New results on Barlow-Proschans type measures of component importance in nonrepairable and repairable systems

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In this paper the Barlow and Proschan (1975) measure of the importance of a component in a binary coherent system is revisited. This measure is for the case of components not undergoing repair equal to the probability of the component causing system failure. It is a useful guide during the system development phase as to which components should receive more urgent attention in achieving system reliability growth. Here we suggest some possible extensions of this measure both for nonrepairable and repairable systems, and a series of new results are proved, also on the Natvig (1979) measure. For repairable systems we especially consider taking a dual term into account based on the probability of the component causing system repair.

1. Measures of component importance in non-repairable systems

There seems to be two main reasons for coming up with a measure of importance of system components. Reason 1: it permits the analyst to determine which components merit the most additional research and development to improve overall system reliability at minimum cost or effort. Reason 2: it may suggest the most efficient way to diagnose system failure by generating a repair checklist for an operator to follow.

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Consider a system consisting of \( n \) components. Let \( (i = 1, \ldots, n) \):
\[
X_i(t) = \begin{cases} 
1 & \text{if the } i \text{th component functions at time } t, \\
0 & \text{if the } i \text{th component is failed at time } t.
\end{cases}
\]
Assume also that the stochastic processes \( \{X_i(t), t \geq 0\}, i = 1, \ldots, n, \) are mutually independent. Introduce \( \mathbf{X}(t) = (X_1(t), \ldots, X_n(t)) \) and let:
\[
\varphi(\mathbf{X}(t)) = \begin{cases} 
1 & \text{if the system functions at time } t, \\
0 & \text{if the system is failed at time } t.
\end{cases}
\]
We also assume the \textit{structure function} \( \varphi \) to be coherent. For an excellent introduction to coherent structure theory, we refer to Barlow and Proschan (1981).

In this section we restrict our attention to the case where the components, and hence the system, cannot be repaired. Let the \( i \)th component have an absolutely continuous life distribution \( F_i(t) \) with density \( f_i(t) \). Then the \textit{reliability} of this component at time \( t \) is given by:
\[
P[X_i(t) = 1] = 1 - F_i(t) = \frac{d}{d\mathbf{F}}(t).
\]
Introduce \( \tilde{\mathbf{F}}(t) = (\tilde{F}_1(t), \ldots, \tilde{F}_n(t)) \). Then the reliability of the system at time \( t \) is given by:
\[
P[\varphi(\mathbf{X}(t)) = 1] = h(\tilde{\mathbf{F}}(t)),
\]
where \( h \) is the system’s \textit{reliability function}. The following notation will be used:
\[
(\gamma, \mathbf{x}) = (x_1, \ldots, x_{i-1}, \gamma, x_{i+1}, \ldots, x_n).
\]

1.1. The Birnbaum measure

Birnbaum (1969) defines the importance of the \( i \)th component at time \( t \) by:
\[
I_B^{(i)}(t) = P[\varphi(1_i, \mathbf{X}(t)) - \varphi(0_i, \mathbf{X}(t)) = 1] = h(1_i, \tilde{\mathbf{F}}(t)) - h(0_i, \tilde{\mathbf{F}}(t)),
\]
which is the probability that the system is in a state at time \( t \) in which the functioning of the \( i \)th component is critical, i.e. the system functions if the \( i \)th component functions and is failed otherwise. By pivot decomposition it follows that
\[
I_B^{(i)}(t) = \frac{d h(\tilde{\mathbf{F}}(t))}{d\tilde{F}_i(t)},
\]
indicating that the Birnbaum measure reflects Reason 1. There are two main objections to this measure. Firstly, it gives the importance at fixed points of time leaving for the analyst at the system development phase to determine which points are important. Secondly, the measure does not depend on the reliability of the \( i \)th component, whether good or bad, although the ranking of the importance of the components depends on all component reliabilities.
1.2. The Barlow-Proshchan measure

These objections cannot be raised to the time-independent Barlow and Proshchan (1975) importance of the $i$th component:

$$I_{B-P}^{(i)} = P[\text{The failure of the } i\text{th component coincides with the failure of the system}].$$

Now obviously:

$$I_{B-P}^{(i)} = \int_0^\infty I_{B-P}^{(i)}(t)f_i(t)dt = \int_0^\infty [h(1_i, \bar{F}(t)) - h(0_i, \bar{F}(t))]f_i(t)dt. \quad (2)$$

Hence, the Barlow-Proshchan measure is a weighted average of the Birnbaum measure, the weight at time $t$ being $f_i(t)$. According to this measure a component is more important the more likely it is to be the direct cause of system failure, indicating that it takes well care of both Reasons 1 and 2.

Since a system failure coincides with the failure of exactly one component, we have

$$\sum_{i=1}^n I_{B-P}^{(i)} = 1. \quad (3)$$

This is not true for the Birnbaum measure. The following theorem, with an original proof, generalizing a theorem the proof of which is sketched in Barlow and Proshchan (1975), seems to indicate that this is a sensible measure:

**Theorem 1.** Let the $i$th component be in series (parallel) with the rest of the system. Let for $j \neq i$ $F_i(t) \geq F_j(t)(\bar{F}_i(t) \geq \bar{F}_j(t))$ for all $t \geq 0$. Then

$$I_{B-P}^{(i)} \geq I_{B-P}^{(j)} + \int_0^\infty \frac{f_j(t)}{F_j(t)}h(0_j, \bar{F}(t))dt$$

$$\left(I_{B-P}^{(i)} \geq I_{B-P}^{(j)} + \int_0^\infty \frac{f_j(t)}{F_j(t)}(1 - h(1_j, \bar{F}(t)))dt\right).$$

