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Singular control of SPDEs and backward SPDEs with reflection

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Abstract

In the first part, we consider general singular control problems for random fields given by a stochastic partial differential equation (SPDE). We show that under some conditions the optimal singular control can be identified with the solution of a coupled system of SPDE and *a kind of reflected backward* SPDE (RBSPDE).In the second part, existence and uniqueness of solutions of RBSPDEs are established, which is of independent interest.

Key Words: Stochastic partial differential equations (SPDEs), singular control of SPDEs, maximum principles, comparison theorem for SPDEs, reflected SPDEs, optimal stopping of SPDEs.
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1 Introduction

Let $B_t, t \geq 0$ be an m-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Let D be a bounded smooth domain in $\mathbb{R}^d$. Fix $T > 0$ and let $\phi(\omega, x)$ be an $\mathcal{F}_T$-measurable $H = L^2(D)$-valued random variable. Let

$$k : [0,T] \times D \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$$

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be a given measurable mapping and $L(t,x) : [0,T] \times D \to \mathbb{R}$ a given continuous function. Consider the problem to find $\mathcal{F}_t$-adapted random fields $u(t,x) \in \mathbb{R}, Z(t,x) \in \mathbb{R}^m, \eta(t,x) \in \mathbb{R}^+$ left-continuous and increasing w.r.t. t, such that

$$\begin{aligned} du(t,x) &= -Au(t,x)dt - k(t,x,u(t,x),Z(t,x))dt + Z(t,x)dB_t, t \in (0,T) \\ &\quad -\eta(dt,x), t \in (0,T), \qquad (1.1) \\ u(t,x) &\geq L(t,x), \\ \int_0^T \int_D & (u(t,x) - L(t,x))\eta(dt,x) = 0, \\ u(T,x) &= \phi(x) \quad a.s, \qquad (1.2) \end{aligned}$$

where A is a second order linear partial differential operator. This is a backward stochastic partial differential equation (BSPDE) with reflection.

The maximum principle method for solving a stochastic control problem for stochastic partial differential equations involves a BSPDE for the adjoint processes $p(t,x), q(t,x)$.See [ØPZ].

The purpose of this paper is twofold: (i) We study a class of singular control problems for SPDEs and prove a maximum principle for the solution of such problems. This maximum principle leads to a kind of reflected backward stochastic partial differential equations. (ii) We study backward stochastic *partial* differential equations (BSPDEs) with reflection. This means that we solve the BSPDE with the constraint that the solution must stay in a pre-described region.

2 Singular control of SPDEs

Suppose the state equation is an SPDE of the form

$$\begin{aligned} dY(t,x) &= \{AY(t,x) + b(t,x,Y(t,x))\}dt + \sigma(t,x,Y(t,x))dB(t) \\ &\quad + \lambda(t,x,Y(t,x))\xi(dt,x) \; ; \; (t,x) \in [0,T] \times D \end{aligned} \tag{2.1}$$

$$\begin{cases} Y(0,x) &= y_0(x) \; ; \; x \in D \\ Y(t,x) &= y_1(t,x) \; ; \; (t,x) \in (0,T) \times \partial D. \end{cases} \tag{2.2}$$

Here A is a given linear second order partial differential operator.

The *performance functional* is given by

$$\begin{aligned} J(\xi) = E\Bigg[\int_D \int_0^T & f(t,x,Y(t,x))dtdx + \int_D g(x,Y(T,x))dx \\ &+ \int_D \int_0^T h(t,x,Y(t,x))\xi(dt,x)\Bigg], \end{aligned} \tag{2.3}$$

where $f(t,x,y), g(x,y)$ and $h(t,x,y)$ are bounded measurable functions which are differentiable in the argument y and continuous w.r.t. t.

We want to maximize $J(\xi)$ over all $\xi \in \mathcal{A}$, where $\mathcal{A}$ is a given family of adapted processes $\xi(t,x)$, which are non-decreasing and left-continuous w.r.t. t for all x, $\xi(0,x) = 0$. We call $\mathcal{A}$ the set of admissible singular controls. Thus we want to find $\xi^* \in \mathcal{A}$ (called an optimal control) such that

$$\sup_{\xi \in \mathcal{A}} J(\xi) = J(\xi^*)$$

Define the *Hamiltonian* H by

$$\begin{aligned} H(t,x,y,p,q)(dt,\xi(dt,x)) &= \{f(t,x,y) + b(t,x,y)p + \sigma(t,x,y)q\}dt \\ &\quad + \{\lambda(t,x,y)p + h(t,x,y)\}\xi(dt,x). \end{aligned} \tag{2.4}$$

To this Hamiltonian we associate the following *backward* SPDE (BSPDE) in the unknown process $(p(t,x), q(t,x))$:

$$\begin{aligned} dp(t,x) &= -\left\{ A^* p(t,x)dt + \frac{\partial H}{\partial y}(t,x,Y(t,x),p(t,x),q(t,x))(dt,\xi(dt,x)) \right\} \\ &\quad + q(t,x)dB(t) \ ; \ (t,x) \in (0,T) \times D \end{aligned} \tag{2.5}$$

with boundary/terminal values

$$p(T,x) = \frac{\partial g}{\partial y}(x, Y(T,x)) \ ; \ x \in D \tag{2.6}$$

$$p(t,x) = 0 \ ; \ (t,x) \in (0,T) \times \partial D. \tag{2.7}$$

Here A^* denotes the adjoint of the operator A.

Theorem 2.1 (Sufficient maximum principle for singular control of SPDE) *Let* $\hat{\xi} \in \mathcal{A}$ *with corresponding solutions* $\hat{Y}(t,x)$, $\hat{p}(t,x)$, $\hat{q}(t,x)$. *Assume that*

$$y \to h(x,y) \ \textit{is concave} \tag{2.8}$$

and

$$\begin{aligned} &(y,\xi) \to H(t,x,y,\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x)) \\ &\qquad \textit{is concave.} \end{aligned} \tag{2.9}$$

Assume that

$$\begin{aligned} E[\int_D (\int_0^T &\{(Y^\xi(t,x) - \hat{Y}(t,x))^2 \hat{q}^2(t,x) + \hat{p}^2(t,x)(\sigma(t,x,Y^\xi(t,x)) - \sigma(t,x,\hat{Y}(t,x))^2\} \\ &dt)dx] < \infty, \quad \textit{for all} \quad \xi \in \mathcal{A}. \end{aligned} \tag{2.10}$$

Moreover, assume that the following maximum condition holds:

$$\hat{\xi}(dt,x) \in \operatorname*{argmax}_{\xi\in\mathcal{A}} H(t,x,\hat{Y}(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x)), \tag{2.11}$$

i.e.

$$\begin{aligned}&\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x))\}\xi(dt,x)\\ &\quad\le \{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) \text{ for all } \xi\in\mathcal{A}.\end{aligned} \tag{2.12}$$

Then $\hat{\xi}$ is an optimal singular control.

Proof of Theorem 2.1 Choose $\xi\in\mathcal{A}$ and put $Y=Y^{\xi}$. Then by (2.3) we can write

$$J(\xi)-J(\hat{\xi}) = I_1+I_2+I_3, \tag{2.13}$$

where

$$I_1 = E\left[\int_0^T\int_D \left\{f(t,x,Y(t,x))-f(t,x,\hat{Y}(t,x))\right\}dxdt\right] \tag{2.14}$$

$$I_2 = E\left[\int_D \left\{g(x,Y(T,x))-g(x,\hat{Y}(T,x))\right\}dx\right] \tag{2.15}$$

$$I_3 = E\left[\int_0^T\int_D \left\{h(t,x,Y(t,x))\xi(dt,x)-h(t,x,\hat{Y}(t,x))\hat{\xi}(dt,x)\right\}\right]. \tag{2.16}$$

By our definition of H we have

$$\begin{aligned}I_1 = E\Bigg[&\int_0^T\int_D \{H(t,x,Y(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x))\\ &\quad -H(t,x,\hat{Y}(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\hat{\xi}(dt,x))\Big\}\\ &-\int_0^T\int_D\{b(t,x,Y(t,x))-b(t,x,\hat{Y}(t,x)\}\hat{p}(t,x)dxdt\\ &-\int_0^T\int_D\{\sigma(t,x,Y(t,x))-\sigma(t,x,\hat{Y}(t,x))\}\hat{q}(t,x)dxdt\\ &-\int_0^T\int_D\hat{p}(t,x)\{\lambda(t,x,Y(t,x))\xi(dt,x)-\lambda(t,x,\hat{Y}(t,x))\hat{\xi}(dt,x)\}dx\\ &-\int_0^T\int_D\{h(t,x,Y(t,x))\xi(dt,x)-h(t,x,\hat{Y}(t,x))\hat{\xi}(dt,x)\}dx\Bigg].\end{aligned} \tag{2.17}$$

