On the Wave Field due to a Moving Body Performing Oscillations in the Vicinity of the Critical Frequency

by

Enok Palm and John Grue

Mechanics Division, Department of Mathematics,
University of Oslo, Norway

We consider a two-dimensional submerged body translating under a free surface with steady velocity U while performing small oscillations with frequency ω. It has been known for a long time that for a single source the solution becomes unbounded at the critical frequency, which is given by $\tau = U\omega/g = 1/4$ where g is the acceleration of gravity. It was therefore believed that also the motion due to an oscillating body was unbounded at this frequency. It has, however, in the last few years been shown that this motion is bounded for $\tau = 1/4$. In this paper previous results are discussed, and the strong variation of the forces with respect to ω close to $\tau = 1/4$ is examined. Recently, a mathematical argument was given that the motion at the critical frequency is bounded for bodies with non-zero cross-section area. We prove that also the motion generated by a thin foil with zero cross-section area is bounded at $\tau = 1/4$.

1 Introduction

The problem of a body translating on or beneath a free surface while performing an oscillating motion is of fundamental interest in marine hydrodynamics. This topic is of practical importance to seakeeping of ships and in the study of wave loads on offshore structures and devices for exploiting wave energy. The oscillations are often of small amplitudes such that the conditions required for linearization of the problem is fulfilled. It is then appropriate to solve the problem by using a Green function. For a body moving with a constant horizontal velocity U, or equivalently, a body embedded in a uniform current $-U$, the Green function due to an oscillating concentrated source is well-known (Haskind [1], Wehausen & Laitone [2]).

This Green function is, however, unbounded for a certain value of the frequency ω, corresponding to the non-dimensional number $\tau = U\omega/g = 1/4$ where g is the acceleration of gravity. Physically speaking, in the two-dimensional case four waves are generated in the far-field when τ is less than 1/4 (the k_1-, k_2-, k_3- and k_4-waves, defined in equation (2.19)). Three of these waves have negative group velocities and one of them, the k_2-wave, has positive group velocity. The three first waves are located downstream and the last wave upstream. When $\tau \to 1/4$, two of the waves, the k_1- and the k_2-wave, merge into one wave which has zero group velocity. This wave is not able to transport wave energy and we get a wave cut-off such that the two merged waves do not exist for $\tau > 1/4$. The singularity in the Green function for $\tau = 1/4$ has therefore two causes: two of the waves merge into one which is expected to give a resonance situation, and the resulting wave is not able to transport wave energy.

The motion generated by a body of non-zero volume, oscillating or exposed to an incoming wave, may be found by using a distribution of sources located at the body surface. Since a single source is unbounded at $\tau = 1/4$, it was long believed that this is so also for a body (e.g. Dagan & Miloh [3]). Uncertainty on this point has also been noted by e.g. Wu & Eatock Taylor [4] in three dimensions. Grue & Palm [5] found, however, for a submerged circular cylinder in two dimensions that the motion and physical forces are bounded as $\tau \to 1/4$. The result was shown numerically as well as from the mathematical equations. Similar numerical results were obtained by Mo & Palm [6] for a submerged elliptical cylinder and by Grue, Mo & Palm [7] for a submerged foil.

Liu & Yue [8] brought the theory an important step forwards. They were able to show in two dimensions that the motion at $\tau = 1/4$ is bounded for a submerged body of arbitrary form, provided that the body has a non-zero cross-section area. They also extended their theory to floating two-dimensional bodies and three-dimensional submerged bodies. The paper was followed up by a new paper, Liu & Yue [9] where their result on the motion being finite at $\tau = 1/4$, is coupled to the study of the time dependence of the wave resistance of a body starting from rest. It is known that if the motion is started impulsively from rest to a constant translating velocity, the transient Green function decays slowly, viz. as $t^{-1/2}$ in two dimensions and t^{-1} in three dimensions, where t is time. The reason for this slow decay is the occurrence of the singularity at the frequency corresponding to $\tau = 1/4$. It is shown in that paper that for bodies with non-zero volumes the transient motion decays an order faster: as $t^{-3/2}$ and t^{-2} in two and three dimensions, respectively. For bodies of zero volume they find that the decay is of the same order as for the single source, however.

We now return to the problem in the frequency domain with submerged bodies in two dimensions. There are still shortcomings with the mathematical description of the physical problem for τ close to $1/4$. These shortcomings have prompted this contribution and are shortly described as follows: The first relates specifically to the work by Liu & Yue [8] who claim that a finite solution exists if and only if the cross-section area is non-zero. We prove here that a finite solution of the problem exists for the motion near the singularity also when the body has zero cross-section area, namely for a thin two-dimensional foil. The result is independent of the value of the velocity circulation around the foil. Secondly, for a body with finite submergence the mathematical solution of the physical problem is bounded for $\tau = 1/4$. This solution tends, however, to infinity as the submergence of the body tends to infinity, as noted by Zhang & Zhu [10]. Such a behaviour is of course meaningless from a physical point of view. The latter authors argued that the problem can be avoided by incorporating the effect of non-linearity in the boundary conditions at the free surface (still with large submergence of the body). The wavenumbers k_1 and k_2 then always differ, also at $\tau = 1/4$ (see [3]), and the singularity is removed.