**Proof.** When the $i$th component is in series with the rest of the system, we have noting that $h(\bar{F}(t))$ is nonincreasing in $t$:

$$I_{B-P}^{(i)} = \int_0^\infty f_i(t)h(1_i, \bar{F}(t))dt = \int_0^\infty \frac{f_i(t)}{F_i(t)}h(\bar{F}(t))dt$$

$$= \left[-\ln(\bar{F}_i(t))h(\bar{F}(t))\right]^\infty_0 + \int_0^\infty \ln(\bar{F}_i(t))\frac{d}{dt}h(\bar{F}(t))dt$$
\[ \begin{aligned}
\int_0^\infty \ln(F_j(t)) \frac{d}{dt} h(\bar{F}(t)) dt &= \int_0^\infty \frac{f_j(t)}{F_j(t)} h(\bar{F}(t)) dt \\
&= \int_0^\infty f_j(t) \frac{1}{F_j(t)} [\bar{F}(t)h(1_j, \bar{F}(t)) + (1 - \bar{F}_j(t))h(0_j, \bar{F}(t))] dt \\
&= I_{B-P}^{(j)} + \int_0^\infty \frac{f_j(t)}{F_j(t)} h(0_j, \bar{F}(t)) dt.
\end{aligned} \]

Similarly, when the \( i \)th component is in parallel with the rest of the system:

\[ \begin{aligned}
I_{B-P}^{(i)} &= \int_0^\infty f_i(t)[1 - h(0_i, \bar{F}(t))] dt \\
&= \int_0^\infty \frac{f_i(t)}{F_i(t)} [1 - h(\bar{F}(t))] dt = \int_0^\infty (\log F_i(t)) \frac{d}{dt} h(\bar{F}(t)) dt \\
&\geq \int_0^\infty (\log F_j(t)) \frac{d}{dt} h(\bar{F}(t)) dt \\
&= \int_0^\infty f_j(t) \frac{1}{F_j(t)} [1 - (1 - F_j(t))h(1_j, \bar{F}(t)) - F_j(t)h(0_j, \bar{F}(t))] dt \\
&= I_{B-P}^{(j)} + \int_0^\infty \frac{f_j(t)}{F_j(t)} (1 - h(1_j, \bar{F}(t))) dt.
\end{aligned} \]

The result in Barlow and Proschan (1975) follows by noting that the second terms on the right hand side of the inequalities in Theorem 1 are nonnegative. Note that these terms are zero when the \( j \)th component is respectively in series and in parallel with the rest of the system. This theorem gives lower bounds for how much larger \( I_{B-P}^{(j)} \) is than \( I_{B-P}^{(j)} \), not just that it is larger.

\section{1.3. The Natvig measure}

Intuitively it seems that components that by failing strongly reduce the remaining system lifetime are very important. This seems at least true during the system development phase. This is the motivation for the Natvig (1979) measure. Introduce the random variable:

\[ Z_i = \text{reduction in remaining system lifetime due to the failure of the } i \text{th component.} \]
Then the Natvig measure of the importance of the $i$th component is given by:

$$I_N^{(i)} = \frac{EZ_i}{\sum_{j=1}^{n} EZ_j},$$

(4)
tacitly assuming $EZ_i < \infty$, $i = 1, \ldots, n$. Obviously

$$0 \leq I_N^{(i)} \leq 1, \quad \sum_{i=1}^{n} I_N^{(i)} = 1.$$

(5)

In Natvig (1985) the following surprising relation was proved:

$$EZ_i = \int_{0}^{\infty} I_B^{(i)}(t)\tilde{F}_i(t)(- \ln \tilde{F}_i(t))dt.$$

(6)

Hence, as for the Barlow-Proshan measure $EZ_i$ is a weighted average of the Birnbaum measure. The weight at time $t$, $\tilde{F}_i(t)[- \ln \tilde{F}_i(t)]$, is the improvement in the reliability of the $i$th component at time $t$ due to the allowance of one minimal repair at failure.

The following very similar theorem to Theorem 1 is an essential improvement of a theorem in Natvig (1985):

**Theorem 2.** Let the $i$th and $j$th component be in series (parallel) with the rest of the system. Let for $j \neq i$ $F_i(t) \geq F_j(t)(\tilde{F}_i(t) \geq \tilde{F}_j(t))$ for all $t \geq 0$. Then $I_N^{(i)} \geq I_N^{(j)}$.

$$EZ_i \geq EZ_j + \int_{0}^{\infty} (- \ln \tilde{F}_j(t))h(0_j, \tilde{F}(t))dt$$

$$= \left( EZ_i \geq EZ_j + \int_{0}^{\infty} [( - \ln \tilde{F}_j(t))\tilde{F}_j(t)/F_j(t)](1 - h(1_j, \tilde{F}(t)))dt \right).$$