By (2.10) and concavity of g we have, with $\tilde{Y} = Y - \hat{Y}$,

$$
\begin{aligned}
I_2 &\leq E\left[\int_D \frac{\partial g}{\partial y}(x, \hat{Y}(T,x))(Y(T,x) - \hat{Y}(T,x))dx\right] = E\left[\int_D \hat{p}(T,x)\tilde{Y}(T,x)dx\right] \\
&= E\left[\int_D \int_0^T \tilde{Y}(t,x)d\hat{p}(t,x)dx + \int_D \int_0^T \hat{p}(t,x)d\tilde{Y}(t,x)dx \right. \\
&\quad \left. + \int_D \int_0^T \{\sigma(t,x,Y(t,x)) - \sigma(t,x,\hat{Y}(t,x))\}\hat{q}(t,x)dtdx\right] \\
&= E\left[\int_D \int_0^T \tilde{Y}(t,x)\left\{-A^*\hat{p}(t,x)dt - \frac{\partial H}{\partial y}(t,x,\hat{Y},\hat{p},\hat{q})(dt,\hat{\xi}(dt,x))\right\}dx \right. \\
&\quad + \int_D \int_0^T \hat{p}(t,x)\{A\tilde{Y}(t,x) + b(t,x,Y(t,x)) - b(t,x,\hat{Y}(t,x))\}dtdx \\
&\quad + \int_D \int_0^T \hat{p}(t,x)\{\lambda(t,x,Y(t,x))\xi(dt,x) - \lambda(t,x,\hat{Y}(t,x))\hat{\xi}(dt,x)\}dx \\
&\quad \left. + \int_D \int_0^T \{\sigma(t,x,Y(t,x)) - \sigma(t,x,\hat{Y}(t,x))\}\hat{q}(t,x)dtdx\right].
\end{aligned}
\tag{2.18}
$$

Using integration by parts we get, since $\tilde{Y}(t,x) = \hat{p}(t,x) = 0$ for all $(t,x) \in (0,T) \times \partial D$,

$$
\int_D \tilde{Y}(t,x)A^*\hat{p}(t,x)dx = \int_D \hat{p}(t,x)A\tilde{Y}(t,x)dx. \tag{2.19}
$$

Hence, combining (2.13)-(2.19) and concavity of H,

$$
\begin{aligned}
J(\xi) - J(\hat{\xi}) &\leq E\left[\int_D \int_0^T \{H(t,x,Y(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x)) \right. \\
&\quad \left. -H(t,x,\hat{Y}(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\hat{\xi}(dt,x)) - \tilde{Y}(t,x)\frac{\partial H}{\partial y}(t,x,\hat{Y},\hat{p},\hat{q})(dt,\hat{\xi}(dt,x))\right\}dx\right] \\
&\leq \left[\int_D \int_0^T \nabla_\xi H(t,x,\hat{Y}(t,x),\hat{p}(t,x),\hat{q}(t,x))(\xi(dt,x) - \hat{\xi}(dt,x))dx\right] \\
&= E\left[\int_D \int_0^T \{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}(\xi(dt,x) - \hat{\xi}(dt,x))dx\right] \\
&\leq 0 \text{ by (2.12).}
\end{aligned}
$$

This proves that $\hat{\xi}$ is optimal. □

For $\xi \in \mathcal{A}$ we let $\mathcal{V}(\xi)$ denote the set of adapted processes $\zeta(t,x)$ of finite variation w.r.t. t such that there exists $\delta = \delta(\xi) > 0$ such that $\xi + y\zeta \in \mathcal{A}$ for all $y \in [0,\delta]$.

Proceeding as in [ØS] we prove the following useful result:

Lemma 2.2 *The inequality (2.12) is equivalent to the following two variational inequalities:*

$$
\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x)) \leq 0 \textit{ for all } \quad t,x \tag{2.20}
$$

$$
\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) = 0 \textit{ for all } t,x \tag{2.21}
$$

Proof. (i). Suppose (2.12) holds. Choosing $\xi = \hat{\xi} + y\zeta$ with $\zeta \in \mathcal{V}(\hat{\xi})$ and $y \in (0, \delta(\hat{\xi}))$ we deduce that

$$\{\lambda(s,x,\hat{Y}(s,x))\hat{p}(s,x) + h(s,x,\hat{Y}(s,x))\}\zeta(ds,x) \le 0; (s,x) \in (0,T)\times D \tag{2.22}$$

for all $\zeta \in \mathcal{V}(\hat{\xi})$.

In particular, this holds if we fix $t \in (0,T)$ and put

$$\zeta(ds,x) = a(\omega)\delta_t(ds)\phi(x); (s,x,\omega) \in (0,T)\times D\times\Omega,$$

where $a(\omega) \ge 0$ is $\mathcal{F}_t$- measurable and bounded, $\phi(x) \ge 0$ is bounded, deterministic and $\delta_t(ds)$ denotes the Dirac measure at t. Then we get

$$\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x)) \le 0 \text{ for all } \quad t,x \tag{2.23}$$

which is (2.20).

On the other hand, clearly $\zeta(dt,x) := \hat{\xi}(dt,x) \in \mathcal{V}(\hat{\xi})$ and this choice of ζ in (2.22) gives

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) \le 0; (t,x) \in (0,T)\times D \tag{2.24}$$

Similarly, we can choose $\zeta(dt,x) = -\hat{\xi}(dt,x) \in \mathcal{V}(\hat{\xi})$ and this gives

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) \le 0; (t,x) \in (0,T)\times D \tag{2.25}$$

combining (2.24) and (2.25) we get

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) = 0$$

which is (2.21). Together with (2.23) this proves (i).

(ii). Conversely, suppose (2.20) and (2.21) hold. Since $\xi(dt,x) \ge 0$ for all $\xi \in \mathcal{A}$ we see that (2.12) follows. □

We may formulate what we have proved as follows:

Theorem 2.3 *(Sufficient maximum principle II) Suppose the conditions of Theorem 2.1 hold. Suppose $\xi \in \mathcal{A}$, and that ξ together with its corresponding processes $Y^\xi(t,x), p^\xi(t,x), q^\xi(t,x)$ solve the coupled SPDE-RBSPDE system consisting of the SPDE* (2.1)-(2.2) *together with the reflected backward SPDE (RBSPDE) given by*

$$\begin{aligned}
dp^\xi(t,x) = &-\Big\{A^*p^\xi(t,x) + \frac{\partial f}{\partial y}(t,x,Y^\xi(t,x)) + \frac{\partial b}{\partial y}(t,x,Y^\xi(t,x))p^\xi(t,x) \\
&\quad + \frac{\partial \sigma}{\partial y}(t,x,Y^\xi(t,x))q^\xi(t,x)\Big\}dt \\
&- \Big\{\frac{\partial \lambda}{\partial y}(t,x,Y^\xi(t,x))p^\xi(t,x) + \frac{\partial h}{\partial y}(t,x,Y^\xi(t,x))\Big\}\xi(dt,x)\ ;\ (t,x) \in [0,T]\times D \\
&\lambda(t,x,Y^\xi(t,x))p^\xi(t,x) + h(t,x,Y^\xi(t,x)) \le 0\ ;\ \textit{for all } t,x,a.s. \\
&\{\lambda(t,x,Y^\xi(t,x))p^\xi(t,x) + h(t,x,Y^\xi(t,x))\}\xi(dt,x) = 0\ ;\ \textit{for all} \quad t,x,a.s. \\
&p(T,x) = \frac{\partial g}{\partial y}(x,Y^\xi(T,x))\ ;\ x \in D \\
&p(t,x) = 0\ ;\ (t,x) \in (0,T)\times\partial D.
\end{aligned}$$

Then ξ maximizes the performance functional $J(\xi)$.