Finally, we note that the physical problem has a discontinuity at the critical frequency, since four travelling waves are present for $\tau < 1/4$, while two of these disappear for $\tau > 1/4$. This may lead to quite rapid variations close to $\tau = 1/4$. We show here that some of the (non-zero) physical forces may have infinite derivatives with respect to ω at the point.

The paper is organized as follows: In §2 we provide the mathematical and physical background of the problem. §3 is a review which connects previous findings for submerged bodies of finite cross-section area. The variation of the forces with respect to ω close to $\tau = 1/4$ is also considered. In §4 we provide a rigorous mathematical argument that a finite solution also exists for a body of zero cross-section area. Finally, §5 is a conclusion.

2 Mathematical Formulation

We consider a body in two dimensions performing small oscillations in heave or sway with a frequency ω, embedded in a uniform current beneath a free surface. There may in general also be an incoming wave of the same frequency. All equations will be linearized. One frequently used approach for solving these type of problems is to apply a source distribution over the body surface S. Making use of the proper Green function and utilizing the body boundary conditions we end up with a Fredholm integral equation of the second kind. In the case of a circular cylinder the equation may be solved analytically as well as numerically. Following [5] the coordinates are taken with the origin in the mean free surface of the fluid. The x-axis is horizontal and the y-axis positive upwards. Unit vectors $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are introduced accordingly. The fluid is assumed incompressible and the motion irrotational. The fluid velocity may then be written

$$\boldsymbol{v} = \nabla\Phi - U\boldsymbol{e}_x, \tag{2.1}$$

where Φ is a velocity potential and U the speed of the current, directed along the negative x-axis. The solution of the problem is divided into one steady solution and one oscillating part. We therefore write

$$\Phi = -U\mathcal{X}(x,y) + \varphi(x,y,t), \tag{2.2}$$

where t denotes time. The potential $\varphi(x,y,t)$ satisfies the two-dimensional Laplace equation

$$\nabla^2\varphi = 0. \tag{2.3}$$

The fluid layer will be assumed to be of infinite depth. The boundary condition at $y = -\infty$ is then

$$\nabla\varphi = 0 \qquad (y = -\infty). \tag{2.4}$$

We shall assume that the boundary conditions may be linearized which means that the oscillatory motion is small. Furthermore, we neglect the effect of $\mathcal{X}(x,y)$ in the free surface boundary condition, which is a good approximation if the body is either slender or not close to the free surface (see Zhao & Faltinsen [11]). This gives

$$\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)^2 \varphi + g\frac{\partial\varphi}{\partial x} = 0 \qquad (y = 0), \tag{2.5}$$

where g is the acceleration of gravity. The kinematic boundary condition applied at the mean position of the body surface S can be written

$$\frac{\partial\varphi}{\partial n} = p(x,y), \tag{2.6}$$

where $\boldsymbol{n}$ is the normal vector of the body, defined positive into the fluid, and $p(x,y)$ is related to the body oscillation and the steady potential $\mathcal{X}$ (see e.g. Newman [12], eq. (3.28) or [5], eq. (2.12)).

The potential is properly divided into two parts

$$\varphi = \varphi_0 + \varphi_1, \tag{2.7}$$

where φ_0 is the potential of a possible incoming wave. We shall assume sinusoidal time dependence with period $2\pi/\omega$. Introducing complex variables we write

$$\varphi = \mathrm{Re}[f_c(z)\cos\omega t + f_s(z)\sin\omega t], \tag{2.8}$$

where $f_c(z)$ and $f_s(z)$ are analytic functions of

$$z = x + iy \tag{2.9}$$

and i is the imaginary unit. Equation (2.8) may be written in shorter notations by introducing a new imaginary unit j, independent of i and connected to the time variable so that

$$\varphi = \mathrm{Re}_j \mathrm{Re}_i f(z) \exp(j\omega t). \tag{2.10}$$

Here Re_i and Re_j denote the real part with respect to i and j, respectively. $f(z)$ is given by

$$f(z) = f_c(z) - j f_s(z). \tag{2.11}$$

Corresponding to (2.10) we write

$$\begin{aligned} \varphi_0 &= \mathrm{Re}_j \mathrm{Re}_i f_0(z) \exp(j\omega t), \\ \varphi_1 &= \mathrm{Re}_j \mathrm{Re}_i f_1(z) \exp(j\omega t), \end{aligned} \tag{2.12}$$

with

$$f(z) = f_0(z) + f_1(z), \tag{2.13}$$

where $f_0(z)$ is the known complex potential for the incoming wave and $f_1(z)$ is the unknown complex potential due to the presence of the body.