**Proof.**

When the $i$th component is in series with the rest of the system we have:

$$EZ_i = \int_{0}^{\infty} (- \ln \tilde{F}_i(t))\tilde{F}_i(t)h(1_i, \tilde{F}(t))dt$$

$$= \int_{0}^{\infty} (- \ln \tilde{F}_i(t))h(\tilde{F}(t))dt$$

$$\geq \int_{0}^{\infty} (- ln \tilde{F}_j(t))h(\tilde{F}(t))dt$$

$$= \int_{0}^{\infty} (- ln \tilde{F}_j(t))\tilde{F}_j(t)[h(1_j, \tilde{F}(t)) - h(0_j, \tilde{F}(t))]dt + \int_{0}^{\infty} (- ln \tilde{F}_j(t))h(0_j, \tilde{F}(t))dt$$

$$= EZ_j + \int_{0}^{\infty} (- ln \tilde{F}_j(t))h(0_j, \tilde{F}(t))dt.$$
Similarly, when the $i$th component is in parallel with the rest of the system:

\[
EZ_i = \int_0^\infty (-\ln \tilde{F}_i(t)) \tilde{F}_i(t)(1 - h(0_i, \tilde{F}(t)))dt \\
= \int_0^\infty [(-\ln \tilde{F}_i(t))\tilde{F}_i(t)/F_i(t)](1 - h(\tilde{F}(t)))dt \\
\geq \int_0^\infty [(-\ln \tilde{F}_j(t))\tilde{F}_j(t)/F_j(t)][1 - \tilde{F}_j(t)h(1_j, \tilde{F}(t)) \\
- F_j(t)h(0_j, \tilde{F}(t)) + F_j(t)h(1_j, \tilde{F}(t)) - F_j(t)h(1_j, \tilde{F}(t))]dt \\
= EZ_j + \int_0^\infty [(-\ln F_j(t))F_j(t)/F_j(t)](1 - h(1_j, \tilde{F}(t)))dt.
\]

The first part of the theorem follows by noting that the second terms on the right hand side of the inequalities are nonnegative.

Note again that these terms are zero when the $j$th component is respectively in series and in parallel with the rest of the system. This theorem gives lower bounds for how much larger $EZ_i$ is than $EZ_j$, not just that it is larger.

In a way the Natvig measure can be considered as a more complex cousin of the Barlow-Proshan measure. A comparison of the two for different lifetime distributions are given in Natvig (1985).

1.4. The Gåsemyr extension of the Barlow-Proshan measure

In our view the Barlow-Proshan measure gives a somewhat incomplete picture of the importance of the $i$th component by only taking into account the probability that the component is the direct cause of system failure. The basic idea of the Natvig measure is that a component can be important even by failing before the system. Along these lines of thought a component that could have saved the system from failing at a specific time, but fails before, also is important. Denote by $D_{1}^{(i)}$ the event that the $i$th component fails at a time when it is critical, thereby being the direct cause of system failure. Hence, $P(D_{1}^{(i)}) = I_{B-P}^{(i)}$. Furthermore, let $D_{2}^{(i)}$ be the event that the $i$th component becomes critical at a time when it has failed, thereby being an indirect cause of system failure. The first present author has suggested the following extension of the Barlow-Proshan measure:

\[
I_{B-P,G}^{(i)} = \frac{J^{(i)}}{\sum_{j=1}^{n} J^{(j)}}, \tag{7}
\]

where

\[
J^{(i)} = P(D_{1}^{(i)} \cup D_{2}^{(i)}) = P(D_{1}^{(i)}) + P(D_{2}^{(i)}) = I_{B-P}^{(i)} + P(D_{2}^{(i)}).
\]
If the $i$th component is not in series with the rest of the system denote by $f_{-i}(t)$ the density of the random variable:

$$S_i = \inf\{t : \varphi(1_i, X(t)) - \varphi(0_i, X(t)) = 1\},$$

which is the time that the $i$th component gets critical. In this case:

$$J^{(i)} = \int_0^\infty f_{-i}(t)F_i(t)dt + I^{(i)}_{B-P}.$$  \hspace{1cm} (9)

We have:

$$f_{-i}(t) = \sum_{j=1}^nf_j(t)E\{[\varphi(1_i, 0_j, X(t)) - \varphi(0_i, 0_j, X(t))] \times [\varphi(0_i, 1_j, X(t)) - \varphi(0_i, 0_j, X(t))]\}. \hspace{1cm} (10)$$

If the $i$th component is in series with the rest of the system, clearly $P(D^{(i)}_2) = 0$ and $J^{(i)} = I^{(i)}_{B-P}$. In particular, in a series system $I^{(i)}_{B-P,G} = I^{(i)}_{B-P}, i = 1, \ldots, n$. On the other hand, if the $i$th component is in parallel with the rest of the system, one realizes that $J^{(i)} = 1$. Technically, this follows since:

$$E\{[\varphi(1_i, 0_j, X(t)) - \varphi(0_i, 0_j, X(t))][\varphi(0_i, 1_j, X(t)) - \varphi(0_i, 0_j, X(t))]\}F_i(t)$$

$$= E\{\varphi(0_i, 1_j, X(t)) - \varphi(0_i, 0_j, X(t))\}F_i(t)$$

$$= \tilde{F}_i(t) + F_i(t)h(0_i, 1_j, \tilde{F}(t)) - [\tilde{F}_i(t) + F_i(t)h(0_i, 0_j, \tilde{F}(t))]$$

$$= h(1_j, \tilde{F}(t)) - h(0_j, \tilde{F}(t)) = I^{(j)}_B(t).$$

Hence from (9) and (10):

$$J^{(i)} = \sum_{j=1}^n \int_0^\infty f_j(t)I^{(j)}_B(t)dt + I^{(i)}_{B-P} = \sum_{j=1}^n I^{(j)}_{B-P} - 1,$$

having also applied (2) and (3). Accordingly, for a parallel system $I^{(i)}_{B-P,G} = 1/n$ for $i = 1, \ldots, n$. That all components in a parallel system, whether good or bad, are equally important according to this measure is somewhat discomforting.