The concavity conditions of Theorem 2.1 are not always satisfied in applications, and it is of interest to have a maximum principle which does not need these assumptions. Moreover, it is useful to have a version which is of so called "necessary type". To this end, we first prove some auxiliary results:

Lemma 2.4 *Let $\xi(dt,x) \in \mathcal{A}$ and choose $\zeta(dt,x) \in \mathcal{V}(\xi)$. Define the derivative process*

$$\mathcal{Y}(t,x) = \lim_{y\to 0^+} \frac{1}{y}(Y^{\xi+y\zeta}(t,x) - Y^{\xi}(t,x)). \tag{2.26}$$

Then $\mathcal{Y}$ satisfies the SPDE

$$\begin{aligned} d\mathcal{Y}(t,x) &= A\mathcal{Y}(t,x)dt + \mathcal{Y}(t,x)[\frac{\partial b}{\partial y}(t,x,Y(t,x))dt \\ &\quad + \frac{\partial \sigma}{\partial y}(t,x,Y(t,x))dB(t) + \frac{\partial \lambda}{\partial y}(t,x,Y(t,x))\xi(dt,x)] \\ &\quad + \lambda(t,x,Y(t,x))\zeta(dt,x)\ ; \quad (t,x) \in [0,T]\times D \\ \mathcal{Y}(t,x) &= 0; \quad (t,x) \in (0,T)\times \partial D \\ \mathcal{Y}(0,x) &= 0\ ; \quad x \in D \end{aligned} \tag{2.27}$$

Proof. This follows from the equation (2.1)-(2.2) for $Y(t,x)$. We omit the details. □

Lemma 2.5 *Let $\xi(dt,x) \in \mathcal{A}$ and $\zeta(dt,x) \in \mathcal{V}(\xi)$. Put $\eta = \xi + y\zeta; y \in [0,\delta(\xi)]$. Assume that*

$$\begin{aligned} E[\int_D(\int_0^T &\{(Y^{\eta}(t,x) - Y^{\xi}(t,x))^2 q^2(t,x) \\ &+ p^2(t,x)(\sigma(t,x,Y^{\eta}(t,x)) - \sigma(t,x,Y^{\xi}(t,x)))^2\}dt)dx] < \infty \quad \textit{for all } y \in [0,\delta(\xi)], \end{aligned} \tag{2.28}$$

where $(p(t,x),\ q(t,x))$ is the solution of (2.5)-(2.7) corresponding to $Y^{\xi}(t,x)$. Then

$$\begin{aligned} &\lim_{y\to 0^+}\frac{1}{y}(J(\xi+y\zeta) - J(\xi)) \\ &\quad = E[\int_D(\int_0^T \{\lambda(t,x,Y(t,x))p(t,x) + h(t,x,Y(t,x))\}\zeta(dt,x))dx]. \end{aligned} \tag{2.29}$$

Proof. By (2.3) and (2.26), we have

$$\begin{aligned} &\lim_{y\to 0^+}\frac{1}{y}(J(\xi+y\zeta) - J(\xi)) \\ &\quad = E[\int_D\{\int_0^T \frac{\partial f}{\partial y}(t,x,Y(t,x))\mathcal{Y}(t,x)dt + \frac{\partial g}{\partial y}(x,Y(T,x))\mathcal{Y}(T,x)\}dx \\ &\quad + \int_D\int_0^T \frac{\partial h}{\partial y}(t,x,Y(t,x))\mathcal{Y}(t,x)\xi(dt,x)dx + \int_D\int_0^T h(t,x,Y(t,x))\zeta(dt,x)dx]. \end{aligned} \tag{2.30}$$

By (2.4) and (2.27) we obtain

$$E[\int_D \int_0^T \frac{\partial f}{\partial y}(t,x,Y(t,x))\mathcal{Y}(t,x)dtdx]$$

$$= E[\int_D (\int_0^T \mathcal{Y}(t,x)\{\frac{\partial H}{\partial y}(dt,\xi(dt,x)) - p(t,x)\frac{\partial b}{\partial y}(t,x)dt$$

$$-q(t,x)\frac{\partial \sigma}{\partial y}(t,x)dt - (p(t,x)\frac{\partial \lambda}{\partial y}(t,x) + \frac{\partial h}{\partial y}(t,x))\xi(dt,x)\})dx, \qquad (2.31)$$

where we have used the abbreviated notation

$$\frac{\partial H}{\partial y}(dt,\xi(dt,x)) = \frac{\partial H}{\partial y}(t,x,Y(t,x),p(t,x),q(t,x))(dt,\xi(dt,x))$$

etc.

By the Itô formula and (2.5), (2.28) we see that

$$E[\int_D \frac{\partial g}{\partial y}(x)\mathcal{Y}(T,x)dx]$$

$$= E[\int_D p(T,x)\mathcal{Y}(T,x)dx]$$

$$= E[\int_D (\int_0^T \{p(t,x)d\mathcal{Y}(t,x) + \mathcal{Y}(t,x)dp(t,x)\} + [p(\cdot,x),\mathcal{Y}(\cdot,x)](T))dx]$$

$$= E[\int_D (\int_0^T [p(t,x)\{A\mathcal{Y}(t,x)dt + \mathcal{Y}(t,x)\frac{\partial b}{\partial y}(t,x)dt$$

$$+ \mathcal{Y}(t,x)\frac{\partial \lambda}{\partial y}(t,x)\xi(dt,x) + \lambda(t,x)\zeta(dt,x)\}$$

$$+ \mathcal{Y}(t,x)\{-A^*p(t,x)dt - \frac{\partial H}{\partial y}(dt,\xi(dt,x))\}$$

$$+ \mathcal{Y}(t,x)\frac{\partial \sigma}{\partial y}(t,x)q(t,x)]dt)dx], \qquad (2.32)$$

where $[p(\cdot,x),\mathcal{Y}(\cdot,x)](t)$ denotes the covariation process of $p(\cdot,x)$ and $\mathcal{Y}(\cdot,x)$.

Since $p(t,x) = \mathcal{Y}(t,x) = 0$ for $x \in \partial D$, we deduce that

$$\int_D p(t,x)A\mathcal{Y}(t,x)dx = \int_D A^*p(t,x)\mathcal{Y}(t,x)dx. \qquad (2.33)$$

Therefore, substituting (2.31) and (2.32) into (2.30), we get

$$\lim_{y\to 0^+} \frac{1}{y}(J(\xi + y\zeta) - J(\xi))$$

$$= E[\int_D (\int_0^T \{\lambda(t,x)p(t,x) + h(t,s)\}\zeta(dt,x))dx].$$

□

We can now state our necessary maximum principle:

Theorem 2.6 *[Necessary maximum principle]*

(i) Suppose $\xi^* \in \mathcal{A}$ is optimal, i.e.

$$\max_{\xi \in \mathcal{A}} J(\xi) = J(\xi^*). \tag{2.34}$$

Let $Y^*, (p^*, q^*)$ be the corresponding solution of (2.1)-(2.2) and (2.5)-(2.7), respectively, and assume that (2.28) *holds with $\xi = \xi^*$. Then*

$$\lambda(t,x,Y^*(t,x))p^*(t,x) + h(t,x,Y^*(t,x)) \leq 0 \quad \text{for all } t,x \in [0,T] \times D, a.s. \tag{2.35}$$

and

$$\{\lambda(t,x,Y^*(t,x))p^*(t,x) + h(t,x,Y^*(t,x))\}\xi^*(dt,x) = 0 \quad \text{for all } t,x \in [0,T] \times D, a.s. \tag{2.36}$$

(ii) Conversely, suppose that there exists $\hat{\xi} \in \mathcal{A}$ such that the corresponding solutions $\hat{Y}(t,x), (\hat{p}(t,x), \hat{q}(t,x))$ of (2.1)-(2.2) and (2.5)-(2.7), respectively, satisfy

$$\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x)) \leq 0 \quad \text{for all } t,x \in [0,T] \times D, a.s. \tag{2.37}$$

and

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) = 0 \quad \text{for all } t,x \in [0,T] \times D, a.s. \tag{2.38}$$

Then $\hat{\xi}$ is a directional sub-stationary point for $J(\cdot)$, in the sense that

$$\lim_{y \to 0^+} \frac{1}{y}(J(\hat{\xi} + y\zeta) - J(\hat{\xi})) \leq 0 \quad \text{for all } \zeta \in \mathcal{V}(\hat{\xi}). \tag{2.39}$$

Proof. This is proved in a similar way as in Theorem 2.4 in [ØS]. For completeness we give the details:

(i) If $\xi \in \mathcal{A}$ is optimal, we get by Lemma 2.5

$$\begin{aligned} 0 &\geq \lim_{y \to 0^+} \frac{1}{y}(J(\xi + y\zeta) - J(\xi)) \\ &= E[\int_D \int_0^T \{\lambda(t,x)p(t,x) + h(t,x)\}\zeta(dt,x)dx], \quad \text{for all } \zeta \in \mathcal{V}(\xi). \end{aligned} \tag{2.40}$$

In particular, this holds if we choose ζ such that

$$\zeta(ds,x) = a(\omega)\delta_t(s)\phi(x) \tag{2.41}$$

for some fixed $t \in [0,T]$ and some bounded $\mathcal{F}_t$-measurable random variable $a(\omega) \geq 0$ and some bounded, deterministic $\phi(x) \geq 0$, where $\delta_t(s)$ is Dirac measure at t. Then (2.40) gets the form

$$E[\int_D \{\lambda(t,x)p(t,x) + h(t,x)\}a(\omega)\phi(x)dx] \leq 0.$$

Since this holds for all such $a(\omega), \phi(x)$ we deduce that

$$\lambda(t,x)p(t,x) + h(t,x) \leq 0 \quad \text{for all } t, x, a.s. \tag{2.42}$$

Next, if we choose $\zeta(dt,x) = \xi(dt,x) \in \mathcal{V}(\xi)$, we get from (2.40)

$$E[\int_D \int_0^T \{\lambda(t,x)p(t,x) + h(t,x)\}\xi(dt,x)dx] \leq 0. \tag{2.43}$$

On the other hand, we can also choose $\zeta(dt,x) = -\xi(dt,x) \in \mathcal{V}(\xi)$, and this gives

$$E[\int_D \int_0^T \{\lambda(t,x)p(t,x) + h(t,x)\}\xi(dt,x)dx] \geq 0. \tag{2.44}$$

Combining (2.43) and (2.44) we get

$$E[\int_D \int_0^T \{\lambda(t,x)p(t,x) + h(t,x)\}\xi(dt,x)dx] = 0. \tag{2.45}$$

Combining (2.42) and (2.45) we see that

$$\{\lambda(t,x)p(t,x) + h(t,x)\}q(dt,x) = 0 \quad \text{for all } t, x, a.s. \tag{2.46}$$

as claimed. This proves (i).

(ii) Conversely, suppose $\hat{\xi} \in \mathcal{A}$ is as in (ii). Then (2.39) follows from Lemma 2.5.

□

3 Existence and Uniqueness

In this section, we will prove the main existence and uniqueness result for reflected backward stochastic partial differential equations. For notational simplicity, we choose the operator A to be the Laplacian operator Δ. However, our methods work equally well for general second order differential operators like

$$A = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}),$$

where $a = (a_{ij}(x)) : D \to \mathbb{R}^{d\times d}$ $(d > 2)$ is a measurable, symmetric matrix-valued function which satisfies the uniform elliptic condition

$$\lambda|z|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)z_i z_j \leq \Lambda|z|^2, \ \forall\, z \in \mathbb{R}^d \ \text{ and } \ x \in D$$

for some constant λ, $\Lambda > 0$

First we will establish a comparison theorem for BSPDEs, which is of independent interest. Consider two backward SPDEs:

$$\begin{aligned} du_1(t,x) &= -\Delta u_1(t)dt - b_1(t,u_1(t,x),Z_1(t,x))dt + Z_1(t,x)dB_t, t\in(0,T) \\ u_1(T,x) &= \phi_1(x) \quad a.s. \end{aligned} \tag{3.1}$$

$$\begin{aligned} du_2(t,x) &= -\Delta u_2(t)dt - b_2(t,u_2(t,x),Z_2(t,x))dt + Z_2(t,x)dB_t, t\in(0,T) \\ u_2(T,x) &= \phi_2(x) \quad a.s. \end{aligned} \tag{3.2}$$

From now on, if $u(t,x)$ is a function of (t,x), we write $u(t)$ for the function $u(t,\cdot)$.

The following result is a comparison theorem for backward stochastic partial differential equations.

Theorem 3.1 *(Comparison theorem for BSPDEs) Suppose $\phi_1(x)\le\phi_2(x)$ and $b_1(t,u,z)\le b_2(t,u,z)$. Then we have $u_1(t,x)\le u_2(t,x), x\in D$, a.e. for every $t\in[0,T]$.*

Proof. For $n\ge 1$, define functions $\psi_n(z)$, $f_n(x)$ as follows (see [DP1]).

$$\psi_n(z)=\begin{cases} 0 & \text{if } z\le 0,\\ 2nz & \text{if } 0\le z\le \frac{1}{n},\\ 2 & \text{if } z>\frac{1}{n}. \end{cases} \tag{3.3}$$

$$f_n(x)=\begin{cases} 0 & \text{if } x\le 0,\\ \int_0^x dy\int_0^y \psi_n(z)dz & \text{if } x>0. \end{cases} \tag{3.4}$$

We have

$$f_n'(x)=\begin{cases} 0 & \text{if } x\le 0,\\ nx^2 & \text{if } x\le\frac{1}{n},\\ 2x-\frac{1}{n} & \text{if } x>\frac{1}{n}. \end{cases} \tag{3.5}$$

Also $f_n(x)\uparrow(x^+)^2$ as $n\to\infty$. For $h\in K:=L^2(D)$, set

$$F_n(h)=\int_D f_n(h(x))dx.$$

F_n has the following derivatives for $h_1,h_2\in K$,

$$F_n'(h)(h_1)=\int_D f_n'(h(x))h_1(x)dx, \tag{3.6}$$

$$F_n''(h)(h_1,h_2)=\int_D f_n''(h(x))h_1(x)h_2(x)dx. \tag{3.7}$$

Applying Ito's formula we get

$$
\begin{aligned}
& F_n(u_1(t)-u_2(t)) \\
= \; & F_n(\phi_1-\phi_2)+\int_t^T F_n'(u_1(s)-u_2(s))(\Delta(u_1(s)-u_2(s)))ds \\
& +\int_t^T F_n'(u_1(s)-u_2(s))(b_1(s,u_1(s),Z_1(s))-b_2(s,u_2(s),Z_2(s)))ds \\
& -\int_t^T F_n'(u_1(s)-u_2(s))(Z_1(s)-Z_2(s))dB_s \\
& -\frac{1}{2}\int_t^T F_n''(u_1(s)-u_2(s))(Z_1(s)-Z_2(s),Z_1(s)-Z_2(s))ds \\
=: \; & I_n^1+I_n^2+I_n^3+I_n^4+I_n^5, \qquad (3.8)
\end{aligned}
$$

where,

$$
\begin{aligned}
I_n^2 &= \int_t^T F_n'(u_1(s)-u_2(s))(\Delta(u_1(s)-u_2(s)))ds \\
&= \int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))(\Delta(u_1(s,x)-u_2(s,x)))dxds \\
&= -\int_t^T f_n''(u_1(s,x)-u_2(s,x))|\nabla(u_1(s,x)-u_2(s,x)|^2dxds\le 0, \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
I_n^5 &= -n\int_t^T\int_D \chi_{\{0\le u_1(s,x)-u_2(s,x)\le\frac{1}{n}\}}(u_1(s,x)-u_2(s,x))|Z_1(s,x)-Z_2(s,x)|^2dxds \\
&\quad -\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>\frac{1}{n}\}}|Z_1(s,x)-Z_2(s,x)|^2dxds. \qquad (3.10)
\end{aligned}
$$

