A note on the contributions due to the steady second order potential in the slow-drift problem

John Grue
Mechanics Division, Department of Mathematics,
University of Oslo, Norway

This note is devoted to the role of the time-averaged second order potential in the slow-drift problem. The purpose is to describe some new relations, point out a simple procedure to evaluate the resulting formulae and give some illustrative examples that complement previous work, Grue & Palm (1993), where the effect of the time-averaged second order potential was pointed out.

We consider a floating body moving with slow velocities in the three horizontal modes of motion while responding to incoming monochromatic waves with amplitude $A$. The fluid has a mean depth $h$. We define a relative frame of reference $O - xyz$, with unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ accordingly, that follows the slow motion of the body. The $z$-axis is vertical, with $z = 0$ defining the mean free surface, and with $z = -h$ defining the bottom of the fluid. Let $U\mathbf{i} + V\mathbf{j}$ denote the slow horizontal velocity of the body and $\Omega \mathbf{k}$ its slow angular velocity about the vertical axis. Potential theory is applied to describe the motion of the fluid which is assumed to be incompressible and homogeneous. The fluid velocity $\mathbf{v}$ in the relative frame of reference may be decomposed by $\mathbf{v} = \nabla \Phi' + \mathbf{w}$, where $\Phi'$ is a potential that governs the motion due to the presence of the waves and the oscillatory motions of the body, and $\mathbf{w}$ denotes the velocity field when no waves are present. The potential $\Phi'$ may be decomposed by $\Phi' = \Phi + \psi^{(2)}$, where $\Phi$ contains the oscillatory parts of $\Phi'$, and $\psi^{(2)}$ denotes the time-averaged part of $\Phi'$. The potential $\psi^{(2)}$ is proportional to the wave amplitude squared, to leading approximation.

We shall be concerned with the contributions from $\psi^{(2)}$ to the wave-drift damping matrix, which emanates from the time-averaged force and moment acting on the floating body. More precisely, let $F_x\mathbf{i} + F_y\mathbf{j}$ denote the time-averaged horizontal force and let $M_z\mathbf{k}$ denote the time-averaged moment with respect to the vertical axis. Expanding the force and moment in the slow velocities we obtain

$$
\begin{pmatrix}
F_x \\
F_y \\
M_z
\end{pmatrix} =
\begin{pmatrix}
F_{x0} \\
F_{y0} \\
M_{z0}
\end{pmatrix} -
\begin{pmatrix}
B_{11} & B_{12} & B_{16} \\
B_{21} & B_{22} & B_{26} \\
B_{31} & B_{32} & B_{36}
\end{pmatrix}
\begin{pmatrix}
U \omega / g \\
V \omega / g \\
\Omega / \omega
\end{pmatrix}
$$

(1)

where $\{B_{ij}\}$ denotes the wave-drift damping matrix and $(F_{x0}, F_{y0}, M_{z0}) = (F_x, F_y, M_z)$ for $U = V = \Omega = 0$. Furthermore, $\omega$ denotes the wave frequency in the fixed frame of reference and $g$ the acceleration due to gravity. $(F_{x0}, F_{y0}, M_{z0})$ and $\{B_{ij}\}$ are proportional to the wave amplitude squared and are independent of $U$, $V$ and $\Omega$, to leading approximation. It may be demonstrated that the contributions from the potentials $\Phi$ and $\psi^{(2)}$ may be divided into two separate contributions, i.e.

$$B_{ij} = B_{ij}(\psi^{(2)}) + B_{ij}(\Phi)$$

(2)

where we primarily shall be interested in $B_{ij}(\psi^{(2)})$. References describing how to obtain $B_{ij}(\Phi)$ are given below.

Before we proceed by discussing the wave-drift damping matrix, we note that the velocity field $\mathbf{w}$ is conveniently decomposed by

$$
\mathbf{w} = U \mathbf{w}_1 + V \mathbf{w}_2 + \Omega \mathbf{w}_6
$$

(3)

$$
\mathbf{w}_i = \nabla(-x_i + \chi_i), \quad i = 1, 2, \quad \mathbf{w}_6 = -\mathbf{k} \times \mathbf{x} + \nabla \chi_6
$$

(4)
The potentials $\chi_1$, $\chi_2$, $\chi_6$ in (4) satisfy the Laplace equation in the fluid and

$$\frac{\partial \chi_i}{\partial n} = n_i \quad \text{at} \quad S_B \quad \text{with} \quad n_i = \frac{\partial \chi_i}{\partial n}$$

$$\frac{\partial \chi_i}{\partial z} = 0 \quad \text{at} \quad z = 0, -h \quad \text{for} \quad |x| \to \infty$$

where $n = (n_1, n_2, n_3)$ denotes the unit normal, pointing out of the fluid, at the mean position of the wetted body surface, $S_B$, and $n_6 = \chi_2 - \chi_1$.