However, clearly if either of the events $D^{(i)}_1$ and $D^{(i)}_2$ occurs, the occurrence is at the time of system failure. Hence:

$$J^{(i)} = P[\inf_t [\varphi(1_i, X(t)) = 0] > \inf_t [\varphi(X(t)) = 0]].$$

This leads to two different interpretations of $J^{(i)}$ that motivates the $I^{(i)}_{B-P,G}$ measure. Interpretation 1: $J^{(i)}$ is the probability that the lifetime of the
system would have been prolonged if the \( i \)th component had been replaced by a perfect one. Interpretation 2: \( J^{(i)} \) is the probability that the system becomes operative immediately after system failure by the immediate repair of a failed \( i \)th component.

Interpretation 1 indicates that Reason 1 is reflected in the \( I_{B-P,G}^{(i)} \) measure if the scope of possible improvements admits the extreme case of improvement to perfect functioning. With this interpretation in mind, the measure is analogous to the \( I_{N_4}^{(i)} \) measure given in Natvig (1985) which is based on the expected increase in system lifetime by replacing the \( i \)th component by a perfect one. Interpretation 2 indicates that also Reason 2 is reflected in the \( I_{B-P,G}^{(i)} \) measure.

The \( I_{B-P,G}^{(i)} \) measure behaves in a sensible way when the \( i \)th component is in series with the rest of the system as shown by the following theorem:

**Theorem 3.** Let the \( i \)th component be in series with the rest of the system. Let for \( j \neq i \) \( F_i(t) \geq F_j(t) \) for all \( t \geq 0 \). Then \( I_{B-P,G}^{(i)} \geq I_{B-P,G}^{(j)} \).

**Proof.** Since the \( i \)th component is in series with the rest of the system, applying Theorem 1, we get if the \( j \)th component is not in series with the rest of the system:

\[
J^{(i)} = I_{B-P}^{(i)} \geq I_{B-P}^{(j)} + \int_0^\infty \frac{f_j(t)}{F_j(t)} h(0_j, \mathbf{F}(t)) dt \\
\geq I_{B-P}^{(j)} + \int_0^\infty f_j(t) h(0_j, \mathbf{F}(t)) dt \\
= I_{B-P}^{(j)} + [F_j(t) h(0_j, \mathbf{F}(t))]_0^\infty - \int_0^\infty F_j(t) \frac{d}{dt} h(0_j, \mathbf{F}(t)) dt \\
= I_{B-P}^{(j)} + \int_0^\infty F_j(t) \sum_{k=1\atop k \neq j}^n - \frac{\partial}{\partial F_k(t)} h(0_j, \mathbf{F}(t)) f_k(t) dt \\
= I_{B-P}^{(j)} + \int_0^\infty F_j(t) \sum_{k=1\atop k \neq j}^n [h(0_j, 1_k, \mathbf{F}(t)) - h(0_j, 0_k, \mathbf{F}(t))] f_k(t) dt \\
\geq I_{B-P}^{(j)} + \int_0^\infty F_j(t) f_{-j}(t) dt = J^{(j)},
\]

having applied (10) and (9). If the \( j \)th component is in series with the rest of the system, we get from Theorem 1:

\[
J^{(i)} = I_{B-P}^{(i)} \geq I_{B-P}^{(j)} = J^{(j)}.
\]

Parallel to \( D_2^{(i)} \) let \( U_2^{(i)} \) be the event that the \( i \)th component becomes critical at a time when it is still functioning, thereby saving the system from failing. If the \( i \)th component is not in series with the rest of the system, let
$J^{(i)} = P(U_2^{(i)})$, otherwise let $J^{(i)} = 1$. The following measure parallels $I_{B-P,G}^{(i)}$:

$$I_{B-P,G}^{(i)} = \frac{J^{(i)}}{\sum_{i=1}^{n} J^{(i)}}.$$

If the $i$th component is not in series with the rest of the system:

$$J^{(i)} = \int_0^{\infty} f_{-i}(t) \hat{F}_i(t) dt.$$  

For a series system $I_{B-P,G}^{(i)} = 1/n$ for $i = 1, \ldots, n$. We also have:

$$J^{(i)} = P[\inf_t [\varphi(0,t, X(t)) = 0] < \inf_t [\varphi(X(t)) = 0]].$$

Hence, parallel to Interpretation 1 of $J^{(i)}$, we can interpret $\bar{J}^{(i)}$ as the probability that the lifetime of the system would have been shortened if the $i$th component had been replaced by a defect one. The $I_{B-P,G}^{(i)}$ measure behaves in a sensible way when the $i$th component is in parallel with the rest of the system as shown by the following theorem:

**Theorem 4.** Let the $i$th component be in parallel with the rest of the system. Let for $j \neq i$ $\hat{F}_i(t) \geq \hat{F}_j(t)$ for all $t \geq 0$. Then $I_{B-P,G}^{(i)} \geq I_{B-P,G}^{(j)}$.