For I_n^3, we have

$$
\begin{aligned}
I_n^3 &= \int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))(b_1(s,u_1(s,x),Z_1(s,x))-b_2(s,u_2(s,x),Z_2(s,x)))dxds \\
&= \int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))(b_1(s,u_1(s,x),Z_1(s,x))-b_2(s,u_1(s,x),Z_1(s,x)))dxds \\
&\quad +\int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))(b_2(s,u_1(s,x),Z_1(s,x))-b_2(s,u_2(s,x),Z_1(s,x)))dxds \\
&\quad +\int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))(b_2(s,u_2(s,x),Z_1(s,x))-b_2(s,u_2(s,x),Z_2(s,x)))dxds \\
&\le \int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))(b_2(s,u_2(s,x),Z_1(s,x))-b_2(s,u_2(s,x),Z_2(s,x)))dxds \\
&\quad +C\int_t^T\int_D((u_1(s,x)-u_2(s,x)^+)^2dxds := I_{n,1}^3+I_{n,2}^3, \qquad (3.11)
\end{aligned}
$$

where the Lipschiz condition of b and the assumption $b_1 \leq b_2$ have been used. $I^3_{n,1}$ can be estimated as follows:

$$
\begin{aligned}
I^3_{n,1} &\leq C\int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))|Z_1(s,x)-Z_2(s,x)|dxds \\
&= C\int_t^T\int_D \chi_{\{0\leq u_1(s,x)-u_2(s,x)\leq\frac{1}{n}\}}n(u_1(s,x)-u_2(s,x))^2|Z_1(s,x)-Z_2(s,x)|dxds \\
&\quad +C\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>\frac{1}{n}\}}[2(u_1(s,x)-u_2(s,x))-\frac{1}{n}]|Z_1(s,x)-Z_2(s,x)|dxds \\
&\leq C\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>\frac{1}{n}\}}(2(u_1(s,x)-u_2(s,x))-\frac{1}{n})^2dxds \\
&\quad +\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>\frac{1}{n}\}}|Z_1(s,x)-Z_2(s,x)|^2dxds \\
&\quad +\frac{1}{4}C^2\int_t^T\int_D \chi_{\{0\leq u_1(s,x)-u_2(s,x)\leq\frac{1}{n}\}}n(u_1(s,x)-u_2(s,x))^3dxds \\
&\quad +\int_t^T\int_D \chi_{\{0\leq u_1(s,x)-u_2(s,x)\leq\frac{1}{n}\}}n(u_1(s,x)-u_2(s,x))^2|Z_1(s,x)-Z_2(s,x)|^2dxds \\
&\leq C'\int_t^T\int_D ((u_1(s,x)-u_2(s,x))^+)^2dxds \\
&\quad +\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>\frac{1}{n}\}}|Z_1(s,x)-Z_2(s,x)|^2dxds \\
&\quad +\int_t^T\int_D \chi_{\{0\leq u_1(s,x)-u_2(s,x)\leq\frac{1}{n}\}}n(u_1(s,x)-u_2(s,x))^2|Z_1(s,x)-Z_2(s,x)|^2dxds
\end{aligned}
\tag{3.12}
$$

(3.10),(3.11) and (3.12) imply that

$$I^3_n + I^5_n \leq C\int_t^T\int_D((u_1(s,x)-u_2(s,x))^+)^2dxds \tag{3.13}$$

Thus it follows from (3.8), (3.9) and (3.13) that

$$
\begin{aligned}
&F_n(u_1(t)-u_2(t)) \\
\leq\ & F_n(\phi_1-\phi_2)+C\int_t^T\int_D((u_1(s,x)-u_2(s,x))^+)^2dxds \\
&-\int_t^T F_n'(u_1(s)-u_2(s))(Z_1(s)-Z_2(s))dB_s
\end{aligned}
\tag{3.14}
$$

Take expectation and let $n\to\infty$ to get

$$E[\int_D((u_1(t,x)-u_2(t,x))^+)^2dx] \leq \int_t^T dsE[\int_D((u_1(s,x)-u_2(s,x))^+)^2dx] \tag{3.15}$$

Gronwall's inequality yields that

$$E[\int_D ((u_1(t,x) - u_2(t,x))^+)^2 dx] = 0, \tag{3.16}$$

which completes the proof of the theorem. ■

Remark. After this paper was written we became aware of the paper [MYZ], where a similar comparison theorem is proved. However, the theorems are not identical and the proofs are quite different.

We now proceed to prove existence and uniqueness of the reflected BSPDEs. Let $V = W_0^{1,2}(D)$ be the Sobolev space of order one with the usual norm $||\cdot||$. Consider the reflected backward stochastic partial differential equation:

$$du(t) = -\Delta u(t)dt - b(t, u(t,x), Z(t,x))dt + Z(t,x)dB_t, t \in (0,T) \tag{3.17}$$

$$-\eta(dt,x), t \in (0,T), \tag{3.18}$$

$$u(t,x) \geq L(t,x),$$

$$\int_0^T \int_D (u(t,x) - L(t,x))\eta(dt,x)dx = 0,$$

$$u(T,x) = \phi(x) \quad a.s. \tag{3.19}$$

Theorem 3.2 *Assume that* $E[|\phi|_K^2] < \infty$. *and that*

$$|b(s,u_1,z_1) - b(s,u_1,z_1)| \leq C(|u_1 - u_2| + |z_1 - z_2|).$$

Let $L(t,x)$ *be a measurable function which is differentiable in* t *and twice differentiable in* x *such that*

$$\int_0^T \int_D L'(t,x)^2 dxdt < \infty, \int_0^T \int_D |\Delta L(t,x)|^2 dxdt < \infty.$$

Then there exists a unique $K \times L^2(D, \mathbb{R}^m) \times K$*-valued progressively measurable process* $(u(t,x), Z(t,x), \eta(t,x))$ *such that*

(i) $E[\int_0^T ||u(t)||_V^2 dt] < \infty, \quad E[\int_0^T |Z(t)|^2_{L^2(D,R^m)} dt] < \infty.$

(ii) η *is a* K*-valued continuous process, non-negative and nondecreasing in* t *and* $\eta(0,x) = 0$.

(iii) $u(t,x) = \phi(x) + \int_t^T \Delta u(t,x)ds + \int_t^T b(s, u(s,x), Z(s,x))ds - \int_t^T Z(s,x)dB_s$ $+\eta(T,x) - \eta(t,x); \quad 0 \leq t \leq T,$ (3.20)

(iv) $u(t,x) \geq L(t,x) \quad a.e. \quad x \in D, \forall t \in [0,T].$

(v) $\int_0^T \int_D (u(t,x) - L(t,x))\eta(dt,x)dx = 0$

where $u(t)$ stands for the K-valued continuous process $u(t,\cdot)$ and (iii) is understood as an equation in the dual space V^* of V.

For the proof of the theorem, we introduce the penalized BSPDEs:

$$\begin{aligned} du^n(t) &= -\Delta u^n(t)dt - b(t, u^n(t,x), Z^n(t,x))dt + Z^n(t,x)dB_t \\ &\quad -n(u^n(t,x) - L(t,x))^- dt, \quad t \in (0,T) \qquad (3.21) \\ u^n(T,x) &= \phi(x) \quad a.s. \qquad (3.22) \end{aligned}$$

According to [ØPZ], the solution (u^n, Z^n) of the above equation exists and is unique. We are going to show that the sequence (u^n, Z^n) has a limit, which will be a solution of the equation (3.20). First we need some a priori estimates.

Lemma 3.3 *Let (u^n, Z^n) be the solution of equation (3.21). We have*

$$\sup_n E[\sup_t |u^n(t)|_K^2] < \infty, \qquad (3.23)$$

$$\sup_n E[\int_0^T ||u^n(t)||_V^2] < \infty, \qquad (3.24)$$

$$\sup_n E[\int_0^T |Z^n(t)|^2_{L^2(D,R^m)}] < \infty. \qquad (3.25)$$

Proof. Take a function $f(t,x) \in C_0^{2,2}([-1, T+1] \times D)$ satisfying $f(t,x) \geq L(t,x)$. Applying Itô's formula, it follows that

$$\begin{aligned} |u^n(t) - f(t)|_K^2 &= |\phi - f(T)|_K^2 + 2\int_t^T < u^n(s) - f(s), \Delta u^n(s) > ds \\ &\quad +2\int_t^T < u^n(s) - f(s), b(s, u^n(s), Z^n(s)) > ds \\ &\quad -2\int_t^T < u^n(s) - f(s), Z^n(s) > dB_s \\ &\quad +2n\int_t^T < u^n(s) - f(s), (u^n(s) - L(s))^- > ds - \int_t^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)} ds \\ &\quad +2\int_t^T < u^n(s) - f(s), f'(s) > ds, \quad a.s. \qquad (3.26) \end{aligned}$$

where $<,>$ denotes the inner product in K. Now we estimate each of the terms on the right hand side.