The potential $f_1(z)$, which must satisfy the radiation conditions at $x = \pm\infty$ and the boundary conditions at $y = 0$ and $y = -\infty$, is then written

$$f_1(z) = \int_S \sigma(s) G_\sigma(z, \zeta(s)) ds. \tag{2.14}$$

Here $G_\sigma(z, z_0)$ is the Green function for the problem, i.e. the velocity potential due to a concentrated source at the point z_0 embedded in the current and oscillating with a frequency ω. Furthermore, the contour is determined by $z = \zeta(s)$ where s is the arclength, and $\sigma(s)$ is the (unknown) source strength being real with respect to i, but complex in j. The Green function is given by

$$G_\sigma(z, z_0) = \frac{1}{2\pi}[\ln(z - z_0) - g(z, z_0)], \tag{2.15}$$

where

$$\begin{aligned} g(z, z_0) &= \ln(z - \overline{z}_0) \\ &+ \frac{1 - ij}{(1 - 4\tau)^{1/2}}[F_1(z, z_0) - F_2(z, z_0)] + \frac{1 + ij}{(1 + 4\tau)^{1/2}}[F_3(z, z_0) - F_4(z, z_0)], \end{aligned} \tag{2.16}$$

$$F_n(z, z_0) = \exp(-ik_n z) \int_{C_n}^{z} \frac{\exp(ik_n u)}{u - \overline{z}_0} du \qquad (n = 1, 2, 3, 4). \tag{2.17}$$

The function $g(z, z_0)$ is non-singular in the fluid (except at $\tau = 1/4$), a bar denotes complex conjugate and C_n is defined by

$$C_n = \infty \ (n = 1, 3, 4), \ C_2 = -\infty \ \ (\text{For } \tau > 1/2\text{: } C_1 = i\infty/k_1, \ C_2 = i\infty/k_2). \tag{2.18}$$

The wavenumbers are determined by

$$k_{1,2} = \frac{\nu}{2\tau^2}[1 - 2\tau \pm (1-4\tau)^{1/2}], \quad k_{3,4} = \frac{\nu}{2\tau^2}[1 + 2\tau \pm (1+4\tau)^{1/2}], \tag{2.19}$$

where

$$\nu = \frac{\omega^2}{g}, \qquad \tau = \frac{U\omega}{g}. \tag{2.20}$$

The integral equation becomes (see [5])

$$\sigma(s') + \frac{1}{\pi}\int_s \sigma(s)L(s',s)ds = H(s'), \tag{2.21}$$

where

$$\begin{aligned} L = \mathrm{Im}_i\Big\{ \frac{d\zeta(s')}{ds'} \Big(& \frac{1}{\zeta(s') - \zeta(s)} - \frac{1}{\zeta(s') - \overline{\overline{\zeta(s)}}} \\ & + \frac{i+j}{(1-4\tau)^{1/2}}(k_1F_1(\zeta(s'),\zeta(s)) - k_2F_2(\zeta(s'),\zeta(s))) \\ & \quad + \frac{i-j}{(1+4\tau)^{1/2}}(k_3F_3(\zeta(s'),\zeta(s)) - k_4F_4(\zeta(s'),\zeta(s)))\Big)\Big\}, \end{aligned} \tag{2.22}$$

and

$$H(s') = 2\frac{\partial\varphi(s')}{\partial n} - 2\mathrm{Im}_i\Big(f_0'(\zeta(s'))\frac{d\zeta(s')}{ds}\Big) \tag{2.23}$$

which is known.

It is noted from (2.15)–(2.19) that the Green function gives rise to waves with wavenumbers k_1, k_2, k_3 and k_4. From (2.19) it follows that for $\tau < 1/4$ all four wavenumbers are real. The k_1-, k_3- and k_4-waves then have negative group velocities and propagate downstream, whereas the k_2-wave with positive group velocity propagates upstream (see equation (3.8)). As $\tau \uparrow 1/4$, the k_1-wave and the k_2-wave merge into one wave with zero group velocity. These two effects, the merging of the two waves and the fact that their joint group velocity becomes zero such that no wave energy can be transported away, cause the Green function to become infinite at $\tau = 1/4$.

For $\tau > 1/4$, the wavenumbers k_1 and k_2 become complex and the waves disappear. The wavenumbers k_3 and k_4 are real and these waves still exist.

3 Submerged Body

3.1 Circular Cylinder

It was long accepted in the literature that also the (linearized) velocity potential generated by an arbitrary body is infinite at $\tau = 1/4$ (e.g. [3]). This may seem reasonable since the solution may be given as a distribution of sources over the body surface. It was shown, however, by Grue & Palm [5], numerically as well as from the mathematical equations, that for a submerged circular cylinder the source strength σ and the far field amplitudes of all four waves are finite as $\tau \to 1/4$. The variation of σ and some physical forces with respect to the frequency of oscillation, was, however, often found to be very strong near the critical point.

It is of interest to closer examine the variation of the forces at $\tau = 1/4$. The pressure force on the body in the radiation problem has the form

$$\boldsymbol{F} = \mathrm{Re}_j \boldsymbol{P} \exp(j\sigma t), \tag{3.1}$$

where in sway the force may be written by

$$\boldsymbol{P} = (\widehat{\mu}_{11} - j\widehat{\lambda}_{11})\boldsymbol{e}_x + (\widehat{\mu}_{21} - j\widehat{\lambda}_{21})\boldsymbol{e}_y, \tag{3.2}$$

and in heave by

$$\boldsymbol{P} = (\widehat{\mu}_{12} - j\widehat{\lambda}_{12})\boldsymbol{e}_x + (\widehat{\mu}_{22} - j\widehat{\lambda}_{22})\boldsymbol{e}_y. \tag{3.3}$$