**Proof.** Since the $i$th component is in parallel with the rest of the system applying Theorem 1 we get because the $j$th component cannot be in series with the rest of the system:

$$J^{(i)} = 1 - J^{(i)} + I_{B-P}^{(i)} = I_{B-P}^{(i)}$$

$$\geq I_{B-P}^{(j)} + \int_0^{\infty} \frac{f_j(t)}{\hat{F}_j(t)}(1 - h(1_j, \hat{F}(t))) dt$$

$$\geq \int_0^{\infty} f_j(t)(h(1_j, \hat{F}(t)) - h(0_j, \hat{F}(t))) dt + \int_0^{\infty} f_j(t)(1 - h(1_j, \hat{F}(t))) dt$$

$$= \int_0^{\infty} f_j(t)(1 - h(0_j, \hat{F}(t))) dt$$

$$= [\hat{F}_j(t)(1 - h(0_j, \hat{F}(t)))]_0^{\infty} - \int_0^{\infty} \hat{F}_j(t) \frac{d}{dt} h(0_j, \hat{F}(t)) dt$$

$$= (1 - h(0_j, 1)) + \int_0^{\infty} \hat{F}_j(t) \sum_{k=1}^{n} [h(0_j, 1_k, \hat{F}(t)) - h(0_j, 0_k, \hat{F}(t))] f_k(t) dt$$

$$\geq \int_0^{\infty} \hat{F}_j(t) f_{-j}(t) dt = \bar{J}^{(j)},$$

noting that $h(0_j, 1) = 1$ since the $i$th component is in parallel with the rest of the system.
The intuitive requirements that if the $i$th and $j$th components are in series (parallel) with the rest of the system and $F_i(t) \geq F_j(t)$ ($\bar{F}_i(t) \geq \bar{F}_j(t)$) for all $t \geq 0$, then the former should be the most important, are in a way based on opposite perspectives focusing on the contributions of the components to system failure (survival). $I_{B-P,G}^{(i)}$ ($I_{B-P,G}^{(i)}$) is focusing exclusively on system failure (survival) which explains the shortcomings for the parallel (series) system.

2. The Barlow-Proshkan measure of component importance in repairable systems and its dual extension

In this and the subsequent section we consider the case where the components, and hence the system, can be repaired. Let the $i$th component have an absolutely continuous repair time distribution with mean $\nu_i$, and let the mean time to failure of the $i$th component be $\mu_i$, $i = 1, \ldots, n$. Let $A_i(t)$ be the availability of the $i$th component at time $t$, i.e. the probability that the component is functioning at time $t$. The corresponding stationary availabilities are given by:

$$A_i = \lim_{t \to \infty} A_i(t) = \frac{\mu_i}{\mu_i + \nu_i}, \quad i = 1, \ldots, n.$$  \hspace{1cm} (11)

Introduce $A(t) = (A_1(t), \ldots, A_n(t))$ and $A = (A_1, \ldots, A_n)$.

2.1. The Barlow-Proshkan measure

Let $(i = 1, \ldots, n)$:

$N_i(t)$ = the number of failures of the $i$th component in $[0, t]$

$\bar{N}_i(t)$ = the number of system failures caused by the $i$th component in $[0, t]$.

Finally, denote $E N_i(t)$ by $M_i(t)$. In Barlow and Proshkan (1975) the following relation is proved somewhat heuristically $(i = 1, \ldots, n)$:

$$E \bar{N}_i(t) = \int_0^t I_{B}^{(i)}(u) dM_i(u),$$ \hspace{1cm} (12)

where

$$I_{B}^{(i)}(u) = h(1, A(u)) - h(0, A(u)).$$ \hspace{1cm} (13)
However, a rigorous proof can be given. A time dependent Barlow-Proshcan measure of the importance of the \( i \)th component in the time interval \( [0, t] \) in repairable systems is given by:

\[
I^{(i)}_{B-P}(t) = \frac{\sum_{j=1}^{n} E N_j(t)}{E N_i(t)},
\]

(14)

although not explicitly mentioned in Barlow and Proshcan (1975). By a renewal theory argument they arrive at the corresponding stationary measure:

\[
I^{(i)}_{B-P} = \lim_{t \to \infty} I^{(i)}_{B-P}(t) = \frac{I^{(i)}_{B} / (\mu_i + \nu_i)}{\sum_{j=1}^{n} I^{(j)}_{B} / (\mu_j + \nu_j)},
\]

(15)

where

\[
I^{(i)}_{B} = \lim_{t \to \infty} I^{(i)}_{B}(t) = h(1_i, A) - h(0_i, A).
\]

(16)

\( I^{(i)}_{B-P} \) is the stationary probability that the failure of the \( i \)th component is the cause of system failure, given that system failure has occurred.

The following results given in Barlow and Proshcan (1975) are straightforward from (15) and (16):

**Theorem 5.** For a series system we have:

\[
I^{(i)}_{B-P} = \frac{\prod_{k\neq i} \mu_k}{\sum_{j=1}^{n} \prod_{k\neq j} \mu_k},
\]

whereas for a parallel system we have the dual expression:

\[
I^{(i)}_{B-P} = \frac{\prod_{k\neq i} \nu_k}{\sum_{j=1}^{n} \prod_{k\neq j} \nu_k}.
\]

As noted by the authors the former does not depend on component mean repair times, whereas the latter does not depend on component mean times to failure. We feel this is somewhat discomforting.