$$\begin{aligned} & 2\int_t^T < u^n(s) - f(s), \Delta u^n(s) > ds \\ = & -2\int_t^T ||u^n(s)||_V^2 ds + 2\int_t^T < \frac{\partial f(s)}{\partial x}, \frac{\partial u^n(s)}{\partial x} > ds \\ \leq & -\int_t^T ||u^n(s)||_V^2 ds + \int_t^T ||f(s)||_V^2 ds \qquad (3.27) \end{aligned}$$

$$
\begin{aligned}
& 2\int_t^T < u^n(s)-f(s), b(s,u^n(s),Z^n(s)) > ds \\
= & \ 2\int_t^T < u^n(s)-f(s), b(s,u^n(s),Z^n(s)) - b(s,f(s),Z^n(s)) > ds \\
& +2\int_t^T < u^n(s)-f(s), b(s,f(s),Z^n(s)) - b(s,f(s),0) > ds \\
& +2\int_t^T < u^n(s)-f(s), b(s,f(s),0) > ds \\
\leq & \ C\int_t^T |u^n(s)-f(s)|_H^2 ds + \frac{1}{2}\int_t^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)} ds \\
& +C\int_t^T |b(s,f(s),0)|_H^2 ds
\end{aligned} \tag{3.28}
$$

$$
\begin{aligned}
& 2n\int_t^T < u^n(s)-f(s), (u^n(s)-L(s))^- > ds \\
= & \ 2n\int_t^T \int_D (u^n(s,x)-f(s,x))\chi_{\{u^n(s,x)\leq L(s,x)\}}(L(s,x)-u^n(s,x)) ds dx \leq 0
\end{aligned} \tag{3.29}
$$

Substituting (3.27),(3.28) and (3.29) into (3.26) we obtain

$$
\begin{aligned}
& |u^n(t)-f(t)|_K^2 + \int_t^T ||u^n(s)||_V^2 ds + \frac{1}{2}\int_t^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)} ds \\
\leq & \ |\phi - f(T)|_K^2 + C\int_t^T |u^n(s)-f(s)|_K^2 ds + C\int_t^T |b(s,f(s),0)|_K^2 ds \\
& + \int_t^T ||f(s)||_V^2 ds - 2\int_t^T < u^n(s)-f(s), Z^n(s) > dB_s
\end{aligned} \tag{3.30}
$$

Take expectation and use the Gronwall inequality to obtain

$$
\sup_n \sup_t E[|u^n(t)|_K^2] < \infty \tag{3.31}
$$

$$
\sup_n E[\int_0^T ||u^n(t)||_V^2] < \infty \tag{3.32}
$$

$$
\sup_n E[\int_0^T |Z^n(t)|^2_{L^2(D,\mathbb{R}^m)} dt] < \infty \tag{3.33}
$$

By virtue of (3.33), (3.31) can be further strengthened to (3.23). Indeed, by Burkholder inequality,

$$
\begin{aligned}
& E\Big[2\sup_{v\le t\le T}|\int_t^T <u^n(s)-f(s),Z^n(s)>dB_s|\Big]\\
\le\ & CE\Big[(\int_v^T |u^n(s)-f(s)|_K^2|Z^n(s)|^2_{L^2(D,\mathbb{R}^m)}ds)^{\frac{1}{2}}\Big]\\
\le\ & CE\Big[\sup_{v\le s\le T}(|u^n(s)-f(s)|_K)(\int_v^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)}ds)^{\frac{1}{2}}\Big]\\
\le\ & \frac{1}{2}E\Big[\sup_{v\le s\le T}(|u^n(s)-f(s)|_K^2)\Big]+CE\Big[\int_v^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)}ds\Big] \qquad (3.34)
\end{aligned}
$$

With (3.34), taking superum over $t\in[v,T]$ on both sides of (3.26) we obtain (3.23). ■

We need the following estimates.

Lemma 3.4 *Suppose the conditions in Theorem 3.2 hold. Then there is a constant C such that*

$$E[\int_0^T\int_D((u^n(t,x)-L(t,x))^-)^2dxdt]\le\frac{C}{n^2}. \qquad (3.35)$$

Proof. Let f_m be defined as in the proof of Theorem 3.1. Then $f_m(x)\uparrow(x^+)^2$ and $f'_m(x)\uparrow 2x^+$ as $m\to\infty$. For $h\in K$, set

$$G_m(h)=\int_D f_m(-h(x))dx.$$

It is easy to see that for $h_1,h_2\in K$,

$$G'_m(h)(h_1)=-\int_D f'_m(-h(x))h_1(x)dx, \qquad (3.36)$$

$$G''_n(h)(h_1,h_2)=\int_D f''_m(-h(x))h_1(x)h_2(x)dx. \qquad (3.37)$$

Applying It$\hat{o}$'s formula we get

$$\begin{aligned}
& G_m(u^n(t)-L(t)) \\
= \quad & G_m(\phi - L(T)) + \int_t^T G'_m(u^n(s)-L(s))(\Delta u^n(s)))ds \\
& + \int_t^T G'_m(u^n(s)-L(s))(b(s,u^n(s),Z^n(s)))ds \\
& +n \int_t^T G'_m(u^n(s)-L(s))((u^n(s)-L(s))^-)ds \\
& + \int_t^T G'_m(u^n(s)-L(s))(L'(s))ds \\
& - \int_t^T G'_m(u^n(s)-L(s))(Z^n(s))dB_s \\
& -\frac{1}{2} \int_t^T G''_m(Z^n(s),Z^n(s))ds \\
=: \quad & I_m^1 + I_m^2 + I_m^3 + I_m^4 + I_m^5 + I_m^6 + I_m^7. \qquad (3.38)
\end{aligned}$$

Now,

$$\begin{aligned}
& I_m^2 = \int_t^T G'_m(u^n(s)-L(s))(\Delta u^n(s)))ds \\
= \quad & - \int_t^T \int_D f'_m(L(s,x)-u^n(s,x))(\Delta(u^n(s,x)-L(s,x)))dxds \\
& - \int_t^T \int_D f'_m(L(s,x)-u^n(s,x))(\Delta L(s,x))dxds \\
\leq \quad & - \int_t^T \int_D f''_m(L(s,x)-u^n(s,x))|\nabla(u^n(s,x)-L(s,x))|^2 dxds \\
& +\frac{1}{4}n \int_t^T \int_D f'_m(L(s,x)-u^n(s,x))^2 xds \\
& +\frac{C}{n} \int_t^T \int_D (\Delta L(s,x))^2 dxds, \qquad (3.39)
\end{aligned}$$

$$\begin{aligned}
I_m^3 \quad = \quad & - \int_t^T \int_D f'_m(L(s,x)-u^n(s,x))b(s,u^n(s,x),Z^n(s,x))dxds \\
\leq \quad & \frac{1}{4}n \int_t^T \int_D f'_m(L(s,x)-u^n(s,x))^2 ds \\
+ \quad & \frac{C}{n} \int_t^T \int_D (b(s,u^n(s,x),Z^n(s,x)))^2 dxds, \qquad (3.40)
\end{aligned}$$

$$
\begin{aligned}
I_m^5 &= -\int_t^T \int_D f_m'(L(s,x)-u^n(s,x))(L'(s,x))dxds \\
&\le \frac{1}{4}n\int_t^T \int_D f_m'(L(s,x)-u^n(s,x))^2 ds \\
&+ \frac{C}{n}\int_t^T \int_D (L'(s,x))^2 dxds.
\end{aligned}
\tag{3.41}
$$

Combining (3.38)–(3.41) and taking expectation we obtain

$$
\begin{aligned}
& E[G_m(u^n(t)-L(t))] \\
\le\ & E[G_m(\phi - L(T))] + \frac{3}{4}n\int_t^T \int_D f_m'(L(s,x)-u^n(s,x))^2 ds \\
& + \frac{C}{n}E[\int_t^T \int_D (L'(s,x))^2 dxds] + \frac{C}{n}E[\int_t^T \int_D (\Delta L(s,x))^2 dxds] \\
& + \frac{C}{n}E[\int_t^T \int_D (b(s,u^n(s,x),Z^n(s,x)))^2 dxds] \\
& - nE[\int_t^T \int_D f_m'(L(s,x)-u^n(s,x))((u^n(s,x)-L(s,x))^-)ds].
\end{aligned}
\tag{3.42}
$$