Here $\widehat{\mu}_{mn}$ and $\widehat{\lambda}_{mn}$ are real, being the components of the added-mass force and the damping force, respectively. Non-dimensional forces are obtained by

$$\mu_{mn} = \widehat{\mu}_{mn}/\rho g R \varepsilon_b, \qquad \lambda_{mn} = \widehat{\lambda}_{mn}/\rho g R \varepsilon_b, \tag{3.4}$$

where ρ is the density, R the radius of the circular cylinder and ε_b the amplitude of the body oscillation. The damping coefficients λ_{mn} may be determined by the energy equation in the form (see [5])

$$\begin{aligned} D &= \frac{\rho g}{2\varepsilon_b}\left\{\left(\frac{a_1^2}{k_1} + \frac{a_2^2}{k_2}\right)(1-4\tau)^{1/2} + \left(-\frac{a_3^2}{k_3} + \frac{a_4^2}{k_4}\right)(1+4\tau)^{1/2}\right\} \quad & \tau \leq 1/4, \\ D &= \frac{\rho g}{2\varepsilon_b}\left\{\left(-\frac{a_3^2}{k_3} + \frac{a_4^2}{k_4}\right)(1+4\tau)^{1/2}\right\} & \tau > 1/4, \end{aligned} \tag{3.5}$$

where D denotes $\widehat{\lambda}_{11}$ in the sway motion and $\widehat{\lambda}_{22}$ in the heave motion. Furthermore, a_i $(i = 1, 2, 3, 4)$ are the wave amplitudes (of the elevation) in the far field. These equations are valid for an arbitrary two-dimensional body.

It is seen immediately from the formula that $\widehat{\lambda}_{11}$ and $\widehat{\lambda}_{22}$ are finite and continuous at $\tau = 1/4$, since a_i are finite. By taking the derivative of (3.5) with respect to ω, we obtain that the derivatives of $\widehat{\lambda}_{11}$ and $\widehat{\lambda}_{22}$ become $-\infty$ for $\tau \uparrow 1/4$ and finite for $\tau \downarrow 1/4$, provided that $a_1, a_2 \neq 0$. (If a_1 and a_2 are zero we have $\partial D/\partial\omega = 0$ at $\tau = 1/4$.) These results are confirmed numerically for a circular cylinder for different values of the Froude number and submergences of the body in [5] and by Grue [13] who also has obtained the off-diagonal damping force λ_{21} and the added-mass forces μ_{11}, μ_{21} and μ_{22}. His results are displayed in figures 1a, 1b and 1c. It is seen that all quantities are finite and continuous at $\tau = 1/4$. It is in particular noted that μ_{11}, λ_{21} and μ_{22} have a cusp at the critical point. In this example $\mu_{12} = -\mu_{21}$, $\lambda_{12} = -\lambda_{21}$, see [13]. (The parameter d in figure 1 denotes the submergence of the cylinder centre.)

A similar result is obtained for the mean horizontal second-order force in the radiation problem, $\overline{F}_x$, from the momentum equation ([5])

$$\overline{F}_x = -\frac{E_2}{c_2}c'_{g2} + \frac{E_1}{c_1}c'_{g1} + \frac{E_3}{c_3}c'_{g3} + \frac{E_4}{c_4}c'_{g4}, \tag{3.6}$$

where

$$E_i = \frac{1}{2}\rho g a_i^2 \qquad (i = 1, 2, 3, 4), \tag{3.7}$$

$$c_i = \left(\frac{g}{k_i}\right)^{1/2} \quad (i = 1, 2, 3), \qquad c_4 = -\left(\frac{g}{k_4}\right)^{1/2}, \quad c'_{gi} = \frac{1}{2}c_i - U \quad (i = 1, 2, 3, 4). \tag{3.8}$$

It follows from (3.6) that $\overline{F}_x$ is bounded and continuous at $\tau = 1/4$. Furthermore, by taking the derivative of (3.6) with respect to ω we find that the derivative of $\overline{F}_x$ becomes infinite for $\tau \uparrow 1/4$ and finite for $\tau \downarrow 1/4$. Also these results which are valid for an arbitrary two-dimensional body, are confirmed numerically in [5] for a circular cylinder. (If a_1 and a_2 are zero we have $\partial\overline{F}_x/\partial\omega = 0$ at $\tau = 1/4$.)

$\frac{R\omega^2}{g} \longrightarrow$ $\frac{R\omega^2}{g} \longrightarrow$

$\frac{R\omega^2}{g} \longrightarrow$

Figure 1: Added-mass and damping forces, $U\sqrt{gR} = 0.4$, $d/R = 2$. Adapted from [13].