**Conservation of linear and angular momentum in the far-field**

The force and moment may be obtained either by integrating the pressure over the wetted body surface or by evaluating the linear and angular momentum flux at a control surface. By using the latter method at a control surface in the far-field we find (Grue & Palm, 1993; Newman, 1993; Grue & Palm, 1996; Finne & Grue, 1998)

$$B_{61}^{\text{far}}(\psi^{(2)}) = \frac{\rho g}{\omega} M_2$$

$$B_{62}^{\text{far}}(\psi^{(2)}) = -\frac{\rho g}{\omega} M_1$$

$$B_{66}^{\text{far}}(\psi^{(2)}) = \rho \omega \left(-\frac{\partial M_1}{\partial \beta} - M_2 \right)$$

$$B_{26}^{\text{far}}(\psi^{(2)}) = \rho \omega \left(-\frac{\partial M_2}{\partial \beta} + M_1 \right)$$

$$M_i = \int_{S_B + S_F} (x_i - x_i) \frac{\partial \psi^{(2)}}{\partial n} dS, \quad i = 1, 2$$

where $\rho$ denotes the density of the fluid, $\beta$ denotes the wave angle of the incoming waves, $S_F$ denotes the mean free surface and super-index 'far' denotes that the far-field method is used. Otherwise $B_{ij}^{\text{far}}(\psi^{(2)})$ are zero. How to obtain $B_{ij}^{\text{far}}(\Phi)$ is described e.g. in the references cited previous to eq. (8).

**Pressure integration in the near-field**

Next we consider the contributions from $\psi^{(2)}$ when we integrate the pressure over the instantaneous wetted body surface. The pressure is given by

$$p = -\rho \frac{\partial \psi}{\partial t} + \mathbf{w} \cdot \nabla \psi + \frac{1}{2} |\nabla \psi|^2 + C(t)$$

where $C(t)$ is independent of $\mathbf{x}$, and gives rise to terms

$$\alpha \int_{S_B} \mathbf{w}_j \cdot \nabla \psi^{(2)} n_j dS, \quad i, j = 1, 2, 6$$

in the wave-drift damping matrix, where $\alpha = \rho g/\omega$ for $j = 1, 2$, $\alpha = \rho \omega$ for $j = 6$ and the integrals may be evaluated at the mean position of the body, $S_B$, within the present approximation. The case of a translation along the $x$-axis ($j = 1$) was considered by Grue & Palm (1993). Following their procedure also for $j = 2$ ($w_2$) we obtain

$$B_{ij}^{\text{near}}(\psi^{(2)}) = \rho g \int_{S_B} \mathbf{w}_j \cdot \nabla \psi^{(2)} n_i dS = \int_{S_B} \mathbf{w}_j \mathbf{n}_i dS \quad \text{with} \quad \mathbf{n}_i = \frac{\partial \psi^{(2)}}{\partial \mathbf{n}}$$

$$i, j = 1, 2$$

$$\quad i, j = 1, 2$$

$$B_{61}^{\text{near}}(\psi^{(2)}) = \rho g \int_{S_B} \mathbf{w}_1 \cdot \nabla \psi^{(2)} n_6 dS = \int_{S_B} \mathbf{w}_1 \mathbf{n}_6 dS \quad \text{with} \quad \mathbf{n}_6 = \frac{\partial \psi^{(2)}}{\partial \mathbf{n}}$$

$$i, j = 1, 2$$

$$B_{62}^{\text{near}}(\psi^{(2)}) = \rho g \int_{S_B} \mathbf{w}_2 \cdot \nabla \psi^{(2)} n_6 dS = \int_{S_B} \mathbf{w}_2 \mathbf{n}_6 dS \quad \text{with} \quad \mathbf{n}_6 = \frac{\partial \psi^{(2)}}{\partial \mathbf{n}}$$

$$i, j = 1, 2$$
where $\partial/\partial \theta = x \partial/\partial y - y \partial/\partial x$ and super-index ‘near’ denotes that pressure integration over the wetted body surface is used.