Letting $m \to \infty$ we conclude that

$$
\begin{aligned}
& E[\int_D ((u^n(t,x)-L(t,x))^-)^2 dx] \\
\le\ & \frac{3}{4}nE[\int_t^T \int_D ((u^n(s,x)-L(s,x))^-)^2 dxds] \\
& - nE[\int_t^T \int_D ((u^n(s,x)-L(s,x))^-)^2 dxds] + \frac{C'}{n},
\end{aligned}
\tag{3.43}
$$

where the Lipschiz condition of b and Lemma 3.3 have been used. In particular we have

$$
E[\int_t^T \int_D ((u^n(s,x)-L(s,x))^-)^2 dxds] \le \frac{C'}{n^2}. \tag{3.44}
$$

■

Lemma 3.5 *Let (u^n, Z^n) be the solution of equation (3.21). We have*

$$
\lim_{n,m\to\infty} E[\sup_{0\le t\le T} |u^n(t)-u^m(t)|_K^2] = 0, \tag{3.45}
$$

$$
\lim_{n,m\to\infty} E[\int_0^T ||u^n(t)-u^m(t)||_V^2 dt] = 0. \tag{3.46}
$$

$$
\lim_{n,m\to\infty} E[\int_0^T |Z^n(t)-Z^m(t)|_{L^2(D,\mathbb{R}^m)}^2 dt] = 0. \tag{3.47}
$$

Proof. Applying Itô's formula, it follows that

$$
\begin{aligned}
& |u^n(t)-u^m(t)|_K^2 \\
= \ & 2\int_t^T < u^n(s)-u^m(s), \Delta(u^n(s)-u^m(s)) > ds \\
& +2\int_t^T < u^n(s)-u^m(s), b(s,u^n(s),Z^n(s)) - b(s,u^m(s),Z^m(s)) > ds \\
& -2\int_t^T < u^n(s)-u^m(s), Z^n(s)-Z^m(s) > dB_s \\
& +2\int_t^T < u^n(s)-u^m(s), n(u^n(s)-L(s))^- - m(u^m(s)-L(s))^- > ds \\
& -\int_t^T |Z^n(s)-Z^m(s)|^2_{L^2(D,\mathbb{R}^m)} ds \qquad (3.48)
\end{aligned}
$$

Now we estimate each of the terms on the right side.

$$
\begin{aligned}
& 2\int_t^T < u^n(s)-u^m(s), \Delta(u^n(s)-u^m(s)) > ds \\
= \ & -2\int_t^T ||u^n(s)-u^m(s)||_V^2 ds. \qquad (3.49)
\end{aligned}
$$

By the Lipschitz continuity of b and the inequality $ab \leq \varepsilon a^2 + C_\varepsilon b^2$, one has

$$
\begin{aligned}
& 2\int_t^T < u^n(s)-u^m(s), b(s,u^n(s),Z^n(s)) - b(s,u^m(s),Z^m(s)) > ds \\
\leq \ & C\int_t^T |u^n(s)-u^m(s)|_K^2 ds + \frac{1}{2}\int_t^T |Z^n(s)-Z^m(s)|^2_{L^2(D,\mathbb{R}^m)} ds. \qquad (3.50)
\end{aligned}
$$

In view of (3.44),

$$
\begin{aligned}
& 2E[\int_t^T < u^n(s) - u^m(s), n(u^n(s) - L(s))^- - m(u^m(s) - L(s))^- > ds] \\
= \ & 2nE[\int_t^T < u^n(s) - L(s), (u^n(s) - L(s))^- > ds] \\
& +2mE[\int_t^T < L(s) - u^n(s), (u^m(s) - L(s))^- > ds] \\
& +2mE[\int_t^T < u^m(s) - L(s), (u^m(s) - L(s))^- > ds] \\
& +2nE[\int_t^T < L(s) - u^m(s), (u^n(s) - L(s))^- > ds] \\
\leq \ & 2mE[\int_t^T < L(s) - u^n(s), (u^m(s) - L(s))^- > ds] \\
& +2nE[\int_t^T < L(s) - u^m(s), (u^n(s) - L(s))^- > ds] \\
\leq \ & 2mE[\int_t^T \int_D (u^n(s,x) - L(s,x))^- (u^m(s,x) - L(s,x))^- dxds] \\
& +2nE[\int_t^T \int_D (u^m(s,x) - L(s,x))^- (u^n(s,x) - L(s,x))^- dxds] \\
\leq \ & 2m(E[\int_t^T \int_D ((u^n(s,x) - L(s,x))^-)^2 dxds])^{\frac{1}{2}} (E[\int_t^T \int_D ((u^m(s,x) - L(s,x))^-)^2 dxds])^{\frac{1}{2}} \\
& +2n(E[\int_t^T \int_D ((u^n(s,x) - L(s,x))^-)^2 dxds])^{\frac{1}{2}} (E[\int_t^T \int_D ((u^m(s,x) - L(s,x))^-)^2 dxds])^{\frac{1}{2}} \\
\leq \ & C'(\frac{1}{n} + \frac{1}{m}).
\end{aligned}
\tag{3.51}
$$

It follows from (3.48) and (3.49) that

$$
\begin{aligned}
& E[|u^n(t) - u^m(t)|_K^2] + \frac{1}{2}E[\int_t^T |Z^n(s) - Z^m(s)|^2_{L^2(D,\mathbb{R}^m)} ds] \\
& +E[\int_t^T ||u^n(s) - u^m(s)||_V^2 ds] \\
\leq \ & C \int_t^T E[|u^n(s) - u^m(s)|_K^2] ds + C'(\frac{1}{n} + \frac{1}{m}).
\end{aligned}
\tag{3.52}
$$

Application of the Gronwall inequality yields

$$
\lim_{n,m\to\infty} \{E[|u^n(t) - u^m(t)|_K^2] + \frac{1}{2}E[\int_t^T |Z^n(s) - Z^m(s)|^2_{L^2(D,\mathbb{R}^m)} ds]\} = 0, \tag{3.53}
$$

$$\lim_{n,m\to\infty} E[\int_t^T ||u^n(s)-u^m(s)||_V^2 ds] = 0. \tag{3.54}$$

By (3.53) and the Burkholder inequality we can further show that

$$\lim_{n,m\to\infty} E[\sup_{0\le t\le T} |u^n(t)-u^m(t)|_K^2] = 0. \tag{3.55}$$

The proof is complete. ■

Proof of Theorem 3.2. From Lemma 3.5 we know that $(u^n, Z^n), n \ge 1$, forms a Cauchy sequence. Denote by $u(t,x)$, $Z(t,x)$ the limit of u^n and Z^n. Put

$$\bar{\eta}^n(t,x) = n(u^n(t,x) - L(t,x))^-$$

Lemma 3.4 implies that $\bar{\eta}^n(t,x)$ admits a non-negative weak limit, denoted by $\bar{\eta}(t,x)$, in the following Hilbert space:

$$\bar{K} = \{h; \quad \text{h is a K-valued adapted process such that} \quad E[\int_0^T |h(s)|_K^2 ds] < \infty\}$$

with inner product

$$< h_1, h_2 >_{\bar{K}} = E[\int_0^T \int_D h_1(t,x)h_2(t,x)dtdx].$$

Set $\eta(t,x) = \int_0^t \bar{\eta}(s,x)ds$. Then η is a continuous K-valued process which is increasing in t. Keeping Lemma 3.5 in mind and letting $n \to \infty$ in (3.21) we obtain

$$\begin{aligned} & u(t,x) \\ = \ & \phi(x) + \int_t^T \Delta u(t,x)ds + \int_t^T b(s,u(s,x),Z(s,x))ds - \int_t^T Z(s,x)dB_s \\ & +\eta(T,x) - \eta(t,x); \quad 0 \le t \le T. \end{aligned} \tag{3.56}$$

Recall from Lemma 3.4 that

$$E[\int_t^T \int_D ((u^n(s,x) - L(s,x))^-)^2 dxds] \le C' \frac{1}{n^2}$$

By the Fatou Lemma, this implies that $E[\int_t^T \int_D ((u(s,x) - L(s,x))^-)^2 dxds] = 0$. In view of the continuity of u in t, we conclude $u(t,x) \ge L(t,x)$ a.e. in x, for every $t \ge 0$. Combining the strong convergence of u^n and the weak convergence of $\bar{\eta}^n$, we also have

$$\begin{aligned} & E[\int_0^T \int_D (u(s,x) - L(s,x))\eta(dt,x)dx] \\ = \ & E[\int_0^T \int_D (u(s,x) - L(s,x))\bar{\eta}(t,x)dtdx] \\ \le \ & \lim_{n\to\infty} E[\int_0^T \int_D (u^n(s,x) - L(s,x))\bar{\eta}^n(t,x)dtdx] \le 0 \end{aligned} \tag{3.57}$$

Hence,

$$\int_0^T \int_D (u(s,x) - L(s,x))\eta(dt,x)dx = 0, \quad a.s.$$

We have shown that (u, Z, η) is a solution to the reflected BSPDE (3.17).