3.2 Submerged Body of Arbitrary Form

We now consider a submerged two-dimensional body of arbitrary cross-section. We first note that radiated and scattered waves from a submerged elliptical cylinder was discussed in [6], using the methods described in the previous section. It was found in this paper that also for this body at $\tau = 1/4$ the motion and the source strength are bounded and that the derivative of D and $\overline{F}_x$ with respect to the frequency has the same behaviour as found for the circular cylinder. A simplified version of the integral equation (2.21)–(2.23), valid near the critical point, was derived. In a slightly more developed form this equation may be written

$$\sigma(s') + \frac{2k}{\delta}[(n_x(s') + jn_y(s'))\exp(-jk\zeta(s'))] \int_S \sigma(s)\exp(jk\overline{\zeta}(s)) + \int_S \sigma(s)M(s,s')ds + O(\delta) = H(s'). \tag{3.9}$$

Here

$$\delta = (1-4\tau)^{1/2}, \qquad k_1, k_2 \to k = \omega/U \quad \text{for } \tau \to 1/4. \tag{3.10}$$

Furthermore, n_x and n_y are the x- and y-components of the normal vector of the body, and $M(s,s')$ is the part of the Green function which is bounded at $\tau = 1/4$. It was concluded from (3.9) that, since the far-field amplitude for the k_1- and k_2-wave at $\tau = 1/4$ is given by (see [5])

$$\frac{1}{\delta}\int_S \sigma(s)\exp(jk\overline{\zeta}(s))ds,$$

this amplitude would be finite if σ is finite. This is valid for submerged bodies of arbitrary form.

An important contribution to the study of the effect of the singularity at $\tau = 1/4$ was given by Liu & Yue [8]. They derived a form of the integral equation similar to (3.9) and were able to prove easily that in the two-dimensional case σ is finite at the critical point for *all* submerged bodies with non-zero cross-section area. Their proof is simply obtained by multiplying (3.9) with $\exp(jk\overline{\zeta}(s'))$ and integrating over S, whereby an expression for the integral $\int_S \sigma(s)\exp(jk\overline{\zeta}(s))ds$ is obtained. Substituting this expression into (3.9) it is obtained that

$$\sigma(s') - \frac{2k(n_x + in_y)\exp(-jk\zeta(s'))}{\delta + 2jk\Gamma}\int_S \sigma(s)ds \int_S M(s,s')\exp(jk\overline{\zeta}(s'))ds' + \int_S \sigma(s)M(s,s')ds = H(s') - \frac{(n_x + in_y)\exp(-jk\zeta(s'))}{\delta + 2jk\Gamma}\int_S H(s')\exp(jk\overline{\zeta}(s'))ds', \tag{3.11}$$

where

$$\Gamma = \int_S (-jn_x + n_z)\exp(2ky(s))ds = 2k\int_B \exp(2ky)dB. \tag{3.12}$$

To obtain the latter integral we have used Gauss' theorem, and B denotes the body section. Equation (3.11) is (in our notations) the equation derived in [8].

We notice that if $\Gamma \neq 0$, the equation is non-singular at $\tau = 1/4$. It is seen from (3.12) that Γ, which is a function of the frequency ω, is only zero when the cross-section area is zero. It was also shown in [8] that if $\Gamma \neq 0$, the velocity potential $f_1(z)$ is bounded at $\tau = 1/4$, also for $x \to \pm\infty$. However, from equations (3.11) and (3.12) is drawn the strong conclusion that the necessary and sufficient condition for a finite solution to exist at $\tau \to 1/4$, is that $\Gamma \neq 0$. As will be shown in the next section, this statement is not a necessary condition.

In [8] is also considered a two-dimensional surface piercing body and a submerged body in three dimensions. In the former case a sufficient condition for the solution to be bounded is derived, and in the latter case a criterion similar to that found in two dimensions is obtained.

It is seen, however, from (3.12) that Γ is proportional to $\exp(-2kd)$ where d is the submergence of the cylinder centre. Hence, when $\tau \to 1/4$, the far-field amplitudes become proportional to $\exp(kd)$ (see [8], eq. (5.13)) which implies that a deeply submerged body results in large effects at the free surface. The reason for this is most likely a resonance effect. This is an unphysical result and makes the model not valid for deep submergences, as pointed out by Zhang & Zhu [10]. These authors therefore propose to solve the problem by introducing non-linear effects. They exploit a quasi-linear model using a Green function, originally derived by Dagan & Miloh [3], which satisfies the free surface condition up to third order in the amplitude. The wavenumbers k_1 and k_2 then always differ, and the Green function becomes regular at $\tau = 1/4$. The theory is quasi-linear in that respect that non-linear effects are only introduced in the derivation of the Green function. By this method they obtain that the wave motion set up at the free surface by a deeply situated body *decays* exponentially with the submergence of the body at $\tau = 1/4$ which is a resonable result.

We are lead to the conclusion that the theories based on linearized equations, e.g. [5] and [8], are not valid for bodies which are situated close to the free surface due to the requirement of linear free boundary condition, nor are they valid for deeply situated bodies due to the resonance effect. It is believed, however, that for moderate submergence of the body the theories give useful approximations.

The theory of Zhang & Zhu has the merit that, in principle, it becomes better the deeper the body is situated. It may, however, be objected to their theory that even though the Green function satisfies the free surface boundary condition up to third order in the small parameter ε, this may not be true for the velocity potential. Hence we believe that their Green function may be replaced by another Green function having the merit that for $\varepsilon \to 0$, the classical Green function is recovered, and for deeply submerged bodies the wave motion at the free surface decays rapidly with the depth of the body.