The contributions due to the coupling between $\psi^{(2)}$ and $w_6$ may be obtained similarly. In the first step we make use of (see Finne & Grue, 1997, eqs. 3.25, 3.26, 5.10)

$$\int_{S_B} w_6 \cdot \nabla \psi^{(2)} n_1 dS = - \int_{S_B} \psi^{(2)} m_1 dS \quad (17)$$

where the generalized $m_i$-terms for the rotational mode, with $\nabla \times w_6 \neq 0$, are given by

$$(m_1, m_2, m_3) = -\frac{\partial w_6}{\partial n} - 2k \times n = -\frac{\partial}{\partial n} \chi_6 + n_2 \mathbf{i} - n_1 \mathbf{j} \quad (18)$$

$$(m_4, m_5, m_6) = -\frac{\partial}{\partial n} (\mathbf{x} \times w_6) - 2 \mathbf{x} \times (\mathbf{k} \times n) \quad (19)$$

We note that $m_6 = -\partial(\partial \chi_6/\partial \theta)/\partial n$. We then use Green’s theorem to $\chi_i$ and $\psi^{(2)}$, giving $\int_{S_B} \psi^{(2)} n_1 dS = \int_{S_B + S_F} \chi_i \partial \psi^{(2)}/\partial n dS$. Next, applying (17) and using Green’s theorem, similarly as in the derivations of (14)–(16), we find

$$B^{\text{near}}_{16}(\psi^{(2)}) = \rho \omega \int_{S_B} w_6 \cdot \nabla \psi^{(2)} n_1 dS = \rho \omega \int_{S_B + S_F} \left( \frac{\partial \chi_6}{\partial x} - \chi_2 \right) \frac{\partial \psi^{(2)}}{\partial n} dS \quad (20)$$

$$B^{\text{near}}_{26}(\psi^{(2)}) = \rho \omega \int_{S_B} w_6 \cdot \nabla \psi^{(2)} n_2 dS = \rho \omega \int_{S_B + S_F} \left( \frac{\partial \chi_6}{\partial y} + \chi_1 \right) \frac{\partial \psi^{(2)}}{\partial n} dS \quad (21)$$

$$B^{\text{near}}_{66}(\psi^{(2)}) = \rho \omega \int_{S_B} w_6 \cdot \nabla \psi^{(2)} n_6 dS = \rho \omega \int_{S_B + S_F} \frac{\partial \chi_6}{\partial \theta} \frac{\partial \psi^{(2)}}{\partial n} dS \quad (22)$$

We observe that $B^{\text{far}}_{ij}(\psi^{(2)})$ and $B^{\text{near}}_{ij}(\psi^{(2)})$ are different. Both matrices are, however, expressed in terms of integrals over $S_B$ and $S_F$ of products between $\partial \psi^{(2)}/\partial n$ and $\chi_i$, and derivatives of $\chi_i$. It suffices to determine $\partial \psi^{(2)}/\partial n$ from the nonlinear boundary conditions for $U = V = \Omega = 0$. The boundary conditions for $\psi^{(2)}$ are, to leading approximation, determined from the linear wave potential $\Phi^{(1)}$ at zero speed, which may be written

$$\Phi^{(1)} = \text{Re}\{A \text{ig}/\omega) e^{i \omega t}\} \quad (23)$$

where $\phi = \phi(x)$. We then have (Grue & Palm, 1993, eqs. 10–12)

$$\frac{\partial \psi^{(2)}}{\partial z} = -\frac{g A^2}{2 \omega^2} \text{Im}(\phi \phi^{*}) \quad \text{at} \quad S_F \quad (24)$$

$$\frac{\partial \psi^{(2)}}{\partial n} = \frac{g A^2}{2 \omega} \text{Im}\{B \cdot (n \cdot \nabla) \nabla \phi^{*} - C \cdot [(\omega^2/\gamma) \mathbf{B}^{*} - \nabla \phi^{*}]\} \quad \text{at} \quad S_B \quad (25)$$

where an asterisk denotes complex conjugate, $(\ )_{zz} = \partial^2(\ )/\partial z^2$, $B = [(\xi_1, \xi_2, \xi_3) + (\xi_4, \xi_5, \xi_6) \times x]/A$, $C = (\xi_4, \xi_5, \xi_6) \times n/A$, and $\xi_i$ denote the complex amplitudes of the linear body responses at zero speed. ($A$ denotes the wave amplitude.)