We have arrived at the following theorem parallel to Theorem 1:

**Theorem 6.** Let the \( i \)th component be in series (parallel) with the rest of the system. Let for \( j \neq i \mu_i \leq \mu_j (\nu_i \leq \nu_j) \). Then \( I^{(i)}_{B-P} \ngeq I^{(j)}_{B-P} \).

\[
\left( \frac{I^{(i)}_{B}}{\mu_i + \nu_i} \geq \frac{I^{(j)}_{B}}{\mu_j + \nu_j} + \frac{h(0_j, A)}{\mu_i} \right),
\]

\[
\left( \frac{I^{(i)}_{B}}{\mu_i + \nu_i} \geq \frac{I^{(j)}_{B}}{\mu_j + \nu_j} + \frac{1 - h(1_j, A)}{\nu_i} \right).\]
Proof. When the $i$th component is in series with the rest of the system we have:

$$
\frac{I_B^{(i)}}{\mu_i + \nu_i} = \frac{h(1, A)}{\mu_i + \nu_i} = \frac{1}{\mu_i} \left[ \frac{\mu_j}{\mu_j + \nu_j} \left( h(1, A) - h(0, A) \right) + h(0, A) \right]
$$

$$
\geq \frac{I_B^{(i)}}{\mu_j + \nu_j} + \frac{h(0, A)}{\nu_i}.
$$

When the $i$th component is in parallel with the rest of the system we have:

$$
\frac{I_B^{(i)}}{\mu_i + \nu_i} = \frac{1 - h(0, A)}{\mu_i + \nu_i} = \frac{1 - h(0, A)}{\nu_i} \left[ \frac{\nu_j}{\mu_j + \nu_j} \left( h(1, A) - h(0, A) \right) + 1 - h(1, A) \right]
$$

$$
\geq \frac{I_B^{(i)}}{\mu_j + \nu_j} + \frac{1 - h(1, A)}{\nu_i}.
$$

It is still discomforting that the assumption in the first (second) inequality does not depend on the component mean times to repair (failure).

2.2. The dual extension of the Barlow-Proshcan measure

As an attempt to improve the Barlow-Proshcan measures (14) and (15) for repairable systems the second present author has suggested taking a dual term into account based on the probability that the repair of the $i$th component is the cause of system repair, given that system repair has occurred. Let $(i = 1, \ldots, n)$:

$V_i(t) =$ the number of repairs of the $i$th component in $[0, t]$ 

$\tilde{V}_i(t) =$ the number of system repairs caused by the $i$th component in $[0, t]$.

Denote $EV_i(t)$ by $R_i(t)$.

Note that

$$A_i(t) = P[V_i(t) - N_i(t) = 0] = E[V_i(t) - N_i(t) + 1] = R_i(t) - M_i(t) + 1.$$

A more complex expression is given in Aven and Jensen (1999).

Parallel to (12) we get $(i = 1, \ldots, n)$:

$$E\tilde{V}_i(t) = \int_0^t I_B^{(i)}(u)dR_i(u). \quad (17)$$

An extended version of (14) is given by:

$$I_{B-P,N}^{(i)}(t) = \frac{E\tilde{N}_i(t) + E\tilde{V}_i(t)}{\sum_{j=1}^n [E\tilde{N}_j(t) + E\tilde{V}_j(t)]} = \frac{\int_0^t I_B^{(i)}(u)d(M_i(u) + R_i(u))}{\sum_{j=1}^n \int_0^t I_B^{(j)}(u)d(M_j(u) + R_j(u))}. \quad (18)$$
However, since from renewal theory:
\[
\lim_{t \to \infty} \frac{M_i(t)}{t} = \lim_{t \to \infty} \frac{R_i(t)}{t} = \frac{1}{\mu_i + \nu_i},
\]
it turns out that for the corresponding stationary measure:
\[
I_{B-P,N}^{(i)} = \lim_{t \to \infty} I_{B-P,N}^{(i)}(t) = I_{B-P}^{(i)}.
\]
Hence, Theorems 5 and 6 are also valid for \( I_{B-P,N}^{(i)} \) which is disappointing.

3. Barlow-Proshan type measures of component importance in repairable systems

3.1. The Gåsemyr extension of the Barlow-Proshan measure

For the case where the components, and hence the system, can be repaired a time dependent generalization of the \( I_{B-P,G}^{(i)} \) measure in (7) for nonrepairable systems, is given by:
\[
I_{B-P,G}^{(i)}(t) = \frac{\tilde{J}^{(i)}(t)}{\sum_{j=1}^{n} \tilde{J}^{(j)}(t)},
\]
where
\[
\tilde{J}^{(i)}(t) = \int_{0}^{t} \alpha_{-i}(u)(1 - A_i(u))du + E\tilde{N}_i(t)
\]
with
\[
\alpha_{-i}(u) = \sum_{j \neq i}^{n} \frac{d}{du} M_j(u) E\{[\varphi(1_i, 0_j, X(u)) - \varphi(0_i, 0_j, X(u))] \\
\times [\varphi(0_i, 1_j, X(u)) - \varphi(0_i, 0_j, X(u))]\}
\]
and \( E\tilde{N}_i \) given by (12). The corresponding stationary measure is given by:
\[
I_{B-P,G}^{(i)} = \lim_{t \to \infty} I_{B-P,G}^{(i)}(t) = \frac{\tilde{J}^{(i)}}{\sum_{j=1}^{n} \tilde{J}^{(j)}}
\]
where
\[
\tilde{J}^{(i)} = \alpha_{-i}(1 - A_i) + I_B^{(i)} / (\mu_i + \nu_i)
\]
with
\[
\alpha_{-i} = \sum_{j=1}^{n} \frac{1}{\mu_j + \nu_j} E\{[\varphi(1_i, 0_j, X) - \varphi(0_i, 0_j, X)] \times [\varphi(0_i, 1_j, X) - \varphi(0_i, 0_j, X)]\},
\]
(22)
and $X_j$, $j = 1, \ldots, n$ are independent Bernoulli distributed with success probabilities $A_j$.