In [10] the theory is applied on submerged circular cylinders and the results are compared with those obtained in [5]. The authors choose their small parameter as $\varepsilon = R/d$ and consider first the far-field amplitudes for $\varepsilon = 0.5$ and Froude number $F = U/\sqrt{gR} = 0.4$ and 1. The agreement is fairly good, except very close to $\tau = 1/4$. They claim, however, that the agreement becomes better for greater submergences. This seems reasonable since ε is assumed small in both theories. They also compare the damping force D for $F = 0.4$ and $\varepsilon = 0.5$ and $1/3$. The agreement is good for $\varepsilon = 0.5$, except close to $\tau = 1/4$. For $\varepsilon = 1/3$, the curves are virtually identical right up to resonance. These results suggest that for moderate submergence the linearized theory give reliable values for the physical forces, also very close to $\tau = 1/4$. For small submergences, non-linear effects may become important, especially close to the critical point.

4 Submerged Two-Dimensional Body of Zero Cross-Section

We now consider a thin oscillating foil submerged in a uniform current under a free surface. A thin moving foil may be used to extract wave energy for propulsion of ships, and the motion near the critical frequency is of practical interest. It is assumed that the foil has a small camber and angle of attack. For the oscillatory part of the flow the effects of camber and thickness are only secondary and the foil may mathematically be replaced by a flat plate. Furthermore, the amplitude of the oscillations of the foil and the amplitude of the incoming waves are assumed small. Hence, the boundary conditions at the free surface and at the foil may be linearized, even if the foil is placed close to the free surface.

Coordinates are taken as in §2, with the origin located above the center of the foil. We may also use the same formalism as in the previous sections and write

$$\boldsymbol{v} = \nabla\varphi - U\boldsymbol{e}_x. \tag{4.1}$$

In the present case we may set $\mathcal{X}(x, y) = 0$. The potential φ satisfies the equations (2.3)–(2.5) and we also make use of (2.7)–(2.13). The boundary conditions at the body is, however, now given by

$$\frac{\partial\varphi}{\partial y} = \left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)\zeta \qquad (y = -d,\ \ |x| < l), \tag{4.2}$$

where $\zeta(x, t)$ denotes the vertical displacement of the foil and $2l$ and d are the chord length and depth, respectively.

This problem, a foil submerged in a uniform flow beneath a free surface, was discussed by Grue, Mo & Palm [7] in the context of exploiting water waves for propulsion of ships. Following this paper we derive an integral equation for the motion by expressing $f_1(z)$ (defined by (2.12)) as a continuous distribution of vortices. The velocity circulation around the foil will oscillate in time due to the periodic motion of the foil or the harmonic incoming waves. Hence, vortices will be shed at the trailing edge, and a vortex wake will be formed behind the foil, extending from the trailing edge to $x = -\infty$ as time goes towards infinity. In the first approximation the wake may be considered to be located along the line $y = -d$, and $f_1(z)$ is therefore expressed as an integral from $x = -\infty$ to $x = l$. Let $G_\gamma(z, z_0)$ denote the complex potential for a vortex of strength unity located at $z = z_0$. Then $G_\gamma(z, z_0)$ fulfills the boundary condition at the free surface, the radiation conditions at $x = \pm\infty$ and (2.4) at $y = -\infty$. We may then write $f_1(z)$ by

$$f_1(z) = \int_{-\infty}^{l} \gamma(\xi) G_\gamma(z, \xi - id) d\xi\,. \tag{4.3}$$

$G_\gamma(z, z_0)$ was derived in [7] by an analogous procedure as used in [1] in deriving the Green function for a source. It was found that

$$G_\gamma(z, z_0) = \frac{1}{2\pi i}[\ln(z - z_0) + g(z, z_0)], \tag{4.4}$$

where $g(z, z_0)$ is non-singular in the fluid (except at $\tau = 1/4$) and defined by (2.16)–(2.20). In this problem $\gamma = \Delta u$ where Δ denotes the difference between the lower and upper value along the cut $-\infty < x < l$, $y = -d$, and u is the horizontal velocity. The velocity circulation around the foil is then given by

$$\Gamma_s = \int_{-l}^{l} \gamma(x) dx \tag{4.5}$$

and the vortex strength γ in the wake by

$$\gamma = \gamma_0 \exp(jkx), \qquad k = \omega/U, \tag{4.6}$$

where γ_0 is a complex constant with respect to j. The velocity circulation Γ_s is associated with γ_0 by

$$j\omega\Gamma_s = -U\gamma_0 \exp(-jkl). \tag{4.7}$$

The integral equation determining γ may be transformed to a Fredholm equation of second kind [7, eq. (3.25)]

$$\gamma(x) = \frac{1}{\pi^2}(l^2 - x^2)^{-1/2}\Big(⨍_{-l}^{l} \frac{(l^2-\eta^2)^{1/2}}{x-\eta}[H(\eta) + F(\eta)]d\eta + \pi\Gamma_s\Big), \tag{4.8}$$

where a bar through the integral sign indicates the principal value, $H(\eta)$ is obtained by multiplying (2.23) with $-\pi$, and $F(x)$ is given by

$$F(x) = F_0(x) - \gamma_0 \int_{-\infty}^{-l} \exp(jk\xi)K(x,\xi)d\xi - \int_{-l}^{l} \gamma(\xi)K(x,\xi)d\xi. \tag{4.9}$$

Here

$$F_0(x) = \gamma_0 \int_{-\infty}^{-l} \frac{\exp(jk\xi)}{x-\xi} d\xi, \tag{4.10}$$

$$K(x,\xi) = -\mathrm{Re}_i \frac{\partial g}{\partial z}(z, \xi - id), \qquad z = x - id. \tag{4.11}$$