**Computational procedure**

It may be convenient to avoid evaluating the second derivative in the body boundary condition for $\psi^{(2)}$, i.e. the first term on the right of (25). By using a variant of Stokes’ theorem we may show that

$$\int_{S_B} \psi \frac{\partial \psi^{(2)}}{\partial n} dS = -\frac{g A^2}{2 \omega} \text{Im}\left\{ \int_{S_B} \nabla \Psi \cdot \nabla \phi^{*} \mathbf{B} \cdot n dS + \frac{\omega^2}{\gamma} \int_{C_B} \Psi \mathbf{B}^{*} \cdot n dl \right\} \quad (26)$$
where $\Psi$ is an arbitrary function and $C_B$ denotes the water line of $S_B$. The proof is given in Finne & Grue (1998, eqs. 7.9-7.12) and is not repeated here. Thus, the integrals in (12), (14)–(16) and (20)–(22) may be obtained by applying

$$\int_{S_B+S_F} \Psi \frac{\partial \phi_s^{(2)}}{\partial n} dS = -\frac{gA^2}{2 \omega} \text{Im} \left\{ \int_{S_F} \Psi \phi_s^{*} dS + \int_{S_B} \nabla \Psi \cdot \nabla \phi^{*} B \cdot n dS + \frac{\omega^2}{g} \int_{C_B} \Psi \phi B^* \cdot n d\ell \right\} \quad (27)$$

**Numerical example**

To illustrate the analysis we consider an example where a freely floating half-immersed sphere with radius $R$ and center located in $x = y = 0$ is exposed to incoming waves with wavenumber $k$ propagating along the $y$-axis. The wavenumber satisfies the usual dispersion relation $\omega^2 = gk \tanh kh$. In this example $\chi_3 = 0$, $\partial \chi_1 / \partial \theta + \chi_2 = 0$ and $\partial \chi_2 / \partial \theta - \chi_1 = 0$. (When $h = \infty$ we also have $\chi_1 = -(xR^3)/(2|x|^3)$ and $\chi_2 = -(yR^3)/(2|x|^3).$) The only nonzero components of the wave-drift damping matrix are $B_{11}$ and $B_{22}$. Otherwise we have that $B_{ij} = B_{ij}(\psi^{(2)}) + B_{ij}(\Phi) = 0$.

The contributions from $\psi^{(2)}$ to the wave-drift damping matrix using the near-field method gives

$$B_{16}^{\text{near}}(\psi^{(2)}) = -\rho \omega \int_{S_B+S_F} \chi_2 \frac{\partial \psi^{(2)}}{\partial n} dS, \quad B_{ij}^{\text{near}}(\psi^{(2)}) = 0 \text{ otherwise} \quad (28)$$

Since $B_{16}^{\text{near}}(\psi^{(2)}) + B_{16}^{\text{near}}(\Phi) = 0$, we have that $B_{16}^{\text{near}}(\Phi) = -B_{16}^{\text{near}}(\psi^{(2)})$ in this example.

If we instead use the far-field method we obtain

$$B_{61}^{\text{far}}(\psi^{(2)}) = \frac{\partial \Psi}{\partial \omega} M_2, \quad B_{ij}^{\text{far}}(\psi^{(2)}) = 0 \text{ otherwise} \quad (29)$$

where $M_2$ is given by (12). Since $B_{61}^{\text{far}}(\psi^{(2)}) + B_{61}^{\text{far}}(\Phi) = 0$, the component $B_{61}^{\text{far}}(\Phi)$ has a value which equals $-B_{61}^{\text{far}}(\psi^{(2)})$. We then compute for the sphere $B_{61}^{\text{far}}(\psi^{(2)})$, using (8), (12), and $B_{61}^{\text{far}}(\Phi)$, using an expansion of eqs. (54)–(55) in Grue & Biberg (1993). The numerical results presented in the figure show that $B_{61}^{\text{far}}(\psi^{(2)})$ and $B_{61}^{\text{far}}(\Phi)$ both are large. Their sum is very close to zero, however. (In this example $B_{61}^{\text{near}}(\psi^{(2)}) = B_{61}^{\text{near}}(\Phi) = 0$.)

This work was conducted under the Joint Industry Project ‘The complete wave drift damping matrix and applications’ supported by Det Norske Veritas, Mobil, Norsk Hydro and Statoil.
Figure 1: Wave drift damping coefficient $B_{61}$ vs. wavenumber $kR$. Far-field method. Freely floating half immersed sphere with radius $R$ and mass equals its bouyancy. Center located in $x = y = 0$. Incoming waves propagating along the $y$-axis. Squares: $B_{61}^{\text{far}}(\psi^{(2)})$, triangles: $B_{61}^{\text{far}}(\Phi)$, Solid line without squares or triangles: total $B_{61}^{\text{far}}$. a) $h = \infty$, b) $h/R = 1.1$. The numerical results are extrapolated from computations using (784,3136) and (1600,6400) panels on $(S_B, S_F)$. Outer discretization radius of $S_F$ is $10R$. 

$a$ 

$b$
References