As for the nonrepairable system case if the $i$th component is in series with the rest of the system, we get $\tilde{J}^{(i)}(t) = E\tilde{N}_i(t)$. Again in a series system

$I^{(i)}_{B-P,G}(t) = I^{(i)}_{B-P}(t)$, $i = 1, \ldots, n$. If the $i$th component is in parallel with the rest of the system, we similarly get:

$$\tilde{J}^{(i)}(t) = \sum_{j=1}^{n} E\tilde{N}_j = \text{the number of system failures in } [0, t].$$

Accordingly, for a parallel system we still get $I^{(i)}_{B-P,G}(t) = 1/n$ for $i = 1, \ldots, n$.

### 3.2. The dual extension of the Gâsemyr extension of the Barlow-Proshan measure

Hence, we are inspired to try to improve the measures (19) and (21) for repairable systems taking a dual term into account where a component that could have obstructed the system from being repaired at a specific time, but is functioning, also is important.

Let

$$\tilde{K}^{(i)}(t) = \int_0^t \beta_{-i}(u)A_i(u)du + E\tilde{V}_i(t)$$

with

$$\beta_{-i}(u) = \sum_{j=1}^{n} \frac{d}{du}R_j(u)E\{[\varphi(1_i, 1_j, \mathbf{X}(u)) - \varphi(0_i, 1_j, \mathbf{X}(u))] \\
\times [\varphi(1_i, 1_j, \mathbf{X}(u)) - \varphi(1_i, 0_j, \mathbf{X}(u))]\}$$

and $E\tilde{V}_i(t)$ given by (17). Then an extended version of (19) is given by:

$$I^{(i)}_{B-P,G,N}(t) = \frac{\tilde{J}^{(i)}(t) + \tilde{K}^{(i)}(t)}{1 - \sum_{j=1}^{n} [\tilde{J}^{(j)}(t) + \tilde{K}^{(j)}(t)]} \quad (24)$$

$$= \frac{\int_0^t [\alpha_{-i}(u)(1 - A_i(u)) + \beta_{-i}(u)A_i(u)]du + \int_0^t I^{(i)}_{B}(u)d(M_i(u) + R_i(u))} {\sum_{j=1}^{n} [\int_0^t [\alpha_{-j}(u)(1 - A_j(u)) + \beta_{-j}(u)A_j(u)]du + \int_0^t I^{(j)}_{B}(u)d(M_j(u) + R_j(u))]}.$$  

The corresponding stationary measure is given by:

$$I^{(i)}_{B-P,G,N} = \lim_{t \to \infty} I^{(i)}_{B-P,G,N}(t) = \frac{\tilde{J}^{(i)}(t) + \tilde{K}^{(i)}(t)}{\sum_{j=1}^{n} (\tilde{J}^{(j)} + \tilde{K}^{(j)})}$$
where $\alpha_{-i}$ is given by (22) and:

$$
\beta_{-i} = \sum_{j=1}^{n} \frac{1}{\mu_j + \nu_j} E[\{(\varphi(1_i, 1_j, X) - \varphi(0_i, 1_j, X)) \times [\varphi(1_i, 1_j, X) - \varphi(1_i, 0_j, X)]\},
$$

with again $X_j$, $j = 1, \ldots, n$ independent Bernoulli distributed with success probabilities $A_j$.

Now it is easy to see that for the series system we have:

$$
I_{B-P,G,N}^{(i)}(t) = \frac{\sum_{j=1}^{n} \int_{0}^{t} I_{B}^{(j)}(u)dR_j(u) + \int_{0}^{t} I_{B}^{(j)}(u)dM_i(u)}{n \sum_{j=1}^{n} \int_{0}^{t} I_{B}^{(j)}(u)dR_j(u) + \sum_{j=1}^{n} \int_{0}^{t} I_{B}^{(j)}(u)dM_j(u)}
$$

(25)

and

$$
I_{B-P,G,N}^{(i)} = \left[1 + \frac{\prod_{k\neq i} \mu_k}{\sum_{j=1}^{n} \prod_{k
ot= j} \mu_k}\right] / (n + 1).
$$

(26)

For a parallel system we get dual results by interchanging $R_j(u)$ and $M_j(u)$ in (25) and replacing $\mu_j$ by $\nu_j$ in (26) for $j = 1, \ldots, n$. Hence, we have proved the following result:

**Theorem 7.** For a series and parallel system we have:

$$
I_{B-P,G,N}^{(i)} = \left[1 + I_{B-P}^{(i)}\right] / (n + 1).
$$

Hence, the objections to $I_{B-P}^{(i)}$ are inherited by $I_{B-P,G,N}^{(i)}$.

3.3. The Huseby measures

Introduce the random variable ($i = 1, \ldots, n$):

$$
Y_i(t) = \text{the downtime for the system in } [0, t] \text{ in which the } i\text{th component is critical and failed}.
$$

A time dependent version of a measure suggested in Huseby (2004) is given by:

$$
I_{H}^{(i)}(t) = \frac{EY_i(t)}{\sum_{j=1}^{n} EY_j(t)}.
$$

(27)

Obviously:

$$
EY_i(t) = \int_{0}^{t} I_{B}^{(i)}(u)(1 - A_i(u))du.
$$

(28)
In a way the \( I_{B-P_G}^{(i)}(t) \) measure can be considered as a more complex cousin to the \( I_H^{(i)}(t) \) one measuring the contribution of the \( i \)th component to the expected number of system failures in \([0, t]\) rather than to the expected system downtime in \([0, t]\). If costs are mainly linked to the expected downtime for the system, \( I_H^{(i)}(t) \) is preferable. However, if costs are more associated with the expected number of system failures, \( I_{B-P_G}^{(i)}(t) \) should be applied. For some cases a linear combination of these two measures may be a sensible choice.