(Note that $g(z, z_0)$ here has opposite sign of that in [7].) Near $\tau = 1/4$, the kernel $K(x,\xi)$ may be written

$$K(x,\xi) = \frac{A}{\delta}\exp(-jk_1(x-\xi)) + K_1(x,\xi) + O(\delta), \tag{4.12}$$

where

$$A = 2\pi k_1 \exp(-2k_1 d), \quad \delta = (1-4\tau)^{1/2}, \tag{4.13}$$

and $K_1(x,\xi)$ is the regular part of $K(x,\xi)$. Equation (4.8) may then be written

$$\begin{aligned}\gamma(x) = \frac{1}{\pi^2}(l^2-x^2)^{-1/2}\Big\{ -\frac{A\mathcal{R}}{\delta} ⨍_{-l}^{l} \frac{(l^2-\eta^2)^{1/2}}{x-\eta}\exp(-jk\eta)d\eta + \\ ⨍_{-l}^{l} \frac{(l^2-\eta^2)^{1/2}}{x-\eta}\Big(\int_{-l}^{l} -\gamma(\xi)K_1(\eta,\xi)d\xi + H(\eta) + F_0(\eta) + \tilde{F}_1(\eta)\Big)d\eta - \pi\Gamma_s\Big\},\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}\mathcal{R} &= \int_{-\infty}^{l} \exp(jk_1x)\gamma(x)dx = \int_{-l}^{l}[\exp(jk_1x) - \frac{k}{k+k_1}\exp(-jk_1l)]\gamma(x)dx, \\ \tilde{F}_1(x) &= -\gamma_0 \int_{-\infty}^{-l} \exp(jk\xi)K_1(x,\xi)d\xi.\end{aligned} \tag{4.15}$$

To obtain a non-singular equation, we multiply (4.14) by $\exp(jk_1x)$ and integrate from $-l$ to l, which gives

$$\int_{-l}^{l} \gamma(x)\exp(jk_1x)dx = T_1 + \frac{\Gamma_s}{\pi}\int_{-l}^{l}\frac{\exp(jk_1x)}{(l^2-x^2)^{1/2}}dx$$

$$-\frac{A\mathcal{R}}{\pi^2\delta}\int_{-l}^{l}(l^2-\eta^2)^{1/2}\exp(-jk_1\eta) {⨍}_{-l}^{l} \frac{\exp(jk_1x)}{(l^2-x^2)^{1/2}(x-\eta)}dxd\eta, \qquad (4.16)$$

where in the last term on the right the order of integration is changed. T_1 is given by

$$T_1 = \frac{1}{\pi^2}\int_{-l}^{l}\frac{\exp(jk_1x)}{(l^2-x^2)^{1/2}} {⨍}_{-l}^{l} \frac{(l^2-\eta^2)^{1/2}}{x-\eta}\cdot$$

$$\Big(\int_{-l}^{l} -\gamma(\xi)K_1(\eta,\xi)d\xi + H(\eta) + F_0(\eta) + \tilde{F}_1(\eta)\Big)d\eta\, dx. \qquad (4.17)$$

Noting that ([14], eqs. (44)–(47))

$$\int_{-l}^{l}(l^2-\eta^2)^{1/2}\exp(-jk_1\eta) {⨍}_{-l}^{l} \frac{\exp(jk_1x)}{(l^2-x^2)^{1/2}(x-\eta)}dx\, d\eta$$

$$= \frac{j\pi}{l}\int_0^{k_1l} J_0(\kappa)\int_{-l}^{l}(l^2-\eta^2)^{1/2}\exp(-j\kappa\eta)d\eta\, d\kappa$$

$$= j\pi^2 l\int_0^{k_1l}\frac{J_0(\kappa)J_1(\kappa)}{\kappa}d\kappa$$

$$= j\pi^2 l\{k_1l[J_0^2(k_1l)+J_1^2(k_1l)] - J_0(k_1l)J_1(k_1l)\}\,, \qquad (4.18)$$

we find that (4.16) reduces to

$$\int_{-l}^{l}\gamma(x)\exp(jk_1x)dx = T_1 - \frac{R}{\delta}jg_1(k_1l) + J_0(k_1l)\Gamma_s, \qquad (4.19)$$

where

$$g_1(k_1l) = 2\pi k_1 l\exp(-2k_1d)\{k_1l[J_0^2(k_1l)+J_1^2(k_1l)] - J_0(k_1l)J_1(kl)\}, \qquad (4.20)$$

and $J_0(k_1l)$ $(= (1/\pi)\int_{-l}^{l}(l^2-x^2)^{-1/2}\exp(jk_1x)dx)$ and $J_1(k_1l)$ denote Bessel functions of the first kind of order zero and one, respectively.