Uniqueness. Let (u_1, Z_1, η_1), (u_2, Z_2, η_2) be two such solutions to equation (3.20). By Itô's formula, we have

$$\begin{aligned}
& |u_1(t) - u_2(t)|_K^2 \\
= \; & 2\int_t^T < u_1(s) - u_2(s), \Delta(u_1(s) - u_2(s)) > ds \\
& +2\int_t^T < u_1(s) - u_2(s), b(s, u_1(s), Z_1(s)) - b(s, u_2(s), Z_2(s)) > ds \\
& -2\int_t^T < u_1(s) - u_2(s), Z_1(s) - Z_2(s) > dB_s \\
& +2\int_t^T < u_1(s) - u_2(s), \eta_1(ds) - \eta_2(ds) > \\
& -\int_t^T |Z_1(s) - Z_2(s)|^2_{L^2(D,\mathbb{R}^m)} ds \qquad (3.58)
\end{aligned}$$

Similar to the proof of Lemma 3.5, we have

$$2\int_t^T < u_1(s) - u_2(s), \Delta(u_1(s) - u_2(s)) > ds \leq 0, \qquad (3.59)$$

and

$$\begin{aligned}
& 2\int_t^T < u_1(s) - u_2(s), b(s, u_1(s), Z_1(s)) - b(s, u_2(s), Z_2(s)) > ds \\
\leq \; & C\int_t^T |u_1(s) - u_2(s)|_K^2 ds + \frac{1}{2}\int_t^T |Z_1(s) - Z_2(s)|^2_{L^2(D,\mathbb{R}^m)} ds \qquad (3.60)
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& 2E[\int_t^T < u_1(s)-u_2(s), \eta_1(ds)-\eta_2(ds) >] \\
= & \ 2E[\int_t^T \int_D (u_1(s,x)-L(s,x))\eta_1(ds,x)dx] \\
& -2E[\int_t^T \int_D (u_1(s,x)-L(s,x))\eta_2(ds,x)dx] \\
& +2E[\int_t^T \int_D (u_2(s,x)-L(s,x))\eta_2(ds,x)dx] \\
& -2E[\int_t^T \int_D (u_2(s,x)-L(s,x))\eta_1(ds,x)dx] \\
\leq & \ 0
\end{aligned} \tag{3.61}
$$

Combining (3.58)—(3.61) we arrive at

$$
\begin{aligned}
& E[|u_1(t)-u_2(t)|_K^2] + \frac{1}{2}E[\int_t^T |Z_1(s)-Z_2(s)|^2_{L^2(D,\mathbb{R}^m)}ds] \\
\leq & \ C\int_t^T E[|u_1(s)-u_2(s)|_K^2]ds.
\end{aligned} \tag{3.62}
$$

Appealing to Gronwall inequality, this implies

$$
u_1 = u_2, \quad Z_1 = Z_2
$$

which further gives $\eta_1 = \eta_2$ from the equation they satisfy. □

4 Link to optimal stopping

In this section, we provide a link between the solution of a reflected backward stochastic partial differential equation and an optimal stopping problem. Let $u(t,x)$ be the solution of the following reflected BSPDE.

$$
\begin{aligned}
& u(t,x) \\
= & \ \phi(x) + \int_t^T \frac{1}{2}\Delta u(t,x)ds + \int_t^T k(s,x,u(s,x),Z(s,x))ds - \int_t^T Z(s,x)dB_s \\
& +\eta(T,x) - \eta(t,x); \quad 0 \leq t \leq T, \\
& u(t,x) \geq L(t,x), \\
& \int_0^T \int_D (u(s,x)-L(s,x))\eta(dt,x)dx = 0 \quad a.s.
\end{aligned} \tag{4.1}
$$

Let $\mathcal{S}_{t,T}$ be the set of all stopping times τ satisfying $t \le \tau \le T$. For $\tau \in \mathcal{S}_{t,T}$, define

$$R_t(\tau, x) = \int_t^\tau P_{s-t} k(s,x) ds + P_{\tau-t} L(\tau, x)\chi_{\{\tau<T\}} + P_{\tau-t}\phi(x)\chi_{\{\tau=T\}},$$

where $k(s,\cdot) = k(s,\cdot,u(s,\cdot),Z(s,\cdot))$ and P_t denotes the semigroup generated by the Laplacian operator $\frac{1}{2}\Delta$, i.e.

$$P_t f(x) = (2\pi t)^{-d/2} \int_{\mathbb{R}^d} f(y) exp(-\frac{|y-x|^2}{2t}) dy; f \in L^1(\mathbb{R}^d).$$

Here, and in the following we will use the simplified notation $P_t k(s,x) = (P_t k(s,\cdot))(x)$ etc.

Theorem 4.1 *$u(t,x)$ is the value function of the the optimal stopping problem associated with $R_t(\tau,x)$, i.e.,*

$$u(t,x) = esssup_{\tau \in \mathcal{S}_{t,T}} E[R_t(\tau,x)|\mathcal{F}_t] \tag{4.2}$$

Proof. Observe that u admits the following mild representation:

$$\begin{aligned} & u(t,x) \\ = \ & P_{T-t}\phi(x) + \int_t^T P_{s-t}(k(s,u(s,x),Z(s,x)))ds - \int_t^T P_{s-t}(Z(s,x))dB_s \\ & + \int_t^T P_{s-t}\eta(ds,x); \quad 0 \le t \le T. \end{aligned} \tag{4.3}$$

This implies that for any stopping time τ with $t \le \tau \le T$, we have

$$\begin{aligned} & u(t,x) \\ = \ & P_{\tau-t}(u(\tau,x)) + \int_t^\tau P_{s-t}(k(s,x))ds - \int_t^\tau P_{s-t}(Z(s,x))dB_s \\ & + \int_t^\tau P_{s-t}\eta(ds,x); \quad 0 \le t \le \tau. \end{aligned} \tag{4.4}$$

Since $\eta(s,x)$ is increasing in s and $u(s,x) \ge L(s,x)$ for $s < T$, it follows that

$$u(t,x) \ge R_t(\tau,x) - \int_t^\tau P_{s-t}(Z(s,x))dB_s \tag{4.5}$$

Take conditional expectation with respect to $\mathcal{F}_t$ on both sides to get

$$\begin{aligned} & u(t,x) \\ \ge \ & E[R_t(\tau,x)|\mathcal{F}_t] - E[\int_t^\tau P_{s-t}(Z(s,x))dB_s|\mathcal{F}_t] \\ = \ & E[R_t(\tau,x)|\mathcal{F}_t]. \end{aligned} \tag{4.6}$$

As τ is arbitrary, we obtain

$$u(t,x) \geq ess \sup_{\tau \in \mathcal{S}_{t,T}} E[R_t(\tau,x)|\mathcal{F}_t] \tag{4.7}$$

Now, define

$$\hat{\tau}_t = \inf\{s \in [t,T)|u(s) = L(s)\} \wedge T$$

From the property of η, it is not increasing on the interval $[t,\hat{\tau}_t]$. Therefore, $\int_t^{\hat{\tau}_t} P_{s-t}\eta(ds,x) = 0$. Thus we have from (4.4) that

$$\begin{aligned} & u(t,x) \\ = \; & P_{\hat{\tau}_t - t}(u(\hat{\tau}_t),x) + \int_t^{\hat{\tau}_t} P_{s-t}(k(s,x))ds - \int_t^{\hat{\tau}_t} P_{s-t}(Z(s,x))dB_s \\ = \; & R_t(\hat{\tau}_t,x) - \int_t^{\hat{\tau}_t} P_{s-t}(Z(s,x))dB_s. \end{aligned} \tag{4.8}$$

Taking conditional expectation yields that

$$u(t,x) = E[R_t(\hat{\tau}_t,x)|\mathcal{F}_t]$$

Combining this with (4.7) we obtain the theorem. ■

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