By linking the importance of a component to its contribution to the uptime for the system instead, let \((i = 1, \ldots, n)\):

\[
\bar{Y}_i(t) = \text{the uptime for the system in } [0, t] \text{ in which the } i \text{th component is critical and functioning.}
\]

Then the corresponding dual Huseby measure is given by:

\[
\bar{I}_H^{(i)}(t) = \frac{E\bar{Y}_i(t)}{\sum_{j=1}^{n} E\bar{Y}_j(t)}.
\] (29)

Obviously:

\[
E\bar{Y}_i(t) = \int_0^t I_B^{(i)}(u) A_i(u) du.
\] (30)

If the \( i \)th component is in parallel with the rest of the system, we get:

\[
EY_i(t) = \int_0^t (1 - h(A(u))) du.
\]

Accordingly, for a parallel system \( I_B^{(i)}(t) = 1/n \) for \( i = 1, \ldots, n \). Similarly, if the \( i \)th component is in series with the rest of the system:

\[
E\bar{Y}_i(t) = \int_0^t h(A(u)) du.
\]

Hence, for a series system \( \bar{I}_H^{(i)}(t) = 1/n \) for \( i = 1, \ldots, n \).

For the stationary Huseby measure we have the following theorem:

**Theorem 8.** Let the \( i \)th component be in series (parallel) with the rest of the system. Let for \( j \neq i \) \( A_i \leq A_j \) (\( A_i \geq A_j \)). Then \( I_H^{(i)} \geq \bar{I}_H^{(i)} \) (\( I_H^{(i)} \geq \bar{I}_H^{(i)} \)).

The proof is parallel to the one of Theorem 6 and is left to the reader.
3.4. The standardized Birnbaum measure

Accordingly, we are inspired to add the terms (28) and (30) and then standardizing. This leads to the following standardized Birnbaum measure of the importance of the $i$th component in the time interval $[0, t]$ in repairable systems:

$$\hat{I}_B^{(i)}(t) = \frac{\int_0^t I_B^{(i)}(u)du}{\sum_{j=1}^n \int_0^t I_B^{(j)}(u)du}.$$  \hspace{1cm} (31)

The corresponding standardized stationary measure is given by:

$$\hat{j}_B^{(i)} = \lim_{t \to \infty} \hat{I}_B^{(i)}(t) = \frac{I_B^{(i)}}{\sum_{j=1}^n I_B^{(j)}},$$  \hspace{1cm} (32)

where $I_B^{(i)}$ is given by (16). The following result is straightforward.

**Theorem 9.** For a series system we have

$$\hat{j}_B^{(i)} = \frac{1 + \nu_i/\mu_i}{\sum_{j=1}^n (1 + \nu_j/\mu_j)},$$

whereas for a parallel system we have the dual expression:

$$\hat{j}_B^{(i)} = \frac{1 + \mu_i/\nu_i}{\sum_{j=1}^n (1 + \mu_j/\nu_j)}.$$  

According to this measure the importance of a component in a series system is increasing in mean time to repair and decreasing in mean time to failure, i.e. the poorer the more important. For a parallel system it is the other way round. This seems perfectly sensible.

Furthermore, we have the following more satisfactory theorem than Theorems 6 and 8:

**Theorem 10.** Let the $i$th component be in series (parallel) with the rest of the system. Let for $j \neq i$ $A_i \leq A_j$ ($A_i \geq A_j$). Then $\hat{j}_B^{(i)} \geq \hat{j}_B^{(j)}$.

$$I_B^{(i)} \geq I_B^{(j)} + \frac{h(0_j, A)}{A_i},$$

$$\left( I_B^{(i)} \geq I_B^{(j)} + \frac{1 - h(0_j, A)}{1 - A_i} \right).$$

Again the proof is left to the reader.
4. Concluding remarks

In this paper we have first considered measures of component importance in nonrepairable systems. In this case Theorems 1 and 2 respectively indicate that the Barlow-Proshcan and the Natvig measures are sensible. On the other hand the two Gåsemyr extensions of the Barlow-Proshcan measure are based on opposite perspectives focusing respectively on the contributions of the components to system failure and system survival. This is reflected in Theorems 3 and 4.

Reasonable measures of component importance for repairable systems represent a challenge. Theorems 5 and 6 covering the stationary Barlow-Proshcan measure in this case are not satisfactory. Furthermore, the dual extension of this measure gives exactly the same results. Theorem 7 shows that the objections to these measures are inherited by the dual extension of the Gåsemyr extension of the Barlow-Proshcan measure.

The two Huseby measures are again based on opposite perspectives focusing respectively on the contributions of the components to system downtime and system uptime. This is reflected in Theorem 8. Theorems 9 and 10 indicate that the standardized Birnbaum measure is sensible for repairable systems. This measure considers contributions to system downtime and uptime as equally important which, however, may be questionable when for instance mean times to failure of the components are large compared to mean times to repair.

We have not considered extensions of the Natvig measure to repairable systems in the present paper. This is ongoing research based on advanced simulation technology and will be reported in due time.

References