We then subtract $\exp(-jk_1l)\Gamma_s k/(k+k_1)$ from both sides of (4.19) and obtain

$$\frac{\mathcal{R}}{\delta}(\delta + jg_1(k_1l)) = T_1 + \Gamma_s\Big(J_0(k_1l) - \frac{k}{k+k_1}\exp(-jk_1l)\Big), \qquad (4.21)$$

giving

$$\frac{\mathcal{R}}{\delta} = \frac{T_1 + \Gamma_s[J_0(k_1l) - k\exp(-jk_1l)/(k+k_1)]}{\delta + jg_1(k_1l)}. \qquad (4.22)$$

By introducing (4.22) into (4.16) we obtain an integral equation where all terms are bounded as $\delta \to 0$, provided that $g_1(k_1 l) \neq 0$. To show that $g_1(k_1 l)$ always is positive we first note that

$$\begin{aligned}&\hat{x}[J_0^2(\hat{x}) + J_1^2(\hat{x})] - J_0(\hat{x})J_1(\hat{x}) \\ &\qquad = (\hat{x} - \tfrac{1}{2})[J_0^2(\hat{x}) + J_1^2(\hat{x})] + \tfrac{1}{2}[J_0(\hat{x}) - J_1(\hat{x})]^2 > 0 \quad \text{for } \hat{x} \geq \tfrac{1}{2}. \end{aligned} \tag{4.23}$$

For $0 < \hat{x} < \frac{1}{2}$, we have that $J_1(\hat{x}) < \hat{x}J_0(\hat{x})$ which gives

$$\hat{x}[J_0^2(\hat{x}) + J_1^2(\hat{x})] - J_0(\hat{x})J_1(\hat{x}) > \hat{x}J_1^2(\hat{x}) > 0 \quad \text{for } \hat{x} < \tfrac{1}{2}. \tag{4.24}$$

Since $k_1 l \neq 0$ for $\tau \to 1/4$ it follows that $g_1(k_1 l) \neq 0$ for all values of the argument.

We have hereby shown that all terms in the integral equation are finite for $\delta \to 0$, even though the cross-section area is zero. The mathematical arguments leading to (4.22) are independent of the circulation Γ_s. The result is in conflict with the claim in [8] that a finite solution exists as $\tau \to 1/4$ if and only if $\Gamma \neq 0$, i.e. the cross-section area is different from zero. The reason for the disagreement is obviously that for bodies with zero cross-section area the proper way of solving the problem is to use a distribution of vortices instead of sources.

It follows now immediately that the far-field amplitude is finite. From [7, eqs. (5.2)–(5.4)] we obtain for the k_1-wave

$$a_1 = \frac{\mathcal{R}}{\delta}\left(\frac{k_1}{g}\right)^{1/2} \exp(-k_1 d), \tag{4.25}$$

and a similar expression for the k_2-wave. We may also use the energy equation (3.5) and the momentum equation (3.6) to derive similar results for the behaviour of the damping forces and the second order force at the critical point as for submerged bodies of non-zero cross-section.

Finally, as $\tau \to 1/4$, the velocity potential $f_1(z)$ is given by

$$\mathrm{Re}_i f_1(z) = -\frac{\mathcal{R}}{\delta} \exp(k_1(y - d - jx)) + O(1). \tag{4.26}$$

which shows that also $f_1(z)$ is finite at the critical frequency.

It is noted from (4.20), (4.22), (4.25) and (4.26) that for deep submergence of the body, the velocity potential and the far-field amplitude are proportional to $\exp(k_1 d)$, similar to what was obtained for a body of finite cross-section. In the present case the body may be situated close to the free surface, without violating the requirements for linarization of the problem. We therefore conclude that the results obtained in this section are valid for bodies situated near or at moderate distances below the free surface.

5 Conclusion

In the present study we consider the wave motion generated by a submerged two-dimensional body, which is translating and performing small oscillations under a free surface. We focus on oscillations with frequencies close to the critical frequency ($\tau = 1/4$). It has been known for some years that the motion due to a physical body is bounded at the critical frequency, in contrast to the motion generated by a moving, oscillating singularity (source, dipole etc.). Also the physical forces are bounded. Their variation with the frequency close to $\tau = 1/4$ may, however, be extremely large. Such behaviour has also been found experimentally by Maruo & Matsunaga [15] in three dimensions for a Lewis hull moving in heave and pitch.

The wave motion may be found by applying a source distribution over the body surface. There are three different routes which may be taken. The first one is to integrate the integral equation directly through the singularity ([5], [6], [7], [13]), which can be performed without difficulty. The second one is to develop the integral equation to a form without any singularity [8] and the third one is to use a Green function which is non-singular at $\tau = 1/4$ and approaches the ordinary Green function away from the singularity. Each of these methods have their advantages and disadvantages. In the first method it is necessary to compute with small steps close to $\tau = 1/4$. In the second method the integral equation is somewhat more complicated than in the first method. This method has, however, the merit that it includes a proof that the solution is bounded at the critical point. These two methods fail for very deep submergences of the body. The third method leads to a similar integral equation as the first method, but is not singular for $\tau = 1/4$. The solution is valid also for deep submergence. The solution contains, however, a small parameter ε, of order the wave amplitude, and is therefore not unique. It seems likely that the result is not sensitively dependent on the choice of ε.

It is proved here in §4 that the motion is finite at $\tau = 1/4$ also for a moving thin foil (flat plate). Since the cross-section of this body is zero, this result is in contradiction to the claim in [8] that the motion is bounded if and only if the cross-section area is non-zero. The reason for this disagreement is that for a body of zero cross-section a distribution of vortices must be used instead of a distribution of sources.

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