LC \int_0^t ||X^{k+1} - X^k||^p_{L^p_A(\Omega,L^p([-r,T],\mathbb{R}^d))}\, du
\end{aligned}$$

where we have used $(H)(i)$, the global Lipschitziniaty of the coefficients. Then, using $P(ii)(k)$, we finally have

$$||X^{k+2}-X^{k+1}||^p_{L^p_A(\Omega,L^p([-r,T],\mathbb{R}^d))} \leq LC\frac{(LC)^{k-1}}{(k-1)!}\int_0^t u^{k-1}\,du||X^2-X^1||^p_{L^p_A(\Omega,L^p([-r,T],\mathbb{R}^d))}$$
$$= \frac{(LCt)^k}{k!}||X^2-X^1||^p_{L^p_A(\Omega,L^p([-r,T],\mathbb{R}^d))}$$

which proves $(P)(ii)(k+1)$ and hence $(P)(ii)$. Hence, by completeness of $L^p_A(\Omega, L^p([-r,T],\mathbb{R}^d))$, the Cauchy sequence $\{X^k\}_{k\geq 1}$ to some $X \in L^p_A(\Omega, L^p([-r,T],\mathbb{R}^d))$. It remains to show that the limit process X is indeed a strong solution of the SFDE.
This is can be done by showing that

$$\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big|X(t)-\eta(0)-\int_0^t f(u,X_u)\,du+\int_0^t g(u,X_u)\,dB(u)+\int_0^t\int_{\mathbb{R}_0} h(u,X_u)(z)\,\tilde{N}(du,dz)\Big|^p\Big]^{\frac{1}{p}}=0$$

Denote the expression above by δ. Hence, our goal is to prove that $\delta=0$. Adding and subtracting X^{k+1} to separate δ into two terms, and using Kunita's inequality in the second term, we have

$$\begin{aligned}\delta \leq &||X-X^{k+1}||_{L^p_A(\Omega,L^p([-r,T],\mathbb{R}^d))}+\Big(C\int_0^T\big(||f(t,X^k(u),X^k_t)-f(t,X^k(u),X_t)||^p_{L^p(\Omega;\mathbb{R}^d)}\\ &+||g(t,X^k(u),X^k_t)-g(t,X_t)||_{L^p(\Omega;\mathbb{R}^{d\times n})}+||h(t,X^k(u),X^k_t)-h(t,X^k(u),X_t)||^p_{L^p(\Omega,L^p(\nu))}\\ &+||h(t,X^k(u),X^k_t)-h(t,X^k(u),X_t)||^p_{L^p(\Omega,L^2(\nu))}\big)\,dt\Big)^{\frac{1}{p}}\end{aligned}$$

By the Lipschitz condition, we finally have that

$$\delta \leq ||X-X^{k+1}||_{L^p_A(\Omega,L^p([-r,T],\mathbb{R}^d))}+(CTL)^{\frac{1}{p}}||X-X^k||_{L^p_A(\Omega,L^p([-r,T],\mathbb{R}^d))}\overset{k\to\infty}{\to}$$

which shows that $delta$, being independent of k, is indeed null and hence X is a solution of the SFDE.
Suppose now that X and Y are two solutions of the same SFDE. Then, by a similar argument, by Kunita's inequality and the Lipschitz condition, for all $t\in[0,T]$, we have

$$||X-Y||^p_{L^p_A(\Omega,L^p([-r,t],\mathbb{R}^d))}\leq CL\int_0^t ||X-Y||^p_{L^p_A(\Omega,L^p([-r,u],\mathbb{R}^d))}\,du$$

Let $\phi(t) := ||X-Y||^p_{L^p_A(\Omega,L^p([-r,t],\mathbb{R}^d))}$. Then the above expression can be rewritten as

$$\phi(t)\leq CL\int_0^t \phi(u)\,du$$

which is a clear application of Grönwall's inequality with $\alpha(t)=0$ and $\beta(t)=CL$, i.e $\phi(t)\leq 0$ for all $t\in[0,T]$. By positivity of the norm, we conclude that $\phi(t)=0$ for all $t\in[0,T]$ which tells us that $X=Y$ a.s, which completes the proof. □

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